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AN EXTENSION OF ELTON'S ℓ_1^n THEOREM TO COMPLEX BANACH SPACES

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ABSTRACT. Let $\varepsilon > 0$ be sufficiently small. Then, for $\theta = 0.225\sqrt{\varepsilon}$, there exists $\delta := \delta(\varepsilon) < 1$ such that if $(e_i)_{i=1}^n$ are vectors in the unit ball of a complex Banach space X which satisfy

$$\mathbb{E}\left\|\sum_{i=1}^{n} Z_i e_i\right\| \geq \delta n$$

(where (Z_i) are independent complex Steinhaus random variables), then there exists a set $B \subseteq \{1, \dots, n\}$, with $|B| \geq \theta n$, such that

$$\left\|\sum_{i\in B} z_i e_i\right\| \geq (1-\varepsilon)\sum_{i\in B}|z_i|$$

for all $z_i \in \mathbb{C}$ $(i \in B)$. The $\sqrt{\varepsilon}$ dependence on ε of the threshold proportion θ is sharp.

1. INTRODUCTION

A well-known theorem of Elton [5, Th. 1] on ℓ_1^n subsystems has an 'isomorphic' and an 'almost isometric' version. For the isomorphic version and for related results, we refer the reader to [5, 8, 13]. The isomorphic result was extended to complex Banach spaces in [7, 8].

In this paper we are concerned with the almost isometric version of Elton's theorem, which may be formulated as follows.

Theorem (Elton). *Suppose that $\theta \in (0, 1/2)$ and that $\varepsilon \in (0, 1)$. There exists $\delta := \delta(\theta, \varepsilon) < 1$ such that if $(e_i)_{i=1}^n$ are vectors in the unit ball of a real Banach space X such that*

$$\underset{\pm}{\text{average}}\left\|\sum_{i=1}^{n} \pm e_i\right\| \geq \delta n$$

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(the average taken over all choices of $\pm$), then there exists a set $B \subseteq \{1, \ldots, n\}$, with $|B| \geq \theta n$, such that

$$\left\| \sum_{i \in B} a_i e_i \right\| \geq (1 - \varepsilon) \sum_{i \in B} |a_i|$$

for all real scalars $(a_i)_{i \in B}$.

A surprising and interesting feature of the above result is that the 'threshold' proportion $\theta = 1/2$ is independent of ε. An example due to Szarek [5, p. 121] shows that this is the optimal threshold, and recently it was shown that it is still the optimal threshold even if the hypothesis is strengthened substantially, e.g. by replacing the average value of $\| \sum_{i=1}^{n} \pm e_i \|$ by the minimum value instead [4].

In this paper we prove a *complex* version of the above result. Accordingly, we now assume that $(e_i)_{i=1}^{n}$ are vectors in the unit ball of a complex Banach space X, and we seek a large set $B \subseteq \{1, \ldots, n\}$ such that

$$\left\| \sum_{i \in B} z_i e_i \right\| \geq (1 - \varepsilon) \sum_{i \in B} |z_i|$$

for all complex scalars $(z_i)_{i \in B}$. It is easy to see that the hypothesis of Elton's theorem—that the average of $\| \sum \pm e_i \|$ over all choices of $\pm$ signs is large—is not powerful enough to obtain the desired conclusion. In the complex setting we should instead consider the average of $\| \sum e^{i\theta_j} e_j \|$ over all complex signs $(e^{i\theta_j})$. This can sometimes give rise to interesting phenomena which arise specifically in complex Banach spaces: a good example of this is the property of *complex uniform convexity* studied in [3].

In the language of probability theory, this means that we should replace *Bernoulli* averages by *Steinhaus* averages. Let $(Z_i)_{i=1}^{\infty}$ be a sequence of independent complex Steinhaus random variables (defined on a probability space (Ω, Σ, P)) uniformly distributed on $\{z : |z| = 1\}$, i.e. $P(a \leq \arg(Z_i) \leq b) = (b - a)/(2\pi)$ for $0 \leq a \leq b < 2\pi$.

Now we can state the complex analogue of Elton's theorem. (Here $\mathbb{E}$ denotes *expected value* as usual.)

