On the Atomic Decomposition of $H^1$ and Interpolation

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In [1] Coifman used the Fefferman-Stein theory of $H^p$ spaces [4] to decompose the functions of these spaces into basic building blocks (atoms), further clarifying their real variable nature. Coifman and Weiss have provided a comprehensive treatment of these ideas and many applications to harmonic analysis in [2]. In this note, we use the nontangential maximal function $Nf$ to give an elementary proof of the decomposition of $H^1$ functions on the line and then characterize the Peetre $K$-functional for $H^1$ and $L^\infty$ in terms of $Nf$.

Let $u$ be the harmonic extension [5] of $f$ to the upper half plane $\mathbb{R}^+$. For $x \in \mathbb{R}$, denote by $\Gamma_x = \{(z, y) \in \mathbb{R}^2 : |x - z| \leq y\}$ the cone with vertex at $x$. The nontangential maximal function of $f$ is defined by $Nf(x) = \sup\{|u(z, y)| : (z, y) \in \Gamma_x\}$. We define the (real) $H^1$ norm of $f$ to be the standard $H^1$ norm of $u + iv$, where $v$ is the harmonic conjugate of $u$ which satisfies $v(0) = 0$. A classical result of Hardy and Littlewood asserts that $\|Nf\|_{L^1} < c\|f\|_{H^1}$. For an interval $I$ an $H^1$-atom is any function $a_I$ such that $\int a_I = 0$ and $|a_I| \leq |I|^{-1}\chi_I$ a.e.

PROPOSITION. Suppose $u$ is harmonic on an open square $S$ and continuous on $S$. Then its average on $\partial S$ equals the average over the two diagonals.

PROOF. By dilating to a subcube of $S$ and then expanding back, we may assume that both $u$ and its harmonic conjugate $v$ are continuous on $S$. Now $S$ is composed of four $15^\circ$ right triangles with common vertex the center of $S$. Let $T$ be the lower triangle and denote its edges by $L, B$ and $R$, where $B$ is the hypotenuse. Applying Cauchy’s theorem to $u + iv$ on $T$ and taking real parts of the integrals gives

$$0 = \int_{\partial T} u \, dx - v \, dy = \int_B u - \frac{1}{\sqrt{2}} \left( \int_{R+L} u + \int_R v - \int_L v \right).$$

Using rotations and symmetry, applying this argument to the three remaining subtriangles of $S$, and summing the resulting equations, we see that the terms involving $v$ cancel and we are left with our stated result. □

THEOREM. If $Nf \in L^1$, then we may write $f = \sum_i \lambda_i a_i$ so that the $a_i$'s are atoms and the coefficients $\lambda_i$ satisfy

$$\sum_i |\lambda_i| \leq 42\|Nf\|_{L^1}.$$
PROOF. Since $u$ is continuous in $\mathbb{R}^2$, then $E_k = \{x : Nf(x) > 2^k\}$ is open in $\mathbb{R}$. Let $I(f) = \int f \, dx / |I|$ and define $F_k$ as the complement in $\mathbb{R}$ of $E_k$. We write $E_k$ as the disjoint union of its collection $C_k$ of open components and then decompose $f$ as a sum, $f = g_k + h_k$, where

$$g_k = \sum_{I \in C_k} [f - I(f)] \chi_I, \quad h_k = \int_{F_k} f \chi_I.$$

We claim that $|h_k Z| \leq 7 \times 2^k$ a.e. Clearly, the estimate holds on $F_k$ since $Nf \leq 2^k$ there and $|f| \leq Nf$ a.e. For the remaining set $E_k$, we fix an interval $I$ in $C_k$ and show that

$$|I(f)| \leq 7 \times 2^k.$$

Let $S_k$ be the open square $I \times (\varepsilon, |I| + \varepsilon)$ in $\mathbb{R}^2$. By the Proposition and letting $\varepsilon \downarrow 0$, $I(f)$ is seen to equal four times the average of $u$ over the union of the two main diagonals less the sum of its averages over the three remaining sides. But the endpoints of $I$ belong to $F_k$, so the diagonals, sides and top of $S$ all belong to the “good” set for $u$, namely $\Gamma = \{(z, y) \in \Gamma_z : x \in F_k\}$. The definitions of $F_k$ and $Nf$ imply that $u$ is bounded by $2^k$ on $\Gamma$ which establishes (3).

