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Generalizations of the Graham-Pollak Tree Theorem

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by

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Dedication

To Kay Menchel.

You are dearly missed every day.

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Abstract

Graham and Pollak showed in 1971 that the determinant of a tree's distance matrix depends only on its number of vertices, and, in particular, it is always nonzero. This dissertation will generalize their result via two different directions: Steiner distance k-matrices and distance critical graphs. The Steiner distance of a collection of k vertices in a graph is the fewest number of edges in any connected subgraph containing those vertices; for $k = 2$, this reduces to the ordinary definition of graphical distance. Here, we show that the hyperdeterminant of the Steiner distance k-matrix is always zero if k is odd and nonzero if k is even, extending the result beyond $k = 2$. We conjecture that not just the vanishing, but the value itself, of the Steiner distance k-matrix hyperdeterminant of an n-vertex tree depends only on k and n. We further introduce new techniques to prove that the distance matrix (the $k = 2$ case) of a tree has a nonzero determinant, thus providing weaker versions of the Graham-Pollak Theorem. In the second half of the dissertation, we focus on distance critical graphs. In a tree, standard distance is measured by the unique path connecting two vertices. The distance measure, however, may be obtained from multiple paths if the graph is not a tree. A distance critical graph is a connected graph such that no vertex can be deleted without altering the distance metric on the remaining vertices. We generalize this unique path distance concept that holds within a tree by introducing and discussing properties of distance critical graphs.

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Chapter 1

INTRODUCTION

Distance matrices have served as an integral tool in graph theory since their introduction in 1971 by Graham and Pollak. These matrices helped solve the loop switching problem in which messages pass through a communication network by following the shortest path, thus reducing the hold time. Since then, distance matrices have found their way into spectral graph theory and have helped determine important structural properties of graphs by simply analyzing their spectrum. These results have then been applied to chemistry, network analysis, computer science, and combinatorics.

A *tree* is defined as an undirected graph in which any two vertices are connected by exactly one path, or equivalently a connected acyclic undirected graph. In addition to solving the loop switching problem, Graham and Pollak showed that the determinant of the distance matrix of a tree T on n vertices – the $n \times n$ matrix whose each $(v, w) \in V(T) \times V(T)$ entry is the ordinary graph distance between v and w – depends only on n. In fact, they gave a formula: $-(n-1)(-2)^{n-2}$ [16]. Others have viewed this problem and come up with different proofs. One such example was discovered by Yan and Yeh [28] that includes an elegant proof relying on the fact that two pendant vertices can be deleted from a tree and the remaining entries in the distance matrix are unchanged. Further research led to the conclusion that weighted trees could be considered. Bapat, Kirkland, and Neumann [2] proved that if T is a weighted tree on n vertices with edge weights α_i, for $i = 1, ..., n-1$ and D is the distance matrix of

T, then

$$\det(D) = (-1)^{n-1} 2^{n-2} \left(\prod_{i=1}^{n-1} \alpha_i \right) \left(\sum_{i=1}^{n-1} \alpha_i \right).$$

These techniques were then expanded upon to analyze the distance spectra of other types of graphs besides trees. See [1] for more of these results.

The first half of this dissertation will answer Y. Mao's question [1] as to whether the Graham-Pollak result can be extended to "Steiner distance", a generalization of distance introduced by Hakimi [18] and popularized by [5]. The *Steiner distance* $d_G(S)$ of a set $S \subseteq V(G)$ of vertices is the fewest number of edges in a connected subgraph of G containing all of S. Note that, if $S = \{v, w\}$, this reduces to the classical definition of the distance from v to w, since a connected graph of smallest size containing v and w is a path of length $d_G(v, w) := d_G(\{v, w\})$. (See [22] for an extensive survey on Steiner distance.)

Just as all pairwise distances in a graph can be represented by a symmetric matrix, we can write the Steiner distances of all k-tuples of vertices as an order-k *hypermatrix* (sometimes referred to as a *tensor*): the $\overbrace{[n] \times \cdots \times [n]}^{k}$ (super-)symmetric integer array whose $(v_1, \ldots, v_k)$ entry is the Steiner distance of $\{v_1, \ldots, v_k\}$. We sometimes refer to such hypermatrices as "cubical" since all the index sets are identical. There is a notion of hyperdeterminant that generalizes determinant, and shares many of its properties, though in general is much harder to compute. See, for example, [23] for discussion of the symmetric hyperdeterminant. For our purposes, what will matter about the hyperdeterminant is that it detects nontrivial simultaneous vanishing of a system of degree-$(k-1)$ homogeneous polynomials (a.k.a. $(k-1)$-forms) in n variables. We will discuss the techniques for computing this system of polynomials in Chapter

[1] Personal communication.

2 as well as how we use this to classify the hyperdeterminant of the Steiner distance k-matrix.

In Chapter 3, we prove the results for the odd order Steiner distance k-matrices as well as go into the specific structure of the nullvariety for the case $k = 3$. Chapter 4 will prove the results for the even order Steiner distance k-matrices as well as introduce some new variations for the $k = 2$ case to append to the list of proofs for the Graham-Pollak theorem. Distance critical graphs will be introduced and discussed in Chapter 5 and will compose the second half of the dissertation. We named these graphs in an attempt to use the techiniques of Yan and Yeh to classify the determinant of the distance matrix for some other class of graphs besides trees. We will leave the reader in Chapter 6 with open problems in these areas of research.

Chapter 2

Hypermatrix Preliminaries

The hypermatrix is a generalization of the standard order-2 (or $n \times n$) matrix. In 1843, Cayley introduced the idea of calculating the hyperdeterminant [3]; however, this definition can only be applied to hypercubes having an even number of dimensions. Cayley expanded his definition two years later to eliminate these restrictions and introduce what is commonly referred to as Cayley's second hyperdeterminant [4]. For the remainder of this paper, we will be referencing this second hyperdeterminant. While groundbreaking, much of Cayley's work was limited to the $2 \times 2 \times 2$ hypermatrix and was largely forgotten for about 140 years until the release of [14]. This book reignited interest in hyperdeterminants and specifically discovered a condition for when a hyperdeterminant is non-trivial for higher order hypermatrices.

2.1 Background

Before introducing this result, we define some terms using much of the notation from [8].

Definition 2.1. *A (cubical)* hypermatrix $\mathcal{A}$ *over a set* $\mathbb{S}$ *of dimension-n and order-k is a collection of* n^k *elements* $a_{i_1 i_2 \ldots i_k} \in \mathbb{S}$ *where* $i_j \in [n]$.

Definition 2.2. *A hypermatrix is said to be* symmetric *if entries which use the same index sets are the same. That is,* $\mathcal{A}$ *is symmetric if* $a_{i_1 i_2 \ldots i_k} = a_{i_{\sigma(1)} i_{\sigma(2)} \cdots i_{\sigma(k)}}$ *for all permutations* σ *of* $[k]$.

An order-k dimension-n symmetric hypermatrix $\mathcal{A}$ uniquely defines a homogeneous degree k polynomial in n variables (a.k.a. a "k-form") by

$$f_{\mathcal{A}}(\mathbf{x}) = \sum_{i_1,i_2,\ldots,i_k=1}^{n} a_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

If we write $\mathbf{x}^k$ for the order-k dimension-n hypermatrix with $i_1, i_2, \ldots, i_k$ entry $x_{i_1} x_{i_2} \ldots x_{i_k}$, then the expression above can be written as $f_{\mathcal{A}}(\mathbf{x}) = \mathcal{A}\mathbf{x}^k$. We next introduce the resultant which will be required to compute Cayley's second hyperdeterminant.

Theorem 2.3 (The Resultant, [14])**.** *Fix degrees $d_1, d_2, \ldots, d_n$. For $i \in [n]$, consider all monomials $\mathbf{x}^{\alpha}$ of total degree d_i in $x_1, \ldots, x_n$. For each such monomial, define a variable $u_{i,\alpha}$. Then there is a unique polynomial RES $\in \mathbb{Z}[\{u_{i,\alpha}\}]$ with the following three properties:*

(1) If $F_1, \ldots, F_n \in \mathbb{C}[x_1, \ldots, x_n]$ are homogeneous polynomials of degrees $d_1, \ldots, d_n$, respectively, then the polynomials have a non-trivial common root in $\mathbb{C}^n$ exactly when RES$(F_1, \ldots, F_n) = 0$. Here, RES$(F_1, \ldots, F_n)$ is interpreted to mean substituting the coefficient of $\mathbf{x}^{\alpha}$ in F_i for the variable $u_{i,\alpha}$ in RES.

(2) RES$(x_1^{d_1}, \ldots, x_n^{d_n}) = 1$.

(3) RES is irreducible, even in $\mathbb{C}[\{u_{i,\alpha}\}]$.

Moreover, for $i \in [n]$, RES is homogeneous in the variable $\{u_{i,\alpha}\}$ with degree

$$\prod_{j \in [n], j \neq i} d_i.$$

Definition 2.4. *The* symmetric hyperdeterminant *of $\mathcal{A}$, denoted* $\det(\mathcal{A})$*, is the resultant of the polynomials in $\nabla f_{\mathcal{A}}(\mathbf{x})$ and $f_{\mathcal{A}}(\mathbf{x})$.*

Definition 2.5. *The* characteristic polynomial *of a hypermatrix $\mathcal{A}$ is calculated as $\phi_{\mathcal{A}}(\lambda) = \det(\lambda\mathcal{I} - \mathcal{A})$ where λ is an indeterminate and $\mathcal{I}$ is the identity hypermatrix.*

See [23] for more of a discussion on the symmetric hyperdeterminant.

While the hyperdeterminant is much more complex to evaluate than the standard determinant, there are some similarities between the two. For example, Qi shows in [23] that the hyperdeterminant is the constant term of the characteristic polynomial. Other similarities are described in the following theorem. The term *slice* is used to represent the indices of the hypermatrix with one fixed component.

Theorem 2.6 ([14])**.** *Let $\mathcal{A}$ be a hypermatrix.*

(a) Interchanging two parallel slices leaves the hyperdeterminant invariant up to sign (which may equal 1).

(b) The hyperdeterminant is a homogeneous polynomial in the entries of each slice. The degree of homogeneity is the same for parallel slices.

(c) The hyperdeterminant does not change if we add to some slice a scalar multiple of a parallel slice.

(d) The hyperdeterminant of a matrix having two parallel slices proportional to each other is equal to 0. In particular, $\det(\mathcal{A}) = 0$ if $\mathcal{A}$ has a zero slice.

With this background, we next introduce the result from [14] stating when a hyperdeterminant is non-trivial. This theorem will be critical for the results in the remainder of this paper.

Theorem 2.7. *The hyperdeterminant $\det(\mathcal{A})$ of the order-k, dimension-n hypermatrix $\mathcal{A} = (a_{i_1,\ldots,i_k})_{i_1,\ldots,i_k=1}^{n}$ is a monic irreducible polynomial which evaluates to zero iff* *there is a nonzero simultaneous solution to* $\nabla f_{\mathcal{A}} = \vec{0}$*, where*

$$f_{\mathcal{A}}(x_1, \ldots, x_n) = \sum_{i_1,\ldots,i_k} a_{i_1,\ldots,i_k} \prod_{j=1}^{k} x_{i_j}.$$

2.2 Computing Hyperdeterminant

While it is beneficial to know when a hyperdeterminant is non-trivial, we also hoped to calculate this value. Cox, Little, and O'Shea provide an algorithm in [10] to calculate the characteristic polynomial of a hypermatrix which in turn will allow us to determine the hyperdeterminant from the constant term. We present the algorithm here.

To compute $\mathrm{RES}(F_1, F_2, \ldots, F_n)$ for an order-k hypermatrix, let $d = n(k-1)-n+1$ and S be the set of all monomials of degree d in the variables $x_1, \ldots, x_n$. (We denote such a monomial $\mathbf{x}^\alpha$, where $\mathbf{x}$ stands for a variable vector, and α stands for an exponent vector.) Let

$$\begin{aligned} S_1 &= \{\mathbf{x}^\alpha \in S | x_1^{k-1} \text{ divides } \mathbf{x}^\alpha\} \\ S_2 &= \{\mathbf{x}^\alpha \in S \backslash S_1 | x_2^{k-1} \text{ divides } \mathbf{x}^\alpha\} \\ &\vdots \\ S_n &= \{\mathbf{x}^\alpha \in S \backslash \cup_{i=1}^{n-1} S_i | x_n^{k-1} \text{ divides } \mathbf{x}^\alpha\}. \end{aligned}$$

This collection forms a partition of S (using the pigeon-hole principle). Fix an arbitrary ordering on S, and define the $|S| \times |S|$ matrix M as follows. The (α, β) entry of M is the coefficient of $\mathbf{x}^\beta$ in the polynomial $F_i(x)\dfrac{\mathbf{x}^\alpha}{x_i^{k-1}}$, where i is the unique index such that $\mathbf{x}^\alpha \in S_i$. In particular, any non-zero (α, β) entry is one of the coefficients of F_i where i has $\mathbf{x}^\alpha \in S_i$.

Call a monomial $\mathbf{x}^\alpha \in S$ reduced if there is exactly one i so that x_i^{k-1} divides $\mathbf{x}^\alpha$. Form the matrix M' by deleting the rows and columns of M that correspond to reduced monomials. The resultant of the system is then $\det(M)/\det(M')$, provided that the denominator does not vanish. In our case, each determinant is actually a characteristic polynomial, so this is never an issue.

Using Dutle's code made available in [13] as a framework, we evaluated the characteristic polynomials of specific hypermatrices at $\lambda = 0$ to determine the hyperdeterminant as desired.

Chapter 3

Odd Order Steiner k-matrices

In an attempt to generalize the Graham-Pollak theorem, we extend the notion of regular distance to consider Steiner distances. The *Steiner distance* $d_G(S)$ of a set $S \subseteq V(G)$ of vertices is the fewest number of edges in a connected subgraph of G containing all of S. Note that there is a choice to be made in generalizing distance matrices: instead of $d_G(S)$, we could also simply set the entries corresponding to vertex sets S of cardinality less than k to zero. However, doing so yields a hyperdeterminant of zero *irrespective of the non-degenerate entries*, as we now show. For a hypermatrix $\mathcal{A} \in \mathbb{C}^{S \times \cdots \times S}$, call an entry $\mathcal{A}(i_1, \ldots, i_k)$ "degenerate" if $|\{i_1, \ldots, i_k\}| < k$.

Theorem 3.1 ([9])**.** *Let $\mathcal{A}$ be any cubical hypermatrix with all degenerate entries set equal to* 0*. Then the hyperdeterminant of $\mathcal{A}$ is* 0*.*

Proof. To prove the hyperdeterminant is 0, we exhibit a nontrivial simultaneous zero of the partial derivatives of the k-form

$$f_{\mathcal{A}}(\mathbf{x}) = \sum_{i_1, i_2, \ldots, i_k = 1}^{n} a_{i_1 i_2 \cdots i_k} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Since $\mathcal{A}$ has degenerate entries set equal to zero, any term that has $i_p = i_q$ for some $p, q \in [k]$ will have a matrix entry of zero and thus will not appear in the polynomial. Therefore, the only terms that will appear are $x_{i_1} x_{i_2} \cdots x_{i_k}$ with each i_p distinct. The gradient vector of these polynomials will consist of terms of degree $k - 1$ where once again each i_p is distinct. Therefore, choose $x_1 = x_2 = \cdots = x_{n-1} = 0$ and let x_n be any nonzero value; this is a nontrivial point where all partial derivatives vanish, so that the hyperdeterminant is 0. □

So, instead, we use positive Steiner distance to populate all entries of the hypermatrix. This is made precise as follows.

Definition 3.2. *Given a graph G and a subset S of the vertices, the* Steiner distance *of S, written $d_G(S)$ or $d_G(v_1, \dots, v_k)$ where $S = \{v_1, \dots, v_k\}$, is the number of edges in the smallest connected subgraph of G containing $S = \{v_1, \dots, v_k\}$. Since such a connected subgraph of G witnessing $d_G(S)$ is necessarily a tree, it is called a* Steiner tree *of S.*

Definition 3.3. *Given a graph G, the* Steiner polynomial *of G is the k-form*

$$p_G^{(k)}(\mathbf{x}) = \sum_{v_1,\dots,v_k \in V(G)} d_G(v_1, \dots, v_k) x_1 \cdots x_k$$

where we often suppress the subscript and/or superscript on $p_G^{(k)}$ if it is clear from context.

Equivalently, we could define the Steiner k-form to be the k-form associated with the Steiner distance hypermatrix:

Definition 3.4. *Given a graph G, the* Steiner k-matrix *(or just "Steiner hypermatrix" if k is understood) of G is the order-k, cubical hypermatrix $\mathcal{S}_G$ of dimension-n whose $(v_1, \dots, v_k)$ entry is $d_G(v_1, \dots, v_k)$.*

Throughout the sequel, we write D_r for the operator $\partial/\partial x_r$, and we always assume that T is a tree.

Definition 3.5. *Given a graph G on n vertices, the* Steiner k-ideal *– or just "Steiner ideal" if k is clear – of G is the ideal in $\mathbb{C}[x_1, \dots, x_n]$ generated by the polynomials $\{D_j p_G\}_{j=1}^n$.*

Thus, the Steiner ideal is the *Jacobian ideal* of the Steiner polynomial of G.

Definition 3.6. *A* Steiner nullvector *is a point where all the polynomials within the Steiner ideal vanish. The set of all Steiner nullvectors – a projective variety – is the* Steiner nullvariety*.*

3.1 $k =$ Odd Proof

While the problem of finding the Steiner distance of a set of vertices in a graph is NP-complete [22], we can easily determine that distance when the graph is a tree. Definition 3.2 states $d_G(S)$ is witnessed when the subgraph containing the vertices of S is a tree. The smallest subtree containing S will then occur when the leaf vertices of the subtree are all elements in S. Therefore, we can iterate through the leaves of the original tree and remove every leaf that does not lie within S until we obtain the smallest subtree, thus also obtaining the Steiner distance.

Using this method, we adjusted Dutle's code [13] to create the Steiner distance k-matrix of a tree and evaluate the characteristic polynomial when $\lambda = 0$. The results led to the following conjecture.

Conjecture 1. *The order-k Steiner distance hypermatrix of a tree T on $n \geq 3$ vertices has a hyperdeterminant that only depends on T through n, and is 0 iff k is odd.*

Below, we show that this conjecture holds for all odd k, when the hyperdeterminant is 0 irrespective of the choice of T. We then go on to describe the Steiner nullvariety for $k = 3$. These results appear in [9].

