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On the Existence of Non-Free Totally Reflexive Modules

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Abstract

For a standard graded Cohen-Macaulay ring R, if the quotient $R/(\underline{x})$ admits non-free totally reflexive modules, where $\underline{x}$ is a system of parameters consisting of elements of degree one, then so does the ring R. A non-constructive proof of this statement was given in [16]. We give an explicit construction of the totally reflexive modules over R obtained from those over $R/(\underline{x})$.

We consider the question of which Stanley-Reisner rings of graphs admit non-free totally reflexive modules and discuss some examples. For an Artinian local ring $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ and containing the complex numbers, we describe an explicit construction of uncountably many non-isomorphic indecomposable totally reflexive modules, under the assumption that at least one such non-free module exists. In addition, we generalize Rangel-Tracy rings. We prove that her results do not generalize. Specifically, the presentation of a totally reflexive module cannot be choosen generically in our generalizations.

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Chapter 1

Introduction

Totally reflexive modules were introduced by Auslander and Bridger in [1], under the name of modules of Gorenstein dimension zero. These modules were used as a generalization of free modules, in order to define a new homological dimension for finitely generated modules over Noetherian rings, called the G-dimension.

In this paper, R and S will denote commutative Noetherian rings. The following theorems are well known [10].

Theorem 1.0.1. *Let R be a commutative Noetherian local ring with residue field k. Then the following conditions are equivalent.*

1. *R is regular*
2. *every finitely generated R module has finite projective dimension*
3. *k has finite projective dimension*

Theorem 1.0.2 (Auslander-Buchsbaum)**.** *Let R be a commutative Noetherian local ring and M an R module with finite projective dimension. Then* $\operatorname{depth} M + \operatorname{pd} M = \operatorname{depth} R$.

Auslander and Bridger generalized these theorems with G-dimension [1].

Theorem 1.0.3. *Let R be a commutative Noetherian local ring with residue field k. Then the following conditions are equivalent.*

1. *R is Gorenstein.*

2. *every finitely generated R module has finite G-dimension*

3. *k has finite G-dimension*

Theorem 1.0.4 (Auslander-Bridger)**.** *Let R be a commutative Noetherian local ring and M an R module with finite G-dimension. Then* $\operatorname{depth} M + G\text{-}\dim M = \operatorname{depth} R$*.*

From Auslander's and Bridger's work it is clear that G-dimension is a more robust invariant for modules in the sense that some modules could have finite G-dimension while having infinite projective dimension. They further showed that $\operatorname{pd} M \leq G\text{-}\dim M$ and equality holds $\operatorname{pd} M$ is finite. It is also not difficult to find modules with finite G-dimension but having infinite projective dimension. For instance over a Gorenstein non-regular ring, all finitely generated modules have finite G-dimension, but not all have finite projective dimension, specifically the residue field.

In [7] it was shown that one can use totally reflexive modules to give a characterization of simple hypersurface singularities among all complete local algebras. It was also shown in [7] that if a local ring is not Gorenstein, then it either has infinitely many indecomposable pairwise non-isomorphic totally reflexive modules, or else it has none other than the free modules. This dichotomy points out that it is important to understand which non-Gorenstein rings admit non-free totally reflexive modules and which do not. This answer is not well understood at the present time. In this paper we study this issue from the point of view of reducing to the case of Artinian rings, and we use this technique to study a class of rings obtained from a graph.

Definition 1.0.5. A finitely generated module M is called *totally reflexive* if there exists an infinite complex of finitely generated free R-modules

$$F: \qquad \cdots \to F_1 \to F_0 \to F_{-1} \to \cdots$$

such that M is isomorphic to $\operatorname{Coker}(F_1 \to F_0)$, and such that both the complex F and the dual $F^* = \operatorname{Hom}_R(F, R)$ are exact.

Such a complex F is called a *totally acyclic complex*. We say that F is a *minimal totally acyclic complex* if the entries of the matrices representing the differentials are in the maximal ideal (or homogeneous maximal ideal in the case of a graded ring). In the case when R does not admit minimal totally acyclic complexes, we say R is *G-regular*.

It is obvious that free modules are totally reflexive. The next easiest example is provided by exact zero divisors, studied under this name in [3]:

Definition 1.0.6. A pair of elements $a, b \in R$ is called a *pair of exact zero divisors* if $\mathrm{Ann}_R(a) = (b)$ and $\mathrm{Ann}_R(b) = (a)$.

Note that if R is an Artinian ring, then one of these conditions implies the other (as it implies that $l((a)) + l((b)) = l(R)$).

If a, b is a pair of exact zero divisors, then the complex

$$\cdots R \xrightarrow{a} R \xrightarrow{b} R \xrightarrow{a} \cdots$$

is a totally acyclic complex, and $R/(a)$, $R/(b)$ are totally reflexive modules.

More complex totally reflexive modules can be constructed using a pair of exact zero divisor, see [9] and [6].

Many properties of commutative Noetherian rings can be reduced to the case of Artinian rings, via specialization. We use this approach in order to study the existence of non-free totally reflexive modules. The following observation is well-known (see Proposition 1.5 in [5]).

Observation 1.0.7. *Let R be a Cohen-Macaulay ring and let M be a non-free totally reflexive R-module. If $\underline{x}$ is a system of parameters in R, then $M/(\underline{x})M$ is a non-free totally reflexive $R/(\underline{x})$-module.*

On the other hand, it follows from [2] that any ring which is an embedded deformation admits non-free totally reflexive modules:

Theorem 1.0.8. *Let $(R, \mathfrak{m})$ be a Cohen-Macaulay module, and let $\underline{x} \subseteq \mathfrak{m}^2$ be a (part of a) system of parameters. Then the quotient $R/(\underline{x})$ admits non-free totally reflexive modules.*

Given a Cohen-Macaulay standard graded or local ring $(R, \mathfrak{m})$, one would like to investigate whether R admits non-free totally reflexive modules via investigating the same issue for specializations $R/(\underline{x})$. In order to use this approach, one needs a converse of (1.0.7). In light of Theorem 1.0.8, such a converse cannot be true if the system of parameters $\underline{x}$ is contained in $\mathfrak{m}^2$ (as in this case $R/(\underline{x})$ always has non-free totally reflexive modules, even if R does not). This converse is proved in ([16], Proposition 4.6)

Theorem 1.0.9. *[16] Let $(S, \mathfrak{m})$ be a local ring, and let $x_1, \ldots, x_d$ be a regular sequence such that $x_i \in \mathfrak{m} \setminus \mathfrak{m}^2$. Let $R = S/(x_1, \ldots, x_d)$. If R has non-free totally reflexive modules, then so does S.*

The proof in ([16]) is non-constructive; in Chapter 2 we give a constructive approach to this result in the graded case, where we indicate how a minimal totally acyclic complex over R can be used to build a minimal totally acyclic complex over S.

Once we have reduced to an Artinian ring, the easiest way to detect totally reflexive modules is when they are given by pairs of exact zero-divisors. Rings with positive dimension usually do not admit a pair of exact zero-divisors. We will give examples where they can be found in specializations, thus allowing us to conclude that the original ring also had non-free totally reflexive modules. The examples that we focus on in Chapter 3 are Stanley-Reisner rings of connected graphs. These are two-dimensional Cohen-Macaulay rings, and, after modding out by a linear system of parameters, they satisfy $\mathfrak{m}^3 = 0$. We will give some necessary conditions for the existence of non-free totally reflexive modules, as well as examples where we can find pairs of exact zero divisors in the specialization.

Most of the constructions of totally reflexive modules in the literature start with a pair of exact zero divisors, which can then be used to construct more complicated modules. We are only aware of one example (Proposition 9.1 in [6]) of a ring which admits non-free totally reflexive modules, but does not have exact zero divisors. This example occurs over a characteristic two field, and can be considered a pathological case (the ring defined by the same equations over a field of characteristic different from two will have exact zero divisors). In Section 4 we provide another, characteristic-free example of a ring that does not have exact zero divisors, but has non-free totally reflexive modules (Example 3.5.1). Moreover, we indicate how to construct infinitely many non-indecomposable non-isomorphic totally reflexive modules over this ring. It is likely that our example can be generalized to a family of rings with these properties.

We consider Artinian local rings $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ which contains the complex numbers, and we describe a construction that gives rise to uncountably many non-isomorphic indecomposable totally reflexive modules, under the assumption that one such non-free module exists. It has been known from [7] that, under the assumptions above, there would be infinitely many such modules, but this is the first time that an explicit construction is provided that does not use a pair of exact zero-divisors.

Chapter 2

Existence of Totally Reflexive Modules under Specializations

2.1 Overview

In this chapter we give techniques for verifying and producing totally reflexive modules under specializations. If x is a regular sequence and M is a totally reflexive R-module, it is known that M/xM is a totally reflexive R/xR-module. The question arises in the other direction. That is, if in the specialization R/xR admits non-free totally reflexive modules, then is it true that R has non-free totally reflexive R-modules? Takashi's paper describes precisely when this phenomenon occurs; however, the proof is non-constructive. We give an algorithm that produces non-free totally reflexive modules in the scenario Takashi describes.

2.2 Algorithm

We begin by giving an algorithmic proof of Takashi's result. This construction will build totally reflexive modules.

Theorem 2.2.1. *Let $S = k \oplus S_1 \oplus S_2 \oplus \cdots$ be a standard graded k-algebra, and let $x_1, \ldots, x_d \in S_1$ be a regular sequence. If the ring $R = S/(x_1, \ldots, x_d)$ admits a minimal totally acyclic complex $(C., \phi.)$, then S also admits a minimal totally acyclic complex.*

Proof. It is enough to prove the case $d = 1$, then induct on d. Let $x_1 := x$. For an element $u \in S$, we will use $\overline{u}$ to denote the image of u in R.

Given

$$\cdots \longrightarrow R^{b_{i+1}} \xrightarrow{\delta_{i+1}} R^{b_i} \xrightarrow{\delta_i} \cdots \tag{2.1}$$

a doubly infinite totally acyclic R-complex, we will construct a doubly infinite totally acyclic S- complex

$$\cdots \longrightarrow S^{2b_{i+1}} \xrightarrow{\epsilon_{i+1}} S^{2b_i} \xrightarrow{\epsilon_i} \cdots \tag{2.2}$$

Let $\tilde{\delta}_i : S^{b_i} \to S^{b_{i-1}}$ denote a lifting of δ_i to S, for all $i \in \mathbb{Z}$. We will view these maps as matrices with entries in S. Since $\delta_i \delta_{i+1} = 0$, it follows that there exists a matrix M_{i+1} with entries in S such that

$$\tilde{\delta}_i \tilde{\delta}_{i+1} = x M_{i+1}. \tag{2.3}$$

We define ϵ_i as follows: if i is even, then

$$\epsilon_i = \begin{bmatrix} \tilde{\delta}_i & x I_{b_{i-1}} \\ M_i & \tilde{\delta}_{i-1} \end{bmatrix},$$

If i is odd,

$$\epsilon_i = \begin{bmatrix} \tilde{\delta}_i & -x I_{b_{i-1}} \\ -M_i & \tilde{\delta}_{i-1} \end{bmatrix}$$

Note that if all the entries of δ_i are in the homogeneous maximal ideal of R for all i, then all the entries of ϵ_i will be in the homogeneous maximal ideal of S (since x has degree one and the entries of $\tilde{\delta}_i \tilde{\delta}_{i+1}$ have degree at least two, equation (2.3) shows that the entries of M_i cannot be units). We check that (4.1) is a complex. Let i be even. We have

$$\epsilon_i \epsilon_{i+1} = \begin{bmatrix} \tilde{\delta}_i \tilde{\delta}_{i+1} - x M_{i+1} & -x\tilde{\delta}_i + x\tilde{\delta}_i \\ M_i \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1} M_{i+1} & -x M_i + \tilde{\delta}_{i-1} \tilde{\delta}_i \end{bmatrix}$$

Using equation (2.3), we see that all entries are zero except possibly $M_i \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1} M_{i+1}$. However, we have

$$x(M_i \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1} M_{i+1}) = (x M_i) \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1}(x M_{i+1}) = \tilde{\delta}_{i-1} \tilde{\delta}_i \tilde{\delta}_{i+1} - \tilde{\delta}_{i-1} \tilde{\delta}_i \tilde{\delta}_{i+1} = 0.$$

The calculation is similar if i is odd.

