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Involution on Banach Spaces and Reflexivity

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1. Notation and results. Let $E$ and $F$ be (real or complex) Banach spaces. $E$ is said to be finitely representable in $F$ if, given $\epsilon > 0$ and a finite dimensional subspace $E_0$ of $E$, there exists a subspace $F_0$ of $F$ such that 
\[ d(E_0, F_0) \leq 1 + \epsilon, \]
where
\[ d(E_0, F_0) = \inf \{ T^{-1} \text{ is an isomorphism from } E_0 \text{ onto } F_0 \} \]
denotes the Banach-Mazur distance coefficient. $E$ is said to be super-reflexive if every Banach space which is finitely representable in $E$ is reflexive. Super-reflexivity has been characterized in terms of the notion of $J$-convexity: suppose that $n \geq 1$ and that $\epsilon > 0$; $E$ is said to be $J(n, \epsilon)$-convex if, for all $x_1, \ldots, x_n$ in the unit ball of $E$, we have
\[ \inf_{1 \leq k \leq n-1} \| x_1 + \cdots + x_k - x_{k+1} - \cdots - x_n \| \leq n - \epsilon. \]
The "if" part of the following theorem was proved in [12] and [5], and the "only if" part was proved in [10].

**Theorem A.** $E$ is super-reflexive if and only if $E$ is $J(n, \epsilon)$-convex for some $n \geq 1$ and $\epsilon > 0$.

The main purpose of this article is to extend Theorem A to a certain class of operators. To this end we introduce some new definitions: an operator $T$ on $E$ will be said to be $J(n, \epsilon)$-convexifying ($n \geq 1$ and $\epsilon > 0$) if, for all $x_1, \ldots, x_n$ in the unit ball of $E$, we have
\[ \inf_{0 \leq k \leq n} \| x_1 + \cdots + x_k + T(x_{k+1} + \cdots + x_n) \| \leq n - \epsilon. \]
When no importance is placed on $\epsilon$ or $n$ we shall say that $T$ is $J(n)$-convexifying or simply $J$-convexifying. $T$ will be said to be an involution (of order $n \geq 1$) if $T^n = I$, where $I$ denotes the identity operator on $E$. The following main result is proved in Section 4 below.
THEOREM 1.1. Suppose that $E$ admits a $J$-convexifying involution. Then either $c_0$ is finitely representable in $E$ or $E$ is super-reflexive.

Combining Theorem 1.1 with the "only if" part of Theorem A gives rise to the following characterization of super-reflexive Banach spaces.

THEOREM 1.2. $E$ is super-reflexive if and only if $c_0$ is not finitely representable in $E$ and $E$ admits a $J$-convexifying involution.

Theorem 1.1 gives the following geometrical characterization of super-reflexive complex Banach spaces.

THEOREM 1.3. Suppose that $E$ is a complex Banach space, that $|\lambda| = 1$ and that $\lambda \neq 1$. Then $E$ is super-reflexive if and only if there exist $n \geq 2$ and $\epsilon > 0$ such that for all $x_1, \ldots, x_n$ in the unit ball of $E$, we have

$$\inf_{1 \leq k \leq n} \|x_1 + \cdots + x_k + \lambda x_{k+1} + \lambda x_n\| \leq n - \epsilon.$$

PROOF: Necessity is proved in Corollary 2.3 below. Sufficiency follows from Theorem 1.1 when $\lambda$ is a root of unity (and so multiplication by $\lambda$ is an involution) by observing that $c_0(C)$ does not satisfy the hypothesis. The case for general $\lambda \neq 1$ is simply a consequence of the density of the roots of unity in the unit circle.

It is not known to me whether the possibility of $c_0$ being finitely representable in $E$ in the conclusion of Theorem 1.1 may be eliminated, but when $E$ is a complex Banach space this can be done.

THEOREM 1.4. Suppose that $E$ is a complex Banach space. Then $E$ is super-reflexive if and only if $E$ admits a $J$-convexifying involution.