Theorem 3.7. *For k odd, the Steiner distance k-matrix of a tree T with at least 3 vertices has a hyperdeterminant equal to zero.*

Proof. Since T has at least 3 vertices, let u be a leaf, w a neighbor of u, and $v \neq u$ a neighbor of w. Let $\mathbf{x}$ denote the vector whose z coordinate x_z is given by

$$x_z = \begin{cases} 1 & \text{if } z = u \\ \zeta & \text{if } z = v \\ -1-\zeta & \text{if } z = w \\ 0 & \text{otherwise,} \end{cases}$$

where $\zeta = \exp(\pi i/(k-1))$, a $(2k-2)$-root of unity. By Theorem 2.7, it suffices to show that $D_z p_T(\mathbf{x}) = 0$ for each $z \in V(T)$. First, suppose v is not on the $u - z$ path in T and $z \neq u$ (which includes the case $z = w$). Let $\alpha = d_T(z,u,v,w)$, so that

$$\begin{aligned}
\frac{1}{k} D_z p_T(\mathbf{x}) &= \sum_{a+b+c=k-1} x_u^a x_v^b x_w^c \binom{k-1}{a,b,c} d_T(z,u,v,w) \\
&+ \sum_{a+c=k-1} x_u^a x_w^c \binom{k-1}{a,c} (d_T(z,u,w) - d_T(z,u,v,w)) \\
&+ \sum_{b+c=k-1} x_v^b x_w^c \binom{k-1}{b,c} (d_T(z,v,w) - d_T(z,u,v,w)) \\
&+ x_w^{k-1}(d_T(z,u,v,w) - d_T(z,u,w) - d_T(z,v,w) + d_T(z,w)) \\
&= \alpha(x_u + x_v + x_w)^{k-1} - (x_u + x_w)^{k-1} - (x_v + x_w)^{k-1} \\
&= 0 - (-\zeta)^{k-1} - (-1)^{k-1} = 0.
\end{aligned}$$

Next, if v is on the $u - z$ path in T and $z \notin \{u, w\}$, we obtain

$$\begin{aligned}
\frac{1}{k} D_z p_T(\mathbf{x}) &= \sum_{a+b+c=k-1} x_u^a x_v^b x_w^c \binom{k-1}{a,b,c} d_T(z,u,v,w) \\
&+ \sum_{b+c=k-1} x_v^b x_w^c \binom{k-1}{b,c} (d_T(z,v,w) - d_T(z,u,v,w)) \\
&+ x_v^{k-1}(d_T(z,v) - d_T(z,v,w)) \\
&= \alpha(x_u + x_v + x_w)^{k-1} - (x_v + x_w)^{k-1} - x_v^{k-1} \\
&= 0 - (-1)^{k-1} - (\zeta)^{k-1} = 0.
\end{aligned}$$

Finally, if $z = u$, then

$$\begin{aligned}
\frac{1}{k} D_z p_T(\mathbf{x}) &= \sum_{a+b+c=k-1} x_u^a x_v^b x_w^c \binom{k-1}{a,b,c} d_T(u,v,w) \\
&+ \sum_{a+c=k-1} x_u^a x_w^c \binom{k-1}{a,c} (d_T(u,w) - d_T(u,v,w)) \\
&+ x_u^{k-1}(d_T(u) - d_T(u,w)) \\
&= 2(x_u + x_v + x_w)^{k-1} - (x_u + x_w)^{k-1} - x_u^{k-1} \\
&= 0 - (-\zeta)^{k-1} - 1^{k-1} = 0.
\end{aligned}$$

□

To help visualize the Steiner distance k-matrix, we include the example of the unique tree T on 3 vertices, a path of length 2 with leaves 1 and 3, and $k = 3$. Let

$$\mathcal{S}_T = \left[M_1 \,\middle|\, M_2 \,\middle|\, M_3 \right]$$

represent the Steiner distance 3-matrix of T. With this notation, M_i represents the i^{th} "slice" of the hypermatrix, the matrix whose (j_1, j_2) entry is $d_T(i, j_1, j_2)$. Therefore,

$$M_1 = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 2 \end{bmatrix}, \quad M_2 = \begin{bmatrix} 1 & 1 & 2 \\ 1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}, \quad M_3 = \begin{bmatrix} 2 & 2 & 2 \\ 2 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}.$$

Now, we consider the Steiner 3-form:

$$p_T = 3x_1^2 x_2 + 6x_1^2 x_3 + 3x_1 x_2^2 + 3x_2^2 x_3 + 6x_1 x_3^2 + 3x_2 x_3^2 + 12x_1 x_2 x_3,$$

which has a gradient given by

$$\nabla p_T = \begin{bmatrix} 6x_1 x_2 + 12x_1 x_3 + 3x_2^2 + 6x_3^2 + 12x_2 x_3 \\ 3x_1^2 + 6x_1 x_2 + 6x_2 x_3 + 3x_3^2 + 12x_1 x_3 \\ 6x_1^2 + 3x_2^2 + 12x_1 x_3 + 6x_2 x_3 + 12x_1 x_2 \end{bmatrix}.$$

It is easy to check the vector $\mathbf{x} = [1, -1-i, i]$ is a zero of every polynomial in the gradient; therefore, $\mathcal{S}_T$ has a hyperdeterminant of 0.

Note that the hyperdeterminant of a tree on one vertex is also zero. This is because the Steiner k-form, $p_T^{(k)}(\mathbf{x})$, only contains one monomial: $d_T(1, \ldots, 1)x_1^k$. Since $d_T(1, \ldots, 1) = 0$, the Steiner k-form as well as the partial derivative is automatically 0, and so the Steiner nullvector $\mathbf{x} = [x_1]$ can be set to anything.

For the tree on two vertices and odd k, the hyperdeterminant is not always zero. It is straightforward to write the Steiner k-form as $p_T^{(k)}(\mathbf{x}) = (x_1 + x_2)^k - x_1^k - x_2^k$. The partial derivatives are therefore $D_j p_T^{(k)}(\mathbf{x}) = k(x_1 + x_2)^{k-1} - kx_j^{k-1}$ for $j = 1, 2$, so $D_1 p_T^{(k)}(\mathbf{x}) = D_2 p_T^{(k)}(\mathbf{x}) = 0$ implies $x_1^{k-1} = (x_1 + x_2)^{k-1} = x_2^{k-1}$. Then, since $k-1$ is even, $x_2 = \pm\zeta x_1$, where $\zeta^{k-1} = 1$, but if $x_1 \neq 0$ this implies

$$1 = (x_1 + x_2)^{k-1}/x_1^{k-1} = (1 \pm \zeta)^{k-1}.$$

This holds true only when $k \equiv 1 (\bmod 6)$. This can be viewed geometrically by considering which roots of unity are the same when 1 is added to them. In other words, we are looking for the intersections between the unit circle and the unit circle with 1 added to every value. A picture of this is shown below, proving that the equality only holds if $k \equiv 1 (\bmod 6)$.

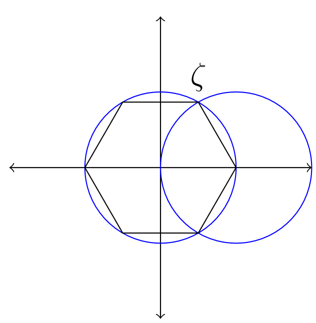

Figure 3.1 Intersection of unit circles

Therefore, for $k \equiv 1 (\mathrm{mod}\ 6)$, $\mathbf{x} = [x_1, \pm\zeta x_1]$ is a nontrivial nullvector for $x_1 \neq 0$ and the hyperdeterminant must be zero. For $k \not\equiv 1 (\mathrm{mod}\ 6)$, the hyperdeterminant is nonzero since the only nullvector is the trivial one.

3.2 $k = 3$ Nullvariety

Now that we have established that all Steiner hyperdeterminants of odd order with $n \geq 3$ are zero, we describe in more detail the corresponding Steiner nullvariety, at least for order $k = 3$. The following simple lemma appears as a special case of Theorem 3 in [17].

Lemma 3.8. *For any distinct vertices i, j, k of a tree T, we have*

$$2d_T(i,j,k) = d_T(i,j) + d_T(i,k) + d_T(j,k).$$

Proof. It is easy to check the formula for each of the two cases: either the Steiner tree of $\{i, j, k\}$ is a path or a tree with three leaves. $\square$

The following result shows that $p_T^{(3)}(\mathbf{x})$ is divisible by the elementary symmetric polynomial of degree 1, which we refer to as s.

Proposition 3.9. *The Steiner 3-form $p_T^{(3)}(\mathbf{x})$ is divisible by $s = \sum_r x_r$.*

Proof. Let $p = p_T^{(3)}(\mathbf{x})$. If s divides p, then $p = sg$ for some polynomial g. We claim $g = 3\sum_{i<j} d_T(i,j)x_ix_j$. We show that

$$p = sg = \sum_r x_r \left(3 \sum_{i<j} d_T(i,j)x_ix_j\right) = \sum_{r,i<j} 3d_T(i,j)x_ix_jx_r$$

holds by classifying summands according to the triple (r, i, j).

- If $r = i$, the contribution is $3\sum_{i<j} d_T(i,j)x_i^2x_j$.

- If $r = j$, the contribution is $3\sum_{i<j} d_T(i,j)x_i x_j^2$.

- If $r \neq i, j$, then the contribution becomes

$$\begin{aligned}
3\sum_{\substack{i<j\\ r\neq i,j}} d_T(i,j)x_i x_j x_r &= 3\sum_{i<j<k}[d_T(i,j)+d_T(i,k)+d_T(j,k)]x_i x_j x_k \\
&= 6\sum_{i<j<k} d_T(i,j,k)x_i x_j x_k \\
&= \sum_{i,j,k \text{ distinct}} d_T(i,j,k)x_i x_j x_k
\end{aligned}$$

where the second equality follows from Lemma 3.8. On the other hand,

$$p = 3\sum_{i\neq j} d_T(i,j)x_i^2 x_j + \sum_{i,j,k \text{ distinct}} d_T(i,j,k)x_i x_j x_k,$$

which agrees with the sum of the three types of terms in sg. □

Theorem 3.10. *If $\boldsymbol{x} = (x_1, \cdots, x_n)$ is a Steiner nullvector of order 3 and $s = \sum_{i=1}^n x_i$, then s^3 lies within the Steiner ideal J.*

Proof. We can write $p = gs$, where $p = p_T^{(3)}(\mathbf{x})$, $s = \sum_i x_i$, and $g = 3\sum_{i<j} d_T(i,j)x_i x_j$. Thus, writing D_r for differentiation with respect to x_r, we obtain

$$D_r p = g + sD_r g.$$

Then

$$\begin{aligned}
\sum_r x_r D_r p &= \sum_r x_r(g + sD_r g) \\
&= g\sum_r x_r + s\sum_r x_r D_r g \\
&= s(g + \sum_r x_r D_r g).
\end{aligned}$$

Now,

$$\begin{aligned}\sum_r x_r D_r g &= 3\sum_r x_r D_r \left(\sum_{i<j} d_T(i,j)x_i x_j\right)\\ &= 3\sum_r x_r \sum_j d_T(r,j)x_j\\ &= 6\sum_{r<j} d_T(r,j)x_r x_j = 2g.\end{aligned}$$

Putting these together gives that $\sum_r x_r D_r p = s(g+2g) = 3sg$. So sg is in the Steiner ideal $J = \langle \{D_r p\}_r \rangle$. Since $D_r p = g + sD_r g \in J$, we also have $s(g + sD_r g) = sg + s^2 D_r g \in J$, and so also $sg + s^2 D_r g - sg = s^2 D_r g \in J$.

Now, $D_r g = 3\sum_j d_T(j,r)x_j$. In other words, $\nabla g = Mx$, where M denotes the (symmetric) distance matrix of the tree and x is the vector of all variables. By the Graham-Pollak Theorem, M is invertible for trees, so $yM = \vec{1}$ has a solution (where $\vec{1}$ is the all-ones row vector). Let the solution be $y = (c_1, \ldots, c_n)$. Then

$$y\nabla g = yMx = \vec{1}x = s$$

i.e., $\sum_r c_r D_r g = s$. Thus, $\sum_r c_r s^2 D_r g = s^3 \in J$. $\square$

In fact, tracing back through the computation gives $s^3 = \sum_r f_r D_r p$ where

$$f_r = c_r s - \frac{x_r}{3}\sum_j c_j.$$

It is not hard to deduce from Proposition 3.15 below that $s^2 \notin J$.

Corollary 3.11. *If $\boldsymbol{x}$ is a Steiner nullvector, then the sum of the coordinates of $\boldsymbol{x}$ is 0.*

Proof. Since $s^3 \in J$, we have $s \in \sqrt{J}$. Therefore, if $\mathbf{x}$ is in the Steiner nullvariety, then $s(\mathbf{x}) = 0$, i.e., the coordinates of $\mathbf{x}$ sum to 0. $\square$

Theorem 3.12 ([15] Lemma 1)**.** *Let T be a tree with vertex set $[n]$, let d_j be the degree of vertex j, and let a_{ij} be the indicator function that $ij \in E(T)$. If D is the distance matrix of T and the ij-entry of D^{-1} is d^*_{ij}, then*

$$d^*_{ij} = \frac{(2-d_i)(2-d_j)}{2(n-1)} + \begin{cases} -d_i/2 & \text{if } i = j \\ a_{ij}/2 & \text{if } i \neq j \; . \end{cases}$$

Proposition 3.13. *$c_r = (2-d_r)/(n-1)$ and $\sum_r c_r = 2/(n-1)$.*

Proof. Let M be the distance matrix of T. Since $yM = \vec{1}$ and M is invertible,

$$y = \vec{1}M^{-1}.$$

Therefore, applying Theorem 3.12, we can write

$$\begin{aligned} c_r &= \sum_j \left(\frac{(2-d_r)(2-d_j)}{2(n-1)} + \begin{cases} -d_r/2 & \text{if } r = j \\ a_{rj}/2 & \text{if } r \neq j \end{cases} \right) \\ &= \frac{2-d_r}{2(n-1)} \sum_j (2-d_j) - \frac{d_r}{2} + \frac{d_r}{2} \\ &= \frac{2-d_r}{2(n-1)}(2n - 2(n-1)) = \frac{2-d_r}{n-1}. \end{aligned}$$

Thus,

$$\sum_r c_r = \sum_r \frac{2-d_r}{n-1} = \frac{1}{n-1}(2n-2(n-1)) = \frac{2}{n-1}.$$

□

Corollary 3.14. *$s^3 = \sum_r f_r D_r p$ where*

$$f_r = \frac{1}{n-1}\left((2-d_r)s - \frac{2}{3}x_r \right).$$

So, s is in the radical $\sqrt{J}$ of J, and we can write s^3 (but not s^2) in terms of the generators of J. In particular, the codimension of the Steiner nullvariety is at least one. The next few results show that the codimension is in fact, 2.

Proposition 3.15. *The polynomials $D_r p$ are not divisible by s.*

Proof. Suppose $s|D_r p$. Then, since $p = gs$, we have $D_r p = g + sD_r g$, so $s|g$ as well. But, g is quadratic, so there exist $a_1, \ldots, a_n \in \mathbb{C}$ so that $g = s \sum_r a_r x_r$, i.e.,

$$g = \sum_{i,j} a_i x_i x_j.$$

The x_i^2 term on the right-hand side is $a_i x_i^2$, but the corresponding coefficient on the left-hand side is 0, so $a_i = 0$ for each i. Then $\sum_r a_r x_r = 0$, so $g = 0$, a contradiction. $\square$

Theorem 3.16. *The codimension of an order-3 Steiner nullvariety of a tree is 2.*

Proof. If J is the Steiner ideal, then, by the previous result, $\langle s \rangle \subsetneq \langle s, g \rangle \subseteq \sqrt{J}$. On the other hand, $D_r p = g + sD_r g \in \langle g, s \rangle$, so $\sqrt{J} = \langle s, g \rangle$. $\square$

In fact, we can go even further: for every assignment of values to $n-2$ vertices, there is an assignment to the last two vertices that yields a Steiner nullvector:

Proposition 3.17. *For any tree T on n vertices and $n-2$ values $a_3, \ldots, a_n \in \mathbb{C}$, there exist a_1, a_2 so that $(a_1, \ldots, a_n)$ is a Steiner nullvector: a_1 is any solution to*

$$Aa_1^2 + Ba_1 + C = 0,$$

where $A = d_T(1,2)$, $B = \sum_{j \geq 3}(d_T(1,2) - d_T(1,j) + d_T(2,j))a_j$, and C can be calculated as $\sum_{j,k \geq 3}(d_T(2,j) - \frac{1}{2}d_T(j,k))a_j a_k$; and $a_2 = -a_1 - \sum_{j=3}^n a_j$.

Proof. Assume $v = (a_1, \cdots, a_n)$ is a nullvector where $a_3, \cdots, a_n$ are arbitrary complex number. Since v is a nullvector, Corollary 3.11 states that $\sum_{j=1}^n a_j = 0$. Therefore, $a_2 = -a_1 - \sum_{j=3}^n a_j$.

Also, since v is a nullvector, by definition all partial derivatives to the Steiner 3-form must vanish. Notice by Proposition 3.9, $D_r p = D_r(sg) = sD_r g + g$ where

$g = 3\sum_{j<k} d_T(j,k)a_ja_k$. Since $s = \sum_{j=1}^n a_j = 0$, this means that we only need to show that $g = 3\sum_{j<k} d_T(j,k)a_ja_k = 0$. Rewriting g to pull out any terms involving a_1 or a_2, we see that

$$3d_T(1,2)a_1a_2 + 3a_1\sum_{j=3}^n d_T(1,j)a_j + 3a_2\sum_{j=3}^n d_T(2,j)a_j + 3\sum_{\substack{j<k\\ j\geq 3}} d_T(j,k)a_ja_k = 0.$$

Plugging in $a_2 = -a_1 - \sum_{j=3}^n a_j$ and simplifying yields

$$\begin{aligned} a_1^2 d_T(1,2) + a_1 &\left[\sum_{j\geq 3}(d_T(1,2) - d_T(1,j) + d_T(2,j))a_j\right] \\ &+ \sum_{j,k\geq 3}(d_T(2,j) - \frac{1}{2}d_T(j,k))a_ja_k = 0, \end{aligned}$$

which has a solution for every choice of $a_3,\ldots,a_n$. $\square$

Chapter 4

Even Order Steiner k-matrices

Conjecture 1 states that the Steiner distance k-matrix of a tree on at least 3 vertices has a zero hyperdeterminant if and only if k is odd. The backwards direction was proved in Chapter 3, and the forwards direction is proved here along with the case $n = 2$. The same definitions and notation will be used that were presented in Chapter 3 in our consideration of even order Steiner distance k-matrices. We introduce some additional notation that will be used for the even case.