Now we check that the complex (4.1) is exact. Let i be even, and let $c = [c_1, c_2]^t \in \ker(\epsilon_i)$, where $c_1 \in S^{b_i}$ and $c_2 \in S^{b_{i-1}}$. We have $\tilde{\delta}_i c_1 + x c_2 = 0$, and therefore $\delta_i(\overline{c_1}) = 0$. The exactness of (2.1) implies that there are elements $d_1, d_2 \in S$ such that $c_1 = \tilde{\delta}_{i+1} d_1 - x d_2$.

Define

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} := \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \epsilon_{i+1} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

It is clear that $c_1' = 0$ and $[c_1', c_2']^t \in \ker(\epsilon_i)$. It follows that $x c_2' = 0$. Since $x \in S$ is a regular element, we must have $c_2' = 0$. In other words, $[c_1, c_2]^t = \epsilon_{i+1}[d_1, d_2]^t \in \operatorname{im}(\epsilon_{i+1})$, which is what we wanted to show.

The calculation is similar if i is odd.

We also need to check that the dual of the complex (4.1) is exact. Let i be even and let $[c_1, c_2]^t \in \ker(\epsilon_{i+1}^t)$, where $c_1 \in S^{b_i}$ and $c_2 \in S^{b_{i-1}}$. We have

$$\epsilon_{i+1}^t = \begin{bmatrix} \tilde{\delta}_{i+1}^t & -M_{i+1}^t \\ -x I_{b_i} & \tilde{\delta}_i^t \end{bmatrix}.$$

It follows that $-x c_1 + \tilde{\delta}_i^t c_2 = 0$, so $\delta_i^t(\overline{c_2}) = 0$. Due to the exactness of the dual of (2.1), we have $c_2 = x d_1 + \delta_{i-1}^t d_2$ for some $d_1, d_2 \in S$. Define

$$\begin{bmatrix} c_1' \\ c_2' \end{bmatrix} := \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} - \epsilon_i^t \begin{bmatrix} d_1 \\ d_2 \end{bmatrix}.$$

It is clear that $c_2' = 0$ and $[c_1', c_2']^t \in \ker(\epsilon_{i+1}^t)$. Therefore, we have $x c_1' = 0$. Since $x \in S$ is a regular element, it follows that $c_1' = 0$, and thus $[c_1, c_2]^t = \epsilon_i^t [d_1, d_2]^t \in \operatorname{im}(\epsilon_i^t)$, which is what we wanted to show.

The calculation is similar if i is odd.

□

Specializing a ring is a useful technique and it makes verifying equations easier. The following theorem reduces checking acyclicity of a bounded complex to first specializing and then checking acyclicity.

Theorem 2.2.2. *Let S be a ring, x be a non-zero divisor, $R = S/x$, F be a bounded S-complex of flat modules, $F \otimes_S R$ be acyclic. Then F is acyclic.*

Proof. From the given hypotheses, we can see the Künneth's formula [17] should be applied. We consider the spectral sequence, $E^2_{pq} = Tor^S_p(H_q(F), R)$, and observe that E^2_{pq} converges to $H_{p+q}(F \otimes R) = 0$. Also we have the short exact sequence

$$0 \to S \xrightarrow{x} S \to R \to 0.$$

Therefore, we can conclude $E^2_{pq} = 0$ when $p > 1$. It follows that there is a short exact sequence

$$0 \to E^2_{1,n-1} \to H_n(F \otimes R) \to E^2_{0,n} \to 0.$$

By assumption, we have $H_n(F \otimes R) = 0$. It follows that $E^2_{1,n-1} = E^2_{0,n} = 0$ for all n. We conclude that $E^2_{pq} = 0$ for all p and q; thus, F is acyclic as desired.

□

Chapter 3

Stanley-Reisner Rings for Graphs

3.1 Overview

Stanley-Reisner rings have many easily computable invariants. In the case when the simplex is a graph, we can specialize the Stanley-Reisner ring to a ring with the property $\mathfrak{m}^3 = 0 \neq \mathfrak{m}^2$. Such a ring is called a short-ring. Yoshino gives necessary conditions for when short-rings admit totally reflexive modules [18]. In this chapter we specialize Stanley-Reisner rings of graphs to short rings. Then we convert Yoshino's conditions into necessary properties for graphs for which the corresponding Stanley-Reisner rings admit totally reflexive modules.

An easy way to find totally reflexive modules is to produce exact-zero divisors; albeit it is not necessary for a non G-regular ring to admit exact-zero divisors. In the last section we construct examples of non G-regular rings including one which does not admit exact-zero divisors.

3.2 Properties of Stanley Reisner Rings for Graphs

Let $\Gamma = (V, E)$ be a connected graph, where $V = \{x_1, \ldots, x_n\}$ is the set of vertices, and E is the set of edges. Let k be an infinite field. The Stanley-Reisner ring of Γ over k is

$$R_\Gamma = \frac{k[X_1, \ldots, X_n]}{I_\Gamma}$$

where I_Γ is the ideal generated by all the monomials X_iX_j for which $\{x_i, x_j\} \notin E$, and all monomials $X_iX_jX_k$ with distinct i, j, k. The general theory of Stanley-

Reisner rings (see [4], Corollary 5.3.9) shows that, under the assumption that Γ is connected, R_Γ is a two-dimensional Cohen-Macaulay ring. We investigate the existence of non-free totally reflexive modules for R_Γ via reducing modulo a linear system of parameters. We denote $|V| = n$ and $|E| = e$.

Lemma 3.2.1. *Let $l_1, l_2 \in k[X_1, \ldots, X_n]$ be general linear forms. Then $R := R_\Gamma/(l_1, l_2)$ is an Artinian ring with maximal ideal $\mathfrak{m}$. We have $\mathfrak{m}^3 = 0$, $\dim_k \mathfrak{m}/\mathfrak{m}^2 = n-2$, and $\dim_k \mathfrak{m}^2 = e - n + 1$.*

Proof. Note that the degree two component of R_Γ is generated by $X_1^2, \ldots, X_n^2$, and X_iX_j with $\{x_i, x_j\} \in E$, and the degree three component is generated by $X_1^3, \ldots, X_n^3$, and $X_i^2X_j, X_iX_j^2$ with $\{x_i, x_j\} \in E$. Therefore, the Hilbert series of R_Γ has the form

$$H_{R_\Gamma}(t) = 1 + nt + (n+e)t^2 + (n+2e)t^3 + \cdots$$

Since the images of two general linear form l_1, l_2 are a regular sequence in R_Γ, we have

$$H_R(t) = (1-t)^2 H_{R_\Gamma}(t) = 1 + (n-2)t + (e-n+1)t^2 + 0t^3$$

which proves the claim. □

Observation 3.2.2. *In order for Γ to be connected, we must have $e \geq n-1$. We have $\mathfrak{m}^2 = 0 \Leftrightarrow e = n-1 \Leftrightarrow \Gamma$ is a tree.*

Note that if $\mathfrak{m}^2 = 0$, then it is known that if R is not Gorenstein, then R does not have non-free totally reflexive modules (see [18]). In [18], Yoshino gives the following necessary conditions for an Artinian ring with $\mathfrak{m}^3 = 0$ to have non-free totally reflexive modules:

Theorem 3.2.3. *([18], Theorem 3.1) Let $(R, \mathfrak{m})$ be a non-Gorenstein local ring with $\mathfrak{m}^3 = 0$. Assume that R contains a field k isomorphic to $R/\mathfrak{m}$, and assume that there is a non-free totally reflexive R-module M. Then:*

(1) R has a natural structure of homogeneous graded ring with $R = R_0 \oplus R_1 \oplus R_2$ with $R_0 = k$, $\dim_k(R_1) = r+1$, and $\dim_k(R_2) = r$, where r is the type of R. Moreover, $(0 :_R \mathfrak{m}) = \mathfrak{m}^2$.

(2) R is a Koszul algebra.

(3) M has a natural structure of graded R-module, and, if M is indecomposable, then the minimal free resolution of M has the form

$$\cdots \to R(-n-1)^b \to R(-n)^b \to \cdots \to R(-1)^b \to R^b \to M \to 0.$$

In other words, the resolution of M is linear with constant betti numbers.

Based on Yoshino's result, we conclude that the following are necessary conditions for R_Γ to have non-free totally reflexive modules:

Proposition 3.2.4. *If R_Γ has non-free totally reflexive modules, then the following must hold:*

(a) $e = 2n - 4$.

(b) Γ does not have any cycles of length 3.

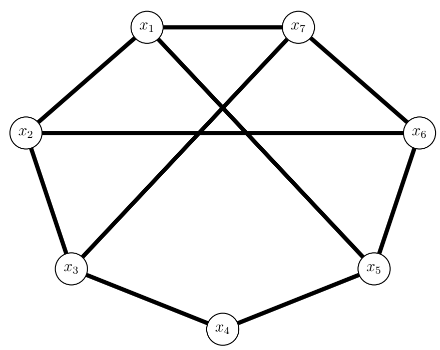
(c) Γ does not have leaves (a leaf is a vertex which belongs to only one edge).

Proof. If R_Γ has non-free totally reflexive modules, then so does $R_\Gamma/(l_1, l_2)$. We apply the necessary conditions from Theorem (3.2.3) to the ring $R = R_\Gamma/(l_1, l_2)$. Part (a) is immediate using the calculation from Lemma (3.2.1). Part (b) is a consequence of the requirement that R is a Koszul algebra, which in particular implies that it has to be defined as a quotient of a polynomial ring by quadratic equation. If the graph Γ has a cycle consisting of vertices x_k, x_l, x_j, then $x_k x_l x_j$ would be one of the defining equations of the Stanley-Reisner ring, and also one of the defining equations of $R_\Gamma/(l_1, l_2)$ (viewed as a quotient of a polynomial ring in two fewer variables). To see (c), assume that there is a vertex x_k of Γ that belongs to only one edge, say $\{x_k, x_j\}$. We can use the equations l_1, l_2 to replace the variables X_j, X_k by linear combinations of the remaining variables, and view $R = R_\Gamma/(l_1, l_2)$ as a quotient of a polynomial ring in these remaining variables. Then the image of X_k annihilates the images of all the variables, and therefore $\overline{X_k} \in (0 :_R \mathfrak{m})$. This contradicts the condition $(0 :_R \mathfrak{m}) = \mathfrak{m}^2$ from Theorem (3.2.3).

