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Poset Ramsey Numbers for Boolean Lattices

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Dedication

To my parents, my brother, and Joseph Babčanec.

Abstract

For each positive integer n, let Q_n denote the Boolean lattice of dimension n. For posets P, P', define the *poset Ramsey number* $R(P, P')$ to be the least N such that for any red/blue coloring of the elements of Q_N, there exists either a subposet isomorphic to P with all elements red, or a subposet isomorphic to P' with all elements blue.

Axenovich and Walzer introduced this concept in *Order* (2017), where they proved $R(Q_2, Q_n) \leq 2n + 2$ and $R(Q_n, Q_m) \leq mn + n + m$. They later proved $2n \leq R(Q_n, Q_n) \leq n^2 + 2n$. Walzer later proved $R(Q_n, Q_n) \leq n^2 + 1$. We provide some improved bounds for $R(Q_n, Q_m)$ for various $n, m \in \mathbb{N}$. In particular, we prove that $R(Q_n, Q_n) \leq n^2 - n + 2$, $R(Q_2, Q_n) \leq \frac{5}{3}n + 2$, and $R(Q_3, Q_n) \leq \lceil \frac{37}{16}n + \frac{55}{16} \rceil$. We also prove that $R(Q_2, Q_3) = 5$, and $R(Q_m, Q_n) \leq \left\lceil \left(m - 1 + \frac{2}{m+1}\right) n + \frac{1}{3}m + 2 \right\rceil$ for all $n > m \geq 4$.

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Chapter 1

Introduction

Ramsey theory roughly says that any 2-coloring of elements in a sufficiently large discrete system contains a monochromatic system of given size. In the domain of complete graphs, the classical Ramsey theorem states that for any two graphs G and H there is a integer N_0 such that if the edges of a complete graph K_N with $N \geq N_0$ are colored in two colors then there exists either a red copy of G or a blue copy of H in K_N. The least such number N_0 is called the Ramsey number $R(G, H)$. This theorem was proved by Ramsey [10] in 1930, but the problem of exactly determining Ramsey numbers remains open and is the subject of continuing research. For examples, see the dynamic survey by Radziszowski [9].

In this paper, we will consider the poset Ramsey number instead of the graph Ramsey number. Given two posets $(P, \leq)$ and $(Q, \leq')$, we say $(P, \leq)$ is an *induced subposet* of $(Q, \leq')$ if there is an injective mapping $f\colon P \to Q$ such that for any $x, y \in P$, we have

$$x \leq y \text{ if and only if } f(x) \leq' f(y).$$

We call $f : P \to Q$ an *embedding* of P into Q, and the image $f(P)$ is called a *copy* of P in Q. A *Boolean lattice* of dimension n, denoted Q_n, is the power set of an n-element ground set X equipped with the inclusion relation.

For posets P and P', let the **poset Ramsey number** $R(P, P')$ be the least integer N such that whenever the elements of Q_N are colored red or blue, then Q_N contains either a red copy of P or a blue copy of P'. The focus of this paper is the case where P and P' are Boolean lattices Q_m and Q_n for $m, n \in \mathbb{N}$. Axenovich and Walzer [1] give upper bounds and lower bounds for $R(Q_m, Q_n)$ for various values of $m, n \in \mathbb{N}$. In particular, they prove the following.

Theorem 1.1. *(Thm. 1 in [1]) For any integers* $n, m \geq 1$*,*

(i) $2n \leq R(Q_n, Q_n) \leq n^2 + 2n$*,*

(ii) $R(Q_1, Q_n) = n + 1$*,*

(iii) $R(Q_2, Q_n) \leq 2n + 2$*,*

(iv) $n + m \leq R(Q_n, Q_m) \leq mn + n + m$*,*

(v) $R(Q_2, Q_2) = 4, R(Q_3, Q_3) \in \{7, 8\}$*.*

Walzer, in his master's thesis [11], improved the upper bound in Theorem 1.1, part (i) to the following.

Theorem 1.2. *(Thm. 64 in [11])* $R(Q_n, Q_n) \leq n^2 + 1$*.*

Given two posets $(P, \leq)$ and $(Q, \leq')$, we say $(P, \leq)$ is a *weak subposet* of $(Q, \leq')$ if there is an injective mapping $f\colon P \to Q$ such that for any $x, y \in P$, we have

$$f(x) \leq' f(y) \text{ whenever } x \leq y.$$

The image $f(P)$ is called a *weak copy* of P in Q. For posets P and P', let the **weak poset Ramsey number** $R_w(P, P')$ be the least integer N such that whenever the elements of Q_N are colored red or blue, then Q_N contains either a weak red copy of P or a weak blue copy of P'. Observe that $R(P, P') \geq R_w(P, P')$ for all posets P and P'.

Cox and Stolee [4] showed that $R_w(Q_n, Q_n) \geq 2n + 1$ for $n \geq 13$ using a probabilistic construction. Recently, in the induced case, Bohman and Peng [2] gave an explicit construction showing the bound $R(Q_n, Q_n) \geq 2n + 1$. Grósz, Methuku, and Tompkins [5] gave an explicit construction which yields the following lower bound on $R_w(Q_m, Q_n)$ for all m and $n \geq 68$, generalizing the results of Bohman and Peng to the weak poset case, and extending their results and those of Cox and Stolee to the off-diagonal case.

Theorem 1.3. *For any $m \geq 2$ and $n \geq 68$, we have $R_w(Q_m, Q_n) \geq m + n + 1$.*

This implies that for any $m \geq 2$ and $n \geq 68$, we have $R(Q_m, Q_n) \geq m + n + 1$.

A *chain* of length k is a poset of k distinct, pairwise comparable elements and is denoted by C_k. Cox and Stolee [4] showed that $R_w(C_k, Q_n) = n + k - 1$. Since Q_m is a weak subposet of C_{2^m}, this implies that $R_w(Q_m, Q_n) \leq n + 2^m - 1$. Combining this result with Theorem 1.3, we find that $R_w(Q_2, Q_n) = n + 3$.

Axenovich and Walzer also studied Ramsey numbers for Boolean algebras in [1]. A Boolean algebra $\mathcal{B}_n$ of dimension n is a set system $\{X_0 \cup \bigcup_{i \in I} X_i : I \subseteq [n]\}$, where $X_0, X_1, \ldots, X_n$ are pairwise disjoint sets and, for $i = 1, \ldots, n$, $X_i \neq \emptyset$. Boolean algebras have a more restrictive structure than Boolean lattices. If a subset of Q_N contains a Boolean algebra of dimension n, then it contains a copy of Q_n. The converse, however, is not always true. Consider, for example, $\{\emptyset, \{1, 2\}, \{2, 3\}, \{1, 2, 3\}\}$. Gunderson, Rödl, and Sidorenko [6] first considered the number $R_{Alg}(n)$, defined to be the smallest N such that any red/blue coloring of subsets of $[N]$ contains a red or a blue Boolean algebra of dimension n. Here, "contains" means subset containment in $2^{[N]}$, not containment as a subposet. Axenovich and Walzer were the first to use the notation $R_{Alg}(n)$ in [1].

Let $K^n(s, \ldots, s)$ be the complete n-uniform n-partite hypergraph with parts of size s and $R_h(K^n(2, \ldots, 2))$ be the smallest N' such that any 2-coloring of $K^n(N', \ldots, N')$ contains a monochromatic $K^n(2, \ldots, 2)$. The following theorem states the best known bounds on $R_{Alg}(n)$. The lower bound is given without proof by Brown, Erdős, Chung, and Graham [3], and the upper bound was proved by Axenovich and Walzer [1].

