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Some Properties and Applications of Spaces of Modular Forms With ETA-Multiplier

Cuyler Daniel Warnock

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by

Cuyler Daniel Warnock

Bachelor of Arts
Georgia College & State University, 2017

Master of Science
University of South Carolina, 2019

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Accepted by:

Matthew Boylan, Major Professor

Alexander Duncan, Committee Member

Michael Filaseta, Committee Member

Andrew Kustin, Committee Member

Ognian Trifonov, Committee Member

Subrahmanyam Bulusu, Committee Member

Tracy L. Weldon, Interim Dean, Vice Provost for Graduate Education

Dedication

To my beautiful baby girl. Even though your mother and I have not met you yet, we pray for you every day and look forward to holding you in our arms. We love you!

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Most importantly, I would like to start by thanking my Lord and Savior, Jesus Christ. He has been with me throughout this entire journey and has brought me peace and joy every step of the way. I am forever grateful for how He has orchestrated all the events in my life and has prepared me for the road ahead.

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Abstract

This dissertation considers two topics. In the first part of the dissertation, we prove the existence of fourteen congruences for the p-core partition function of the form given by Garvan in [14]. Different from the congruences given by Garvan, each of the congruences we give yield infinitely many congruences of the form

$$a_p(\ell \cdot p^{t+1} \cdot n + p^t \cdot k - \delta_p) \equiv 0 \pmod{\ell}.$$

For example, if $t \geq 0$ and $\left(\frac{m}{n}\right)$ is the Jacobi symbol, then we prove

$$a_7(7^t \cdot n - 2) \equiv 0 \pmod{5}, \quad \text{if } \left(\frac{n}{5}\right) = 1 \text{ and } \left(\frac{n}{7}\right) = -1.$$

It follows that for all natural numbers n and for $k \in \{6, 19, 24, 26, 31, 34\}$,

$$a_7(5 \cdot 7^{t+1} \cdot n + 7^t \cdot k - 2) \equiv 0 \pmod{5}.$$

In the second part of the dissertation, we give results on where Hecke operators map spaces of modular forms which arise as multiples of eta-quotients. Let $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$ and let $f(z)$ be a level N holomorphic eta quotient with integer weight. Then we precisely describe how T_n with $\gcd(n, 6) = 1$ permutes subspaces of the form

$$\{f(Dz)F(Dz) : F(z) \in M_w(\Gamma_0(N), \chi)\}.$$

Subspaces of this type play a significant role in recent works [1, 2, 6, 7, 13, 30, 31, 32], primarily for $N = 1$ and with applications, for example, to congruences for partition functions.

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Chapter 1

Introduction

Let $\mathfrak{h} = \{z \in \mathbb{C} : \mathrm{Im} z > 0\}$ and for $N \in \mathbb{N}$, define

$$\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod{N} \right\}.$$

Definition 1.0.1. *Let $k \in \mathbb{Z}$ and χ be a Dirichlet character modulo N. Then a modular form of weight k, level N, with Nebentypus character χ is a holomorphic function $f : \mathfrak{h} \to \mathbb{C}$ which satisfies the following two properties:*

1. *For all $z \in \mathfrak{h}$ and for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have that*

$$f\left(\frac{az+b}{cz+d}\right) = \chi(d)(cz+d)^k f(z).$$

2. *The function $f(z)$ is holomorphic at the cusps: for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,*

$$\lim_{z \to i\infty} f\left(\frac{az+b}{cz+d}\right) < \infty$$

   exists.

The set of modular forms of weight k, level N, and characer χ form a finite-dimensional vector space denoted $M_k(\Gamma_0(N), \chi)$. The *Dedekind eta-function* is defined as an infinite product by

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n), \tag{1.0.1}$$

where $q := e^{2\pi iz}$. Even though $\eta(z)$ is not a modular form, it transforms under the action of $\mathrm{SL}_2(\mathbb{Z})$ on $\mathfrak{h}$ in weight $1/2$ with respect to a multiplier system ν_η which takes values in μ_{24}, the complex 24th roots of unity. For details on ν_η, see Definition 2.3.2. Let $N \geq 1$. An *eta-quotient* of level N is a function of the form

$$f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}, \tag{1.0.2}$$

where $\delta, r_\delta \in \mathbb{Z}$ with $\delta \geq 1$. From (1.0.1) and (1.0.2), an eta-quotient is meromorphic with poles, if any, supported at cusps. The following proposition gives criteria for an eta-quotient to be a weakly holomorphic modular form.

