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Nonlinear Stochastic Integral-Equation of Hammerstein Type

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ON A NONLINEAR STOCHASTIC INTEGRAL EQUATION
OF THE HAMMERSTEIN TYPE

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ABSTRACT. A nonlinear stochastic integral equation of the
Hammerstein type in the form

\[ x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega)f(s, x(s; \omega)) \, d\mu(s) \]

is studied where \( t \in S \), a \( \sigma \)-finite measure space with certain prop-
erties, \( \omega \in \Omega \), the supporting set of a probability measure space
\((\Omega, A, P)\), and the integral is a Bochner integral. A random sol-
uition of the equation is defined to be a second order vector-valued
stochastic process \( x(t; \omega) \) on \( S \) which satisfies the equation almost
certainly. Using certain spaces of functions, which are spaces of
second order vector-valued stochastic processes on \( S \), and fixed
point theory, several theorems are proved which give conditions
such that a unique random solution exists.

1. Introduction. The purpose of this note is to study the existence and
uniqueness of a random solution of a nonlinear stochastic integral equation
of the Hammerstein type of the form

\[ x(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega)f(s, x(s; \omega)) \, d\mu(s), \]

where

(i) \( S \) is a locally compact metric space with metric \( d \) defined on \( S \times S \)
and \( \mu \) is a complete \( \sigma \)-finite measure defined on the collection of Borel
subsets of \( S \);
(ii) \( \omega \in \Omega \), where \( \Omega \) is the supporting set of the probability measure
space \((\Omega, A, P)\);
(iii) \( x(t; \omega) \) is the unknown vector-valued random variable for each \( t \in S \);
(iv) \( h(t; \omega) \) is the stochastic free term defined for \( t \in S \);
(v) \( k(t, s; \omega) \) is the stochastic kernel defined for \( t \) and \( s \) in \( S \); and
(vi) \( f(t, x) \) is a vector-valued function of \( t \in S \) and \( x \).
The integral in equation (1.1) is interpreted as a Bochner integral [12].
Further assumptions concerning the functions in (1.1) will be stated in §2.

The equation (1.1) is a generalization of stochastic integral equations studied by Padgett and Tsokos [9], Tsokos [11], and Anderson [1]. Also, equation (1.1) is a stochastic version of the deterministic integral equations which were investigated by Petryshyn and Fitzpatrick [10], Browder and Gupta [5], Browder, de Figueiredo, and Gupta [6], among others.

In order to investigate the stochastic integral equation (1.1), we will define several spaces of functions which are spaces of second order vector-valued stochastic processes on S and will use certain aspects of the “theory of admissibility” of Banach spaces as introduced into the study of integral equations by Corduneanu [7] and the methods of “probabilistic functional analysis” [3].

2. Preliminaries. We will further assume that S is the union of a countable family of compact subsets \( \{ C_n \} \) having the properties that \( C_1 \subset C_2 \subset C_3 \subset \cdots \) and that for any other compact set in S there is a \( C_i \) which contains it [2].

We define \( C = C(S, L^2(\Omega, A, P)) \) to be the space of all continuous functions from S into the space \( L^2(\Omega, A, P) \) with the topology of uniform convergence on compacta. That is, for each fixed \( t \in S \), \( x(t; \omega) \) is a vector-valued random variable such that

\[
\| x(t; \omega) \|_{L^2(\Omega, A, P)}^2 = \int_{\Omega} |x(t; \omega)|^2 dP(\omega) < \infty.
\]

It may be noted that \( C(S, L^2(\Omega, A, P)) \) is a locally convex space [12, pp. 24–26] whose topology is defined by the countable family of seminorms given by

\[
\| x(t; \omega) \|_n = \sup_{\omega \in C_n} \| x(t; \omega) \|_{L^2(\Omega, A, P)}, \quad n = 1, 2, \cdots.
\]

Moreover, \( C(S, L^2(\Omega, A, P)) \) is complete relative to this topology since \( L^2(\Omega, A, P) \) is complete.

