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On Two Function-Spaces which are Similar to L0

S J. Dilworth

University of South Carolina - Columbia, dilworth@math.sc.edu

D A. Trautman

ON TWO FUNCTION SPACES WHICH ARE SIMILAR TO L_0

S. J. DILWORTH AND D. A. TRAUTMAN

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ABSTRACT. Let Λ_0 consist of all functions f measurable on $(0,\infty)$ with

$$\lambda\{s\colon |f(s)| > t\} < \infty$$

for all $t > 0$, where λ is Lebesgue measure, and let $L_0(0,\infty)$ consist of all measurable functions f with

$$\lim_{t\to\infty} \lambda\{s\colon |f(s)| > t\} = 0.$$

Let each space have the topology induced by convergence in measure. We show that every infinite-dimensional Banach subspace of Λ_0 contains c_0 or l_p for some $1 \le p < \infty$. We also identify the duals of Λ_0 and $L_0(0,\infty)$.

1. INTRODUCTION AND NOTATION

Let $\mathscr{L}_0(0,\infty)$ denote the collection of all (equivalence classes of) almost everywhere finite measurable functions on $(0,\infty)$, and let $L_0(0,\infty)$ denote the collection of all $f \in \mathscr{L}_0(0,\infty)$ satisfying $\lambda(\{s\colon |f(s)| > t\}) \to 0$ as $t \to \infty$, where λ is Lebesgue measure. $L_0(0,\infty)$ is a topological vector space under the topology of convergence in measure, and it is the largest linear subspace of $\mathscr{L}_0(0,\infty)$ with this property. Routine calculations show that $L_0(0,\infty)$ is a non-separable complete metric linear space, with the metric given by $d(f,g) = \|f - g\|$, where

$$\|f\| = \sup_E \int_E \frac{|f|}{1 + |f|} d\lambda,$$

the supremum being taken over all $E \subset (0,\infty)$ of measure one.

The distribution function $d_f(t)$ of a measurable function f on $(0,\infty)$ is defined by

$$d_f(t) = \lambda(\{s\colon |f(s)| > t\}) \qquad (0 < t < \infty),$$

and the decreasing rearrangement $f^*(t)$ by

$$f^*(t) = \inf\{s > 0\colon d_f(s) \le t\}.$$

We shall study the separable subspace of $L_0(0,\infty)$ consisting of all $f \in L_0(0,\infty)$ for which $d_f(t) < \infty$ for all $t > 0$. In view of the classical identity $\|f\|_{\Lambda_\alpha} = \|d_f\|_{L_\alpha}^\alpha$ $(0 < \alpha < \infty)$, where L_α and Λ_α are the standard Lebesgue and Lorentz function spaces (see [10] for the original definition of Λ_α), it seems to be appropriate to denote the collection of all functions on $(0,\infty)$ with a finite distribution function by the symbol Λ_0, and we have chosen to do so.

In §2 the Banach subspaces of Λ_0 are investigated. It is proved that Λ_0 contains an isomorphic copy of every Orlicz function space on $(0,\infty)$. The main result of the paper is Theorem 2.3, which says that every infinite-dimensional Banach subspace of Λ_0 contains a subspace isomorphic to c_0 or to l_p $(1 \leq p < \infty)$. The prototype for results of this kind, of course, is the famous theorem of D. J. Aldous [1] that every infinite-dimensional subspace of $L_1(0,1)$ contains l_p for some $1 \leq p \leq 2$. Subsequently, various generalizations of Aldous's theorem have been obtained (see e.g. [5]). Our strategy in proving Theorem 2.3 is perhaps the obvious one of reducing the problem to the consideration of Banach spaces whose subspace structure is already clearly understood.

In the short third section the dual spaces of Λ_0 and $L_0(0,\infty)$ are computed. It is proved that Λ_0 has trivial dual and that the dual of $L_0(0,\infty)$ may be identified with an infinite-dimensional subspace of the dual of $L_\infty(0,\infty)$.

Let us now state some definitions and notation which we shall use in the remainder of the paper. An Orlicz function $\phi\colon [0,\infty) \to [0,\infty)$ is a strictly increasing function, continuous at 0, with $\phi(0) = 0$. We say that ϕ satisfies the Δ_2-condition at 0 if $\sup_{0<x\leq 1}\phi(2x)/\phi(x) < \infty$. We define the Orlicz function space $L_\phi(0,\infty)$ as the space of all functions f measurable on $(0,\infty)$ with

$$\int_0^\infty \phi\left(\frac{|f(t)|}{\alpha}\right) dt < \infty$$

for all $\alpha > 0$. The topology is given by the quasi-norm

$$\|f\|_\phi = \inf\{\alpha > 0\colon \int_0^\infty \phi\left(\frac{|f(t)|}{\alpha}\right) dt \leq 1\}.$$

The Orlicz sequence space l_ϕ is defined as the space of all sequences (a_n) such that

$$\sum_{n=1}^\infty \phi\left(\frac{|a_n|}{\alpha}\right) < \infty$$

for all $\alpha < \infty$. We use a quasi-norm similar to that of $L_\phi(0,\infty)$ to define the topology.