Following Coifman [1] and Herz [6f], the atoms are defined by

$$a_I \lambda_I^{-1} (g_k - g_{k+1}) \chi_I, \quad \lambda_I = 21 \times 2^k |I|,$$

for each $I \in C_k$ and all integers $k$. By telescoping and using both that $g_k - g_{k+1} = h_{k+1} - h_k$ and that $g_{k+1}$ is supported in $E_{k+1} \subset E_k$, it follows that

$$f = \sum_{k=0}^{\infty} (g_k - g_{k+1}) = \sum_k \sum_{I \in C_k} \lambda_I a_I.$$

Each $a_I$ is an atom since it is supported in $I$ and the estimate $\|a_I\|_{\infty} \leq |I|^{-1}$ follows from our $L^\infty$ estimate on the $h_k$,

$$\|g_k - g_{k+1}\|_{\infty} = \|h_{k+1} - h_k\|_{\infty} \leq 7(2^{k+1} + 2^k) = 21 \times 2^k.$$

To see that $a_I$ has mean value zero, it suffices to write it in the form

$$a_I = \lambda_I^{-1} \left( [f - I(f)] \chi_I - \sum_{J \in C_{k+1}} [f - J(f)] \chi_J \right).$$

To establish inequality (1) (subject to relabeling) we use

$$\sum_{k} \sum_{I \in C_k} |\lambda_I| + 21 \sum_{k} 2^k \sum_{I \in C_k} |I| = 21 \sum_{k} 2^k |E_k| = 21 \sum_{k} (2^{k+1} - 2^k)|E_k|.$$

Indeed by (5), summation by parts, and the fact $Nf > 2^k$ on $E_k$, we have

$$\sum_{k} \sum_{I \in C_k} |\lambda_I| \leq 42 \sum_{k} 2^k |E_k \setminus E_{k+1}| \leq 42 \int Nf(x) \, dx. \quad \Box$$

Fefferman, Rivière and Sagher [3] estimated the $K$-functional

$$K(f, t) = \inf \{ \|g\|_{H^1} + t \|h\|_{L^\infty} : g \in H^1, h \in L^\infty, f = g + h \}$$

in terms of the “grand maximal” operator to describe interpolation spaces for the pair. We provide a description in terms of $Nf$. Let $g^*$ denote the decreasing rearrangement of $|g|$. 

COROLLARY (OF THE PROOF OF THE THEOREM). If $f$ belongs to $H^1 + L^\infty$, then

\begin{equation}
K(f,t) \sim \int_0^t (Nf)^*(s) \, ds, \quad t > 0.
\end{equation}

PROOF. The subadditivity of the integral operator in (7) implies that it is dominated by $K(f,t)$. To establish the opposite estimate, we fix $t > 0$ and select an integer $j$ so that $2^{j-1} < (Nf)^*(t) \leq 2^j$. From the constructions in (2), we see

\begin{equation}
g_j = \sum_{k=j}^{\infty} (g_k - g_{k+1}) = \sum_{k=j}^{\infty} \left( \sum_{I \in C_k} \lambda_I a_I \right).
\end{equation}

The estimate $\|h_j\|_\infty \leq 14 (Nf)^*(t)$ follows by our selection of the index $j$, while

\begin{equation}
\|g_j\|_{H^1} \leq 42c \int_{E_j} Nf(x) \, dx \leq 42c \int_0^t (Nf)^*(s) \, ds
\end{equation}

is derived as in (5)–(6) using the identity (8). Combining these estimates completes the proof. □

Minor modifications using $p$-atoms permit extension of these results to $H^p$ spaces ($\frac{1}{2} < p < 1$) on $\mathbb{R}$. Beginning with $Nf$ and using classical techniques (theorems of Spanne-Stein and Hardy-Littlewood), these results provide a simplified approach to the various descriptions of $H^p(\mathbb{R})$ (duality, grand maximal operator). By conformally mapping the unit disc onto $\mathbb{R}^2$ and estimating the appropriate integrals obtained from the Proposition, one obtains the expected results for the circle. Exploiting a Fourier analytical technique of Calderón, Wilson has given a proof of the atomic decomposition into $L^2$ atoms for higher dimensions in [8], while the condition $N(u + iv) \in L^1$ is required in [7]. Finally, the author extends his thanks to Colin Bennett and Guido Weiss for valuable discussions related to this paper.

REFERENCES

7. J. M. Wilson, A simple proof of the atomic decomposition for $H^p(R^n)$, $0 < p \leq 1$, Studia Math. 74 (1982), 25–33.