Theorem 1.4. *There is a positive constant c such that*

$$2^{cn} \leq R_{Alg}(n) \leq \min\{2^{2^{n+1} n \log n}, nR_h(K^n(2, \ldots, 2))\}.$$

Gunderson, Rődl, and Sidorenko [6] also considered the number $b(n,d)$, defined to be the maximum cardinality of a $\mathcal{B}_d$-free family contained in $2^{[n]}$. They proved the following bounds:

$$n^{-\frac{(1+o(1))d}{2^{d+1}-2}} \cdot 2^n \leq b(n,d) \leq 10^d 2^{-2^{1-d}} d^{d-2^{-d}} n^{-1/2^d} \cdot 2^n.$$

Johnston, Lu, and Milans [7] later used the Lubell function to improve the upper bound to the following, where C is a constant independent of d and n:

$$b(n,d) \leq Cn^{-1/2^d} \cdot 2^n.$$

In this paper, we improve the upper bounds on the poset Ramsey numbers $R(Q_m, Q_n)$ given by Axenovich and Walzer in [1]. In Sections 2.2-2.4, the following theorems are proved.

Theorem 1.5. *For any integer* $n \geq 1$, $R(Q_2, Q_n) \leq \frac{5}{3}n + 2$.

Theorem 1.6. *For any integer* $n \geq 1$, $R(Q_n, Q_n) \leq n^2 - n + 2$.

Theorem 1.7. *For any integer* $n \geq 1$, $R(Q_3, Q_n) \leq \lceil \frac{37}{16}n + \frac{55}{16} \rceil$.

In Section 2.5, we also prove the following.

Theorem 1.8. *For all* $n > m \geq 4$, $R(Q_m, Q_n) \leq \left\lceil \left(m - 1 + \frac{2}{m+1}\right) n + \frac{1}{3}m + 2 \right\rceil$.

Additionally, we are now able to identify the following previously unknown poset Ramsey number.

Theorem 1.9. $R(Q_2, Q_3) = 5$.

Later, Grósz, Methuku, and Tompkins [5] determined the value of $R(Q_2, Q_n)$ asymptotically by proving the following theorem.

Theorem 1.10. *For every* $c > 2$*, there exists an integer* n_0 *such that for all* $n \geq n_0$*, we have* $R(Q_2, Q_n) \leq n + c\frac{n}{\log_2 n}$.

Combining Theorem 1.10 with the lower bound $R(Q_2, Q_n) \geq n + 2$, we find that $R(Q_2, Q_n)$ is asymptotically equal to n.

In Section 2.1, we give more definitions and introduce notation. Also in Section 2.1, we state and prove Lemma 2.1, the key embedding lemma we use to prove Theorems 1.5, 1.6, 1.7, and 1.8. We prove Theorems 1.5, 1.6, 1.7, 1.8, and 1.9 in Sections 2.2, 2.3, 2.4, 2.5, and 3.1, respectively.

Chapter 2

The Generalized Blob Lemma

2.1 Notation and Key Lemma

A *partially ordered set*, or *poset*, consists of a set S together with a partial order $\leq$, which is a binary relation on S satisfying

Reflexive Property: $x \leq x$, for any $x \in S$.

Transitive Property: If $x \leq y$ and $y \leq z$ then $x \leq z$ for any $x, y, z \in S$.

Antisymmetric Property: If $x \leq y$ and $y \leq x$ then $x = y$ for any $x, y \in S$.

Let $[n]$ denote the set $\{1, 2, \ldots, n\}$ and $Q_n = (2^{[n]}, \subseteq)$ be the poset over the family of all subsets of $[n]$. The k-th level of Q_n, denoted by $\binom{[n]}{k}$, is the set of all k-element subsets of the ground set $[n]$, where $0 \leq k \leq n$. For any two subsets (of $[n]$) $S \subset T$, let $Q_{[S,T]}$ be the induced poset of Q_n over all sets F such that $S \subseteq F \subseteq T$. Let $Q_n^* := Q_n \setminus \{\emptyset, [n]\}$. Let $\hat{R}(Q_m, Q_n)$ denote the smallest N such that any red/blue coloring of Q_N^* contains either a red copy of Q_m^* or a blue copy of Q_n^*. Equivalently, $\hat{R}(Q_m, Q_n)$ is the least N such that if $\emptyset$ and $[N]$ are assumed to be both red and blue while the rest of Q_N is colored either red or blue, then Q_N contains either a red copy of Q_m or a blue copy of Q_n. For a subset $S \subseteq N$, let $\bar{S}$ denote the complement set of S in $[N]$. When $S = \{x\}$, we simply write $\bar{x}$ for $\overline{\{x\}}$.

The following key lemma generalizes the blob lemma of Axenovich and Walzer (see [1], Lemma 3). The special case $a = b = 0$ of the following lemma gives the blob lemma.

Lemma 2.1. *For any nonnegative integers N, m, n, n', a, b satisfying $n' \geq n \geq a+b$ and $N \geq n' + (n+1-a-b) \cdot m$, suppose the Boolean lattice Q_N on the ground set $[N]$ is colored in two colors red and blue, and there is an embedding $I\colon Q_n \to Q_{n'} \subset Q_N$ with the following properties.*

1. *I maps bottom a-levels of Q_n (i.e. sets in $\bigcup_{i=0}^{a-1} \binom{[n]}{i}$) to blue sets,*

2. *For all sets S in the top b levels of Q_n, $I(S) \cup ([N] \setminus [n'])$ is blue.*

Then either a blue copy of Q_n or a red copy of Q_m exists in Q_N.

Proof of Lemma 2.1: Let Q_N be the Boolean lattice on the ground set $[N]$ colored red and blue with the properties listed above.

Let $k = n+1-(a+b)$. Since $N \geq n' + (n+1-(a+b)) \cdot m = n' + k \cdot m$, partition $[N]$ like so:

$$[N] = [n'] \cup X_1 \cup X_2 \cup \cdots \cup X_k,$$

where $|X_i| \geq m$ for all $i \in [k]$. With this partition in mind, create an embedding f of Q_n into the blue sets of Q_N. Consider the map $f : Q_n \to Q_N$ defined by

$$f(S) = \begin{cases} I(S) & \text{if } |S| \leq a-1, \\ I(S) \cup X_1 \cup X_2 \cup \cdots \cup X^*_{|S|-a+1} & \text{if } a \leq |S| \leq n-b \\ I(S) \cup X_1 \cup X_2 \cup \cdots \cup X_k & \text{if } |S| \geq n-b+1. \end{cases}$$

Let $j = |S| - a + 1$. Here, $I(S) \cup X_1 \cup X_2 \cup \cdots \cup X^*_j$ denotes an arbitrarily chosen blue element from the subposet with bottom element $S \cup X_1 \cup X_2 \cup \ldots X_{j-1} \cup \emptyset$ and top element $I(S) \cup X_1 \cup X_2 \cup \cdots \cup X_{j-1} \cup X_j$. If no such blue element exists, this entire subposet is red and Q_N contains a red Q_m.

If such a blue element always exists, this function is well-defined and preserves all the subset relations found in Q_n. Its image consists entirely of blue elements, so Q_N contains a blue Q_n. □

Many times when applying Lemma 2.1 with $n' = n$, we omit the trivial identity mapping I. However, we will explicitly define the mapping I if $n' > n$. In some applications, we will exchange the colors red and blue, resulting in an exchange of m and n.

2.2 Upper bound for $R(Q_2, Q_n)$

Proof of Theorem 1.5: The result is known to hold for $n = 1$ and $n = 2$, so let $n \geq 3$. Let $N \in \mathbb{N}$ be such that there exists a red/blue coloring of Q_N containing no red copy of Q_2 and no blue copy of Q_n. Consider such a red/blue coloring c of Q_N. Let T be a red element such that $\min\{N - |T|, |T|\} \leq \min\{N - |T'|, |T'|\}$ for all red elements $T' \in Q_N$. Without loss of generality, let $N - |T| \leq |T|$. Let $t := N - |T|$. Let S be a red element such that $|S| \leq |S'|$ for all red elements $S' \in Q_{[\emptyset,T]}$. Let $s := |S|$.

Claim I: $|T| - |S| \leq n + 1$.