A Steiner k-ideal (Definition 3.5) can be interpreted in the context of hypermatrices. We use the notation $\mathcal{S}(x^{k-1}, *)$ to represent $\mathcal{S}(\overbrace{x, x, \ldots, x}^{k-1}, *)$. Notice that $S(x^{k-1}, *) = S(x^{k-2}, *, *)x$ where $S(x^{k-2}, *, *)$ is an $n \times n$ matrix whose entries are homogeneous polynomials of degree $k-2$. Then the *Steiner k-ideal* of a graph G on n vertices is the ideal in $\mathbb{C}[x_1, \ldots, x_n]$ generated by the components of $\mathcal{S}(x^{k-1}, *)$.

4.1 k = Even Proof

We show that the Steiner distance k-matrix of a tree T on $n \geq 2$ vertices has a nonzero hyperdeterminant for even k. Therefore, for the remainder of this chapter, we assume T is a tree on at least 2 vertices with vertex set $\{v_0, \ldots, v_n\}$ and k is even. To condense notation, let $S = \mathcal{S}(x^{k-2}, *, *)$ as defined in the introduction. Further, we use only the subscripts of the vertices when referencing the Steiner sets, i.e. $d_T(0, \ldots, k-1)$ represents the Steiner distance among vertices $v_0, \ldots, v_{k-1}$.

Proposition 4.1. *If vertex v_u is adjacent to vertex v_{u+1} in a tree T on $n+1$ vertices, then $S_u - S_{u+1}$ is a vector of the form $[x_0, x_1, \ldots, x_n]$ where $x_0 = x_1 = \ldots = x_u = -\left(\sum_{i=0}^{u} x_i\right)^{k-2}$ and $x_{u+1} = \ldots = x_n = \left(\sum_{i=u+1}^{n} x_i\right)^{k-2}$ with $v_0, \ldots, v_u$ lying in one component of $T - v_u v_{u+1}$ and $v_{u+1}, \ldots, v_n$ lying in the other component.*

Proof. Assume vertex v_u is adjacent to vertex v_{u+1}. Notice that

$$S_{uw} = \sum_{v_{i_1}, \ldots, v_{i_{k-2}} \in V(T)} d_T(u, w, i_1, \ldots, i_{k-2}) x_{i_1} \ldots x_{i_{k-2}}.$$

It is sufficient to show that for any v_w and $v_{w'}$ that lie within the same component of $T - v_u v_{u+1}$,

$$d_T(u, w, i_1, \ldots, i_{k-2}) - d_T(u+1, w, i_1, \ldots, i_{k-2}) = d_T(u, w', i_1, \ldots, i_{k-2}) - d_T(u+1, w', i_1, \ldots, i_{k-2})$$

holds for all $v_{i_1}, \ldots, v_{i_{k-2}} \in V(T)$. For every case, let $d = d_T(u, w, i_1, \ldots, i_{k-2})$ and $d' = d_T(u, w', i_1, \ldots, i_{k-2})$.

Case 1: Let $v_{i_1}, \ldots, v_{i_{k-2}}$ lie in the same component as v_u.

(a) Assume v_w and $v_{w'}$ also lie within the same component of v_u. Then,

$$d_T(u, w, i_1, \ldots, i_{k-2}) - d_T(u+1, w, i_1, \ldots, i_{k-2}) = d - (d+1) = -1$$

$$d_T(u, w', i_1, \ldots, i_{k-2}) - d_T(u+1, w', i_1, \ldots, i_{k-2}) = d' - (d'+1) = -1.$$

(b) Assume instead that v_w and $v_{w'}$ lie in the same component as v_{u+1}. Then,

$$d_T(u, w, i_1, \ldots, i_{k-2}) - d_T(u+1, w, i_1, \ldots, i_{k-2}) = d - d = 0$$

$$d_T(u, w', i_1, \ldots, i_{k-2}) - d_T(u+1, w', i_1, \ldots, i_{k-2}) = d' - d' = 0.$$

Case 2: Let $v_{i_1}, \ldots, v_{i_{k-2}}$ lie in the same component as v_{u+1}.

(a) Assume v_w and $v_{w'}$ also lie within the same component of v_{u+1}. Then,

$$d_T(u, w, i_1, \ldots, i_{k-2}) - d_T(u+1, w, i_1, \ldots, i_{k-2}) = d - (d-1) = 1$$

$$d_T(u, w', i_1, \ldots, i_{k-2}) - d_T(u+1, w', i_1, \ldots, i_{k-2}) = d' - (d'-1) = 1.$$

(b) Assume instead that v_w and $v_{w'}$ lie in the same component as v_u. Then,

$$d_T(u, w, i_1, \ldots, i_{k-2}) - d_T(u+1, w, i_1, \ldots, i_{k-2}) = d - d = 0$$

$$d_T(u, w', i_1, \ldots, i_{k-2}) - d_T(u+1, w', i_1, \ldots, i_{k-2}) = d' - d' = 0.$$

Case 3: Suppose some of the vertices among $v_{i_1}, \ldots, v_{i_{k-2}}$ lie in the same component as v_u and some lie in the same component as v_{u+1}. Then,

$$d_T(u, w, i_1, \ldots, i_{k-2}) - d_T(u+1, w, i_1, \ldots, i_{k-2}) = d - d = 0$$

$$d_T(u, w', i_1, \ldots, i_{k-2}) - d_T(u+1, w', i_1, \ldots, i_{k-2}) = d' - d' = 0.$$

Therefore, the vector $S_u - S_{u+1}$ has equal entries among the locations of vertices that lie in the same component of $T - v_u v_{u+1}$.

To calculate these entry values, we combine the different cases. Notice that Case 3 will always give a zero contribution; therefore, we can omit those entries in the calculations. We first calculate $x_u = S_{uu} - S_{u+1,u}$. Since v_u lies in the same component as itself, we are working with Cases 1(a) and 2(b). Case 2(b) contributes a zero to the summation; therefore, we need only consider the vertices in Case 1(a), giving us $x_u = -\sum_{i_1, i_2, \ldots i_{k-2}=0}^{u} x_{i_1} x_{i_2} \ldots x_{i_{k-2}} = -\left(\sum_{i=0}^{u} x_i\right)^{k-2}$.

We now calculate $x_{u+1} = S_{u,u+1} - S_{u+1,u+1}$. Since v_{u+1} lies in the same component as itself, we are working with Cases 1(b) and 2(a). Case 1(b) contributes a zero to the summation; therefore, we need only consider the vertices in Case 2(a), giving us $x_{u+1} = \sum_{i_1, i_2, \ldots, i_{k-2}=u+1}^{n} x_{i_1} x_{i_2} \ldots x_{i_{k-2}} = \left(\sum_{i=u+1}^{n} x_i\right)^{k-2}$. $\square$

Let $\mathbf{h}_u(a,b) = [h_0, h_1, \ldots, h_n] \in \mathbb{C}^{n+1}$ be a vector with at most two distinct coordinates: $h_0 = \ldots = h_u = a$ and $h_{u+1} = \ldots = h_n = b$.

Theorem 4.2. *Let $\mathbf{x} = [x_0, x_1, \ldots, x_n]$ be a Steiner nullvector of a tree T. All entries that correspond to leaf vertices are constant and equal to the summation of every vertex except itself.*

Proof. Since T is a tree on at least 2 vertices, it must have a leaf vertex, call it v_0. Let v_1 be the unique neighbor of v_0. Theorem 4.1 ensures that

$$\begin{aligned}(D_0 p_T - D_1 p_T)(\mathbf{x}) &= \mathbf{h}_0(a,b) \cdot \mathbf{x} \\ &= \mathbf{h}_0\left(-x_0^{k-2}, \left(\sum_{i=1}^{n} x_i\right)^{k-2}\right) \cdot \mathbf{x} \\ &= -x_0^{k-1} + \left(\sum_{i=1}^{n} x_i\right)^{k-1} = 0\end{aligned}$$

so that $x_0^{k-1} = (\sum_{i=1}^{n} x_i)^{k-1}$. Since $k-1$ is odd,

$$x_0 = \sum_{i=1}^{n} x_i.$$

From here, we see that $\sum_{i=0}^{n} x_i = x_0 + x_0$ so that $x_0 = \frac{1}{2}\sum_{i=0}^{n} x_i$. The choice of v_0 was made arbitrarily among the leaf vertices; therefore, this must be true for all leaf vertices so that they have constant value in the Steiner nullvector. The value of these leaf vertices can then be determined by $\sum_{i=1}^{n} x_i$. $\square$

Theorem 4.3. *Let v_u and v_{u+1} be adjacent vertices in the tree T on $n+1$ vertices. Let $v_0, \ldots, v_u$ lie in one component of $T - v_u v_{u+1}$ and $v_{u+1}, \ldots, v_n$ lie in the other component. Let $\mathbf{x} = [x_0, x_1, \ldots, x_n]$ be a Steiner nullvector. Then $\sum_{i=0}^{u} x_i = x_0$ where x_0 is the entry of a leaf vertex.*

Proof. Theorem 4.1 ensures that

$$\begin{aligned}
(D_u p_T - D_{u+1} p_T)(\mathbf{x}) &= \mathbf{h}_u(a,b)\cdot\mathbf{x} \\
&= \mathbf{h}_u\left(-\left(\sum_{i=0}^{u} x_i\right)^{k-2}, \left(\sum_{i=u+1}^{n} x_i\right)^{k-2}\right)\cdot\mathbf{x} \\
&= -\left(\sum_{i=0}^{u} x_i\right)^{k-1} + \left(\sum_{i=u+1}^{n} x_i\right)^{k-1} = 0
\end{aligned}$$

so that $\left(\sum_{i=0}^{u} x_i\right)^{k-1} = \left(\sum_{i=u+1}^{n} x_i\right)^{k-1}$. Since $k-1$ is odd,

$$\sum_{i=0}^{u} x_i = \sum_{i=u+1}^{n} x_i. \tag{4.1}$$

We assumed x_0 is the entry of a leaf vertex; therefore, Theorem 4.2 tells us that $x_0 = \sum_{i=1}^{n} x_i$. We can rewrite (4.1) to be $x_0 + \sum_{i=1}^{u} x_i = x_0 - \sum_{i=1}^{u} x_i$ so that

$$\sum_{i=1}^{u} x_i = 0.$$

Therefore, $\sum_{i=0}^{u} x_i = x_0$ as desired. □

Given these restrictions on the entries of a Steiner nullvector of a tree T, we prove the necessary structure of such a nullvector.

Theorem 4.4. *The Steiner nullvector entries of a tree T on $n+1$ vertices are given by $x_0(2-\deg(v_t))$ where v_t is the vertex to which the entry corresponds and x_0 is the entry of the leaf vertices.*

Proof. We first consider a leaf vertex, v_0. Since $\deg(v_0) = 1$, the nullvector entry is $x_0 = x_0(2-\deg(v_0))$ as desired. Theorem 4.2 tells us that all leaf vertices have a constant value in the nullvector; therefore, x_0 can represent all entries for leaf vertices.

For the remaining vertices, choose an arbitrary interior vertex, v_t, with degree d. Since T is a tree on at least 2 vertices, v_t has at least one neighbor, call it v_{t+1}.

Let $\mathcal{G}$ be the set of all vertices that lie in the same component as v_t in $T - v_t v_{t+1}$. Theorem 4.3 tells us that $\sum_{\{i:v_i \in \mathcal{G}\}} x_i = x_0$. We can further group the vertices of $\mathcal{G}$ into subgroups $\mathcal{G}_0, \ldots, \mathcal{G}_{d-2}$ so that every branch leading away from v_t contains vertices that lie in the same group. (Note that there are only $d-1$ subgroups since the branch containing vertex v_{t+1} is not included in $\mathcal{G}$). Once again, we can apply Theorem 4.3 to say that $\sum_{\{i:v_i \in \mathcal{G}_j\}} x_i = x_0$ for $0 \leq j \leq d-2$. Therefore,

$$\begin{aligned} x_0 &= \sum_{\{i:v_i \in \mathcal{G}\}} x_i \\ &= x_t + \sum_{j=0}^{d-2} \sum_{\{i:v_i \in \mathcal{G}_j\}} x_i \\ &= x_t + \sum_{j=0}^{d-2} x_0 \\ &= x_t + x_0(\deg(v_t) - 1) \end{aligned}$$

and $x_t = x_0(2 - \deg(v_t))$ as desired. $\square$

We are now equipped to prove the main result. Within the following proof, let $\mathbf{i} \in \{0, \ldots, n\}^k$ represent a k-dimensional vector with entries taken from the set $\{0, \ldots, n\}$. Further, we define $\mathbf{x}^{\mathbf{i}} := \prod_{j=1}^{\dim(\mathbf{i})} x_{\mathbf{i}_j}$ and $\nu_t(\mathbf{i}) := |\{j : \mathbf{i}_j = t\}|$.

Theorem 4.5. *The Steiner k-matrix of a tree has a nonzero hyperdeterminant.*

Proof. Theorem 4.4 tells us that any Steiner nullvector $\mathbf{x} = [x_0, \ldots, x_n]$ has entries given by $x_0(2 - \deg(v_t))$ where x_0 is a leaf vertex entry and v_t is the vertex to which the entry corresponds. We prove the result by contradiction.

Consider any tree T on $n+1$ vertices where $n \geq 1$. Let v_n be a leaf vertex of T whose neighbor is v_{n-1}, and let $\mathbf{x} = [x_0, \ldots, x_{n-1}, x_n]$ be a Steiner nullvector of T. Assume the hyperdeterminant is zero. Theorem 2.7 states that there must exist a nontrivial solution to the Steiner k-ideal. Since the generators of the Steiner k-ideal are homogeneous, and $x_n = 0$ implies that $x_t = 0$ for all $v_t \in V(T)$ by Theorem 4.4, we may assume $x_n = 1$. Notice that

$$\begin{aligned}
D_{n-1}p_T(\mathbf{x}) &= \sum_{\mathbf{i}\in\{0,\ldots,n\}^{k-1}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \\
&= \sum_{a=0}^{k-1} \sum_{\substack{\mathbf{i}\in\{0,\ldots,n\}^{k-1} \\ \nu_n(\mathbf{i})=a}} d_T(n-1,\mathbf{i},\overbrace{n,n,\ldots,n}^{a})\mathbf{x}^{\mathbf{i}} \\
&= \sum_{a=0}^{k-1} \binom{k-1}{a} x_n^a \sum_{\mathbf{i}\in\{0,\ldots,n-1\}^{k-1-a}} d_T(n-1,\mathbf{i},\overbrace{n,n,\ldots,n}^{a})\mathbf{x}^{\mathbf{i}} \\
&= \sum_{\mathbf{i}\in\{0,\ldots,n-1\}^{k-1}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \\
&\qquad + \sum_{a=1}^{k-1} \binom{k-1}{a} \sum_{\mathbf{i}\in\{0,\ldots,n-1\}^{k-1-a}} [d_T(n-1,\mathbf{i})+1]\mathbf{x}^{\mathbf{i}}
\end{aligned}$$

by partitioning the vectors $\mathbf{i}$ according whether $\nu_n(\mathbf{i}) = 0$ or a for some $a > 0$. Continuing to rewrite the quantity $D_{n-1}p_T(\mathbf{x})$,

$$\begin{aligned}
&= \sum_{a=0}^{k-1} \binom{k-1}{a} \sum_{\mathbf{i}\in\{0,\ldots,n-1\}^{k-1-a}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} + \sum_{a=1}^{k-1} \binom{k-1}{a} \sum_{\mathbf{i}\in\{0,\ldots,n-1\}^{k-1-a}} \mathbf{x}^{\mathbf{i}} \\
&= \sum_{a=0}^{k-1} \binom{k-1}{a} \sum_{\mathbf{i}\in\{0,\ldots,n-1\}^{k-1-a}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} + \left(1+\sum_{i=0}^{n-1} x_i\right)^{k-1} - \left(\sum_{i=0}^{n-1} x_i\right)^{k-1} \\
&= \left(\sum_{a=0}^{k-1} \binom{k-1}{a} \sum_{b=0}^{k-1-a} \sum_{\substack{\mathbf{i}\in\{0,\ldots,n-1\}^{k-1-a} \\ \nu_{n-1}(\mathbf{i})=b}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \right) + 2^{k-1} - 1 \\
&= \left(\sum_{a=0}^{k-1} \binom{k-1}{a} \sum_{b=0}^{k-1-a} \binom{k-1-a}{b} x_{n-1}^b \sum_{\mathbf{i}\in\{0,\ldots,n-2\}^{k-1-a-b}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \right) + 2^{k-1} - 1 \\
&= \left(\sum_{a+b=0}^{k-1} \binom{k-1}{a,b,k-1-a-b} x_{n-1}^b \sum_{\mathbf{i}\in\{0,\ldots,n-2\}^{k-1-a-b}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \right) + 2^{k-1} - 1 \\
&= \left(\sum_{j=0}^{k-1} \binom{k-1}{j} \sum_{b=0}^{j} \binom{j}{b} x_{n-1}^b \sum_{\mathbf{i}\in\{0,\ldots,n-2\}^{k-1-j}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \right) + 2^{k-1} - 1
\end{aligned}$$

$$= \left(\sum_{j=0}^{k-1} \binom{k-1}{j} (x_{n-1}+1)^j \sum_{\mathbf{i} \in \{0,\ldots,n-2\}^{k-1-j}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \right) + 2^{k-1} - 1$$

where j has been introduced to replace $a+b$. If T is a tree on 2 vertices, then $D_{n-1}p_T(x) = 2^{k-1} - 1 \neq 0$. Therefore, the only Steiner nullvector of a tree on 2 vertices is the trivial one, and the hyperdeterminant must be nonzero. We continue by assuming that T is a tree on at least 3 vertices.