We further define \( BC = BC(S, L^2(\Omega, A, P)) \) to be the Banach space of all bounded continuous functions from S into \( L^2(\Omega, A, P) \) with norm

\[
\| x(t; \omega) \|_{BC} = \sup_{t \in S} \| x(t; \omega) \|_{L^2(\Omega, A, P)}.
\]

The space \( BC \subset C \) is the space of all second order vector-valued stochastic processes defined on S which are bounded and continuous in mean-square.

We will consider the functions \( h(t; \omega) \) and \( f(t, x(t; \omega)) \) to be in the space \( C(S, L^2(\Omega, A, P)) \). With respect to the stochastic kernel we assume that for each pair \( (t, s) \), \( k(t, s; \omega) \in L^\infty(\Omega, A, P) \) and denote the norm by

\[
\| k(t, s; \omega) \| = \| k(t, s; \omega) \|_{L^\infty(\Omega, A, P)} = P\text{-ess sup}_{\omega \in \Omega} |k(t, s; \omega)|.
\]
Also, we will suppose that $k(t, s; \omega)$ is such that
\[ \|k(t, s; \omega)\| \cdot \|x(s; \omega)\|_{L_2(\Omega, A, P)} \]
is $\mu$-integrable with respect to $s$ for each $t \in S$ and $x(s; \omega)$ in $C(S, L_2(\Omega, A, P))$, and that there exists a real-valued function $G$ defined $\mu$-a.e. on $S$ so that $G(s) \|x(s; \omega)\|_{L_2(\Omega, A, P)}$ is $\mu$-integrable and, for each pair $(t, s) \in S \times S$,
\[ \|k(t, u; \omega) - k(s, u; \omega)\| \cdot \|x(u; \omega)\|_{L_2(\Omega, A, P)} \leq G(u) \|x(u; \omega)\|_{L_2(\Omega, A, P)} \]
$\mu$-a.e. Further, for almost all $s \in S$, $k(t, s; \omega)$ will be continuous in $t$ from $S$ into $L_2(\Omega, A, P)$.

We now define the integral operator $T$ on $C(S, L_2(\Omega, A, P))$ by
\[ (Tx)(t; \omega) = \int_S k(t, s; \omega)x(s; \omega) \, d\mu(s), \]
where the integral is a Bochner integral. From the conditions on $k(t, s; \omega)$, we have that for each $t \in S$, $(Tx)(t; \omega) \in L_2(\Omega, A, P)$ and that $(Tx)(t; \omega)$ is continuous in mean square by Lebesgue's dominated convergence theorem. That is, $(Tx)(t; \omega) \in C(S, L_2(\Omega, A, P))$.

**Lemma 2.1.** The linear operator $T$ defined by equation (2.1) is continuous from $C(S, L_2(\Omega, A, P))$ into itself.

**Proof.** Note that $C(S, L_2(\Omega, A, P))$ is a Fréchet space with metric $d^*$ defined by the Fréchet combination of the sequence of seminorms $\| \cdot \|_n$, $n=1, 2, \cdots$.

Define the sequence of linear operators $\{T_M\}$, $M=1, 2, \cdots$, by
\[ (T_Mx)(t; \omega) = \int_{C_M} k(t, s; \omega)x(s; \omega) \, d\mu(s). \]

Hence, as $M \to \infty$ we have $(T_Mx)(t; \omega) \to (Tx)(t; \omega)$.

Let $\{x_j(t; \omega)\}$ be a sequence of functions converging to $x(t; \omega)$ in $C(S, L_2(\Omega, A, P))$. Then by definition of the seminorms, for each $M$
\[ \|(T_Mx)(t; \omega) - (T_Mx_j)(t; \omega)\|_n \leq \sup_{t \in C_M} \int_{C_M} \|k(t, s; \omega)\| \cdot \|x(s; \omega) - x_j(s; \omega)\|_{L_2(\Omega, A, P)} \, d\mu(s). \]