PROOF: We need only prove sufficiency. It follows from the theory of algebraic operators (e.g., [11]) that if $T$ is an involution on $E$, then $E$ may be written as a direct sum of closed subspaces $E_i$ on which $T$ acts as multiplication by a root of unity. If $c_0(C)$ is finitely representable in $E$, then $c_0(C)$ is finitely representable in some $E_i$, but this means that $T$ is not $J$-convexifying.

We conclude by stating a special case of Theorem 1.3 which may be regarded as a complex version of a theorem of R. C. James on uniformly non-square Banach spaces ([8]).
THEOREM 1.5. Suppose that $E$ is a complex Banach space, that $\epsilon > 0$, and that for all $x, y$ in the unit ball of $E$, we have

$$\min\{\|x + y\|, \|x + iy\|\} \leq 2 - \epsilon.$$  

Then $E$ is reflexive.

2. J-convexifying operators. In this section we shall make use of the notion of the numerical range of an operator, which we now define. Suppose that $E$ is a Banach space (either real or complex); the collection $\Pi$ is defined by

$$\Pi = \{(x, f) : \|f\| = \|x\| = f(x) = 1\} \subset E \times E^*.$$  

The numerical range of an operator $T$ on $E$, denoted $W(T)$, is defined by

$$W(T) = \{f(Tx) : (x, f) \in \Pi\}.$$  

PROPOSITION 2.1. Suppose that $T$ is a J-convexifying operator on $E$.

(a) $W(T) \subseteq \{z : \Re(z) < 1\}$.

(b) If $\|T\| = 1$ then $\|I + T\| < 2$.

PROOF: (a) Let $\epsilon > 0$ and $n \geq 1$ be given. If $\overline{W(T)} \nsubseteq \{z : \Re(z) < 1\}$ then there exists $(x, f) \in \Pi$ such that $\Re(f(Tx)) \geq 1 - \epsilon/2n$. It follows that for each $0 \leq k \leq n$, we have

$$\|kx + (n - k)Tx\| \geq |f(kx + (n - k)Tx)| \geq n - \frac{\epsilon}{2}.$$  

Hence $T$ is not J-convexifying, and the result follows.

(b) By a theorem of Lumer (e.g., [2, page 82]), we have

$$\sup\{\Re(z) : z \in W(T)\} = \lim_{\alpha \to 0} \frac{1}{\alpha} (\|I + \alpha T\| - 1);$$

but by (a) there exists $t < 1$ such that $\sup\{\Re(z) : z \in W(T)\} < t$, and so there exists $0 < \alpha < 1$ such that $\|I + \alpha T\| < 1 + \alpha t$. Since $\|T\| = 1$ it follows that $\|I + T\| < 2$.

The next result concerns J-convexifying operators on super-reflexive spaces. It generalizes one of the implications in Theorem A and serves as a partial converse to Proposition 2.1(b).
**PROPOSITION 2.2.** Suppose that $E$ is super-reflexive and that $T$ is a norm one operator on $E$ with $\|I + T\| < 2$. Then $T$ is $J$-convexifying.

**PROOF:** Select $\epsilon > 0$ and $\delta > 0$ such that $\|I + T\| + \epsilon + \delta < 2$. If $T$ is not $J$-convexifying, then for each $n \geq 1$ there exist $x_1, \ldots, x_n$ in the unit ball of $E$ such that

$$\inf_{0 \leq k \leq n} \|x_1 + \cdots + x_k + T(x_{k+1} + \cdots + x_n)\| > n - \epsilon.$$ 

