A Scale of Linear Spaces Related to the Lp Scale

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A SCALE OF LINEAR SPACES RELATED TO THE 
$L_p$ SCALE

BY

S.J. DILWORTH

1. Introduction and preliminary results

In [7] the Banach space $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ was investigated. In the present paper we consider a range of spaces of which $L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ is one member. This scale of Banach spaces is closely related both to the $L_p$ scale and to Hilbert space. Subspace structure and other linear topological properties of the scale are investigated.

Let $(\Omega, \Sigma, m)$ be a measure space and let $L_p(\Omega)$ be the usual Lebesgue space with the norm

$$
\|f\|_p = \left( \int_\Omega |f|^p \, dm \right)^{1/p} \quad (0 < p < \infty)
$$

and

$$
\|f\|_\infty = \text{ess sup} \{ |f(\omega)| : \omega \in \Omega \}.
$$

For $0 < p \leq \infty$, let $Y_p(\Omega)$ be the collection of all measurable $f$ such that

$$
\|f\|_{Y_p} = \|f^*I(0,1)\|_p + \|f^*I(1,\infty)\|_2 < \infty
$$

(here $f^*$ denotes the decreasing rearrangement of $|f|$), and for $0 < n < \infty$ let $M_p(\Omega)$ consist of all $f$ such that

$$
\|f\|_{M_p} = \|f^*I(0,1)\|_2 + \|f^*I(1,\infty)\|_p < \infty
$$

Finally, let $M_\infty(\Omega)$ be the closure of $L_2(\Omega)$ with respect to the norm

$$
\|f\|_{M_\infty} = \|f^*I(0,1)\|_2.
$$

Observe that if $\Omega$ is a probability space then

$$
Y_p(\Omega) = L_p(\Omega) \quad \text{and} \quad M_p(\Omega) = L_2(\Omega),
$$
and if $\Omega = \{1, 2, \ldots \}$ with the counting measure then
\[ Y_p(\Omega) = l_2 \quad \text{and} \quad M_p(\Omega) = l_p. \]

The main purpose of this paper is to investigate the $M_p$ scale for $\Omega = (0, \infty)$ with Lebesgue measure. In this case
\[ M_p(\Omega) = L_p(0, \infty) \cap L_2(0, \infty) \quad \text{for} \quad 0 < p \leq 2 \]
and
\[ M_p(\Omega) = L_p(0, \infty) + L_2(0, \infty) \quad \text{for} \quad 2 \leq p < \infty, \]
while $Y_p(\Omega)$ is just the reverse. Henceforth we shall write $L_p$ for $L_p(0, \infty)$ etc., because there will be no ambiguity.

The study of $Y_p$ as a Banach space has been effectively reduced to the study of $L_p$ by the following fundamental theorem.

**Theorem A [8]**. Let $1 < p < \infty$. Then the Banach space $L_p$ has exactly two representations as a rearrangement invariant function space on $(0, \infty)$, namely $L_p$ and $Y_p$.

The $M_p$ spaces, however, are isomorphically distinct from the $L_p$ scale, though there are many similarities in their Banach space structure. In particular, many results about $L_p$ have a counterpart for $M_q$, where $p$ and $q$ are Hölder conjugate indices. It turns out, for instance, that the space $M_\infty$ resembles $L_1$: for example, in its subspace structure and in the fact that $M_\infty$ does not embed into a separable dual space or a space with unconditional basis. The linear structure of the $M_p$ spaces is examined in Sections 4 to 7 below, with the ranges $0 < p < 1$, $1 \leq p < 2$, $2 < p < \infty$, and $p = \infty$ receiving separate treatment.

There is also a close affinity between the $M_p$ spaces and Hilbert space. One aspect of this is brought out in the second section, in which some members of the $M_p$ scale are constructed by complex interpolation between the two representations of $L_p$ as an r.i. space on $(0, \infty)$. To be precise, if the parameter $\theta$ of the method is chosen so that $[L_{p_0}, L_{p_1}]_\theta = L_2$, with $p_0 < 2 < p_1 < \infty$, then $[L_{p_0}, L_{p_1}]_\theta = M_p$ for some $p$.

Two embedding theorems are proved in the third section. In the main result a generalization of an inequality of Rosenthal for sums of independent random variables is used to construct an embedding of $M_p$ into the Lebesgue-Bochner space $L_2(l_p)$. The impossibility of such an embedding into $l_2(l_p)$ is also demonstrated.

We conclude this section with some easy or standard facts about the $M_p$ spaces which are needed later. The reader is referred to [16] and [17] for the Banach space and Banach lattice terminology used throughout the article.
**Proposition 1.1.** Let $1 < p < \infty$. Then up to an equivalent norm $(M_p)^* = M_q$, where $1/p + 1/q = 1$, with the usual duality $(f, g) = \int_0^\infty fg \, dt$.

**Proposition 1.2.** (a) Let $0 < p \leq 2$. Then $M_p$ is the $(2/p)$-concavification of $Y_{a/p}$.

(b) Let $2 \leq p < \infty$. Then $M_p$ is the $(p/2)$-convexification of $Y_{a/p}$.

**Proof.** Plainly, $\|f\|_{M_p} \sim \|f\|^{p/2}_{L_{a/p}}$.

**Proposition 1.3.** Let $0 < p < \infty$ and let

$$\phi_p(t) = \begin{cases} t^p & (0 \leq t \leq 1) \\ t^2 & (1 \leq t < \infty) \end{cases}.$$

Then $M_p$ is equal to the Orlicz space $L_{\phi_p}(0, \infty)$.