Since $\|x(s; \omega) - x_j(s; \omega)\|_{L_2(\Omega, A, P)} \to 0$ uniformly on the compact set $C_M$, for $\varepsilon > 0$ there exists a positive integer $N_M$ such that $j \geq N_M$ implies
\[ \|(T_Mx)(t; \omega) - (T_Mx_j)(t; \omega)\|_n < \varepsilon \sup_{t \in C_M} \int_{C_M} \|k(t, s; \omega)\| \, d\mu(s). \]
Now, by the conditions on $k(t, s; \omega)$, there exists a constant $K_n$ such that $\|k(t, s; \omega)\| \leq K_n$ for all $t \in C_n$ and almost all $s$. Hence, for $j \geq N_M$

$$\|\left(T_M X_j\right)(t; \omega) - \left(T_M X_j\right)(t; \omega)\|_n < \varepsilon K_n\mu(C_M).$$

Since convergence in every seminorm is equivalent to convergence in the metric $d^*$, $(T_M X_j)(t; \omega)$ converges to $(T_M X)(t; \omega)$ in $C(S, L_2(\Omega, A, P))$ for each $M$. Therefore, by [8, p. 54], $T$ is continuous from $C(S, L_2(\Omega, A, P))$ into itself.

Let $B$ and $D$ be Banach spaces. The pair $(B, D)$ is said to be admissible with respect to a linear operator $T$ if $T(B) \subset D$.

**Lemma 2.2.** If $T$ is a continuous linear operator from $C(S, L_2(\Omega, A, P))$ into itself and $B, D \subset C(S, L_2(\Omega, A, P))$ are Banach spaces stronger than $C(S, L_2(\Omega, A, P))$ such that $(B, D)$ is admissible with respect to $T$, then $T$ is continuous from $B$ into $D$.

The lemma follows from the closed-graph theorem.

From Lemmas 2.1 and 2.2 it follows that $T$ defined by equation (2.1) is a bounded linear operator from $B$ into $D$.

By a random solution of the equation (1.1) we will mean a function $x(t; \omega)$ in $C(S, L_2(\Omega, A, P))$ which satisfies the equation $P$-a.e.

**3. Existence of a random solution.** We now present theorems concerning the existence and uniqueness of a random solution of the equation (1.1).

**Theorem 3.1.** We consider the stochastic integral equation (1.1) subject to the following conditions:

(i) $B$ and $D$ are Banach spaces stronger than $C(S, L_2(\Omega, A, P))$ such that $(B, D)$ is admissible with respect to the integral operator defined by equation (2.1);

(ii) $x(t; \omega) \rightarrow f(t, x(t; \omega))$ is an operator from the set

$$Q(\rho) = \{x(t; \omega) : x(t; \omega) \in D, \|x(t; \omega)\|_D \leq \rho\}$$

into the space $B$ satisfying the Lipschitz condition

$$\|f(t, x(t; \omega)) - f(t, y(t; \omega))\|_B \leq \lambda \|x(t; \omega) - y(t; \omega)\|_D$$

for $x(t; \omega), y(t; \omega) \in Q(\rho)$, where $\rho$ and $\lambda$ are constants;

(iii) $h(t; \omega) \in D$.

Then there exists a unique random solution of (1.1) in $Q(\rho)$, provided $\lambda K < 1$ and

$$\|h(t; \omega)\|_D + K \|f(t, 0)\|_B \leq \rho(1 - \lambda K),$$

where $K$ is the norm of $T$. 

**PROOF.** Define the operator $U$ from $Q(p)$ into $D$ by

$$(Ux)(t; \omega) = h(t; \omega) + \int_S k(t, s; \omega)f(s, x(s; \omega)) \, d\mu(s).$$

Then from the conditions of the theorem

$$\| (Ux)(t; \omega) \|_D \leq \| h(t; \omega) \|_D + K \| f(t, x(t; \omega)) \|_B$$

$$\leq \| h(t; \omega) \|_D + K \| f(t, 0) \|_B + K\lambda \| x(t; \omega) \|_D \leq \rho.$$ 

Hence, $(Ux)(t; \omega) \in Q(p)$.

Now, for $x(t; \omega), y(t; \omega) \in Q(p)$ we have by condition (ii) that

$$\| (Ux)(t; \omega) - (Uy)(t; \omega) \|_D$$

$$= \left\| \int_S k(t, s; \omega)[f(s, x(s; \omega)) - f(s, y(s; \omega))] \, d\mu(s) \right\|_D$$

$$\leq K \| f(t, x(t; \omega)) - f(t, y(t; \omega)) \|_B$$

$$\leq \lambda K \| x(t; \omega) - y(t; \omega) \|_D.$$ 

Since $\lambda K < 1$, $U$ is a contraction on $Q(p)$.

