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Inequalities Relating Sectional Curvatures of a Submanifold to the Size of its 2nd Fndamental Form and Applications to Pinching Theorems for Submanifolds

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INEQUALITIES RELATING SECTIONAL CURVATURES OF A SUBMANIFOLD TO THE SIZE OF ITS SECOND FUNDAMENTAL FORM AND APPLICATIONS TO PINCHING THEOREMS FOR SUBMANIFOLDS

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ABSTRACT. The Gauss curvature equation is used to prove inequalities relating the sectional curvatures of a submanifold with the corresponding sectional curvature of the ambient manifold and the size of the second fundamental form. These inequalities are then used to show that if a manifold $\overline{M}$ is δ-pinched for some $\delta > \frac{1}{4}$, then any submanifold M of $\overline{M}$ that has small enough second fundamental form is δ_M-pinched for some $\delta_M > \frac{1}{4}$. It then follows from the sphere theorem that the universal covering manifold of M is a sphere. Some related results are also given.

1. Introduction. This note is motivated by questions of the following type: Let $\overline{M}$ be a complete Riemannian manifold and M a compact immersed submanifold of $\overline{M}$; how then is the topology of M affected by placing a sufficiently small upper bound on the size of the second fundamental form of M in $\overline{M}$? For example, when $\overline{M}$ is isometric to a standard sphere, Lawson and Simons [**L-S**] show that if the length of the second fundamental form of M is small enough, then M is a homotopy sphere. If $\overline{M}$ is the product of two spheres, then the second author has shown in [**Wei**] that the submanifolds of $\overline{M}$ with sufficiently small second fundamental are homeomorphic to totally geodesic submanifolds of $\overline{M}$.

Here we will consider the case that $\overline{M}$ is δ-pinched for some $\delta > \frac{1}{4}$. That is, all sectional curvatures of $\overline{M}$ are in the closed interval $[\delta K_0, K_0]$ for some constant $K_0 > 0$. In this case the well-known sphere theorem of Berger, Klingenberg, Rauch and Toponogov implies that the universal covering manifold of $\overline{M}$ is homeomorphic to a sphere. If $\overline{M}$ and M are both simply connected and M has codimension one, then Flaherty has given conditions (cf. §3 below) on the second fundamental form of M which forces M to be a homotopy sphere.

In this note we will extend this to higher codimensions and at the same time weaken the assumptions on the second fundamental form of M and drop the assumption of simple connectivity on $\overline{M}$.

Our method is to use the Gauss curvature equation to prove inequalities relating the sectional curvatures of a submanifold with the corresponding sectional curvatures of the ambient manifold and the size of the second fundamental form of the submanifold. These inequalities then imply that a submanifold of a pinched manifold is also pinched (with a slightly worse pinching constant) provided that its second fundamental form is small enough. The proofs of these inequalities are elementary; they only involve completing the square.

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2. The inequalities. Let M be an n-dimensional ($n \geq 2$) submanifold isometrically immersed in the Riemannian manifold $\overline{M}$. At each point $x \in M$ the tangent space to M at x will be written as TM_x and the normal space to M at x as $T^\perp M_x$. The second fundamental form h_x of M in $\overline{M}$ at x is a symmetric bilinear form $TM_x \times TM_x$ to $T^\perp M_x$. If $e_1, \ldots, e_n$ is any orthonormal basis on TM_x, then the length of h_x is defined by

$$\|h_x\|^2 = \sum_{1 \leq i,j \leq n} \|h_x(e_i, e_j)\|^2. \tag{1}$$

If P is a plane section of M at x, i.e. a two-dimensional subspace of TM_x, then denote by $\overline{K}(P)$ the sectional curvature of $\overline{M}$ at P, by $K(P)$ the sectional curvature of M at P and by $h|_P$ the symmetric bilinear from $P \times P$ to $T^\perp M_x$ obtained by restricting h_x to $P \times P$. Let e_1, e_2 be any orthonormal basis of P. Then the Gauss curvature equation can be written as

$$K(P) = \overline{K}(P) + \langle h(e_1, e_1), h(e_2, e_2)\rangle - \|h(e_1, e_2)\|^2 \tag{2}$$

and the length of $h|_P$ is

$$\begin{aligned} \|h|_P\|^2 &= \sum_{1 \leq i,j \leq 2} \|h(e_i, e_j)\|^2 \\ &= \|h(e_1, e_1)\|^2 + 2\|h(e_1, e_2)\|^2 + \|h(e_2, e_2)\|^2. \end{aligned} \tag{3}$$