Proposition 1.0.2 ([25], Theorem 1.6.4)**.** *Let $N \geq 1$, and let $f(z)$ be an eta-quotient of level N with $k = \frac{1}{2}\sum_{\delta \mid N} r_\delta \in \mathbb{Z}$ and $s = \prod_{\delta \mid N} \delta^{r_\delta}$. Suppose that $f(z)$ satisfies*

$$\sum_{\delta \mid N} \delta r_\delta \equiv 0 \pmod{24} \quad \textit{and} \quad N \sum_{\delta \mid N} \frac{r_\delta}{\delta} \equiv 0 \pmod{24}.$$

Then we have $f(z) \in M_k^!\left(\Gamma_0(N), \left(\frac{(-1)^k s}{\cdot}\right)\right)$: for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, we have

$$f\left(\frac{az+b}{cz+d}\right) = \left(\frac{(-1)^k s}{d}\right)(cz+d)^k f(z).$$

In order for an eta-quotient to be holomorphic, we need that its order of vanish at each of the cusps is nonnegative. We can calculate the order of vanish of an eta-quotient at the cusps of $\Gamma_0(N)$ by using the following formula.

Proposition 1.0.3 ([22], Corollary 2.2)**.** *Let c, d, and N be integers with $d \mid N$, $N \geq 1$, and $\gcd(c,d) = 1$; let $f(z)$ be a level N eta-quotient. Then the order of vanish of $f(z)$ at the cusp $\frac{c}{d}$ of $\Gamma_0(N)$ is*

$$\frac{1}{24} \sum_{\delta \mid N} \frac{\gcd(d,\delta)^2 r_\delta}{\delta}.$$

We weight the order of vanish at the cusp $\frac{c}{d}$ of $\Gamma_0(N)$ by its width, $\frac{N}{\gcd\left(d,\frac{N}{d}\right)d}$:

$$\mathrm{ord}_{\frac{c}{d}}(f) = \frac{N}{24\gcd\left(d^2,N\right)}\sum_{\delta|N}\frac{\gcd(d,\delta)^2 r_\delta}{\delta}. \tag{1.0.3}$$

Note that this definition gives $24\cdot\mathrm{ord}_{\frac{c}{d}}(f)\in\mathbb{Z}$. Also observe that $\mathrm{ord}_{\frac{c}{d}}(f)=\mathrm{ord}_{\frac{1}{d}}(f)$; hence, the eta-quotient f is holomorphic if and only if for all $d \mid N$, we have $\mathrm{ord}_{\frac{1}{d}}(f)\geq 0$.

In this dissertation, we study interesting properties held by modular forms which are holomorphic eta-quotients. In Chapter 3, we analyze the coefficients of eta-quotients of the form $\frac{\eta(pz)^p}{\eta(z)}$ for primes $5\leq p\leq 23$ and obtain congruences for the p-core partition function of the form given by Garvan in [14]. These congruences have the form

$$a_p(n-\delta_p)\equiv 0 \pmod{\ell},$$

where $\delta_p=\frac{p^2-1}{24}$, ℓ is a prime divisor of $\frac{p-1}{2}$, $n\not\equiv 0 \pmod{\ell}$ and $\left(\frac{n}{p}\right)=\epsilon_p\in\{1,-1\}$. For example, if $p=13$ and $\ell=2$, then

$$a_{13}(n-7)\equiv 0 \pmod{2} \tag{1.0.4}$$

for all odd natural numbers n with $\left(\frac{n}{13}\right)=1$. Note that (1.0.4) can instead be expressed as the six congruences

$$a_{13}(26n+2,10,16,18,20,22)\equiv 0 \pmod{2}. \tag{1.0.5}$$

Theorem 3.2.4 gives the existence of fourteen Garvan type congruences. Our results differ from those given by Garvan, Radu and Sellers [26], and Chen [8], since each congruence listed yields infinitely many congruences. For instance, we prove that for all $t\geq 0$,

$$a_{13}(13^t\cdot n-7)\equiv 0 \pmod{2}$$

for all odd natural numbers n with $\left(\frac{n}{13}\right)=1$. When $t=0$, we obtain the same congruences as in (1.0.5). When $t=1$, we obtain the congruences

$$a_{13}(2\cdot 13^2\cdot n+6,32,110,214,292,318)\equiv 0 \pmod{2}.$$

Therefore, for each choice of t, we obtain six new congruences for $a_{13}(n)$. Similar to Garvan, Radu and Sellers, and Chen, we prove the congruences using the theory of modular forms.