Now suppose that $1 \leq k \leq n$, that $\alpha_i \geq 0$ and that $\sum_{i=1}^{k} \alpha_i = \sum_{i=k+1}^{n} \alpha_i = 1$. Then

$$\|\alpha_1 x_1 + \cdots + \alpha_k x_k + T(\alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n)\|$$

$$\geq \|x_1 + \cdots + x_k + T(x_{k+1} + \cdots + x_n)\| - \|(1 - \alpha_1)x_1 + \cdots + (1 - \alpha_k)x_k + T((1 - \alpha_{k+1})x_{k+1} + \cdots + (1 - \alpha_n)x_n)\|$$

$$\geq n - \epsilon - \sum_{i=1}^{k} (1 - \alpha_i) - \sum_{i=k+1}^{n} (1 - \alpha_i)\|T\|$$

$$\geq n - \epsilon - (k - 1) - (n - k - 1)$$

$$= 2 - \epsilon.$$ 

Using the fact that $\|I + T\| + \epsilon + \delta < 2$, we obtain

$$\|\alpha_1 x_1 + \cdots + \alpha_k x_k - (\alpha_{k+1} x_{k+1} + \cdots + \alpha_n x_n)\| \geq 2 - \epsilon - \|I + T\| \geq \delta.$$ 

So for each $1 \leq k \leq n$, we have

$$d(\text{conv}(x_1, \ldots, x_k), \text{conv}(x_{k+1}, \ldots, x_n)) \geq \delta.$$ 

It follows from a characterization of super-reflexivity in [9] that $E$ is not super-reflexive. This contradiction proves that $T$ is $J$-convexifying.

The following immediate consequence of Proposition 2.2 completes the proof of Theorem 1.3.

**COROLLARY 2.3.** Suppose that $E$ is a super-reflexive complex Banach space, that $|\lambda| = 1$, and that $\lambda \neq 1$. Then there exist $n \geq 2$ and $\epsilon > 0$ such that for all $x_1, \ldots, x_n$ in the unit ball of $E$, we have

$$\inf_{1 \leq k \leq n} \|x_1 + \cdots + x_k + \lambda x_{k+1} + \cdots + \lambda x_n\| \leq n - \epsilon.$$ 

When $E$ is uniformly convex the converse of Proposition 2.1(a) is also true.
THEOREM 2.4. Suppose that $T$ is an operator on a uniformly convex space $E$. Then $T$ is $J$-convexifying if and only if $\overline{W(T)} \subset \{ z : \text{Re}(z) < 1 \}$.

PROOF: Necessity is proved in Proposition 2.1 To prove the converse we shall suppose that $\overline{W(T)} \subset \{ z : \text{Re}(z) < k \}$, where $k < 1$. Select $\eta > 0$ so that $k + 2\eta \|T\| = m < 1$ and let $\epsilon = \eta^2/8$. Since $E$ is uniformly convex there exists $\delta \in (0, \frac{1}{2}(1 - m))$ such that if $\|x\| \leq 1$, $\|y\| \leq 1$ and $\|x + y\| \geq 2 - \delta$, then $\|x - y\| \leq \epsilon$. Now select $n \geq 1$ such that $(\|T\| + \delta - 1)/(n - 1) < \epsilon$. Suppose that $x_1, \ldots, x_n$ lie in the unit ball of $E$ and that $\|x_1 + \cdots + x_n\| \geq n - \delta$. Then $\|x_i + x_j\| \geq 2 - \delta$ $(1 \leq i < j \leq n)$, and so $\|x_i - x_j\| \leq \epsilon$. To obtain a contradiction we shall suppose that $x_1 + \cdots + x_n + T(x_n) \leq n - 5$. Select $f \in E^*$ such that $f = 1$ and $f(x_1 + \cdots + x_n + T(x_n)) = \|x_1 + \cdots + x_n + T(x_n)\|$. It follows that

$$\max_{1 \leq j \leq n} \text{Re}(f(x_j)) \geq \frac{n - \delta - \|T\|}{n - 1} \geq 1 - \epsilon,$$

and so $\text{Re}(f(x_n)) \geq 1 - 2\epsilon \geq 1 - \eta^2/4$. By the Bishop-Phelps-Bollobás Theorem (e.g., [2]) there exists $(x, g) \in \Pi$ such that $\|x - x_n\| < \eta$ and $\|g - f\| < \eta$. So

$$\text{Re}(f(Tx_n)) \leq \text{Re}(g(Tx_n)) + \eta \|T\|$$

$$\leq \text{Re}(g(Tx)) + 2\eta \|T\|$$

$$\leq k + 2\eta \|T\|$$

$$< 1 - 2\delta.$$ 

Hence $\|x_1 + \cdots + x_n + T(x_n)\| = f(x_1 + \cdots + x_n + T(x_n)) \leq n - 2\delta$, which is the desired contradiction. It follows that $T$ is $J(n, \delta)$-convexifying.