Theorem. *Let $\varepsilon > 0$ be sufficiently small. Then, for $\theta = (0.99/(\pi\sqrt{2}))\sqrt{\varepsilon}$, there exists $\delta := \delta(\varepsilon) < 1$ such that if $(e_i)_{i=1}^{n}$ are vectors in the unit ball of a complex Banach space X which satisfy*

$$\mathbb{E} \left\| \sum_{i=1}^{n} Z_i e_i \right\| \geq \delta n,$$

then there exists a set $B \subseteq \{1, \ldots, n\}$, with $|B| \geq \theta n$, such that

$$\left\| \sum_{i \in B} z_i e_i \right\| \geq (1 - \varepsilon) \sum_{i \in B} |z_i|$$

for all complex scalars $(z_i)_{i \in B}$. Moreover, we may take $\delta = 1 - 4.45 \cdot 10^{-5} \varepsilon^{3/2}$ for all $n \geq N(\varepsilon)$.

Remarks. 1. Note that the 'threshold' now depends on ε.

2. The number 0.99 may be replaced by any number less than unity provided the coefficient of $\varepsilon^{3/2}$ in the estimate for δ is adjusted accordingly. Note that the

coefficient of $\sqrt{\varepsilon}$, when 0.99 is replaced by unity, is approximately 0.225. We do not know the best value for this coefficient, but Example 1 below shows that it cannot exceed 1.3862.

3. The estimate for δ is a byproduct of the proof. We do not know whether the $\varepsilon^{3/2}$ dependence is of the correct order.

4. Obviously, there exists $c > 0$ such that, for $\theta = c\sqrt{\varepsilon}$, the Theorem is valid for all $\varepsilon \in (0,1)$.

The Theorem is reasonably sharp. Indeed, for *any* fixed $\delta < 1$, the $\sqrt{\varepsilon}$ dependence of $|A|$ on ε is of the correct order as the following example shows. This example is based on the aforementioned example given by Szarek.

Example 1. Let $q \geq 2$ and $m \geq 1$ be positive integers and set $n := mq$. Let $\mathcal{S}$ be the collection of all n-tuples $(\xi_i)_{i=1}^n$ of unimodular complex numbers such that exactly m of the ξ_i's fall into each of the following q arcs of the unit circle:

$$A_j = \left\{ e^{i\theta} : (j-1)\left(\frac{2\pi}{q}\right) \leq \theta < j\left(\frac{2\pi}{q}\right) \right\} \quad (1 \leq j \leq q).$$

We define a norm $\|\cdot\|$ on $\mathbb{C}^n$ in the following way (here $(e_i)_{i=1}^n$ is the standard basis of $\mathbb{C}^n$):

$$\left\| \sum_{i=1}^n z_i e_i \right\| = \max_{(\xi_i)\in\mathcal{S}} \left| \sum_{i=1}^n \xi_i z_i \right| .$$

Note that $\|e_i\| = 1$ $(1 \leq i \leq n)$. By the Strong Law of Large Numbers,

$$\mathbb{E}\frac{1}{n}\left\| \sum_{i=1}^n Z_i e_i \right\| \longrightarrow 1 \quad \text{as } n \longrightarrow \infty.$$

In particular, if $\delta < 1$ is fixed, we have

$$\mathbb{E}\left\| \sum_{i=1}^n Z_i e_i \right\| > \delta n$$

for all sufficiently large n. Now fix $y \in (0, 1/2)$ and suppose that $|A| = 2m(1+y)$. Then we have

$$\begin{aligned}\left\| \sum_{i\in A} e_i \right\| &= 2m\left|1 + y e^{\frac{2\pi i}{q}}\right| \\ &= |A|\left(1 - \frac{2\pi^2 y}{(1+y)^2}\frac{1}{q^2} + O\left(\frac{1}{q^4}\right)\right).\end{aligned}$$

So, for large q, we have

$$\left\| \sum_{i\in A} e_i \right\| \sim |A|\left(1 - \frac{2\pi^2 y}{(1+y)^2}\frac{1}{q^2}\right).$$

Thus, making the change of variable $\varepsilon := (2\pi^2 y/(1+y)^2)/q^2$, we have

$$|A| = 2m(1+y) = \frac{2n(1+y)}{q} = \left(\frac{\sqrt{2}(1+y)^2}{\pi\sqrt{y}}\sqrt{\varepsilon}\right) n.$$

The coefficient of $\sqrt{\varepsilon}$ is minimized when $y = 1/3$. Setting $y = 1/3$, we get

$$|A| = \left(\frac{16\sqrt{2}}{3\pi\sqrt{3}} \sqrt{\varepsilon} \right) n.$$

This example shows that the *best value* for the coefficient of $\sqrt{\varepsilon}$ in the Theorem is no greater than $(16\sqrt{2})/(3\pi\sqrt{3}) \approx 1.38612$.