□

Before we restrict our attention to bipartite graphs we do point out that it is not necessary for a Stanley-Reisner ring to be bipartite for the ring to admit minimal totally reflexive modules.

Example 3.2.5. Let Γ be the following non-bipartite graph.



Let $a = -16244x_1 + 10394x_2 + 10605x_3 + 9302x_4 + 5353x_5 - 564x_6 - 7086x_7$ and $b = 13837x_1 - 3193x_2 - 7618x_3 + 13982x_4 - 1593x_5 + 9010x_6 + 12626x_7$. These two elements form a regular $R_\Gamma-$sequence. The pair $(x_3 + 10407x_4 + 4274x_5 - 9267x_6 + 1303x_7, x_3 + 6868x_4 - 12743x_5 - 6663x_6 - 10604x_7)$ form a pair of exact zero-divisors in $R_\Gamma/(a, b)$. All these calculations can be verified using Macaulay2.

3.3 Bipartite Graphs and Sufficient Condition for Existence of Exact Zero Divisors

We will be able to obtain better results when the graph Γ is bipartite, i.e. the vertices can be labeled $x_1, \ldots, x_k, y_1, \ldots, y_l$, and all the edges are of the form $\{x_i, y_j\}$ for some

i, j. This in particular implies that the graph does not have cycles of length three (in fact, a graph is bipartite if and only if it does not have any cycles of odd length).

Lemma 3.3.1. *Let* Γ *be a bipartite graph, and let*

$$l_1 = \sum_{i=1}^{k} X_i, l_2 = \sum_{j=1}^{l} Y_j.$$

Then l_1, l_2 *is a system of parameters.* $R = R_\Gamma/(l_1, l_2)$ *can be regarded as a quotient of* $k[X_1, \ldots, X_{k-1}, Y_1, \ldots, Y_{l-1}]$*, and it satisfies*

$$(\overline{X}_{,} \ldots, \overline{X}_{k-1})^2 = (\overline{Y}_1, \ldots, \overline{Y}_{l-1})^2 = 0 \tag{3.1}$$

where $\overline{X_i}, \overline{Y_j}$ *denote the images of* X_i, Y_j *in* R.

For every $u \in R_1$*, we write* $u := x + y$*, where* x *is a linear combination of* $\overline{X}_1, \ldots, \overline{X}_{k-1}$*, and* y *is a linear combination of* $\overline{Y}_1, \ldots, \overline{Y}_{l-1}$*. We define* $u' := x - y$*, and observe*

$$uu' = 0. \tag{3.2}$$

Proof. Since all the edges of the graph are of the form $\{x_i, y_j\}$, it follows that the products of the images in R_Γ of any two distinct X_i, X_j is zero. Moreover, the equation $X_iX_k = 0$ in R_Γ translates to $\overline{X}_i(\sum_{j=1}^{k-1} \overline{X}_j) = 0$ in R, and thus we obtain $\overline{X}_i^2 = 0$ in R. A similar argument shows that $\overline{Y}_j^2 = 0$, and we obtain (3.1). Now (3.1) implies that the product of the images of any three variables in R is zero, and therefore R satisfies $\mathfrak{m}^3 = 0$. Since $\dim(R_\Gamma) = 2$ and R is Artinian, it follows that l_1, l_2 is a system of parameters for R_Γ. The claim (3.2) is obvious. $\square$

We observe that in the case of graded rings with $R_3 = 0$ and $\dim_k(R_2) = \dim_k(R_1) - 1$, there is a connection between existence of exact zero divisors and the Weak Lefschetz Property (WLP). See ([11]) for the general definition and more information regarding WLP. For the case of graded ring with $R_3 = 0$, WLP simply means that there exists an element $x \in R_1$ such that the multiplication by x map

$: R_1 \to R_2$ has maximal rank (if $\dim_k(R_2) \leq \dim_k(R_1)$, maximal rank means that this map is surjective).

Observation 3.3.2. *(a.) Let $R = k \oplus R_1 \oplus R_2$ be a standard graded ring with $R_3 = 0$ and $\dim_k R_2 = \dim_k R_1 - 1$. If R admits a pair of exact zero divisors x, y, then R has WLP.*

(b.) Assume that $R = R_\Gamma/(l_1, l_2)$, where Γ is a bipartite graph, and l_1, l_2 are as in Lemma (3.3.1). If R has WLP, then R admits a pair of exact zero divisors.

Proof. (a). Assume that (x, y) is a pair of exact zero-divisors. By Theorem (3.2.3), we have $x, y \in R_1$. Then the kernel of the map $\cdot x : R_1 \to R_2$ is generated by y, and is therefore 1-dimensional as a k-vector space. It follows that the dimension of the image is $\dim_k R_1 - 1 = \dim_k(R_2)$, and therefore the map is surjective.

(b) Assume that $z \in R_1$ is such that $\cdot z : R_1 \to R_2$ is surjective. Equivalently, the kernel of this map is a one-dimensional vector space. Using the notation from Lemma (3.3.1), we have $zz' = 0$. Therefore, every element in R_1 that annihilates z must be a scalar multiple of z'. We claim that $\mathrm{Ann}_R(z) = (z')$, which will then imply that z, z' is a pair of exact zero divisors. It suffices to prove that $R_2 \subseteq (z')$, or, equivalently, the map $\cdot z' : R_1 \to R_2$ is surjective. We can write $z = x + y$ and $z' = x - y$ as in Lemma (3.3.1). We observe that the map $\cdot z : R_1 \to R_2$ is surjective if and only if R_2 is spanned by $x\overline{Y}_1, \ldots, x\overline{Y}_{l-1}, y\overline{X}_1, \ldots, y\overline{X}_{k-1}$, and the same conclusion holds for the map $\cdot z' : R_1 \to R_2$. Therefore, the multiplication by z map is surjective if and only if the multiplication by z' map is.

□

One might hope that the converse of the statement in Observation (3.3.2) (a) above is true without the extra assumptions we made in Part (b). The example below shows that this is not the case.

Example 3.3.3. Let

$$R = \frac{k[X,Y]}{(X^2 - Y^2, X^2 - XY, X^3)}.$$

Then R satisfies the assumptions from Observation (3.3.2) (a), and R has WLP since the multiplication by $ax + by : R_1 \to R_2$ is surjective as long as $a + b \neq 0$. However, R has a linear socle element, namely $x + y$, which implies that R cannot have exact zero divisors; in fact it cannot have totally reflexive modules, by Theorem (3.2.3)(1).

Now we give a sufficient condition on a graph Γ with $e = 2n-4$ for $R = R_\Gamma/(l_1, l_2)$ to have WLP. When Γ is a bipartite graph satisfying this condition, Observation (3.3.2)(b) implies that R will have a pair of exact zero divisors, and Theorem (2.2.1) will then allow us to conclude that R_Γ has non-free totally reflexive modules.

Proposition 3.3.4. *a. Let Γ be a graph with vertex set $V = \{x_1, \dots, x_n\}$. Assume that $e = 2n-4$ and the vertices can be ordered in such a way that for each $i \geq 3$, there are at least two edges connecting x_i to $\{x_1, \dots, x_{i-1}\}$. Then $R = R_\Gamma/(l_1, l_2)$ has WLP for l_1, l_2 a system of parameters consisting of linear forms with generic coefficients.*

b. Assume moreover that Γ is bipartite with vertex set $\{x_1, \dots, x_k, y_1, \dots, y_l\}$ (where $n = k + l$), and $l_1 = \sum_{i=1}^k x_i, l_2 = \sum_{j=1}^l y_j$. Then $R = R_\Gamma/(l_1, l_2)$ admits a pair of exact zero-divisors.

Proof. a. The calculation of the Hilbert function of R from Lemma (3.2.1) shows that the map $\cdot l : R_1 \to R_2$ has maximal number of generators if and only if it is surjective, which is equivalent to having one dimensional kernel. Fix $l_1 = \sum_{i=1}^n \alpha_i x_i, l_2 = \sum_{i=1}^n \beta_i x_i$, and $l = \sum_{i=1}^n a_i x_i$, where the coefficients α_i, β_i, a_i are generic in k^{3n}. We consider the linear forms $f_1 = \sum_{i=1}^n u_i x_i, f_2 = \sum_{i=1}^n v_i x_i$, and $f = \sum_{i=1}^n w_i x_i$ satisfying

$$l_1 f_1 + l_2 f_2 + l f = 0 \text{ in } R_\Gamma \tag{3.3}$$

Equation (3.3) translates into a system of $e + n$ equations in $3n$ unknowns. The unknowns are the coefficients u_i, v_i, w_i for $i = 1, \dots, n$, and we get one equation corresponding to each edge $\{x_i, x_j\}$ of Γ:

$$\alpha_j u_i + \alpha_i u_j + \beta_j v_i + \beta_i v_j + a_j w_i + a_i w_j = 0, \tag{3.4}$$

obtained by setting the coefficient of $x_i x_j$ in equation (3.3) (these account for $2n-4$ equations), and one equation for each $i = 1, \dots, n$:

$$\alpha_i u_i + \beta_i v_i + a_i w_i = 0, \tag{3.5}$$

which is obtained by setting the coefficient of x_i^2 in equation (3.3) equal to zero. We claim that if the coefficients α_i, β_i, a_i are chosen generically, then the vector space of solutions this system of linear equations is four dimensional. Equations (3.5) give $w_i = -\frac{\alpha_i}{a_i} u_i - \frac{\beta_i}{a_i} v_i$. Plugging this into the equations (3.4), we obtain $2n-4$ equations with $2n$ unknowns, of the form

$$\alpha_{ji} u_i + \alpha_{ij} u_j + \beta_{ji} v_i + \beta_{ij} v_j = 0 \tag{3.6}$$

for each edge $\{x_i, x_j\}$ in Γ, where

$$\alpha_{ij} = \frac{\alpha_i a_j - \alpha_j a_i}{a_j}, \alpha_{ji} = \frac{\alpha_j a_i - \alpha_i a_j}{a_i}, \beta_{ij} = \frac{\beta_i a_j - \beta_j a_i}{a_j}, \beta_{ji} = \frac{\beta_j a_i - \beta_i a_j}{a_i}.$$

By assumption $\{x_1, x_3\}$ and $\{x_2, x_3\}$ are edges. The two equations corresponding to these edges involve 6 unknowns, u_i, v_i for $i = 1, 2, 3,$. The two equations in (3.6) corresponding to the edges $\{x_1, x_3\}, \{x_2, x_3\}$ allow us to solve for u_3, v_3 as linear combinations of u_1, v_1, u_2, v_2 (using Cramer's rule, provided the determinant $\alpha_{13}\beta_{23} - \beta_{13}\alpha_{23}$ is nonzero). Now let $i \geq 3$. By induction, we may assume that u_j, v_j can be expressed as linear combinations of u_1, v_1, u_2, v_2 for all $j \leq i-1$. By assumption, there are two edges that connect x_i to the set $\{x_1, \dots, x_{i-1}\}$. Say that these edges are $\{x_{i_1}, x_i\}$ and $\{x_{i_2}, x_i\}$. The equations in (3.6) corresponding to these edges allow us to solve for u_i, v_i in terms of $u_{i_1}, v_{i_1}, u_{i_2}, v_{i_2}$ (using Cramer's rule, provided that the determinant $\alpha_{i_1 i}\beta_{i_2 i} - \beta_{i_1 1}\alpha_{i_2 i}$ is nonzero), and therefore in terms of u_1, v_1, u_2, v_2 using the inductive hypothesis. It is immediate to see that the conditions

$$\alpha_{i_1 i}\beta_{i_2 i} - \beta_{i_1 i}\alpha_{i_2 i} \neq 0 \tag{3.7}$$

translate into non-vanishing of certain non-trivial polynomials in α_i, β_i, a_i, and thus there is a non-empty open set in k^{3n} such that for any choice of α_i, β_i, a_i in this open set, the vector space of solutions of (3.3) is four dimensional.