COROLLARY 2.5. The following are equivalent:

(i) $E$ is super-reflexive;

(ii) if $T$ is an operator on $E$ such that $\overline{W(T)} \subset \{ z : \text{Re}(z) < 1 \}$ then $E$ can be renormed so that $T$ is a $J$-convexifying operator with respect to the new norm.

PROOF: (ii) implies (i) follows at once from Theorem A by considering $T = -I$. To prove that (i) implies (ii) we recall that a super-reflexive Banach space admits an equivalent uniformly convex norm ([7]). Moreover,
it is well known (see e.g., [1, page 211]) that if \((E, \| \cdot \|)\) admits an equivalent uniformly convex norm then, given \(\epsilon > 0\), there exists a uniformly convex norm \(\| \cdot \|\) on \(E\) such that \((1 - \epsilon)\|x\| \leq \|x\| \leq (1 + \epsilon)\|x\|\) for all \(x \in E\). A straightforward perturbation argument involving the Bishop-Phelps-Bollobas theorem proves that provided \(\epsilon\) is sufficiently small the numerical range of \(T\) with respect to \(\| \cdot \|\) still satisfies \(W(T) \subseteq \{z : \text{Re}(z) < 1\}\). It now follows from Theorem 2.4 that \(T\) is \(J\)-convexifying with respect to \(\| \cdot \|\).

We conclude this section by recalling the notion of an ultrapower of a Banach space, which will be needed in Section 4. Let \(F\) denote the collection of all bounded sequences \(x = (x_n)_{n=1}^\infty\) in \(E\), and let \(U\) be a non-trivial ultrafilter on \(N\). A semi-norm on \(F\) is defined by \(\|x\| = \lim_U \|x_n\|\). Quotienting \(F\) by the kernel of this semi-norm and taking the completion gives rise to the ultrapower \(E^N/U\). An operator \(T\) on \(E\) induces an operator \(\hat{T}\) on \(E^N/U\) in the obvious way. The following proposition, whose straightforward proof is omitted, will be needed in Section 4.

**Lemma.** If \(T\) is a \(J\)-convexifying operator on \(E\), then \(\hat{T}\) is \(J\)-convexifying on \(E^N/U\).

We shall also need to use the fact that \(E\) is super-reflexive if and only if every ultrapower \(E^N/U\) is reflexive.

3. Generalization of the Brunel-Sucheston technique. To prove Theorem 1.1 we need to develop the machinery of the Brunel-Sucheston procedure ([4], [5]) in the more general setting of an algebra of operators acting on a normed space. Once the definitions have been decided upon much of the theory carries across from [4], [5] with only minor modifications; when this is so we shall merely state the corresponding result without proof.

Suppose that \(A\) is a real or complex algebra with identity and that \(E\) is a normed space. We shall say that \(E\) is an \(A\)-module if \(A\) acts as an algebra of bounded operators on \(E\). Let \(N = \{\alpha \in A : \alpha x = 0 \text{ for all } x \in E\}\); then \(A/N\) may and shall be regarded as a subalgebra of the algebra of all operators on \(E\). Let \(S\) denote the space of all sequences \(a = (a_i)_{i=1}^\infty\) of elements of \(A\) with only finitely many non-zero terms.

**Theorem 3.1.** Let \((y_n)_{n=1}^\infty\) be a bounded sequence in \(E\) and suppose that \(A/N\) is separable in the operator norm topology. There exists a semi-norm \(\| \cdot \|\) on \(S\) and a subsequence \((x_n)_{n=1}^\infty\) such that, for all \(a \in S\) and \(\epsilon > 0\),
there exists a positive integer \( v \) such that

\[
\left| \left\| \sum_{i=1}^{\infty} a_i(x_{n_i}) \right\| - \|a\| \right| \leq \epsilon
\]

for all integers \( \nu \leq n_1 < n_2 < \cdots \).

