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Thomas L. Markham
University of South Carolina - Columbia, markham@math.sc.edu

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AN APPLICATION OF THEOREMS OF SCHUR AND ALBERT

THOMAS L. MARKHAM

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ABSTRACT. Suppose Π_n is the cone of $n \times n$ positive semidefinite matrices, and $\operatorname{int}(\Pi_n)$ is the set of positive definite matrices. Theorems of Schur and Albert are applied to obtain some elements of Π_n and $\operatorname{int}(\Pi_n)$. Then an analogue of Albert's theorem is given for M-matrices, and finally a generalization is given for matrices of class P.

I. **Introduction**. Suppose Π_n is the cone of $n \times n$ positive semidefinite matrices over the complex field. The interior of Π_n, denoted $\operatorname{int}(\Pi_n)$, is the set of $n \times n$ positive definite matrices.

If A and B are arbitrary matrices of the same size, the Hadamard product of A and B is the matrix $A*B$ whose (i,j) entry is $a_{ij}b_{ij}$. A rather comprehensive account of this product is given in [9].

J. Schur proved the following theorem.

THEOREM 1.1 [8]. *If A, $B \in \Pi_n$, then $A*B \in \Pi_n$. Further, if A, $B \in \operatorname{int}(\Pi_n)$, then $A*B \in \operatorname{int}(\Pi_n)$.*

This theorem is easily proved by noting $A*B$ is a principal submatrix of the tensor product of A and B.

Now suppose M is a matrix partitioned in the form

$$M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}. \tag{1.1}$$

In [2], the generalized Schur complement of A in M is defined as

$$M|A = D - CA^{+}B, \tag{1.2}$$

where A^{+} is the Moore-Penrose inverse of A. Similarly, we define

$$M|D = A - BD^{+}C. \tag{1.3}$$

If M given in (1.1) is hermitian and is partitioned symmetrically, then $C = B^*$. For this case, Albert [1] has proved the following theorem, which was generalized in [2, Theorem 2].

THEOREM 1.2. *Suppose M is hermitian and partitioned symmetrically in* (1.1). *Then $M \in \Pi_n$ if and only if $A \in \Pi_k$, $M|A \in \Pi_{n-k}$ and the null space of A is*

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contained in the null space of B^* *(i.e.* $N(A) \subseteq N(B^*)$*). Further,* $M \in \text{int}(\Pi_n)$ *if and only if* $A \in \text{int}(\Pi_k)$, $M|A \in \text{int}(\Pi_{n-k})$, *and* $M|D \in \text{int}(\Pi_k)$.

We shall utilize Theorems 1.1 and 1.2 to obtain some new results on positive semidefinite matrices.

II. **Some elements of** Π_n. As in §I, $N(A)$ will denote the null space of the matrix A.

THEOREM 2. *Suppose each of* A, B, C, D *is in* Π_n, *and* $N(A) \subseteq N(B)$, $N(C) \subseteq N(D)$. *Then*

$$BA^+B * DC^+D - (B * D)(A * C)^+ (B * D) \in \Pi_n.$$

PROOF. Let

$$M = \begin{pmatrix} A & B \\ B & BA^+B \end{pmatrix}, \qquad N = \begin{pmatrix} C & D \\ D & DC^+D \end{pmatrix}.$$

Both M and N are in Π_{2n} by Albert's theorem. Then applying Schur's theorem, we get

$$M * N = \begin{pmatrix} A * C & B * D \\ B * D & (BA^+B) * (DC^+D) \end{pmatrix} \in \Pi_{2n}. \tag{2.1}$$

Now we reapply Theorem 1.2 to (2.1) and obtain $(BA^+B) * (DC^+D) - (B * D)(A * C)^+(B * D) \in \Pi_n$. □

Note that as a consequence of Theorem 1.2, using the assumptions of the above theorem, we obtain that $N(A * C) \subseteq N(B * D)^*$.

One can obtain readily now a number of corollaries; we shall mention a few of these.

COROLLARY 2.1. *If* A, $C \in \text{int}(\Pi_n)$, *then* $A^{-1} * C^{-1} - (A * C)^{-1} \in \Pi_n$.

PROOF. Let $B = I_n = D$ in Theorem 2, and use the fact that $A^+ = A^{-1}$ if A is invertible.

COROLLARY 2.2. *Suppose* A, $B \in \text{int}(\Pi_n)$; C, $D \in \Pi_n$. *Then* $(A * B^{-1} + C) - (A^{-1} * B + D)^{-1} \in \Pi_n$.