If we delete the leaf vertex v_n, we obtain a new tree T' on n vertices. Let $\mathbf{x}' = [x'_0, \ldots, x'_{n-2}, x'_{n-1}]$ be a Steiner nullvector of T'. Notice that for $0 \leq i \leq n-2$, $\deg_T(v_i) = \deg_{T'}(v_i)$, and $\deg_T(v_{n-1}) = \deg_{T'}(v_{n-1}) + 1$. Using Theorem 4.4 and the definition of $\mathbf{x}$, we can say that $\mathbf{x}' = [x_0, \ldots, x_{n-2}, x_{n-1}+x_n] = [x_0, \ldots, x_{n-2}, x_{n-1}+1]$. Notice that

$$\begin{aligned}
D_{n-1}p_{T'}(\mathbf{x}') &= \sum_{\mathbf{i} \in \{0,\ldots,n-1\}^{k-1}} d_T(n-1,\mathbf{i})\mathbf{x}'^{\mathbf{i}} \\
&= \sum_{j=0}^{k-1} \sum_{\substack{\mathbf{i} \in \{0,\ldots,n-1\}^{k-1} \\ \nu_{n-1}(\mathbf{i})=j}} d_T(n-1,\mathbf{i})\mathbf{x}'^{\mathbf{i}} \\
&= \sum_{j=0}^{k-1} \binom{k-1}{j} \left(x'_{n-1}\right)^j \sum_{\mathbf{i} \in \{0,\ldots,n-2\}^{k-1-j}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \\
&= \sum_{j=0}^{k-1} \binom{k-1}{j} (x_{n-1}+1)^j \sum_{\mathbf{i} \in \{0,\ldots,n-2\}^{k-1-j}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}}.
\end{aligned}$$

Since we assumed $\mathbf{x}$ and $\mathbf{x}'$ were Steiner nullvectors, both $D_{n-1}p_T(\mathbf{x}) = 0$, and $D_{n-1}p_{T'}(\mathbf{x}') = 0$. Therefore,

$$\begin{aligned}
0 &= D_{n-1}p_T(\mathbf{x}) - D_{n-1}p_{T'}(\mathbf{x}') \\
&= \left(\sum_{j=0}^{k-1} \binom{k-1}{j} (x_{n-1}+1)^j \sum_{\mathbf{i} \in \{0,\ldots,n-2\}^{k-1-j}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \right) + 2^{k-1} - 1 \\
&\qquad - \sum_{j=0}^{k-1} \binom{k-1}{j} (x_{n-1}+1)^j \sum_{\mathbf{i} \in \{0,\ldots,n-2\}^{k-1-j}} d_T(n-1,\mathbf{i})\mathbf{x}^{\mathbf{i}} \\
&= 2^{k-1} - 1 \neq 0.
\end{aligned}$$

Therefore, the only solution must be the trivial one, and the hyperdeterminant must be nonzero. □

4.2 $k = 2$ Variations

We attempted many different methods as we tried to prove the even k case. These methods, while unsuccessful for $k > 2$, provided new insight into the $k = 2$ case. The results presented next prove that the standard distance matrix for a tree has nonzero determinant. The Graham-Pollak theorem is more precise since it provides an exact formula for the determinant; however, these new results generalize the theorem utilizing new techniques. Note that in all cases, T is a tree on at least 2 vertices.

4.2.1 Polynomial Method

Proposition 4.6. *The 2-Steiner nullity of a tree T is zero.*

Proof. We show this by arguing that any Steiner 2-nullvector must be the all-zeroes vector. Suppose $\mathbf{x} \in \mathbb{C}^{V(T)}$ is a Steiner 2-nullvector, and for each $u \in V(G)$, write x_u for its u-coordinate. Then $\mathbf{x}$ must be a common zero of the polynomial system $\{D_u p_T^{(2)}\}_{u \in V(G)}$. Note that

$$D_v p_T^{(2)}(\mathbf{x}) = 2 \sum_{u \in V(T)} d(u,v) x_u.$$

Consider two adjacent vertices $v, w \in V(T)$. Then

$$\sum_{u \in V(T)} d(u,v) x_u = 0 = \sum_{u \in V(T)} d(u,w) x_u.$$

Subtracting the right-hand side from the left-hand side yields

$$\sum_{u \in V(T)} (d(u,v) - d(u,w)) x_u = 0. \tag{4.2}$$

The vertices of T can be partitioned into two sets, $V(T_j)$ for $j = 1, 2$, where $T_j = T_j(v, w)$ are the two components of $T - vw$ indexed so that $v \in T_1$ and $w \in T_2$. Note that $d(u, v) - d(u, w) = (-1)^j$ if $u \in T_j$. Therefore, abbreviating $\sum_{u \in U} x_u$ by $\Sigma(U)$ and $S(v, w) := V(T_1(v, w))$, we may rewrite (4.2) as

$$\Sigma(S(v, w)) = \Sigma(S(w, v)). \tag{4.3}$$

Let $v \in V(T)$ be arbitrary, and suppose $w_1, \ldots, w_{\deg_T(v)}$ are the neighbors of v in T. Applying (4.3) for an edge vw_j yields

$$\Sigma(S(w_j, v)) = \Sigma(S(v, w_j)) = x_v + \sum_{i \neq j} \Sigma(S(w_i, v)) = \Sigma(V(T)) - \Sigma(S(w_j, v)). \tag{4.4}$$

Therefore,

$$2\Sigma(S(w_j, v)) = \Sigma(V(T)),$$

so we may rewrite (4.4) as

$$x_v = (1 - \deg_T(v)/2)\Sigma(V(T)).$$

Because the system $\{D_u p_T^{(2)}\}_{u \in V(G)}$ is homogeneous, we may assume $\Sigma(V(T)) = 1$, i.e., $x_v = 1 - \deg_T(v)/2$ for every v. Thus, for each $v \in V(G)$, we have

$$\begin{aligned} 0 &= 2 \sum_{u \in V(T)} d(u, v) x_u \\ &= \sum_{u \in V(T)} d(u, v) \left(2 - \deg_T(u)\right). \end{aligned}$$

From here, there are two ways with which we can obtain the result.

- Proof 1: We prove by induction that there exists a vertex v in every tree such that $\sum_{u \in V(T)} d(u, v) \left(2 - \deg_T(u)\right) > 0$.

  For a tree on 2 vertices, note that $\sum_{u \in V(T)} d(u, v) \left(2 - \deg_T(u)\right) = 1 > 0$.

  Assume the result holds for a tree on $n - 1$ vertices.

Consider a tree, T, on n vertices. Remove a leaf to make a new tree, T' on $n-1$ vertices. By induction, there exists a $v \in V(T')$ such that

$$\sum_{u \in V(T')} d(u,v)\,(2 - \deg_{T'}(u)) > 0.$$

Attach an additional vertex w to some vertex $z \in V(T')$. Now, $\deg_T(z) = \deg_{T'}(z) + 1$ and $d(w,v) = d(z,v) + 1$. Let $S' = \sum_{u \in V(T')} d(u,v)\,(2 - \deg_{T'}(u))$ where $S' > 0$ by induction. Now, let

$$\begin{aligned}
S &= \sum_{u \in V(T)} d(u,v)\,(2 - \deg_T(u)) \\
&= S' - d(z,v)(2 - \deg_{T'}(z)) + d(z,v)(2 - \deg_T(z)) + d(w,v)(2 - \deg_T(w)) \\
&= S' - d(z,v) + d(w,v) \\
&= S' + 1 > 0.
\end{aligned}$$

Therefore, the only solution must be the trivial one so that the nullity is zero.

- Proof 2: We still prove that there exists a vertex v in every tree such that

$$\sum_{u \in V(T)} d(u,v)(2 - \deg_T(u)) > 0,$$

but not by induction. Let H be the set of high degree vertices (vertices with degree at least 3), and let L be the set of low degree vertices (the leaves). Then $\sum_{u \in H} d(u,v)(2 - \deg(u)) + \sum_{w \in L} d(w,v)(2 - \deg(w)) > 0$ so that

$$\sum_{w \in L} d(w,v) > \sum_{u \in H} d(u,v)(-2 + \deg(u)) \geq \sum_{u \in H} d(u,v).$$

We show there exists a matching between the high degree vertices and the low degree vertices that proves the inequality. Choose a leaf, w, and let P be the path used to calculate $d(w,v)$. If there exists a high degree vertex on P, call it u that has not been previously matched, match the leaf with the vertex u that minimizes $d(w,u)$. Then $d(w,v) = d(w,u) + d(u,v)$ so that $d(w,v) \geq d(u,v)$. If no such u exists, then the summation on the left hand side (i.e. the low degree vertices) receives an additional entry while the right hand side does not. This method ensures that every high degree vertex gets paired with a leaf. Indeed, assume a high degree vertex u did not receive a matching with a leaf. This would imply there were more high degree vertices than leaves, which is impossible. Therefore, with each pairing w and u, we have $d(w,v) \geq d(u,v)$ with at least one pairing where $d(w,v) > d(u,v)$. This ensures $\sum_{w \in L} d(w,v) > \sum_{u \in H} d(u,v)$ as desired.

$\square$

Proposition 4.6 ensures the only solution to the Steiner 2-ideal is the trivial one. Therefore, by Theorem 2.7, the determinant must be nonzero.

4.2.2 Kernel Method

We first define some terms using the same vocabulary as in [21].

Definition 4.7. *A map $K : X \times X \to \mathbb{C}$ is called a* kernel *on X. We say that a kernel K is* hermitian *if $K(x,y) = \overline{K(y,x)}$. A hermitian kernel K will be called* Schoenberg kernel *if $K(x,y) = K(y,x) \geq 0$ and $K(x,x) = 0$.*

Let $c_c(X)$ denote the set of all complex-valued functions on a set X with finite supports.

Definition 4.8. *We say that a hermitian kernel K is* conditionally (strictly) negative definite *if*

$$\forall \lambda \in c_c(X) \backslash \{0\} \sum_{x,y \in V} \lambda(x)\overline{\lambda(y)}K(x,y) \leq 0 \ (< 0) \textit{ provided that } \sum_{x \in X} \lambda(x) = 0.$$

Definition 4.9. *A map α is called the* quadratic embedding *of a metric space into a Hilbert space if* $d(x, y) = \|\alpha(x) - \alpha(y)\|^2$.

Definition 4.10. *A system of vectors $\{v_0, \ldots, v_n\} \in V$ is* affinely independent *if the system of vectors $\{v_1 - v_0, \ldots, v_n - v_0\}$ is linearly independent.*

Knowing these definition, we next introduce two theorems that will serve as an outline to prove our result.

Theorem 4.11. *[24] A graph G admits a quadratic embedding if and only if the kernel formed from its' distance matrix is conditionally negative definite.*

Theorem 4.12. *[21] Let (X, K) be a finite set with a conditionally negative definite Schoenberg kernel, let $\alpha : V \to \mathcal{H}$ be its quadratic embedding. The following conditions are equivalent:*

(a) The matrix defined by K is invertible.

(b) The set of vectors $\{\alpha(x) : x \in X\}$ is affinely independent.

(c) K is conditionally strictly negative definite.

Our goal is then to prove that a tree can be applied to these theorems. We prove this below.

Theorem 4.13. *Any tree, T, admits a quadratic embedding into a Hilbert space.*

Proof. We will label the vertices of T with n-tuples consisting of 1's and 0's. Every time a vertex, j, is labeled, we add e_j to the label of its neighbor. We prove this labeling is a quadratic embedding recursively.

Consider when $n = 1$. The vertex is labeled as (0). Clearly, $\|\alpha(v) - \alpha(v)\|^2 = \|(0) - (0)\|^2 = 0 = d_T(v, v)$ so there exists a quadratic embedding.

Assume the result holds on $n-1$ vertices.

Consider a tree on n vertices. Identify a leaf vertex, v, and its neighbor, w. Delete vertex v to obtain a tree on $n-1$ vertices. We know by induction that there exists a quadratic embedding of this tree. Now, append v back to the tree by attaching it to vertex w, and append 0 to the label of every labeled vertex. Note this does not change the quadratic embedding for the first $n-1$ vertices since the 0 in the last spot of the labeling will not affect the distance calculation. Vertex w; therefore, has some labeling $(w_1, \ldots, w_{n-1}, 0)$. Label vertex v as $(w_1, \ldots, w_{n-1}, 0) + e_n$. We know that for any vertex j in the tree, $d_T(w, j) = \|\alpha(w) - \alpha(j)\|^2$. Further, by the structure of a tree, $d_T(v, j) = d_T(w, j) + 1$. For every $j \in V(T) \backslash \{v\}$, $\alpha(j)$ has a 0 in the last spot of its labeling. Therefore, $\|\alpha(v) - \alpha(j)\|^2 = \|\alpha(w) - \alpha(j)\|^2 + 1$. Therefore, a quadratic embedding exists as desired. □

Theorem 4.14. *The determinant of the distance matrix of any tree, T, is nonzero.*

Proof. Theorem 4.13 states that T can be quadratically embedded into a Hilbert space. Therefore, by Theorem 4.11, the kernel formed from the distance matrix of T is conditionally negative definite. If we can prove that the set of vectors created from the quadratic embedding is affinely independent, Theorem 4.12 states that the distance matrix is invertible. From our mapping, take v_0 to be the trivial vector. We prove $v_1, \ldots, v_n$ are linearly independent by induction.

Clearly, a single vector, v_1 is linearly independent.

Assume the result holds on a tree on $n-1$ vertices.

By construction of the embedding, the nth vertex is assigned a vector that has a 1 in the last entry. Every other vector has a zero in the last entry. By induction, the first $n-1$ vectors are linearly independent. This additional vector has an entry that cannot be obtained by any linear combination of the previous vectors. Therefore, the system is linearly independent as desired. Theorem 4.12 then says that the kernel is conditionally strictly negative definite and the determinant of the distance matrix must be nonzero.

□

4.2.3 Elementary Vector Method

Linear algebra tells us that a matrix is invertible if it's columns (or rows) are linearly independent. We use this fact and reduce the columns of the distance matrix for a tree into the elementary vectors, thus proving the result.

Theorem 4.15. *The determinant of the distance matrix of any tree T on at least two vertices is nonzero.*

Proof. We prove this by showing there are a series of steps that allow us to obtain the elementary vectors from the columns of the distance matrix. Therefore, the columns span $\mathbb{R}^n$ so that the determinant of the distance matrix is nonzero.

First, we consider the vertices that have degree 2 in the tree. Let K_{v_i} be the vector with entries representing the distance from vertex v_i to the other vertices. Further, assume v_k is the vertex of degree 2 with v_{k-1} and v_{k+1} as its neighbors. Let $v_0, v_1, \ldots, v_{k-1}$ lie in one component of $T - v_{k-1}v_k$ and $v_k, v_{k+1}, \ldots, v_{n-1}$ lie in the other component. Then

$$2K_{v_k} - K_{v_{k-1}} - K_{v_{k+1}} = [0, 0, \ldots, 0, -2, 0, \ldots, 0] \tag{4.5}$$

where the only nonzero entry is in position v_k. Therefore, we have a multiple of the elementary vector e_k.

We consider the leaf vertices next. Let $\mathcal{L}$ represent the set containing all the leaves of the tree. We know $|\mathcal{L}| = x \geq 2$; therefore, without loss of generality assume $v_0 \in \mathcal{L}$

and v_1 is the unique neighbor of v_0. Then

$$K_{v_0} - K_{v_1} = [-1, 1, 1, \ldots, 1]. \tag{4.6}$$

Since $K_{v_\ell} - K_{v_{\ell+1}}$ has the same form for all $v_\ell \in \mathcal{L}$ and $v_{\ell+1}$ the unique neighbor of v_ℓ, we can conclude that

$$\sum_{v_\ell \in \mathcal{L}} \Big(K_{v_\ell} - K_{v_{\ell+1}}\Big) = [x-2, x-2, \ldots, x-2, x, \ldots, x] \tag{4.7}$$

where the $x-2$ entries are in leaf positions and the x entries are in positions of vertices of degree 2 or greater. Note that if every non-leaf vertex was degree 2, then

$$\sum_{v_\ell \in \mathcal{L}} \Big(K_{v_\ell} - K_{v_{\ell+1}}\Big) - 2 \sum_{\{k:\deg(v_k)=2\}} e_k = [x-2, x-2, \ldots, x-2]. \tag{4.8}$$

Therefore, we have obtained the constant vector so that

$$K_{v_0} - K_{v_1} - \frac{1}{x-2} \sum_{v_\ell \in \mathcal{L}} \Big(K_{v_\ell} - K_{v_{\ell+1}}\Big) = [-2, 0, \ldots, 0] \tag{4.9}$$

where the only nonzero entry is in position v_0, a leaf vertex. Note, however, this only holds when there are no vertices of degree 3 or greater. If these high degree vertices exist in the tree, then

$$\sum_{v_\ell \in \mathcal{L}} \Big(K_{v_\ell} - K_{v_{\ell+1}}\Big) - 2 \sum_{\{k:\deg(v_k)=2\}} e_k = [x-2, x-2, \ldots, x-2, x, \ldots, x] \tag{4.10}$$

where the x entries are in the positions of the high degree vertices. Therefore, we move on to the vertices of degree at least 3 to obtain the constant vector from Equation 4.10.

Consider a vertex, v_t, of degree $d \geq 3$ and let $\mathcal{N} = \{v_{t+1}, v_{t+2}, \ldots, v_{t+d}\}$ be the set of neighbors of v_t. Each neighbor creates a sub-tree that leads away from vertex v_t. Then

$$2K_{v_t} - K_{v_{t+1}} - K_{v_{t+2}} = [0, 0, \ldots, 0, -2, -2, \ldots, -2] \tag{4.11}$$

where the nonzero entries are at locations v_t and the vertices in the sub-trees not attached to vertices v_{t+1} or v_{t+2}. Notice that

$$2dK_{v_t} - 2\sum_{v_i \in \mathcal{N}} K_{v_i} = [-2(d-2), \dots, -2(d-2), -2d, -2(d-2), \dots, -2(d-2)] \quad (4.12)$$

where every entry is $-2(d-2)$ except the v_t entry. Notice that in Equation 4.10, the v_t entry is x since it is a vertex of degree at least 3. Therefore, we can add a multiple of Equation 4.12 to Equation 4.10 so that it has the same entry in position v_t as the entries in the leaf positions. Indeed,

$$\sum_{v_\ell \in \mathcal{L}} \left(K_{v_\ell} - K_{v_{\ell+1}}\right) - 2\sum_{\{k:\deg(v_k)=2\}} e_k + \frac{1}{2}\left(2dK_{v_t} - 2\sum_{v_i \in \mathcal{N}} K_{v_i}\right) =$$
$$[x-d, \dots, x-d, x-d+2, \dots, x-d+2]$$

where now the $x-d$ entries are in positions of leaf vertices, degree 2 vertices, as well as the vertex v_t while the $x-d+2$ entries are in locations of all other high degree vertices. The same technique can be applied so that the other high degree vertices' entries match the remaining entries, thus obtaining the constant vector as desired.

