4-1-1995

A Reverse Isoperimetric Inequality, Stability and Extremal Theorems For Plane-Curves With Bounded Curvature

Ralph Howard
University of South Carolina - Columbia, howard@math.sc.edu

ANDREJS TREIBERGS

Publication Info

Rocky Mountain Journal of Mathematics, Volume 25, Issue 2, Spring 1995, pages 635-684.

ROCKY MOUNTAIN
JOURNAL OF MATHEMATICS
Volume 25, Number 2, Spring 1995

A REVERSE ISOPERIMETRIC INEQUALITY, STABILITY AND EXTREMAL THEOREMS FOR PLANE CURVES WITH BOUNDED CURVATURE

RALPH HOWARD AND ANDREJS TREIBERGS

0. Introduction. In this note we discuss some elementary theorems about the relation between area and length of closed embedded plane curves with bounded curvature. Our main result (see Theorem 4.1) solves the extremal problem of which domain has *largest* boundary length among embedded disks in the plane whose boundary curvatures are uniformly bounded and whose area is fixed and sufficiently small.

Reverse Isoperimetric Inequality. *If M is an embedded closed disk in the plane $\mathbf{R}^2$ whose boundary curvature satisfies $|\kappa| \leq 1$ and with area $A \leq \pi + 2\sqrt{3}$ then the length of ∂M is bounded by*

$$\frac{L - 2\pi}{4} \leq \operatorname{Arcsin}\left(\frac{A - \pi}{4}\right).$$

If equality holds then M is congruent to a peanut-shaped domain as in Figure 1.

This gives an estimate in the reverse direction to the classical isoperimetric inequality. There is also a threshold phenomenon: if the area is larger than $\pi + 2\sqrt{3}$ then there is no upper bound for the length of ∂M. This is the area of the pinched peanut domain $P_{\sqrt{3}}$. Examples can be found by breaking a thin peanut and connecting the ends with a long narrow strip. In fact, the set of possible points (A, L) for embedded disks whose boundary satisfies $|\kappa| \leq 1$ is further restricted (Theorem 4.1). There is a suggestive physical interpretation of the equivalent dual problem, where the length is fixed and the minimal area disk is sought. One may imagine the cross section of a hose in which the inside pressure is smaller than the outside. If the hose has limited flexibility, modelled by a uniform bound on the curvature, then the equilibrium section is again the peanut shape.

Received by the editors on December 14, 1992.

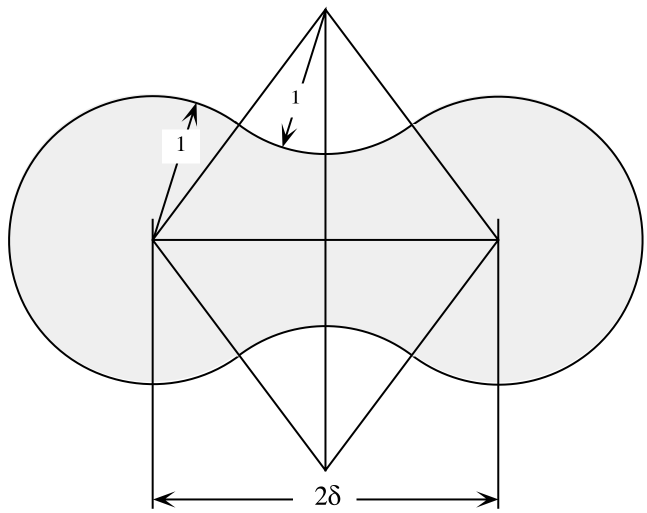

FIGURE 1. "Peanut" domain P_δ.

In Section 4 we prove existence and uniqueness of the extremal figures. We use a replacement argument to show that extremals are piecewise circular arcs. Compactness depends on a priori length bounds. In Section 3 we consider length estimates and some related stability results in the class of embedded disks whose boundary curvature is uniformly bounded. Our results say that if area or some other quantity is small, such as the circumradius, then the curve must be near the circle. This gives a preliminary reverse isoperimetric inequality which improves with the addition of extra information, say on the circumradius, for this class of curves. The results depend on a theorem of Pestov and Ionin [**20**] on the existence of a large disk in a domain with uniformly bounded curvature (see e.g. [**5**].) In Section 2 we include an argument for Pestov and Ionin's theorem along the lines of Lagunov's [**16**] proof of the higher dimensional generalization using analysis of the structure of the cut set of such a domain. Lagunov gives a sharp lower bound for the radius of the biggest ball enclosed within hypersurfaces all of whose principal curvatures are bounded $|\kappa_i| \leq 1$. Lagunov and Fet [**17**] show that the bound is increased if additional topological hypotheses are imposed. It is noteworthy that the examples which show the sharpness of the Lagunov and Lagunov-Fet bounds for dimension greater than one are not unit spheres. We indicate how the argument carries over to general Riemannian surfaces. In higher dimensions, Alexander and Bishop [**1**] have found inradius bounds depending on the curvatures and topology of the manifold and its boundary. Our results use both the existence of a disk and structure of the cut set. In Section 1. we consider curves which are only continuously differentiable and whose curvature is bounded in an appropriate weak sense which is suitable to extremal problems.

We call curves whose curvature is bounded by $|\kappa| \le 1$ in this weak sense *curves of class* $\mathcal{K}$. Some other extremal problems for such curves have been studied previously. For example, the problem of finding the shortest plane curve of class $\mathcal{K}$ with given endpoint and starting line element (position and direction) was solved by Markov [**21**]. The problem of finding the shortest plane curve of class $\mathcal{K}$ given starting and ending line elements was solved by Dubins [**10**].

1. Curves with weakly bounded curvature. Let Σ be a 2-manifold of class C^2. We will usually assume that our curves $c : (a, b) \to \Sigma$ are C^1, parameterized by arclength, such that the tangent vector $\mathbf{t} = c'$ is absolutely continuous. Let $\mathbf{n}$ be the unit normal, chosen so that $\{\mathbf{t}, \mathbf{n}\}$ is a right handed system. Then the defining equation for geodesic curvature $\nabla_{\mathbf{t}}\mathbf{t}(s) = \kappa_g(s)\mathbf{n}(s)$ implies that κ_g exists almost everywhere as an L^1_{loc} function which we shall assume satisfies the L^∞ bound

$$\|\kappa_g\|_\infty \le 1. \tag{1.1}$$

Let $\mathcal{K}$ denote the class of C^1 curves of Σ, parameterized by arclength, whose c' is absolutely continuous, and whose geodesic curvature satisfies (1.1).

If $\Sigma = \mathbf{R}^2$ then $c \in \mathcal{K}$ is equivalent to the condition $c \in C^1$ and

$$|c'(s) - c'(t)| \le |s - t| \quad \text{for all } s,\, t \tag{1.2}$$

which is called a *constraint on average curvature* by Dubins [**10**]. Since (1.2) implies that the c' is Lipschitz and thus absolutely continuous, by a theorem of Lebesgue c' is differentiable almost everywhere, and is the integral of its derivative in the sense that

$$c'(b) - c'(a) = \int_a^b c''(s)\,ds.$$

By looking at difference quotients and using (1.2) we see that $|c''(s)| \leq 1$ at all points where it exists. As $\langle c', c' \rangle \equiv 1$ we also have that $c'' = \mathbf{t}' \perp \mathbf{t} = c'$ at all points where $c'' = \mathbf{t}'$ exists. Thus κ, given by $c'' = \mathbf{t}' = \kappa \mathbf{n}$ is defined almost everywhere. As $|c''| \leq 1$ this implies $|\kappa| \leq 1$ at all points where c'' exists, hence (1.1) holds.

Plane curves $c \in \mathcal{K}$ have a well defined direction angle from which the position can be recovered. Because the rotation J by 90° is linear, $\mathbf{n} = J\mathbf{t}$ is differentiable at exactly the same points that $\mathbf{t}$ is, and $\mathbf{n}' = J\mathbf{t}' = J\kappa\mathbf{n} = -\kappa\mathbf{t}$. Thus the usual $c' = \mathbf{t}$ and $\mathbf{t}' = \kappa\mathbf{n}$ hold at all points where c'' exists, and thus almost everywhere. Now define a function $\vartheta(s)$ by

$$\vartheta(s) = \int_0^s \kappa(t)\,dt$$

where the integral is in the sense of Lebesgue. By another theorem of Lebesgue the function ϑ is absolutely continuous and has derivative κ almost everywhere. This implies $|d\vartheta/ds| = |\kappa| \leq 1$ almost everywhere. Define functions $\mathbf{R} \to \mathbf{R}^2$ by

$$\mathbf{t}^*(s) = (\cos\vartheta(s), \sin\vartheta(s)), \qquad \mathbf{n}^*(s) = (-\sin\vartheta(s), \cos\vartheta(s)).$$

A unit speed curve c^* is defined by

$$c^*(s) = \int_0^s \mathbf{t}^*(t)\,dt.$$

Then c^*, $\mathbf{t}^*$, and $\mathbf{n}^*$ satisfy

$$c^{*\prime} = \mathbf{t}^*, \qquad \mathbf{t}^{*\prime} = \kappa\mathbf{n}^*, \qquad \mathbf{n}^{*\prime} = -\kappa\mathbf{t}^* \tag{1.3}$$

almost everywhere. We can rotate and translate c^* so that $c(0) = c^*(0)$ and $\mathbf{t}(0) = \mathbf{t}^*(0)$.

Consider the function $f(s) = |\mathbf{t}(s) - \mathbf{t}^*(s)|^2 + |\mathbf{n}(s) - \mathbf{n}^*(s)|^2$. Then f is absolutely continuous and (1.1), (1.3) and a calculation imply that $f'(s) = 0$ almost everywhere. As f is absolutely continuous this implies

f is constant. But $f(0) = 0$ so $f \equiv 0$. Therefore $c' = c^{*\prime}$ which in turn implies $c = c^*$. So $c'(s) = (\cos\vartheta(s), \sin\vartheta(s))$ where $d\vartheta/ds = \kappa$ almost everywhere, and

$$\vartheta(s) - \vartheta(0) = \int_0^s \kappa(t)\, dt. \tag{1.4}$$

Proposition 1.1 and Proposition 1.2, which generalize the classical lemmata of Schur-Schmidt, continue to hold assuming only regularity of class $\mathcal{K}$. This is the desirable hypothesis since it is the expected regularity for solutions of optimal control problem for curves of maximal length with the control curvature $\kappa \in [-1, 1]$ and the constraints that the area be fixed and the curve be embedded. Part (3) is a special case of a theorem due to A. Schur and E. Schmidt [**2, 7, 9, 11**], which says the distance between endpoints of a convex planar curve is smaller than the distance between endpoints of a second curve with smaller curvatures at corresponding points.

Proposition 1.1. *Let $\gamma : [0, L] \to \mathbf{R}^2$ be a curve in $\mathcal{K}$ (i.e. γ is parameterized by arclength, $\mathbf{t} = \gamma'$ is absolutely continuous and $|\nabla_{\mathbf{t}}\mathbf{t}| = |\kappa| \le 1$ a.e.). Let $\gamma(0) = 0$ and $\gamma'(0) = \partial/\partial x$. Denote the coordinates of $\gamma(s) = (x(s), y(s))$. Then*

(1) *$x(s) \ge \sin s$ for all $0 \le s \le \pi$. Equality holds if and only if γ is an arc of a unit circle.*

(2) *$|y(s)| \le 1 - \cos s$ for $0 \le s \le \pi/2$ with equality if and only if γ is an arc of a unit circle.*

(3) *$|\gamma(s)| \ge 2\sin(s/2)$ for $0 \le s \le 2\pi$ with equality if and only if γ is an arc of the unit circle.*

(4) *In particular, if $\gamma : [0, \pi] \to \mathbf{R}^2$ is tangent to a unit circle at $\gamma(0)$ then $\gamma(s)$ is disjoint from the interior of the circle for $0 \le s \le \pi$.*

Proof. This is a standard fact from elementary differential geometry for C^2 curves and given by Dubins [**10**] for curves in $\mathcal{K}$. We give the proof for completeness sake. By the assumption on curvature and the representation (1.3), the direction of the tangent vector of the curve $\vartheta(s) = \int_0^s \kappa(s)\, ds$ is less than the corresponding angle of a circular arc with unit curvature $|\vartheta(s)| \leq s$. Hence

$$x(s) = \int_0^s \cos(\vartheta(s))\, ds \geq \int_0^s \cos s\, ds = \sin s$$

for $0 \leq s \leq \pi$ with equality if and only if $|\vartheta| = s$ almost everywhere. Similarly

$$|y(s)| = \left|\int_0^s \sin(\vartheta(s))\, ds\right| \leq \int_0^s \sin s\, ds = 1 - \cos s$$

for $0 \leq s \leq \pi/2$ with equality if and only if $|\vartheta| = s$ almost everywhere. This implies (1) and (2). Now orient the curve so that $\gamma(s/2) = 0$ and $\gamma'(s/2) = \partial/\partial x$. By (1) $x(s) \geq \sin(s/2)$ and $x(0) \leq -\sin(s/2)$ which implies (3). Equality implies γ is the arc of a unit circle. □

This implies a related result which is occasionally useful.

Proposition 1.2. *Let $\gamma : [0, L] \to \mathbf{R}^2$ be of class $\mathcal{K}$. Suppose the curve has endpoints on the boundary of a disk B_R of radius $R \leq 1$, lies outside B_R and has length*

$$L \leq 2\pi - 2\sin^{-1} R.$$

Then γ is an arc of a unit circle and either

(1) $L = 2\pi - 2\sin^{-1} R$; *or*

(2) $R = 1$.

Proof. By the Schur-Schmidt Proposition 1.1, the endpoints of γ are a distance $D = \operatorname{dist}(\gamma(0), \gamma(L)) \geq 2\sin(L/2)$ apart. If $L > 2\sin^{-1} R$ then $D > 2R$, the diameter of the circle unless (1) holds. On the other hand, a curve of length $L \leq 2\sin^{-1} R \leq \pi R$ whose endpoints are on a circle of radius R can be at most the distance of the chord along the circle apart, namely $\operatorname{dist}(\gamma(0), \gamma(L)) \leq 2R\sin(L/2R)$ which is a contradiction unless $R = 1$ and γ is the arc of a unit circle. □

The strong maximum principle holds for curves $\gamma \in \mathcal{K}$. Although this follows from the maximum principle for weak solutions of an elliptic equation, in the curve case it also follows immediately from Proposition 1.1. Suppose $\gamma \in \mathcal{K}$ so that $\gamma(0) \in \partial B_1(O)$ and $\gamma(-\varepsilon, \varepsilon) \subset \overline{B_1}(O)$ for some $0 < \varepsilon < \pi$. Then $\gamma((-\varepsilon, \varepsilon)) \subset \partial B_1(O)$ because γ is tangent to the disk at $\gamma(0)$.

2. The theorem of Pestov and Ionin and the structure of the cut locus. Let M be a simply connected plane domain with C^1 boundary which satisfies a one-sided condition on the curvature. Let the boundary curve of M be positively oriented, parameterized by arclength, γ' absolutely continuous and $\langle \gamma'(s+h) - \gamma'(s)), \mathbf{n}(s) \rangle \leq h$ for all s and $0 < h < \pi$. Equivalently, the boundary ∂M has curvature satisfying $\kappa_g \leq 1$ almost everywhere. We denote the class of all such curves by $\mathcal{K}^+$.

Proposition 2.1. (Pestov and Ionin [20]). *Let $M \subset \mathbf{R}^2$ be an embedded disk whose boundary is of class $\mathcal{K}^+$. Then M contains a disk of radius one. In particular the area of M is at least π with equality if and only if M is a disk of radius one.*

Outline of the proof. For $X \in \partial M$ let $C(X)$ be the first point P along the inward normal to ∂M at X where the segment $[X, P(X)]$ stops minimizing $\operatorname{dist}(P, \partial M)$. Call this the cut point of $X \in \partial M$ in M. From the definition it is clear that M contains a disk of radius $\operatorname{dist}(X, C(X))$ about $C(X)$. Lemma 2.2 shows that if $C(X)$ is the cut point of $X \in \partial M$, then at least one of the following two conditions holds

(1) $C(X)$ is a focal point of ∂M along the normal line to ∂M at X, or

(2) there is at least one other point $Y \in \partial M$ so that $C(Y) = C(X)$ and

$$|C(X) - X| = |C(X) - Y| = \operatorname{dist}(C(X), \partial M).$$

(For example, if the boundary were C^2, see [**6**, Lemma 5.2 p. 93].) If $C(X)$ is a focal point of ∂M then the curvature condition implies $|X - C(X)| \geq 1$ by Lemma 2.3 and we are done. However, if $\mathcal{C}$ denotes the set of all cut points then we will show that $\mathcal{C}$ contains at least one focal point in Lemma 2.7. □

We spell out these notions for $M \subset \mathbf{R}^2$ with boundaries of class $\mathcal{K}^+$. In fact, the results of this section apply almost directly for M which is a C^2 compact two dimensional Riemannian manifold with C^1 nonempty boundary ∂M. For any $X \in \partial M$ let $\eta_X(s)$ be the unit speed geodesic, $\eta_X(0) = X$ with $\eta_X{}'(0)$ equal to the inward unit normal to ∂M. The *cut point* of $X \in \partial M$ is the point $\eta_X(s_0)$ where s_0 is the supremum of all $s > 0$ so that the segment $\eta_X([0, s])$ realizes the distance $\operatorname{dist}(\eta_X(s), \partial M)$. The *focal point* of $X \in \partial M$ is the point $\eta_X(s_1)$ where s_1 is supremum of values $s > 0$ so that the function on ∂M defined by $Y \mapsto \operatorname{dist}(\eta_X(s), Y)$ has a local minimum at $Y = X$. If ∂M is C^2 at X then s_1 is the first s where $Y \mapsto \operatorname{dist}(\eta_X(s), Y)$ ceases to have a positive second derivative at $Y = X$. It is possible that no such s_1 exists; in this case we say that the *focal distance* is $s_1 = \infty$. Clearly $s_0 \le s_1$.