Proof of Claim I: Otherwise, suppose $|T| - |S| \geq n + 2$. Let u, v be two red elements in $Q_{[S,T]}$. If u and v are incomparable, $\{S, u, v, T\}$ form a red Q_2. So every red element in $Q_{[S,T]}$ lies on the same maximal chain. With the exception of this maximal chain, the rest of $Q_{[S,T]}$ is blue. Suppose $x, y \in T \setminus S$ are two elements such that $S \cup \{x\}$ and $T \setminus \{y\}$ are on the maximal chain. Then all elements in $Q_{[S\cup\{y\}, T\setminus\{x\}]}$ are blue. Since $|T| - |S| - 2 \geq n$, we find a copy of blue Q_n. □

Claim II: $N \leq 3n + 1 - 2(s + t)$.

Proof of Claim II: Otherwise, assume $N \geq 3n + 2 - 2(s + t)$. We have $N \geq n + (n + 1 - (s + t)) \cdot 2$. There exists a copy Q'_n of Q_n in $Q_{[\emptyset,T]}$ so that the bottom s-levels of Q_n are all colored blue. The top t-levels of Q_N are all colored blue. If we let $n' = n$, $a = s$, $b = t$, and the embedding I be the canonical mapping from Q_n to Q'_n, by Lemma 2.1, Q_N contains either a blue copy of Q_n or a red copy of Q_2. □

From Claim I, we have

$$s + t = N - (|T| - |S|) \geq N - (n + 1). \tag{2.1}$$

Combining (2.1) with Claim II, we have

$$N \leq 3n + 1 - 2[N - (n + 1)] = 5n + 3 - 2N. \tag{2.2}$$

We get

$$N \leq \frac{5n}{3} + 1,$$

which gives us the desired result and concludes the proof of Theorem 1.5. □

2.3 Upper bound for $R(Q_n, Q_n)$

Let $n \in \mathbb{N}$. The result is known to hold for $n = 1$ and $n = 2$, so let $n \geq 3$. Recall the definition of $\hat{R}(Q_m, Q_n)$ from Section 2.1. To prove the theorem, we first prove the following lemma.

Lemma 2.2. *For all $n \geq 3$, $\hat{R}(Q_n, Q_n) \leq n^2 - n$.*

Proof of Lemma 2.2: By way of contradiction, suppose there is a red/blue coloring c of Q_N (with $N = n^2 - n$) such that $\emptyset$ and $[N]$ are colored both red and blue while all other elements of Q_N only receive one color. Since $N = n^2 - n$, there are $n^2 - n \geq 2n$ singleton sets in the first row of Q_N. By the Pigeonhole Principle, there are n sets in the first row of Q_N with the same color. Without loss of generality, suppose at least n of these sets are blue. Then there is a copy of Q'_n in Q_N such that level 1 of Q'_n consists of some subset of these blue sets.

Consider an embedding $I : Q_n \to Q'_n \subset Q_N$, which maps the bottom $a = 2$ levels of Q_n to blue sets. Also, consider the top $b = 1$ level of Q_N to be colored blue. By Lemma 2.1, since $N \geq n^2 - n = n + (n - 2) \cdot n = n + (n + 1 - a - b) \cdot n$, either a blue copy of Q_n or a red copy of Q_n exists in Q_N. □

Having proved Lemma 2.2, we now prove Theorem 1.6.

Proof of Theorem 1.6: Let $N = n^2 - n + 2$. Consider a Q_N, and let the elements of Q_N be colored red or blue. Consider the following cases.

Case 1. *Sets $\emptyset$ and $[N]$ are the same color.* Without loss of generality, assume both $\emptyset$ and $[N]$ are colored in red. We have three subcases:

Subcase 1a: All level $N - 1$ sets are red.

Subcase 1b: There exists a blue set T at level $N - 1$ and a blue subset S (of T) at level 1.

Subcase 1c: There exists a blue set T at level $N - 1$ such that all subsets of T at level 1 are red.

In Subcases 1a, apply Lemma 2.1 to Q_N with $a = 1$, $b = 2$, and $n' = n$.

In Subcase 1b, we consider $Q_{[S,T]}$. Since $|T| - |S| = N - 2 \geq n^2 - n$, by Lemma 2.2, $Q_{[S,T]}$ contains either a red or blue copy of $Q_n \setminus \{\emptyset, [n]\}$, which can be extended to a red or blue copy of Q_n.

In Subcase 1c, apply Lemma 2.1 to $Q_{[\emptyset,T]}$ with $a = 2$, $b = 1$, and $n' = n$.

If there exist two blue sets S and T with $|S| = 1$, $|T| = N - 1$, and $S \subset T$, then consider $Q_{[S,T]}$. Since $|T| - |S| = N - 2 \geq n^2 - n$, by Lemma 2.2, $Q_{[S,T]}$ contains either a red or blue copy of $Q_n \setminus \{\emptyset, [n]\}$, which can be extended to a red or blue copy of Q_n.

If there do not exist such two blue sets S and T, there are only three subcases:

Subcase 1a: All level 1 sets are red.

Subcase 1b: All level $N - 1$ sets are red.

Subcase 1c: There exists an element $x \in N$ such that $\{x\}$ and $[N] \setminus \{x\}$ are blue, but all other sets in level 1 and level $N - 1$ are red.

In Subcases 1a and 1c, apply Lemma 2.1 with $a = 2$, $b = 1$, and $n' = n$. In Subcase 1b, apply Lemma 2.1 with $a = 1$, $b = 2$, and $n' = n$.

Case 2. *Sets $\emptyset$ and $[N]$ are not the same color. Without loss of generality, suppose $\emptyset$ is red and $[N]$ is blue.* We have four subcases:

Subcase 2a: All level 1 sets are red.

Subcase 2b: All level $N-1$ sets are blue.

Subcase 2c: There exist a blue set S at level 1 and a red set at $N-1$ such that $S \subset T$.

Subcase 2d: There exists an element $x \in N$ such that all level 1 sets except $\{x\}$ are red and all level $N-1$ sets except $\bar{x}$ are blue.

In Subcase 2a, if there is a red set T at level $N-1$ or $N-2$, we apply Lemma 2.1 to $Q_{[\emptyset,T]}$ with colors exchanged and parameters $n' = n, a = 2, b = 1$. Since $|T| \geq N - 2 = n + (n+1-2-1) \cdot n$, $Q_{[\emptyset,T]}$ contains either a red copy of Q_n or a blue copy of Q_n, as does Q_N. Otherwise, all sets in the top 3 levels of Q_N are blue. We apply Lemma 2.1 to Q_N with parameters $n' = n, a = 0, b = 3$. Since $N \geq n + (n+1-0-3) \cdot n$, Q_N contains either a red copy of Q_n or a blue copy of Q_n.

Subcase 2b is symmetric to Subcase 2a.

In Subcase 2c, since $\emptyset$ is red and S is blue, and $[N]$ is red and T is blue, the poset $Q_{[S,T]}$ of dimension $n^2 - n$ can be viewed as having bottom and top elements colored both red and blue. By Lemma 2.2, $Q_{[S,T]}$ contains a red copy of Q_n or a blue copy of Q_n.

In Subcase 2d, we apply Lemma 2.1 to $Q_{[\emptyset,\bar{x}]}$ with colors exchanged and parameters $n' = n, a = 2, b = 1$. Since $|T| \geq N - 1 > n + (n+1-2-1) \cdot n$, $Q_{[\emptyset,\bar{x}]}$ contains either a red copy of Q_n or a blue copy of Q_n, as does Q_N.

Suppose there is a pair S, T of comparable elements, where S is blue, T is red, $|S| = 1$, and $|T| = N - 1$. Since $\emptyset$ is red and S is blue, and $[N]$ is red and T is blue, the poset $Q_{[S,T]}$ of dimension $n^2 - n$ can be viewed as having bottom and top elements colored both red and blue. By Lemma 2.2, $Q_{[S,T]}$ contains a red Q_n or a blue Q_n.

Otherwise, there are only four subcases:

Subcase 2a: All level 1 sets are red and all level $N - 1$ sets are blue.