**Proposition 1.4** [7]. (a) Let $1 \leq p \leq \infty$. Then $H$ is a Schauder basis for $M_p$.

(b) Let $0 < p \leq \infty$ and suppose that $M_p$ is isomorphic to a subspace of a quasi-Banach space $X$ having a Schauder basis $(x_n)_{n=1}^\infty$. Then $G$ is equivalent to a block basis of $(x_n)_{n=1}^\infty$.

2. The $M_p$ spaces and complex interpolation

In this section we show how the $M_p$ spaces arise as interpolation spaces in the ordinary method of complex interpolation. In particular, if the parameter $0$ is chosen so that $L_2$ is the result of interpolating between $L_{p_0}$ and $L_{p_1}$, then $M_p$ (for some $p$) is the corresponding interpolation space between $L_{p_0}$ and $Y_{p_1}$.

The reader is referred to [3] and [23] for the method of complex interpolation and the Riesz convexity theorem.
Theorem 2.1. Let $0 < p_0 < 2 < p_1 < \infty$ and $0 < \theta < 1$. Then

$$
\left[ L_{p_0}, Y_{p_1} \right]_\theta = M_p
$$

where

$$
\frac{1 - \theta}{p_0} + \frac{1}{p_1} = \frac{1}{2} \quad \text{and} \quad \frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{2}.
$$

In particular, $M_p$ admits such a representation provided $1 < p < 2$.

Proof. $Y_{p_1} = L_{p_1} \cap L_2$ for $p_1 \geq 2$, and so

$$
\left[ L_{p_0}, Y_{p_1} \right]_\theta = \left[ L_{p_0}, L_{p_1} \cap L_2 \right]_\theta \subseteq \left[ L_{p_0}, L_{p_1} \right]_\theta \cap \left[ L_{p_0}, L_2 \right]_\theta = L_2 \cap L_p = M_p.
$$

To complete the proof we must show that $M_p \subseteq \left[ L_{p_0}, Y_{p_1} \right]_\theta$. Let

$$
g = \sum_{j=1}^n a_j I(E_j)
$$

be a simple integrable function (here $|E|$ denotes the Lebesgue measure of a set $E$) with $a_1 \geq a_2 \geq \cdots \geq a_n \geq 0$ and $|E_1| \geq 1$: call such a function $g$ "flat". Plainly, $\|g\|_{M_p} \sim \|g\|_p$. Consider the vector-valued analytic function

$$
g(z) = \sum_{j=1}^n a_j^{\alpha(z)} I(E_j) \quad \text{where} \quad \alpha(z) = p \left( \frac{1 - z}{p_0} + \frac{z}{2} \right).
$$

Since $g(z)$ is always flat it follows that

$$
\|g(1 + it)\|_{Y_{p_1}} \sim \|g(1 + it)\|_2 \quad \text{for} \quad -\infty < t < \infty.
$$

The usual proof of the Riesz convexity theorem now gives

$$
\max \left\{ \|g(it)\|_{p_0}, \|g(1 + it)\|_{Y_{p_1}} : -\infty < t < \infty \right\} \sim \max \left\{ \|g(it)\|_{p_0}, \|g(1 + it)\|_2 : -\infty < t < \infty \right\} \leq C\|g\|_p \sim C\|g\|_{M_p}.
$$
Now suppose that
\[ h = \sum_{j=1}^{n} b_j I(E_j) \]
is a non-negative simple function with \( \sum_{j=1}^{n} |E_j| \leq 1 \), so that \( \|h\|_{M_p} \sim \|h\|_2 \).

Consider the vector-valued analytic function
\[ g(z) = \sum_{j=1}^{n} b_j^{\beta(z)} I(E_j) \]where \( \beta(z) = 2 \left( \frac{1 - z}{p_0} + \frac{z}{p_1} \right) \).

Using the fact that \( \|h(1 + it)\|_{Y_{p_1}} \sim \|h(1 + it)\|_{p_1} \) we obtain
\[ \max \{ \|h(it)\|_{p_0}, \|h(1 + it)\|_{p_1} : -\infty < t < \infty \} \leq C \|h\|_{M_p}. \]

Finally, since any non-negative integrable simple function \( f \) can be expressed in the form \( f = g + h \), with \( g \) and \( h \) as above, the inclusion \( M_p \subset [L_{p_0}, Y_{p_1}]_\theta \) follows easily.

The following corollary is an immediate consequence of the duality theorem for complex interpolation [3, p. 98].

**Corollary 2.2.** Suppose that \( 1 < p_0 \leq 2 \leq p_1 < \infty \) and that
\[ \frac{1}{2} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad (0 < \theta < 1). \]

Then
\[ [Y_{p_0}, L_{p_1}]_\theta = M_p \quad \text{with} \quad \frac{1}{p} = \frac{1 - \theta}{2} + \frac{\theta}{p_1}. \]

In particular, \( M_p \) admits such a representation for \( 2 < p < 4 \).

The proofs of the following two interpolation theorems are similar to that of Theorem 2.1 and have been omitted. The duality statements have been left to the reader.

**Proposition 2.3.** Suppose that \( 0 < p_0 \leq 2 \leq p_1 \leq \infty \) and that
\[ \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} = \frac{1}{2} \quad (0 < \theta < 1). \]

Then
\[ [M_{p_0}, L_{p_1}]_\theta = Y_p \quad \text{where} \quad \frac{1}{p} = \frac{1 - \theta}{2} + \frac{\theta}{p_1}. \]
Proposition 2.4. Suppose that $0 < p_0 \leq p_1 \leq 2$ and that

$$\frac{1}{p} = \frac{1 - \theta}{p_0} + \frac{\theta}{p_1} \quad (0 < \theta < 1).$$

Then

$$[M_{p_0}, M_{p_1}]_\theta = M_p.$$

3. A probabilistic interpretation

This section is mainly devoted to a proof of the following theorem. Recall that if $X$ is a Banach space, then $L_p(X)$ is the usual space of $X$-valued Bochner-measurable functions $f$ on $[0, 1]$ with the norm

$$\|f\|_{L_p(X)} = \left( \int_0^1 \|f(t)\|^p \, dt \right)^{1/p}.$$

Theorem 3.1. Let $0 < p < \infty$. Then $M_p$ is isomorphic to a subspace of $L_2(I_p)$. If $1 < p < \infty$ then this subspace may be taken to be complemented in $L_2(I_p)$.