This constant vector can be used to complete the remaining cases. Equation 4.6 applies to any leaf vertex; therefore, subtracting off a multiple of the constant vector allows us to obtain the elementary vector e_ℓ for any leaf, v_ℓ. Equation 4.12 applies to any vertex of degree 3 or more; therefore, subtracting off a multiple of the constant vector allows us to obtain the elementary vector e_t for any high degree vertex, v_t.

We have now accounted for all vertices producing a multiple of their corresponding elementary vector; therefore, the columns span $\mathbb{R}^n$ and the determinant of the distance matrix is nonzero as desired. $\square$

4.2.4 Poisson-Product Formula Method

The last method we considered was applying the Poisson-Product Formula to the distance matrix of a star graph. We include the following description of the product formula from [10].

Theorem 4.16 (Poisson Product Formula)**.** *Let $F_1, \ldots, F_n$ be homogeneous polynomials of respective degrees $d_1, \ldots, d_n$ in $K[x_1, \ldots, x_n]$ where K is an algebraically closed field. For $1 \leq i \leq n$, let $\overline{F_i}$ be the homogeneous polynomial in $K[x_2, \ldots, x_n]$ obtained by substituting $x_1 = 0$ in F_i, and let f_i be the polynomial in $K[x_2, \ldots, x_n]$ obtained by substituting $x_1 = 1$ in F_i. Let V be the set of simultaneous zeros of the system of polynomials $\{f_2, \ldots, f_n\}$, that is, V is the affine variety defined by the polynomials. If $Res(\overline{F_2}, \ldots, \overline{F_n}) \neq 0$, then V is a zero-dimentional variety (a finite set of points), and*

$$Res(F_1, \ldots, F_n) = Res(\overline{F_2}, \ldots, \overline{F_n})^{d_1} \prod_{p \in V} f_1(p)^{m(p)}$$

where $m(p)$ is the multiplicity of a point $p \in V$.

Cooper and Dutle utilized this method in [7] by applying it to the all-ones hypermatrix, so we had hoped to extend this to the Steiner distance k-matrix. While this did not prove fruitful for even $k > 2$, it had not been applied to the $k = 2$ case (the standard distance matrix) to the best of our knowledge. Further, we could only apply it to the star graph since it has predictable entries in its' distance matrix. With this method, we were actually able to prove the formula in the Graham-Pollak theorem for a star rather than just proving that the determinant was nonzero like the previous cases.

Theorem 4.17. *Let G be a star graph on n vertices and M be its Steiner distance 2-matrix. Then* $\det(G) = -(n-1)(-2)^{n-2}$.

Proof. Let 1 be the center vertex and $2, \ldots, n$ be the leaf vertices of G. We proceed by induction using the Poisson Product Formula. The base case when $n = 2$ is trivial since $M = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ so that $\det(M) = -1 = -(n-1)(-2)^{n-2}$.

Assume the result holds for a star on $n-1$ vertices.

Now, consider a star on n vertices. In this case, $M = \begin{bmatrix} 0 & 1 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 2 & 2 & \cdots & 2 \\ 1 & 2 & 0 & 2 & \cdots & 2 \\ \vdots & \vdots & 2 & \ddots & \ldots & \vdots \\ 1 & 2 & 2 & 2 & \ldots & 0 \end{bmatrix}$ so that $F_1 = \sum_{i=2}^{n} x_i$ and $F_j = x_1 + 2\sum_{i=2}^{n} x_i - 2x_j$ for $2 \leq j \leq n$. The Poisson Product Formula requires substituting 0 for some variable. Slight variations of the proof occur depending on which variable is chosen.

- Proof 1: If we set $x_1 = 0$, we are left with the matrix $2J - 2I$ of size $n-1$. Therefore, $\operatorname{Res}(\overline{F_2}, \cdots, \overline{F_n}) = \det(2J - 2I) = 2^{n-1}(-1)^{n-2}(n-2)$. Applying this to the Poisson Product formula yields

$$\operatorname{Res}(F_1, \ldots, F_n) = 2^{n-1}(-1)^{n-2}(n-2) \prod_{p \in V} f_1(p)^{m(p)}.$$

To find the solutions to $\{f_2, \ldots, f_n\}$, we solve

$$(2J - 2I) \begin{bmatrix} p_2 & p_3 & p_4 \ldots & p_n \end{bmatrix}^T = \begin{bmatrix} -1 & -1 & -1 & \ldots & -1 \end{bmatrix}^T.$$

This gives the solution $\mathbf{p} = \left[-\frac{1}{2(n-2)}, -\frac{1}{2(n-2)}, \ldots, -\frac{1}{2(n-2)} \right]$. Lastly, applying this to the Poisson Product formula gives us

$$\operatorname{Res}(F_1, \ldots, F_n) = 2^{n-1}(-1)^{n-2}(n-2) \left(-\frac{n-1}{2(n-2)} \right) = -(n-1)(-2)^{n-2}$$

as desired.

- If we instead set $x_n = 0$, then the resulting graph G is still a star on $n-1$ vertices. Applying the inductive hypothesis to the Poisson Product formula then yields

$$\operatorname{Res}(F_1, \ldots, F_n) = -(n-2)(-2)^{n-3} \prod_{p \in V} f_n(p)^{m(p)}.$$

Let $D = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 2 & \cdots & 2 \\ 1 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 1 & 2 & 2 & \cdots & 0 \end{bmatrix}$. To find the solutions to $\{f_1, \ldots, f_{n-1}\}$, we solve

$$D \begin{bmatrix} p_1 & p_2 & p_3 & \ldots & p_{n-1} \end{bmatrix}^T = \begin{bmatrix} -1 & -2 & -2 & \ldots & -2 \end{bmatrix}^T.$$

This gives the solution

$$\mathbf{p} = \begin{bmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & 0 & 2 & \cdots & 2 \\ 1 & 2 & 0 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & 2 \\ 1 & 2 & 2 & \cdots & 0 \end{bmatrix}^{-1} \begin{bmatrix} -1 \\ -2 \\ -2 \\ \vdots \\ -2 \end{bmatrix}$$

$$= \frac{1}{\det(D)} \begin{bmatrix} 2^{n-2}(-1)^{n-3}(n-3) & -(-2)^{n-3} & \cdots & -(-2)^{n-3} \\ -(-2)^{n-3} & 2^{n-4}(-1)^{n-3}(n-3) & -(-2)^{n-4} & \cdots \\ \vdots & -(-2)^{n-4} & \ddots & -(-2)^{n-4} \\ -(-2)^{n-3} & \vdots & -(-2)^{n-4} & 2^{n-4}(-1)^{n-3}(n-3) \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -2 \\ \vdots \\ -2 \end{bmatrix}$$

where $\det(D) = 2^{n-3}(-1)^{n-2}(n-2)$. Continuing with this substitution,

$$= \begin{bmatrix} -\dfrac{2(n-3)}{n-2} & \dfrac{1}{n-2} & \cdots & \dfrac{1}{n-2} \\ \dfrac{1}{(n-2)} & -\dfrac{n-3}{2(n-2)} & \dfrac{1}{2(n-2)} & \cdots \\ \vdots & \dfrac{1}{2(n-2)} & \ddots & \dfrac{1}{2(n-2)} \\ \dfrac{1}{n-2} & \vdots & \dfrac{1}{2(n-2)} & -\dfrac{n-3}{2(n-2)} \end{bmatrix} \begin{bmatrix} -1 \\ -2 \\ -2 \\ \vdots \\ -2 \end{bmatrix}$$

$$= \begin{bmatrix} \dfrac{-2}{n-2} \\ -\dfrac{1}{n-2} \\ \vdots \\ -\dfrac{1}{n-2} \end{bmatrix}$$

Lastly, applying this to the Poisson Product formula gives us

$$\begin{aligned} \operatorname{Res}(F_1, \ldots, F_n) &= -(n-2)(-2)^{n-3} \left(-\frac{2}{n-2} + 2(n-2) \left(-\frac{1}{n-2} \right) \right) \\ &= -(n-1)(-2)^{n-2} \end{aligned}$$

as desired.

$\square$

Chapter 5

Distance Critical Graphs

Before extending the Graham-Pollak theorem to higher orders, we began by analyzing the order-2 case (the general definition of distance) for trees. While many proofs exist in the literature, the simplest proof was perhaps given by Yan-Yeh. This proof combines different linear algebra techniques to calculate this determinant. One such technique is the following:

Proposition 5.1 (Dodgson's Rule [12])**.** *Let A be a matrix of order $n > 2$, A_{ij} the minor of A formed by deleting the i^{th} row and j^{th} column, and A_2 the minor of A by deleting rows 1 and n. Then*

$$\det(A)\det(A_2) = \det(A_{11})\det(A_{nn}) - \det(A_{1n})\det(A_{n1}).$$

Using this result, we present the Yan-Yeh proof since the basic idea for defining distance critical graphs relies on it's technique.

5.1 Yan-Yeh Proof

Theorem 5.2 ([16])**.** *Suppose T is a tree with vertex set $V(T) = \{v_1, \ldots, v_n\}$. Let $D = (d_{ij})_{n\times n}$ be the distance matrix of T, where D_{ij} equals the distance between vertices v_i and v_j. Then $\det(D) = -(n-1)(-2)^{n-2}$.*

Proof. [28] The theorem is proved through induction. First, the base cases $n \leq 3$ are trivial to show.

Assume the result holds for a tree on $n-1$ vertices.

Consider a tree, T, on n vertices with $n \geq 4$. Note that T has at least two pendant, or leaf, vertices. Without loss of generality, we assume both v_1 and v_n are two pendant vertices of T. Further, assume v_p is the unique neighbor of v_1, and v_q is the unique neighbor of v_n. Let d_i denote the i^{th} column of D. By definition of v_1, v_p, v_q, and v_n, $(d_1 - d_p)^T = (-1, 1, 1, \ldots, 1)$ and $(d_n - d_q)^T = (1, 1, \ldots, 1, -1)$. Therefore,

$$\det(D) = \det(d_1 - d_p + d_q - d_n, d_2, \ldots, d_{n-1}, d_n).$$

Notice that $(d_1 - d_p + d_q - d_n)^T = (-2, 0, 0, \ldots, 0, 2)$ so that

$$\det(D) = -2\det(D_{11}) + 2(-1)^{n+1}\det(D_{n1}).$$

Dodgson's determinant-evaluation rule also states that

$$\det(D)\det(D_2) = \det(D_{11})\det(D_{nn}) - \det(D_{1n})\det(D_{n1}).$$

Since the distance matrix, D, is symmetric, $\det(D_{1n}) = \det(D_{n1})$. Also, note that D_2, D_{11}, and D_{nn} respectively represent the distance matrices of the trees $T - v_1 - v_n$, $T - v_1$ and $T - v_n$. The induction hypothesis allows us to conclude that

$$\begin{cases} \det(D) = -2[-(n-2)(-2)^{n-3}] + 2(-1)^{n+1}\det(D_{n1}), \\ \det(D)[-(n-3)(-2)^{n-4}] = [-(n-2)(-2)^{n-3}]^2 - [\det(D_{n1})]^2 \end{cases}$$

from which the result immediately follows. $\square$

5.2 Definitions/Computations

The elegance of the Yan-Yeh proof lies in the fact that trees always have at least 2 leaves which can be deleted such that the remaining entries in the distance matrix are unchanged. This holds because the interior (or non-leaf) vertices have a unique path by which the distance metric is obtained. This led us to question what types of graphs have the restriction that no vertex can be deleted without altering the distance metric on the remaining vertices. The question can also be viewed as determining whether there exists a maximal proper induced subgraph H of a graph G such that H embeds isometrically into G. Djokovic studied this idea specifically when G is a hypercube [11] and Winkler expanded on this to prove that isometric embeddings of H into any product of complete graphs are unique [27]. Our goal, however, was to study these isometric embeddings among all graphs. To begin, we define this new class of graphs.

Definition 5.3. *Distance Critical (DC) Graph: A connected graph such that no vertex can be deleted without altering the distance metric on the remaining vertices.*

The existence of leaf vertices in a graph, G, have already been shown to eliminate G from being distance critical. We require the following definition to determine whether the vertices in a graph classify it as distance critical.

Definition 5.4. *Determining Pair for vertex v: a pair of nonadjacent vertices, a and b, with exactly one common neighbor, v.*

These definitions immediately result in the following property.

Proposition 5.5. *A connected graph G is distance critical if and only if for all $v \in V(G)$, v admits a determining pair $\{a, b\}$ where $a, b \in V(G)$.*

Proof. First we prove the forward direction. Note that, if G is complete, then it is not distance critical, and no vertex admits a determining pair. If G is not complete but is connected and distance critical, then let P be a path that witnesses the shortest distance between two nonadjacent vertices, x and y. Let v be an interior vertex of P, and let a and b be the neighbors of v in P. First, notice that a is not adjacent to b; otherwise, v would not be included in the shortest path. Assume there exists another vertex, w, such that a is adjacent to w and b is adjacent to w. Since G is assumed to be distance critical, no vertex can be deleted without altering the distance metric on the remaining vertices. However, by deleting vertex v, the distance from x to y remains unchanged since the path $xPawbPy$ has the same length as P. This is a contradiction. Therefore, v must be the unique common neighbor of some two nonadjacent vertices, as desired.

Now for the reverse direction. Let v be some vertex of a connected graph G and choose two other vertices, x and y, from $G-v$. If there exists a shortest path between x and y that does not contain v, then the distance between x and y in $G-v$ and G is the same. Now, consider when v lies on every shortest path between vertices x and y. Choose one of these shortest paths and call it P. Let a and b be the neighbors of v that lie on P. First, notice that a is not adjacent to b; otherwise, v would not be included in the shortest path. Assume there exists another vertex, $w \neq v$, such that a is adjacent to w and b is adjacent to w. However, by deleting vertex v, the distance from x to y remains unchanged in $G-v$ since the path $xPawbPy$ is present in $G-v$ and has the same length as P. Therefore, deleting vertex v will never affect the distance metric and G is distance critical by contraposition. □

For an example of a distance critical graph, see Figure 5.1.

Proposition 5.5 immediately implies the following corollary.

Corollary 5.6. *Distance critical implies minimum degree at least* 2*.*

We were able to compute the number of distance critical graphs up until 11 vertices using Sage. See Table 5.1 for these results. This became our database from which to derive conjectures for distance critical graphs. Clearly, there are no distance critical graphs on one or two vertices since there are not enough vertices to form a determining

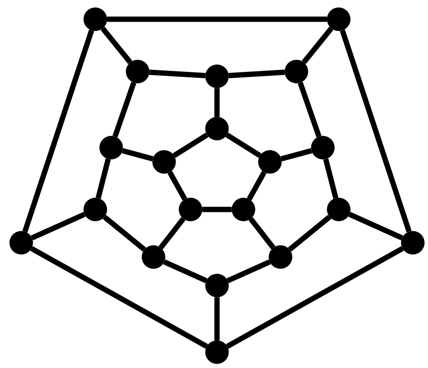

Figure 5.1 Dodecahedron

pair. Further, there are no distance critical graphs on three or four vertices given the restrictions imposed by Proposition 5.5. The first distance critical graph that occurs is the cycle on five vertices, as shown in Figure 5.2.

Table 5.1 The total number of distance critical graphs on n vertices.

n	Number of DC graphs
1	0
2	0
3	0
4	0
5	1
6	1
7	4
8	15
9	168
10	2,252
11	94,504

The maximum possible number of edges in a graph on n vertices is known to be $n(n-1)/2$. If $N = n(n-1)/2$, then the *edge density* of a graph G is the ratio of the number of edges in the graph and the total possible number of edges in the graph, i.e. $|E(G)|/N$. After analyzing our database up to 11 vertices, it appeared that the edge densities of the graphs were rather sparse, leading to the following conjecture.

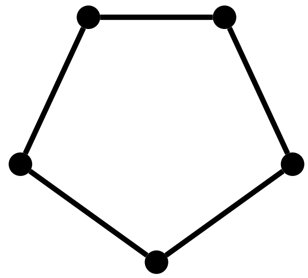

Figure 5.2 The cycle on 5 vertices (C_5)

Conjecture 2. *Distance critical graphs have a maximum edge density of* $1/2$.

The definition of distance critical graphs relies on a local property (requiring that each vertex has a determining pair) which did not prove useful in proving this edge density hypothesis. We therefore tried to narrow our search to detect some other property that inhibits high density distance critical graphs. Notice that a graph being distance critical does not require that it has maximum density. Consider, for example, the cycle on 8 vertices as shown in Figure 5.3.

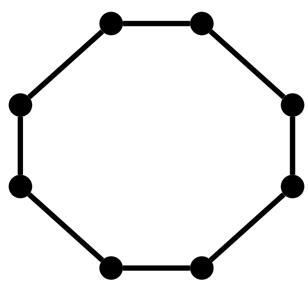

Figure 5.3 The cycle on 8 vertices (C_8)

C_8 is clearly a distance critical graph; however, edges can be added without breaking it's distance criticality. A graph G with a given graph property P is said to be *edge-maximal* if adding any additional edge to G destroys P. Consider the example presented in Figure 5.4 that depicts the maximal distance critical graph on 8 vertices that contains C_8 as a subgraph.

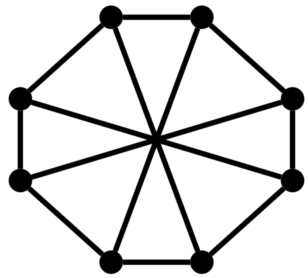

Figure 5.4 Maximal DC cycle on 8 vertices

With this restriction, we adjusted our previous Sage code to display only the maximal distance critical graphs. See Table 5.2 for these results.