Subcase 2b: All level 1 sets are red, and there exists a red $N - 1$-set.

Subcase 2c: All level $N - 1$ sets are blue, and there exists a blue 1-set.

Subcase 2d: There exists an element $x \in N$ such that all level 1 sets except $\{x\}$ are red and all level $N - 1$ sets except $\bar{x}$ are blue.

A similar argument works for Subcases 2b, 2c, and 2d since there exists a $Q_{[N-1]}$ so that there are three levels of one color.

In Subcase 2a, suppose there exists a blue set in level 2. Then there exists a blue $Q_{[N-2]}$ and a similar argument works. If there does not exist such a blue set, the bottom three levels of Q_N are red.

In this case, since

$$n^2 - n + 2 \geq n + (n - 2) \cdot n,$$

partition $[N] = [n] \cup X_1 \cup \cdots \cup X_{n-2}$ so that $|X_i| \geq n$. Map the first three levels of Q_n into Q_N to get a red copy of Q_n. Applying Lemma 2.1 with $a = 3$ and $b = 0$, we get the desired monochromatic copy of Q_n.

In any case where $N = n^2 - n + 2$, we have shown Q_N contains a red Q_n or a blue Q_n. It follows that $R(Q_n, Q_n) \leq n^2 - n + 2$, the desired result. $\square$

2.4 Upper bound for $R(Q_3, Q_n)$

Recall the definition of $\hat{R}(Q_m, Q_n)$ from Section 2.1. To prove the theorem, we first prove the following lemma.

Lemma 2.3. *For all integers $n \geq 1$, $\hat{R}(Q_3, Q_n) \leq \lceil \frac{7}{4}n + \frac{9}{4} \rceil$.*

Proof of Lemma 2.3: By way of contradiction, suppose there is a red/blue coloring c of Q_N (with $N = \lceil \frac{7}{4}n + \frac{9}{4} \rceil$) such that $\emptyset$ and $[N]$ are colored both red and blue, all other elements of Q_N receive one color, and Q_N contains neither a red copy of Q_3 nor a blue copy of Q_n.

Let $\ell = \lceil \frac{3}{8}n + \frac{5}{8} \rceil$ be a fixed integer. Consider the following cases.

Case 1. *There exist red sets A_1, A_2, A_3 in the bottom ℓ levels of Q_N with the following property.*

$$\forall i \in [3], \exists x_i \in [N] \text{ such that } x_i \in A_i, \text{ but } x_i \notin A_j \quad \forall j \in [3] \backslash i. \tag{2.3}$$

Note that, for all $i \in [3]$, $|A_i| \leq \ell - 1$, and for all $\{i, j\} \subset [3]$, $|A_i \cup A_j| \leq 2(\ell - 1)$. Let $X^*_{i,j}$ denote an arbitrarily chosen red element from the subposet with bottom element $A_i \cup A_j$ and top element $\bar{x}_k$, where $\{i, j, k\} = [3]$. Since

$$\ell = \left\lceil \frac{3}{8}n + \frac{5}{8} \right\rceil = \left\lceil \frac{\frac{7}{4}n - n + \frac{5}{4}}{2} \right\rceil = \left\lceil \frac{N - n - 1}{2} \right\rceil \leq \frac{N - n + 1}{2},$$

$$N + 1 \geq 2\ell + n,$$

$$\text{and} \quad N + 1 \geq \ell + (\ell - 1) + n + 1,$$

we are able to define the following embedding of Q_3 into the red sets of Q_N. Consider the map $f : Q_3 \to Q_N$ defined by

$$f(\emptyset) = \emptyset,$$

$$f(\{i\}) = A_i \text{ for all } i \in [3],$$

$$f(\{i,j\}) = X^*_{i,j} \text{ for all } \{i,j\} \subset [3],$$

$$f([3]) = [N].$$

If no such red element $X^*_{i,j}$ exists, the entire n-dimensional subposet with bottom element $A_i \cup A_j$ and top element $\bar{x}_k$ is blue and Q_N contains a blue Q_n. If a red element $X^*_{i,j}$ exists, the function f is well-defined and preserves all the subset relations found in Q_3. The range of f consists entirely of red elements, so Q_N contains a red Q_3, a contradiction.

Case 2. *There exist red sets B_1, B_2, B_3 in the top ℓ levels of Q_N with the following property.*

$$\forall i \in [3], \exists x_i \in [N] \text{ such that } x_i \notin B_i, \text{ but } x_i \in B_j \;\; \forall j \in [3] \backslash i. \tag{2.4}$$

This case is the same is as Case 1, except everything is flipped over the middle level(s) of Q_N. Using a similar argument, we show that Q_N contains a blue Q_n or a red Q_3.

Case 3. *There do not exist sets A_1, A_2, A_3 with property (2.3) or sets B_1, B_2, B_3 with property (2.4).*

Since there do not exist sets A_1, A_2, A_3 with property (2.3), we make the following claim.

Claim I: There exists a set L with cardinality at most ℓ such that all subsets in the family $\bigcup_{i=1}^{\ell-1}\binom{[N]\setminus L}{i}$ are blue.

We prove Claim I by contradiction. Assume Claim I does not hold.

Pick a nonempty red set A_1 with minimum cardinality. We have $|A_1| \leq \ell - 1$. Otherwise, Claim I holds with $L = \emptyset$.

Pick an element $x_1 \in A_1$. Let A_2 be a nonempty red subset of $[N]\setminus\{x_1\}$ of minimum cardinality. We have $|A_2| \leq \ell - 1$. Otherwise, Claim I holds with $L = \{x_1\}$.

Observe that $A_2 \not\subset A_1$ since A_1 is a red set with minimum cardinality. Thus A_2 must contain an element x_2 such that $x_2 \notin A_1$. Let A_3 be a minimal red non-empty subset in $[N]\setminus(A_1 \cup \{x_2\})$. We have $|A_3| \leq \ell - 1$. Otherwise, Claim I holds with $L = A_1 \cup \{x_2\}$. Observe that $A_3 \not\subset A_2$ since A_2 red subset of $[N]\setminus\{x_1\}$ of minimum cardinality. Thus, A_3 must contain an element x_3 such that $x_3 \notin A_2$. The sets A_1, A_2, A_3, along with the elements x_1, x_2, x_3, respectively, satisfy property (2.3), a contradiction.

By symmetry, we prove the following claim since there do not exist exist B_1, B_2, B_3 with property (2.4).

Claim II: There exists a set L' with cardinality at most ℓ such that all subsets in the family $\{S \cup L' \colon \forall S \in \bigcup_{i=1}^{\ell-1}\binom{[N]\setminus L'}{N-|L'|-i}\}$ are blue.

Apply Lemma 2.1 with $n' = n$, $m = 3$, and $a = b = \ell$. Since $N > 2(\ell - 1) + n$, we can find a set S of size n such that $S \cap (L \cup L') = \emptyset$. Let $I : Q_n \to Q_S$ be the canonical mapping. By Claim I and II, both items 1 and 2 in Lemma 2.1 are satisfied. The inequality $n' \geq n \geq a + b$ is trivially true. The other inequality can be verified as follows:

$$\begin{aligned} n' + (n + 1 - a - b)m &= n + (n + 1 - 2\ell) \cdot m \\ &\leq n + \left(n + 1 - 2\left(\frac{3}{8}n + \frac{5}{8}\right) + 1\right) \cdot 3 \\ &= \frac{7}{4}n + \frac{9}{4} \end{aligned}$$

$$\leq N.$$

All conditions of Lemma 2.1 are verified. Thus, Q_N contains either a red copy of Q_3 or a blue copy of Q_n. The proof of Lemma 2.3 is finished. $\square$

Having proved Lemma 2.3, we now prove Theorem 1.7.