We present the proof of the Theorem in Section 3. The basic argument is similar to the proof of Elton's theorem in [5], but the details are considerably more complicated. The reason for the extra complication can be traced to the aforementioned fact that the threshold proportion is no longer independent of ε.

The proof of Elton's theorem uses the combinatorial result known as the Sauer-Shelah Lemma [9, 10, 11]. Besides Elton's theorem, this result has found many other applications in Banach space theory, e.g. [12]. Our result requires a nontrivial extension of the Sauer-Shelah Lemma which is due to Karpovsky and Milman [6]. For completeness we include a direct proof of this result in Section 2.

Notation and terminology are standard. Since our results are asymptotic in nature, we make the standing assumption that certain quantities, such as θn, are positive integers. This helps to simplify the notation. We also use the following notation for asymptotic comparisons: $f(n) \sim g(n)$ means $\lim_{n\to\infty} f(n)/g(n) = 1$ and $f(n) \lesssim g(n)$ means $\lim_{n\to\infty} f(n)/g(n) \leq 1$.

2. A theorem of Karpovsky and Milman

In this section we give a direct proof of a combinatorial result of Karpovsky and Milman [6] that is needed for the proof of the Theorem. A more general result, which yields the theorem of Karpovsky and Milman as a special case, was proved by Alon [1] (see also [2] for an exposition).

Notation. Let $q \geq 2$ and $n \geq 1$ be fixed positive integers. Let Φ_n^q be the collection of all n-tuples $(\phi_i)_{i=1}^n$, where $\phi_i \in \{0, 1, \ldots, q-1\}$. For $A \subseteq \{1, \ldots, n\}$, let Φ_A^q be the collection of all tuples $(\phi_i)_{i\in A}$ indexed by A. For any set $\mathcal{S} \subseteq \Phi_n^q$, we define the *projection* $P_A(\mathcal{S})$ in the natural way:

$$P_A(\mathcal{S}) = \{(\phi_i)_{i\in A} : \phi \in \mathcal{S}\}.$$

We say that A has *full density* in $\mathcal{S}$ if $P_A(\mathcal{S}) = \Phi_A^q$. Also, we say that $\mathcal{S}$ *has density* k if there exists a set A of full density in $\mathcal{S}$ such that $|A| = k$. (Note that if $\mathcal{S}$ has density k, then it has density j for all $1 \leq j \leq k$.)

The proof uses the following lemma which can be proved by a simple counting argument which we omit.

Lemma 1. *For all $n > k \geq 1$ and $q \geq 2$ the following combinatorial identity holds:*

$$\sum_{j=0}^{k} \binom{n}{j} (q-1)^{n-j} = \sum_{j=0}^{k-1} \binom{n-1}{j} (q-1)^{n-1-j} + \sum_{j=0}^{k} \binom{n-1}{j} (q-1)^{n-j}.$$

Theorem (Karpovsky and Milman). *If $\mathcal{S} \subseteq \Phi_n^q$, $1 \leq k \leq n$, and*

$$|\mathcal{S}| > \sum_{j=0}^{k-1} \binom{n}{j} (q-1)^{n-j},$$

then $\mathcal{S}$ has density k.

Remark 5. The case $q = 2$ is the Sauer-Shelah Lemma.

Proof. We will prove the result by a double induction argument, first on k and then on n. The inductive hypothesis for k asserts that the result holds for all $n \geq k$. Clearly the result holds for $k = 1$ and all $n \geq 1$, and so the induction on k starts. Fix $p \in \{1, \dots, n\}$ and set $\overline{p} := \{1, \dots, n\} \setminus \{p\}$. For fixed $k > 1$, we will obtain the result for all $n \geq k$ by induction on n. We note that if $n = k$ the result is trivial since

$$\sum_{j=0}^{k-1} \binom{k}{j} (q-1)^{k-j} = q^k - 1.$$

So suppose that $n > k > 1$. Set $F_p = \{\phi \in \mathcal{S} : |P_{\overline{p}}^{-1}(P_{\overline{p}}(\phi)) \cap \mathcal{S}| = q\}$. (Recall that $P_{\overline{p}}(\mathcal{S})$ is the projection of $\mathcal{S}$ onto the set of coordinates $\overline{p}$.) Observe that F_p consists of those $\phi \in \mathcal{S}$ with the property that if the value of ϕ is freely changed at coordinate p, then the new n-tuple so obtained still belongs to $\mathcal{S}$.