Denote by $\mathcal{C}$ the set of all cut points of ∂M in M. Our goal is to understand what the local geometry of $\mathcal{C}$ is like at its "nice" points.

Lemma 2.2. *Any point $P \in \mathcal{C}$ satisfies at least one of the following two conditions*

(1) *P is a focal point of ∂M or*

(2) *there are two or more distance minimizing geodesics from ∂M to P.*

Proof. This is standard. If $P \in \mathcal{C}$ is not a focal point of ∂M then let $r := \operatorname{dist}(P, \partial M)$ and let $X \in \partial M$ be a point with $P = \eta_X(r)$. Then choose a sequence $s_k \searrow r$ such that for each k there is a point $X_k \in \partial M$ so that $\eta_X(s_k) = \eta_{X_k}(r_k)$ for some $r_k < s_k$. By going to a subsequence we can assume that $X_k \to Y$ for some $Y \in \partial M$. Because P is not focal point of ∂M we have $Y \ne X$. It follows that $\eta_Y(r) = P$ and η_Y is a minimizing geodesic from ∂M to P. □

Lemma 2.3. *Let $M \subset \mathbf{R}^2$ be a domain whose boundary is of class $\mathcal{K}^+$. Let $Y \in \mathcal{C}$ be a focal point. Then* $\operatorname{dist}(Y, \partial M) \ge 1$.

Proof. Let $Y = \eta_X(s_0)$ for some point $X \in \partial M$ and $s_0 > 0$. Let $\gamma \in \mathcal{K}^+$ denote the boundary curve ∂M parameterized so that

$\gamma(0) = X$. Since γ is tangent to ∂M at X, by Proposition 1.1, some interval $\gamma((-\varepsilon, \varepsilon))$ is not contained in the open disk $B_s(\eta_X(s))$ for each $0 < s < 1$. Hence $\partial M \ni Z \mapsto \operatorname{dist}(Z, \eta_X(s))$ has a local minimum at $Z = X$. Thus $s_0 \geq 1$. □

Lemma 2.4. (Structure of the cut locus away from focal points.) *Let $P \in \mathcal{C}$ be a cut point that is not a focal point and let $r = \operatorname{dist}(P, \partial M)$. Then there is a finite number of $k \geq 2$ of minimizing geodesics from P to ∂M, and*

Case 1. *If $k = 2$, then there is a neighborhood U of P so that $\mathcal{C} \cap U$ is a C^1 curve and the tangent to $\mathcal{C}$ at P bisects the angle between the two minimizing geodesics from P to ∂M.*

Case 2. *If $k \geq 3$, then the k geodesic segments from P to ∂M split the disk $B_r(P)$ into k sectors $S_1, \ldots, S_k$. There is a small open disk U about P so that in each sector S_i the set $\mathcal{C} \cap U \cap S_i$ is a C^1 curve ending at P and the tangent to this curve at P is the angle bisector of the two sides of the sector S_i at P.*

Remark 2.5. When viewed correctly, this is not a surprising result. In the Euclidean case take k points $X_1, \ldots, X_k$ on the boundary of a disk $B_r(P)$ that divides the disk into sectors $S_1, \ldots, S_k$. The cut set $\mathcal{C}$ of the finite set $\{X_1, \ldots, X_k\}$ is exactly the union of the angle bisectors of the sectors S_i. The theorem just says that away from focal points this model is correct at the infinitesimal level.

Proof. By Lemma 2.2 there are at least two minimizing geodesics from P to ∂M. If there were an infinite number of these segments, then their endpoints would accumulate at some point $Y \in \partial M$. Then P would be a focal point of Y. Thus the number of such segments is finite.

Let $X_1, \ldots, X_k$ be the points in $\partial B_r(P) \cap \partial M$ (so that $\operatorname{dist}(P, X_i) = \operatorname{dist}(P, \partial M)$). Let $\varepsilon > 0$ be small. There is then an $r_1 > r$ so that

$$B_{r_1}(P) \cap \partial M \subseteq \bigcup_{k=1}^{k} B_\varepsilon(X_i). \tag{2.1}$$

Therefore if $\delta = (r_1 - r)/2$ and $Q \in B_\delta(P)$ then the point of ∂M closest to Q is in $\partial M \cap \bigcup_1^k B_\varepsilon(X_i)$. For let $X \in \partial M$ be the point closest to Q, then

$$\operatorname{dist}(Q, X) \le \operatorname{dist}(Q, X_1) \le \operatorname{dist}(Q, P) + \operatorname{dist}(P, X_1) \le \delta + r < r_1.$$

Thus (2.1) implies $X \in \partial M \cap \bigcup_1^k B_\varepsilon(X_i)$. In what follows we will take a point Q close to P and assume that the point of ∂M closest to Q is close to one of the points $X_1, \dots, X_k$. This is justified in light of the remarks just made.

If $k = 2$ then $P = \eta_{X_1}(r) = \eta_{X_2}(r)$ for $X_1, X_2 \in \partial M$. Let c_i be a small piece of ∂M containing X_i and let $\rho_i(Q)$ be the distance of $Q \in M$ from c_i. Then, as P is not a focal point, the function ρ_i is C^1 in a neighborhood of the minimizing geodesic from X_i to P. The gradient of ρ_i at the point $\eta_{X_i}(s)$ (where $0 \le s \le r$) is

$$\nabla \rho_i(\eta_{X_i}(s)) = {\eta_{X_i}}'(s). \tag{2.2}$$

Set $f = \rho_1 - \rho_2$. Thus the zero set of f is the set of points that are at equal distances from c_1 and c_2. This is a C^1 function in a neighborhood of P. The gradient of f at P is

$$\nabla f(P) = \nabla \rho_1(P) - \nabla \rho_2(P) = \eta'_{X_1}(r) - \eta'_{X_2}(r) \tag{2.3}$$

which is not zero (if it were, then the minimizing geodesics from P to ∂M would be equal). Therefore by the implicit function theorem the set $\mathcal{S}$ defined by $f = 0$ is a C^1 curve in some small open disk U about P. The points of $\mathcal{S}$ are all cut points as they can connected to ∂M by two minimizing geodesics. Moreover no other point of U can be a cut point as any point of U is either closer to c_1 than c_2 or the other way around. Thus $\mathcal{C} \cap U = \mathcal{S}$. As $\mathcal{S}$ is a level set of f its tangent at P is orthogonal to $\nabla f = \nabla \rho_1 - \nabla \rho_2$. But each $\nabla \rho_i$ is a unit vector so that $\nabla \rho_1 + \nabla \rho_2$ is orthogonal to ∇f. But $\nabla \rho_1 + \nabla \rho_2$ bisects the angle between $\nabla \rho_1$ and $\nabla \rho_2$. This completes the proof of case 1.

If $k \ge 3$ then choose a sector and reorder things so that this sector is S_1 and so that X_1, X_2 are the points of $\{X_1, \dots, X_k\} = \partial B_r(p) \cap \partial M$ that are on S_1. For each i with $1 \le i \le k$ choose a small piece c_i of ∂M centered at x_i and as in the case of $k = 2$ let ρ_i be the distance from c_i. As before each ρ_i is C^1 in a neighborhood on the minimizing geodesic segment from X_i to P. Again let $f = \rho_1 - \rho_2$. Again as before $\nabla f \ne 0$

near P and so in some small open disk U about P the set defined by $f = 0$ is a C^1 curve through P and the tangent to this curve bisects the angle between the two sides of the sector S_1. Call this curve $\mathcal{S}$. Set

$$\mathcal{S}_+ = \mathcal{S} \cap S_1, \qquad \mathcal{S}_- = \mathcal{S} \setminus S_1.$$

If Q is a point in the interior of $U \cap S_1$, and the disk U is small enough, then the point of ∂M closest to Q will be in c_1 or c_2. To see this let $3 \le i \le k$. Then by the argument above the set F_{i1} of points in U closer to c_i that to c_1 is separated from the set G_{i1} of points in U closer to c_1 than to c_i by a smooth curve γ_{1i} whose tangent at P bisects the angle between the geodesic segments $[P, X_i]$ and $[P, X_1]$. Likewise for the sets F_{2i} and G_{2i}. Therefore the set of points closer to c_i then either c_1 or c_2 is contained in $F_{i1} \cap F_{i2}$ and this set is disjoint from S_1, at least when U is small enough. This implies the statement above about Q. But this makes it clear that the part of $\mathcal{C}$ in S_1 is just $\mathcal{S}_+$. A similar argument applies to the other sectors. This completes the proof. □

We now provide the details of the proof of Proposition 2.1. First we need that M and the cut locus $\mathcal{C}$ have very similar topology.

Proposition 2.6. *For any compact two dimensional Riemannian manifold $(M, \partial M)$ the cut locus $\mathcal{C}$ is a strong deformation retract of M.*

Proof. This is standard. Retraction is accomplished by normal deformation. □

Lemma 2.7. *If M is simply connected, then the cut locus $\mathcal{C}$ contains at least one focal point.*

Proof. Assume, toward a contradiction, that $\mathcal{C}$ has no focal points. Then by the structure theorem $\mathcal{C}$ is a graph in the sense that it is a finite number of points connected by a finite number of C^1 imbedded arcs. (Note that loops, that is arcs that begin and end at the same point, may be possible.) Also by the structure theorem each vertex of the graph has valence at least 3 in the sense that there are at least three arcs ending at the vertex. But by Proposition 2.6, $\mathcal{C}$ has the same homotopy type as M and thus it is also simply connected. Therefore it is a tree. But a tree has at least two vertices that are "ends" in that they have valence one. This contradiction completes the proof. □

Remark 2.8. Thus we have established Proposition 2.1. In fact, a nontrivial cut locus $\mathcal{C}$ must have at least two focal points. To see this note that an easy variant of the structure theorem shows that if the set of focal points is finite, then $\mathcal{S}$ is a graph. Again by the structure theorem any valence one vertex must be a focal point. Thus if the domain is simply connected there are three cases.

Case 1. $\mathcal{S}$ is a one point set in which case the domain is a disk.

Case 2. $\mathcal{S}$ has more than one point and a finite number of focal points. In this case $\mathcal{S}$ is a graph and thus has two or more vertices, which implies it has two or more focal points.

Case 3. $\mathcal{S}$ contains an infinite number of focal points.

Note that we are only assuming a one sided bound on κ, not a bound on $|\kappa|$. The argument applies equally well to Riemannian surfaces. In the Riemannian surface case, the focal distance depends on the upper bound of the curvature K_0 as well as the boundary curvature.

Theorem 2.9. *Let M be simply connected with $\partial M \subset \mathcal{K}^+$. Suppose that the Gauss curvature K of M satisfies $K \leq 0$ and the geodesic curvature κ of ∂M with respect to the inward normal satisfies $\kappa \leq 1$ almost everywhere. Then M contains a disk of radius one. M has area at least π with equality if and only if M is isometric with the standard unit disk in the plane.*

Proof. As in Lemma 2.7 the cut locus $\mathcal{C}$ has at least one focal point P. By adapting standard comparison theorems this focal point has a distance of at least 1 from ∂M. Also (see, e.g., [**15**]) the area inequality holds. This implies the theorem. □

Remark 2.10. For general curvature, the radius estimate of the contained disk has the following form. Let $r_0 > 0$ and let K_0 be any real number. When $K_0 > 0$ we assume $r_0 < \pi/\sqrt{K_0}$. Set

$$\kappa_0 = \begin{cases} \sqrt{K_0}\,\cot\left(\sqrt{K_0}\,r_0\right), & \text{if } K_0 > 0, \\ 1/r_0, & \text{if } K_0 = 0, \\ \sqrt{|K_0|}\coth\left(\sqrt{|K_0|}\,r_0\right), & \text{if } K_0 < 0. \end{cases} \tag{2.4}$$

Then it is straightforward to modify the proof of the last theorem to show that if M is simply connected, the Gauss curvature of M satisfies $K \leq K_0$ and the geodesic curvature of ∂M with respect to the inward normal satisfies $\kappa \leq \kappa_0$ *a.e.*, then M contains a disk of radius r_0. This gives the area of a disk of radius r_0 in the model space of constant curvature K_0 as a lower bound for the area of M with equality if and only if M is isometric to a disk of radius r_0 in the model.

There is an application to minimal (zero mean curvature) surfaces spanning curves in space that follows easily from what we have done.

Corollary 2.11. *Let $c \in \mathcal{K}$ be a curve in $\mathbf{R}^3$ whose curvature as a space curve has $k \leq 1$ almost everywhere. Assume that $X : \overline{D} \to \mathbf{R}^3$ is a minimal immersion of a disk with boundary so that the restriction $X|\partial D$ is a regular parameterization of c and $X \in C^{1,1}(\overline{D}, \mathbf{R}^3)$. Then the area $|D| \geq \pi$.*

Proof. Intrinsically D is a surface with nonpositive Gauss curvature and with boundary geodesic curvature $\kappa \leq 1$ almost everywhere since the geodesic curvature of $\partial D = c$ does not exceed the curvature of c viewed as a space curve. Thus Theorem 2.9 shows the area $|D| \geq \pi$ with equality if and only if D is a flat round disk. □

Corollary 2.12. *Let $c \in \mathbf{R}^3$ be a closed embedded $C^{1,1}$ space curve whose curvature $k \leq 1$ at all points. Then any disk spanning c has area at least π.*

Proof. For a C^1 space curve let $A(c)$ be the infimum of the areas of the disks spanning c. A is a continuous function of c in the C^1 topology. As the real analytic curves are dense in the space of all curves in the C^2

topology we may assume that c is real analytic. If M is the Douglas-Rado solution to the Plateau problem, then the area of M is $A(M)$ and by a theorem of Osserman [**19**] and Gulliver [**12**] this is free of interior branch points, and by a theorem of Gulliver and Lesley [**13**] it is free of boundary branch points. Therefore the last corollary implies that $A(c) \geq \pi$. □

Remark 2.13. Minimizing surfaces in $\mathbf{R}^n$ for $n \geq 4$ can have branch points. We wonder if there is still a way to get a lower bound on the inradius of a disk with a metric that is smooth except for a finite number of singularities of "branch point type."

Remark 2.14. Note that if c is a space curve that is very close to a standard circle double covered, then there is a minimal surface of the type of a Möbius strip that spans c and has small area. However, no lower bound is to be expected if, as for the Möbius strip, the Euler characteristic vanishes. For example if M is the planar region between concentric circles, one of radius 1 and the other of radius $1 + 2r$, then the curvature of boundary of M has $|\kappa| \leq 1$ but the largest disk that can be put in M has radius r. Since r can be taken to be as small as we please this shows there is no lower bound for the inradius.

For higher connectivity, there is another inradius lower bound. The following is a special case of a more general theorem of Alexander and Bishop [**1**] for curved surfaces under curvature bounds. The proof for the case of plane domains follows easily from what we have already done so we include it here for completeness.

Theorem 2.15. *Let $M \subset \mathbf{R}^2$ be a bounded connected domain in the Euclidean plane with boundary of class $\mathcal{K}$ (so that the curvature satisfies $|\kappa| \leq 1$ a.e.) If the Euler characteristic $\chi(M)$ is non-zero, then M contains a disk of radius $r_1 = 2/\sqrt{3} - 1 \approx .15470053838$.*

Lemma 2.16. *If a disk $B_r(p)$ has three points on the boundary so that the unit disks tangent to $B_r(p)$ at these points are disjoint, then $r \geq r_1 = 2/\sqrt{3} - 1$.*

Proof [**16**]. The extremal figure is three unit disks centered at the vertices of an equilateral triangle with sides of length 2 and $B_r(p)$ the disk centered at the center of the triangle that is tangent to the three larger disks. This disk has radius r_1 given above. □

Proof of Theorem 2.15. If the cut locus of M has a focal point, then M contains a disk of radius 1 and we are done. Thus assume that the cut locus $\mathcal{C}$ has no focal points. If $\mathcal{C}$ has no vertices then by the structure theorem $\mathcal{C}$ is a C^1 connected curve, and thus a circle. Thus $\mathcal{C}$ and, by Proposition 2.6, also M have Euler characteristic zero. Therefore by the hypothesis $\mathcal{C}$ has a least one vertex, and by the structure theorem the valence of this vertex P is at least three. Set $r = \text{dist}(P, \partial M)$ and let $\{X_1, \dots, X_k\} = \partial B_r(P) \cap \partial M$.