Therefore, by Banach's fixed point theorem there exists a unique $x^*(t; \omega) \in Q(p)$ which is a fixed point of $U$, that is, $x^*(t; \omega)$ is the unique random solution of equation (1.1).

A similar theorem may be obtained when $f$ is a nonlinear contraction on $Q(p)$ [4].

**Theorem 3.2.** Assume that the stochastic integral equation (1.1) satisfies the following conditions:

(i) same as Theorem 3.1 (i);

(ii) $x(t; \omega) \to f(t, x(t; \omega))$ is an operator from the set $Q(p)$ into the space $B$ satisfying

$$\| f(t, x(t; \omega)) - f(t, y(t; \omega)) \|_B \leq \phi(\| x(t; \omega) - y(t; \omega) \|_D)$$

for $x(t; \omega), y(t; \omega) \in Q(p)$, where $\phi$ is a real-valued continuous function such that $\phi(s) < s$ for $s > 0$;

(iii) $h(t; \omega) \in D$.

Then there exists a unique random solution of (1.1) in $Q(p)$, provided $K \leq 1$ and $\| h(t; \omega) \|_D + K \| f(t, 0) \|_B \leq \rho(1 - K)$, where $K$ is the norm of $T$.

The proof of Theorem 3.2 is similar to that of Theorem 3.1 except that the fixed point theorem of Boyd and Wong [4] is used.

The following is a useful application of Theorem 3.1.
**Corollary 3.1.** Suppose the stochastic integral equation (1.1) satisfies the following conditions:

(i) \( \sup_{t \in S} \int \| k(t, s; \omega) \| \, d\mu(s) < \infty; \)

(ii) \( f(t, x) \) is a continuous function of \( t \in S \) uniformly in \( x \) such that for \( \| x(t; \omega) \| \leq \lambda \), \( \| y(t; \omega) \| \leq \rho \), \( \| f(t, x(t; \omega)) - f(t, y(t; \omega)) \| \leq \lambda \| x(t; \omega) - y(t; \omega) \| \), for each \( t \in S \), where \( \lambda \) and \( \rho \) are constants;

(iii) \( h(t; \omega) \) is a bounded continuous function from \( S \) into \( L_2(\Omega, A, P) \).

Then there exists a unique random solution of equation (1.1), provided \( \sup_{t \in S} \int \| k(t, s; \omega) \| \, d\mu(s), \lambda, \) and \( \| f(t, 0) \| \) are sufficiently small.

**Proof.** We must show that condition (i) implies that the pair \( ( BC, BC ) \) is admissible with respect to the integral operator \( T \) defined by equation (2.1). Let \( x(t; \omega) \in BC(S, L_2(\Omega, A, P)) \). Then by the properties of the Bochner integral

\[
\| (Tx)(t; \omega) \|_{L_2(\Omega, A, P)} \leq \int \| k(t, s; \omega) x(s; \omega) \|_{L_2(\Omega, A, P)} \, d\mu(s)
\]

\[
\leq \sup_{t \in S} \| x(t; \omega) \|_{L_2(\Omega, A, P)} \int \| k(t, s; \omega) \| \, d\mu(s)
\]

\[
\leq \| x(t; \omega) \|_{BC} \sup_{t \in S} \int \| k(t, s; \omega) \| \, d\mu(s).
\]

Hence, \( (Tx)(t; \omega) \in BC(S, L_2(\Omega, A, P)) \), that is, \( ( BC, BC ) \) is admissible with respect to \( T \).

Conditions (ii)–(iii) clearly imply that conditions (ii)–(iii) of Theorem 3.1 hold. Thus, by Theorem 3.1, there exists a unique random solution of equation (1.1).

Other corollaries of Theorems 3.1 and 3.2 may be obtained by choosing different Banach spaces contained in the space \( C(S, L_2(\Omega, A, P)) \) and different conditions on \( f \) and \( k \).

**References**


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