Clearly $\|h|_P\|^2 \leq \|h_x\|^2$. Our estimates are

PROPOSITION 1. *If P is a plane section of M, then*

$$\begin{aligned} \overline{K}(P) - \tfrac{1}{2}\|h\|^2 \leq \overline{K}(P) - \tfrac{1}{2}\|h|_P\|^2 &\leq K(P) \\ &\leq \overline{K}(P) + \tfrac{1}{2}\|h|_P\|^2 \leq \overline{K}(P) + \tfrac{1}{2}\|h\|^2. \end{aligned}$$

PROPOSITION 2. *If M is a minimal surface in $\overline{M}$, then*

$$\overline{K}(P) - \tfrac{1}{2}\|h\|^2 = K(P) \leq \overline{K}(P).$$

PROPOSITION 3. *If M is a totally umbilic surface in $\overline{M}$, then*

$$\overline{K}(P) \leq K(P) = \overline{K}(P) + \tfrac{1}{2}\|h\|^2.$$

PROPOSITION 4. *If $\overline{M}$ is a Kaehler manifold and M is a Kaehler submanifold of $\overline{M}$, then for every holomorphic plane section P of M*

$$\overline{K}(P) - \tfrac{1}{2}\|h\|^2 \leq \overline{K}(P) - \tfrac{1}{2}\|h|_P\|^2 = K(P) \leq \overline{K}(P).$$

REMARKS. Propositions 2 and 3 show that the inequalities in Proposition 1 are sharp in the case that M is two-dimensional. By considering cylinders over minimal surfaces or umbilic surfaces in Euclidean space it is possible to show that the inequalities in Proposition 1 are sharp in all dimensions. Proposition 4 is a restatement of Proposition 9.2 in Volume 2 of [**K-N**]. It is included here because of its relation to the other results.

PROOF. Let e_1, e_2 be an orthonormal basis of P. Let $X = h(e_1, e_1)$, $Y = h(e_1, e_2)$ and $Z = h(e_2, e_2)$. Because of equations (2) and (3), to prove Proposition 1 it is enough to show that

$$-(\|X\|^2 + 2\|Y\|^2 + \|Z\|^2) \leq 2(\langle X, Z\rangle - \|Y\|^2) \leq \|X\|^2 + 2\|Y\|^2 + \|Z\|^2.$$

This follows at once from the identities

$$\|x\|^2 + 2\|Y\|^2 + \|Z\|^2 - 2(\langle X, Z\rangle - \|Y\|^2) = \|X - Z\|^2 + 4\|Y\|^2 \geq 0,$$

$$2(\langle X, Z\rangle - \|Y\|^2) + \|X\|^2 + 2\|Y\|^2 + \|Z\|^2 = \|X + Z\|^2 \geq 0.$$

If M is a minimal surface and $x \in M$, then let e_1, e_2 be an orthonormal basis of TM_x. Because M is minimal the mean curvature vector of M is zero so $0 = h(e_1, e_1) + h(e_2, e_2) = X + Z$ (X, Y, Z as above). Using $Z = -X$ in (2) yields $K(P) = \overline{K}(P) - \|X\|^2 - \|Y\|^2$ and in (1) it yields $\|h\|^2 = 2\|X\|^2 + 2\|Y\|^2$. These two equations imply Proposition 2.

If M is a totally umbilic surface, then by definition $Y = h(e_1, e_2) = 0$ and $X = h(e_1, e_1) = h(e_2, e_2) = Z$. Thus $K(P) = \overline{K}(P) + \|X\|^2$ and $\|h\|^2 = 2\|X\|^2$. This proves Proposition 3.