In a topological vector space X, a sequence (x_n) is a basis for X if, for each $x \in X$, there exist unique scalars (a_n) with

$$x = \sum_{n=1}^\infty a_n x_n.$$

A basic sequence (x_n) is a sequence which is a basis for its closed linear span $\overline{\mathrm{lin}}(x_n)$. Two bases (x_n) of X and (y_n) of Y are equivalent whenever $\sum_{n=1}^{\infty} a_n x_n$ converges if and only if $\sum_{n=1}^{\infty} a_n y_n$ converges. A basis (x_n) of X is symmetric if for each permutation π of $\mathbf{Z}^+$, $(x_{\pi(n)})$ is equivalent to (x_n).

If K is a subset of a topological vector space, $\mathrm{conv}(K)$ will denote its convex hull. In Λ_0 or $L_0(0,\infty)$, for $\varepsilon > 0$ we let

$$B_\varepsilon = \{f \colon \|f\| < \varepsilon\}.$$

The indicator function of a set E is denoted by $I(E)$. The reader should consult [8] and [9] for other standard notation and terminology.

2. SUBSPACES OF Λ_0

Our first proposition shows that Λ_0 has a very large collection of Banach and quasi-Banach subspaces. This observation lends interest to the main result, Theorem 2.3 below, and provides a motivation for wishing to prove it.

Proposition 2.1. *Let ϕ be an Orlicz function. Then $L_\phi(0,\infty)$ is isomorphic to a subspace of Λ_0.*

Proof. Since Λ_0 is clearly isomorphic to the space $\tilde{\Lambda}_0$ consisting of all measurable functions on $(0,\infty) \times (0,\infty)$ possessing a distribution function, it will be enough to show that $\tilde{\Lambda}_0$ contains a subspace isomorphic to $L_\phi(0,\infty)$. To this end, let $f(s) = (\phi^{-1}(s))^{-1}$, and define $T\colon L_\phi(0,\infty) \to \tilde{\Lambda}_0$ by $g(t) \mapsto f(s)g(t)$. To see that T maps $L_\phi(0,\infty)$ into $\tilde{\Lambda}_0$, observe that $g \in L_\phi(0,\infty)$ if and only if, for all $\alpha > 0$, we have

$$\int_0^\infty \phi\left(\frac{|g(t)|}{\alpha}\right) dt < \infty;$$

that is, if and only if

$$\int_0^\infty f^{-1}\left(\frac{\alpha}{|g(t)|}\right) dt < \infty.$$

This is the case if and only if

$$\int_0^\infty \lambda(\{s\colon f(s)|g(t)| > \alpha\}) \, dt < \infty$$

for all $\alpha > 0$. By Fubini's theorem, the latter condition is simply the statement that $f(s)g(t)$ belongs to $\tilde{\Lambda}_0$. It is also easy to check from the above calculation that T is an isomorphism onto its range.

The main calculations used in the proof of Theorem 2.3 are brought together in the following lemma. There is a close affinity between this lemma and [3, Proposition 2.3].

Lemma 2.2. *Let X be a Banach space and let $T: X \to \Lambda_0$ be an isomorphic embedding. For each $M > 0$, let $\psi_M(f) = f \cdot I(0, M)$. If, for each $M > 0$, $\psi_M \circ T$ does not restrict to an isomorphic embedding of any infinite-dimensional subspace of X, then X contains a symmetric basic sequence.*

