An improved Fibonacci inequality

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Fibonacci numbers and Fibonacci sequences play a key role in many areas of mathematics and other sciences. Many inequalities satisfied by Fibonacci sequences have been established. In this paper we prove a new Fibonacci inequality using Candido's identity.

Introduction

In this paper, we consider a problem in [1]. The problem posed is to prove that for every natural number $n$,

$$2 \left( F_n^4 + F_{n+1}^4 + F_{n+2}^4 \right) \left( \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} \right)^2 > 81$$

where $F_n$ is the $n^{th}$ Fibonacci as defined in the section below.

We prove much stronger inequality:

$$2 \left( F_n^4 + F_{n+1}^4 + F_{n+2}^4 \right) \left( \frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} \right)^2 > 100.$$

Existing identity and inequality

Definition 1. The $n^{th}$ Fibonacci number is defined by

$$F_n = F_{n-1} + F_{n-2}$$

where $F_0 = 0$, $F_1 = F_2 = 1$ and $n \geq 2$ is a natural number.

Lemma 2. (Candido’s identity (2), [3], [4])

For every natural number $n$,

$$(F_n^2 + F_{n+1}^2 + F_{n+2}^2)^2 = 2 (F_n^4 + F_{n+1}^4 + F_{n+2}^4).$$

Lemma 3. For every natural number $n$,

$$\frac{F_n}{F_{n+1}} \geq \frac{1}{2}$$

Auxiliary equations

In this section we give a proof of some auxiliary equations that we use to obtain our main result.

Lemma 4. For every natural number $n$,

$$\left( \frac{F_{n+1}}{F_n} \right)^2 = 1 + 2 \left( \frac{F_{n-1}}{F_n} \right) \left( \frac{F_{n-1}}{F_n} \right)^2.$$

Proof.

$$\left( \frac{F_{n+1}}{F_n} \right)^2 = \left( \frac{F_n + F_{n-1}}{F_n} \right)^2 = \left( 1 + \frac{F_{n-1}}{F_n} \right)^2 = 1 + 2 \left( \frac{F_{n-1}}{F_n} \right) + \left( \frac{F_{n-1}}{F_n} \right)^2.$$ (by Lemma 2.)
\[
\begin{align*}
\left(3 + \frac{F_{n+1}^2 + F_{n+2}^2}{F_n^2} + \frac{F_n^2 + F_{n+2}^2}{F_{n+1}^2} + \frac{F_n^2 + F_{n+1}^2}{F_{n+2}^2}\right)^2 \\
= \left(3 + \left(\frac{F_{n+1}}{F_n}\right)^2 + \left(\frac{F_{n+2}}{F_n}\right)^2 + \left(\frac{F_n}{F_{n+1}}\right)^2 + \left(\frac{F_n}{F_{n+2}}\right)^2\right)
\end{align*}
\]

\[
\begin{align*}
\left(9 + \frac{6F_{n-1}}{F_n} + \frac{2F_n}{F_{n+1}} + \frac{2}{F_n} + 2 + \frac{F_n}{F_{n+1}} + \left(\frac{F_n}{F_{n+2}}\right)^2\right)^2 \\
> \left(10 + \frac{6F_{n-1}}{F_n} + \frac{2F_n}{F_{n+1}} + \frac{2}{F_n} + \left(\frac{F_n}{F_{n+1}}\right)^2 + \left(\frac{F_n}{F_{n+2}}\right)^2\right)
\end{align*}
\]

(by lemma 4 – 5.)

(by lemma 3.)

> 100.

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References