Consider unit disks $\{B_1, \dots, B_k\}$ exterior to M and tangent to ∂M at the X_j. We now argue that these unit disks are disjoint. Suppose this is not the case for, say, two consecutive contact points X_1, X_2. Then the two tangent unit disks B_1 and B_2 intersect. Let the piece of boundary curve from X_1 to X_2 be denoted $c(s)$ where $c(0) = X_1$ and $c(L) = X_2$. We first claim that $L \leq \pi$. By Proposition 1.1 (1) and (2) there are $0 < s_1, s_2 < \pi/2$ so that $c(s_1) \in \bar{B}_2$ and $c(L - s_2) \in \bar{B}_1$ Hence $c([0, s_1]) \cap c([L - s_2, L]) \neq \varnothing$, therefore $L \leq s_1 + s_2 \leq \pi$. By Proposition 1.2, (1) is this is impossible. By Lemma 2.16 this implies $r \geq r_1$. □

Remark 2.17. In higher dimensions, Lagunov [**16**] shows that the largest ball contained in a domain of $\mathbf{R}^n$, $n \geq 3$, with connected boundary with principal curvatures satisfying $|\kappa_i| \leq 1$ has radius at least $r_1 = 2/\sqrt{3} - 1$ but gives an example for which this cannot be enlarged. However, for domains satisfying additional topological restrictions, such as the ball, Lagunov and Fet [**17**] show that the least radius of the contained ball is increased to $r_2 = \sqrt{3/2} - 1$ and give examples of surfaces showing this is sharp.

The essence of Lagunov's argument, again, is to study the cut set. Let $\mathcal{C}(X_1, \dots, X_k)$ be the cut locus for finitely many points $\{X_1, \dots, X_k\} \in \partial B_1(0)$ on the surface of the unit ball in $\mathbf{R}^n$. $\mathcal{C}(X_1, \dots, X_k)$ divides $\mathbf{R}^n$ into sectors S_i containing X_i where S_i can be defined as the set of points of $\mathbf{R}^n$ that are closer to X_i than to any of the other points in the set $\{X_1, \dots, X_k\}$. If $k = 2$ then $\mathcal{C}(X_1, X_2)$ is the hyperplane that is a perpendicular bisector of the segment between X_1 and X_2. For a domain M with smooth boundary ∂M let $\mathcal{C}$ be the cut locus of ∂M. Choose $P \in \mathcal{C}$ such that P is not a focal point of ∂M. Set $r = \operatorname{dist}(P, \partial M)$. Lagunov shows that there is a finite number $k \geq 2$ of points $\{X_1, \ldots, X_k\}$ in $\partial B_r(P) \cap \partial M$. For each i let $\mathbf{u}_i$ be the unit vector in $T_P M$ that is tangent to the segment $[P, X_i]$ and is pointing in the direction of X_i. If $k = 2$ then there is a small open ball U about P so $U \cap \mathcal{C}$ is a smooth hypersurface and the tangent space to $\mathcal{C}$ at P is $\mathcal{C}(\mathbf{u}_1, \mathbf{u}_2)$. If $k \geq 3$ then there is a small ball U about P so that $U \cap \mathcal{C}$ is a "nice" stratified set, in particular it has a tangent cone at P and this tangent cone is $\mathcal{C}(\mathbf{u}_1, \ldots, \mathbf{u}_k)$. For general domains, if there is no focal point, the cut set must contain points where $k \geq 3$. Locally this looks like a triple junction graph crossed with an interval, hence it allows balls of radius r_1 as in Theorem 2.15. Under topological hypotheses, Lagunov and Fet [**17**] deduce existence of $k = 4$ points which yields the larger radius r_2 in the analog of Lemma 2.16.

There may be a stronger inradius estimates in the higher dimensional version if topological assumptions are replaced by geometric ones such as a bound on diameter. Also, we suspect that any starlike domain in $\mathbf{R}^3$ with all principle curvatures ≤ 1 has inradius ≥ 1.

3. Gradient estimate and star-shapedness.

The following lemma gives an estimate on the radial component of velocity of a curve in an annulus in the plane. It says that the rate at which a curve approaches the boundary circles cannot be too large near the boundary lest the curve be forced to "drive into the curb". The estimate was found by computing the gradient of the distance function on circular arcs tangent to the bounding circles.

Lemma 3.1. *Suppose $M \subset \mathbf{R}^2$ is an embedded disk with class $\mathcal{K}$ boundary satisfying $|\kappa| \leq 1$ almost everywhere. Let $c(s)$ denote the boundary curve parameterized by arclength and $\rho(s) = \langle c(s), c(s) \rangle$ be the square of the distance to the origin. If the disk of radius r centered at the origin $B_r(0) \subset M$ then*

$$\rho_s{}^2 \leq \left[\rho - r^2\right]\left[(r+2)^2 - \rho\right] \quad \text{whenever } \rho \leq r^2 + 2r. \tag{3.1}$$

If $M \subset B_R(0)$ where $R \leq 2$ then

$$\rho_s{}^2 \leq \left[R^2 - \rho\right]\left[\rho - (R-2)^2\right]. \tag{3.2}$$

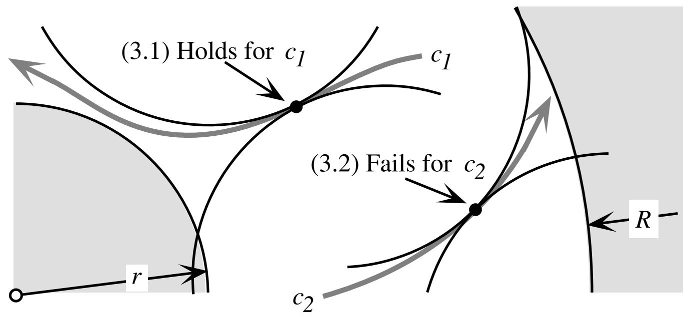

FIGURE 2. A curve with large $|\rho_s|$ cannot avoid "driving into the curb."

Moreover, inequality (3.2) *implies that* $\rho \geq (2-R)^2$. *If* $2 < R$ *then* (3.2) *holds whenever* $\rho \geq R^2 - 2R$.

Proof. For (3.1), we may assume that $c'(0)$ makes an angle of at most 90 degrees with the segment from $c(0)$ to the origin. (If not, replace $c(s)$ with $c(-s)$.) Let $\mathcal{C}_1$ and $\mathcal{C}_2$ be two unit circles tangent to c at $c(0)$ with $\mathcal{C}_1$ the one that is "closest" to the origin. If $\rho \leq r^2 + 2r$ and (3.1) is false, then both $\mathcal{C}_1$ and $\mathcal{C}_2$ must intersect the interior of $B_r(0)$. Proposition 1.1 implies that c will also intersect the interior of $B_r(0)$, which is not the case. Thus $\mathcal{C}_1$ is either tangent to $\partial B_r(0)$ or disjoint from the closure of $B_r(0)$. This implies that if the angle between $c'(0)$ and the segment from $c(0)$ to the origin is at least as large as the angle between this segment and the tangent to the unit circle through $c(0)$ and tangent to $\partial B_r(0)$. A calculation shows that this is equivalent to

$$[\rho'(s)]^2 = [(r+2)^2 - \rho(s)][\rho(s) - r^2]. \tag{3.3}$$

Formula (3.2) can be proven along the same lines; however, we indicate an alternate proof. As before, ρ_s is absolutely continuous and ρ_{ss} is defined almost everywhere. The (weak) equations for ρ on ∂M are

$$\rho_s = 2\langle c, \mathbf{t}\rangle, \qquad \rho_{ss} = 2 - 2\kappa p \tag{3.4}$$

almost everywhere where $p = -\langle X, \mathbf{n}\rangle$ and $\mathbf{t}$ and $\mathbf{n}$ are the tangent and outer unit normal vectors to c. Thus setting $\rho_s^2 + 4p^2 = 4\rho$ we get

$$\rho_{ss} \leq 2 + \sqrt{4\rho - \rho_s^2} \tag{3.5}$$

almost everywhere. This can be integrated to yield (3.2). □

Lemma 3.2. *Suppose $M \subset \mathbf{R}^2$ is an embedded disk with boundary of class $\mathcal{K}(k)$; that is, the boundary curve $c(s)$ is C^1, parameterized by arclength, with c' absolutely continuous with boundary curvature bounded by a constant $|\kappa| \le k$ almost everywhere. Let $p = -\langle c, \mathbf{n}\rangle$ denote the support function to ∂M and assume $p \ge p_0 > -1/k$ at all $x \in \partial M$. Then*

$$L \le \frac{2kA + 2\pi p_0}{1 + kp_0}. \tag{3.6}$$

In particular, if M were star-shaped with respect to the origin (thus $p_0 \ge 0$) then $L \le 2kA$.

Proof. This follows from the Minkowski formulas whose proof we sketch. By Lebesgue's theorem we can recover ρ_s by integration. Equation (3.4) implies

$$L - \int_{\partial M} \kappa p \, ds = \int_{\partial M} 1 - \kappa p \, ds = \tfrac{1}{2} \int_{\partial M} \rho_{ss} \, ds = 0.$$

The function $f(X) = \langle X, X\rangle$ satisfies $\mathbf{n}f = 2\langle c, \nabla_{\mathbf{n}} X\rangle = 2p$ for $X \in \partial M$. Using $p \ge p_0$ and Stokes theorem,

$$\begin{aligned} L - 2\pi p_0 &= \int_{\partial M} \kappa \left(p - p_0\right) \, ds \le k \int_{\partial M} p - p_0 \, ds \\ &= \frac{k}{2} \int_{\partial M} \frac{\partial f}{\partial N} \, ds - kp_0 L = \frac{k}{2} \int_M \Delta f \, dx \wedge dy - kp_0 L \\ &= 2kA - kp_0 L \end{aligned}$$

where Δ is the $\mathbf{R}^2$ Laplacian. Hence (3.6) holds. □

Remark 3.3. In fact, there is a reverse isoperimetric inequality in all dimensions for regions $M \subset \mathbf{R}^n$ which are starlike with respect to the origin and with mean curvature H (normalized so that $H = 1$ on the unit sphere) of ∂M satisfying $|H| \le 1$. To see this let p the support function of ∂M. Then the surface area A of ∂M and the volume V of M are given by the Minkowski formulas

$$A = \int_{\partial M} pH \, dA, \qquad nV = \int_{\partial M} p \, dA.$$

Using that $|H| \leq 1$ and that $p \geq 0$ (as M is starlike with respect to the origin) in these leads at once to the reverse isoperimetric inequality $A \leq nV$. Also the isoperimetric inequality in $\mathbf{R}^n$ is

$$A(\mathbf{S}^{n-1})^n V(M)^{n-1} \leq V(B^n)^{n-1} A(\partial M)^n.$$

Using the relation $A(\mathbf{S}^{n-1}) = nV(B_1^n)$ and the last two inequalities leads to sharp lower bounds for the $A(\partial M)$ and $V(M)$

$$A(\mathbf{S}^{n-1}) \leq A(\partial M), \qquad V(B^n) \leq V(M),$$

in the class of starlike regions with mean curvature having $|H| \leq 1$.

The following lemma is the link between radius bounds and length bounds. Its main conclusion is star-shapedness of the domain.

Theorem 3.4. *Suppose $M \subset \mathbf{R}^2$ is an embedded disk with ∂M of class $\mathcal{K}$. Suppose also that $B_1(0) \subset M \subset B_R(0)$ where $R \leq 3$. Then M is star-shaped with respect to the origin. There is the estimate*

$$L \leq \frac{8A + 2\pi\left(3 + 2R - R^2\right)}{7 + 2R - R^2} \tag{3.7}$$

where L is the length and A is the area of M. In particular, for all $1 \leq R \leq 3$ there holds $L \leq 2A$.

Proof. If $R < 3$ then Lemma 3.1 with $r = 1$ and $R = 3$ imply that ${\rho_s}^2 < 4\rho$ and so $4p^2 = 4\rho - {\rho_s}^2 > 0$ so p cannot change sign and must remain positive. If $R = 3$ the only possibility for $p < 0$ is that at a point $c(s_0) \in \partial M$ there holds $p = 0$ at $\rho = 3$ and ∂M consists of circular arcs near $c(s_0)$ extending the entire range $\rho \in [1, 9]$. But at $\rho \in \{1, 9\}$ we must have $p = +\sqrt{\rho}$ and $P \geq 0$ on the arcs adjacent to $c(s_0)$ or else M is on the "wrong" side of ∂M so at $\rho = 1$ and $1 \neq \inf \rho$, or $\rho = 9$ and $9 \neq \sup \rho$. The remainder of the argument is to give an estimate of p_0. Recall that

$$4p^2 = 4\rho - {\rho_s}^2. \tag{3.8}$$

For $R < 3$ the bounds given by Lemma 3.1 intersect at $\rho = \rho_3 := (3 - 2R + R^2)/2$. Hence, there is an absolute bound

$${\rho_s}^2 \leq (9 - \rho_3)(\rho_3 - 1) = \frac{(5 - R)(3 + R)(1 - R)^2}{4}.$$

Observe that

$$4\rho - \min\left\{(\rho-1)(9-\rho), \left[R^2-\rho\right]\left[\rho-(R-2)^2\right]\right\}$$

is minimized at ρ_3. Thus by (3.8) we get the bound

$$4p^2 \geq 4\rho_3 - (\rho_3-1)(9-\rho_3) = \frac{(3-R)^2(1+R)^2}{4}$$

hence

$$p_0 \geq \frac{(3-R)(1+R)}{4}$$

which gives (3.7) when inserted in Lemma 3.2. □

For $0 < \delta \leq \sqrt{3}$, let $E_\delta = B_1((\delta,0)) \cup B_1((-\delta,0))$ denote the domain consisting of the union of two unit disks whose centers are 2δ apart. Let F denote the bounded region between $\bar{E}_\delta$ and the pair of tangent disks $\bar{B}_1\left((0,\pm(4-\delta^2)^{1/2})\right)$. If $\delta \leq 1$ then the "fillet" F has two triangular components. Consider two unit circular arcs c_1, c_2 of $\partial F - \bar{E}_\delta$ tangent to each of the circular boundary components of ∂E_δ. Let P_δ denote the "peanut-shaped" domain consisting of $P_\delta = \operatorname{int}\left(\bar{E}_\delta \cup \bar{F}\right)$. This is the region pictured in Figure 1.

Lemma 3.5. *Suppose $\delta \leq 1$. Let $\gamma(s) : [0,L] \to \mathbf{R}^2$ be a curve of regularity $\mathcal{K}$ with curvature bounded by $|\kappa| \leq 1$ almost everywhere whose endpoints are on ∂P_δ and which lies exterior to E_δ. Then the entire curve is exterior to P_δ.*

Proof. Proof is by the maximum principle. Let c_1 be the arc $\partial B_1\left(0,(4-\delta^2)^{1/2}\right) \cap \bar{F}$ and suppose that $\gamma(s)$ enters the $y > 0$ component of the fillet F. Consider the foliation of F by arcs of unit circles with centers $(0,\eta)$ where $1 \leq \eta < (4-\delta^2)^{1/2}$. Let η_0 be the smallest η for which the arc and γ intersect. Since the arcs are transverse to ∂E_δ, an intersection point is interior to F and γ and the η_0 arc are tangent there. By Proposition 1.1, γ stays outside the η_0 circle and hence "crashes" into ∂E_δ before it reaches c_1. □

Lemma 3.6. *Suppose M is an embedded disk with area A whose boundary curve is of class $\mathcal{K}$. Assume that M contains two unit disks* *whose centers Z_1, Z_2 lie on the cut locus $\mathcal{C}$ of ∂M. If* $\operatorname{dist}(Z_1, Z_2) = 2\delta > 2$ *then*

$$A > \pi + 2\sqrt{3}. \tag{3.9}$$

Proof. Let $|S|$ denote the area of a set S. Since it is connected, the cut locus connects Z_1 to Z_2. We may suppose that $\mathcal{C} - (B_1(Z_1) \cup B_1(Z_2))$ contains no focal points ζ. Otherwise M contains a unit disk about ζ and

$$|B_1(Z_1) \cup B_1(Z_2) \cup B_1(\zeta)| \geq \pi + \sqrt{3} + 2\pi/3 > \pi + 2\sqrt{3}$$

and we are done. Hence we suppose that there is a piecewise C^1 simple subarc $\gamma : [0, L] \to \mathcal{C}$, connecting $Z_1 = \gamma(0)$ to $Z_2 = \gamma(L)$ so that $\gamma([0, L]) - (B_1(Z_1) \cup B_1(Z_2))$ is without focal points.