Table 5.2 The total number of maximal distance critical graphs on n vertices.

n	Number of Maximal DC graphs
1	0
2	0
3	0
4	0
5	1
6	1
7	2
8	4
9	14
10	82
11	557

5.3 Distance Critical Results

With this new database in hand, we were able to prove the following results for distance critical graphs. We began by considering products of graphs to obtain some larger examples.

Definition 5.7. *The* cartesian product *of two graphs G and H, denoted $G\square H$, is a graph such that $V(G\square H) = V(G) \times V(H)$. Two vertices (x, y) and (x', y') are adjacent in $G\square H$ if and only if either*

- $x = x'$ *and* $y \sim y'$ *or*
- $y = y'$ *and* $x \sim x'$.

Lemma 5.8. *The cartesian product of a distance critical graph and any other graph will be distance critical.*

Proof. Consider $w \in V(H)$ and $(v, w) \in G\square H$ with G a distance critical graph and H be some other graph. Let $v \in V(G)$ with a and b as the determining pair for v. Since a and b are the determining pair for v, this implies that (v, w) is adjacent to (a, w) as well as to (b, w). Further, (a, w) and (b, w) are not adjacent because there is no edge between a and b in G. Assume there is another neighbor (x, y) that is adjacent to (a, w) and (b, w). If (x, y) is adjacent to (a, w) either $x = a$ or $y = w$. If $x = a$, this implies w is adjacent to y. We are also assuming that $(x = a, y)$ is adjacent to (b, w). This only occurs if $y = w$ and a is adjacent to b. This is not possible because a and b are the determining pair of vertex v and not adjacent. Therefore $x \neq a$. The other option is if $y = w$. This would then imply that x is adjacent to b and x is adjacent to a. Since a and b are the determining pair of vertex v, this implies that $x = v$ and thus $(x, y) = (v, w)$. Therefore, every vertex of $G\square H$ has a determining pair and the cartesian product is distance critical as desired. $\square$

Definition 5.9. *The* tensor product *of two graphs G and H, denoted $G \times H$, is a graph such that $V(G \times H) = V(G) \times V(H)$. Two vertices (x, y) and (x', y') are adjacent in $G \times H$ if and only if $x \sim x'$ and $y \sim y'$.*

Lemma 5.10. *If the tensor product of two graphs, G and H, is not distance critical, then either G or H is not distance critical.*

Proof. Assume both G and H are distance critical and consider $(v, w) \in G \times H$. Let $v \in V(G)$ with a and b as the determining pair for v. Let $w \in V(H)$ with c and d as the determining pair for w. Since a and b are adjacent to v and c and d are adjacent to w, this implies that (v, w) is adjacent to (a, c) as well as to (b, d). Further, (a, c) and (b, d) are not adjacent because there is no edge between a and b in G. Assume there is another neighbor (x, y) that is adjacent to (a, c) and (b, d). If (x, y) is adjacent to (a, c), then x is adjacent to a and y is adjacent to c. If (x, y) is adjacent to (b, d), then x is adjacent to b and y is adjacent to d. Therefore, overall x is adjacent to a and b and y is adjacent to c and d. From our assumption, this implies that $x = v$ and $y = w$. Therefore, (v, w) is the unique common neighbor between vertices (a, c) and (b, d) and the tensor product is distance critical as desired. $\square$

An example of non-distance criticality being preserved would be letting $G = C_5$ and $H = C_4$.

Definition 5.11. *The* strong product *of two graphs G and H, denoted $G \boxtimes H$, is a graph such that $V(G \boxtimes H) = V(G) \times V(H)$. Two vertices (x, y) and (x', y') are adjacent in $G \boxtimes H$ if and only if*

- *$x = x'$ and $y \sim y'$ or*
- *$y = y'$ and $x \sim x'$ or*
- *$x \sim x'$ and $y \sim y'$.*

Lemma 5.12. *If the strong product of two graphs, G and H, is not distance critical, then either G or H is not distance critical.*

Proof. Assume G and H are distance critical graphs and consider $(v, w) \in G \boxtimes H$. Let $v \in V(G)$ with a and b as the determining pair for v. Let $w \in V(H)$ with c and d as the determining pair for w. Since a and b are adjacent to v and c and d are adjacent to w, this implies that (v, w) is adjacent to (a, c) as well as to (b, d). Further, (a, c) and (b, d) are not adjacent because there is no edge between a and b in G. Assume there is another neighbor (x, y) that is adjacent to (a, c) and (b, d). If (x, y) is adjacent to (a, c), then either $x = a$ or x is adjacent to a. If $x = a$, then y must be adjacent to c. Further, we are assuming that $(x = a, y)$ is adjacent to (b, d). This implies that a is adjacent to b which is not true since a and b are the determining pair for v. Therefore, $x \neq a$, and we can assume that x is adjacent to a. The same reasoning can be applied to (x, y) and (b, d) to conclude that x must be adjacent to b. Therefore, x is adjacent to a and b, so $x = v$. Once again, the same reasoning applies to the second coordinate so that $y = w$ and (v, w) will be the unique common neighbor between (a, c) and (b, d). Therefore, $G \boxtimes H$ is distance critical as desired. □

We next present results based on the structural properties of distance critical graphs. The *girth* of a graph G is the length of the smallest cycle in the graph.

Lemma 5.13. *Let g represent the girth of a graph. A graph, G, that has minimum degree at least 2 and girth $g > 4$ must be distance critical.*

Proof. Consider the contrapositive of this statement. Let G be a graph of minimum degree at least 2 that is not distance critical. Since G is not distance critical, Proposition 5.5 implies that there exists a vertex v which admits no determining pair. However, v must have at least 2 neighbors, call them a and b. If a and b are adjacent, then the graph contains a triangle, so that $g = 3$. Now, assume a and b are not adjacent, so there must be another vertex w adjacent to both a and b, or else they would be a determining pair for v. Then G contains the 4-cycle $vawbv$. Therefore, $g \leq 4$ in this case as well, so, if G is a 2-connected graph that is not distance critical, then $g \leq 4$ as desired. $\square$

Within the following lemma, a connected graph G is said to be κ-connected if $|V| > \kappa$ and removing any $\kappa - 1$ vertices from G does not disconnect it.

Lemma 5.14. *In a 2-connected distance critical graph, every vertex is contained in a cycle of length at least* 5*.*

Proof. Let v be some vertex in G. By Proposition 5.5, there exist two nonadjacent neighbors, a and b, of v, so that v is their unique common neighbor. Further, Corollary 5.6 guarantees that a has degree at least 2. Since a and b are nonadjacent, there must exist some other vertex, $x \neq v$, that is adjacent to a. This vertex x cannot be adjacent to b; otherwise, v would not be the unique common neighbor of a and b. We can use the same argument to show that there exists a vertex, $y \neq v$, that is adjacent to b but not adjacent to a. It follows also that $x \neq y$. Now, G is 2-connected; therefore, we can delete vertex v and there will still exist a path, P, between x and y in $G-v$. The argument that there exists some cycle of length at least 5 that contains v is made via several cases.

- If a and b do not lie on the path P, then $xavbyPx$ is already a cycle in G with length at least 5.

- Assume without loss of generality that b lies on P but a does not. Then, there must exist some other vertex, call it c, that lies between x and b on P, since x and b are non-adjacent in G and therefore also in $G - v$. Therefore, the graph G contains the cycle $xavbPx$ of length ≥ 5.

- If both a and b lie on P, there must exist some vertex, call it e, that lies between a and b on P (because a is not adjacent to b). However, if this was the only other vertex of aPb, then v would not be the unique common neighbor of a and b. Therefore, the path aPb has length at least 2, and once again, the graph G then contains the cycle $avbPa$ of length at least 5.

This completes all possible cases; therefore, a distance critical graph will have every vertex contained in a cycle of length at least 5 as desired. □

A *dominating vertex* is a vertex that is adjacent to all other vertices.

Lemma 5.15. *Distance critical graphs cannot have a dominating vertex.*

Proof. Assume that this can occur. Let G be a distance critical graph with v a dominating vertex. Let w be some other vertex of the graph. Since G is distance critical, w has two nonadjacent neighbors, call them a and b. These vertices, a and b, are also adjacent to v since v is a dominating vertex. This, however, means that w is not the unique common neighbor of a and b; therefore, the graph cannot be distance critical which contradicts the original assumption. □

For the following three lemmas, let n represent the number of vertices in the graph. A *regular graph* is a graph where all vertices have the same degree.

Lemma 5.16. *Every distance critical graph has a minimum of n edges.*

Proof. In order to be distance critical, every vertex of a graph must have a determining pair; therefore, the degree of every vertex must be at least 2. To minimize the number of edges, we want every vertex to have exactly degree 2. This results in a cycle which has n edges. □

Lemma 5.17. *The vertices of a distance critical graph have maximum degree $n-4$.*

Proof. Proposition 5.15 guarantees that no vertex can have degree $n-1$.

Let v be any vertex of a distance critical graph and assume it has degree $n-2$. Then v is adjacent to every other vertex except one; label this exception as u. Let S be the set of vertices that are adjacent to v. We know that u must have a determining pair, and that u is not adjacent to v. Therefore, u must be adjacent to 2 vertices in S, call them x and w. However, x and w are also both adjacent to v, so u is not their unique neighbor. Therefore, u cannot have a determining pair so v cannot have degree $n-2$.

Consider if v has degree $n-3$. Once again, let S be the set of $n-3$ vertices adjacent to v and label the remaining 2 vertices u_1 and u_2. All vertices in S must have a determining pair. Let w be some vertex in S. The options for a determining pair for w are $vu_1(A), vu_2(B), u_1u_2(C), xu_1(D)$, and $xu_2(E)$ where x is some other vertex in S. Assume w has option A as a determining pair. Then w is adjacent to u_1 and u_1 cannot be adjacent to any other vertex in S; otherwise, w would not be the unique common neighbor of v and u_1. Therefore, the remaining vertices of S must have determining pairs given by either options B or E. In either case, this requires u_2 to be nonadjacent to all other vertices of S due to the same reasoning as the previous case, so the remaining vertices of S cannot have a determining pair. Therefore, no vertex of S can have a determining pair of A or B.

Consider u_1. We know that u_1 is not adjacent to v, therefore, assume the determining pair has both vertices in S. This contradicts that the determining pair has a unique vertex u_1 (because all vertices in S are adjacent to v). Therefore, u_1 must be adjacent to u_2 and some other vertex in S. Therefore, no vertex can have u_1u_2 as a determining pair because they are adjacent, so C is not an option.

Therefore, the vertices of S must have determining pairs with either option D or E. Assume w is a vertex of S with determining pair D. Then w is adjacent to u_1y

where y is another vertex of S with determining pair E. However, y is adjacent to u_2 and u_1 is adjacent to u_2 so w is not a common neighbor. The same argument applies if we started with a vertex in S with determining pair E. Therefore, all the vertices cannot have a determining pair, so the graph cannot be distance critical with $n-3$ degree. □

Lemma 5.18. *The vertices of a regular distance critical graph have maximum degree* $\frac{n-1}{2}$.

Proof. First, we prove the upper bound. Every vertex in a distance critical graph, G, has a determining pair. Let $\{x, y\}$ be the determining pair for some vertex v. We know that the neighborhoods of x and y only intersect at v; otherwise, $\{x, y\}$ would not be a determining pair. Therefore, the remaining $n-3$ vertices lie in the neighborhood of x, the neighborhood of y, or are not adjacent to x nor y. Further, since x and y are both adjacent to v, we have that $\deg(x)+\deg(y) \leq n-3+2 = n-1$. In a regular graph, all vertices have the same degree; therefore, $\deg(x) = \deg(y)$ so that $\deg(x) \leq \frac{n-1}{2}$ as desired.

Now, we construct a $\frac{n-1}{2}$-regular distance critical graph to show that this bound is strict. Let G be a graph with vertex set $\{0, 1, 2, ..., n-1\}$. For every $i \in V(G)$, i is adjacent to $i \pm j \pmod{n}$ for $1 \leq j \leq \frac{n-1}{4}$. Therefore, we have a $\frac{n-1}{2}$-regular graph. Now we show that this graph is indeed distance critical. Choose some $i \in V(G)$. This vertex has the determining pair $\{i + \frac{n-1}{4}, i - \frac{n-1}{4}\}$ because $i + \frac{n-1}{4}$ is not adjacent to $i - \frac{n-1}{4}$ and $i + \frac{n-1}{4}$ is adjacent to $i + \frac{n-1}{4} \pm j$ for $1 \leq j \leq \frac{n-1}{4}$ while $i - \frac{n-1}{4}$ is adjacent to $i - \frac{n-1}{4} \pm j$ for $1 \leq j \leq \frac{n-1}{4}$. Therefore, i is the only common neighbor between $i + \frac{n-1}{4}$ and $i - \frac{n-1}{4}$, and the graph is distance critical as desired. □

The results presented thus far are useful in constructing distance critical graphs. We next move towards determining properties necessary in constructing maximal distance critical graphs in an effort to upper bound the edge density.

Lemma 5.19. *If G is a distance critical graph with $x, y \in V(G)$ such that $d_G(x, y) > 3$, then $G + xy$ is distance critical as well.*

Proof. Suppose G is distance critical and $G+xy$ is not. Then there exists a $v \in V(G)$ such that v has a determining pair, call it (w, z), in G but not in $G + xy$. Therefore, w and z are adjacent to v, but $wz \notin E(G)$. Further, for all $u \in V(G \backslash \{v, w, z\})$, $uw \notin E(G)$ or $uz \notin E(G)$ or both are non-edges of G. We assumed; however, that (w, z) is not a determining pair for v in $G + xy$. Two cases must be considered: (1) $wz \in E(G + xy)$ or (2) there exists another vertex $u \in V(G \backslash \{v, w, z\})$ such that $uw \in E(G + xy)$ and $uz \in E(G + xy)$.

In case (1), $wz \in E(G+xy)$. Without loss of generality, let $w = x$ and $z = y$. In G, (w, z) was a determining pair for v; therefore, $wv \in E(G + xy)$ and $vz \in E(G + xy)$. Since $w = x$ and $z = y$, this means that xvy is a P_3 in G which implies that $d_G(x, y) \leq 2$, a contradiction.

In case (2), both $uw \in E(G + xy)$ and $uz \in E(G + xy)$ which eliminates the possibility of (w, z) being a determining pair for v. Without loss of generality, assume $uw \notin E(G)$ but $uw \in E(G+xy)$. Therefore, $uw = xy$ so that either $u = x$ and $w = y$ (call this case (a)) or $u = y$ and $w = x$ (call this case (b)). Now we assumed $uw \notin E(G)$; therefore, $uz \in E(G)$. First considering case (a), notice that $xzvy$ is a path in G. In case (b), notice again that $xvzy$ is a path in G. Either way, $d_G(x, y) \leq 3$, a contradiction. ☐

Lemma 5.20. *Let G be a distance critical graph and $v \in V(G)$ such that $C = G - v$ is a cycle. Then v must be included in a determining pair for some other vertex.*

Proof. Assume that v is not part of a determining pair for any other vertex. If G is distance critical, then v must have a determining pair itself. We consider the possible locations of where this determining pair could occur to reach a contradiction. Label the vertices of C as $0, 1, ..., n-1$. Without loss of generality, let 0 be one of the vertices of the determining pair for v.

Clearly, the other vertex of the determining pair, call it j, cannot be adjacent to 0; therefore, $j \geq 2$.

Consider the case when $j = 2$, i.e. $(0, 2)$ is a determining pair for v. The vertices 0 and 2, however, are the only choice for the determining pair of vertex 1; therefore, they cannot also have a common neighbor with vertex v. This implies that $j \geq 3$.

Consider the case when $j = 3$. We know that $2v \notin E(G)$; otherwise, 1 would not have a determining pair since its neighbors, 0 and 2, would have another common neighbor with vertex v. The same type of reasoning can be applied to vertex 2 to say that $1v \notin E(G)$. We see then, that $01 \in E(G)$ and $0v \in E(G)$ such that 0 is the only common neighbor of vertices 1 and v; therefore, $(1, v)$ is a determining pair for 0, a contradiction. Therefore, $j \geq 4$.

Consider the case when $4 \leq j \leq n-1$. We know that $2v \notin E(G)$, otherwise, 1 would not have a determining pair. If $1v \in E(G)$, then $(2, v)$ would be a determining pair for 1 unless $3v \in E(G)$. This, however, results in the fact that $12 \in E(G)$, $23 \in E(G)$, and both 2 and v are common neighbors of vertices 1 and 3. Therefore, vertex 2 cannot have a determining pair so G cannot be distance critical, a contradiction. Therefore, $1v \notin E(G)$. This implies, however, that $(1, v)$ is a determining pair for 0, a contradiction. $\square$

Lemma 5.21. *Let G be a distance critical graph and $v \in V(G)$ such that $F = G - v$ is also distance critical. If v is not part of a determining pair for any vertex, then $deg(v) > 3$.*

Proof. Corollary 5.6 tells us that $\deg(v) \geq 2$. Assume that $\deg(v) = 2$, and label the neighbors of v as x and y. Since G is distance critical, (x, y) must be the determining pair of v; therefore, $xy \notin E(G)$. Since F is distance critical, we know that x must have a determining pair that does not include v. We know that x is not adjacent to y, so x must have two additional neighbors that are a determining pair, call them u and w. We claimed that $\deg(v) = 2$; therefore, $uv \notin E(G)$. This implies that (u, v) is a determining pair for x, unless $uy \in E(G)$. This contradicts the fact that (x, y) is a determining pair for v, so $\deg(v) \neq 2$.

Assume that $\deg(v) = 3$, and label the neighbors of v as x, y, and z. Since G is distance critical, v has a determining pair. Without loss of generality, let (x, y) be the determining pair for v; therefore, $xy \notin E(G)$. Further, x has a determining pair that does not include vertex v. Therefore, x is adjacent to at least 2 other vertices and since $xy \notin E(G)$, at least one of these vertices must be distinct. Label this new vertex as u. This vertex, u, must be nonadjacent to y; otherwise, (x, y) would not be a determining pair for v.

The same type of reasoning implies that y has a distinct neighbor, call it w such that $xw \notin E(G)$.

Now, we assumed that $\deg(v) = 3$; therefore, $uv \notin E(G)$ and $wv \notin E(G)$. We also assumed that v is not included in any determining pairs; therefore, u must be adjacent to a neighbor of v so that (u, v) is not a determining pair for x. The only possibility is if $uz \in E(G)$. The same type of reasoning implies that $wz \in E(G)$.