First suppose that

$$|F_p| > q \sum_{j=0}^{k-2} \binom{n-1}{j} (q-1)^{n-1-j}.$$

Then

$$|P_{\overline{p}}(F_p)| > \sum_{j=0}^{k-2} \binom{n-1}{j} (q-1)^{n-1-j}.$$

So, by our inductive hypothesis on k applied to $k-1$, we see that $P_{\overline{p}}(F_p)$ has density $k-1$ in $\overline{p}$. But this implies that F_p (and hence $\mathcal{S}$) has density k in $\{1, \dots, n\}$. Now suppose that

$$|F_p| \leq q \sum_{j=0}^{k-2} \binom{n-1}{j} (q-1)^{n-1-j}.$$

Then

$$|P_{\overline{p}}(\mathcal{S})| \geq \frac{|F_p|}{q} + \frac{|\mathcal{S}| - |F_p|}{q-1} = \frac{|\mathcal{S}|}{q-1} - \left(\frac{1}{q-1}\right) \frac{|F_p|}{q},$$

since $P_{\overline{p}}$ is a q-to-1 mapping on F_p and at most a $(q-1)$-to-1 mapping on $\mathcal{S} \setminus F_p$. Thus,

$$\begin{aligned} |P_{\overline{p}}(\mathcal{S})| &> \left(\frac{1}{q-1}\right) \left[\sum_{j=0}^{k-1} \binom{n}{j} (q-1)^{n-j} - \sum_{j=0}^{k-2} \binom{n-1}{j} (q-1)^{n-1-j} \right] \\ &= \sum_{j=0}^{k-1} \binom{n-1}{j} (q-1)^{n-1-j} \end{aligned}$$

by an application of Lemma 1 with k replaced by $k-1$. Now by our inductive hypothesis on n applied to $n-1$ we see that $P_{\overline{p}}(\mathcal{S})$ has density k in $\overline{p}$, and thus $\mathcal{S}$ has density k in $\{1, \dots, n\}$. $\square$

Example 2. For a fixed k and n, with $1 \leq k \leq n$, let $\mathcal{S}$ consist of all n-tuples which have at most $k-1$ coordinates equal to 0. Clearly $\mathcal{S}$ has density $k-1$ but does not have density k. Also

$$|\mathcal{S}| = \sum_{j=0}^{k-1} \binom{n}{j} (q-1)^{n-j}.$$

This shows that the Karpovsky-Milman theorem is best possible.

3. The main result

We shall break the proof down into a long chain of lemmas. But first we must redefine some of the notation introduced in Section 2.

Notation. Fix positive integers $q \geq 2$ and $n \geq 1$. Let $\phi = (\phi_i)_{i=1}^n$ be an n-tuple of arcs, where each arc ϕ_i is of the form

$$\phi_i = \left\{ e^{i\theta} : \left(\frac{j-1}{q}\right) 2\pi \leq \theta < \left(\frac{j}{q}\right) 2\pi \right\}$$

for some $1 \leq j \leq q$. Let Φ_n^q be the collection of all such n-tuples.

Now fix $\alpha, \beta \in (0,1)$; their precise values, depending on ε, will be chosen later.

Recalling that $(Z_i)_{i=1}^n$ is a sequence of independent Steinhaus random variables defined on a probability space (Ω, Σ, P), let

$$E := \left\{ \omega \in \Omega : \left\| \sum_{i=1}^n Z_i e_i \right\| \geq (1 - \alpha\beta)\, n \right\}$$

and, for each $\phi \in \Phi_n^q$, let

$$E^\phi := E \cap \{\omega \in \Omega : Z_i \in \phi_i \text{ for } 1 \leq i \leq n\}.$$

Finally, let $\mathcal{S} := \{\phi \in \Phi_n^q : P(E^\phi) > 0\}$.

Lemma 2. *Suppose that*

$$\mathbb{E} \left\| \sum_{i=1}^n Z_i e_i \right\| \geq (1 - \alpha\beta/2) n. \tag{1}$$

Then $P(E^\phi) > 0$ for at least $q^n/2$ of the ϕ's, i.e. $|\mathcal{S}| \geq q^n/2$.