We will show that either M contains a peanut P_δ or it contains thickening where γ meets the disks. In order to quantify this, consider three disks $B = B_1(0, 0)$, $B' = B_1(\sqrt{3}, 1)$ and $B'' = B_1(\sqrt{3}, -1)$. The triangular region T between the three circles has area $\sqrt{3} - \pi/2$. Let $\sigma(x)$ denote the radius of the circle centered at $(x, 0)$ and tangent to B' and B''. Let $\sigma(x) = 0$ if $x \geq \sqrt{3}$. Let $\tau(x)$ denote the distance of the contact point of $\partial B_{\sigma(x)}(x, 0) \cap \partial B'$ to the origin and $\lambda(\tau(x))$ the length of the arc $\partial B_{\tau(x)}(x, 0) \cap T$. $\sigma(x)$ is a decreasing and $1 \leq \tau(x) \leq \sqrt{3}$ an increasing function of x. We say that γ has *remote ends* if for all $0 \leq x_1, x_2$ there are measurable functions $t_1(x_1), t_2(x_2) \in [0, L]$ so that $\operatorname{dist}\big(\gamma(t_1(x_1)), Z_1\big) = x_1$, $\operatorname{dist}\big(\gamma(t_2(x_2)), Z_2\big) = x_2$ and that $B_{\sigma(x_1)}(\gamma(t_1(x_1))) \cap B_{\sigma(x_2)}(\gamma(t_2(x_1)) = \varnothing$.

First, if γ has remote ends then $|M| > \pi + 2\sqrt{3}$. To see this, observe that for any point $\gamma(t)$ there is an estimate of cut distance $c(\gamma(t))$. In particular, let $X_1, X_2 \in \partial M$ be two distinct points with $c(\gamma(t)) = |c(t) - X_i|$, which by the structure of $\mathcal{C}$ may be chosen on opposite sides of γ. The subsets of ∂M which are within an arclength $\pi/2$ of the X_i must be disjoint by Proposition 1.2. They must also be disjoint of $B_1(Z_1)$. Hence, by Proposition 1.1, the smallest cut distance is possible if the boundary subsets are unit circle arcs and form with $\partial B_1(Z_1)$ a triangle congruent to T. Thus the cut distance is at least that of the distance from $(|Z_1 - \gamma(t)|, 0)$ to ∂T, namely

$c(\gamma(t)) \geq \sigma(|\gamma(t) - Z_1|)$. In particular, there is an arc

$$\mathcal{A}_1(\tau(|Z_1 - \gamma(t)|)) = \partial B_{\tau(|Z_1-\gamma(t)|)}(Z_1) \cap B_{\sigma(|\gamma(t)-Z_1|)}(\gamma(t)) \subset M.$$

There is a similar definition of an arc $\mathcal{A}_2$ near the Z_2 end.

If γ has remote ends, then for all $0 \leq x_i < \sqrt{3}$ the balls $B_{\sigma(|\gamma(t_1)-Z_1|)}(\gamma(t_1))$ and $B_{\sigma(|\gamma(t_2)-Z_2|)}(\gamma(t_2))$ are disjoint, as are the corresponding arcs $\mathcal{A}_i(\tau_i)$. Hence the thickenings

$$\mathcal{H}_i = \bigcup_{1<\tau<\sqrt{3}} \mathcal{A}_i(\tau)$$

are disjoint. Thus

(3.10)
$$\begin{aligned} |M| &\geq |B_1(Z_1)| + |B_1(Z_2)| + |\mathcal{H}_1| + |\mathcal{H}_2| = 2\pi + 2\int_1^{\sqrt{3}} \lambda(\tau)\, d\tau \\ &= 2\pi + 2|T| = \pi + 2\sqrt{3}. \end{aligned}$$

Moreover, since $\gamma(t_1(\sqrt{3})) \notin \mathcal{H}_1 \cup \mathcal{H}_2$ by continuity, but is an interior point of M, the inequality (3.10) must be strict. In particular, if $\delta \geq \sqrt{3}$ then γ has remote ends so (3.9) holds.

From now on, assume $\delta < \sqrt{3}$. Let Λ be the straight line segment from Z_1 to Z_2. By rigid motion we may suppose that Λ is in the y-axis and centered about 0. We deal with the case that $\Lambda \not\subset M$. Choose a point $E \in \Lambda - \bar{M}$. We may also suppose that the bounded part of the complement of $M \cup \Lambda$ lies on the right side of Λ. Let $\chi : [0, \infty) \to \mathbf{R}^2$ in the unbounded part of the complement of $M \cup \Lambda$ be a simple smooth path connecting E to ∞. Let P_δ denote the peanut-shaped region about the balls $B_1(Z_1)$ and $B_1(Z_2)$. Let $Y_1, Y_2 \in \Lambda$ be the endpoints of the interval of $\Lambda - \bar{M}$ containing E. Because of Lemma 3.1, the direction V to ∂M at $X \in \partial M \cap P_\delta$ cannot point at Z_1 nor Z_2 thus must "flow through." For example, foliate $\mathbf{R}^2 - M$ by arcs of the circle $x = +(1 - y^2)^{1/2} + k$ for all constants k so that the semicircle touches $B_1(Z_1)$ and $B_1(Z_2)$. Then $k|_{\partial M}$ cannot have a maximum at one of these arcs, by the maximum principle. In particular, the curves ∂M through Y_i continue to $x > 1$.

Consider the portion β of ∂M starting $\pi/2$ from Y_1, continuing through Y_1, heading in the positive x-direction to Y_2 and ending $\pi/2$

beyond Y_2. By the Schur-Schmidt Proposition 1.1, β starts and ends on the left side of $y = -1$ and passes through P_δ. Let $\mathcal{E}$ denote the "lagoon," the connected component of $\mathbf{R}^2 - (M \cup \Lambda)$ containing E bounded by the union of the arc β and the segment $[Y_1, Y_2]$. Now consider the cut set $\mathcal{C}'$ of the complement of M. Since there are points $(1, y) \in \mathcal{E}$, some point on $\mathcal{E}$ is a cut point of $\mathbf{R}^2 - M$. To see this, let $B_u(Y)$ be the largest ball contained in $\mathbf{R}^2 - M$ with center $Y \in [Y_1, Y_2]$. It must be a cut point of $\mathcal{C}'$ which is not a focal point because, by construction, $u < \sqrt{3} - 1$. Let $\{X_1, \ldots, X_k\}$ be the nearest points of Y in ∂M. It must happen that at least one X_1 is on the boundary near Y_1 and one X_2 is on the boundary near Y_2. Now Y is the only place on the two segments $[X_1, Y] \cup [Y, X_2]$ that meets $\mathcal{C}'$. By the structure of the cut set, at least one continuation of the cut set enters the region $\hat{\mathcal{E}}$ bounded by $[X_1, Y] \cup [Y, X_2] \cup \beta$. But $\hat{\mathcal{E}}$ is simply connected, therefore by the structure theorem of $\mathcal{C}'$, if there are no focal points $\hat{\mathcal{E}} \cap \mathcal{C}'$ is a piecewise C^1 tree that must have an endpoint other than Y. It is a focal point W and therefore $B_1(W) \cap M = \varnothing$. In particular, the intrinsic minimal path from Z_1 to Z_2 within $\mathbf{R}^2 - \left(\chi([0, \infty)) \cup \hat{\mathcal{E}}\right)$ must loop around $B_1(W)$ so must have a length at least $2\sqrt{3} + 4\pi/3 - 2\sin^{-1}(\delta/2) > 2\pi/3 + 2\sqrt{3}$. Thus, γ has remote ends and (3.9) holds.

Finally, consider the case that $\Lambda \subset M$. By considering the foliation of $P_\delta - (B_1(Z_1) \cup B_1(Z_2) \cup \Lambda)$ by the field of extremals again, $x = \pm(1 - y^2)^{1/2} \pm k$, we see that any part of ∂M that begins and ends outside the peanut P_δ must stay outside the peanut. Hence

$$|M| \geq |P_\delta| = \pi + 2\delta\sqrt{4 - \delta^2} > \pi + 2\sqrt{3}$$

whenever $1 < \delta < \sqrt{3}$, which is the present case. □

We now prove the first link of our main estimate. It says that among domains with boundary having bounded curvature, the radius of a disk containing the domain is bounded by area. In fact we give a "stable version": the closer the area is to the area of a disk, the closer the domain itself is to the disk.

Theorem 3.7. [Area Stability Theorem]. *Suppose M is an embedded disk whose boundary curve is of class $\mathcal{K}$ and area A. If $A \leq \pi + 2\sqrt{3}$ then there is a point $P \in M$ so that $B_1(P) \subset M \subset B_R(P)$*

where $R = \zeta(A)$ *and* $\zeta(x)$ *is defined for* $\pi \le x \le \pi + 2\sqrt{3}$ *by the relation*

$$(\zeta - 1)^2 = 8 - 2\sqrt{16 - (x-\pi)^2}. \tag{3.11}$$

Proof. We utilize the structure of the cut locus $\mathcal{C}$ of M. By Proposition 2.1, M contains at least one unit disk which we locate at the origin. $\zeta(x)$ is the radius of the smallest disk at the origin needed to contain a peanut with area x located so the origin coincides with one of the unit disks of the peanut. So $B_1(0) \subset P_\delta \subset B_{\zeta(x)}(0)$ for a given area $x = |P_\delta|$. Denote by

$$\mathcal{G} = \{P \in \mathcal{C} : \operatorname{dist}(P, \partial M) + |P| \le \zeta(A)\}.$$

We claim $\mathcal{G} = \mathcal{C}$ and so $R \le \zeta(A)$. First observe that if $Q \in \mathcal{C}$ and $\operatorname{dist}(Q, \partial M) \ge 1$ then $Q \in \mathcal{G}$. In fact, this implies that there is a unit ball $B_1(P) \subset M$ so that $|P| = |Q| + \operatorname{dist}(Q, \partial M) - 1$. However, as in the proof of Lemma 3.6, $|M| \le \pi + 2\sqrt{3}$ implies that there is a segment from the origin, $[0, Q]$ which is completely contained in $\overline{M}$. Thus also $[O, P] \subset M$ as well as the corresponding peanut so $|P| \le \zeta(A) - 1$ hence $Q \in \mathcal{G}$. Hence all the focal points of $\mathcal{C}$, which have $\operatorname{dist}(Q, \partial M) \ge 1$, are in $\mathcal{G}$. Thus $\mathcal{C} - \mathcal{G}$ is a subset of a tree consisting of C^1 curves joined at vertices with finite valence.

Suppose $\mathcal{C} \ne \mathcal{G}$. Then at some point $X(s_0) \in \partial M$ we have $|X(s_0)| > \zeta(A)$. Consider the function $f(X) = |X|$ restricted to $\mathcal{C}$. At its maximum $Y \in \mathcal{C}$, it exceeds $|Y| \ge |X(s_0)| - |X(s_0) - C(s_0)| > \zeta(A) - 1 > 0$. Because of the structure of the cut locus, there are $2 \le k < \infty$ points $\{X_1, \ldots, X_k\} \subset \partial M \cap \partial B_r(y)$ where $r = \operatorname{dist}(Y, \partial M) < 1$. For any one of the components $(X_i, X_{i+1}) \subset \partial M - \{X_1, \ldots, X_k\}$, the cut set bisects the sector and extends beyond Y in the direction from Y to $(X_i + X_{i+1})/2$. But because f is maximal at Y this implies that $k = 2$ and the cut set directions are at best perpendicular to Y with $Y_1 = (1 + r/|Y|)Y$ and $Y_2 = (1 - r/|Y|)Y$. In particular, the segment $[Y_1, Y_2]$ intersects $\mathcal{C}$ at exactly one point Y where it crosses transversally. Let λ be a simple curve in $\mathbf{R}^2 - M$ connecting Y_1 to Y_2. Now, each connected component $\mathcal{C}_i$, $i = 1, 2$, of $\mathcal{C} - \{Y\}$ must contain a focal point A_i. To see this, suppose $\mathcal{C}_1$ does not. By construction it has the structure of a smoothly embedded connected tree. Hence it must contain at least two vertices of valence 1, thus a focal point. The set

$\mathcal{H} = \mathcal{C} \cup \Gamma_A \cup \gamma - \{Y\} \subset M$ connects both sides the loop $\lambda \cup [Y_1, Y_2]$ without crossing it, which is a contradiction in $\mathbf{R}^2$. □

Corollary 3.8. [Preliminary reverse isoperimetric inequality]. *Suppose M is an embedded disk whose boundary curve is of class $\mathcal{K}$ and with area A. If $A \le \pi + 2\sqrt{3}$ then $L \le 2A$. Moreover M is star-shaped with respect to the center of any unit disk in M. In fact, if M is located so that the origin is the center of such a disk, then the rays through the origin are transverse to ∂M except, possibly, when $A = \pi + 2\sqrt{3}$ and $R = \sqrt{8}$.*

Proof of Corollary 3.8. By Theorem 3.7, $R = R(A) \le 3$. The result follows from Theorem 3.4. In fact, the sharper inequality (3.7) holds for this R. □

We can extend Theorem 3.4 in the following manner. The closer the circumradius of a domain whose boundary has uniformly bounded curvature is to one, the closer the domain is to the unit circle. We obtain a stability estimate provided the original circumradius is at most the circumradius of three touching circles. The case $S \le 2$ also follows from Theorem 3.4.

Theorem 3.9. [Circumradius stability theorem]. *Let $M \subset \mathbf{R}^2$ be an embedded disk with boundary of class $\mathcal{K}$. Suppose in addition that there is a bound on the circumradius $M \subset B_S(0)$ where $1 \le S \le 1 + 2/\sqrt{3}$. Then, in fact, $B_{s_1}(Z) \subset M \subset B_S(Z)$ where*

$$s_1 = \sqrt{3 + 2S - S^2} - 1$$

and some $Z \in M$. M is star-shaped with respect to Z. Moreover, there is an estimate of the length of the form

$$L \le \frac{4A + 2\pi\left(1 + 2S - S^2\right)}{3 + 2S - S^2}. \tag{3.12}$$

Proof. First we show that if two unit disks are in M then so is the peanut between them. Let $B_1(Z_1)$ and $B_2(Z_2)$ two unit disks contained in M. Let $\Lambda = [Z_1, Z_2]$ be the line segment between the centers of length 2δ. Since $\delta < \sqrt{3}$ by assumption, we argue as in Lemma 3.6. If $\Lambda \not\subset M$ then there is a unit disk $B_1(W) \cap M = \varnothing$ which is contained in the convex hull of M. However, the circumradius of three tangent unit disks, $1+2/\sqrt{3}$, is strictly smaller than the circumradius of $B_1(Z_1) \cup B_1(Z_2) \cup B_1(W)$ which is less than S. This is a contradiction. Hence Λ, and therefore as in Lemma 3.6, the peanut $P_\delta \subset M$.

We may assume that B_S is the smallest possible disk that contains M and that its center is at the origin. If we let $\mathcal{Y} = \partial B_S(0) \cap \partial M$ be the contact points then the origin must be in the convex hull of $\mathcal{Y}$. We claim if $C(Y)$ is a cut point for $Y \in \mathcal{Y}$ then $s_0 = |Y - C(Y)| \geq 1$. The argument is similar to the proof of Lemma 3.6. If not, $s_0 < 1$ and the cut point is not a focal point. Hence, for $r(X) = |X|$, the distance function from the origin, and $W \in \mathcal{C}$ where $r|_{\mathcal{C}}$ is maximum, $|W| \geq S - s_0$ so W is not a focal point. It follows that the contact points $\{X_1, \ldots, X_k\} = \partial B_{c(W)}(W) \cap \partial M$ must be $X_1 = (1 + c(W)/|W|)W$ and $X_2 = (1 - c(W)/|W|)W$, because by the structure of the cut set near W, $\mathcal{C}$ has maximal r so must be perpendicular to W there. Now let γ be a curve in $\mathbf{R}^2 - M$ connecting X_1 to X_2. Let $B_1(Z_1)$ and $B_2(Z_2)$ be two unit balls centered at focal points on each component of the sets $\mathcal{C} - \{W\}$. By the previous paragraph, the line segment Λ from Z_1 to Z_2 is in M. Hence the connected set $(\mathcal{C} - \{W\}) \cup \Lambda$ does not intersect the closed path $[X_2, W] \cup [W, X_1] \cup \gamma$ yet connects its two sides. This is a contradiction.