Proof of Theorem 1.7: Let $N = \lceil \frac{37}{16}n + \frac{55}{16} \rceil$. Suppose there exists a red/blue coloring of Q_N containing no red copy of Q_3 and no blue copy of Q_n. Consider a red/blue coloring c of Q_N. Let T be a red element such that $\min\{N - |T|, |T|\} \leq \min\{N - |T'|, |T'|\}$ for all red elements $T' \in Q_N$. Without loss of generality, let $N - |T| \leq |T|$. Let $t := N - |T|$. Let S be a red element such that $|S| \leq |S'|$ for all red elements $S' \in Q_{[\emptyset,T]}$. Let $s := |S|$.

We consider the following cases.

Case 1. $s \neq 0$ *and* $t \neq 0$.

It follows that $|T| - |S| + 1 \leq \lceil \frac{7}{4}n + \frac{9}{4} \rceil$ for all $n \in \mathbb{N}$. We make the following claim.

Claim I: $N \leq n + 3(n + 1 - (s + t)) - 1$.

Proof of Claim I: Otherwise, assume $N \geq n + 3(n + 1 - (s + t))$. Since $|T| \geq n$, we can find a subset $R \subset T$ with cardinality $|R| = n$. Apply Lemma 2.1 to Q_N with $n' = n$, $a = s$, $b = t$ and the canonical map I from Q_n to Q_R. By Lemma 2.1, Q_N contains either a blue copy of Q_n or a red copy of Q_3. $\square$

From Lemma 2.3, we have

$$s + t = N - (|T| - |S|) \geq N - \left\lceil \frac{7}{4}n + \frac{5}{4} \right\rceil. \tag{2.5}$$

Combining (2.5) with Claim I, we have

$$N \leq n + 3(n + 1 - (s + t)) - 1 \leq n + 3\left(n + 1 - \left(N - \left\lceil \frac{7}{4}n + \frac{5}{4} \right\rceil\right)\right) - 1. \tag{2.6}$$

We get

$$N \leq \frac{37}{16}n + \frac{35}{16} < N,$$

a contradiction.

Case 2. $s = 0$ *and* $t = 0$.

In this case, both $\emptyset$ and $[N]$ are necessarily red. Apply Lemma 2.1 with parameters $'m' = n$, $'n' = 3$, $'n'' = 3$, $'a' = 1$, and $'b' = 1$. Since

$$3 + (3 + 1 - 1 - 1)n = 2n + 3 < \frac{37}{16}n + \frac{55}{16} \leq N,$$

Q_N contains either a blue Q_n or a red Q_3, a contradiction.

Case 3. $s \neq 0$ *and* $t = 0$.

In this case, $\emptyset$ is necessarily blue and $[N]$ is necessarily red. We have the following subcases:

Subcase 3a: All level $N-1$ sets are red,

Subcase 3b: There exists an blue element T at level $N-1$ and a red element $S \subset T$ with $|S| \leq \left\lfloor \frac{9}{16}n - \frac{13}{16} \right\rfloor$.

Subcase 3c: There exists an blue element T at level $N-1$ such that all subsets of T at levels $\{0, 1, 2, \ldots, \left\lfloor \frac{9}{16}n - \frac{13}{16} \right\rfloor\}$ are all blue.

In Subcases 3a, apply Lemma 2.1 with parameters $'m' = n$, $'n' = 3$, $'n'' = 3$, $'a' = 0$, and $'b' = 2$. Since

$$3 + (3 + 1 - 0 - 2)n = 2n + 3 < \frac{37}{16}n + \frac{55}{16} \leq N,$$

Q_N contains either a blue Q_n or a red Q_3, a contradiction.

In subcases 3b, since $\emptyset$ is blue and S is red, and T is blue and $[N]$ is red, the poset $Q_{[S,T]}$ of dimension at least

$$N - |S| - 1 = \left\lceil \frac{37}{16}n + \frac{55}{16} \right\rceil - \left\lfloor \frac{9}{16}n - \frac{13}{16} \right\rfloor - 1 \geq \left\lceil \frac{7}{4}n + \frac{9}{4} \right\rceil$$

can be viewed as having bottom and top elements colored both red and blue. By Lemma 2.3, $Q_{[S,T]}$ contains a red Q_3 or a blue Q_n.

In subcase 3c, apply Lemma 2.1 to $Q_{[\emptyset,T]}$ with parameters $'m' = 3$, $'n' = n$, $'n'' = n$, $'a' = \lfloor \frac{9}{16}n - \frac{13}{16} \rfloor + 1$, and $'b' = 1$. Since

$$n + \left(n + 1 - \left\lfloor \frac{9}{16}n - \frac{13}{16} \right\rfloor - 1 - 1 \right) \cdot 3 \leq \frac{37}{16}n + \frac{39}{16} \leq N - 1,$$

Q_N contains either a blue Q_n or a red Q_3, a contradiction. In all cases, we prove Q_N contains either a blue copy of Q_n or a red copy of Q_3. □

2.5 Upper bound for $R(Q_m, Q_n)$

Recall the definition of $\hat{R}(Q_m, Q_n)$ from Section 2.1. Before proving Theorem 1.8, we first prove the following lemma.

Lemma 2.4. *For all $n > m \geq 4$, we have $\hat{R}(Q_m, Q_n) \leq (m-1)\,n + \left\lfloor \frac{1}{3}m \right\rfloor$.*

Proof of Lemma 2.4: By way of contradiction, suppose there is a red/blue coloring c of Q_N (with $N = (m-1)n + \lfloor \frac{1}{3}m \rfloor$) such that $\emptyset$ and $[N]$ are colored both red and blue, all other elements of Q_N receive one color, and Q_N contains neither a red copy of Q_m nor a blue copy of Q_n.

Let $\ell = \lceil \frac{1}{3} + \frac{n}{m} \rceil$ be a fixed integer. Consider the following cases.

Case 1. *There exist red sets $A_1, A_2, \ldots, A_m$ in the bottom ℓ levels of Q_N with the following property.*

$$\forall i \in [m], \exists x_i \in [N] \text{ such that } x_i \in A_i, \text{ but } x_i \notin A_j \ \ \forall j \in [m] \backslash i. \tag{2.7}$$

Apply Lemma 2.1 in an altered way so that the colors "red" and "blue" are exchanged. The parameters of Lemma 2.1 are set as follows: $'n' = m$, $'m' = n$, $'a' = 2$, $'b' = 1$, and $'n'' = |\bigcup_{i=1}^m A_i| \leq m(\ell - 1)$. Let $Q_{n'} = Q_{[\emptyset, \bigcup_{i=1}^m A_i]}$ be the sublattice of Q_N with the minimum element $\emptyset$ and the maximum element $\bigcup_{i=1}^m A_i$. The mapping $I\colon Q_m \to Q_{n'}$ is defined as $I(S) = \bigcup_{j \in S} A_j$ for any $S \in Q_n$. By property (2.7), each A_i has a private representative x_i. Thus I is a poset embedding. Since A_i's are red while $\emptyset$ and $[N]$ are both red and blue, the two items in Lemma 2.1 are satisfied. It is clear that $n' \geq m \geq a + b$. We also have

$$
\begin{aligned}
n' + (m + 1 - a - b)n &\leq m(\ell - 1) + (m - 2)n \\
&\leq m\left(\frac{1}{3} + \frac{n}{m}\right) + (m - 2)n \\
&\leq (m - 1)n + \frac{m}{3}.
\end{aligned}
$$

Note the left hand side is an integer. We have

$$
n' + (m + 1 - a - b)n \leq (m - 1)n + \left\lfloor \frac{m}{3} \right\rfloor = N.
$$

By Lemma 2.1, Q_N contains either a red copy of Q_m or a blue copy of Q_n.

Case 2. *There exist red sets $B_1, B_2, \ldots, B_m$ in the top ℓ levels of Q_N with the following property.*

$$
\forall i \in [m], \exists x_i \in [N] \text{ such that } x_i \notin B_i, \text{ but } x_i \in B_j \ \ \forall j \in [m] \backslash i. \tag{2.8}
$$

This case is the same as Case 1, except everything is flipped over the middle levels(s) of Q_N. Using a similar argument, we show that Q_N contains a blue Q_n or a red Q_m.