Proof.

$$\begin{aligned}
\left(1 - \frac{1}{2}\alpha\beta\right) n &\leq \mathbb{E}\left(\left\| \sum_{i=1}^n Z_i e_i \right\| \right) \\
&= \mathbb{E}\left(\left\| \sum_{i=1}^n Z_i e_i \right\| \Big| E \right) P(E) + \mathbb{E}\left(\left\| \sum_{i=1}^n Z_i e_i \right\| \Big| E^c \right) (1 - P(E)) \\
&\leq nP(E) + (1 - P(E))(1 - \alpha\beta) n.
\end{aligned}$$

Thus, $P(E) \geq 1/2$. Note that, for each $\phi \in \Phi_n^q$, we have

$$P(E^\phi) \leq P\{\omega \in \Omega : Z_i \in \phi_i, 1 \leq i \leq n\} = 1/q^n.$$

Thus,

$$1/2 \leq P(E) = \sum_{\phi \in \mathcal{S}} P(E^\phi) \leq \frac{|\mathcal{S}|}{q^n},$$

and so $|\mathcal{S}| \geq q^n/2$. □

For each $\phi \in \mathcal{S}$ there exists an n-tuple of unimodular complex numbers $(\xi_i^\phi)_{i=1}^n$ such that $\xi_i^\phi \in \phi_i$ for $1 \leq i \leq n$ and

$$\left\| \sum_{i=1}^n \xi_i^\phi e_i \right\| \geq (1 - \alpha\beta)n.$$

Indeed, since $P(E^\phi) > 0$, we can choose any $\omega \in E^\phi$ and then take $\xi_i^\phi = Z_i(\omega)$ for all $1 \leq i \leq n$.

For the rest of the proof we shall assume that (1) is satisfied.

Lemma 3. *For each $\phi \in \mathcal{S}$ there exists f^ϕ, a complex linear functional in the unit ball of X^*, with*

$$f^\phi\left(\sum_{i=1}^n \xi_i^\phi e_i\right) = \left\| \sum_{i=1}^n \xi_i^\phi e_i \right\| \geq (1 - \alpha\beta)n.$$

Proof. The Sobczyk-Bohnenblust theorem, i.e. the Hahn-Banach theorem for complex Banach spaces, guarantees the existence of this linear functional. □

Setting $f_i^\phi := f^\phi(e_i)$, we obtain $\sum_{i=1}^n \Re[\xi_i^\phi f_i^\phi] \geq (1 - \alpha\beta)n$ for each $\phi \in \mathcal{S}$.

Lemma 4. *Let $A^\phi = \{i : \Re[\xi_i^\phi f_i^\phi] \geq 1 - \alpha\}$. Then $|A^\phi| \geq (1 - \beta)n$ for all $\phi \in \mathcal{S}$.*

Proof. For $\phi \in \mathcal{S}$, we have from above that

$$\begin{aligned}
(1 - \alpha\beta)n &\leq \sum_{i=1}^n \Re[\xi_i^\phi f_i^\phi] \\
&= \sum_{i \in A^\phi} \Re[\xi_i^\phi f_i^\phi] + \sum_{i \in (A^\phi)^c} \Re[\xi_i^\phi f_i^\phi] \\
&\leq |A^\phi| + (1 - \alpha)(n - |A^\phi|). \\
&= n - \alpha n + \alpha |A^\phi|.
\end{aligned}$$

Thus, $\left|A^\phi\right| \geq (1 - \beta)n$. □

Lemma 5. *There exist $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \geq |\mathcal{S}|/C(n, \beta n)$, and $A \subseteq \{1, \ldots, n\}$ with $|A| \geq (1 - \beta)n$, such that for each $\phi \in \mathcal{S}'$ and $i \in A$ we have $\Re[\xi_i^\phi f_i^\phi] \geq (1 - \alpha)$.*

Proof. Recall that $\left|A^\phi\right| \geq (1 - \beta)n$ and that $\Re[\xi_i^\phi f_i] \geq (1 - \alpha)$ for all $\phi \in \mathcal{S}$ and all $i \in A^\phi$. By replacing A^ϕ by a smaller set, if necessary, we may assume that $\left|A^\phi\right| = (1 - \beta)n$. Then there are at most $C(n, \beta n)$ possible choices for A^ϕ. By the Pigeonhole Principle there exists a set $A \subseteq \{1, \ldots, n\}$ with $|A| \geq (1 - \beta)n$, and there exists $\mathcal{S}' \subseteq \mathcal{S}$ with $|\mathcal{S}'| \geq |\mathcal{S}|/C(n, \beta n)$, such that $A^\phi = A$ for all $\phi \in \mathcal{S}'$. □