Finally, since we have shown that the cut distances $c(Y) \geq 1$ for all $Y \in \mathcal{Y}$, there is an osculating unit disk $D(Y) \subset M$ centered at $\eta_Y(1)$. Every segment connecting pairs of $\eta_Y(1)$'s for $Y \in \mathcal{Y}$ must be in M as must their peanuts. The origin is in the convex hull of the $\eta_Y(1)$'s of $Y \in \mathcal{Y}$ and as the segments connecting these points are in M which is simply connected, this implies the origin is in M. To estimate $\operatorname{dist}(0, \partial M)$, observe that the origin is in the set consisting of the union of all peanuts with centers at $\eta_Y(1)$'s of $Y \in \mathcal{Y}$, with any possible holes filled in (union in the bounded components of the complement of the union of peanuts). Hence the minimal distance to the boundary is the distance to the boundary of the narrowest possible peanut, namely one with $\operatorname{diam}(P_\delta) = 2\delta + 2 \leq 2S$. Hence $\operatorname{dist}(0, \partial M) \geq \operatorname{dist}(0, P_{(S-1)}) = s_1 = (3 + 2S - S^2)^{1/2} - 1 \geq \sqrt{8/3} - 1$.

Thus $S - s_1 < 2$ and we may continue the argument as in Theorem 3.7.

Let $\rho(X) = \langle X, X\rangle$ be the square of the distance from the origin. We find that the bounds from Lemma 3.1 with $r = s_1$ and $R = S$ agree when $\rho = 1$ so

$$4p^2 = 4\rho - {\rho_s}^2 \geq 4 - \left[S^2 - 1\right]\left[1 - (S-2)^2\right].$$

This lower bound for p is used in Lemma 3.2 to obtain (3.12). ◻

Corollary 3.10. *Let $M \subset \mathbf{R}^2$ be an embedded disk with boundary of class $\mathcal{K}$. Assume there is a bound on the circumradius $M \subset B_S(0)$ where $1 \leq S \leq 1 + 2/\sqrt{3}$. Then*

$$L \leq \frac{2\pi(1+S)}{3-S}; \qquad L \leq \frac{3A+\pi}{2}.$$

Proof. For the first inequality substitute $A \leq \pi S^2$ into Theorem 3.9. For the second substitute $S \leq 1 + 2/\sqrt{3}$. ◻

Remark 3.11. Observe that Theorem 3.4, Theorem 3.7 and Theorem 3.9 are sharp. That is, for every $\varepsilon > 0$ there is an example M that has $A < \pi + 2\sqrt{3} + \varepsilon$, $B_1(Z) \subset M \subset B_{3+\varepsilon}(Z)$ and circumradius $S < 1 + 2/\sqrt{3} + \varepsilon$ but has arbitrarily large length. Take three disjoint circles in $B_{1+2/\sqrt{3}+\varepsilon/2}$ which can be thought of as pulleys. Think of the domain as the region between a long fanbelt (Figure 3). Thread a loop about one of the circles, wind both sides around a second circle, then continue arbitrarily many times about the peanut formed by the first two circles. End by looping about the last circle. By taking the region sufficiently thin, one can obtain all four conditions.

The last result in this section shows that embedded disks with small area but large length have to be thin in a sense appropriate for $\partial M \in \mathcal{K}$. In this way we get a quantitative description of the degeneration of embeddedness or "puckering." A measure of the pinching of a domain is given by

$$\omega(M) = \operatorname{dist}(\mathcal{C}, \partial M), \tag{3.13}$$

where $\mathcal{C} \subset M$ is the cut locus of the boundary. ω is called the *rolling number* of M because it is the largest radius such that at every boundary point the tangent disk of that radius remains inside M.

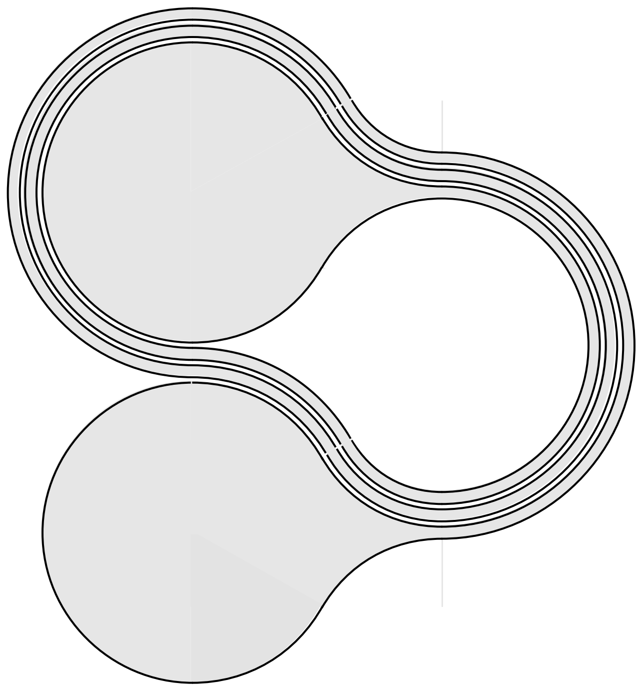

FIGURE 3. "Fanbelt" counterexample.

Theorem 3.12. [Puckering Theorem]. *Let $M \in \mathbf{R}^2$ be an embedded disk with boundary of class $\mathcal{K}$. Then the rolling number* (3.13) *satisfies*

$$\omega(M) \le \frac{L - \sqrt{L^2 - 4\pi A}}{2\pi}. \tag{3.14}$$

If in addition, $L > 2A$ then $A > \pi + 2\sqrt{3}$ and

$$\omega(M) \le \frac{A - \pi - 2\sqrt{3}}{L - 2\pi - 4\sqrt{3}}. \tag{3.15}$$

Proof. Consider the interior parallel domain $M_a = \{P \in M : \operatorname{dist}(p, \partial M) \ge a\}$ for constant $a \ge 0$. If $a < \omega(M)$ then ∂M_a is $C^{1,1}$

embedded circle whose curvature has $\|\tilde{\kappa}\|_\infty \le 1/(1-a)$. Let $c(s)$ denote an arclength parameterization of ∂M. Then the tubular neighborhood $M - M_a$ has a parameterization $\partial M \times [0,a] \ni (s,t) \mapsto \eta_{c(s)}(t)$ with area form $(1-\kappa(s)t)dt\,ds$. Estimating the area for $0 < a < \omega$,

$$\begin{aligned} A \ge |M - M_a| &= \int_{\partial M} \int_0^a (1-\kappa(s)t)\ dt\,ds \\ &= aL - \pi a^2. \end{aligned} \tag{3.16}$$

This holds for small a so the smaller root provides (3.14).

By the assumption $L > 2A$ and Proposition 2.1, we have $A > \pi + 2\sqrt{3}$ and $L > A + \pi + 2\sqrt{3}$ thus, by (3.14), $\omega(M) < 1$. Let $Z \in \mathcal{C}$ be a point where the distance to ∂M is a minimum. We argue that $\mathcal{Y} = \overline{B_{\omega(M)}(Z)} \cap \partial M$ is a doubleton $\mathcal{Y} = \{Y_1, Y_2\}$ using the structure theorem for the cut locus near the nonfocal point Z. Let $U \subset M$ be a small enough disk about Z. If $\sharp\mathcal{Y} \ge 3$ then one of the components of $U - \mathcal{C}$, say corresponding to Y_1, has a vertex angle $< \pi$ at Z. Hence the distance to the cut set as a function on ∂M near Y_1 cannot have a local minimum at Y_1. Thus $\mathcal{Y} = \{Y_1, Y_2\}$. Also the angle between the vectors $Y_1 - Z$ and $Y_2 - Z$ is π, for otherwise the distance to ∂M does not have a local minimum at Z on $\mathcal{C}$. Therefore the midpoint of the segment $[Y_1, Y_2]$ is Z and the tangent line to $\mathcal{C}$ at Z is the perpendicular bisector of $[Y_1, Y_2]$.

We claim that there must be at least two focal disks in M whose centers $Z_1, Z_2 \in \mathcal{C}$ are at least $\tilde{r} = 2(3 - 2\omega - \omega^2)^{1/2}$ apart, measured along $\mathcal{C}$. The set $\mathcal{C} - \{Z\}$ has at least two connected components and by the structure theorem each of these components has at least one focal point. Let Z_1, Z_2 be these focal points. We now show that $\operatorname{dist}_{\mathcal{C}}(Z, Z_1) \ge \tilde{r}/2 = (3 - 2\omega - \omega^2)^{1/2}$. Consider the exterior unit disks D_1, D_2 tangent to ∂M at Y_1, Y_2, respectively. From the last paragraph we see that these disks have their centers on the line through Y_1, Y_2 and that the distance of the centers to Z is $(1+\omega)$. If $\operatorname{dist}(Z, Z_1) < (3 - 2\omega - \omega^2)^{1/2}$ then the disk $B_1(Z_1)$ intersects one of the exterior disks, say D_1. However, Proposition 1.1 implies that the boundary curve near Y_1 must enter $B_1(Z_1)$ which is impossible. Thus $\operatorname{dist}_{\mathcal{C}}(Z, Z_1) \ge |Z - Z_1| \ge (3 - 2\omega - \omega^2)^{1/2} = \tilde{r}/2$. As any curve in $\mathcal{C}$ from Z_1 to Z_2 must pass through Z, $\operatorname{dist}_{\mathcal{C}}(Z_1, Z_2) \ge \tilde{r}$.

We apply a scaled version of the proof Lemma 3.6 to estimate the area of M_a. First note that if $0 < a < \omega$ then the curvature of ∂M_a satisfies

$\|\tilde{\kappa}\|_\infty \le (1-a)^{-1}$ and so if the two $1-a$ disks about $Z_1, Z_2 \in \tilde{\mathcal{C}} = \mathcal{C}$ are at least $2\sqrt{3}(1-a)$ apart along $\tilde{\mathcal{C}}$, then M_a has remote ends and thus area $|M_a| \ge (1-a)^2[\pi + 2\sqrt{3}]$. A calculation shows that for all $a < \omega < 1$ that $\sqrt{3}(1-a) \le \tilde{r}/2 = \left(3 - 2\omega - \omega^2\right)^{1/2}$ so the lower bound on $|M_a|$ will always hold. The total area can now be estimated,

$$\begin{aligned} A &= |M - M_a| + |M_a| \\ &= \int_{\partial M} \int_0^a (1 - \kappa(s)t)\; dt\, ds + |M_a| \\ &\ge La - \pi a^2 + (1-a)^2 A_1 \end{aligned}$$

where $A_1 = \pi + 2\sqrt{3}$. Hence

$$4\sqrt{3}a \le \Big((L - 2A_1)^2 + 8\sqrt{3}(A - A_1)\Big)^{1/2} - (L - 2A_1).$$

Using $A > \pi + 2\sqrt{3}$ in this yields (3.15). □

The method of interior parallels is a familiar theme. Since the reverse inequality to (3.16) holds for general domains [**4, 24**] it is used to deduce isoperimetric inequalities. The well known equality (3.16) and consequently (3.14) holds since $a \le \omega$ and ∂M_a has a nice parameterization. Hadwiger [**14**] gave conditions for equality under related bounds on boundary curvature.

4. A sharp reverse isoperimetric inequality and the extremal figure. In this section we consider the best reverse isoperimetric inequality and the figure which extremizes this inequality. Let $\mathcal{M}(A)$ denote the space of all embedded closed disks $M \subset \mathbf{R}^2$ whose boundary curves are in class $\mathcal{K}$ and whose area is A. Let $\mathcal{N}(L)$ denote the space of all embedded closed disks $M \subset \mathbf{R}^2$ whose boundary curves are in class $\mathcal{K}$ and whose length $|\partial M| = L$. Then we say $E \in \mathcal{M}(A)$ is extremal if $|\partial E| = \sup\{|\partial M| : M \in \mathcal{M}(A)\}$. Similarly, $E \in \mathcal{N}(L)$ is extremal if $|M| = \inf\{|M| : M \in \mathcal{N}(L)\}$. Although these problems are dual, they require slightly different treatment (*e.g.* see Lemma 4.14).

Theorem 4.1. *The set of pairs (A, L) where A is the area and L is the boundary length of $M \subset \mathbf{R}^2$, an embedded closed disk whose*

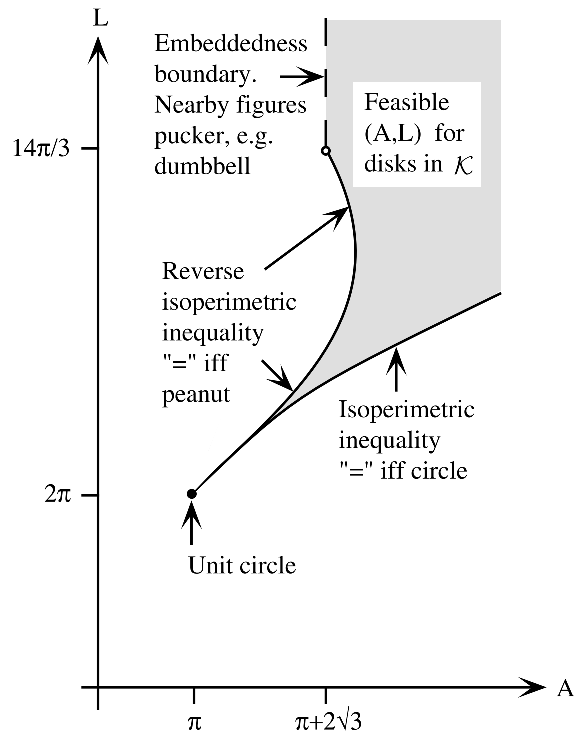

FIGURE 4. Attainable (A, L) for $M \in \mathcal{K}$.

boundary is of class $\mathcal{K}$*, consists exactly of the points in the first quadrant* (*shown in Figure* 4) *satisfying three inequalities*:

(1) *The isoperimetric inequality*

$$4\pi A \leq L^2.$$

Equality holds if and only if M *is a circular disk.*

(2) *The reverse isoperimetric inequality. If* $2\pi \leq L < 14\pi/3$*, then there holds*

$$\sin\left(\frac{L-2\pi}{4}\right) \leq \frac{A-\pi}{4}. \tag{4.1}$$

Equality holds in (4.1) *if and only if M is congruent to the peanut P_δ* (*Figure* 1) *where*

$$\delta = 4\sin\left(\frac{L-2\pi}{8}\right).$$

(3) *Embeddedness border. If $L \geq 14\pi/3$, then*

$$A > \pi + 2\sqrt{3}.$$

Equality cannot hold, although there are arbitrarily nearby regions for which the embeddedness degenerates by "puckering." For example, one can consider a sequence of domains decreasing to the dumbbell region consisting of two unit disks, two triangles with circular sides and a segment of length $L/2 - 7\pi/3$.

First we show that extremal figures exist.

Theorem 4.2. *For any $\pi \leq A \leq \pi + 2\sqrt{3}$ there exists an $E \in \mathcal{M}(A)$ with maximal length, namely, $|\partial E| = \sup\{|\partial M| : M \in \mathcal{M}(A)\}$.*

Proof. For area fixed, consider a maximizing sequence of embedded disks $M_i \subset \mathcal{M}(A)$ such that $|\partial M_i| \nearrow \ell(A) = \sup\{|\partial M| : M \subset \mathcal{M}(A)\}$. Corollary 3.8 shows that if $M_i \in \mathcal{M}(A)$ then it contains a disk of radius 1 and that M_i is star-shaped with respect to the center of this disk. Translating if necessary, we assume that these centers lie on the origin. Corollary 3.8 also shows that the lengths are uniformly bounded $\ell(A) \leq 2A$. By scaling the parameters, the boundary curves $\sigma_i : \mathbf{S}^1 \to \mathbf{R}^2$ are uniformly bounded in $C^{1,1}(\mathbf{S}^1)$. It follows by Arzela's Theorem that a subsequence $\sigma_{i'} \to \sigma$ uniformly in C^1, hence with the same bound on $\mathbf{Lip}(\sigma')$. Thus $\sigma \in \mathcal{K}$. The limit M must also contain $B_1(0)$. Since $(\rho_i)_s$ are bounded as in Corollary 3.8, $\partial M_{i'}$ are uniformly transverse to the rays from the origin, except possibly if $|M| = \pi + 2\sqrt{3}$ and then only for one radius, the same must be true of the limit. Hence we can conclude that M is topologically an embedded closed disk. □

Theorem 4.3. *For any $2\pi \leq L < 14\pi/3$ there exists an $E \in \mathcal{N}(L)$ with least boundary length, namely, $|E| = \inf\{|M| : M \in \mathcal{N}(L)\}$.*

Proof. Again, since length is bounded, after translating if necessary, the same compactness property shows there exists limits of subsequences. It remains to argue that the limiting figures are embedded. If not, there are elements of the approximating minimizing sequence, call them M which are nearly degenerate having a thin waist. Let $\gamma : [0, L] \to \partial M$ denote an arclength parameterization of the boundary curve. There are points on opposite sides of the waist, $s_1 + 2\pi < s_2 < s_1 + L - 2\pi$, so that $d = \text{dist}(\gamma(s_1), \gamma(s_2))$ is minimum among such points and arbitrarily small. Hence $\gamma(s_1)$ and $\gamma(s_2)$ have a common normal line. But the total length must exceed the double of the minimal length for a segment of class $\mathcal{K}$ connecting the line elements $\gamma(s_1)$ and $\gamma(s_2)$. By Dubins' Theorem [**10**] this segment consists of the three unit arcs forming a lightbulb shape with length $\pi + 4\cos^{-1}(1/2 + d/4)$ which exceeds $L/2$ for d sufficiently small. □

First we shall show that extremal figures consist of finitely many unit circular arcs. From the control theory standpoint, viewing the curvature κ as a control, and maximizing the length among curves with curvature κ which are closed, embedded, of class $\mathcal{K}$, enclosing an area A, this shows the *bang–bang* type result that $\kappa = \pm 1$. We shall examine variations of short arcs on the boundary of minimizers. Since the curvature function is not necessarily continuous, to insure that the variations we construct are admissible, we substitute sections of the curve rather than add bump functions. First we give our parameterization of pairs of line elements on the endpoints of a short arc using the shortest path.