From here, we can state that at least one of the pairs, xz or yz is not an edge. Indeed, assume $yz \in E(G)$. Then $xz \notin E(G)$ so that (x, y) is a determining pair for v. Therefore, without loss of generality, assume $xz \notin E(G)$.

We know that x has a determining pair that does not include vertex v; therefore, x has at least one other distinct neighbor. For all $p \in N(x) - v$, $py \notin E(G)$ so that (x, y) is a determining pair for v. Further, $pv \notin E(G)$ since $\deg(v) = 3$.

In order to make (v, p) not a determining pair for x, p must be adjacent to z. This, however, means that for all $p \in N(x)$, $xp \in E(G)$ and $pz \in E(G)$. Therefore, x cannot have a determining pair, a contradiction. We conclude that $\deg(v) > 3$ as desired. □

Lemma 5.22. *Let G be a distance critical graph on n vertices and let S be the set of vertices which are involved in some determining pair. Then $|S| \geq \sqrt{2n}$.*

Proof. Since G is distance critical, each of the n vertices has a determining pair. A determining pair consists of 2 vertices; therefore, $\binom{|S|}{2} \geq n$. Solving, we see that asymptotically, $|S| \geq \sqrt{2n}$ as desired. □

Within the following lemma, $\overline{G}$ represents G complement. The *complement* of a graph G is a graph H on the same vertex set such that two vertices are adjacent in H if and only if they are not adjacent in G.

Lemma 5.23. *Let G be a distance critical graph and $z \in V(G)$. If z has a determining pair in G but z has no determining pair in $G + xy$ for some $xy \in E(\overline{G})$, then the set of z-determining pairs in G is a star in $\overline{G}$ whose center is either x or y.*

Proof. Two cases exist for ways in which the addition of the edge xy can interfere with the fact that z has a determining pair: (1) (x, y) is the only determining pair for z or (2) xy interferes with all determining pairs for z, i.e. for every determining pair (u, v), (u, v) is no longer a determining pair for z in $G + xy$.

In case (1), (x, y) was the only determining pair of z in G. In $G+xy$, this non-edge becomes an edge so that (x, y) is no longer a determining pair. Therefore, the unique determining pair of z forms a star since an edge is itself a star.

In case (2), the edge xy interferes with all determining pairs of z. If the addition of xy destroys the determining pair (u, v), then u and v must have another common neighbor after xy is added besides z. Without loss of generality, consider v within the determining pair of z. There exists some other vertex, $y \in V(G)$ such that $vy \in E(G)$. The addition of xy interferes with all determining pairs; therefore, there exists some $x \in V(G)$ such that $uy = xy$ so that $u = x$. Another way of viewing this is that x is contained in every determining pair of z so that the set of determining pairs of z form a star with center x. $\square$

Corollary 5.24. *Let S be the set of vertices which are involved in some determining pair in an edge-maximal distance critical graph, G. Then every non-edge of G intersects the set S.*

Proof. Let $xy \in E(\overline{G})$. Since G is edge critical distance critical, there exists a $z \in V(G)$ such that z does not have a determining pair in $G + xy$. Lemma 5.23 tells us that the set of determining pairs of z forms a star with center either x or y. Since S is the union of all the determining pairs of all vertices (including z), this implies that xy intersects S as desired. $\square$

Corollary 5.25. *If G is an edge-maximal distance critical graph and S is the set of vertices involved in some determining pair, then the set of vertices $T = V(G) - S$ induces a clique.*

Proof. Assume not, that instead there are two vertices x and y of T which are non-adjacent in G. Since G is edge-maximal, it must also be edge-critical. We assumed $xy \in E(\overline{G})$ and does not intersect S, which contradicts Corollary 5.24. $\square$

A *clique* is a set of vertices such that all vertices are adjacent. If we could determine the maximum clique size in a distance critical graph, then we could obtain a closer estimate of the upper bound for the edge density. Since a determining pair must consist of nonadjacent vertices, the determining pair for vertices within the clique have two options: (1) the determining pair consists of one vertex within the clique and one vertex outside the clique (call these type A vertices), and (2) the determining pair consists of vertices outside the clique (call these type B vertices). This led to the following construction and theorem.

Definition 5.26. *Let G_m be a graph which has a vertex set divided into 3 sets. Set A has $m(m-1)/2$ vertices labeled as a_{ij} for $0 \leq i < j < m$, set B has m vertices labeled as b_j for $0 \leq j < m$, and set C has $2m$ vertices labeled as c_j for $0 \leq j < 2m$ (understood modulo $2m$). The edge set for G_m is obtained by considering the relationship between and among each of the sets. First, every pair of vertices of set A is adjacent forming a clique of size $m(m-1)/2$. Also, edges (a_{ij}, b_i) and (a_{ij}, b_j) exist for every element a_{ij} of set A. For every vertex b_j of set B, edges (b_j, c_j) and (b_j, c_{j+m}) exist. Lastly, for every element c_j of set C, the edge $(c_j, c_{j+1(mod 2m)})$ exists.*

The graph G_5 is shown in Figure 5.5 to help visualize this construction. The red vertices lie in set A, the blue vertices are in set B, and the black vertices are in set C.

Theorem 5.27. *Among distance-critical graphs G on n vertices, the maximum clique number of G is $n - \Theta(\sqrt{n})$.*

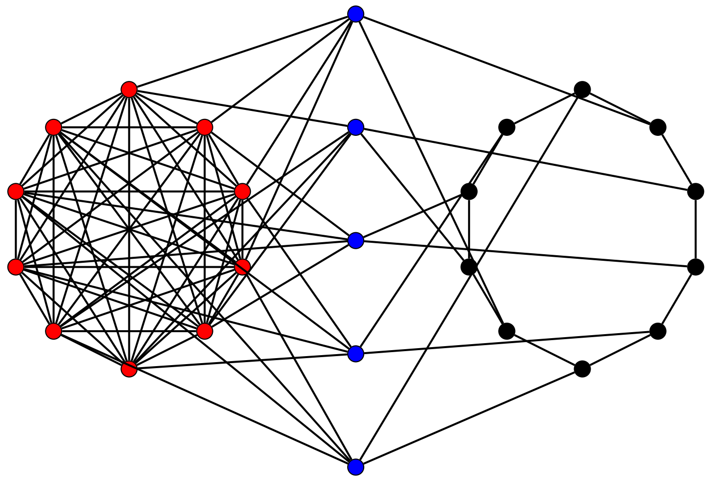

Figure 5.5 G_5

Proof. Consider the graph G_m. First, we argue that G_m is indeed distance critical by noting the determining pairs for each vertex. Consider the vertex a_{ij}. This vertex has the determining pair $\{b_i, b_j\}$ because b_i is not adjacent to b_j and b_i is adjacent to a_{ij}, c_i, and c_{i+m} while b_j is adjacent to a_{ij}, c_j, and c_{j+m}. Therefore, a_{ij} is the only common neighbor between b_i and b_j. Consider vertex b_j. This vertex has determining pair $\{c_j, c_{j+m}\}$ because c_j is not adjacent to c_{j+m} and c_j is adjacent to b_j and c_{j+1} while c_{j+m} is adjacent to b_j and c_{j+m+1}. Therefore, b_j is the only common neighbor between c_j and c_{j+m}. Lastly, consider vertex c_j. This vertex has determining pair $\{c_{j-1}, c_{j+1}\}$ because c_{j-1} is not adjacent to c_{j+1} and c_{j-1} is adjacent to c_j and b_{j-1} while c_{j+1} is adjacent to c_j and b_{j+1}. Therefore, c_j is the only common neighbor between c_{j-1} and c_{j+1}.

Now we establish the clique number. By construction of G_m, a clique of size $m(m-1)/2$ is induced by the vertex set $\{a_{ij}\}_{0\leq i<j<m}$ with a remaining $3m$ vertices of the form b_j or c_j. Therefore, $n = |V(G_m)| = \binom{m}{2} + 3m = m^2/2 + O(m)$ so that $m = \sqrt{2n}(1+o(1))$. From here, we see that $\max_G \omega(G) \geq n-(3+o(1))\sqrt{2n} = n-O\left(\sqrt{n}\right)$.

Now we establish a matching upper bound. Consider a DC graph G on n. Every vertex v in a max clique K of size m must have a determining pair, say, $\{x_v, y_v\}$. Let $S = \bigcup_{v \in K}\{x_v, y_v\}$. Note that, for each $v \in K$, $|\{x_v, y_v\} \setminus K| = 1$ or 2, because if it were zero, then $\{x_v, y_v\} \subset K$ whence $x_v y_v \in E(G)$, contradicting that $\{x_v, y_v\}$ is a determining pair. Let A be the subset of $V(K)$ with $|\{x_v, y_v\} \setminus K| = 1$ and $B = K \setminus A$. For each vertex $v \in A$, wlog we assume $x_v \in K$ and $y_v \notin K$. Note that the y_v are distinct across all $v \in A$, since, if $y_v = y_w$ for some $w \in A$, then x_v and y_v have common neighbors v and w, contradicting that they form a determining pair for v. Thus,

$$|V(G-K)| \geq |\{y_v : v \in A\}| \geq |A|.$$

On the other hand, the pairs $\{x_v, y_v\}$ for $v \in B$ are entirely contained in $V(G-K)$. Since none of these pairs are repeated (or else they could not be determining pairs), the graph $(\bigcup_{v \in B}\{x_v, y_v\}, \{x_v y_v : v \in B\})$ has $|B|$ edges and therefore at least $\sqrt{2|B|}$ vertices, all of which lie outside K. Therefore,

$$|V(G-K)| \geq \max\{|A|, \sqrt{2|B|}\} = \max\{|A|, \sqrt{2(m-|A|)}\} \geq \sqrt{2m},$$

since $0 \leq |A| \leq m$. Then G contains at least $\sqrt{2m}$ vertices in addition to the clique K, and so $n \geq m + \sqrt{2m}$ which implies $m \leq n - \sqrt{(2+o(1))n}$, and we may conclude that $\omega(G) \leq n - \Omega(\sqrt{n})$. $\square$

Within the previous proof, the constants obtained in proving the upper and lower bounds did not match. We were able to narrow the difference between these constants by considering the following construction.

Definition 5.28. *Let G'_m be a graph which has a vertex set divided into 3 sets. Set A has $m(m-1)/2$ vertices labeled as a_{ij} for $0 \leq i < j < m$, set B has m vertices labeled as b_j for $0 \leq j < m$, and set C has m vertices labeled as c_j for $0 \leq j < m$.*

The edge set for $G_{m'}$ is obtained by considering the relationship between and among each of the sets. First, every pair of vertices in set A are adjacent forming a clique of size $m(m-1)/2$. Also, edges (a_{ij}, b_i) and (a_{ij}, b_j) exist for every element a_{ij} of set A. For every vertex b_j of set B, edge (b_j, c_j) exists. Lastly, every pair of vertices in set C are adjacent forming a clique of size m.

The graph G'_5 is shown in Figure 5.6 to help visualize this construction. Once again, the red vertices lie in set A, the blue vertices in lie in set B, and the black vertices lie in set C.

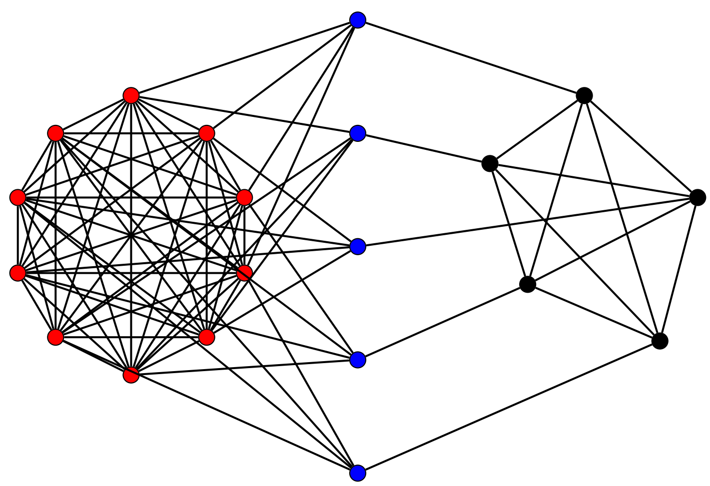

Figure 5.6 G'_5

The constructions of G_m and G'_m led to the following two theorems.

Theorem 5.29. *Every graph is an induced subgraph of some distance critical graph.*

Proof. Let G be any graph on n vertices. If $n = 1$, clearly a single point is an induced subgraph of some distance critical graph. Now consider $n = 2$. Either G is a path of length 1, or G has two disconnected vertices. Every distance critical graph has a pair of adjacent vertices; therefore, it includes an induced subgraph consisting of a path of length 1. Further, every distance critical graph has a pair of nonadjacent vertices that make up a determining pair; therefore, it includes an induced subgraph consisting of two disconnected vertices.

Now, consider when $n \geq 3$. We know that every vertex of G must have a determining pair if it is an induced subgraph of some distance critical graph. Using the construction of G_m, let G be the set of A vertices so that $n = \binom{m}{2}$. We need an additional m vertices to make up the B set where $m^2 - m - 2n = 0$; therefore, add $m = \lceil \frac{1+\sqrt{1+8n}}{2} \rceil$ additional vertices to account for the B set. Label the vertices of G as a_{ij} for $0 \leq i < j < m$ and stop once all n vertices have been labeled. Label the added vertices b_j for $0 \leq j < m$. Add edges (a_{ij}, b_i) and (a_{ij}, b_j) for every element a_{ij} of set A (or the original graph G). We showed when defining G_m that this ensures every element of set A has a determining pair. Similarly, add an additional $2m = \lceil 1 + \sqrt{1+8n} \rceil$ vertices to make up set C and label these vertices as c_j for $0 \leq j < 2m$ (understood modulo $2m$). Add edges (b_j, c_j) and (b_j, c_{j+m}) for $0 \leq j < m$ so that every element in B has a determining pair. Lastly, add edges $(c_j, c_{j+1 \pmod{2m}})$ for $0 \leq j < 2m$ so that every element in C has a determining pair. Therefore, the newly constructed graph is distance critical. We can then conclude that any graph can be input into the G_m construction as set A, thereby proving that it is an induced subgraph of some distance critical graph. $\square$

We conclude the chapter on distance critical graphs by disproving Conjecture 2.

Theorem 5.30. *Among distance critical graphs G on n vertices, the maximal edge density of G is between $1 - O(1/\sqrt{n})$ and $1 - \Omega(1/n)$.*

Proof. G_m and $G_{m'}$ have approximately $N^2 - cN^{3/2}$ edges for $N = \dfrac{m(m+5)}{2}$ and $N = \dfrac{m(m+3)}{2}$ respectively. Further, we know that every vertex of a distance critical graph must have a determining pair which creates at least N non-edges. Therefore, the maximal edge density must be between $1 - O(1/\sqrt{n})$ and $1 - \Omega(1/n)$. $\square$

Chapter 6

Open Problems

Tolkien embodied research when he stated, "You certainly usually find something, if you look, but it is not always quite the something you were after" [26]. Much of the work presented in this dissertation arose from taking detours as we tried to answer other questions. While we were able to pinpoint certain results, many problems remain open to exploration in both these areas of research. We leave the reader by presenting some of these questions.

6.1 Steiner k-matrix

It is now proven that the Steiner distance k-matrix has a zero hyperdeterminant if and only if $n = 1$, k is odd and $n > 2$, or $k \equiv 1 (\text{mod } 6)$ and $n = 2$. We hope, however, to completely generalize the Graham-Pollak theorem to find a formula that only depends on k and n for the Steiner hyperdeterminant.

Question 6.1. *What is the hyperdeterminant of the Steiner distance k-matrix for a tree on n vertices?*

For k odd, the question is mostly answered since the hyperdeterminant is always zero when $n \geq 3$ or when $k \equiv 1 (\text{mod } 6)$ and $n = 2$. The proof for an even k appears to be much harder. Our Sage code database, provides evidence that suggests the result does generalize to even order Steiner distance hypermatrices. These calculations are depicted in Table 6.1 below. Note that the case $k = 2$ is excluded since that is the

Graham-Pollak result stating that the determinant of the distance matrix for any tree is given by $-(n-1)(-2)^{n-2}$.

Table 6.1 The hyperdeterminant of the Steiner k-matrix for trees of order n.

k	n	Hyperdeterminant
4	2	$-1 \cdot 2^2 \cdot 7$
	3	$2^{12} \cdot 7 \cdot 23^4$
	4	$-1 \cdot 2^{38} \cdot 3^{27} \cdot 5^6 \cdot 7 \cdot 13^{12}$
	5	$2^{203} \cdot 5^{32} \cdot 7 \cdot 11^{32} \cdot 23^{24} \cdot 37^8$
6	2	$-1 \cdot 11^2 \cdot 31$
	3	$2^{14} \cdot 3^{16} \cdot 11^4 \cdot 31 \cdot 19231^4$
	4	$-1 \cdot 2^{82} \cdot 3^{17} \cdot 11^8 \cdot 31 \cdot 41^{12} \cdot 71^6 \cdot 89^6 \cdot 151^{24} \cdot 257^{24} \cdot 1511^{12}$
8	2	$-1 \cdot 2^6 \cdot 29^2 \cdot 127$
	3	$2^{56} \cdot 13^{16} \cdot 29^4 \cdot 113^8 \cdot 127 \cdot 1009^8 \cdot 2143^4$

The algorithm provided by Cox, Little, and O'Shea [10] does not seem feasible to help determine the formula for the hyperdeterminant. Instead, we hope to utilize the results presented in [25].

Theorem 6.2 (Theorem 3.2 [25])**.** *Let $\mathbb{A}$ be a tensor with dimension n and order $k \geq 2$. Then we have*

(a) If $\mathbb{B}$ is the tensor obtained from $\mathbb{A}$ by interchanging the i^{th} and j^{th} slices of $\mathbb{A}$ $(i \neq j)$, then $\det(\mathbb{B}) = (-1)^{(k-1)^{n-1}} \det(\mathbb{A})$.

(b) If $\mathbb{B}$ is the tensor obtained from $\mathbb{A}$ by adding c times the i^{th} slice to the j^{th} slice of $\mathbb{A}$ $(i \neq j)$, then $\det(\mathbb{B}) = \det(\mathbb{A})$.

(c) If the i^{th} and j^{th} slices of $\mathbb{A}$ are equal $(i \neq j)$, then $\det(\mathbb{A}) = 0$.