Proof. By passing to a subspace, if necessary, we may assume that X has a normalized Schauder basis $(x_n)_{n=1}^{\infty}$. Let $\varepsilon > 0$ be given. Since $\psi_M \circ T$ is not an isomorphism on any infinite-dimensional subspace of X for any M, one can find by induction a normalized block basic sequence $(y_n)_{n=1}^{\infty}$ of $(x_n)_{n=1}^{\infty}$ and an increasing sequence $(M_n)_{n=1}^{\infty}$ of positive numbers such that $\|T(y_n) \cdot I(0, M_{n-1})\| < \varepsilon/2^n$ and $\|T(y_n) \cdot I(M_n, \infty)\| < \varepsilon/2^n$. Let $z_n = T(y_n) \cdot I(M_{n-1}, M_n)$. Provided $\varepsilon > 0$ is chosen so small that $\inf\{\|z_n\|: 1 \leq n < \infty\} > 0$, [6, Lemma 4.3] shows that $(z_n)_{n=1}^{\infty}$ is a basic sequence in Λ_0 equivalent to $(T(y_n))_{n=1}^{\infty}$. Select measurable subsets $A_n \subset (M_{n-1}, \infty)$ such that $\lambda(A_n) = 1$ and $\operatorname{ess\,inf}\{|z_n(t)|: t \in A_n\} \geq \operatorname{ess\,sup}\{|z_n(t)|: t \in A_n^c\}$, and let $z_n' = z_n \cdot I(A_n)$. It is easily seen that $\|\sum_{n=1}^{\infty} a_n z_n\| = \|\sum_{n=1}^{\infty} a_n z_n'\|$ for all scalars $(a_n)_{n=1}^{\infty}$, and so $(z_n)_{n=1}^{\infty}$ and $(z_n')_{n=1}^{\infty}$ are equivalent basic sequences. Let $\delta = \inf\{\|z_n'\|: 1 \leq n < \infty\}$ and let Z_n be the decreasing rearrangement of z_n'. By passing to a subsequence and relabeling, we may assume by Helly's selection theorem that $Z_n \to Z$ pointwise, where Z is a decreasing function supported on $[0,1]$. By Egorov's theorem we may also assume, after passing to a further subsequence, that $|Z_n - Z| < \delta/4^n$ except on a set of measure less than $\delta/4^n$. By [2, Theorem 7.5], for each n there exists a measure-preserving transformation $\sigma_n: A_n \to [0, 1]$ with $|z_n'| = Z_n \circ \sigma_n$ almost everywhere. Let $W_n = (Z \circ \sigma_n) \cdot \operatorname{sgn}(z_n')$. Then $\|z_n' - W_n\| < 2\delta/4^n$, and so by [6, Lemma 4.3] $(W_n)_{n=1}^{\infty}$ and $(z_n')_{n=1}^{\infty}$ are equivalent basic sequences. Clearly $(W_n)_{n=1}$ is a symmetric basic sequence as each W_n has Z as its decreasing rearrangement.

Remark. It is easy to see that if the function Z of the previous proof is bounded, then $(W_n)_{n=1}^{\infty}$ spans a subspace isomorphic to c_0. Since c_0 does not embed isomorphically into $L_0(0, 1)$ it follows that Λ_0 and $L_0(0, 1)$ are not isomorphic spaces. (The latter fact also follows from Proposition 2.1.) In particular, Λ_0 has at least two non-isomorphic complemented subspaces.

Theorem 2.3. *Let X be a Banach subspace of Λ_0. Then X contains a subspace isomorphic to c_0 or to l_p for some $1 \leq p < \infty$.*

Proof. Suppose that $T: X \to \Lambda_0$ is an isomorphic embedding. If, for some M, $\phi_M \circ T$ restricts to an isomorphic embedding, then some infinite-dimensional subspace of X is by dilation isomorphic to a subspace of $L_0(0, 1)$, and thus isomorphic to a subspace of $L_q(0, 1)$ for all $0 < q < 1$ (see [11]). One can then modify Garling's work ([5, Theorem 16, p. 167, Theorem 7, p. 138 and Theorem 6, p. 136]) to show that $L_q(0, 1)$ is stable, that all stable q-Banach spaces have an l_p-type for some $q \leq p < \infty$, and that all infinite-dimensional stable q-Banach spaces contain an isomorphic copy of l_p for some $q \leq p < \infty$.

Since X is a Banach space it follows that X must contain a copy of l_p for some $1 \le p < \infty$.

Now suppose that X is a Banach subspace of Λ_0 satisfying the hypotheses of Lemma 2.2. Then X contains a subspace isomorphic to the closed linear span of a sequence of functions $(f_n)_{n=1}^{\infty}$, each having a common distribution function $d_f(t)$. By the remark following Lemma 2.2 we may assume that f is an unbounded function. Let $\phi(t) = d_f(1/t)$. A short calculation shows that $\sum_{n=1}^{\infty} a_n f_n$ converges in Λ_0 if and only if $\sum_{n=1}^{\infty} \phi\left(\frac{|a_n|}{\lambda}\right) < \infty$ for all $\lambda > 0$. Hence $\overline{\lim}(f_n)$ and l_ϕ are isomorphic spaces. By aping the proof of [9, Proposition 4.a.4], we see that either l_ϕ contains c_0 or ϕ satisfies the Δ_2-condition at 0. If ϕ satisfies the Δ_2-condition at 0, then by [7, Theorem 3.3] l_ϕ is isomorphic to l_ψ for some convex Orlicz function ψ which satisfies the Δ_2-condition at 0. An appeal to [9, Theorem 4.a.9] shows that l_ψ, and hence both l_ϕ and $\overline{\lim}(f_n)$, contains an isomorphic copy of c_0 or l_p for some $1 \le p < \infty$.