Now observe that three of the solutions of equation (3.3) come from the Koszul relations on l_1, l_2, l, so $(f_1^1, f_2^1, f^1) = (-l_2, l_1, 0), (f_1^2, f_2^2, f^2) = (-l, 0, l_1), (f_1^3, f_2^3, f^3) = (0, -l, l_2)$ are linearly independent solutions. Let (f_1^4, f_2^4, f^4) be such that (f_1^j, f_2^j, f^j) where $j = 1, \ldots, 4$ is a basis for the vector space of solutions of (3.3). Consider the map ϕ given by multiplication by the image of $l : R_1 \to R_2$, where $R = R_\Gamma/(l_1, l_2)$. For a linear form $f \in R_\Gamma$, the image of f is in the kernel of this map if and only if there exist $f_1, f_2 \in R_\Gamma$ such that (f_1, f_2, f) is a solution to (3.3). This implies that $f \in (l_1, l_2, f^4)$. Therefore, the kernel of the ϕ is one-dimensional, spanned by the image of f^4, and thus ϕ is surjective.

(b) We need to check that the choice of $l_1 = \sum_{i=1}^k x_i, l_2 = \sum_{j=1}^l y_j$ allows us to choose $l = \sum_{i=1}^k a_i x_i + \sum_{j=1}^l a'_j y_j$ such that the determinants in (3.7) are non-zero. With notation as above, we have $\alpha_i = 1, \beta_i = 0$ for $1 \leq i \leq k$, $\alpha_i = 0, \beta_i = 1$ for $k+1 \leq i \leq n$. The conditions (3.7) need to be checked whenever $\{i_1, i\}$ and $\{i_2, i\}$ are edges of Γ. Due to the bipartite nature of the graph, this means that we have either $i_1, i_2 \leq k$ and $i \geq k+1$, or $i_1, i_2 \geq k+1$ and $i \leq k$. In the first case, we have $\alpha_{i_1 i} = \dfrac{a_{i_1}}{a_i}, a_{i_2 i} = \dfrac{a_{i_2}}{a_i}, \beta_{i_1 i} = \beta_{i_2 i} = 0$, and (3.7) becomes $a_{i_1} \neq a_{i_2}$. The second case is similar.

□

3.4 Conditions for Bipartites that Do Not Admit Exact Zero Divisors

Now we give a condition for a bipartite graph Γ that implies that the ring $R = R_\Gamma/(l_1, l_2)$ does not have exact zero divisors. This will be used in the next section to construct an example of a ring that has no exact zero divisors, but has non-free totally reflexive modules.

Proposition 3.4.1. *Let* Γ *be a bipartite graph with* $e = 2n - 4$, $n = k + l$, *and a vertex set* $V = \{x_1, \ldots, x_k, y_1, \ldots, y_l\}$. *Assume that there exist* $i \in \{1, \ldots, k\}$ *and* $j \in \{1, \ldots, l\}$ *such that the subgraph induced on* $V \setminus \{x_i, y_j\}$ *is disconnected. Then* $R = R_\Gamma/(l_1, l_2)$ *does not have exact zero divisors.*

Proof. Without loss of generality, we may assume $i = k, j = l$. Then the maximal ideal of R is generated by the images $\overline{X}_1, \ldots, \overline{X}_{k-1}, \overline{Y}_1, \ldots, \overline{Y}_{l-1}$ of the variables corresponding to the vertices in $V \setminus \{x_k, y_l\}$. Since the graph induced on $V \setminus \{x_k, y_l\}$ is disconnected, we may partition the set of vertices into disjoint sets A, B such that there is no edge connecting any vertex of A to any vertex of B. Let $\mathfrak{a}, \mathfrak{b}$ denote the ideals of R generated by the images of the variables corresponding to vertices in A and B respectively. Then we have $\mathfrak{a}\mathfrak{b} = 0$, and $\mathfrak{a} + \mathfrak{b} = \mathfrak{m}$. Assume that u, v is a pair of exact zero divisors consisting of linear elements in R. Then we can write $u = u_\mathfrak{a} + u_\mathfrak{b}$ with $u_\mathfrak{a} \in \mathfrak{a}$ and $u_\mathfrak{b} \in \mathfrak{b}$. Using the notation from Lemma (3.3.1), we also have $u'_\mathfrak{a} \in \mathfrak{a}$, $u'_\mathfrak{b} \in \mathfrak{b}$. Then we have $uu'_\mathfrak{a} = u_\mathfrak{a}u'_\mathfrak{a} + u_\mathfrak{b}u'_\mathfrak{a} = 0$, where the first term is zero from (3.2), and the second term is zero because $\mathfrak{a}\mathfrak{b} = 0$. Similarly, we have $uu'_\mathfrak{b} = 0$. This shows that u cannot be part of a pair of exact zero-divisors.

□

We replace the property that $\mathfrak{m} = \mathfrak{a} + \mathfrak{b}$ and $\mathfrak{a}\mathfrak{b} = 0$ from the proof of Proposition (3.4.1) with the stronger condition that there are non-zero ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{b}$.

It follows from Proposition (3.2) in [13] that R is a fiber product, the results of [12] (see also Corollary 3.8 in [13]) imply that it does not have any non-free totally reflexive modules unless it is Gorenstein. This holds without the assumption that $\mathfrak{m}^3 = 0$, but we provide a more direct proof for the case $\mathfrak{m}^3 = 0$ below.

Observation 3.4.2. *Let R be a non-Gorenstein ring with $\mathfrak{m}^3 = 0$ and assume that there are non-zero ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}$ such that $\mathfrak{m} = \mathfrak{a} \oplus \mathfrak{b}$.*

Then R does not have non-free totally reflexive modules.

Proof. Assume that R has non-free totally reflexive modules. Condition (1) from Theorem (3.2.3) tells us that we may assume $\nu(\mathfrak{m}) \geq 3$. We know from Theorem (3.2.3) that the resolution of a totally reflexive module must have constant Betti numbers B and matrices D_i consisting of linear forms in R. We can write $D_i = D_{i,\mathfrak{a}} + D_{i,\mathfrak{b}}$ where $D_{i,\mathfrak{a}}$ has entries in $\mathfrak{a}$, and $D_{i,\mathfrak{b}}$ has entries in $\mathfrak{b}$. Every vector $\mathfrak{u}$ with entries in $\mathfrak{m}$ can be written as $u_\mathfrak{a} + u_\mathfrak{b}$, where $u_\mathfrak{a}$ has entries in $\mathfrak{a}$ and $u_\mathfrak{b}$ has entries in $\mathfrak{b}$. Note that

$$D_i u = D_{i,\mathfrak{a}} u_\mathfrak{a} + D_{i,\mathfrak{b}} u_\mathfrak{b},$$

and $u \in \ker(D_i) \Leftrightarrow u_\mathfrak{a} \in \ker(D_{i,\mathfrak{a}})$ and $u_\mathfrak{b} \in \ker(D_{i,\mathfrak{b}}) \Leftrightarrow u_\mathfrak{a}, u_\mathfrak{b} \in \ker(D_i)$. Since the columns of D_{i+1} span $\ker(D_i)$, it follows that we can write D_{i+1} as a matrix in which every column has either all entries in $\mathfrak{a}$ or all entries in $\mathfrak{b}$. Say that there are n_i columns of the first type, and n'_i columns of the second type, where $n_i + n'_i = B$. The columns of D_{i+1} that have all entries in $\mathfrak{a}$ are annihilated by evey element of $\mathfrak{b}$, and the columns of D_{i+1} that have all entries in $\mathfrak{a}$ are annihilated by every element of $\mathfrak{b}$. Say that $\nu(\mathfrak{a}) = a$ and $\nu(\mathfrak{b}) = b$. Then we have $bn_i + an'_i$ linearly independent relations on the columns of D_{i+1} described in the previous sentence. It follows that $B \geq bn_i + an'_i$. Since $n_i + n'_i = B$, and $a, b \geq 1$, this is only possible if $a = b = 1$. This would contradict the assumption that $\nu(\mathfrak{m}) \geq 3$.

□

Examples of rings R satisfying the hypothesis in (3.4.2) are obtained from bipartite graphs Γ that satisfy the assumption in Proposition 3.4.1 for some i, j , and also have the property that x_i is connected to all of $y_1, \ldots, y_l$, and y_j is connected to all of $x_1, \ldots, x_k$.

This illustrates the fact that the necessary conditions in Theorem 3.2.3 are far from being sufficient.

Observation 3.4.3. *Given a bipartite graph* Γ *that satisfies* $e = 2n - 4$, *one is led to wonder whether one of the hypothesis in Proposition (3.3.4) or the hypothesis in Proposition (3.4.1) must hold. We have not been able to establish this or find a counterexample.*

3.5 A Non $G-$Regular ring with no exact zero divisors

In this section we study an example of a ring $(R, \mathfrak{m})$ with $\mathfrak{m}^3 = 0$ such that $\mathfrak{m} = \mathfrak{a} + \mathfrak{b}$ for two ideals $\mathfrak{a}, \mathfrak{b} \subseteq \mathfrak{m}$ which satisfy $\mathfrak{a}\mathfrak{b} = (0)$, but $\mathfrak{a} \cap \mathfrak{b} \neq (0)$. Proposition (3.4.1) can be applied to show that this ring does not have exact zero divisors. We will give a construction that produces infinitely many non-isomorphic indecomposable totally reflexive modules over this ring. It is theoretically known for a non-Gorenstein ring that if it has one non-free totally reflexive module, then it must have infinitely many non-isomorphic indecomposable totally reflexive modules; see [7]. However, most concrete constructions that give rise to infinitely many such modules rely on the existence of a pair of exact zero divisors; see [6], [14]. The example we study here shows how such a construction can be achieved in the absence of exact zero divisors.

Example 3.5.1. Let

$$R = \frac{k[x_1, \cdots, x_4, y_1, \cdots, y_4]}{(x_1, \cdots, x_4)^2 + (y_1, \cdots, y_4)^2 + I},$$

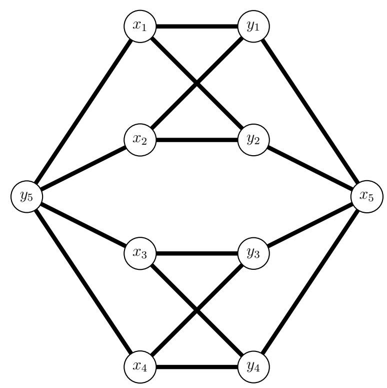
where $I = (x_1, x_2)(y_3, y_4) + (x_3, x_4)(y_1, y_2) + ((\sum_{i=1}^{4} x_i)(\sum_{j=1}^{4} y_j))$.

Then R does not have exact zero divisors, but it has non-free totally reflexive modules.