(d) Moreover, if the slices of $\mathbb{A}$ are linearly dependent, then $\det(\mathbb{A}) = 0$. *But the converse of this property does not hold.*

For the case $n = 2$, Z. Du [1] observed from our data that the Steiner distance k-matrix of a tree has a hyperdeterminant given by the Wendt determinant of k-th circulant matrix $C(k)$ with an adjustment to the sign. The *circulant matrix* $C(k)$ is defined as the matrix whose first row is $[c_1, ..., c_k]$ where $c_i = \binom{n}{i-1}$, and subsequent rows are obtained by cyclically shifting the previous row one place to the right. The Wendt determinant can also be viewed as a resultant of the two polynomials $x^k - 1$ and $(x+1)^k - 1$ [19]. This led to the following theorem by Cooper and Du [6]:

Theorem 6.3. *The hyperdeterminant of the Steiner k-matrix of a tree on 2 vertices is given by the Wendt determinant, W_k, up to sign.*

For larger trees, the hyperdeterminant does not appear to have any entries in OEIS. Rather than trying to determine the entire formula for the hyperdeterminant for larger trees, we could begin by proving the following conjectures.

Conjecture 3. *The sign of the hyperdeterminant of the Steiner distance k-matrix for a tree on n vertices is given by $(-1)^{n-1}$.*

Conjecture 4. *The hyperdeterminant of the Steiner distance k-matrix for a tree has a factor of $2^{k-1} - 1$.*

We can view these conjectures holds true for our data shown in Table 6.1. Interestingly, the factor $2^{k-1} - 1$ also appeared at the end of the proof of Theorem 4.5 where we proved the hyperdeterminant is nonzero by taking the difference between $D_{n-1}p_T(\mathbf{x})$ and $D_{n-1}p_{T'}(\mathbf{x})$ where T is a tree on $n+1$ vertices and T' is a tree on n vertices. While it is unclear how these calculations are related, there appears to be some underlying connection.

[1] Personal communication.

Another question that could be explored with these Steiner distance k-matrices would be the following.

Question 6.4. *Can we extend the kernel method argument (presented in Chapter 4.2.2) to prove that the hyperdeterminant of the Steiner distance k-matrix is nonzero?*

This was our original idea when we approached the problem. If we could show the Steiner distance kernel is conditionally strictly positive definite for all trees, this could potentially lead to similar results to prove the set of vectors is affinely independent and the hypermatrix has a nonzero hyperdeterminant. Greenleaf, Iosevich, and Taylor [20] determine a way to embed a tree into a compact subset of $\mathbb{R}^d$ for $d \geq 2$ such that it's kernel is strictly positive. It has yet to be shown how this could then prove the hyperdeterminant of the hypermatrix formed by this kernel is nonzero.

6.2 Distance Critical Graphs

To the best of our knowledge, distance critical graphs are an entirely new class of graphs that have not been studied. Therefore, many open questions still exist to determine the properties of these types of graphs. One such question is presented below. A graph is considered *Hamiltonian* if it contains a cycle that visits each vertex exactly once.

Question 6.5. *What is the smallest distance critical graph such that its complement is not Hamiltonian?*

The data presented in Table 5.1 showed that the complement of distance critical graphs on $n \leq 11$ vertices was Hamiltonian. The construction of G'_m, however, disproved this result. For $m = 5$, the vertices are divided equally among the large clique and the remainder of the graph and create a Hamiltonian cycle. When $m = 6$, this no longer holds true. Therefore, there exists a distance critical graph on 27 vertices whose complement is not Hamiltonian. The smallest one must then occur between order 12 and 27, but we have not been successful in narrowing this bound yet.

Question 6.6. *What fraction of graphs are distance critical?*

We proved that every graph is a subgraph of some distance critical graph. This proof, however, relied on the structure of the distance critical graph G_m and shows that on n vertices, there are at least $2^{\binom{n-O(\sqrt{n})}{2}}$ distance critical graphs. However, we hope to determine the ratio of all distance critical graphs (not just ones with the G_m structure) compared to all possible graphs. One method to possibly accomplish this would be calculating the probability that a random graph is distance critical.

Question 6.7. *We have a sequence for the number of distance critical graphs and the number of maximal distance critical graphs. To what does this enumeration correspond?*

These sequences did not appear in the OEIS; therefore, it would be interesting to first find some other mathematical concept it applies to and then prove its connection to distance critical graphs. Hopefully, determining the fraction of all graphs that are distance critical will give us more insight into this question.

Question 6.8. *Is there some formula for the chromatic number of distance critical graphs?*

To feasibly examine this, we would have to consider specifically the maximal distance critical graphs to avoid the already known cases such as the cycles. Since we do not know much about the global properties of these graphs, it would probably be best to analyze the star graphs formed by the determining pairs as described in

Lemma 5.23. Since each edge in this star graph is actually a non-edge in the original graph, all the vertices within the star could be colored using the same color. The complication would be to consider how these stars overlap.

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Appendix A

SageMath Code

This appendix is included to share the code we used in SageMath 9.7 to calculate the Steiner distance k-matrix of a tree and its hyperdeterminant. We further include the code we used to calculate the Steiner polynomials and Steiner k-ideals. Within the following, '#' indicates a comment to the code and '<<' is used to indicate a line-break and means that the following text should be appended to the previous line. The first block will calculate the Steiner distance among vertices in a tree.

```
# input: G = tree graph, S = list of vertices
# output: Steiner distance of S in G
def steiner_distance(G,S):
    final = False
    Gc = copy(G)
    while not final:
        pendant = []
        count = 0
        for vertex in Gc.vertices():
            if Gc.degree(vertex) == 1:
                pendant.append(vertex)
                if vertex not in S:
                    Gc.delete_vertex(vertex)
```

```
                else:
                    count += 1
        if count == len(pendant):
            final = True
    return Gc.size()
```

The next few blocks will construct the k-Steiner distance hypermatrix of a tree for $2 \leq k \leq 8$.

```
# input: G = tree graph
# output: Steiner 2-matrix
def two_steiner_matrix_tree(G, verbose=False):
    n = G.order()
    if not G.is_tree():
        print('Graph must be a tree.')
        return False
    else:
        distance_matrix = [[0 for col in range(n)] for
        << row in range(n)]
        for x in range(n):
            for y in range(x,n):
                distance = steiner_distance(G,[x,y])/2
                distance_matrix[x][y] = distance
                distance_matrix[y][x] = distance
        if verbose:
            pretty_print(distance_matrix)
        return(distance_matrix)
```

```
# input: G = tree graph
# output: Steiner 3-matrix if G is a tree
def three_steiner_matrix_tree(G, verbose=False):
    n = G.order()
    if not G.is_tree():
        print('Graph␣must␣be␣a␣tree.')
        return False
    else:
        distance_matrix = [[[0 for col in range(n)] for
        << row in range(n)] for x in range(n)]
        for x in range(n):
            for y in range(x,n):
                for z in range(y,n):
                    distance = steiner_distance(G,[x,y,
                    << z])/3
                    distance_matrix[x][y][z] = distance
                    distance_matrix[x][z][y] = distance
                    distance_matrix[y][x][z] = distance
                    distance_matrix[y][z][x] = distance
                    distance_matrix[z][x][y] = distance
                    distance_matrix[z][y][x] = distance
        if verbose:
            pretty_print(distance_matrix)
        return(distance_matrix)
```

```
# input: G = tree graph
# output: Steiner 4-matrix if G is a tree
def four_steiner_matrix_tree(G, verbose=False):
    n = G.order()
    if not G.is_tree():
        print('Graph must be a tree.')
        return False
    else:
        distance_matrix = [[[[0 for col in range(n)] for
        << row in range(n)] for x in range(n)] for y in
        << range(n)]
        for x in range(n):
            for y in range(x,n):
                for w in range(y,n):
                    for z in range(w, n):
                        distance_matrix[x][y][w][z] =
                        << steiner_distance(G,[x,y,w,z])
                        << /4
        if verbose:
            pretty_print(distance_matrix)
        return(distance_matrix)
```

```
# input: G = tree graph
# output: Steiner 5-matrix if G is a tree
def five_steiner_matrix_tree(G, verbose=False):
    n = G.order()
```

```
    if not G.is_tree():
        print('Graph␣must␣be␣a␣tree.')
        return False
    else:
        distance_matrix = [[[[[0 for col in range(n)]
        << for row in range(n)] for x in range(n)] for y
        << in range(n)] for z in range(n)]
        for x in range(n):
            for y in range(x,n):
                for w in range(y,n):
                    for z in range(w, n):
                        for l in range(z,n):
                            distance_matrix[x][y][w][z]
                            << [l] = steiner_distance(G,
                            <<[x,y,w,z,l])/5
        if verbose:
            pretty_print(distance_matrix)
        return(distance_matrix)
```

```
# input: G = tree graph
# output: Steiner 6-matrix if G is a tree
def six_steiner_matrix_tree(G,verbose=False):
    n = G.order()
    if not G.is_tree():
        print('Graph␣must␣be␣a␣tree.')
        return False
```

```
    else:
        distance_matrix = [[[[[[0 for col in range(n)]
        << for row in range(n)] for x in range(n)] for y
        << in range(n)] for z in range(n)] for w in
        << range(n)]
        for x in range(n):
            for y in range(n):
                for w in range(n):
                    for z in range(n):
                        for l in range(n):
                            for m in range(n):
                                distance_matrix[x][y][w]
                                << [z][l][m] = steiner_
                                << distance(G,[x,y,w,z,
                                << l,m])/6
        if verbose:
            pretty_print(distance_matrix)
        return(distance_matrix)
```

```
# input: G = tree graph
# output: Steiner 7-matrix if G is a tree
def seven_steiner_matrix_tree(G,verbose=False):
    n = G.order()
    if not G.is_tree():
        print('Graph must be a tree.')
        return False
```

```
    else:
        distance_matrix = [[[[[[[0 for col in range(n)]
        << for row in range(n)] for x in range(n)] for y
        << in range(n)] for z in range(n)] for w in
        << range(n)] for m in range(n)]
        for x in range(n):
            for y in range(n):
                for w in range(n):
                    for z in range(n):
                        for l in range(n):
                            for m in range(n):
                                for p in range(n):
                                    distance_matrix[x]
                                    << [y][w][z][l][m]
                                    << [p] = steiner_
                                    << distance(G,[x,y,
                                    << w,z,l,m,p])/7
        if verbose:
            pretty_print(distance_matrix)
        return(distance_matrix)
```

```
# input: G = tree graph
# output: Steiner 8-matrix if G is a tree
def eight_steiner_matrix_tree(G,verbose=False):
    n = G.order()
    if not G.is_tree():
```

```
        print('Graph␣must␣be␣a␣tree.')
        return False
    else:
        distance_matrix = [[[[[[[[0 for col in range(n)]
        << for row in range(n)] for x in range(n)] for y
        << in range(n)] for z in range(n)] for w in
        << range(n)] for m  in range(n)] for p in
        << range(n)]
        for x in range(n):
            for y in range(n):
                for w in range(n):
                    for z in range(n):
                        for l in range(n):
                            for m in range(n):
                                for p in range(n):
                                    for q in range(n):
                                        distance_matrix
                                        << [x][y][w][z]
                                        << [l][m][p][q]
                                        << = steiner_
                                        << distance(G,
                                        << [x,y,w,z,l,m,
                                        << p,q])/8
        if verbose:
            pretty_print(distance_matrix)
```

```
        return(distance_matrix)
```

Note that in each of these functions, we must divide the entries in the Steiner distance k-matrix by k. This is due to the following proposition in which $\mathcal{A}\mathbf{x}^{k-1}$ denotes a vector whose i^{th} component is $\sum_{i_2,\ldots i_k=1}^{n} a_{i,i_2,\ldots,i_k} x_{i_2} \cdots x_{i_k}$. In other words, the i^{th} component of $\mathcal{A}\mathbf{x}^{k-1}$ is given by $D_i f_{\mathcal{A}}(\mathbf{x})/k$.

Proposition A.1. *[23] The symmetric hyperdeterminant of $\mathcal{A}$, $\det(\mathcal{A})$, is the resultant of $\mathcal{A}\mathbf{x}^{k-1} = 0$, and is a homogeneous polynomial in the entries of $\mathcal{A}$.*

Using Definition 2.4, the symmetric hyperdeterminant is the resultant of $f_{\mathcal{A}}(\mathbf{x})$ and $\nabla f_{\mathcal{A}}(\mathbf{x})$. Notice that $f_{\mathcal{A}}(\mathbf{x}) = 0$ when $\mathcal{A}\mathbf{x}^{k-1} = 0$. Therefore, $\mathrm{RES}(\mathcal{A}\mathbf{x}^{k-1}) = 0$ if and only if $\det(\mathcal{A}) = 0$. Since both are monic polynomials, $\mathrm{RES}(\mathcal{A}\mathbf{x}^{k-1}) = \det(\mathcal{A})$. We must therefore divide each of the polynomials within $\nabla f_{\mathcal{A}}(\mathbf{x})$ by k to create the polynomials $\mathcal{A}\mathbf{x}^{k-1}$ as desired, thus dividing each entry in the Steiner distance k-matrix by k.

The next block calculates the Steiner k-polynomial given a tree graph G.

```
# input: G = tree graph, k = order
# output: Steiner polynomial of G
def steiner_poly(G,k):
    v = G.num_verts()
    varnames = [var('a_%s' % i) for i in G.vertices()]
    R = PolynomialRing(QQ, names=varnames, k='degrevlex')
    s = R.gens()
    outpoly = R(0)
    V = G.vertices()
    N = len(V)
```

```
    for t in range(N^k):
        u = Integer(t).digits(N,padto=k)
        varprod = prod([s[i] for i in u])
        outpoly += varprod * steiner_distance(G,[V[i]
        << for i in u])
    return outpoly
```

We can also calculate the Steiner k-polynomial directly from the hypermatrix.

```
# input: T = Steiner hypermatrix given as a list of
#lists
# output: Steiner polynomial of T
def build_form(T):
    outpoly = 0
    #get the dimensions of the array
    n=len(T)
    current = deepcopy(T)
    k=0
    while type(current)==list:
        k+=1
        current= current[0]
    # get list of variables
    xvars = list(var('x_%d' % i) for i in range(n))
    # iterate over all tuples of indices
    for R in range(n^k):
        indextuple = Integer(R).digits(base=n,padto=k)
        monomial = prod([xvars[j] for j in indextuple])
```

```
        Tentry = deepcopy(T)
        for j in range(k):
            Tentry = Tentry[indextuple[j]]
        outpoly += Tentry*monomial
    return(outpoly)
```

Similarly, we can calculate the Steiner k-ideal either from the graph itself or it's hypermatrix. Both codes are shown below.

```
# input: G = graph, k = order
# output: Steiner k-ideal
def steiner_ideal(G,k):
    p = steiner_poly(G,k)
    R = p.parent()
    J = R.ideal(gradvec(p))
    return J
```

```
# input: T = Steiner hypermatrix given as a list of
#lists
# output: Steiner k-ideal
def gradvec(T):
    n=len(T)
    current = deepcopy(T)
    k=0
    while type(current)==list:
        k+=1
        current= current[0]
```

```
    # get list of variables
    xvars = list(var('x_%d' % i) for i in range(n))
    F = build_form(T)
    outvec = []
    for j in range(n):
        outvec.append(F.derivative(xvars[j]))
    return outvec
```

Lastly, we include the code to calculate the hyperdeterminant of the Steiner distance k-matrix. The framework of this code was taken from Dutle [13] and adjusted to calculate only the hyperdeterminant (not the characteristic polynomial). The algorithm itself comes from Cox, Little, and O'Shea as described in Chapter 2.2.

```
# input: T = Steiner hypermatrix given as a list of
#lists
# output: Steiner hyperdeterminant
def Steiner_hyperdeterminant(T):
    #get the dimensions of the array
    n=len(T)
    current = deepcopy(T)
    k=0
    while type(current)==list:
        k+=1
        current= current[0]
    #get S (the monomials of total degree d) and
    #initialize the matrix for the resultant,
    d= n*(k-2)+1
```

```
R = []
for i in range(n):
    for j in range(d):
        R.append(i)
L=Subsets(R, d, submultiset=True)
S=[]
for l in L:
    s=[]
    for i in range(n):
        s.append(l.count(i))
    S.append(deepcopy(s))
row = []
for r in S:
    row.append(0)
M = []
for r in S:
    M.append(deepcopy(row))
RED=[]
#for each monomial, determine the correct partial to
#multiply by, and determine if it is 'reduced'
for r in range(len(S)):
    count=0
    ind=0
    red=0
    while ind==0:
```

```
    while S[r][count] < k-1:
        count +=1
    ind = 1
ind = deepcopy(count)
for i in range(d-k+2):
    red = red+S[r].count(k-1+i)
#find the monomials that appear in the partial
#equation, change them by s, and put the entry
#into our matrix
VAR=[]
for i in range(n):
    for j in range(k-1):
        VAR.append(i)
mons = Subsets(VAR, k-1, submultiset = True).
<< list()
for l in mons:
    expvec=deepcopy(S[r])
    expvec[ind]=expvec[ind]-k+1
    for m in l:
        expvec[m]+=1
    Tensentry = deepcopy(T[ind])
    for m in l:
        Tensentry = Tensentry[m]
    if ind in l:
        number = l.count(ind)
```

```
            Tensentry = Tensentry * (number + 1)
        s=[]
        for i in range(n):
            if i == ind:
                num = l.count(i)
                s.append(num+1)
            else:
                s.append(l.count(i))
        num_perm = 1
        choices = deepcopy(k)
        for elem in s:
            num_perm = num_perm * binomial(choices,
            << elem)
            if elem != 0:
                choices -= elem
        Tensentry = Tensentry * num_perm
        M[r][S.index(expvec)] = Tensentry
    if red ==1:
        RED.append(r)
    else:
        red = 0
RED.reverse()
#get the determinant of this matrix, get the reduced
#matrix, and return the hyperdeterminant
P1=Matrix(M).charpoly()
```

```
    for j in range(len(RED)):
        del M[RED[j]]
    for i in range(len(M)):
        for j in range(len(RED)):
            del M[i][RED[j]]
    P0=Matrix(M).charpoly()
    ratio = P1/P0
    return ratio.subs(x=0)*(-1)**(n*(k-1)**(n-1))
```