PROOF. As in the proof of Theorem 2, let

$$M = \begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix}, \qquad N = \begin{pmatrix} B^{-1} & I \\ I & B \end{pmatrix}$$

and put

$$P = \begin{pmatrix} C & 0 \\ 0 & D \end{pmatrix}.$$

Then $M * N + P \in \Pi_{2n}$, and the result follows by the technique used previously.

From Corollary 2.2, one obtains immediately the result that if C, $D \in \Pi_n$, then $(I + C) - (I + D)^{-1} \in \Pi_n$. Simply choose $A = B = I_n$ above.

COROLLARY 2.3. *Let* $A \in \text{int}(\Pi_n)$. *Then*

$$A * A - (A * I)(A^{-1} * A + I)^{-1}(A * I)$$

is in Π_n.

PROOF. Let

$$M = \begin{pmatrix} A & I \\ I & A^{-1} \end{pmatrix}, \quad N = \begin{pmatrix} A & A \\ A & A \end{pmatrix}, \quad \text{and} \quad P = \begin{pmatrix} 0 & 0 \\ 0 & I_n \end{pmatrix}.$$

Then $M * N + P \in \Pi_{2n}$, and the result follows as in the previous corollary.

In fact, even more is known concerning Corollary 2.3. In [9, Corollary 4.3, p. 236], Styan shows that $A * A - 2(A * I)(A^{-1} * A + I)^{-1}(A * I) \in \Pi_n$ using a technique based on probabilistic methods.

We also would like to point out that Theorem 2 is an analogue for the Schur product of Theorem 5 of [2]. There it is shown that if $A, C \in \Pi_n$, and if B, D are chosen so that $N(A) \subseteq N(B^*)$, $N(C) \subseteq N(D^*)$, then

$$B^*A^+B + D^*C^+D - (B + D)^*(A + C)^+ (B + D) \in \Pi_n.$$

From Corollary 2.2, if $A, B \in \operatorname{int}(\Pi_n)$, then it follows that $A * B - (A^{-1} * B^{-1})^{-1} \in \Pi_n$. There is an analogue of this result for matrix addition, i.e. $A + B - (A^{-1} + B^{-1})^{-1} \in \operatorname{int}(\Pi_n)$. This is a consequence of the previously mentioned result of Carlson, Haynsworth and Markham [2]; we offer a simple proof of this fact.

Let

$$M = \begin{pmatrix} A & \frac{1}{2}I \\ \frac{1}{2} & A^{-1} \end{pmatrix} \quad \text{and} \quad N = \begin{pmatrix} B & \frac{1}{2}I \\ \frac{1}{2}I & B^{-1} \end{pmatrix}.$$

By Theorem 1.2, both M and N belong to $\operatorname{int}(\Pi_{2n})$. Now

$$M + N = \begin{pmatrix} A + B & I \\ I & A^{-1} + B^{-1} \end{pmatrix} \in \operatorname{int}(\Pi_{2n}).$$

Apply Theorem 1.2 again. Then $M + N|A^{-1} + B^{-1} \in \operatorname{int}(\Pi_n)$. But $M + N|A^{-1} + B^{-1} = A + B - (A^{-1} + B^{-1})^{-1}$. □

III. M-**matrices**. Suppose A is a square matrix over the real field. Let Z_n denote the class of $n \times n$ matrices whose off-diagonal entries are nonpositive. Assume $A \in Z_n$. A is called an M-matrix, see [6], if and only if A is invertible and A^{-1} is a nonnegative matrix (each entry is nonnegative). Let

$$G = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \tag{3.1}$$

where A and D are square matrices of order k and $n - k$, respectively.

If G is an M-matrix, then it is well known that A and D are M-matrices. Fan [5] proved that if D has order 1, then $G|D$ is an M-matrix. Crabtree [3, Lemma 1] extended this result to D of arbitrary order. Watford [10], in turn, proved this result for generalized M-matrices with respect to a cone.

These results are useful in obtaining an analogue of Albert's Theorem 1.2 for M-matrices.

THEOREM 3. *Suppose G is an $n \times n$ matrix partitioned as in* (3.1), *and G is in Z_n. Then G is an M-matrix if and only if A, D, $G|A$, and $G|D$ are M-matrices.*

PROOF. If G is an M-matrix, then A, D, $G|A$, and $G|D$ are M-matrices by the comments preceding Theorem 3.

Now suppose A, D, $G|A$, $G|D$ are M-matrices. Let

$$\bar{G} = \begin{bmatrix} (G|D)^{-1} & -A^{-1}B(G|A)^{-1} \\ -D^{-1}C(G|D)^{-1} & (G|A)^{-1} \end{bmatrix}. \tag{3.2}$$

It is easy to verify $G \cdot \bar{G} = I$, so G^{-1} exists. Further, G^{-1} is nonnegative since each of A^{-1}, D^{-1}, $(G|A)^{-1}$, and $(G|D)^{-1}$ is nonnegative, and B and C are nonpositive. Thus G is an M-matrix. □

Theorem 3 offers a practical procedure for determining if a given matrix is an M-matrix.