The proof is very similar to the proof of the isomorphism between $Y_p$ and $L_p$ given in [5]. In particular, we require two preliminary results (cf. [5, Prop. 3.1]).

Proposition 3.2. Let $1 < p < \infty$ and let $(\mathcal{F}_k)_{k=1}^n$ be increasing sub-$\sigma$-fields. There exists $C_p$ such that

$$\left\| \left( E\left( f_k|\mathcal{F}_k \right) \right)_{k=1}^n \right\|_{L_2(I_p)} \leq C_p \left\| \left( f_k \right)_{k=1}^n \right\|_{L_2(I_p)}$$

for all $n \geq 1$ and for all $f_1, f_2, \ldots, f_n$ in $L_2(0,1)$.

Proof. This is a minor variation of a theorem of Stein ([22] or [8, p. 243]), but for completeness we give the idea of the proof. The operator

$$(f_k)_{k=1}^n \rightarrow \left( E\left( f_k|\mathcal{F}_k \right) \right)_{k=1}^n$$

is plainly bounded from $L_p(I_p^n)$ to $L_p(I_p^n)$ for $1 \leq p < \infty$. An argument involving Doob's maximal inequality proves that the operator is bounded from $L_p(I_\infty^n)$ to $L_p(I_\infty^n)$ for $1 < p < \infty$. Now a standard interpolation theorem shows that the operator is bounded from $L_2(I_p^n)$ to $L_2(I_p^n)$ for $p > 2$. By duality it is also bounded for $1 < p < 2$. 
Corollary 3.3. Let $1 < p < \infty$ and let $(\mathcal{F}_k)_{k=1}^n$ be independent sub-σ-fields. Then there exists $C_p$ such that

$$\left\| (E(f_k | \mathcal{F}_k))_{k=1}^n \right\|_{L_2(l_p^p)} \leq C_p \left\| (f_k)_{k=1}^n \right\|_{L_2(l_p^p)}$$

for all $n \geq 1$ and for all $f_1, f_2, \ldots, f_n$ in $L_2(0,1)$.

Proof. Let $G_k$ and $\tilde{G}_k$ be the sub-σ-fields generated by $\bigcup_{i=1}^k \mathcal{F}_i$ and $\bigcup_{i=k+1}^n \mathcal{F}_i$ respectively. Applying Proposition 3.2 first with respect to $(G_k)_{k=1}^n$ and then with respect to $(\tilde{G}_k)_{k=1}^n$ gives the result.

Let $(f_n)_{n=1}^\infty$ be a sequence of functions on $[0,1]$. In the following proof $\sum_{n=1}^\infty \Theta f_n$ denotes the function $f$ on $(0, \infty)$ defined by $f(n - 1 + t) = f_n(t)$ for $n \geq 1$ and for $0 < t \leq 1$.

Proof of Theorem 3.1. It is convenient to replace $[0,1]$ by the measure-equivalent space $\Omega = [0,1]^N$ and to denote a typical element of $\Omega$ by $s = (s_1, s_2, \ldots)$. Consider the linear mapping $T: M_p \rightarrow L_2(l_p)$ defined by

$$T\left( \sum_{n=1}^\infty \Theta f_n \right) = (f_n(s_n))_{n=1}^\infty.$$

We show that $T$ is an isomorphic embedding for $0 < p < 2$. First observe that

$$\left\| \sum_{n=1}^\infty \Theta f_n \right\|_{M_p} \sim \left\| \sum_{n=1}^\infty \Theta |f_n|^{p/2} \right\|_{Y_{4/p}}^{2/p} \sim \left\| \left( \sum_{n=1}^\infty |f_n(s_n)|^p \right)^{1/2} \right\|_{L_{4/p}(0,1)}$$


$$\left\| \left( \sum_{n=1}^\infty |f_n(s_n)|^p \right)^{1/2} \right\|_{L_{4/p}}^{p/2} = \left\| \left( \sum_{n=1}^\infty |f_n(s_n)|^p \right)^{1/p} \right\|_2$$

$$= \left\| (f_n(s_n))_{n=1}^\infty \right\|_{L_2(l_p)}.$$

Consider the mapping $P: L_2(l_p) \rightarrow L_2(l_p)$ defined by

$$P((f_n)_{n=1}^\infty) = (E(f_n | s_n))_{n=1}^\infty.$$  

Then $P$ is a self-adjoint projection onto the range of $T$, and by Corollary 3.3,
$P$ is bounded for $1 < p < \infty$. It now follows from the fact that $P$ is self-adjoint that $T$ is an isomorphic embedding for the whole range $0 < p < \infty$ and that $P$ is a bounded projection onto a subspace isomorphic to $M_p$ for $1 < p < \infty$.

**Corollary 3.4.** Let $0 < p < \infty$ and let $(X_n)_{n=1}^{\infty}$ be non-negative independent random variables in $L_2(\Omega)$. Then

$$\left\| \left( \sum_{n=1}^{\infty} X_n^p \right)^{1/p} \right\|_2 \sim \left\| \sum_{n=1}^{\infty} \oplus X_n \right\|_{M_p}.$$

The latter result is actually equivalent to the following extension of Rosenthal's inequality to the range $0 < p < \infty$ discovered by Johnson and Schechtman.