Throughout this section, R will denote the ring described in Example (3.5.1). The proof of the fact that R does not have exact zero divisors is shown in Construction

3.5.2, and examples of non-free totally reflexive R- modules are shown in Construction 3.5.5. Lemmas 3.5.3 and 3.5.4 prove that the modules constructed in Construction 3.5.5 are indeed totally reflexive.

Construction 3.5.2. Let Γ be the bipartite graph with vertices $\{x_1, \cdots, x_5, y_1, \cdots, y_5\}$ and edges $\{x_1, y_1\}$, $\{x_1, y_2\}$, $\{x_2, y_1\}$, $\{x_2, y_2\}$, $\{x_3, y_3\}$, $\{x_3, y_4\}$, $\{x_4, y_3\}$, $\{x_4, y_4\}$, and $\{x_i, y_5\}, \{x_5, y_j\}$ for all $i, j \in \{1, 2, 3, 4\}$.



Note that removing the vertices x_5, y_5 yields a disconnected graph, with connected components $\{x_1, x_2, y_1, y_2\}$ and $\{x_3, y_3, x_4, y_4\}$ (which are complete bipartite subgraphs).

We have $R = R_\Gamma/(l_1, l_2)$, where $l_1 = \sum_{i=1}^{5} x_i, l_2 = \sum_{j=1}^{5} y_j$.

Proposition (3.4.1) shows that R does not have exact zero divisors.

Letting $\mathfrak{a} := (x_1, x_2, y_1, y_2)$ and $\mathfrak{b} := (x_3, x_4, y_3, y_4)$, we have

$$\mathfrak{m} = \mathfrak{a} + \mathfrak{b}, \mathfrak{a}\mathfrak{b} = (0), \text{and } \mathfrak{a} \cap \mathfrak{b} = (\delta), \tag{3.8}$$

where $\delta = (\sum_{i=1}^{4} x_i)(\sum_{j=1}^{4} y_j)$. The number of vertices of Γ is 10 and the number of edges is 16, so the requirement $e = 2n - 4$ is satisfied. This means that $\dim_k(R_2) = \dim_k(R_1) - 1$.

We let $\mathfrak{a}_i$ denote the vector space spanned by monomials of degree i in x_1, x_2, y_1, y_2, and $\mathfrak{b}_i$ the vector space spanned by the monomials of degree i in x_3, x_4, y_3, y_4 for $i = 1, 2$.

Now, let's construct infinitely many totally reflexive modules.

Lemma 3.5.3. *Let A_0, B_0 denote 2×2 matrices of linear forms such that the entries of A_0 are in $\mathfrak{a}$ and the entries of B_0 are in $\mathfrak{b}$. Assume that the maps $\tilde{A}_0 : (\mathfrak{a}_1)^2 \to (\mathfrak{a}_2)^2$ induced by multiplication by A_0 and $\tilde{B}_0 : (\mathfrak{b}_1)^2 \to (\mathfrak{b}_2)^2$ induced by multiplication by B_0 are injective.*

Consider the map $\tilde{A}_0 + \tilde{B}_0 : (R_1)^2 \to (R_2)^2$. Then $\ker(\tilde{A}_0 + \tilde{B}_0)$ is generated by two vectors $\boldsymbol{c}_1 + \boldsymbol{d}_1$ and $\boldsymbol{c}_2 + \boldsymbol{d}_2$ with linear entries,where $\boldsymbol{c}_1, \boldsymbol{c}_2$ have entries in $\mathfrak{a}$, and $\boldsymbol{d}_1, \boldsymbol{d}_2$ have entries in $\mathfrak{b}$.

Let A_1, B_1 denote the matrices with columns $\boldsymbol{c}_1, \boldsymbol{c}_2$ and $\boldsymbol{d}_1, \boldsymbol{d}_2$ respectively. If the maps $\tilde{A}_1 : (\mathfrak{a}_1)^2 \to (\mathfrak{a}_2)^2, \tilde{B}_1 : (\mathfrak{b}_1)^2 \to (\mathfrak{b}_2)^2$ are also injective, then we have an exact complex

$$R^2 \xrightarrow{A_1+B_1} R^2 \xrightarrow{A_0+B_0} R^2. \tag{3.9}$$

Note: We view $\tilde{A}_0, \tilde{B}_0$, etc. as maps of vector spaces, and A_0, B_0, etc. as maps of free R-modules.

Proof. Note that $\mathfrak{a}_i, \mathfrak{b}_i$ have vector space dimension 4 for $i = 1, 2$. Therefore the injectivity assumption implies that $\tilde{A}_0, \tilde{B}_0$ are bijective. An arbitrary vector in R^2 with entries consisting of linear forms can be written as $\mathbf{c} + \mathbf{d}$, with $\mathbf{c} \in (\mathfrak{a}_1)^2$ and $\mathbf{d} \in (\mathfrak{b}_1)^2$. Since $A_0\mathbf{d} = B_0\mathbf{c} = 0$, we have

$$\mathbf{c} + \mathbf{d} \in \ker(A_0 + B_0) \Leftrightarrow A_0\mathbf{c} = -B_0\mathbf{d},$$

and if that is the case, then the entries of $A_0\mathbf{c}$ and $B_0\mathbf{d}$ must be in (δ), and we have

$$A_0\mathbf{c} = -B_0\mathbf{d} = \begin{pmatrix} \alpha\delta \\ \beta\delta \end{pmatrix}$$

with $\alpha, \beta \in k$. The injectivity assumptions imply that there are unique $\mathbf{c}_1, \mathbf{c}_2, \mathbf{d}_1, \mathbf{d}_2$ such that

$$A_0\mathbf{c}_1 = -B_0\mathbf{d}_1 = \begin{pmatrix} \delta \\ 0 \end{pmatrix}, \qquad A_0\mathbf{c}_2 = -B_0\mathbf{d}_2 = \begin{pmatrix} 0 \\ \delta \end{pmatrix} \tag{3.10}$$

It is now easy to check that $\ker(\tilde{A}_0 + \tilde{B}_0)$ is spanned by $\mathbf{c}_1 + \mathbf{d}_1, \mathbf{c}_2 + \mathbf{d}_2$.

It is clear from construction that (3.9) is a complex. Recall that $\dim_k(R_2) = \dim_k(R_1) - 1$. As above, the injectivity assumptions for $\tilde{A}_1$ and $\tilde{B}_1$ imply that $\tilde{A}_1 + \tilde{B}_1 : (R_1)^2 \to (R_2)^2$ has a two dimensional kernel. Since $\dim_k((R_1)^{\oplus 2}) = \dim_k((R_2)^{\oplus 2}) + 2$, it follows that $\tilde{A}_1 + \tilde{B}_1$ is surjective. On the other hand, $\ker(A_0 + B_0)$ consists of $\ker(\tilde{A}_0 + \tilde{B}_0)$ in degree one, and all of $R_2^{\oplus 2}$ in degree two. Therefore the surjectivity of $\tilde{A}_1 + \tilde{B}_1$, together with the fact that the image of $A_1 + B_1$ contains the kernel of $\tilde{A}_0 + \tilde{B}_0$ by construction show the exactness of (3.9). □

Lemma 3.5.4. *Assume that there is a doubly infinite sequence of* 2×2 *matrices* A_n, B_n *for* $n \in \boldsymbol{Z}$ *with the entries of* A_n *in* $\mathfrak{a}_1$ *and the entries of* B_n *in* $\mathfrak{b}_1$*, such that* $\tilde{A}_n, \tilde{A}_n^t : (\mathfrak{a}_1)^2 \to (\mathfrak{a}_2)^2$ *and* $\tilde{B}_n, \tilde{B}_n^t : (\mathfrak{b}_1)^2 \to (\mathfrak{b}_2)^2$ *are injective maps, and* $(A_n + B_n)(A_{n+1} + B_{n+1}) = 0$ *for all* $n \in \boldsymbol{Z}$*.*

Then we have a doubly infinite acyclic complex

$$\mathcal{F}. \qquad \cdots R^2 \xrightarrow{A_{n+1}+B_{n+1}} R^2 \xrightarrow{A_n+B_n} R^2 \xrightarrow{A_{n-1}+B_{n-1}} \cdots$$

whose dual is also acyclic. Any cokernel module in $\mathcal{F}.$ *will be a non-free totally reflexive* R*-module.*

Proof. The acyclicity of the complex $\mathcal{F}.$ was proved in Lemma (3.5.3). In order to see that the dual is also acyclic, note that Lemma (3.5.3) applies to A_{n+1}^t, B_{n+1}^t used in the roles of A, B, and therefore the kernel of $\tilde{A}_{n+1}^t + \tilde{B}_{n+1}^t$ is spanned by two vectors with linear entries. Since we know $(A_{n+1}^t + B_{n+1}^t)(A_n^t + B_n^t) = 0$, it follows that A_n^t, B_n^t can be used in the roles of A_1, B_1. □

Construction 3.5.5. Now we provide an explicit construction that satisfies all the required conditions in Lemma (3.5.4). Let

$$A_n = \begin{pmatrix} x_1 + x_2 + y_1 + y_2 & x_1 - x_2 + y_1 - y_2 \\ x_1 - x_2 + y_1 - y_2 & x_1 + x_2 - y_1 - y_2 \end{pmatrix},$$

$$B_n = \begin{pmatrix} x_3 + x_4 + y_3 + y_4 & x_3 - x_4 + y_3 - y_4 \\ x_3 - x_4 + y_3 - y_4 & x_3 + x_4 - y_3 - y_4 \end{pmatrix}$$

when n is even, and

$$A_n = \begin{pmatrix} x_1 + x_2 + y_1 + y_2 & x_1 - x_2 - y_1 + y_2 \\ x_1 - x_2 - y_1 + y_2 & x_1 + x_2 - y_1 - y_2 \end{pmatrix},$$

$$B_n = \begin{pmatrix} x_3 + x_4 + y_3 + y_4 & x_3 - x_4 - y_3 + y_4 \\ x_3 - x_4 - y_3 + y_4 & x_3 + x_4 - y_3 - y_4 \end{pmatrix}$$

when n is odd.

All the requirements can be checked by direct calculation.

3.6 Constructing uncountably many totally reflexive modules

In this section we consider graded Cohen-Macaulay rings $(R, \mathfrak{m})$ that contain the complex numbers and specialize to a ring with $\mathfrak{m}^3 = 0$ when modding out a linear regular sequence. For such rings, we show that if there are non-free totally reflexive modules, then there are uncountably many non-isomorphic indecomposable ones. Moreover, totally reflexive modules over such rings can be constructed as cokernels of matrices with generic linear entries Note that the existence of infinitely many such modules was known from [7]; the uncountability is an improvement of that statement. This improvement is relevant in view of the theory Dao and Takahashi of radius of a category, applied to the theory of totally reflexive modules over the rings under consideration (see Corollary 3.6.3).