Let $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$ and $D \mid 24$. In Chapter 4, we study how Hecke operators T_n for $\gcd(n, 6) = 1$ act on subspaces of $M_k(\Gamma_0(ND^2), \chi)$ of the form

$$\{f(Dz)F(Dz) : F(z) \in M_w(\Gamma_0(N), \chi)\},$$

where $f(z)$ is a minimal holomorphic level N eta-quotient of denominator D. The following example is one of several which motivated this work. Consider the following eight holomorphic level 2 eta-quotients :

$$f_1(z) = \tfrac{\eta(z)^{15}}{\eta(2z)^7}, \quad f_5(z) = \tfrac{\eta(z)^{11}}{\eta(2z)^3} \quad f_7(z) = \tfrac{\eta(z)^9}{\eta(2z)}, \quad f_{11}(z) = \eta(z)^5\eta(2z)^3,$$
$$f_{13}(z) = \eta(z)^3\eta(2z)^5, \quad f_{17}(z) = \tfrac{\eta(2z)^9}{\eta(z)}, \quad f_{19}(z) = \tfrac{\eta(2z)^{11}}{\eta(z)^3}, \quad f_{23}(z) = \tfrac{\eta(2z)^{15}}{\eta(z)^7}.$$

For all $r \in \{1, 5, 7, 11, 13, 17, 19, 23\}$, note that

$$f_r(24z) = \sum a_r(n)q^n = q^r + \cdots \tag{1.0.6}$$

is a modular form in $M_4\left(\Gamma_0(1152), \left(\frac{2}{\cdot}\right)\right)$ with support on exponents $n \equiv r \pmod{24}$. Let $w \in \mathbb{Z}$, and define

$$A_{r,w} = \{f_r(24z)F(24z) : F(z) \in M_w(\Gamma_0(2))\} \subseteq M_{w+4}\left(\Gamma_0(1152), \left(\frac{2}{\cdot}\right)\right).$$

For all $a \in \mathbb{Z}$, define $\overline{a}$ to be the least non-negative residue of a modulo 24. For all primes ℓ, the Hecke operator T_ℓ on $\sum a(n)q^n \in M_4\left(\Gamma_0(1152), \left(\frac{2}{\cdot}\right)\right)$ acts by

$$\sum a(n)q^n \mid T_\ell = \sum \left(a(\ell n) + \left(\frac{2}{\ell}\right)\ell^3 a\left(\frac{n}{\ell}\right)\right)q^n, \tag{1.0.7}$$

where $a\left(\frac{n}{\ell}\right) = 0$ if $\ell \nmid n$. The following theorem is a special case of Theorem 4.1.8.

Theorem 1.0.4. *Let $\ell \geq 5$ be prime, let $w \in \mathbb{Z}$, and let $r \in \{1, 5, 7, 11, 13, 17, 19, 23\}$. Then we have*

$$T_\ell : A_{r,w} \to A_{\overline{r\ell},w} \text{ and } T_\ell^2 = T_\ell \circ T_\ell : A_{r,w} \to A_{r,w}.$$

Note that for all $s \in \{1,5,7,11,13,17,19,23\}$, the subspace $A_{s,w}$ is relatively small in the ambient subspace $M_{w+4}\left(\Gamma_0(1152), \left(\frac{2}{\cdot}\right)\right)$. It is isomorphic to $M_w(\Gamma_0(2))$, whose dimension is $\left\lfloor \frac{w}{4} \right\rfloor + 1$. On the other hand, [9] implies that $M_{w+4}\left(\Gamma_0(1152), \left(\frac{2}{\cdot}\right)\right)$ has dimension $192w + 608$.

For all $r \in \{1,5,7,11,13,17,19,23\}$, we have $A_{r,0} = \mathbb{C}f_r(24z)$. In this setting, Theorem 1.0.4 and (1.0.7) imply the following corollary, which is an explicit special case of Corollary 4.1.11.

Corollary 1.0.5. *Let $\ell \geq 5$ be prime, let $r \in \{1,5,7,11,13,17,19,23\}$, and for all n, let $a_r(n)$ be as in* (1.0.6). *Then we have*

$$f_r(24z) \mid T_\ell = a_r(\ell(\overline{r\ell}))f_{\overline{r\ell}}(24z), \textit{ and } f_r(24z) \mid T_\ell \mid T_\ell = a_r(\ell(\overline{r\ell}))a_{\overline{r\ell}}(\ell r)f_r(24z).$$

It follows that we can create Hecke eigenforms by taking specific linear combinations of the $f_r(24z)$. For example, the function

$$f_1(z) + \sqrt{-140}f_5(z) - \sqrt{440}f_7(z) + \sqrt{-2464}f_{11}(z) + \sqrt{-4928}f_{13}(z)$$

$$+\sqrt{14080}f_{17}(z) + \sqrt{-17920}f_{19}(z) - \sqrt{2048}f_{23}(z)$$

is a Hecke eigenform.