We shall assume throughout the remainder of this section that \((x_n)_1^\infty, (y_n)_1^\infty\) and the seminorm \(\| \cdot \|\) are fixed. If \(\tilde{K}\) is the kernel of this semi-norm then \(S/\tilde{K}\) is itself a normed \(A\)-module with the action defined coordinate-wise.

**Proposition 3.2.** \(S/\tilde{K}\) is finitely representable in \(E\).

We now introduce a type of finite representability which appropriately reflects the \(A\)-module structure. Suppose that \(E\) and \(F\) are normed \(A\)-modules. Then \(E\) will be said to be \(A\)-finitely representable in \(F\) if, for all positive integers \(n\) and \(N\), for all \(z,...,Z_n\) in \(E\), and for all \(n\)-tuples \((\alpha_1^k,...,\alpha_n^k)\) \((1 \leq k \leq N)\) of elements of \(A\), there exist \(W_1,...,W_n\) in \(F\) such that

\[
\left| \left\| \sum_{i=1}^{n} \alpha_i^k z_i \right\| - \left\| \sum_{i=1}^{n} \alpha_i^k w_i \right\| \right| < \epsilon \quad (1 \leq k \leq N).
\]

A standard compactness argument shows that the above definition coincides with the usual notion of finite representability when \(E\) and \(F\) are just normed spaces.

**Proposition 3.3.** \(S/\tilde{K}\) is \(A\)-finitely representable in \(E\).

**Proof:** Suppose that \(\hat{z}_1,...,\hat{z}_n\) are any vectors in \(S/\tilde{K}\) and that \((\alpha_1^k,...,\alpha_n^k)\) \((1 \leq k \leq N)\) are \(n\)-tuples of elements of \(A\). Let \(z_1,...,z_n\) be representatives from \(S\) of \(\hat{z}_1,...,\hat{z}_n\). For \(m \geq 1\), let \(R_m : S \to E\) be the \(A\)-module homomorphism uniquely defined by \(R_m(e_k) = x_{m+k}\) (here \((e_k)_{k=1}^\infty\) is the canonical basis of \(S\) as a free \(A\)-module). Given \(\epsilon > 0\), there exists \(m \geq 1\) such that

\[
\left| \left\| \sum_{i=1}^{n} \alpha_i^k z_i \right\| - \left\| R_m(\sum_{i=1}^{n} \alpha_i^k z_i) \right\| \right| < \epsilon.
\]
for each $1 \leq k \leq N$. Setting $w_i = R_m(z_i)$ and using the fact that $R_m$ is an $A$-module homomorphism, we obtain

$$\left\| \sum_{i=1}^{n} \alpha_i^k z_i \right\| - \left\| \sum_{i=1}^{n} \alpha_i^k w_i \right\| < \epsilon$$

for each $1 \leq k \leq N$. This completes the proof.

$(S, \| \cdot \|)$ has the property that, for all $k \geq 1$, for all natural numbers $n_1 < n_2 < \cdots < n_k$, and for all $a = \sum_{i=1}^{k} a_i e_i$ in $S$, we have $\| \sum_{i=1}^{k} a_i e_i \| = \| \sum_{i=1}^{k} a_i e_{n_i} \|$. In accordance with [4] such a semi-norm on $S$ will be called “invariant under spreading” (or I.S.). We turn now to define an analogue of the “equal signs additive” norm of [5], [6]. For each $n \geq 1$, the averaging operator $A_n : S \to S$ is defined by $A_n(e_k) = \frac{1}{n} (e_k + e_{k+1} + \cdots + e_{k+n-1})$ with extension to $S$ by $A$-linearity. Given $a = a_1 e_1 + \cdots + a_r e_r$ in $S$, we consider the vector $\sum a_i A_{n_i} (e_{s_1}) + \cdots + a_r A_{n_r} (e_{s_r})$, where $s_1 > 0$, $s_2 \geq s_1 + n_1, \ldots, s_r \geq s_{r-1} + n_{r-1}$. The I.S. property of $\| \cdot \|$ guarantees that the semi-norm of this vector does not depend on the choice of $s_1, \ldots, s_r$; it shall be denoted by $F(a; n_1, \ldots, n_r)$.