Lemma 6. *Let $P_A(\mathcal{S}')$ be the projection of $\mathcal{S}'$ onto A. Then*

$$|P_A(\mathcal{S}')| \geq \frac{|\mathcal{S}'|}{q^{\beta n}}.$$

Proof. Since $|A| \geq (1-\beta)n$, it follows that at most $q^{\beta n}$ elements of $\mathcal{S}'$ project onto each element of $P_A(\mathcal{S}')$. Hence the result follows. □

We now address the following question: for $\theta \in (0,1)$, does $P_A(\mathcal{S}')$ have density θn in A? By the Karpovsky-Milman Theorem, this question has an affirmative answer if

$$|P_A(\mathcal{S}')| > \sum_{k=0}^{\theta n} \binom{|A|}{k} (q-1)^{n-k}. \tag{2}$$

We shall show that (2) is satisfied for appropriate choices of α, β, and θ. First we estimate the right-hand side of (2) from above. Note that, since $|A| \leq n$, the following lemma yields such an upper estimate.

Lemma 7. *Suppose that $\theta < 1/q$. Then*

$$\sum_{k=0}^{\theta n} \binom{n}{k} (q-1)^{n-k} \lesssim \frac{1}{1-\left(\frac{\theta}{1-\theta}\right)(q-1)} \frac{1}{\sqrt{2\pi n\theta(1-\theta)}} \left(\frac{(q-1)^{1-\theta}}{\theta^\theta(1-\theta)^{1-\theta}}\right)^n. \tag{3}$$

Proof. We show that the sum can be dominated by a convergent geometric series and then apply Stirling's Formula. Set $a_k := C(n,k)(q-1)^{n-k}$. Then

$$\frac{a_{k-1}}{a_k} = \left(\frac{k}{n+1-k}\right)(q-1) \leq \left(\frac{k}{n-k}\right)(q-1).$$

So, for $k \leq \theta n$, we have

$$\frac{a_{k-1}}{a_k} \leq \left(\frac{\theta}{1-\theta}\right)(q-1) := \rho(\theta).$$

Note that $\rho(\theta) < 1$ since $\theta < 1/q$. Hence

$$\begin{aligned}\sum_{k=0}^{\theta n} \binom{n}{k} (q-1)^{n-k} &\leq \binom{n}{\theta n}(q-1)^{n(1-\theta)} \sum_{k=0}^{\infty} \rho(\theta)^k \\ &\leq \frac{1}{1-\rho(\theta)} \binom{n}{\theta n} (q-1)^{n(1-\theta)}.\end{aligned}$$

Using Stirling's formula now to estimate $C(n,\theta n)$ gives the result. □

Next we estimate the left-hand side of (2) from below.

Lemma 8.

$$|P_A(\mathcal{S}')| \geq \frac{1}{2}\sqrt{\beta(1-\beta)2\pi n}\left[\beta^\beta(1-\beta)^{(1-\beta)}q^{1-\beta}\right]^n. \tag{4}$$

Proof.

$$|P_A(\mathcal{S}')| \geq \frac{|\mathcal{S}'|}{q^{\beta n}}$$

(by Lemma 6)

$$\geq \frac{|\mathcal{S}|}{C(n,\beta n)q^{\beta n}}$$

(by Lemma 5)

$$\geq \frac{\frac{1}{2}q^n}{C(n,\beta n)q^{\beta n}}$$

(by Lemma 2)

$$\gtrsim \frac{1}{2}\sqrt{\beta(1-\beta)2\pi n}\left[\beta^{\beta}(1-\beta)^{(1-\beta)}q^{1-\beta}\right]^n$$

by Stirling's Formula. □

Lemma 9. *Fix $\gamma \in (0,1)$ and let $\theta := \gamma/q$. Then, for all sufficiently large q, there exists $\beta \in (0,1)$ such that, for all sufficiently large n, $P_A(\mathcal{S}')$ has density θn in A. Moreover, we can choose $\beta \leq \theta$ and $\beta \sim \theta$ as $\theta \to 0$, i.e. as $q \to \infty$.*

Proof. If (2) is satisfied for $\theta = \gamma/q$, then $P_A(\mathcal{S}')$ will have density θn in A. From Lemmas 7 and 8, if β satisfies

$$\beta^{\beta}(1-\beta)^{(1-\beta)}q^{(1-\beta)} > \frac{(q-1)^{1-\theta}}{\theta^{\theta}(1-\theta)^{(1-\theta)}},$$

then (2) will be satisfied for all sufficiently large n. This is simply because, for fixed q and γ, the dominant terms in (3) and (4) are the exponential terms.