We structure the rest of the paper as follows. In Chapter 2, we give background on modular forms which will be used to prove both of our results. In Chapter 3, we define what it means for a partition of n to be p-core and prove Theorem 3.2.4. In Chapter 4, we define terms relating to eta-quotients such as minimal and denominator and state our results. Additionally in Chapter 4, we work out an extended example of our work for $N = 2$ and provide the proof of Theorem 4.1.8. In Appendix A we give the Maple code used to perform our calculations. We finish by listing all minimal eta-quotients of levels $1, 2, 3, 4, 5, 6, 8$ and 9 in Appendix B.

Chapter 2

Background on Modular Forms

In this chapter, we give a brief overview of the theory of modular forms. For a more detailed background, see [10, 11, 21, 25].

2.1 Modular Forms

The cusps of $\mathrm{SL}_2(\mathbb{Z})$ are defined by

$$\left\{ \lim_{z\to\infty} \frac{az+b}{cz+d} : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \right\} = \mathbb{P}^1(\mathbb{Q}) = \mathbb{Q} \cup \{\infty\}.$$

Modular forms can be categorized into broader or narrower categories by strengthening or relaxing the condition of holomorphy at the cusps. If a modular form $f(z)$ vanishes at the cusps, that is, if for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$,

$$\lim_{z\to i\infty} f\left(\frac{az+b}{cz+d}\right) = 0, \tag{2.1.1}$$

then we say that $f(z)$ is a *cusp form of weight k, level N, with Nebentypus character χ*. The set of cusp forms is denoted $S_k(\Gamma_0(N),\chi) \subseteq M_k(\Gamma_0(N),\chi)$. If $f(z)$ is a modular form which is allowed to have poles at the cusps, that is, if the limit in (2.1.1) does not exist for some matrix in $\mathrm{SL}_2(\mathbb{Z})$, then f is a *weakly holomorphic modular form of weight k, level N, with Nebentypus character χ*. We denote this set by $M_k^!(\Gamma_0(N),\chi) \supset M_k(\Gamma_0(N),\chi)$. This set is an infinite dimensional complex vector space.

The set $M_k(\Gamma_0(N),\chi)$ is a finite-dimensional complex vector space. If $M \mid N$, then $\Gamma_0(N) \subseteq \Gamma_0(M)$ which implies $M_k(\Gamma_0(M),\chi) \subseteq M_k(\Gamma_0(N),\chi)$. The dimension of $M_k(\Gamma_0(N),\chi)$ is bounded by the index of $\Gamma_0(N)$ in $\mathrm{SL}_2(\mathbb{Z})$:

$$\dim_{\mathbb{C}} M_k(\Gamma_0(N),\chi) \leq \frac{k}{12}[\Gamma_0(N) : \mathrm{SL}_2(\mathbb{Z})] = \frac{kN}{12} \prod_{\substack{p \mid N \\ p \text{ prime}}} \left(1 + \frac{1}{p}\right). \tag{2.1.2}$$

Let $f(z) \in M_k(\Gamma_0(N),\chi)$. Since the matrix $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \Gamma_0(N)$ for all $N \geq 1$, the first condition of Definition 1.0.1 gives $f(z) = f(z+1)$. It follows that $f(z)$ has a Fourier expansion, that is

$$f(z) = \sum_{n \in \mathbb{Z}} a(n) e^{2\pi i n z}.$$

The part of the Fourier series supported on negative exponents is the *principal part of f*. The second part of Definition 1.0.1 gives that $f(z)$ has no principal part. Let D^* denote the punctured open unit disk in the complex plane and define $\mathfrak{h} \to D^*$ by $z \mapsto q := e^{2\pi i z}$. After applying this mapping, we may identify $f(z)$ by its *q-expansion*:

$$f(z) = \sum_{n=0}^{\infty} a(n) q^n.$$

The second part of Definition 1.0.1 allows us to define $f(i\infty) = a(0)$. It follows that if $f \in S_k(\Gamma_0(N),\chi)$, then $a(0) = 0$. We have the following lemma.