Case 3. *There do not exist sets $A_1, A_2, \ldots, A_m$ with property (2.7) or sets $B_1, B_2, \ldots, B_m$* with property (2.8).

Since there do not exist sets $A_1, A_2, \ldots, A_m$ with property (2.7), we make the following claim.

Claim I: There exists a set L with cardinality at most $(m-2)(\ell-1)+1$ such that all subsets in the family $\bigcup_{i=1}^{\ell-1}\binom{[N]\setminus L}{i}$ are blue.

We prove Claim I by contradiction. Assume Claim I does not hold.

Pick a nonempty red set A_1 with minimum cardinality. We have $|A_1| \leq \ell - 1$. Otherwise, Claim I holds with $L = \emptyset$. Let x_1 be any element in A_1.

Now construct the pairs (A_i, x_i) for $i = 2, \ldots, m$ by iterations. Suppose we have already constructed the pairs (A_j, x_j) for $j = 1, \ldots, i-1$. Pick a nonempty red set $A_i \subset [N] \setminus (\bigcup_{j=1}^{i-2} A_j \cup \{x_{i-1}\})$ of minimum cardinality. We must have $|A_i| \leq \ell - 1$. Otherwise, Claim I holds with $L = \bigcup_{j=1}^{i-2} A_j \cup \{x_{i-1}\}$. Since A_{i-1} is minimal, A_i is not a proper subset of A_{i-1}. Let x_i be an element in $A_i \setminus A_{i-1}$.

By the construction, we have

$$A_i \cap A_j = \emptyset \text{ for all } j = 1, \ldots, i-2,$$

$$x_{i-1} \notin A_i \text{ and } x_i \notin A_{i-1}.$$

This implies, for all $j \neq i$, $x_i \in A_i$, but $x_i \notin A_j$. The constructed sets $A_1, A_2, \ldots, A_m$, along with the elements $x_1, x_2, \ldots, x_m$, respectively, satisfy property (2.7), a contradiction.

By symmetry, we prove the following claim since there do not exist sets $B_1, B_2, \ldots, B_m$ with property (2.8).

Claim II: There exists a set L' with cardinality at most $(m-2)(\ell-1)+1$ such that all subsets of the family $\bigcup_{i=1}^{\ell-1}\binom{[N]\setminus L'}{N-|L'|-i} \cup L'$ are blue.

Apply Lemma 2.1 with $n' = n$ and $a = b = \ell$. Since $n < N - 2((m-2)(\ell-1)+1)$ for all $m, n \geq 4$, we can find a set S of size n such that $S \cap (L \cup L') = \emptyset$. Let $I : Q_n \to Q_S$ be the canonical mapping. By Claim I and II, both items 1 and 2 in Lemma 2.1 are satisfied. The inequality $n' \geq n \geq a + b$ is trivially true. The other inequality can be verified as follows:

$$\begin{aligned} n' + (n+1-a-b)m &= n + (n+1-2\ell)m \\ &\leq n + \left(n + 1 - 2\left(\frac{1}{3} + \frac{n}{m}\right)\right)m \\ &= (m-1)n + \frac{m}{3}. \end{aligned}$$

Note the left hand side is an integer. We have

$$n' + (n+1-a-b)m \leq (m-1)n + \left\lfloor \frac{m}{3} \right\rfloor = N.$$

All conditions of Lemma 2.1 are verified. Thus, Q_N contains either a red copy of Q_m or a blue copy of Q_n. The proof of Lemma 2.4 is finished. □

Having proved Lemma 2.4, we now prove Theorem 1.8.

Proof of Theorem 1.8: For any integers $m, n \in \mathbb{N}$ with $n > m \geq 4$, let $N = \left\lceil (m - 1 + \frac{2}{m+1})n + \frac{1}{3}m + 2 \right\rceil$. Suppose there exists a red/blue coloring of Q_N containing no red copy of Q_m and no blue copy of Q_n. Consider a red/blue coloring c of Q_N. Let T be a red element such that $\min\{N - |T|, |T|\} \leq \min\{N - |T'|, |T'|\}$ for all red elements $T' \in Q_N$. Without loss of generality, let $N - |T| \leq |T|$. Let $t := N - |T|$. Let S be a red element such that $|S| \leq |S'|$ for all red elements $S' \in Q_{[\emptyset, T]}$. Let $s := |S|$. We consider the following cases.

Case 1. $s \neq 0$ *and* $t \neq 0$.

It follows from Lemma 2.4 that

$$N - (s+t) + 1 = |T| - |S| + 1 \leq \hat{R}(Q_m, Q_n) \leq (m-1)n + \frac{1}{3}m. \tag{2.9}$$

We make the following claim.

Claim I: $N \leq n + m(n+1-(s+t)) - 1$.

Proof of Claim I: Otherwise, assume $N \geq n + m(n+1-(s+t))$. Let R be any subset of T with cardinality n. Applying Lemma 2.1 on Q_N with $n' = n$, $a = s$, $b = t$, and the canonical mapping I from Q_n to Q_R, we conclude that Q_N contains either a blue copy of Q_n or a red copy of Q_m, a contradiction. $\square$

From Inequality (2.9), we have

$$s + t = N - (|T| - |S|) \geq N - \left((m-1)\,n + \frac{1}{3}m - 1\right). \tag{2.10}$$

Combining (2.10) with Claim I, we have

$$\begin{aligned} N &\leq n + m(n+1-(s+t)) - 1 \\ &\leq n + m\left(n + 1 - \left(N - (m-1)\,n - \frac{1}{3}m + 1\right)\right) - 1. \end{aligned}$$

Solving for N, we get

$$N \leq \left(m - 1 + \frac{2}{m+1}\right) n + \frac{1}{3}m - \frac{1}{3} - \frac{2}{3m+3} < N, \tag{2.11}$$

a contradiction.

Case 2. $s = 0$ *and* $t = 0$.

In this case, both $\emptyset$ and $[N]$ are necessarily red. Consider levels 1, 2, $N-2$, and $N-1$. We have three subcases:

Subcase 2a: There exist two comparable blue sets S and T with $|S| = 1$, $|T| = N-1$.

Subcase 2b: All sets in levels 1 are red.

Subcase 2c: There exists a blue set S at level 1, but no blue set containing S at level $N-1$.

In Subcase 2a, we consider $Q_{[S,T]}$. Since $\emptyset$ is red and S is blue, consider the bottom element of $Q_{[S,T]}$ to be both red and blue. Since $[N]$ is red and T is blue, consider the top element of $Q_{[S,T]}$ to be both red and blue. By Lemma 2.4, we have

$$N \le \hat{R}(Q_m, Q_n) + 2 \le (m-1)n + \left\lfloor \frac{1}{3}m \right\rfloor + 2 < \left(m - 1 + \frac{2}{m+1}\right)n + \frac{1}{3}m + 1 \le N,$$

a contradiction.

In Subcase 2b, we apply Lemma 2.1 with parameters $'m' = n$, $'n' = m$, $'n'' = m$, $'a' = 2$, and $'b' = 1$. Since $N \ge (m - 1 + \frac{2}{m+1})n + \frac{1}{3}m + 2 > m + (m-2)\cdot n$. By Lemma 1, Q_N contains either a blue Q_n or a red Q_m.

In Subcase 2c, since $\emptyset$ is red, we can view S is both red and blue. We apply Lemma 2.1 to $Q_{[S,[N]]}$ with parameters $'m' = n$, $'n' = m$, $'n'' = m$, $'a' = 1$, and $'b' = 2$. Since $N - 1 \ge (m - 1 + \frac{2}{m+1})n + \frac{1}{3}m + 2 - 1 > m + (m-2)\cdot n$. By Lemma 1, $Q_{[S,[N]]}$ contains either a blue Q_n or a red Q_m, as does Q_N.

Case 3. $s \neq 0$ *and* $t = 0$.