Theorem 3.6.1. *Let $(R, \mathfrak{m})$ be a standard graded non-Gorenstein ring with $\mathfrak{m}^3 = 0$ and containing $\mathbb{C}$, the field of complex numbers. Assume that R admits non-free totally reflexive modules. Then there are uncountably many non-isomorphic indecomposable totally reflexive modules.*

More precisely, let $R = k[x_1, \ldots, x_n]/I$ and let b be the smallest number of generators of a non-free totally reflexive R-module. Think of the set of all $b \times b$ matrices with linear entries in R as being parametrized by $\mathbb{C}^{nb^2}$ (each matrix corresponds to the vector which records the coefficients of the linear entries). Then there are countably many Zariski open sets $\mathcal{U}_k$ in $\mathbb{C}^{nb^2}$ such that if $A \in \cap_k \mathcal{U}_k$, then $\operatorname{coker}(A)$ *is a totally reflexive R-module.*

Proof. We explain how the first claim in the statement follows from the second. The assumption that R admits a non-free totally reflexive module will imply that the Zariski open sets $\mathcal{U}_k$ are non-empty. A countable union of proper Zariski closed sets in $\mathbb{C}^{nb^2}$ is a set of measure zero, and therefore its complement is uncountable. The modules we will construct in the proof of the second claim will be syzygies in a totally acyclic complex with constant betti numbers b. Thus they are indecomposable (since they have a minimal number of generators). We claim that there are uncountably many choices of A that give rise to mutually non-isomorphic cokernels. Let A, A' be $b \times b$ matrices with linear entries. Then $\operatorname{coker}(A) \cong \operatorname{coker}(A')$ if and only if there exist invertible $b \times b$ matrices U, V such that $UA = A'V$. Let $(u)_{ij}, (v)_{ij}$ denote the degree zero components of U, V. Let the (i, j) entry of A be $\sum_{k=1}^{n} a_{ij}^{k} x_k$ and the (i, j) entry of A' be $\sum_{k=1}^{n} a_{ij}^{\prime k} x_k$. Setting the linear components of the entries of UA equal to those of the entries of $A'V$ and identifying the coefficients of each x_k gives rise to equations

$$\sum_{l=1}^{b} u_{il} a_{lj}^{k} = \sum_{l=1}^{b} a_{il}^{\prime k} v_{lj}$$

for every $i, j \in \{1, \ldots, b\}$ and every $k \in \{1, \ldots, n\}$. View u_{ij}, v_{ij} as unknown; there are nb^2 equations and $2b^2$ unknowns. Since U, V are invertible, this system must have nontrivial solutions. We have $n \geq 3$, since otherwise the Hilbert function of R given by Theorem (3.2.3) would force R to be Gorenstein. Therefore, the minors of size $(2b^2 + 1) \times (2b^2 + 1)$ of the resulting matrix of coefficients must be zero. These minors are polynomials in $a_{ij}^{k}, a_{ij}^{\prime k}$. Therefore, if we fix a matrix A, then the set of all the matrices A' that have $\operatorname{coker}(A') \cong \operatorname{coker}(A)$ belong to a Zariski closed set in $\mathbb{C}^{nb^2}$. If there were only countably many isomorphism classes of modules obtained as cokernels of matrices in $\cap_k \mathcal{U}_k$, it would follows that $\mathbb{C}^{nb^2}$ can be obtain as a union of countably many proper Zariski closed sets. This is a contradiction.

Now we prove the second claim. A $b \times b$ matrix with entries in R can be viewed as a R-module homomorphism $A : R^b \to R^b$. Since the entries of A are linear, it also gives rise to a linear map of vector spaces which we denote $\tilde{A} : (R_1)^b \to (R_2)^b$. We have $n = \dim_{\mathbf{C}}(R_1)$, and, by Theorem (3.2.3), $\dim_{\mathbb{C}}(R_2) = n - 1$. The Zariski open set $\mathcal{U}_0$ is defined as the set of matrices A such that $\tilde{A} : (R_1)^b \to (R_2)^b$ is surjective, and the columns of A are linearly independent in $(R_1)^b$. We check that this is indeed a Zariski open set. The linear independence of the columns is clearly an open condition. We know from Theorem (3.2.3) that we can write $R = \dfrac{k[x_1, \ldots, x_n]}{(p_1, \ldots, p_r)}$, where $p_1, \ldots, p_r$ are polynomials of degree two. Let P_2 denote the degree two component of the polynomial ring $P = k[x_1, \ldots, x_n]$ and let $E_1, \ldots, E_b$ denote the standard basis vectors in $\mathbb{C}^b$.

$\tilde{A}$ is surjective if and only if the vectors

$$\{x_i \operatorname{col}_j(\tilde{A}),\ p_l E_j \,|\, i = 1, \ldots, n, j = 1, \ldots, b, l = 1, \ldots, r\} \subseteq P_2^b \tag{3.11}$$

span P_2^b. We identify each vector in P_2^b with a vector in $\mathbb{C}^{Nb}$ (by choosing an ordering of the monomials in P_2 and recording the coefficients of each component), where $N = \begin{pmatrix} n+1 \\ 2 \end{pmatrix}$. Note that $\dim_{\mathbb{C}}(R_2) = n - 1 = N - r$. Form a matrix with $bn + br = b(N+1)$ columns and bN rows with entries in $\mathbb{C}$ by recording each vector in (3.11) as a vector in $\mathbb{C}^{Nb}$ via this identification. Surjectivity of $\tilde{A}$ translates into the condition that this matrix has maximal number of generators. This is obviously an open condition in the coefficients of the entries of A.

The Hilbert function of the ring R shows that the surjectivity of $\tilde{A}$ is equivalent to $\dim_{\mathbb{C}}(\ker(\tilde{A})) = b$. Form a matrix A_1 by using a spanning set of $\ker(\tilde{A})$ in R_1^b as columns. A_1 is a $b \times b$ matrix with entries consisting of linear forms in R. The

Zariski open set $\mathcal{U}_1$ is defined as the set of matrices A such that $A_1 \in \mathcal{U}_0$. In order to see that this is a Zariski open set, it is enough to check that the entries of A_1 are obtained as polynomials in the entries of A. This is obvious since the entries of A_1 are obtained by solving linear equations with coefficients obtained from the entries of A. The Zariski open sets $\mathcal{U}_k$ for $k \geq 2$ are defined recursively as the set of matrices A such that $A_k \in \mathcal{U}_0$, where A_k is defined recursively as the matrix whose columns are a spanning set for $\ker(\tilde{A}_{k-1})$. Also consider the Zariski open sets $\mathcal{U}'_k$ obtained from the transposes of these matrices: $A \in \mathcal{U}'_k \Leftrightarrow A^t_k \in \mathcal{U}_0$.

Now we construct matrices A_k for $k \leq -1$. Assume that $A \in \mathcal{U}'_0$, so the transpose $\tilde{A}^t : (R_1)^b \to (R_2)^b$ is surjective, and let A'_{-1} denote the $b \times b$ matrix with columns equal to a spanning set for $\ker(\tilde{A}^t)$. The coefficients in the linear entries of A'_{-1} can be obtained as polynomials in the entries of A. Let $A_{-1} := (A'_{-1})^t$. Let $\mathcal{U}_{-1}$ denote the Zariski open set of matrices A that yield $A_{-1} \in \mathcal{U}_0$ and let $\mathcal{U}'_{-1}$ denote the Zariski open set of matrices A that yield $A'_{-1} \in \mathcal{U}_0$. The construction is continued recursively: in order to construct A_{-k-1}, assume that there are Zariski open sets $\mathcal{U}_{-k}$ and $\mathcal{U}'_{-k}$ such that if $A \in \mathcal{U}_{-k} \cup \mathcal{U}'_{-k}$, then we have $\tilde{A}_{-k}, \tilde{A}^t_{-k} : (R_1)^b \to (R_2)^b$ surjective. We let A'_{-k-1} denote the matrix with columns obtained as a spanning set of $\ker(A_{-k})^t$, and $A_{-k-1} := (A'_{-k-1})^t$.

We claim that if $A \in \cap_{k \in \mathbf{Z}}(\mathcal{U}_k \cap \mathcal{U}'_k)$, then there is a doubly infinite complex consisting of the free modules R^b and differentials given by the matrices A_k , and this complex is totally acyclic. Let $k \geq 0$. We have

$$\ker(A_k) = \ker(\tilde{A}_k) \cup (R_2)^b,$$

and

$$\operatorname{im}(A_{k+1}) = R \text{ span of } \operatorname{im}(\tilde{A}_{k+1}) = R \text{ span of } \ker(\tilde{A}_k).$$

It follows that $A_k A_{k+1} = 0$, and exactness follows since the surjectivity of $\tilde{A}_{k+1}$ implies that $(R_2)^b \subseteq \operatorname{im}(A_{k+1})$.

Now let $j := -k - 1 \leq -1$. We want to see that $A_j A_{j+1} = 0$. This is equivalent to $A^t_{-k} A^t_{-k-1} = 0$. By construction, $A^t_{-k-1} = A'_{-k-1}$, and we have $\text{im}(A'_{-k-1}) \subseteq \ker(A^t_{-k})$, which gives us the desired conclusion. To prove exactness, we note as before that $\ker(A_j) = \ker(\tilde{A}_j) \cup (R_2)^b$. The choice of A guarantees that $\tilde{A}_{j+1} : (R_1)^n \to (R_2)^n$ is surjective, so $(R_2)^b \subseteq \text{im}(A_{j+1})$. Moreover, the subspace of $(R_1)^b$ spanned by the columns of A_{j+1} contains $\ker(\tilde{A}_j)$, and they are both b-dimensional.

□

Corollary 3.6.2. *Let $(R, \mathfrak{m})$ be a d dimensional graded Cohen-Macaulay ring over the field of complex numbers $\mathbb{C}$. Assume that $\mathfrak{m}^3 \subseteq (x_1, \ldots, x_d)$, where $x_1, \ldots, x_d$ is a linear system of parameters. If R admits non-free totally reflexive modules, then there are uncountably many mutually non-isomorphic indecomposable totally reflexive modules.*

Proof. The case $d = 0$ is the content of Theorem (3.6.1). We will prove the case $d = 1$. The general case will then follow by induction on d. Let x denote a linear parameter and let $R' = R/(x)$. If M is a non-free totally reflexive R-module, then $M' = M/(x)M$ is a non-free totally reflexive $R/(x)$-module. Choose M to have the smallest number of generators among non-free indecomposable totally reflexive R-modules. Then M' will also have number of generators b. The proof of theorem (3.6.1) shows that we can construct uncountably many mutually non-isomorphic totally reflexive modules with number of generators b. We use the construction (3.5.2) from Section 2 to build totally reflexive R-modules from each such R' module. We claim that there are uncountably many choices of M' that give rise to mutually non-isomorphic R-modules via construction (3.5.2). Let A, B denote two presentation matrices of non-isomorphic totally reflexive R'-modules with number of generators b. Assume that A occurs as part of a totally acyclic complex as the map $\delta_1 : F_1 \to F_0$, and B occurs as part of a totally acyclic complex as the map $\delta'_0 : F'_0 \to F'_{-1}$. Let $\tilde{A}, \tilde{B}$ denote liftings of A, B to R.