Lemma 4.4. *Suppose σ is an arc of class $\mathcal{K}$ and length $\ell = |\sigma| \le \pi/2$. Then the shortest arc λ of class $\mathcal{K}$ connecting the starting and ending line elements of σ consists of a unit circular arc of angle (integral curvature) α followed by a line segment of length $\beta \ge 0$ followed by a unit circular arc of angle γ with total length $|\lambda| = |\alpha| + \beta + |\gamma| \le \ell$. Any of the α, β or γ may be zero. A negative angle corresponds to a concave arc.*

Proof. Dubins [**10**] showed that the minimizer consists of either three unit circular arcs or arc-segment-arc or a subarc of these. By Dubins' Proposition 6, the osculating disks on the opposite side of γ at $\gamma(0)$

and $\gamma(\ell)$ are either tangent, in which case the shortest connecting arc of class $\mathcal{K}$ consists of a subarc of the first circle followed by a subarc of the second; or they are disjoint. By Proposition 1.1 neither osculating disk at one endpoint contains the other endpoint in the interior. Hence, among all straight line segments and circular arcs tangent to an osculating disk at one endpoint and to another at the other endpoint, the straight line segment between closest disks gives the shortest curve between elements. □

Our local considerations will examine which boundary subarcs of $M \in \mathcal{M}(A)$ can be lengthened. To do this, we describe various pairs of endpoint line elements. Let $\sigma \subset \partial M$ be a subarc of length $\ell = |\sigma| \le \pi/3$ oriented in the usual direction of ∂M. Let (α, β, γ) parameterize the shortest arc of class $\mathcal{K}$ connecting endpoint elements of σ. If $\alpha \ge 0$ and $\gamma \ge 0$ we say that (α, β, γ) is a *convex* pair of line elements. If $\alpha \le 0$ and $\gamma \le 0$ we say *concave* and if neither we say *mixed*. We will adorn quantities with "+" whenever referring to the interior side of ∂M.

Lemma 4.5. *Given $0 < \ell \le \pi/3$, let $\sigma : [0, \ell] \to \mathbf{R}^2$ be a unit speed arc of class $\mathcal{K}$, and let $\{\mathbf{t}(t), \mathbf{n}(t)\}$ be an orthonormal frame at $\sigma(t)$ so that $\sigma' = \mathbf{t}$ and $\mathbf{n}$ is the inner normal. Let $D^+(t)$ and $D^-(t)$ be the closed osculating unit disks to σ at $\sigma(t)$ on the $\pm\mathbf{n}$ sides of σ. Then*

(1) *σ is an embedded arc*;

(2) *Disks on the same side always intersect*: $D^+(0) \cap D^+(\ell) \ne \varnothing$ *and* $D^-(0) \cap D^-(\ell) \ne \varnothing$;

(3) *Disks on the opposite sides don't intersect*: *For $0 \le \ell \le \pi/2$, either $D^-(0) \cap D^+(\ell) = \varnothing$ or $D^-(0)$ is tangent to $D^+(\ell)$ (or $D^+(0)$ to $D^-(\ell)$) and λ is the class $\mathcal{K}$ curve consisting of an arc of the first circle followed by an arc of the second.*

Proof. The embeddedness follows from the Schur-Schmidt Proposition 1.1. That intersection occurs is a geometric exercise. For example, by moving $D^+(0)$ and $D^+(\ell)$ as far apart as possible, we may assume that σ is the shortest arc λ in $\mathcal{K}$ between the line elements $(\sigma(0), \sigma'(0))$ and $(\sigma(\ell), \sigma'(\ell))$. By Lemma 4.4 we may suppose λ consists of at most three pieces consisting of (a subset) of a unit circular subarc of angle α, a line segment of length $\beta \ge 0$, and another unit circular arc of angle

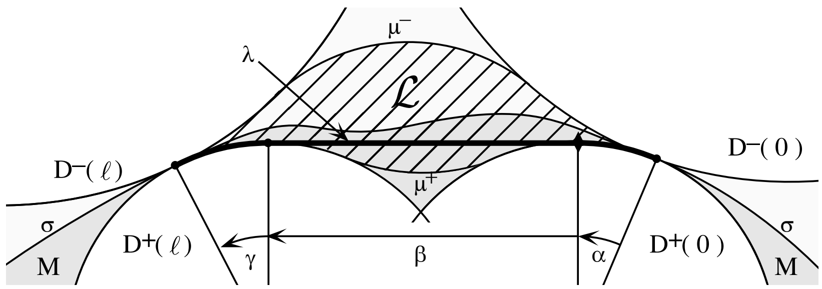

FIGURE 5. "Lips" domain.

γ. If, *e.g.*, $\alpha \geq 0$ and turning so the segment is in the $\partial/\partial x$ direction, we see that the centers of $D^+(0)$ and $D^+(\ell)$ are at most

$$d^2 = (\Delta x)^2 + (\Delta y)^2 = (2\sin\alpha + \beta + 2\sin\gamma)^2 + (2\cos\alpha - 2\cos\gamma)^2$$

apart, where $\alpha+\beta+\gamma \leq \pi/3$ and $\gamma \geq 0$. Because, *e.g.*, $\cos(\alpha) \geq 1/2$, by examining the gradient, we see that the maximum value occurs when $\beta = 0$ making $d = 2$. In other words, $D^+(0) \cap D^+(\ell) \neq \varnothing$. Similarly for D^-. (3) is Dubins' Proposition 6, [**10**]. ◻

Thus, by Proposition 1.1, the segment $\sigma \in \mathcal{K}$ of length $\ell \leq \pi/3$ must lie in the region N determined by the endpoints of σ and their directions, where N is the bounded component of the complement of $D^+(0) \cup D^-(0) \cup D^+(\ell) \cup D^-(\ell)$.

Lemma 4.6. *In the notation of Lemma* 4.5*, there are unit circles which are tangent to both* $D^+(0)$ *and* $D^+(\ell)$*. Let* μ^+ *be the short subarc between the points of tangency which touches* N*.* μ^- *is defined similarly. Then* $\mu^+, \mu^- \subset \bar{N}$*.* (*See Figure* 5)

Proof. Notice that by Lemma 4.5 (3), either σ consists of two circular segments, in which case μ^+ coincides with σ, or $D^-(0) \cap D^+(\ell) = \varnothing$ and $D^-(\ell) \cap D^+(0) = \varnothing$. Then a unit disk rolled from $D^-(0)$ along $D^+(0)$ bumps into $D^+(\ell)$ on the "inside" of N. ◻

Denote the "lips shaped" subregion of $\overline{N}$ between and including μ^+ and μ^- by $\mathcal{L}$ as shown in Figure 5. Denote the boundary arc from $\sigma(0)$ to $\sigma(\ell)$ consisting of concatenations of the subarcs of $\partial D^+(0)$, μ^+ and $\partial D^+(\ell)$ by λ^+. Define λ^- analogously. Then $\lambda^\pm \in \mathcal{K}$ have the same starting and ending line elements as σ. Let λ be the shortest curve of class $\mathcal{K}$ between the starting and ending line elements of σ with parameters (α, β, γ). If $\ell \le \pi/3$ then by Lemma 4.4, λ consists of a unit circle arc of angle α followed by a straight segment of length $\beta \ge 0$ followed by an arc of angle γ so that (up to reflection) $|\gamma| \le |\alpha|$ and $|\alpha| + \beta + |\gamma| \le \pi/3$. We call (α, β, γ) the *parameters* of λ.

Lemma 4.7. *Let $\sigma : [0, \ell] \to \mathbf{R}^2$ be an arclength parameterized curve of class $\mathcal{K}$. We suppose that the osculating disks $D^+(0) \cap D^+(\ell) \neq \varnothing$ and $D^-(0) \cap D^-(\ell) \neq \varnothing$, for example if $0 < \ell \le \pi/3$. Let $\mathcal{L}$ be the lips-shaped region determined by the starting and ending line elements of σ, and $\lambda^\pm$ the corresponding boundary arcs of $\mathcal{L}$. First $\sigma \subset \mathcal{L}$. Moreover, then either*

(1) *σ consists of one or two circular arcs in which case $\lambda^+ = \lambda = \lambda^-$. This is the case if and only if the curvature is almost everywhere piecewise equal to ± 1 with at most one essential sign change.*

(2) *$\sigma = \lambda^+$ (or $\sigma = \lambda^-$, respectively) in which case the curvature is almost everywhere $\kappa = 1$ for $0 \le t < \ell^- + 1$, $\kappa = -1$ for $\ell_1^+ \le t < \ell_2^+$ and $\kappa = 1$ for $\ell_2^+ \le t \le \ell$ (or its negative, respectively) where ℓ_1^+ is the length of the first arc and $\ell - \ell_2^+$ the length of the last. Note that $\ell_i^\pm$ are determined by the starting and ending line elements of λ.*

(3) *$\lambda^+ \not\equiv \sigma \not\equiv \lambda^-$ which is the case if κ is any other function. Let λ be the shortest curve of class $\mathcal{K}$ between starting and ending line elements of σ with parameters (α, β, γ). If σ has convex ending line elements $\alpha, \gamma \ge 0$ then $|\sigma| < |\lambda^-|$.*

Proof. Case (1) occurs if the osculating circles $D^+(0)$ and $D^-(\ell)$ (or $D^-(0)$ and $D^+(\ell)$) are tangent and Proposition 1.1 (3) applies and so $\sigma \in \bar{\mathcal{L}}$. Suppose this is not the case for the rest of the proof. By constructing a field of semicircles by translating $\mu^\pm$ inside $N - \mathcal{L}$ one sees by the maximum principle that $\sigma \subset \overline{\mathcal{L}}$.

Case (3) will be demonstrated in two steps. First we claim that λ^- and σ have a common perpendicular. To see this, orient σ so that

$\lambda(|\alpha|) = 0$ and $\lambda'(|\alpha|) = \partial/\partial x$. By the intermediate value theorem there is a point on σ so that $\sigma'(t_0) = \partial/\partial x$. Put $\sigma'(s) = \exp(i\vartheta(s))$; similarly define $\vartheta^{\pm}$ for $\lambda^{\pm}$. Because $\|\vartheta(t) - \vartheta(t_0)\| \le |t - t_0| \le \pi/3$ we have that σ is a graph over the x-axis. Similarly, so are $\lambda^{\pm}$. Thus we may think of quantities depending on x. Let $a < b$ be the x-coordinates of the endpoints of σ and let $a < x_1 < x_2 < b$ be the x-coordinates of the first and second jumps of the curvature κ^- of λ^- (so they are the endpoints of μ^-). We have

$$\begin{aligned} &\frac{d\vartheta}{dx} = \frac{d\vartheta}{ds}\frac{ds}{dx} = \kappa \sec\vartheta \\ &\frac{d\vartheta^-}{dx} = \frac{d\vartheta^-}{ds^-}\frac{ds^-}{dx} = \begin{cases} +\sec\vartheta, & \text{if } x_1 < x < x_2, \\ -\sec\vartheta, & \text{if } a < x < x_1 \text{ or } x_2 < x < b. \end{cases} \end{aligned} \tag{4.2}$$

Since $\vartheta(a) = \vartheta^{\pm}(a)$ and $\vartheta(b) = \vartheta^{\pm}(b)$, the comparison theorem for (4.2) implies $\vartheta^-(x_1) \le \vartheta(x_1)$ and $\vartheta^-(x_2) \ge \vartheta(x_2)$. Because μ^- is a circular arc, for each $\vartheta^-(x_1) \le \zeta \le \vartheta^-(x_2)$ there is a unique x-coordinate $\xi(\zeta)$ where $\vartheta^-(\xi(\zeta)) = \exp(i\zeta)$. Consider the continuous function $f(x) = \langle \sigma'(x), \mu^-(\xi(\vartheta(x))) - \sigma(x)\rangle$ which measures the distance between the normal lines thru $\sigma(x)$ and μ^- at points with parallel tangents. Observe that $f(x_1) \ge 0$ and $f(x_2) \le 0$. By the intermediate value theorem, there is an $x_3 \in [x_1, x_2]$ where the normal lines of σ and μ^+ coincide. Hence by a rotation, we may assume that this line is the y-axis and that $\sigma'(0) = (\mu^-)'(0) = \partial/\partial x$.

Next we show that $|\lambda^-| > |\sigma|$. It suffices to compare the lengths of the parts when $x \ge 0$ and $x \le 0$ separately. Let $b \ge 0$ again be the ending x-coordinate of σ and $0 \le x_2 \le b$ the coordinate of the jump in κ^-. So $x_2 < 1$. Comparison of solutions of (4.2) using $\vartheta(0) = \vartheta^-(0)$ on the interval $[0, x]$ for $x \le x_2$ and using $\vartheta^-(b) = \vartheta(b)$ on the interval $[x, b]$ shows that $|\vartheta(x)| \le \vartheta^-(x)$ for all $0 \le x \le b$. Hence

$$ds^-/dx = \sec\vartheta^-(x) \ge \sec\vartheta(x) = ds/dx \tag{4.3}$$

on $[0, b]$. Since $\sigma \ne \lambda^-$, the inequality (4.3) is strict at some points so $|\mu^-| > |\sigma|$ follows. □

The key step in the arguments of this section is to replace convex pieces of the boundary by extremal convex curves with the same ending elements. The following lemma identifies the local extremal curves.

Lemma 4.8. *Let* $\sigma \subset \mathbf{R}^2$ *be a convex arc of class* $\mathcal{K}$ *and length* $\ell = |\sigma| \le \pi/3$. *Let* λ *be the shortest arc of class* $\mathcal{K}$ *with the same ending elements as* σ *and let* (α, β, γ) *be the parameters of* λ. *Suppose that* $\alpha \ge \gamma$. *Let* $\psi[t]$ *be the one parameter family of curves of class* $\mathcal{K}$ *having the same end elements as* σ, *consisting in order, of a straight line segment of length* t *tangent to* $\sigma(0)$, *a unit circular arc, a straight segment, and a subarc of* λ *ending at* $\sigma(\ell)$. *Let* $\mathrm{a}(\sigma)$ *denote the area enclosed by* $\lambda \cup \sigma$. *There are unique* $t_1 \le t_2$ *so that length* $|\psi[t_1]| = |\sigma|$ *and area* $\mathrm{a}(\psi[t_2]) = \mathrm{a}(\sigma)$. *Then* $\mathrm{a}(\psi[t_1]) \le \mathrm{a}(\sigma)$ *and* $|\psi[t_2]| \ge |\sigma|$. *The inequalities are strict unless* $\psi[t_1] = \sigma = \psi[t_2]$.

Proof. By definition $\psi[0] = \lambda$ and $t \le T$ where the middle arc of $\psi[T]$ is tangent to the supporting line of σ at $\sigma(\ell)$. Thus any σ is contained within $\psi[T]$. Since area and length are continuous strictly increasing functions of t, the existence and uniqueness of the t_i is assured.