In this case, $\emptyset$ is necessarily blue and $[N]$ is necessarily red. We have three subcases:

Subcase 3a: There is a pair S, T of comparable elements, where S is red, T is blue, $1 \le |S| \le \left\lfloor \frac{2n}{m+1} \right\rfloor - 1$ and $|T| = N-1$ or $N-2$.

Subcase 3b: All sets in levels $N-1$ and $N-2$ are red.

Subcase 3c: There is a blue set T in level $N-1$ or $N-2$ such that all subsets of T in levels $\{1, 2, \ldots, \left\lfloor \frac{2n}{m+1} \right\rfloor\}$ are blue.

In Subcase 3a, since $\emptyset$ is blue and S is red, and T is blue and $[N]$ is red, consider the top and bottom elements of $Q_{[S,T]}$ to be both red and blue. By Lemma 2.4, we have

$$\begin{aligned} N \le \hat{R}(Q_m, Q_n) + |S| + N - |T| &\le (m-1)n + \left\lfloor \frac{1}{3}m \right\rfloor + \left\lfloor \frac{2n}{m+1} - 1 \right\rfloor + 2 \\ &< \left(m - 1 + \frac{2}{m+1}\right)n + \frac{1}{3}m + 2 \le N, \end{aligned}$$

a contradiction.

In Subcase 3b, we apply Lemma 2.1 to Q_N with parameters $'m' = n$, $'n' = m$, $'n'' = m$, $'a' = 0$, and $'b' = 3$. Since $N \geq (m-1+\frac{2}{m+1})n+\frac{1}{3}m+2 > m+(m+1-0-3)\cdot n$. By Lemma 1, Q_N contains either a blue copy of Q_n or a red copy of Q_m.

In Subcase 3c, we apply Lemma 2.1 to $Q_{[\emptyset,T]}$ with parameters $'m' = m$, $'n' = n$, $'n'' = m$, $'b' = 1$, and $'a' = \frac{2n}{m+1}$. Since $N - 2 \geq (m-1+\frac{2}{m+1})n + \frac{1}{3}m > n + (n+1-1-\frac{2n}{m+1}) \cdot m$, by Lemma 1, $Q_{[\emptyset,T]}$ contains either a blue copy of Q_n or a red copy of Q_m, as does Q_N.

In all cases, we prove Q_N contains either a blue copy of Q_n or a red copy of Q_m. □

Chapter 3

Small Poset Ramsey Numbers

3.1 Determining the value $R(Q_2, Q_3)$

Proof of Theorem 1.9: Consider a coloring c of Q_4 defined by

$$c(S) = \begin{cases} \text{blue} & \text{if } |S| \text{ is even} \\ \text{red} & \text{if } |S| \text{ is odd} \end{cases}$$

for all sets S in Q_4. This coloring of Q_4 contains no red copy of Q_2 and no blue copy of Q_3. Thus, $R(Q_2, Q_3) > 4$. Now we need only show $R(Q_2, Q_3) \leq 5$.

Consider a red/blue coloring of Q_5 containing no red Q_2 and no blue Q_3. Consider the following cases.

Case 1. *Both $\emptyset$ and $[5]$ are colored red.*

Let u, v be two red elements in Q_5. If u and v are incomparable, $\{\emptyset, u, v, [5]\}$ form a red Q_2. So every red elements in Q_5 lies on the same maximal chain. With the exception of this maximal chain, the rest of Q_5 is blue, and we have a blue Q_3, a contradiction.

Case 2. *One of $\emptyset$ and $[5]$ is colored red, and the other is blue.*

Without loss of generality, suppose $\emptyset$ is red and $[5]$ is blue. Suppose there exists a red set T with $|T| = 4$. Without loss of generality, let T be $\{1, 2, 3, 4\}$. Consider $Q_{[\emptyset,T]}$, and let U, V be two red elements in $Q_{[\emptyset,T]}$. If U and V are incomparable, $\{\emptyset, U, V, T\}$ form a red Q_2. So every red element in $Q_{[\emptyset,T]}$ lies on the same maximal chain. Without loss of generality, suppose this maximal chain is $\{\emptyset, \{1\}, \{1,2\}, \{1,2,3\}, \{1,2,3,4\}\}$. Then the sets $\{4\}, \{1,4\}, \{2,4\}, \{3,4\}, \{1,2,4\}, \{1,3,4\}, \{2,3,4\}$ are all blue. These sets, along with $[5]$, form a blue Q_3. Thus, every set in level 4 of Q_5 is blue.

Suppose there exist two red sets S_1 and S_2 with $|S_1| = |S_2| = 1$. Then $S_1 \cup S_2$ are blue. Moreover, every set in $Q_{[S_1 \cup S_2, [5]]}$ is blue. Then $Q_{[S_1 \cup S_2, [5]]}$ is a blue copy of Q_3, a contradiction. Thus, Q_5 has at most one red level 1 set.

Without loss of generality, suppose $\{1\}$ is the only red level 1 set in Q_5. Note that $\bar{2}, \bar{3}$, and $\bar{4}$ are all blue. Consider $Q_{[\{5\}, \bar{2} \cap \bar{3}]}$. If $\bar{2} \cap \bar{3} = \{1,4,5\}$ and $\{4,5\}$ are both red, then $\{\emptyset, \{1\}, \{4,5\}, \{1,4,5\}\}$ is a red copy of Q_2. Thus, at least one of $\{4,5\}$ and $\{1,4,5\}$ is blue. Similarly, when we consider $Q_{[\{5\}, \bar{2} \cap \bar{4}]}$ and $Q_{[\{5\}, \bar{3} \cap \bar{4}]}$, we conclude that at least one of $\{3,5\}$ and $\{1,3,5\}$ is blue and at least one of $\{2,5\}$ and $\{1,2,5\}$ is blue. These blue sets, along with $\{5\}$, $\bar{2}$, $\bar{3}$, $\bar{4}$, and $[5]$ form a blue copy of Q_3. Thus, Q_5 has no red level 1 set.

Now, note that $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$ cannot all be blue. Otherwise,

$\{\{1\}, \{1,2\}, \{1,3\}, \{1,4\}, \bar{2}, \bar{3}, \bar{4}, [5]\}$ is a blue copy of Q_3. Suppose, without loss of generality, that $\{1,2\}$ is red. Consider $Q_{[\{1\}, \{1,2,3\}]}$. If $\{2,3\}$ and $\{1,2,3\}$ are both red, then $\{\emptyset, \{1,2\}, \{2,3\}, \{1,2,3\}\}$ is a red copy of Q_2. Thus, at least one of $\{2,3\}$ and $\{1,2,3\}$ is blue. Similarly, when we consider $Q_{[\{1\}, \{1,2,4\}]}$ and $Q_{[\{1\}, \{1,2,5\}]}$, we conclude that at least one of $\{2,4\}$ and $\{1,2,4\}$ is blue and at least one of $\{2,5\}$ and $\{1,2,5\}$ is blue. These blue sets, along with $\{1\}, \bar{3}, \bar{4}, \bar{5}$, and $[5]$ form a blue copy of Q_3, a contradiction.

Case 3. *Both $\emptyset$ and $[5]$ are colored blue.*

Suppose Q_5 has at most 2 red level 1 sets. In other words, Q_5 has at least 3 blue level 1 sets. Without loss of generality, suppose $\{1\}$, $\{2\}$, and $\{3\}$ are all blue. Consider $Q_{[\{1,2\}, \bar{3}]}$. If every set in $Q_{[\{1,2\}, \bar{3}]}$ is red, $Q_{[\{1,2\}, \bar{3}]}$ is a red copy of Q_2. Thus, there is at least one blue set in $Q_{[\{1,2\}, \bar{3}]}$. Similarly, there is at least one blue set in

$Q_{[\{1,3\},\bar{2}]}$ and at least one blue set in $Q_{[\{2,3\},\bar{1}]}$. These sets, along with $\emptyset, \{1\}, \{2\}, \{3\}$, and $[5]$, form a blue copy of Q_3. Thus, Q_5 has at least 3 red level 1 sets. By a similar argument, Q_5 also has at least 3 red level 4 sets.