Taking logarithms of both sides, we require

$$\beta\ln(\beta) + (1-\beta)\ln(1-\beta) + (1-\beta)\ln(q)$$

$$> (1-\theta)\ln(q-1) - \theta\ln(\theta) - (1-\theta)\ln(1-\theta).$$

Note that $\theta \to 0$ as $q \to \infty$. Hence, if $\beta \leq \theta$, then

$$\lim_{q\to\infty} \beta\ln(\beta) = 0,$$

$$\lim_{q\to\infty} (1-\beta)\ln(1-\beta) = 0,$$

$$\lim_{q\to\infty} \theta\ln(\theta) = 0,$$

and

$$\lim_{q\to\infty} (1-\theta)\ln(1-\theta) = 0.$$

Hence, as $q \to \infty$, we simply require

$$(1-\beta)\ln(q) \gtrsim (1-\theta)\ln(q-1),$$

which is satisfied by some $\beta \leq \theta$ with $\beta \sim \theta$ as $q \to \infty$. □

For the rest of the proof we shall assume that $\theta = \gamma/q$, where $\gamma \in (0,1)$ is fixed, and that $\beta \sim \theta$ has been chosen in accordance with the previous lemma so that $P_A(\mathcal{S}')$ has density θn in A. Let $B \subseteq A$ be a set of full density for $P_A(\mathcal{S}')$ satisfying $|B| \geq \theta n$.

We require the following simple lemma about complex numbers.

Lemma 10. *Suppose that $|z| \leq 1$ and that $\Re[z] \geq (1-\alpha)$. Suppose also that $\xi = e^{i\phi}$, where $-2\pi/q \leq \phi \leq 2\pi/q$. Then*

$$\Re[z\xi] \geq (1-\alpha)\cos\left(\frac{2\pi}{q}\right) - \sqrt{2\alpha-\alpha^2}\sin\left(\frac{2\pi}{q}\right).$$

Proof. Let $z = x + iy$ and $\xi = \cos\phi + i\sin\phi$. Then $x = \Re[z] \geq (1-\alpha)$ and $y^2 \leq 1 - x^2 \leq 2\alpha - \alpha^2$. So

$$\Re[z\xi] = x\cos\phi - y\sin\phi \geq (1-\alpha)\cos\phi - \sqrt{2\alpha - \alpha^2}\sin\phi.$$

The worst case clearly occurs when $\phi = 2\pi/q$, which gives the result. □

Lemma 11. *Suppose that $(\xi_i)_{i\in B}$ are unimodular complex numbers, i.e. $|\xi_i| = 1$ for all $i \in B$. Then there exists $f^\phi \in S'$ such that, for all $i \in B$, we have*

$$\Re[f^\phi(\xi_i e_i)] \geq (1-\alpha)\cos\left(\frac{2\pi}{q}\right) - \sqrt{2\alpha - \alpha^2}\sin\left(\frac{2\pi}{q}\right).$$

Proof. Since B has full density in $\mathcal{S}'$ there exists $\phi \in \mathcal{S}'$ such that $\xi_i \in \phi_i$ for all $i \in B$. Thus,

$$\Re[f^\phi(\xi_i^\phi e_i)] = \Re[\xi_i^\phi f_i^\phi] \geq 1 - \alpha$$

for all $i \in B$. Since $|\arg(\xi_i) - \arg(\xi_i^\phi)| \leq 2\pi/q$, Lemma 10 gives

$$\Re[\xi_i f_i^\phi] \geq (1-\alpha)\cos\left(\frac{2\pi}{q}\right) - \sqrt{2\alpha - 2\alpha^2}\sin\left(\frac{2\pi}{q}\right)$$

for all $i \in B$. □

Now suppose that $(z_i)_{i\in B}$ is any collection of complex numbers. Let $z_i = |z_i|\xi_i$ be the polar decomposition of z_i. Then by Lemma 11

$$\begin{aligned}
\left\|\sum_{i\in B} z_i e_i\right\| &\geq \Re\left[f^\phi\left(\sum_{i\in B}|z_i|\xi_i e_i\right)\right] \\
&= \sum_{i\in B}\Re[\xi_i f_i^\phi]|z_i| \qquad (5)\\
&\geq \sum_{i\in B}\left[(1-\alpha)\cos\left(\frac{2\pi}{q}\right) - \sqrt{2\alpha-\alpha^2}\sin\left(\frac{2\pi}{q}\right)\right]|z_i|.
\end{aligned}$$

Now for the proof of the Theorem.