**Corollary 3.5 [9].** Let $0 < p < \infty$ and let $(X_n)_{n=1}^{\infty}$ be independent symmetric random variables in $L_p(\Omega)$. Then

$$\left\| \sum_{n=1}^{\infty} X_n \right\|_p \sim \left\| \sum_{n=1}^{\infty} \oplus X_n \right\|_{Y_p}.$$

**Remark 3.6.** In [1] Aldous proved that if $L_p(X)$ embeds into a Banach space with unconditional basis then $X$ is a UMD space, thus disproving the conjecture (based on Paley's theorem on the unconditionality of the Haar system) that for $1 < p < \infty$, $L_p(X)$ has an unconditional basis if $X$ has one. Since $M_1$ does not embed into a Banach space with unconditional basis [7, Theorem 6], it follows from Theorem 3.1 that $L_2(l_1)$ has the same property, thus verifying Aldous' theorem in the special case $X = l_1$.

We show next that there is no isomorphic embedding from $M_p$ into $l_2(l_p)$.

**Theorem 3.7.** Let $0 < p < \infty$ ($p \neq 2$). Then $M_p$ is not isomorphic to a subspace of $(\Sigma \oplus l_p)_2$.

The case $2 < p < \infty$ will follow easily from later results (see Corollary 5.6), and so we shall assume that $0 < p < 2$.

**Preliminary Lemma.** Suppose that $(x_i)_{i=1}^{N}$ is a sequence in $(\Sigma \oplus l_p)_2$ such that

$$C^{-1} \left( \sum_{i=1}^{N} a_i^2 \right)^{1/2} \leq \left\| \sum_{i=1}^{N} a_i x_i \right\| \leq C \left( \sum_{i=1}^{N} a_i^2 \right)^{1/2}.$$
Suppose further that $x_i = (x_{i,1}, x_{i,2}, \ldots)$ for $1 \leq i \leq N$, where $x_{i,j}$ and $x_{i,k,j}$ are disjointly supported vectors in $l_p$ whenever $i_1 \neq i_2$ and $j \geq 1$. Then given $\varepsilon > 0$ and $M \geq 1$ there exists $N_1$, depending only on $\varepsilon$ and $M$, and a subset $I$ of \{1, 2, \ldots, N\} of cardinality $N - N_1$ such that $\|x_{i,j}\| \leq \varepsilon$ for all $i \in I$ and for all $i \leq j \leq M$.

**Proof of Lemma.** Fix $j$ with $1 \leq j \leq M$ and suppose that $\|x_{i,j}\| > \varepsilon$ for all $i \in I(j)$. Let $\text{card}(A)$ denote the cardinality of a finite set $A$. Then

$$C(\text{card}(I(j)))^{1/2} \geq \left\| \sum_{i \in I(j)} x_i \right\|$$

$$\geq \left( \sum_{i \in I(j)} \|x_{i,j}\|^p \right)^{1/p}$$

$$\geq (\text{card}(I(j)))^{1/p} \varepsilon.$$

Thus

$$\text{card}(I(j)) \leq \left( \frac{C}{\varepsilon} \right)^{2p/(2-p)}.$$

Take $I$ to be $\{1, 2, \ldots, N\} \setminus (\bigcup_{j=1}^M I(j))$. Thus

$$\text{card}(I) \geq N - N_1 \left( \frac{C}{\varepsilon} \right)^{2p/(2-p)}.$$

**Proof of Theorem 3.7.** Suppose that $M_p$ embeds isomorphically into $(\Sigma \oplus l_p)_2$. By Proposition 1.4, $G$ is equivalent to a block basis with respect to the obvious diagonally ordered basis of $(\Sigma \oplus l_p)_2$; to avoid awkward notation identify $G$ with this block basis. Suppose that $h_k = (x_{i,1}, x_{i,2}, \ldots)$ and note that, for each $k \geq 1$, $(h_k^{i=2m+2})_{i=2m+1+1}$ is isometric to the unit vector basis of $l_2^{2m+1}$. Suppose that $k \geq 1$, $M \geq 1$, and $\varepsilon > 0$ are given. Then by the Preliminary Lemma we may choose $m_k$ and

$$I(k) \subseteq \{2^{m_k+1} + 1, \ldots, 2^{m_k+2}\}$$

such that $\text{card}(I(k)) = 2^{m_k}$ and $\|x_{i,j}\| < 2^{-m_k/2} \varepsilon$ for all $i \in I(k)$ and $1 \leq j \leq M$. A straightforward inductive argument now shows that $m_k$ and $I(k)$ may be chosen so that the closed linear span of \{ $h_k^i$: $1 \leq k < \infty$, $i \in I(k)$ \} is isomorphic to $(\Sigma \oplus l_2^{2m_k})_2 = l_2$. But the sequence $(\Sigma_{i \in I(k)} h_k^i)_{k-1}$ is equivalent in $M_p$ to the unit vector basis of $l_p$, which is a contradiction because $l_p$ does not embed into $l_2$. 
Corollary 3.8. Let $0 < p < \infty$ ($p \neq 2$). Then $L_2(l_p)$ is not isomorphic to a subspace of $l_2(l_p)$.

Proof. This is immediate from Theorems 3.1 and 3.7. For $0 < p < 1$ the result is a very special case of a theorem of Kalton [12, Theorem 4.2].

Remark 3.9. The proof of Theorem 3.7 was inspired by the proof of Theorem 6.1 in [15].

In the light of Theorem 3.1 one could be excused for conjecturing that $M_p$ and $L_2(l_p)$ are isomorphic Banach spaces. This is not the case, however, as we shall now show. I am indebted to the referee for this observation.

Proposition 3.10. Let $0 < p < 2$. Then $(\Sigma \oplus l_p)_2$ is not isomorphic to a subspace of $M_p$.