Let S_1, S_2, S_3 be 3 red level 1 sets, and let T_1, T_2, T_3 be 3 level 4 sets. Consider the following subcases.

Subcase 3a. *At least one of S_1, S_2, and S_3 is a subset of T_1, T_2, and T_3.*

Without loss of generality, let $S_1 = \{1\}$ be red and a subset of $T_1 = \bar{3} = \{1, 2, 4, 5\}$, $T_2 = \bar{4} = \{1, 2, 3, 5\}$, and $T_3 = \bar{5} = \{1, 2, 3, 4\}$, all of which are red. Note that no two of $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 2, 5\}$ can be red without creating a red copy of Q_2. Also, no two of $\{1, 2\}$, $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$ can be red without creating a red copy of Q_2.

Suppose $\{1, 2\}$ is red, which means $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$ are all blue, and $\{1, 4, 5\}$, $\{1, 3, 5\}$, and $\{1, 3, 4\}$ are all blue. These 6 sets, along with $\emptyset$ and $[5]$, form a blue copy of Q_3, a contradiction.

Suppose exactly one of $\{1, 2, 3\}$, $\{1, 2, 4\}$, and $\{1, 2, 5\}$ is red. Without loss of generality, suppose $\{1, 2, 3\}$ is red. Neither $\{1, 4\}$ nor $\{1, 5\}$ can be red without creating a red copy of Q_2 with $\{1\}$, $\{1, 2, 3\}$, and $\{1, 2, 3, 4\}$. Suppose $\{1, 3\}$ is red, which means $\{1, 2\}$, $\{1, 4\}$, and $\{1, 5\}$ are all blue. Then $\{1, 4, 5\}$ is red. If $\{1, 3, 4, 5\}$ is red, it forms a red copy of Q_2 with $\{1\}$, $\{1, 3\}$, and $\{1, 4, 5\}$. If $\{1, 3, 4, 5\}$ is blue, it forms a blue copy of Q_3 with $\emptyset$, $\{1, 2\}$, $\{1, 4\}$, $\{1, 5\}$, $\{1, 2, 4\}$, $\{1, 2, 5\}$, and $[5]$. Thus, Q_5 contains a red copy of Q_2 or a blue copy of Q_3, a contradiction.

Suppose $\{1, 2, 3\}$ is red and none of $\{1, 3\}$, $\{1, 4\}$, and $\{1, 5\}$ are red. Then $\{1, 4, 5\}$ is red, and $\{1, 3, 4\}$ and $\{1, 3, 5\}$ are blue. Then $\{2, 4\}$, $\{2, 5\}$, $\{3, 4\}$, and $\{3, 5\}$ are blue, and $\{2, 4, 5\}$ is red. Then $\{4\}$, $\{5\}$, and $\{4, 5\}$ are blue. Then $\{4\}$, $\{5\}$, $\{1, 2\}$, $\{4, 5\}$, $\{1, 2, 4\}$, and $\{1, 2, 5\}$, along with $\emptyset$ and $[5]$, form a blue copy of Q_3, a contradiction.

Now suppose none of $\{1,2,3\}$, $\{1,2,4\}$, or $\{1,2,5\}$ are red. Again, no two of $\{1,2\}$, $\{1,3\}$, $\{1,4\}$ and $\{1,5\}$ are red. Suppose one of $\{1,3\}$, $\{1,4\}$, and $\{1,5\}$ is red. Without loss of generality, suppose $\{1,3\}$ is red. Then $\{1,4,5\}$ is red, and $\{1,3,4,5\}$ is blue. Then $\{1,2\}$, $\{1,4\}$, $\{1,5\}$, $\{1,2,3\}$, $\{1,2,5\}$, and $\{1,3,4,5\}$, along with $\emptyset$ and $[5]$, form a blue copy of Q_3, a contradiction.

Suppose none of $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, or $\{1,5\}$ are red. Then $\{1,4,5\}$, $\{1,3,4\}$, and $\{1,3,5\}$ are all red, and $\{1,3,4,5\}$ is blue. Then $\{1,2\}$, $\{1,3\}$, $\{1,4\}$, $\{1,2,3\}$, $\{1,2,4\}$, and $\{1,3,4,5\}$, along with $\emptyset$ and $[5]$, form a blue copy of Q_3, a contradiction.

In any case where at least one of S_1, S_2, and S_3 is a subset of T_1, T_2, and T_3, Q_5 contains a red copy of Q_2 or a blue copy of Q_3.

Subcase 3b. *None of S_1, S_2, and S_3 is a subset of T_1, T_2, and T_3.*

Without loss of generality, let $S_1 = \{1\}$, $S_2 = \{2\}$, $S_3 = \{3\}$, $T_1 = \bar{1} = \{2,3,4,5\}$, $T_2 = \bar{2} = \{1,3,4,5\}$, and $T_3 = \bar{3} = \{1,2,4,5\}$ all be red. Certainly, if every level 2 set and every level 3 set is blue, or if one or both of $\{4,5\}$ and $\{1,2,3\}$ are the only red sets, then Q_5 contains a blue copy of Q_3.

Suppose one of $\{1,2\}$, $\{1,3\}$ and $\{2,3\}$ is red. Without loss of generality, suppose $\{1,2\}$ is red. Then $\{1,4\}$, $\{1,5\}$, $\{2,4\}$, $\{2,5\}$, $\{1,4,5\}$, and $\{2,4,5\}$ are all blue. Suppose either $\{1,2,3,4\}$ or $\{1,2,3,5\}$ is red. Without loss of generality, suppose $\{1,2,3,4\}$ is red. Then $\{1,3\}$, $\{2,3\}$, $\{1,3,4\}$, and $\{2,3,4\}$ are all blue, and $\{1,2,3,5\}$ is red. Then $\{1,3,5\}$ and $\{2,3,5\}$ are blue. The sets $\{1,4\}$, $\{1,5\}$, $\{1,3\}$, $\{1,4,5\}$, $\{1,3,4\}$, and $\{1,3,5\}$, along with $\emptyset$ and $[5]$, form a blue copy of Q_3, a contradiction.

Now suppose $\{1,2\}$ is red and $\{1,2,3,4\}$ and $\{1,2,3,5\}$ are both blue. Then $\{1,3\}$ is red, and $\{1,2,3\}$ is blue. Then $\{1,4\}$, $\{1,5\}$, $\{1,2,3\}$, $\{1,4,5\}$, $\{1,2,3,4\}$, and $\{1,2,3,5\}$, along with $\emptyset$ and $[5]$, form a blue copy of Q_3, a contradiction. The argument is similar if any one of $\{1,4,5\}$, $\{2,4,5\}$, and $\{3,4,5\}$ is red.

Suppose any level 2 set other than $\{1,2\}$, $\{1,3\}$, $\{2,3\}$, or $\{4,5\}$ is red. Without loss of generality, suppose $\{1,4\}$ is red. Then $\{1,2\}$, $\{1,3\}$, $\{1,5\}$, $\{1,2,5\}$, and $\{1,3,5\}$ are all blue. Then $\{1,2,3\}$ is red, and $\{1,2,3,4\}$ is blue. Then $\{1,2\}$, $\{1,3\}$, $\{1,5\}$, $\{1,2,5\}$, $\{1,3,5\}$, and $\{1,2,3,4\}$, along with $\emptyset$ and $[5]$, form a blue copy of Q_3, a contradiction. The argument is similar if any level 3 set other than $\{1,4,5\}$, $\{2,4,5\}$, $\{3,4,5\}$, or $\{1,2,3\}$ is red.

In any case where none of S_1, S_2, and S_3 is a subset of T_1, T_2, and T_3, Q_5 contains a red copy of Q_2 or a blue copy of Q_3. This concludes the proof of Theorem 1.9. $\square$

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