Construction (**??**) gives matrices

$$E = \begin{bmatrix} \tilde{A} & -xI_b \\ -C & \tilde{\delta}_0 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} \tilde{\delta}_1' & -xI_b \\ -D & \tilde{B} \end{bmatrix},$$

each of which is part of a totally acyclic complex of R-modules. As in the proof of theorem (3.6.1), if $\operatorname{coker}(E) \cong \operatorname{coker}(F)$, then there exist invertible matrices U, V such that $EU = VF$. Representing U, V in block form, we can write

$$U = \begin{bmatrix} U_1 & U_2 \\ U_2 & U_4 \end{bmatrix}, \qquad V = \begin{bmatrix} V_1 & V_2 \\ V_3 & V_4 \end{bmatrix}$$

and the requirement that $EU = VF$ implies that $xV_1 + V_2\tilde{B} = \tilde{A}U_2 + xU_4$. Modulo x, we have $\overline{V_2}B = A\overline{U_2}$. The first part of the proof of Theorem (3.6.1) shows that if we choose A, B sufficiently general, then we must have $\overline{\mathbf{v}}_2 = \overline{\mathbf{u}}_2 = 0$, where $\overline{\mathbf{u}}_2, \overline{\mathbf{v}}_2$ denote degree zero components of U_2, V_2. We also have $-CU_2 + \tilde{\delta}_0 U_4 = -xV_3 + V_4\tilde{B}$. Identifying the linear parts after modding out by x, we have $\delta_0\overline{U}_4 = \overline{V}_4 B$. Since the coefficients of the linear entries entries of δ_0 can be obtained as polynomials in terms of the entries of A, we may also assume that δ_0 and B are sufficiently general so that this implies that $\overline{\mathbf{u}}_4 = \overline{\mathbf{v}}_4 = 0$. The fact that $\overline{\mathbf{u}_2} = \overline{\mathbf{u}_4} = 0$ contradicts the assumption that the matrix U is invertible.

Note that the modules given by the construction (**??**) might not be indecomposable. Nevertheless, having uncountably many mutually non-isomorphic totally reflexive modules of number of generators b implies that there must be uncountably many non-isomorphic indecomposable totally reflexive modules. Otherwise, countably many indecomposables can only give rise to countably many totally reflexive modules. □

Corollary 3.6.3. *Let R be as in Corollary (3.6.2). Then the category of totally reflexive R-modules is either trivial (consists of just the free modules), or else it has positive radius (see [8] for the definition of radius of a category).*

Proof. The definition of radius of a category in ([8], Definition 2.1) counts the minimal number of extensions needed in order to obtain all objects in the category from a single object, via taking direct sums, direct summands, syzygies, and extensions. The existence of uncountably many totally reflexive R-modules implies that they cannot be all obtained from a single object via direct sums, direct summands, and syzygies. Thus, at least one extension is necessary. □

Chapter 4

Generalizations of Rangel-Tracy's Rings

4.1 Overview

In this chapter we explore generalizations of Rangel-Tracy's rings [14]. Rangel-Tracy gives a complete description of all totally acyclic complexes for her rings. It turns out that entries of the differentials for any totally acyclic complex for these rings can be chosen generically. Her rings are also short rings, and her results follow from the work of Yoshino [18]. In this chapter we explore two generalizations of Rangel-Tracy's rings. These generalizations are not short rings, so we cannot utilize Yoshino.

In the first generalization, we use techniques from Striuli and Vraciu in [15]. By using Matlis Duality and studying syzygies of the canonical module we find certain constraints on differentials for totally acyclic complexes. It turns out that unlike Rangel-Tracy's rings, these differentials cannot be chosen generically. The second generalization is a non-Cohen-Macaulay example. There are not a lot of non-Cohen-Macaulay examples in the literature, so we record one here.

4.2 Rangel-Tracy's Rings

We begin by defining Rangel-Tracy's rings.

Definition 4.2.1. Let k be a field of characteristic zero and for $n \geq 2$ set

$$R = k[x, y_1, y_2, \ldots, y_n]/(x^2, (y_1, y_2, \ldots, y_n)^2).$$

Such a ring will be referred as a *Rangel-Tracy ring*.

We note that the hilbert function $HF_R(t) = 1 + (n+1)t + nt$, and R/x is totally reflexive R-module. Therefore, the Rangel-Tracy's rings can utilize Yoshino's work. She gives the following results about presentations and resolutions of totally reflexive modules.

Theorem 4.2.2. *([14], Theorem 3.2) If T is a totally reflexive R-module which is minimally generated by m elements, then there exists a presentation matrix of T of the form*

$$xI_m + \sum_{i=1}^{n} y_n B_{ij},$$

where B_{ij} is a $m \times m$ matrix of scalars from k.

Lemma 4.2.3. *([14], Corollary 3.3) If T is a totally reflexive R-module, then a resolution of T is periodic of period 1 or 2.*

Corollary. *([14], Corollary 3.5) If T is a totally reflexive R-module with minimal presentation*

$$A = xI_n + \sum_{i=1}^{n} y_n B_{ij},$$

then the cooresponding totally acyclic complex is

$$\cdots \xrightarrow{\overline{A}} R^m \xrightarrow{A} R^m \xrightarrow{\overline{A}} \cdots \tag{4.1}$$

where $\overline{A} = xI_m - \sum_{i=1}^{n} y_n B_{ij}$.

4.3 Constructing Rangel-Tracy's Rings

For our first generalization we would like to give a description on how we construct Rangel-Tracy's rings.

Construction 4.3.1. Let $G = k[x]/x^2$ and $R_0 = k[y_1, ..., y_n]/(y_1, .., y_n)^2$, for $n \geq 2$. Then the Rangel-Tray's ring can be viewed as

$$R = G \otimes_k R_0.$$

Here we see Rangel-Tracy's rings are a construction of two simpler rings. The first ring, G, is an artinian Gorenstein ring. The second ring, R_0, has the property that some syzygy of ω_{R_0} is a vector space (in fact when $R_0 = k[y_1, y_2, \ldots, y_n]/(y_1 \ldots, y_n)^2$ all syzgies of any R_0-modules are vector spaces). From this description of Rangel-Tracy's ring we would like to vary which Gorenstein ring we use. We also observe that if $R_0 = k[y_1, y_2, \ldots, y_n]/(y_1, \ldots, y_n)^2$ for $n \geq 2$ and the artinian Gorenstein ring G is not $k[x]/x^2$, then $G \otimes_k R_0$ is not a short ring. In which case we cannot use Yoshino's work to recover results similar to Rangel-Tracy's [14].

Our approach to obtain similar results is to apply a technique from Striuli and Vraciu [15]. Striuli and Vraciu proved certain non-Gorenstein rings were G-regular by investigating the presentation of the canonical module. They showed that if some syzygy of the canonical module splits with a summand isomorphic to the residue field, then the ring is G-regular. We apply a similar approach on our generalization to obtain restraints on the differentials of totally acyclic complexes.

Let's recall a theorem to see how the canonical module can be computed in our generalization.

Theorem 4.3.2. *([4] Proposition 3.3.20) Let* $\varphi : R_0 \to R$ *be a flat homomorphism of Noetherian rings whose fibers* $R \otimes_{R_0} k(\mathfrak{p})$ *are Gorenstein for all* $\mathfrak{p} \in \mathit{Spec} R_0$ *for which there exists a maximal ideal* $\mathfrak{q}$ *in* R *with* $\mathfrak{p} = \mathfrak{q} \cap R_0$*. If* ω_{R_0} *is a canonical module of* R_0 *then* $\omega_{R_0} \otimes_{R_0} R$ *is a canonical module of* R*.*

Observation 4.3.3. *Let* $R = G \otimes_k R_0$ *where* G *is an artinian Gorenstein ring,* $(R_0, \mathfrak{n})$ *is an aritinian ring, and they are both* k*-algebras. Then we have* $\omega_R = \omega_{R_0} \otimes_{R_0} R$*.*

Proof. We utilize the previous observation to compute the canonical module. We note that R is a flat R_0 module. Since R is artinian we only need to verify that $R \otimes_{R_0} k(\mathfrak{n}) = R \otimes_k k$ is Gorenstein to obtain our desired result. Observe that

$$R \otimes_{R_0} k = (G \otimes_k R_0) \otimes_{R_0} k = G \otimes_k (R_0 \otimes_{R_0} k) = G \otimes_k k = G$$

Since G is Gorenstein we have $\omega_R = \omega_{R_0} \otimes_{R_0} R$. $\square$

Lemma 4.3.4. *Let $R = G \otimes_k R_0$ where G is an artinian Gorenstein ring, $(R_0, \mathfrak{n})$ is an aritinian ring such that the syzygy of the canonical module has a direct summand isomorphic to k, and they are both k-algebras. Then for any totally reflexive R-module M, we have $M/\mathfrak{n}M$ is a totally reflexive G-module. Further, the totally acyclic complex of freeR-modules of which M is a syzygy remains exact (and it fact totally acyclic) when tensored with G, and $M/\mathfrak{n}M$ is a syzygy of this new complex.*

Proof. Let $\Omega^i(\omega_{R_0}) = R_0/\mathfrak{n}R_0 \oplus N$. Observe that

$$\begin{aligned}\Omega^i_R(\omega_R) &= \Omega^i_{R_0}(\omega_{R_0}) \otimes_{R_0} R \\ &= (R_0/\mathfrak{n}R_0 \otimes_{R_0} R) \oplus (N \otimes_{R_0} R) \\ &= R/\mathfrak{n}R \oplus (N \otimes_{R_0} R) \\ &= G \oplus (N \otimes_{R_0} R).\end{aligned}$$

By definition we have that $Ext^i_R(M, R) = Ext^i_R(M^*, R) = 0$ for all $i > 0$. It follows from Matlis Duality that $Tor^R_i(M, \omega_R) = Tor^R_i(M^*, \omega_R) = 0$. Since G is a summand of some syzygy of ω_R we see that $Tor^R_i(M, G) = Tor^R_i(M^*, G) = 0$ for all i. This yields $M/\mathfrak{n}M$ is a totally reflexive G-module. In addition, let F be the totally acyclic complex of free R-modules of which M is a syzygy. Since $Tor^R_i(M, G) = Tor^R_i(M^*, G) = 0$, we have that $F \otimes_R G$ the totally acyclic complex of free G-modules of which $M/\mathfrak{n}M$ is a syzygy. $\square$

Let's be more specific in our generalization. Let $R = G \otimes_k R_0$ where $R_0 = k[z_1, z_2, \dots, z_n]/(z_1, z_2, \dots, z_n)^2$ with $n \geq 2$ and G be artinian Gorenstein. We can also view R as $G[z_1, z_2, \dots, z_n]/(z_1, z_2, \dots, z_n)^2$. The following lemma gives some intuition on how we can construct totally acyclic R complexes.

Lemma 4.3.5. *Let G be a ring and $R = G[x_1, x_2, \dots, x_n]/(x_1, x_2, \dots, x_n)^2$. Let*

$$\cdots \longrightarrow G^{n_i} \xrightarrow{\delta_i} G^{n_{i-1}} \xrightarrow{\delta_{i-1}} G^{n_{i-2}} \longrightarrow \cdots$$

be a totally acyclic G complex. Let

$$\cdots \longrightarrow R^{n_i} \xrightarrow{\delta'_i} R^{n_{i-1}} \xrightarrow{\delta'_{i-1}} R^{n_{i-2}} \longrightarrow \cdots \tag{4.2}$$

be an $R-$complex such that for each i, $\delta'_i = \delta_i + \sum_{j=1}^{n} x_j \delta_{i,j}$ where the entries of each $\delta_{i,j}$ come from G. Then (4.2) is a totally acyclic R complex.