Proof. Suppose on the contrary that $(\Sigma \oplus l_p)_2$ may be identified with a subspace of $M_p$. Then the $L_2(0, \infty)$ and the $M_p$ topologies do not coincide on $(\Sigma^\infty_{k=n+1} \oplus l_p)_2$ for any $n \geq 1$. Let $(\epsilon_k)_{k=1}^\infty$ be a null sequence of positive numbers. By an obvious inductive argument there exist an increasing sequence of integers $(n_k)_{k=1}^\infty$ and vectors

$$b_k \in \left( \sum_{n_{k+1}}^{n_k+1} \oplus l_p \right)_2$$

such that $\|b_k\|_p = 1$ and $\|b_k\|_2 < \epsilon_k$. Clearly, $(b_k)_{k=1}^\infty$ spans a subspace of $(\Sigma \oplus l_p)_2$ isomorphic to $l_2$. Yet by the argument of Theorem 4.1 below, $(b_k)_{k=1}^\infty$ spans a subspace isomorphic to $l_p$ provided $(\epsilon_k)_{k=1}^\infty$ decreases rapidly to zero. This contradiction completes the proof.

Corollary 3.11. Let $0 < p < \infty$ ($p \neq 2$). Then $M_p$ is not isomorphic to $L_2(l_p)$.

4. The spaces $M_p$ ($1 \leq p < 2$)

Many of the results in this section about the space $M_p$ ($1 < p < 2$) correspond to well-known theorems about the space $L_q$ ($2 < q < \infty$).

Theorem 4.1. Let $1 \leq p \leq 2$ and let $X$ be a subspace of $M_p$. Then $X$ is isomorphic to a Hilbert space and complemented in $M_p$ or $X$ contains a subspace isomorphic to $l_p$ and complemented in $M_p$. 
Proof. First suppose that the $M_p$ and the $L_2(0, \infty)$ topologies agree on $X$. Then $X$ is isomorphic to a Hilbert space and the orthogonal projection from $L_2(0, \infty)$ onto $X$ restricts to a bounded projection from $M_p$ onto $X$. If the topologies do not agree, then given $\varepsilon > 0$ there exists $f \in X$ such that $\|f\|_p = 1$ and $\|f\|_2 < \varepsilon$. By Hölder’s inequality

$$\|fI(0, M)\|_p^p \leq M^{(2-p)/2}\|f\|_2^p \leq M^{(2-p)/2}\varepsilon^p.$$ 

So by an inductive procedure one can construct functions $(f_n)_{n=1}^{\infty}$ in $X$ and disjoint compactly supported functions $(g_n)_{n=1}^{\infty}$ in $M_p$ such that

$$\|g\|_p = 1, \quad \|g\|_2 \leq \varepsilon_n \quad \text{and} \quad \|f_n - g_n\|_p \leq \varepsilon_n,$$

where $(\varepsilon_n)_{n=1}^{\infty}$ is any decreasing sequence of positive numbers. Then $(g_n)_{n=1}^{\infty}$ is equivalent in $M_p$ to the unit vector basis of $l_p$, and its closed linear span is the range of a contractive projection on $L_2(0, \infty)$ whose restriction to $M_p$ is a bounded projection. A standard perturbation argument now shows that $(f_n)_{n=1}^{\infty}$ spans a complemented subspace of $M_p$ isomorphic to $l_p$ provided $\varepsilon_n$ decreases rapidly to zero.

Theorem 4.2. Let $1 \leq p \leq 2$. Then the only rearrangement invariant (r.i.) function space on $[0,1]$ to embed isomorphically into $M_p$ is $L_2(0,1)$.

Proof. Suppose that $X$ is an r.i. space on $[0,1]$ that is isomorphic to a subspace of $M_p(1 \leq p \leq 2)$. Since $M_p$ is a 2-concave Banach lattice it follows that $X$ is also 2-concave, and so $\|f\|_X \leq C\|f\|_2$ for all $f \in L_2(0,1)$ and some constant $C$ (e.g., [17, p. 133]). Now recall that $M_p = L_{\phi_p}(0,\infty)$ (Proposition 1.3), and so by [8, p. 169 and p. 198] either $X = L_2(0,1)$ or

$$\|f\|_X \geq \frac{1}{K}\|f\|_{M_p} = \frac{1}{K}\|f\|_2$$

for all $f \in L_2(0,1)$ and some constant $K$. In the latter case, we have

$$\frac{1}{K}\|f\|_2 \leq \|f\|_X \leq C\|f\|_2.$$

Thus $X = L_2(0,1)$ in any case.

Corollary 4.3. Let $1 \leq p \leq \infty$. Then $M_p$ is not isomorphic to an r.i. space on $[0,1]$.

Proposition 4.4. Let $1 \leq p < \infty$. Then the only complemented subspaces of $M_p$ with a symmetric Schauder basis are (up to isomorphism) $l_2$ and $l_p$. 

Proof. First suppose that $1 < p < 2$. Let $(f_n)_{n=1}^\infty$ be disjoint functions in $M_p$ and suppose that $\|f_n\|_p = 1$ for all $n$. If the sequence $(f_n)_{n=1}^\infty$ is bounded in $L^2(0, \infty)$, then $(f_n)_{n=1}^\infty$ is equivalent in $M_p$ to the unit vector basis of $l_p$. If $(f_n)_{n=1}^\infty$ is unbounded then we may pass to a subsequence $(f_k)_{k=1}^\infty$ such that $\|f_k\|_2 \geq k^{1/p}$. Then by Hölder’s inequality,

$$\left\| \sum_{k=1}^\infty a_k f_{nk} \right\|_p \leq \left( \sum_{k=1}^\infty \left( \frac{1}{k} \right)^{2(2-p)/(2-p)} \right)^{(2-p)/2} \left( \sum_{k=1}^\infty k^{2/p} |a_k|^2 \right)^{1/2} \leq C_p \left\| \sum_{k=1}^\infty a_k f_{nk} \right\|_2,$$

and so $(f_{nk})_{k=1}^\infty$ is equivalent in $M_p$ to the unit vector basis of $l_2$. It follows that the only disjoint symmetric sequences in $M_p$ are $l_p$ and $l_2$. The desired conclusion now follows in the case $1 < p < 2$ from [8, Lemma 8.10]. The case $2 < p < \infty$ follows by duality.