We may assume λ is oriented so that $\lambda(\alpha) = (\beta, 1)$ and $\lambda'(\beta) = -\partial/\partial x$. We consider, as in Theorem 3.12, the interior parallel curve $\xi_\varepsilon(s) = \sigma(s) + \eta_{\sigma(s)}(\varepsilon)$ to σ at a distance ε, $0 \le \varepsilon \le 1$. Since $\sigma \in \mathcal{K}$, the curve ξ_ε is convex with length $\ell(\varepsilon) = \ell - (\alpha + \gamma)\varepsilon$ and area $\tilde{a}(\varepsilon) = a + (\alpha+\gamma)/2 + \beta - \ell\varepsilon - (\alpha+\gamma)\varepsilon^2/2$ where $\tilde{a}$ means the area inside the region bounded by ξ_ε swept out by the unit normals along σ plus the area enclosed by $\xi = \xi_1$ and the segment between its endpoints. Note that the numbers $\ell(1) = \ell - (\alpha+\gamma)$ and $\tilde{a}(1) = a + \beta - \ell$ determine ℓ and $a(\sigma)$. Thus, ξ is a convex curve with $\xi(0) = (\beta, 0)$, $\xi(\ell(1)) = (0, 0)$ and in the triangle $\mathcal{T}$ between the lines $y = 0$, $y = x \tan\gamma$ and $y = (\beta - x)\tan\alpha$. The lemma is equivalent to showing that among all convex curves in $\mathcal{T}$ starting and ending at the $y = 0$ corners, the one with greatest length for fixed area is the piecewise linear curve χ with single kink $(\hat{x}, \hat{y})$ on the line $y = (\beta - x)\tan\alpha$. To see this, note that χ is the parallel curve at distance 1 from $\psi[t_2]$. Thus if $|\xi_1| \le |\chi|$ then using the relationship between a and the length of a curve to its parallel curve, it follows that $|\psi[t_1]| = |\sigma| \le |\psi[t_2]|$.

This can be proved using a Steiner desymmetrization procedure as for the proof of Sylvester's inequality [**3**]. By approximation, it suffices to show that any piecewise polygonal convex curve in $\mathcal{T}$ with area $\tilde{a}(1)$ has length less than for χ with the same area. We may assume that the x-coordinates for vertices of ξ are $0 = x_0 < x_1 < x_2 < \cdots < x_n \le x'_n < x'_{n-1} < \cdots < x'_1 < x'_0 = \beta$ and corresponding y-coordinates $0 = y_0 =$

$y_0' < y_1 = y_1' < \cdots < y_n = y_n$. Let $P_k = (x_k, y_k)$ and $P_k' = (x_k', y_k')$. Since areas for ξ and χ are equal we assume $y_n < \hat{y}$, for if $y_n \geq \hat{y}$ then ξ must be a triangle of the same height as χ and a calculation shows that the length is maximized when $\chi = \xi$. Let (z_k, y_k) be the point on the right edge of $\mathcal{T}$, given by $y_k = (\beta - z_k)\tan\alpha$. Consider the new convex polygonal curve $\tilde{\xi}$ whose vertices are $\tilde{P}_k = (x_k - x_k' + z_k, y_k)$ and $\tilde{P}_k' = (z_k, y_k)$ obtained by translating the segments at heights y_k to the right as much as possible in $\mathcal{T}$. The areas of ξ and $\tilde{\xi}$ are equal since all the trapezoids $P_k P_{k+1} P_{k+1}' P_k'$ and $\tilde{P}_k \tilde{P}_{k+1} \tilde{P}_{k+1}' \tilde{P}_k'$ have the same area. On the other hand, as in Steiner symmetrization, the lengths $|P_k P_{k+1}| + |P_{k+1}' P_k'| < |\tilde{P}_k \tilde{P}_{k+1}| + |\tilde{P}_{k+1}' \tilde{P}_k'|$ as all the slopes of ξ are less than $\tan\alpha$. Thus $|\tilde{\xi}| > |\xi|$. One can repeat the process with the curve $\tilde{P}_0 \tilde{P}_1 \ldots \tilde{P}_n$ in the new triangle, the subregion of $\mathcal{T}$ above the line $\tilde{P}_0 \tilde{P}_n$. After finitely many iterations, the process stops at χ. □

Lemma 4.9. *For $A \leq \pi + 2\sqrt{3}$, let $M \subset \mathcal{M}(A)$ be an embedded closed disk. Let $\sigma \subset \partial M$ so that its length $|\sigma| \leq \pi/3$. Let λ be the shortest arc of class $\mathcal{K}$ connecting the endpoint line elements of σ and (α, β, γ) its parameters and $\mathcal{L}$ the lips shaped region it determines. Assume that $(\partial M - \sigma) \cap \mathcal{L} = \varnothing$. If σ does not consist of finitely many arcs of unit circles then there is a deformation $M_\varepsilon \in \mathcal{M}(A)$, so that $|\partial M_\varepsilon| > |\partial M|$.*

Proof. We may assume $\beta > 0$ or else we are done by Lemma 4.7. First we remark that it is sufficient to find a deformation which increases the length and merely doesn't increase the area. An outward dilation, which is an admissible deformation, will restore the area and increase the length. Now we consider various cases of σ. Suppose it is possible to find a subarc $\sigma_1 \subset \sigma$ so that the line elements determined by σ_1 are concave, and σ_1 does not coincide with the interior boundary λ_1^+ of the lips shaped region determined by the endpoint elements of σ_1. By Lemma 4.7, replacing σ_1 by λ_1^+ increases the length and decreases the area since $\sigma_1 \neq \lambda_1^+$.

Similarly if a subarc λ_1 is nontrivially mixed, say $\alpha > 0$ and $\gamma \leq 0$. Assume λ is oriented so $\lambda'(\beta) = \partial/\partial x$. The idea is that by continuity, σ must go from nonpositive to nonnegative. Hence there is a point $x \in \sigma \cap \beta$ where the direction of σ is increasing. If $x \neq \beta \cap \gamma$ or $\gamma \neq 0$

then the interval from x to $\sigma(\ell)$ has concave end elements. If $x = \sigma(\ell)$ then there is an x interior to σ with $\sigma(x)$ negative and with tangent line above $D^-(\ell)$. Then the subarc from x to ℓ is concave.

Thus, if an extremal σ has any subarc σ_1 with concave endpoints, σ_1 must coincide with λ_1^+ the inner boundary of the corresponding lips region. If two such subarcs overlap then the boundary coincides with the inner boundary of the lips region for the union. By continuity of the tangent, every arc with concave endpoints is contained in a larger one unless it is the inner boundary of a lips region for an interval. Let $\{\sigma_i\}_{i=1,2,\dots}$ list all such maximal concave intervals whose parameters are $(0, \beta_i, 0)$. There may be many disjoint σ_i if all arcs connecting different $\sigma_i' s$ have convex endpoints. If there are finitely many σ_i then the complementary intervals are convex.

In a complementary arc σ_0 all subarcs have convex endpoints, hence σ_0 is convex and Lemma 4.8 applies. Replacing σ_0 by $\psi[t_2]$ gives at least as much length and the same area. By replacing the straight line segments of ψ by the inner boundary of the corresponding lips regions strictly increases length and decreases area.

It remains to handle the case that σ with parameters (α, β, γ) has convex endpoints but there are infinitely many disjoint $\sigma_i \subset \sigma$. Let $\hat{\sigma}$ denote the outer boundary curve of the convex hull of σ. By Lemma 4.8, there is a convex curve $\hat{\psi}$ with the same end elements and bounding the same area as σ but which is longer. Let ψ denote the curve obtained by replacing the two boundary segments of lengths δ_1,δ_2 of $\hat{\psi}$ by corresponding λ^+'s. Then we may compare the areas and lengths of σ and ψ. Letting $a(\sigma)$ denote the area enclosed by $\sigma \cup \lambda$,

$$a(\psi) + f(\delta_1) + f(\delta_2) = a(\hat{\psi}) = a(\hat{\sigma}) = a(\sigma) + \sum_{i=1}^{\infty} f(\beta_i),$$

$$|\psi| - g(\delta_1) - g(\delta_2) = |\hat{\psi}| \geq |\hat{\sigma}| = |\sigma| - \sum_{i=1}^{\infty} g(\beta_i),$$

where $f(\delta) = \delta - \delta(1 - \delta^2/16)^{1/2}$ is the area of the region between the segment of length δ and its inner lips boundary λ^+ and $g(\delta) = 4\sin^{-1}(\delta/4) - \delta$ is the difference in length between them.

The proof is completed if we can show for any n and $h \in \{f, g\}$,

$$h(\delta_1) + h(\delta_2) \geq \sum_{i=1}^{n} h(\beta_i).$$

Note that both f and g vanish at zero, are strictly increasing and are convex. We may order the sequence so that $\beta_1 \geq \beta_2 \geq \dots$ and $\delta_1 \geq \delta_2$. First $\beta_1 \leq \delta_1$. This is because any convex curve $\hat{\sigma} \in \mathcal{K}$ with a linear segment longer than δ_1 that remains within the supporting lines from the endpoints of σ cannot have the area of $\hat{\psi}$. Second, since length of $|\hat{\psi}| = |\hat{\sigma}|$ but total curvature for both is $\alpha + \gamma = \int_{\hat{\sigma}} \kappa\, ds$ then $0 \leq \kappa \leq 1$ implies

$$\sum_i \beta_i = |\hat{\sigma}| - |\mathrm{spt}\kappa| \leq |\hat{\psi}| - \alpha - \gamma = \delta_1 + \delta_2.$$

Third, $h(x + y) \geq h(x) + h(y)$. By lining x and y along the x-axis, this follows from Lemma 4.7 which says $\lambda^+(x + y)$ is longer than and contains $\lambda^+(x) \cup \lambda^+(y)$. Thus it suffices to let $T = \sum_{i=2}^n \beta_i$ and prove the inequality $h(\delta_1) + h(\delta_2) \geq h(x) + h(y)$ where $x = \max\{T, \beta_1\}$ and $y = \min\{T, \beta_1\}$. If $\beta_1 \leq \delta_2$ we are done since convexity, monotonicity and $h(0) = 0$ implies

$$\sum_{i=1}^{n} h(\beta_i) \leq \frac{h(\beta_1)}{\beta_1} \left(\sum_{i=1}^{n} \beta_i \right) \leq \frac{h(\delta_2)}{\delta_2} (\delta_1 + \delta_2) \leq h(\delta_1) + h(\delta_2).$$

If $\beta_1 \geq \delta_2$ then $\delta_1 + \delta_2 \geq \beta_1 + T$ implies $\delta_1 \geq T$. If also $y \leq \delta_2$ we are done since h is monotone. However, if instead $y \geq \delta_2$, then by convexity,

$$h(y) - h(\delta_2) \leq h(y + x - \delta_2) - h(x) \leq h(\delta_1) - h(x). \qquad \square$$

Lemma 4.10. *For $A \leq \pi + 2\sqrt{3}$, let $E \subset \mathcal{M}(A)$ be an embedded closed disk. Suppose that E has maximal length, that is, $|\partial E| = \sup\{|\partial M| : M \in \mathcal{M}(A)\}$. Then ∂E consists of finitely many unit circular arcs.*

Proof. Take finitely many points in order $x_i \in \partial E$, $i = 1, \dots, N$ so that if σ_i is the arc of ∂E from x_i to x_{i+1}, where $x_{N+1} = x_1$, then

$|\sigma_i| \le \pi/3$ for all i. Because E is an embedded closed disk of class $\mathcal{K}$, it is possible to choose x_i so that the lips shaped domains $\mathcal{L}_i$ determined by the line elements of ∂E at x_i and x_{i+1} are pairwise disjoint. By Lemma 4.9, each σ_i consists of finitely many circular arcs, or else there is a deformation of E to an embedded closed disk of the same area but longer length. □

We now consider the case of fixed length. The first step is to show that unless the boundary curve consists of circular arcs of any radius, the curve cannot enclose the least area.

Lemma 4.11. *Let $M \in \mathcal{N}(L)$ be an embedded closed disk. Let $\sigma \subset \partial M$ so that its length $|\sigma| \le \pi/3$. Let λ be the shortest arc of class $\mathcal{K}$ connecting the endpoint line elements of σ and let $\mathcal{L}$ be the lips region determined by σ. Assume that $(\partial M - \sigma) \cap \mathcal{L} = \varnothing$. If $\sigma \cap \mathcal{L}$ does not consist of finitely many circular arcs and $\sigma \neq \lambda$, then there is a curve σ_1 of class $\mathcal{K}$ in $\mathcal{L}$ with the same length and starting and ending elements as σ but (together with $\partial M - \sigma_1$) bounding less area.*

Proof. Let (α, β, γ) be parameters for λ and suppose $\beta > 0$, otherwise we are done by Lemma 4.7. Let $\sigma_1 \subset \sigma$ be an arbitrary subarc. Denote by λ_1 the shortest arc in $\mathcal{K}$ with the same ending elements as σ_1 and let $(\alpha_1, \beta_1, \gamma_1)$ be its parameters. Let $\mathcal{L}_1$ be the corresponding lips domain and λ_1^+ the corresponding inner boundary of $\mathcal{L}_1$ with the same ending elements as σ_1.

Suppose a subinterval $\sigma_1 \subset \sigma$ has concave end elements so $|\sigma_1| \le |\lambda_1^+|$. Consider the problem of minimizing the area between the fixed curve λ_1^+ and an arbitrary rectifiable curve of length $|\sigma_1|$ with the same endpoints as λ_1^+. This is a special case of the thread problem [**8**]. The minimal area is achieved for a curve that may partially coincide with λ_1^+. The curve will be disjoint from λ_1^+ on at most countably many intervals, all of which have the same constant curvature κ. The endpoints of the disjoint intervals are tangent to λ_1^+. For the special case of λ_1^+, there are very few doubly tangent arcs of constant curvature. Hence the solution of the thread problem must have a single arc with curvature $-1 \le \kappa < 0$ spanning λ_1^+. Let υ denote this curve. It is the least area curve in $\mathcal{L}_1$ for this length.

Any subarc in an extremal σ with concave end elements must be a least area curve υ. As in the proof of Lemma 4.9, σ contains a maximal collection of disjoint arcs σ_i with parameters $(0, \beta_i, 0)$ which are solutions of the thread problem $\sigma_i = \upsilon_i$ or interior lips $\sigma_i = \lambda_i^+$ which is a special case. We argue that there is at most one such σ_i in σ by a method reminiscent of Steiner's four-hinge proof of the isoperimetric inequality [**22**]. If not, choose disjoint $\sigma_1 < \sigma_2 \subset \sigma$ and let P_i and Q_i be the centers of the interior osculating disks D_i^+ at the starting and ending endpoints of σ_i, respectively. The requirement that there are no arcs with concave end elements starting in and ending outside of σ_i implies that the quadrilateral $\mathcal{P} = P_1Q_1P_2Q_2$ is convex. View $\mathcal{P}$ as a linkage with fixed side lengths. Glue the disks $D^+(P_i)$, $D^+(Q_i)$ and the arc υ_i rigidly to the side P_iQ_i. Glue the disks $D^+(Q_1)$, $D^+(P_2)$ and the arc of σ between σ_1 and σ_2 rigidly to the side Q_1P_2. Glue the disks $D^+(P_1)$, $D^+(Q_2)$ and the rest of σ rigidly to the side Q_2P_1. Articulating the quadrilateral gives a deformation of σ which preserves length since the total curvature is preserved. However, the area of $\mathcal{P}$ and hence the area inside the curve is variable and may be made to diminish in an appropriate direction.

In the complementary intervals to σ_1, no subarc has concave endpoints, hence they are convex. Hence Lemma 4.9 implies that we can replace them by $\psi[t_1]$ to strictly diminish area and preserve length unless they agree with $\psi[t_1]$ already. □

Lemma 4.12. *For $2\pi < L < 14\pi/3$, let $E \subset \mathcal{N}(L)$ be an embedded closed disk. Suppose that E has minimal area, that is, $|E| = \inf\{|M| : M \in \mathcal{N}(L)\}$. Then ∂E consists of finitely many unit circular arcs.*

Proof. Similarly to the proof of Lemma 4.9, by covering ∂E with finitely many lips regions, we see that extremal ∂E consists of finitely many circular arcs. The convex arcs have $\kappa = 1$ but the concave ones may have $\kappa > -1$.

E is not extremal unless the concave arcs have $\kappa = -1$. To see this, let γ_0^-, γ_1^+ and γ_1^- be adjacent arcs of curvatures $-1 < \kappa \le 0$, $\kappa = 1$ and $\kappa \le 0$, respectively. If $|\gamma_1^+| < \pi$ then rolling the disk D_1^+ of γ_1^+ along γ_1^- a distance ε remains inside γ_1^+ and the straight segment connecting γ_0^- to the outside of the rolled disk decreases length which can be compensated by bulging γ_0^- inward since $-1 < \kappa \le 0$. This deformation preserves the length of the boundary and decreases the area. Let D_P^+ be an interior unit disk tangent to γ_0^- at P near γ_1^+. If $|\gamma_1^+| \ge \pi$, then the same deformation moves the centers of D_P^+ and D_1^+ closer. By decreasing κ the length remains unchanged but the area decreases. □

Suppose $M \subset \mathbf{R}^2$ is an embedded closed disk whose boundary is of class $\mathcal{K}$ and which has maximal length $|\partial M|$ among disks with area $A \le \pi + 2\sqrt{3}$ or minimal area among disks with length $L < 14\pi/3$. By Lemma 4.10 or Lemma 4.12, ∂M consists of finitely many unit circular arcs . We may write the boundary curve as a sequence of arcs

$$\partial M = \gamma_1^+ \cup \gamma_1^- \cup \gamma_2^+ \cup \gamma_2^- \cup \cdots \cup \gamma_k^+ \cup \gamma_k^- \tag{4.4}$$

where γ_i^+ are *convex arcs*, that is, subarcs of interior osculating circles of ∂M.