**Proposition 3.4.** For each $a = a_1 e_1 + \cdots + a_r e_r$ in $S$, the limit of $F(a; n_1, \ldots, n_r)$ as $\inf\{n_i : 1 \leq i \leq r\} \to \infty$ exists. This limit, denoted $\|a\|$, is a semi-norm on $S$.

If $K$ denotes the kernel of $\| \cdot \|$, then $S/K$ is a normed $A$-module; exactly as in Proposition 3.2 we have the following.

**Proposition 3.5.** $(S/K, \| \cdot \|)$ is both finitely representable and $A$-finitely representable in $E$.

Let $\| \cdot \|$ be a semi-norm on $S$. Then $\| \cdot \|$ will be said to be “equal terms additive” (E.T.A.) if, for each $a = a_1 e_1 + \cdots + a_r e_r$ in $S$ with $a_i = a_{i+1}$ for some $1 \leq i \leq r - 1$, we have

$$|a| = |a_1 e_1 + \cdots + a_{i-1} e_{i-1} + (a_i + a_{i+1}) e_i + a_{i+2} e_{i+2} + \cdots + a_r e_r|.$$ 

It is easily seen that an E.T.A. semi-norm is automatically I.S.

**Proposition 3.6.** $\| \| \cdot \|$ is an E.T.A. semi-norm on $S$.

**Proof:** Suppose that $a = a_1 e_1 + \cdots + a_r e_r$ with $a_i = a_{i+1}$. Let $b = a_1 e_1 + \cdots + a_{i-1} e_{i-1} + (a_i + a_{i+1}) e_i + a_{i+2} e_{i+2} + \cdots + a_r e_r$. The I.S. property implies that $F(a; n_1, \ldots, n_{i-1}N, N, n_{i+2}, \ldots, n_r) = F(b; n_1, \ldots, n_{i-1}N, 2N, n)$.
for all $n_1, \ldots, n_{i-1}, n_{i+2}, \ldots, n_r, N$ and $n$. The result follows at once.

4. $J$-convexifying involutions. This section is devoted to a proof of Theorem 1.1. We shall use the ideas of the previous section and follow the strategy of the proof of Theorem A given in [5]. Anything from [5] which transfers with only minor alteration will be stated without proof. Using the notation of the previous section we shall show that the sequence $(x_n)_{n=1}^\infty$ contains a subsequence which is convergent in Cesaro mean when the hypotheses of Theorem 1.1 are met. This will show that $E$ has the Banach-Saks property and, in particular, that $E$ is reflexive.

**Proposition 4.1 ([5]):** If $||e_1 - e_2|| = 0$ then $(x_n)_{n=1}^\infty$ contains a subsequence which is convergent in Cesaro mean.

**Lemma 4.2.** Suppose that $T$ is a $J$-convexifying involution on a Banach space $E$. Then there exists $k \geq 1$ such that $I + T + T^2 + \cdots + T^k = 0$.

**Proof:** Suppose that $T$ is an involution of order $k + 1$. Then $\|T^n\|^{1/n} \to 1$ as $n \to \infty$, and so $T - \lambda I$ is invertible for all $\lambda > 1$. It follows that either $T - I$ is invertible or that $T - I$ fails to be an isomorphism onto its range: in the complex case this is just the familiar fact that every point in the boundary of the spectrum of $T$ is an approximate eigenvalue. If $T - I$ is not an isomorphism onto its range then, given $\epsilon > 0$, there exists a unit vector $x$ in $E$ with $\|Tx - x\| < \epsilon$. It follows that, for each $n \geq 1$, we have

$$\inf_{0 \leq k \leq n} \|kx + T((n - k)x)\| \geq n - n\epsilon.$$ 

Since $\epsilon$ is arbitrary, this contradicts the fact that $T$ is $J$-convexifying. So $T - I$ is invertible and hence $I + T + T^2 + \cdots + T^k = 0$.