Proof. To verify (4.2) is acyclic, we only need to verify that two adjacent maps form an exact sequence. We relabel the differentials and write an exact sequence

$$G^{n_i} \xrightarrow{B} G^{n_{i-1}} \xrightarrow{A} G^{n_{i-2}},$$

and

$$R^{n_i} \xrightarrow{\overline{B}} R^{n_{i-1}} \xrightarrow{\overline{A}} R^{n_{i-2}} \tag{4.3}$$

a complex where $\overline{A} = A + \sum_{i=1}^{n} x_i A_i$, $\overline{B} = B + \sum_{i=1}^{n} x_i B_i$ with the entries of A_i, B_i come from G. We have that (4.3) is a complex. It follows that $AB_i = -A_i B$ for all $1 \leq i \leq n$. Let $\overline{u} \in ker\overline{A}$, and $\overline{u} = u + \sum_{i=1}^{n} x_i u_i$ where the entries of each u_i come from R. We observe that

$$\overline{A}\overline{u} = Au + \sum_{i=1}^{n} x_i(A_i u + Au_i) = 0.$$

It follows that $Au = 0$ and for each i, $A_i u + Au_i = 0$. By the exactness we can find some w such that $Bw = u$. We observe further that

$$\begin{aligned} 0 &= A_i u + Au_i \\ &= A_i Bw + Au_i \\ &= -AB_i w + Au_i \\ &= -A(B_i w - u_i). \end{aligned}$$

Again by the exactness of the first short exact sequence, we can find some w_i such that $-Bw_i = B_i w - u_i$ or $u_i = B_i w + Bw_i$. Let $\overline{w} = w + \sum_{i=1}^{n} x_i w_i$. It can be verified that $\overline{B}\overline{w} = \overline{u}$; hence, we have that (4.3) is exact; thus, we have that (4.2) is acyclic as desired. □

From this result we have that any totally acyclic G-complex is also a totally acyclic $G[z_1, z_2, \ldots, z_n]/(z_1, z_2, \ldots, z_n)^2$-complex. We notice that if G is a non-regular Gorenstein ring, then we have that $G \otimes_k k[z_1, z_2, \ldots, z_n]/(z_1, z_2, \ldots, z_n)^2$ is not G-regular.

4.4 First Generalization

Let $G = k[x, y]/(x^2, y^2)$ and $R_0 = k[z_1, z_2, \ldots, z_n]/(z_1, z_2, \ldots, z_n)^2$. Then we have $R = G \otimes_k R_0 = k[x, y, z_1, z_2, \ldots, z_n]/(x^2, y^2, (z_1, z_2, \ldots, z_n)^2)$. This will be our first generalization. Let's look at a specific totally acyclic R-complex.

Example 4.4.1. Let's consider

$$\cdots \longrightarrow R^2 \xrightarrow{\begin{bmatrix} x & y \end{bmatrix}} R \xrightarrow{\begin{bmatrix} xy \end{bmatrix}} R \xrightarrow{\begin{bmatrix} x \\ y \end{bmatrix}} R^2 \longrightarrow \cdots$$

It is clear this is a totally acyclic R complex since the differentials form a totally acyclic $k[x, y]/(x^2, y^2)$-complex. We notice that the complex is not periodic and entries of the differentials are not all linear. This shows that Rangel-Tracy's results do not generalize to these rings.

We further want to investigate the entries of the differentials. Specifically, we want to know if we can chose generic linear entries and obtain a totally acyclic complex. The following theorem addresses this question.

Lemma 4.4.2. *Let $R = k[x, y, z_1, z_2, \ldots, z_t]/(x^2, y^2, (z_1, z_2, \ldots, z_t)^2)$. Let $(\mathbb{F}, \delta)$ be a totally acyclic complex where the differentials have the form*

$$\delta = xA + yB + \sum_{i=1}^{n} y_i C_i,$$

with matrices A, B, C_i have entries in k. Then the coefficients of these matrices cannot be chosen generically.

Proof. We suppose for contradiction that we can choose the entries generically. From the previous result we can restrict our attention to matrices of the form $M = xA + yB$. Let A be generic. By possibly considering the dual, we may assume that A has as many columns as rows. With base change we can write

$$A = (I_n|0)$$

where I_n is an identity matrix. Let w be a minimal relation of M and $w = ux + vy$. We compute

$$0 = Mw = ((I_n|0)v + Bu)xy.$$

This gives the relation $(I_n|0)v = -Bu$. We let u vary through the standard basis $\{e_1, e_2, \ldots, e_m\}$ and determine the corresponding v. We also note that $(I_n|0)v$ doesn't incorporate the last $m - n$ entries of v. Therefore we conclude the kernel of M is presented by the matrix

$$\left(\begin{array}{c|c} I_m & 0 \end{array} \right) x - \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I_{m-n} \end{array} \right) y.$$

Now we consider the composition of the differentials in R of the totally acyclic R-complex. We can write

$$\delta_1 = (I_n|0)x + By + \sum_{i=1}^{t} C_i z_i$$

$$\delta_2 = \left(\begin{array}{c|c} I_m & 0 \end{array} \right) x - \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I_{m-n} \end{array} \right) y + \sum_{i=1}^{t} D_i z_i.$$

We compute the composition and see

$$\begin{aligned} \delta_1\delta_2 &= \left((I_n|0)x + By + \sum_{i=1}^{t} C_i z_i\right) \left(\left(\begin{array}{c|c} I_m & 0 \end{array} \right) x - \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I_{m-n} \end{array} \right) y + \sum_{i=1}^{t} D_i z_i \right) \\ &= ((I_n|0)x + By) \left(\sum_{i=1}^{t} D_i z_i\right) + \left(\sum_{i=1}^{t} C_i z_i\right) \left(\left(\begin{array}{c|c} I_m & 0 \end{array} \right) x - \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I_{m-n} \end{array} \right) y \right) \\ &= \sum_{i=1}^{t} ((I_n|0)D_i + C_i(I_m|0)xz_i) + \sum_{i=1}^{t} \left(BD_i - C_i \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I_{m-n} \end{array} \right) yz_i \right). \end{aligned}$$

We observe for each i

$$(I_n|0)D_i = -C_i(I_m|0)), \text{ and}$$

$$BD_i = C_i \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I_{m-n} \end{array}\right).$$

Let $(d_{jk}) = D_i$ and $(c_{jk}) = C_i$. It follows from the first observation that $d_{jk} = -c_{jk}$ when $1 \leq j \leq n$ and $1 \leq k \leq m$, and $d_{jk} = 0$ when $k > m$. Therefore, we can write

$$D_i = -(C_i|0), \text{ and}$$

$$-B(C_i|0) = C_i \left(\begin{array}{c|c} B & 0 \\ \hline 0 & I_{m-n} \end{array}\right).$$

This shows that matrices C and B have to anticommute. Hence, the coefficients cannot be chosen generically. □

4.5 Second Generalization

We note that in the definition of totally acyclic R-complexes it was not necessary for R to be Cohen Macaulay. This generalization turns out to be a non-Cohen Macaulay and non-G-regular ring. This example is purely serendipitous

Construction 4.5.1. Let n be a positive integer,

$$M = \begin{pmatrix} x_{11} & x_{12} & \dots & x_{1n} \\ x_{21} & x_{22} & \dots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \dots & x_{nn} \end{pmatrix},$$

I be the ideal generated by the entries of the matrix M^2, $\mathfrak{m} = (x_{11}, x_{12}, \dots, x_{nn})$, $R_0 = k[y_1, y_2, \dots, y_m]/(y_1, y_2, \dots, y_m)^2$, and $R_n = (k[x_{11}, x_{12}, \dots, x_{nn}]/I)_{\mathfrak{m}}$. Then we define $R := R_n \otimes_k R_0$.

Notice that when $n = 1$ we get $R_1 = k[x]/(x^2)$. Therefore we can recover Rangel-Tracy's rings by $R_1 \otimes_k R_0$. Also, by construction we can view the ring R as $R_n[y_1, y_2, \ldots, y_m]/(y_1, y_2, \ldots, y_m)^2$. Therefore, when constructing totally acyclic R-complexes we can utilize (4.3.5) and focus our attention to totally acyclic R_n-complexes .

Observation 4.5.2. *Let R_n, M, and I be as in construction from (4.5.1). When $n > 1$, R_n is not Cohen Macaulay.*

Proof. To prove the result, we show that R_n has positive dimension and that $\mathfrak{m}$ is an embedded prime.

We claim that the prime ideal $\mathfrak{p} \subset \mathfrak{m}$ that is generated by all the variables except for $x_{n-1,n}$ contains I. But this is clear since the entries of M^2 are all homogeneous of degree 2 and no entry in M^2 has $x_{n-1,n}^2$ as a term. It follows that $\dim R_n \geq 1$.

Let's show that $r := \det M \notin I$. We suppose the contrary that $r \in I$. Let J be the ideal generated by x_{ij} where $i \neq j$. Then it follows that $r \in I + J$. It is seen that $I + J$ is generated by $\{x_{11}^2, x_{22}^2, \ldots, x_{nn}^2\}$ and the generators of J. Hence, $I + J$ is a monomial ideal. We can reduce r to be the monomial $\Pi_{i=1}^n x_{ii}$. It is clear when $n > 1$ that $r \notin I + J$. Therefore, our assumption that $r \in I$ was absurd.

To finish the claim we need to show that $\operatorname{ann} r = \mathfrak{m}$. That is, we need to show for any i and j, we have $x_{ij} r = 0$. By considering a change of basis it suffices to show that $x_{11} r = 0$. We can interpret $x_{11} r$ as

$$x_{11} r = \det \begin{pmatrix} x_{11}^2 & x_{11}x_{12} & \ldots & x_{11}x_{1n} \\ x_{21} & x_{22} & \ldots & x_{2n} \\ \vdots & \vdots & & \vdots \\ x_{n1} & x_{n2} & \ldots & x_{nn} \end{pmatrix}.$$

We would like to conclude that the above determinant is zero. We do this by showing the top row is a combination of the subsequent rows. By considering the top row entries of M^2 we can make a the following substitutions for $x_{1i}x_{i1} = -\sum_{k=2}^{n} x_{1k}x_{ki}$. With these substitutions it is easily verified that

$$\begin{pmatrix} x_{21} & x_{31} & \dots & x_{n1} \\ x_{22} & x_{32} & \dots & x_{n2} \\ \vdots & \vdots & & \vdots \\ x_{2n} & x_{3n} & \dots & x_{nn} \end{pmatrix} \begin{pmatrix} -x_{12} \\ -x_{13} \\ \vdots \\ -x_{1n} \end{pmatrix} = \begin{pmatrix} x_{11}^2 \\ x_{11}x_{12} \\ \vdots \\ x_{11}x_{1n} \end{pmatrix}$$

Thefore, we can conclude that ann$r = \mathfrak{m}$. Thus R_n is not Cohen Macaulay when $n > 1$.

□

Next, we give an example of a minimal totally acyclic R_n-complex.

Example 4.5.3. We would like to construct a totally acyclic R_n complex. In Rangel-Tracy's rings, T, we have that

$$\cdots \xrightarrow{x} T \xrightarrow{x} T \xrightarrow{x} \cdots$$

is a totally acyclic complex. To generalize the above complex into the context of R_n we would like

$$\cdots \xrightarrow{M} R^n \xrightarrow{M} R^n \xrightarrow{M} \cdots \tag{4.4}$$

to be totally acyclic. We verify via Macaulay2 that when $n \leq 4$ we have (4.4) is in fact totally acyclic.

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