3. Submanifolds of pinched manifolds. If M is a Riemannian manifold and $0 < \delta \leq 1$, then M is said to be δ-pinched if and only if there is a positive constant K_0 such that $\delta K_0 \leq K(P) \leq K_0$ for all plane sections P of M. It is clear that the above results can be used to relate pinching (or holomorphic pinching) of a manifold to pinching (or holomorphic pinching) of its submanifolds. For example, Proposition 1 easily implies

PROPOSITION 5. *Let $\overline{M}$ be a Riemannian manifold with $\delta \leq \overline{K}(P) \leq 1$ for all plane sections of P of $\overline{M}$ and let M be a submanifold of $\overline{M}$ so that $\|h|_P\|^2 \leq B^2$ for all plane sections P of M. Then all the sectional curvatures of M are in the interval $[\delta - \frac{1}{2}B^2, \delta + \frac{1}{2}B^2]$. Thus if $B^2 < 2\delta$, then M is δ_M-pinched with*

$$\delta_M = \frac{\delta - B^2/2}{1 + B^2/2} = \frac{2\delta - B^2}{2 + B^2}.$$

COROLLARY. *If $\delta > \frac{1}{4}$ and M is complete with $\|h|_P\|^2 \leq (8\delta - 2)/5$ for all plane sections P of M, then M is δ_M-pinched for some $\delta_M > \frac{1}{4}$ and thus its universal covering manifold is homeomorphic to a sphere.*

We now give a statement and an elementary proof of the theorem of Flaherty mentioned above.

THEOREM [**F**]. *Let $\overline{M}$ be a complete, simply connected, Riemannian manifold of dimension at least three that has all its sectional curvatures in the interval $[\delta, 1]$ with $\delta > \frac{1}{4}$ (this implies $\overline{M}$ is homeomorphic to a sphere). Let M be a simply connected hypersurface of $\overline{M}$ such that the second fundamental forms of M with respect to one of the two outward unit normals have their eigenvalues in $[0, B]$, where $B < \cot(\pi/(4\sqrt{\delta}))$. Then M is a homotopy sphere.*

To prove this theorem we first note that if all of the eigenvalues of the second fundamental form of a hypersurface M are in the interval $[0, B]$ for one of the two choices of the outward normal, then for all plane sections P of M,

(A) $K(P) \geq \overline{K}(P)$,

(B) $\|h|_P\|^2 \leq 2B^2$.

(The first follows from the Gauss equation and the assumption that the eigenvalues are ≥ 0. For the second use that eigenvalues of $h|_P$ are also in the interval $[0, B]$ and so $\|h|_P\|^2 = \lambda_1^2 + \lambda_2^2 \leq 2B^2$.) The conditions (A) and (B) make sense for submanifolds of any codimension.

Proposition 1 now implies

PROPOSITION 6. *Let $\overline{M}$ be a Riemannian manifold with all its sectional curvatures in the interval* $[\delta, 1]$ *with* $\delta > 0$. *Let M be a complete submanifold of $\overline{M}$ that satisfies the conditions* (A) *and* (B). *Then the sectional curvatures of M are in the interval* $[\delta, 1 + B^2]$ *and thus M is δ_M-pinched with* $\delta_M = \delta/(1 + B^2)$.

COROLLARY. *If* $\delta > \frac{1}{4}$ *and* $B^2 < 4\delta - 1$ *in the last proposition, then M is δ_M-pinched for some* $\delta_M > \frac{1}{4}$. *Therefore the universal covering manifold of M is a sphere.*

To show that this corollary implies Flaherty's theorem, it is enough to show that $\frac{1}{4} < \delta \leq 1$ implies $\cot^2(\pi/(4\sqrt{\delta})) < 4\delta - 1$. Since $0 < \cot(\pi/(4\sqrt{\delta})) \leq 1$ for δ in the given interval, the required inequality is implied by $\cot(\pi/4\sqrt{\delta}) < 4\delta - 1$. Letting $x = 1\sqrt{\delta}$ we want $f(x) = 4x^{-2} - \cot(\pi x/4) - 1 > 0$ when $1 \leq x < 2$. It is enough to show f has no zero on $[1, 2)$. At a zero of f, we have $4x^{-2} - 1 = \cot(\pi x/4) \leq 1$. This inequality implies $x \geq \sqrt{2}$. Thus we only need to show $f(x) \neq 0$ on $[\sqrt{2}, 2)$. On this interval

$$f'(x) = -\frac{8}{x^3} + \frac{\pi}{4}\csc^2\left(\frac{\pi}{4}x\right) \leq -\frac{8}{x^3}\bigg|_{x=2} + \frac{\pi}{4}\csc^2\left(\frac{\pi}{4}x\right)\bigg|_{x=\sqrt{2}}$$
$$= -1.0 + .978262725 < 0.$$

Therefore f is decreasing on $[\sqrt{2}, 2)$ and $f(2) = 0$. Consequently, $f(x) > 0$ on $[1, 2)$ as claimed.

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