Now suppose that the element $\alpha$ of $A$ satisfies $I + \alpha + \cdots + \alpha^{k-1} = 0$. We shall say that $\alpha$ is cyclic of order $k$. A sequence of vectors $(f_n)_{n=1}^\infty$ in $S$ is defined by

$$f_n = e_{(n-1)k+1} + \alpha e_{(n-1)k+2} + \cdots + \alpha^{k-1} e_{nk};$$

the real vector subspace spanned by $(f_n)_{n=1}^\infty$ will be denoted $F$. 
PROPOSITION 4.3. If \( \|e_1 - e_2\| \neq 0 \) then \((F, \|\| \|\|)\) is a normed space.

PROPOSITION 4.4. \((f_n)_{n=1}^\infty\) is an orthogonal sequence in \(F\).

PROOF: We have to show that, for all \( m \geq 1 \), for all real \( \lambda_1, \ldots, \lambda_m \), and for each \( 1 \leq r \leq m \), we have

\[
\| \lambda_1 f_1 + \cdots + \lambda_m f_m \| \geq \| \lambda_1 f_1 + \cdots + \lambda_{r-1} f_{r-1} + \lambda_{r+1} f_{r+1} + \cdots + \lambda_m f_m \|.
\]

The I.S. property is used to write the expression on the left hand side as each of the following \( n \) expressions:

\[
\|y + \lambda_r (e_{(r-1)k+1} + \alpha e_{(r-1)k+2} + \cdots + \alpha^{k-1} e_{rk}) + z\|;
\]

\[
\|y + \lambda_r (e_{(r-1)k+2} + \alpha e_{(r-1)k+3} + \cdots + \alpha^{k-1} e_{rk+1}) + z\|;
\]

down to

\[
\|y + \lambda_r (e_{(r-1)k+n} + \alpha e_{(r-1)k+n+1} + \cdots + \alpha^{k-1} e_{rk+n-1}) + z\|,
\]

where \( y = \sum_{i=1}^{r-1} \lambda_i f_i \) and \( z = U_n(\sum_{i=r+1}^m \lambda_i f_i) \) (here \( U_n : S \to S \) is the \( A \)-module homomorphism defined by \( U_n(e_k) = e_{n+k} \) for all \( k \geq 1 \)). Taking the average of these, and using the triangle inequality and the fact that \( I + \alpha \cdots + \alpha^{k-1} = 0 \), we obtain

\[
\| \lambda_1 f_1 + \cdots + \lambda_m f_m \| \geq \| y + z \| - \frac{|\lambda_r|}{n} \| a + b \|,
\]

where

\[
a = e_{(r-1)k+1} + (1 + \alpha)e_{(r-1)k+2} + \cdots + (1 + \alpha + \cdots + \alpha^{k-2})e_{(r-1)k+k-1}
\]

and

\[
b = \alpha^{k-1} e_{rk+n-1} + (\alpha^{k-1} + \alpha^{k-2}) e_{rk+n-2} + \cdots + (\alpha^{k-1} + \cdots + \alpha) e_{rk+n-k+1}.
\]

Using the I.S. property and taking the limit as \( n \) tends to infinity gives the required result.
COROLLARY 4.5.  (a) \((f_n)_{n=1}^\infty\) is an unconditional basic sequence in \(F\).
(b) Either \(c_0\) is finitely representable in \(F\) or \(|||f_1 + \cdots + f_n|||\) increases to infinity with \(n\).

PROPOSITION 4.6. Suppose that \(E\) is a normed \(A\)-module and that \(\alpha\) is a \(J\)-convexifying cyclic element of \(A\). Then either \(c_0\) is finitely representable in \(E\) or \(E\) is reflexive.