3. THE DUAL OF $L_0(0, \infty)$

We begin by showing that Λ_0 has trivial dual. The proof is modelled on Day's proof [4] that L_p has trivial dual for $0 < p < 1$.

Proposition 3.1. $\Lambda_0^* = \{0\}$.

Proof. Let $\varepsilon > 0$; we shall show that $\Lambda_0 = \operatorname{conv} B_\varepsilon$. Let $f \in \Lambda_0$ and let $A = \{t: |f(t)| \ge \varepsilon/2\}$. Since $\lambda(A) < \infty$ we may decompose it into finitely many sets A_i $(1 \le i \le n)$ of measure less than $\varepsilon/2$. Let $f_i = nf \cdot I(A_i) + f \cdot I(A^c)$. Then each $f_i \in B_\varepsilon$ and $f = \sum_{i=1}^{n}(1/n)f_i$. It follows that Λ_0 has trivial dual.

Two easy lemmas are required to identify the dual of $L_0(0, \infty)$.

Lemma 3.2. *Let $\mathcal{N}$ be a neighborhood base of the origin for $L_0(0, \infty)$. Then $\Lambda_0 = \cap_{U \in \mathcal{N}} \operatorname{conv} U$.*

Proof. The fact that $\Lambda_0 \subset \cap_{U \in \mathcal{N}} \operatorname{conv} U$ was essentially proved in Proposition 3.1. Let $f \in L_0(0, \infty) \backslash \Lambda_0$; then there exists $a > 0$ such that $\lambda\{t: |f(t)| > a\} = \infty$. Let $0 < \varepsilon < a/(1 + a)$, let $f_1, \ldots, f_n \in B_\varepsilon$, and let $E_i = \{t: |f_i(t)| \ge a\}$. Clearly $\lambda(E_i) \le 1$, and so $\lambda\{t: |(\sum_{i=1}^{n} \alpha_i f_i)(t)| \ge a\} \le n$ whenever $\alpha_i \ge 0$, $\sum_{i=1}^{n} \alpha_i = 1$. Thus $f \ne \sum_{i=1}^{n} a_i f_i$ for any such α_i, and so $f \notin \operatorname{conv}(B_\varepsilon)$. It follows that $\cap_{U \in \mathcal{N}} \operatorname{conv} U \subset \Lambda_0$.

Lemma 3.3. *The space $L_0(0, \infty)/\Lambda_0$ is isomorphic to the Banach space $L_\infty(0, \infty)/(\Lambda_0 \cap L_\infty(0, \infty))$.*

Proof. Let $f \in L_0(0, \infty)$. Then there exists $M > 0$ such that $\lambda\{t: |f(t)| > M\} < \infty$. Clearly $f \cdot I\{t: |f(t)| \le M\} \in L_\infty(0, \infty)$, while $f \cdot I\{t: |f(t)| > M\} \in \Lambda_0$, and so the mapping $T: L_\infty(0, \infty) \to L_0(0, \infty)/\Lambda_0$ given by $T(g) = g + \Lambda_0$

is onto. T is clearly continuous and its kernel is $\Lambda_0 \cap L_\infty(0,\infty)$, whence the result.

Theorem 3.4. *The dual of the space $L_0(0,\infty)$ may be identified with the dual of the Banach space $L_\infty(0,\infty)/(\Lambda_0 \cap L_\infty(0,\infty))$.*

Proof. By Lemma 3.2 every element of $(L_0(0,\infty))^*$ vanishes on Λ_0. The result now follows from Lemma 3.3.

Remarks. 1.   Theorem 3.4 allows an explicit description of the dual of $L_0(0,\infty)$. Let $\psi \in (L_0(0,\infty))^*$. Then there exists a finitely additive signed measure μ_ψ on the Lebesgue σ-field of $(0,\infty)$ such that $\mu_\psi(A) = 0$ for every $A \subset (0,\infty)$ with $\lambda(A) < \infty$, and such that for all $f \in L_0(0,\infty)$ we have

$$\psi(f) = \lim_{M\to\infty} \int (f \wedge M)\,d\mu_\psi,$$

where $(f \wedge M)(t) = \min(f(t),M)$. Conversely, every such measure μ defines a continuous linear functional on $L_0(0,\infty)$.

2. It is possible to show that there is no continuous linear projection from $L_0(0,\infty)$ onto Λ_0 by adapting Whitley's proof [12] of the non-existence of a bounded projection from l_∞ onto c_0. We omit the details of this result.

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UNIVERSITY OF SOUTH CAROLINA, DEPARTMENT OF MATHEMATICS, COLUMBIA, SOUTH CAROLINA 29208

THE CITADEL, DEPARTMENT OF MATHEMATICS AND COMPUTER SCIENCE, CHARLESTON, SOUTH CAROLINA 29409