Proof of the Theorem. From the Taylor expansion we see that for large q and small α, we have

$$\begin{aligned}
(1-\alpha)\cos\left(\frac{2\pi}{q}\right) &- \sqrt{2\alpha-\alpha^2}\sin\left(\frac{2\pi}{q}\right) \\
&= 1 - \frac{2\pi^2}{q^2} - \alpha - \sqrt{2\alpha}\left(\frac{2\pi}{q}\right) + \text{smaller terms}.
\end{aligned}$$

Recall that $\beta \sim \theta = \gamma/q$ as $q \to \infty$. Hence

$$1 - \frac{2\pi^2}{q^2} - \alpha - \sqrt{2\alpha}\left(\frac{2\pi}{q}\right) = 1 - \frac{2\pi^2}{\gamma^2}\beta^2 - \alpha - \frac{2\pi\sqrt{2}}{\gamma}\beta\sqrt{\alpha} + \text{smaller terms}.$$

Now we choose values for our parameters. Set $\gamma = 0.999$. Provided ε is sufficiently small, we may choose $q := q(\varepsilon) \in \mathbb{N}$ such that Lemma 9 is satisfied and such that $\theta = \gamma/q$ and the β given by Lemma 9 satisfy the following:

$$\frac{0.99}{\pi\sqrt{2}}\sqrt{\varepsilon} \leq \beta \leq \theta \leq \frac{0.991}{\pi\sqrt{2}}\sqrt{\varepsilon}.$$

Set $\alpha = 4 \cdot 10^{-5}\varepsilon$. For these choices of parameters, we have

$$1 - \frac{2\pi^2}{\gamma^2}\beta^2 - \alpha - \frac{2\pi\sqrt{2}}{\gamma}\beta\sqrt{\alpha} \geq 1 - 0.999\varepsilon.$$

It follows that, for all sufficiently large n (depending on ε),

$$|B| \geq \theta n \geq (0.99/(\pi\sqrt{2}))\sqrt{\varepsilon}\, n,$$

and from (5) that for all $(z_i)_{i\in B}$, we have

$$\left\| \sum_{i\in B} z_i e_i \right\| \geq (1-\varepsilon) \sum_{i\in B} |z_i|$$

for all sufficiently small ε. From Lemma 2 we see that for all sufficiently large n we can take $\delta = 1 - \alpha\beta/2 \leq 1 - 4.45 \cdot 10^{-5}\varepsilon^{3/2}$. Thus we have shown that for all sufficiently small ε there exists an integer $N(\varepsilon)$ such that the Theorem holds for all $n \geq N(\varepsilon)$ with $\delta = 1 - 4.45 \cdot 10^{-5}\varepsilon^{3/2}$.

It remains to dispose of the case $n < N(\varepsilon)$. But in this case an easy triangle inequality calculation which we omit shows that there exists $\delta' := \delta'(\varepsilon) < 1$ such that if $n < N(\varepsilon)$ and $\mathbb{E}\|\sum_{i=1}^n Z_i e_i\| \geq \delta' n$, then $\|\sum_{i=1}^n \xi_i e_i\| \geq n - \varepsilon$ for all unimodular complex numbers $(\xi_i)_{i=1}^n$. Let $(z_i)_{i=1}^n$ be complex numbers satisfying $\sum_{i=1}^n |z_i| = 1$, and let $z_i = |z_i|\xi_i$ be the polar decomposition of z_i. Then

$$\begin{aligned}
\left\| \sum_{i=1}^n z_i e_i \right\| &\geq \left\| \sum_{i=1}^n \xi_i e_i \right\| - \left\| \sum_{i=1}^n (1-|z_i|)\xi_i e_i \right\| \\
&\geq n - \varepsilon - \sum_{i=1}^n (1-|z_i|) \\
&= 1 - \varepsilon = (1-\varepsilon)\sum_{i=1}^n |z_i|.
\end{aligned}$$

Thus, for $n < N(\varepsilon)$, we can take $\delta = \delta'$ and $B = \{1, \ldots, n\}$, which completes the proof. □

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