**Corollary 4.5.** Let $1 \leq p \leq \infty$. Then $M_p$ and $L^2(0, \infty)$ are the only r.i. spaces on $(0, \infty)$ which are isomorphic to complemented subspaces of $M_p$.

Proof. First suppose that $1 \leq p < 2$. Let $X$ be an r.i. space on $(0, \infty)$ which is isomorphic to a complemented subspace of $M_p$. By Theorem 4.2 the restriction of $X$ to $[0,1]$ is $L^2(0,1)$. Thus

$$\|f\|_X \sim \|f^* I(0,1)\|_2 + \psi((f^*(n))_{n=1}^\infty),$$

where $\psi$ is the symmetric sequence norm associated with the sequence $(I(n-1,n))_{n=1}^\infty$ in $X$. This sequence corresponds to a complemented subspace of $M_p$ with a symmetric basis, and so by Proposition 4.4, $\psi$ is equivalent to the $l_p$ or $l_2$ norm. The former easily implies that $X = M_p$ and the latter that $X = L^2(0, \infty)$.

**Corollary 4.6.** Let $1 \leq p \leq \infty$. Then the Banach space $M_p$ has a unique representation as an r.i. function space on $(0, \infty)$.

**Proposition 4.7.** Let $X$ be an r.i. space on $(0, \infty)$, and let $1 \leq p < 2$. If $X$ is order isomorphic to a sublattice of $M_p$, then $X = M_p$ or $X = L^2(0, \infty)$.

Proof. It was shown in the proof of Proposition 4.4 that $l_2$ and $l_p$ are the only disjoint symmetric sequences in $M_p$ for $p < 2$. If $X$ is order isomorphic
to a sublattice of $M_p$, then the sequence $(I(n - 1, n))_{n=1}^\infty$ in $X$ corresponds to a disjoint symmetric sequence in $M_p$. The proof is concluded as in Corollary 4.5.

5. The spaces $M_p$ ($2 < p < \infty$)

As is the case for $L_q$ with $0 < q < 2$ the subspace structure of $M_p$ is quite varied in the range $2 < p < \infty$. We prove below, for example, a counterpart of the well-known embedding of $L_s$ into $L_r$ for $0 < s \leq r \leq 2$. But first we deduce the appropriate version of Theorem 4.1 for this range. The reader is referred to [14] for the theory of "stable" Banach spaces.

**Proposition 5.1.** Let $1 < p < \infty$. Then $M_p$ is isomorphic to a stable Banach space.

**Proof.** By the results of [14] $L_2(l_p)$ is stable for $1 \leq p < \infty$. The result now follows from Theorem 3.1 since stability is inherited by subspaces.

**Corollary 5.2.** Let $2 < p < \infty$ and let $X$ be an infinite-dimensional subspace of $M_p$. Then $X$ contains a subspace isomorphic to $l_r$ for some $r$ in $[2, p]$.

**Proof.** It is easy to verify that $M_p$ has "type 2" and "cotype $p$" in this range (see [17] for the basic facts about type and cotype), whence it follows that if $l_r$ embeds in $M_p$ then $2 \leq r \leq p$. Finally, a stable Banach space necessarily contains a copy of $l_r$ for some $r$.

**Proposition 5.3.** Let $2 < p < \infty$ and let $X$ be a complemented subspace of $M_p$. Then $X$ is isomorphic to $l_2$ or $X$ contains a subspace isomorphic to $l_p$ which is complemented in $M_p$.

**Proof.** This follows by duality from Theorem 4.1.

**Proposition 5.4.** Let $2 < p < q < \infty$. Let

$$f_1(t) = \begin{cases} t^{-1/p} & (0 < t \leq 1) \\ 0 & (t > 1) \end{cases}$$

and let $f_n(t)$ be the translation of $f_1$ to the interval $(n - 1, n)$. Then $(f_n)_{n=1}^\infty$ is equivalent in $M_q$ to the unit vector basis of $l_p$.

**Proof.** $M_q$ contains the function $g(t) = t^{-1/p}$ ($0 < t < \infty$), and any sequence $(g_n)_{n=1}^\infty$ of disjoint functions having the same distribution as $g$ is
obviously equivalent to the $l_p$ basis. Now
\[ \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p} \sim \left\| \sum_{n=1}^{\infty} a_n g_n \right\|_{M_p} \geq \left\| \sum_{n=1}^{\infty} a_n f_n \right\|_{M_p}, \]
and so to conclude the proof it is enough to show that
\[ \left\| \sum_{n=1}^{\infty} a_n f_n \right\|_{M_p} \geq \frac{1}{C} \left( \sum_{n=1}^{\infty} |a_n|^p \right)^{1/p}. \]

Select $a_1, a_2, \ldots, a_n$ such that $\sum_{k=1}^{n} |a_k|^p = 1$ and let $f = \sum_{k=1}^{n} a_k f_k$. Then
\[ \|f^* I(0, 1)\|_{2}^2 \geq \sum_{k=1}^{n} |a_k|^2 \int_{0}^{1} |a_k|^p t^{-2/p} dt \]
\[ = \frac{p}{p-2} \sum_{k=1}^{n} |a_k|^p \]
\[ = \frac{p}{p-2}, \]
and so
\[ \|f\|_{M_p} \geq \sqrt{\frac{p}{p-2}}. \]

**Corollary 5.5.** (a) Let $2 < p < \infty$. Then $M_p$ is not isomorphic to a subspace of $L_p$.
(b) Let $1 \leq p < 2$ or $p = \infty$. Then $M_p$ is not isomorphic to a complemented subspace of $L_p$.