Lemma 4.13. *Fix $2\pi < L < 14\pi/3$. Suppose $M \subset \mathbf{R}^2$ is an embedded closed disk whose boundary is of class $\mathcal{K}$ which has minimal area among disks with length L. Then the boundary consists of at most three positive and three negative unit circle arcs such that $|\gamma_i^+| \ge \pi$ for each convex arc and $|\gamma_i^-| < 2\pi/3$ for each concave arc in the notation* (4.4).

Proof. We begin by showing that each positive arc must have angle at least π. If not, let σ be a positive arc of length less than π and let ν_1 and ν_2 be the negative arcs on either side of σ. Since $|\sigma| < \pi$ we may consider the ε-translate of σ toward the interior of M and the deformation obtained by the class $\mathcal{K}$ curve consisting of a subarcs of ν_1 and ν_2 connected by straight line segments to a subarc of the translate of σ. The resulting curve is $\mathbf{O}(\varepsilon^{3/2})$ shorter and bounds an $\mathbf{O}(\varepsilon)$ smaller area than M. By an outward dilation, this gives a local deformation fixing L and decreasing A for ε small.

The total curvature of ∂M is

$$2\pi = \sum_{i=1}^{k} p_i - \sum_{i=1}^{k} n_i \tag{4.5}$$

where $p_i = |\gamma_i^+|$ and $n_i = |\gamma_i^-|$ are the lengths of the positive and negative arcs of ∂M as in (4.4). Hence using (4.5),

$$L = \sum_{i=1}^{k} p_i + \sum_{i=1}^{k} n_i = 2\sum_{i=1}^{k} p_i - 2\pi \geq 2\pi k - 2\pi. \tag{4.6}$$

Since k is an integer and $14\pi/3 > L$ this yields $k \leq 3$. Suppose $n_i \geq 2\pi/3$ for $i = 1, \dots, \ell$ and $n_i < 2\pi/3$ for $i > \ell$. Then inserting into (4.5) and (4.6),

$$L = 2\pi + 2\sum_{i=1}^{k} n_i \geq 2\pi + \frac{4\pi\ell}{3}.$$

But $L < 14\pi/3$ so $\ell \leq 1$. If $k = 2$ then the only possible way two negative arcs can be tangent to two disks making an embedded disk is $p_1 = p_2$ and $n_1 = n_2 < 2\pi/3$. If $k = 3$ and $\ell = 1$ so $n_1 \geq 2\pi/3$, we estimate the length. Let $B_i = B_1^+(Z_i)$ be the interior osculating disk to γ_i^+. Assume B_1 and B_2 are fixed. Consider all positions of B_3 so that the arc $\zeta = \gamma_2^- \cup \gamma_3^+ \cup \gamma_3^-$ connects B_1 to B_2 but goes around γ_1^-. Letting s_i, $i = 1, 2$ be the distance between the centers of B_i and B_3 then if ζ is to connect then both $s_i \geq 1$. If B_i is to connect to B_3 by a single negative arc then $s_i \leq 4$. Denote the vector connecting the centers $\mathbf{v} = \overrightarrow{Z_2 Z_3}$. Consider the portion of $\gamma_2^+ \cup \gamma_2^- \cup \gamma_3^+$ outside $\mathbf{v}$ which starts and ends in the direction $\mathbf{v}$. Since $n_2 < 2\pi/3$, its length is $4\sin^{-1}(s_2/4)$ and integral curvature is zero. The s_1 side is similar. The $[Z_1, Z_2]$ side has n_1 negative arc, hence the same positive arc. Adding the three sides and joining arcs gives the total length of ∂M to be

$$f(s_1, s_2) = 2n_1 + 4\sin^{-1}\left(\frac{s_1}{4}\right) + 4\sin^{-1}\left(\frac{s_2}{4}\right) + 2\pi.$$

This is a convex function, minimized at $(s_1, s_2) = (1, 1)$ among $\{(s_1, s_2) : 1 \leq s_1, s_2 \leq 4\}$. Since this set includes all feasible points, we have

$$|\partial M| \geq f(1, 1) = 4\pi + n_1 > L$$

which is a contradiction. □

Lemma 4.14. [Duality]. *Consider embedded closed disks in* $\mathbf{R}^2$ *whose boundary is of class* $\mathcal{K}$. *Suppose* M *has area* A *and length* L.

If M has maximal boundary length among such disks with area A for some $A \le \pi + 2\sqrt{3}$ then it has minimal area among such disks whose length is L. In particular ∂M has the form (4.4) *with $k \le 3$, $n_i < 2\pi/3$ and $p_i \ge \pi$.*

Proof. If $M \in \mathcal{N}(L)$ did not have minimal area then a deformation decreasing area for fixed length gives a deformation with increased length for fixed area by outward dilation. Because the dilation is outward, the curvature constraint is preserved. Hence $M \in \mathcal{M}(A)$ would not have maximal boundary length. By Corollary 3.8 we have $L \le 2A < 2\pi + 4\sqrt{3} < 14\pi/3$ so Lemma 4.13 applies. □

Proof of Theorem 4.1. Denote the feasible region by

$$\mathcal{F} = \{(|M|, |\partial M|) : M \subset \mathbf{R}^2 \text{ an embedded disk with } \partial M \in \mathcal{K}\}$$

and the isoperimetric ratio by $I = L^2/A$. The isoperimetric inequality $I \ge 4\pi$ gives one boundary of $\mathcal{F}$. Equality implies M is a circle, e.g., [**23**, p. 119] or [**7**, p. 108]. Outward dilation $M \mapsto tM$ preserves $\partial(tM) \in \mathcal{K}$ so whenever $(A, L) \in \mathcal{F}$ so is (t^2A, tL) for $t \ge 1$. Thus the set $\mathcal{I}(I) = \{(A, L) \in \mathcal{F} : L^2 = IA\}$ is a half parabola. $\mathcal{I}(I) \ne \varnothing$ for all $I \ge 4\pi$ since there is a convex hull of a pair of unit disks for any I. Therefore it remains to describe what happens at $A = \Psi(I) = \inf\{x : (x, y) \in \mathcal{I}(I)\}$. By Corollary 3.8 we know that $A \le \pi + 2\sqrt{3}$ implies $L \le 2A$. If $I \ge (14\pi/3)^2/(\pi + 2\sqrt{3}) =: I_0$ then $L \ge 14\pi/3 > 2\pi + 4\sqrt{3}$ so Theorem 3.12 applies as $A \to \pi + 2\sqrt{3}$ showing that the domains must degenerate by puckering. In fact one can find dumbbells with A_i/L_i^2 fixed with (A_i, L_i) converging to all $(\pi + 2\sqrt{3}, L)$. This describes the embeddedness border (3).

To describe the remaining border of $\mathcal{F}$ corresponding to $I_0 > I > 4\pi$, observe that an embedded disk $M \subset \mathbf{R}^2$ with boundary of class $\mathcal{K}$ which has minimal area among disks with length fixed also is the least point of the dilation parabola, namely $A = \Psi(L^2/A)$. To see this, suppose $(A_0, L_0) \in \mathcal{F}$ is not least. Then for some $0 < x_0 < 1$ and hence all $0 < x < x_0$ there is a domain N whose area and length are $((1-x)^2A_0, (1-x)L_0) \in \mathcal{F}$. We show that there are domains in $\mathcal{F}$ with the same L_0 but less area than A_0. Let $M \in \mathcal{K}$ be an embedded disk with area A and length L. For $y > 0$ small enough, the interior parallel domain M_y, as in Proof 3.12, has area $A - Ly + \pi y^2$ and length $L - 2\pi y$ but a bound on curvature $\|\tilde{\kappa}\|_\infty \le (1-y)^{-1}$. By an outward dilation by factor $(1-y)^{-1}$ we restore the curvature bound but now area is $(1-y)^{-2}(A - Ly + \pi y^2)$ and length is $(1-y)^{-1}(L - 2\pi y)$. The composite $M \mapsto (1-y)^{-1}M_y$ gives a one parameter deformation of embedded disks in $\mathcal{K}$. If one applies this deformation to N for x small and chooses $x = x(y)$ so that $L = L_0$, *i.e.* so $(1-y)L_0 = (1-x)L_0 - 2\pi y$, then the resulting area is

$$A = A_0 + \frac{y(1-y)L_0 + \pi y^2}{(1-y)^2}\left(\frac{4\pi A_0}{L_0^2} - 1\right).$$

Thus $A < A_0$ by the isoperimetric inequality. Hence A_0 is not extremal for $M \in \mathcal{N}(L)$ proving the claim.

By Lemma 4.13 or Lemma 4.14, the extremal figure for either problem consists of a piecewise circular curve (4.4) with $k \le 3$. If $k = 2$ then $M = P_\delta$ as in Lemma 4.13. The centers Z_i of the convex circles of ∂M form a triangle which we denote $\mathcal{P} = \mathcal{P}(Z_1, Z_2, Z_3)$. Let $\delta_i = \operatorname{dist}(Z_i, Z_{i+1})$, where subscripts are taken modulo 3, ϑ_i the direction angle of the vectors $\mathbf{v}_i = \overrightarrow{Z_i Z_{i+1}}$ and $\phi_i = \vartheta_{i+1} - \vartheta_i > 0$ the angle deficit at the Z_i vertex. Let ζ_i be the subarc of the boundary corresponding to $\mathbf{v}_i$ as in Lemma 4.13. The total length

$$L = |\partial M| = \sum_{i=1}^{3} (\phi_i + |\zeta_i|) = 2\pi + 4\sum_{i=1}^{3} \sin^{-1}\left(\frac{\delta_i}{4}\right). \tag{4.7}$$

The area of M between $\mathbf{v}_i$ and ζ_i is $a_i = \delta_i\sqrt{1 - \delta_i^2/16}$.

It remains to show that among all possible triangles with fixed L, the least area occurs if $\mathcal{P}$ has one zero length side. Because of $n_i < 2\pi/3$ and the triangle inequality we have the constraints

$$\delta_i + \delta_{i+1} \ge \delta_{i+2}, \quad 0 \le \delta_i \le 2\sqrt{3}, \quad \text{for } i = 1, 2, 3. \tag{4.8}$$

If $T(\delta_1, \delta_2, \delta_3)$ denotes the area of the triangle $\mathcal{P}$ then the area of M is

$$A = \pi + T(\delta_1, \delta_2, \delta_3) + \sum_{i=1}^{3} \delta_i\sqrt{1 - \frac{\delta_i^2}{16}}. \tag{4.9}$$

We claim that the solution to the nonlinear optimization problem of finding $\delta = (\delta_1, \delta_2, \delta_3)$ to minimize (4.9) subject to the constraints (4.7) and (4.8) occurs if one of the δ_i vanishes and, thus by (4.8) the other δ_i are equal. Perhaps the easiest way to see this is to parameterize the surface satisfying (4.7) in δ-space by the change of variable $\delta_i = 4\sin(\xi_i)$. Thus the optimization problem becomes to minimize

$$A = \pi + T\left(4\sin(\xi_1), 4\sin(\xi_2), 4\sin(\xi_3)\right) + 2\sum_{k=1}^{3}\sin(2\xi_k) \tag{4.10}$$

subject to

$$\frac{L - 2\pi}{4} = \xi_1 + \xi_2 + \xi_3, \tag{4.11}$$

$$\sin\xi_i + \sin\xi_{i+1} \geq \sin\xi_{i+2}, \qquad 0 \leq \xi_i \leq \frac{\pi}{3}, \quad \text{for } i = 1, 2, 3. \tag{4.12}$$

Observe that now (4.10) is to be minimized on the subset satisfying (4.12) of the plane (4.11). Equation (4.10) without the T term is a strictly concave function whose minimum occurs on the vertices of (4.12), namely, where one of the $\delta_i = 0$. Since T is the area of the triangle, it is nonnegative and zero if and only if $\mathcal{P}$ has one zero length side, i.e., when one of the triangle inequalities (4.12) is equality. This happens at the vertices. Hence the vertices provide the minimum of (4.10). Thus, among all triangles of fixed perimeter, the area of corresponding M is smallest for the biangle. Theorem 4.1 is proved. □

Acknowledgment. This study evolved from a question of P. Fife. Also we thank S. Alexander, R. Bishop, N. Korevaar, T. Serbinowski for helpful comments and a referee for an improvement in Proof 4.7.

REFERENCES

1. S. Alexander and R. Bishop, *The simple cut distance of a Riemannian* 2-*manifold with boundary*, manuscript dated Sept. 3, 1991.

2. W. Blaschke and W. Leichtweiß, *Elementare Differentialgeometrie*, Grundlehren Math. Wiss. **1** (1973), 72–77.

3. W. Blaschke and K. Reidemeister, *Vorlesungen Über Differentialgeometrie.* II, *Affine Differentialgeometrie*, Springer, Berlin, 1923; Reprinted in *Differentialgeometrie* I *&* II, Chelsea, New York, 1967, 57–60.

4. G. Bol, *Isoperimetrisches Ungleighung für Berieche auf Flächen*, Jahresbericht der Deut. Math. Ver. **51** (1941), 219–257.

5. J.D. Burago and V.A. Zalgaller, *Geometric inequalities*, "Nauka," Leningrad, 1980 (Russian); English transl., Grundlehren Math. Wiss. **285** (1980), 231.

6. J. Cheeger and D. Ebin, *Comparison theorems in Riemannian geometry*, North Holland, Amsterdam, 1975.

7. S.S. Chern, *Curves and surfaces in Euclidean space*, in *Studies in Mathematics, Global differential geometry* (S.S. Chern, ed.), vol. 27, Mathematical Association of America, Providence, 1989, 99–139.

8. U. Dierkes, S. Hildebrandt, A. Küster and O. Wohlrab, *Minimal Surfaces* II, Grundlehren Math. Wiss. **296** (1992), 250–292.

9. M. do Carmo, *Differential geometry of curves and surfaces*, Prentice-Hall, Englewood Cliffs, 1976.

10. L. Dubins, *Curves with minimal length with constraint on average curvature, and with prescribed initial and terminal tangents*, Amer. J. Math. **79** (1957), 497–516.

11. H. Guggenheimer, *Differential geometry*, McGraw-Hill, 1963; reprinted Dover, New York, 1977, 30–31.

12. R. Gulliver, *Regularity of minimizing surfaces of prescribed mean curvature*, Ann. of Math. **97** (1973), 275–305.

13. R. Gulliver and F. Lesley, *On the boundary branch points of minimal surfaces*, Arch. Rat. Mech. Annal. **52** (1973), 20–25.

14. H. Hadwiger, *Die erweiterten Steinerschen Formeln für ebene und sphärische Bereiche*, Comment. Math. Helv. **18** (1945/46), 59–72.

15. H. Karcher, *Riemannian comparison constructions*, in *Studies in Mathematics, Global Differential Geometry* (S.S. Chern, ed.), vol. 27, Mathematical Association of America, Providence, 1989, 170–222.

16. V.N. Lagunov, *On the largest possible sphere contained in a given closed surface* I, Siberian Math. **1** (1960), 205–236; II **2** (1961), 874–880; (summary), Doklad. Akad. Nauk SSSR **127** (1959), 1167–1169 (Russian).

17. V.N. Lagunov and I.A. Fet, *Extremal problems for surfaces of prescribed topological type* I, Siberian Math. J. **4** (1963), 145–167; II **6** (1965), 1026–1038 (Russian).

18. M. Miranda, *Existence and regularity of hypersurfaces of R^n with prescribed mean curvature*, Proc. Symp. Pure and Appl. Math. **23** (1973), 1–9.

19. R. Osserman, *A proof of the regularity everywhere of the classical solution to Plateau's problem*, Ann. of Math. **91** (1970), 550–569.

20. G. Pestov and V. Ionin, *On the largest possible circle embedded in a given closed curve*, Dokl. Akad. Nauk Ukrain SSSR **127** (1959), 1170–1172 (Russian).

21. J. Petrov, *Variational methods in optimal control theory*, Enerǵija, Moscow-Leningerad, 1965 (Russian); English transl., Academic Press, New York, 1968.

22. G. Pólya, *Mathematics and plausible reasoning*, Princeton U. Press, Princeton, 1954.

23. L. Santaló, *Integral geometry and geometric probability*, Addison-Wesley, Reading, 1976.

24. B. Sz. Nagy, *Über Parallelmengen nichtkonvexer ebener Bereiche*, Acta Sci. Math. (Acta Szeged) **20** (1959), 36–47.

Department of Mathematics, University of South Carolina, Columbia, SC 29208

Department of Mathematics, University of Utah, Salt Lake City, UT 84109 and University of California Irvine, Irvine, CA 92717