Now we will take a closer look at Albert's theorem. First, we need some additional notation. If α and β are strictly increasing sequences on $\{1, 2, \ldots, n\}$ of the same length, then $M(\alpha|\beta)$ will denote the minor of M with rows indexed by α and columns indexed by β. If $\alpha = \beta$, then we write $M(\alpha)$. If M is partitioned as in (1.1), where A is nonsingular of order k, then $M|A = (e_{ij})$, $i, j = k + 1, \ldots, n$, with

$$e_{ij} = \frac{M(1, 2, \ldots, k, i|1, 2, \ldots, k, j)}{M(1, 2, \ldots, k)} = \frac{M(1, 2, \ldots, k, i|1, 2, \ldots, k, j)}{\det(A)};$$

see [4].

If M is hermitian, then M is positive definite if and only if the leading principal minors of M are positive. Hence we can rephrase Albert's theorem for this case.

THEOREM 4. *Suppose M is hermitian, and is partitioned symmetrically in* (1.1). *Then $M \in \mathrm{int}(\Pi_n)$ if and only if $A \in \mathrm{int}(\Pi_k)$ and $M|A \in \mathrm{int}(\Pi_{n-k})$.*

PROOF. It is well known that if $M \in \mathrm{int}(\Pi_n)$, then A and $M|A$ are positive definite.

Conversely, we need only show that the leading principal minors of M are positive. Consider an arbitrary minor, say $M(1, \ldots, i_p)$. If $i_p \leq k$, this minor is positive since it is a principal minor of A. Assume $i_p > k$. Then, using an identity of Sylvester [7, p. 101], we have

$$M|A(k+1, \ldots, i_p) = (\det(A))^{-1} M(1, \ldots, k, k+1, \ldots, i_p).$$

The result now follows. □

If $M \in Z$, then M is an M-matrix if and only if the leading principal minors of M are positive. Thus, Theorem 3 could also be restated in the form of Theorem 4.

DEFINITION [6]. Suppose M is an $n \times n$ matrix. Then M belongs to class P if and only if all principal minors of M are positive.

We can generalize Albert's theorem to class P in the following manner.

THEOREM 5. *Let M be partitioned as in* (1.1), *where the submatrix A has order* 1. *Then*

$$M \in P \text{ if and only if } A \in P,\ M|A \in P, \text{ and } D \in P. \tag{3.3}$$

We omit the proof since the techniques are similar to those of Theorem 4.

Observe the following concerning Theorem 5. On the one hand, to see if $M \in P$, there are $2^n - 1$ principal minors to check. Applying the above result, we obtain a number and two matrices of order $n - 1$ to check the principal minors. Using this equivalence iteratively (to the right-hand side of (3.3)), there are $1 + 2 + \cdots + 2^{n-1}$ numbers which must be verified to be positive. But $1 + 2 + \cdots + 2^{n-1} = 2^n - 1$ for n a positive integer, so, in fact, the same number of elements must be verified. The obvious advantage of the right-hand side of (3.3) lies in the reduction of the order of the matrices at each iteration.

It is possible to reduce the number of minors checked? For example, if M has leading positive principal minors, then M does not necessarily belong to class P. A simple example to illustrate is $M = \begin{pmatrix} 1 & -2 \\ 4 & -2 \end{pmatrix}$.

Does there exist an analogue to Theorem 3 for class P when M is partitioned as in (1.1), with A of order k? If M has order 2 or 3, the result holds. For larger orders, it need not hold. Consider

$$M = \left[\begin{array}{cc|cc} 1 & 1 & 1 & 0 \\ 1 & 2 & 0 & 1 \\ \hline \multicolumn{2}{c|}{\frac{1}{2}\begin{pmatrix} 2 & 3 \\ 3 & 5 \end{pmatrix}} & 1 & 1 \\ & & 1 & 2 \end{array}\right] = \begin{pmatrix} A & B \\ C & D \end{pmatrix}.$$

Here A, $M|A$, D, and $M|D$ are all in class P, but $M(13)$ is zero.

We conclude with the following query. Suppose M is an $n \times n$ matrix. What is the minimal number of principal minors of M that must be positive in order that M belong to class P? Is it necessary to verify that all $2^n - 1$ principal minors, or related minors, are positive?

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Department of Mathematics and Computer Science, University of South Carolina, Columbia, South Carolina 29208