**Proof.** (a) By [10, p. 169], $l_2$ and $l_p$ are the only symmetric basic sequences in $L_p$ for $p > 2$. (Also by [8] the only r.i. spaces on $(0, \infty)$ to embed into $L_p$ ($p > 2$) are $L_p$ and $L_2$.)

(b) The case $1 < p < 2$ follows by duality from (a), and the case $p = 1$ or $p = \infty$ simply form the fact that $l_2$ is not isomorphic to a complemented subspace of $L_1$ or $L_\infty$.

We are now able to conclude the proof of Theorem 3.7.

**Corollary 5.6.** Let $2 < p < \infty$. Then $M_p$ is not isomorphic to a subspace of $(\Sigma \oplus l_p)_2$. 
Proof. By Proposition 5.4 it is enough to show that $L_r$ is not isomorphic to a subspace of $(\Sigma \oplus l_p)_2$ for $2 < r < p$. This is a fairly routine "gliding hump" argument and will therefore be omitted.

We now use Proposition 5.4 to prove an analogue of the "stable embedding" of $L_r$ into $L_p$ for $1 \leq p \leq r \leq 2$.

**Theorem 5.7.** Let $2 < p < q \leq \infty$. Then $M_p$ is order isomorphic to a sublattice of $M_q$.

**Proof.** Let $f_0^{(1)}$ be a countably valued function on $[0,1]$ such that

$$t^{-1/p} \leq f_0^{(1)} \leq 2t^{-1/p}.$$  

We define a mapping $T$ from the simple dyadic functions on $[0,\infty)$ into $M_q$. Put $T(I(0,1)) = f_0^{(1)}$. Now choose disjoint identically distributed functions $f_{1,1}^{(1)}$ and $f_{1,2}^{(1)}$ such that $f_0^{(1)} = f_{1,1}^{(1)} + f_{1,2}^{(1)}$. Let

$$T\left(I\left(0, \frac{1}{2}\right)\right) = f_{1,1}^{(1)} \quad \text{and} \quad T\left(I\left(\frac{1}{2}, 1\right)\right) = f_{1,2}^{(1)}.$$  

Now select identically distributed functions $f_{2,k}^{(1)}$ ($1 \leq k \leq 4$) such that

$$f_{1,1}^{(1)} = f_{2,1}^{(1)} + f_{2,2}^{(1)} \quad \text{and} \quad f_{1,2}^{(1)} = f_{2,3}^{(1)} + f_{2,4}^{(1)}$$  

and define

$$T\left(I\left(\frac{k-1}{4}, \frac{k}{4}\right)\right) = f_{2,k}^{(1)} \quad (1 \leq k \leq 4).$$  

Continue in this manner to define $T$ on all simple dyadic functions on $[0,1]$. Finally, extend $T$ by translation to all simple dyadic functions on $[0,\infty)$. Observe that $T$ has the following properties: (i) if $f$ and $g$ have the same distribution, then $Tf$ and $Tg$ have the same distribution; (ii) if $|f| \leq |g|$ then $|Tf| \leq |Tg|$. Now let $f$ be any simple dyadic function on $(0,\infty)$. Write $f = g + h$, where $g^*(t) = f^*(1)$ for $0 \leq t \leq 1$, $g^*(t) = f^*(t)$ for $t \geq 1$, and $h = f - g$. Then

$$\|Tf\|_{M_q} \sim \max\left(\|Tg\|_{M_q}, \|Th\|_{M_q}\right).$$  

Plainly, $\|Th\|_{M_q} = \|Tg\|_2 = C\|h\|_2$, where $C$ is an absolute constant. To compute $\|Tg\|$, observe that

$$\sum_{n=1}^{\infty} f^*(n)I(n-1, n) \leq g^* \leq f^*(1)I(0,1) + \sum_{n=1}^{\infty} f^*(n)I(n, n+1).$$
So
\[ \sum_{n=1}^{\infty} f^*(n) T(I(n-1, n)) \leq T(g^*) \]
\[ \leq f^*(1) T(I(0,1)) + \sum_{n=1}^{\infty} f^*(n) T(I(n-1, n)), \]
whence by Proposition 5.4,
\[ \|T(g)\|_{M_q} = \|T(g^*)\|_{M_q} \sim \left( \sum_{n=1}^{\infty} |f^*(n)|^p \right)^{1/p} . \]
It follows that
\[ \|Tf\|_{M_q} \sim \max\left( \|f^*I(0,1)\|_2, \|f^*I(1,\infty)\|_p \right), \]
and so \( T \) extends to a lattice isomorphism from \( M_p \) onto a sublattice of \( M_q \).

**Proposition 5.8.** Let \( 2 < r < p \leq \infty \). Then \( L_r(0,1) \) is isometric to a sublattice of \( M_p \).

**Proof.** The function \( t^{-1/r} \) belongs to \( M_p \), and so the result follows at once from [8, p. 222].

6. The space \( M_\infty \)

In this short section we initiate an examination of the Banach space \( M_\infty \). These results are analogous to some well-known facts about \( L_1 \) (cf. [10, 18]).