PROOF: If \(E\) is not reflexive then there exists a bounded sequence \((y_n)_{y=1}^\infty\) in \(E\) which has no Cesaro mean convergent subsequence. It follows from Propositions 4.1 and 4.3 that the space \(F\) constructed above is a normed space. If \(c_0\) is not finitely representable in \(E\), then by Proposition 3.5 \(c_0\) is not finitely representable in \(S/K\), of which \(F\) is a subspace; so by Corollary 4.5(b), \(|||f_1 + \cdots + f_n|||\) increases to infinity with \(n\). Now suppose that \(\alpha\) is cyclic of order \(k\) and is \(J(r,\epsilon)\)-convexifying. For each \(1 \leq j \leq r\) and \(n \geq 1\), we define

\[ v^j_n = (e_{j+r} + \alpha e_{j+2r} + \cdots + \alpha^{k-1} e_{j+kr}) + \cdots + (e_{j+(n-1)kr+r} + \cdots + \alpha^{k-1} e_{j+kr}). \]

Let \(d^s_n = v^1_n + \cdots + v^s_n + \alpha (v^{s+1}_n + \cdots + v^r_n)\) \((0 \leq s \leq r)\); we write \(d^s_n = S_1 + S_2 + S_3\) by grouping the terms as follows:

\[
S_1 = -e_{s+1} - \cdots - e_r;
S_2 = \{(e_{s+1} + \cdots + e_{s+r}) + \alpha(e_{r+s+1} + \cdots + e_{2r+s}) + \alpha^2(e_{2r+s+1} + \cdots + e_{3r+s}) + \cdots + \alpha^{k-1}(e_{s+(k-1)r+1} + \cdots + e_{s+kr})
+ \cdots + \{(e_{s+1+(n-1)kr} + \cdots + e_{s+(n-1)(k+1)r}) + \cdots + \alpha^{k-1}(e_{s+1+(nk-1)r} + \cdots + e_{nk+1})\};
S_3 = e_{nk+1+s} + \cdots + e_{(nk+1)k}.
\]

The I.S. property implies that

\[ |||v^j_n||| = |||f_1 + \cdots + f_n||| \quad (1 \geq j \geq r), \]
and so \(|||S_1|||/|||v^j_n|||\) and \(|||S_3|||/|||v^j_n|||\) both tend to zero as \(n\) tends to infinity. Moreover, the E.T.A. property implies that \(|||S_2||| = r|||v^j_n|||\). Let \(z^j_n = v^j_n/|||v^j_n|||\), so that \(|||z^j_n||| = 1\). Then

\[
\inf_{0 \leq j \leq r} \ |||z^1_n + \cdots + z^j_n + \alpha(z^{j+1}_n + \cdots + z^r_n)||| \geq r - \epsilon_n,
\]

where $\epsilon_n \to 0$ as $n \to \infty$. Hence $\alpha$ is not $J(r)$-convexifying for $(S/K, \|\| \|\|)$. But by Proposition 3.5 $S/K$ is $A$-finitely representable in $E$, and so $\alpha$ is not $J(r)$-convexifying for $E$. This contradiction completes the proof of the proposition.

**Proof of Theorem 1.1:** Let $T$ be a $J$-convexifying involution on a Banach space $E$ and suppose that $c_0$ is not finitely representable in $E$. By Lemma 2.6 the induced operator $\tilde{T}$ on the ultrapower $E^N/\mathcal{U}$ is $J$-convexifying; moreover, $\tilde{T}$ is clearly also an involution. Let $A$ be the subalgebra generated by $\tilde{T}$ of the algebra of all bounded operators on $E^N/\mathcal{U}$. Then $E^N/\mathcal{U}$ is a normed $A$-module, and by Lemma 4.2 $\tilde{T}$ is a $J$-convexifying cyclic element of $A$. It follows from Proposition 4.6 that $E^N/\mathcal{U}$ is reflexive, and so $E$ is super-reflexive.

**References**