**Theorem 6.1.** \( M_\infty \) does not embed into any Banach space with an unconditional basis.

**Proof.** By Proposition 5.8, \( L_r(0,1) \) is isometric to a subspace of \( M_\infty \) for all \( 2 < r < \infty \). The result now follows from the "reproducibility" of the Haar system and from the fact that the constant of unconditionality of the Haar basis in \( L_r(0,1) \) increases without limit as \( r \) increases [15].

**Theorem 6.2.** Let \( X \) be a subspace of \( M_\infty \). Then either \( X \) is reflexive or \( X \) contains a subspace isomorphic to \( c_0 \) and complemented in \( M_\infty \).

**Proof.** Since \( M_\infty \) is an order continuous Banach lattice it follows from [16, p. 35] that every non-reflexive subspace contains \( c_0 \) or \( l_1 \). But \( l_1 \) is not contained in \( M_\infty \) since \( M_\infty^* = M_1 \) is separable. Thus every non-reflexive subspace contains a subspace isomorphic to \( c_0 \), and by Sobczyk's theorem [21] such a subspace must be complemented in \( M_\infty \).
Corollary 6.3. Let $X$ be a complemented subspace of $M_\infty$. Then either $X$ is isomorphic to $l_2$ or $X$ contains a subspace isomorphic to $c_0$ and complemented in $M_\infty$.

Proof. Since $X^*$ is isomorphic to a subspace of $M_1$, it follows from Theorem 4.1 that every reflexive complemented subspace of $M_\infty$ is isomorphic to $l_2$. Theorem 6.2 now concludes the proof.

Corollary 6.4. Let $X$ be an infinite-dimensional complemented subspace of $M_\infty$ with the Radon-Nikodym property. Then $X$ is isomorphic to $l_2$.

7. The spaces $M_p$ ($0 < p < 1$)

The proof of the following result is standard and has been omitted. (See [13] for the definition of Banach envelope.)

Proposition 7.1. Let $0 < p < 1$. Then the Banach envelopes of $M_p$ is isomorphic to $M_1$.

Corollary 7.2. Let $0 < p < 1$ and let $T: M_p \to X$ be a bounded linear operator from $M_p$ into a quasi-Banach space $X$. Then either $T$ factorizes through $M_1$ or $T$ fixes a copy of $l_2$.

Proof. We use the fact that $M_p = L_\phi(0, \infty)$. The result is now an immediate consequence of a theorem of Kalton [11, Theorem 3.4]. (In fact, Kalton's theorem is stated for operators on the Orlicz space $L_\phi(0,1)$, but the proof readily extends to $L_\phi(0, \infty)$.)

The next result is proved by checking the above dichotomy for an operator from $M_p$ into $l_p$ and for a projection on $M_p$.

Corollary 7.3. Let $0 < p < 1$.
(a) Every operator from $M_p$ into $l_p$ is compact.
(b) Every complemented subspace of $M_p$ contains $l_2$.

Proposition 7.4. Let $0 < p < 1$ and let $X$ be a separable Banach space. Then $X$ is isomorphic to a quotient space of $M_p$.

Proof. It is easily verified that the operator $f \to (\int_{n-1}^{n} f(t) \, dt)^{\infty}_{n=1}$ defines a quotient mapping from $M_p$ onto $l_1$. To conclude, recall that every separable Banach space is a quotient of $l_1$.

A weaker version of Theorem 4.1 is also valid.
PROPOSITION 7.5. Let $0 < p < 1$ and let $X$ be a subspace of $M_p$. Then either $X$ is isomorphic to a Hilbert space and complemented in $M_p$, or $X$ contains a subspace isomorphic to $l_p$.

Finally, we state a consequence of a further theorem of Kalton [11, Corollary 2.4].

PROPOSITION 7.6. Let $0 < p < 1$. Then $M_p$ is not isomorphic to a subspace of a quasi-Banach space with a Schauder basis.

Appendix

Recall that a Banach space $X$ is said to be primary if whenever $X$ is isomorphic to $Y \oplus Z$, then either $Y$ or $Z$ is isomorphic to $X$. The fact that the $M_p$ spaces are primary for $1 < p < \infty$ is just a special case of the following theorem. The same result for r.i. spaces on $[0,1]$ was proved in [2].

THEOREM A.1. Let $X$ be a separable r.i. function space on $(0, \infty)$ whose Boyd indices $p_X$ and $q_X$ satisfy $1 < p_X, q_X < \infty$. Then $X$ is a primary Banach space.

The proof of Theorem A.1 closely follows the argument of [2] as it is expounded in [17]. Because the necessary changes are fairly routine it will suffice to state without proof some of the preliminary results that are needed, and to leave the verification of Theorem A.1 to the interested reader. The main idea is to use the system $H$ (see Proposition 1.4 above) in place of the Haar system. Throughout $X$ denotes a separable r.i. space on $(0, \infty)$ whose Boyd indices satisfy $1 < p_X, q_X < \infty$.

PROPOSITION A.2 (cf. [17, p. 172]). $X$ is isomorphic to the Banach space $X(l_2)$.

PROPOSITION A.3 (cf. [17, p. 172]). Let $Y$ be a complemented subspace of $X$ which itself contains a complemented subspace isomorphic to $X$. Then $Y$ is isomorphic to $X$.

Let $(\phi_n)_{n=1}^{\infty}$ be a subsequence of the system $H$ and let

$$\sigma_n = \{ t \in [n - 1, n]: \phi_k(t) \neq 0 \text{ for infinitely many } k \}.$$ 

The following result is the key ingredient in the proof of Theorem A.1.

PROPOSITION A.4 (cf. [17], p. 178). Suppose that there exists $\delta > 0$ such that $|\sigma_n| > \delta$ for infinitely many $n$. Then $(\phi_n)_{n=1}^{\infty}$ is isomorphic to $X$.
REFERENCES


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