Lemma 2.1.1 (Sturm's Bound). *Let $f(z) = \sum a(n)q^n$ and $g(z) = \sum b(n)q^n$ be in $M_k(\Gamma_0(N),\chi)$ and suppose $a(n) = b(n)$ for all n less than or equal to the bound given in* (2.1.2)*. Then $f(z) = g(z)$ and $a(n) = b(n)$ for all n.*

2.2 Eisenstein Series

Consider the following functions $G_k : \mathfrak{h} \to \mathbb{C}$ defined by

$$G_k(z) = \sum_{(m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}} \frac{1}{(mz+n)^k}.$$

For all even $k \geq 2$, we have that $G_k(z)$ satisfies both conditions in Definition 1.0.1 for weight k and level 1. Note that $G_k(z)$ converges absolutely, and is therefore holomorphic, when $k \geq 4$, but fails to do so for $k = 2$. Thus, we have that for all even $k \geq 4$, $G_k(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. If $j, n \in Z$ with $j \geq 1$ and $n \geq 0$, define

$$\sigma_j(n) := \sum_{d|n} d^j.$$

The q-expansion of $G_k(z)$ for $k \geq 4$ is

$$G_k(z) = 2\zeta(k) + \frac{2(2\pi i)^k}{(k-1)!}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n,$$

where $\zeta(k)$ is the Riemann zeta function. We normalize $G_k(z)$ so that $a(0) = 1$ in its q-expansion. Recall the following definition of the Bernoulli numbers.

Definition 2.2.1. *For every nonnegative integer k, the k^{th} Bernoulli number B_k is defined by*

$$\frac{1}{e^t - 1} = \sum_{k=0}^{\infty} B_k \frac{t^k}{k!}.$$

Therefore, we obtain the following normalized modular forms.

Definition 2.2.2. *For all even $k \geq 2$, the Eisenstein series of weight k is*

$$E_k := \frac{1}{2\zeta(k)} G_k = 1 - \frac{2k}{B_k}\sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n.$$

As mentioned earlier, $E_2(z)$ is not a modular form. However, note that for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ we have that

$$E_2\left(\frac{az+b}{cz+d}\right) = (cz+d)^2 E_2(z) + \frac{12c(cz+d)}{2\pi i}.$$

The form $E_2(z)$ is the prototypical example of a *quasi-modular form.* Despite not being a modular form, it plays a crucial role in the theory.

Two Eisenstein series of great importance are

$$E_4(z) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n)q^n = 1 + 240q + 2160q^2 + 6720q^3 + \cdots$$

and

$$E_6(z) = 1 - 504 \sum_{n=1}^{\infty} \sigma_5(n) q^n = 1 - 504q - 16632q^2 - 122976q^3 + \cdots .$$

Note that $E_4(z)^3$ and $E_6(z)^2$ are both forms in $M_{12}(\mathrm{SL}_2(\mathbb{Z}))$ with $a(0) = 1$. It follows that

$$\Delta(z) := \frac{E_4(z)^3 - E_6(z)^2}{1728} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots \in S_{12}(\mathrm{SL}_2(\mathbb{Z})).$$

These three modular forms are instrumental in proving the following precise formula for the dimension of $M_k(\mathrm{SL}_2(\mathbb{Z}))$:

$$\dim_{\mathbb{C}} M_k(\mathrm{SL}_2(\mathbb{Z})) = \begin{cases} \left\lfloor \dfrac{k}{12} \right\rfloor, & \text{if } k \equiv 2 \pmod{12} \\ \left\lfloor \dfrac{k}{12} \right\rfloor + 1, & \text{if } k \not\equiv 2 \pmod{12}. \end{cases}$$

Moreover, a basis for $M_k(\mathrm{SL}_2(\mathbb{Z}))$ is $\{E_4(z)^a E_6(z)^b : a, b \geq 0, 4a + 6b = k\}$. This basis is called the *Eisenstein basis*. It follows that $M_2(\mathrm{SL}_2(\mathbb{Z})) = \emptyset$. However, for every $N > 1$, we have that $E_2(z) - NE_2(Nz) \in M_2(\Gamma_0(N))$.

2.3 The Dedekind Eta-Function

Recall that the *Dedekind eta-function* is

$$\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).$$

Euler's Pentagonal Number Theorem represents $\eta(z)$ as a power series by

$$\eta(z) = q^{1/24} \left(1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{n(3n-1)/2} + q^{n(3n+1)/2} \right) \right). \tag{2.3.1}$$

Recall that $\eta(z)$ is a weight $1/2$ modular form on $\mathrm{SL}_2(\mathbb{Z})$ with multiplier system ν_η. In order to define ν_η, we first require some definitions. When $a \in \mathbb{Z}$ and p is an odd prime, we let $\left(\frac{a}{p}\right)$ denote the Legendre symbol, and we let $\left(\frac{a}{1}\right) = 1$. For all $b \neq 0$ in $\mathbb{Z}$, we denote by $v_p(b)$ the exponent in the power of p dividing b. When $b \geq 1$ is odd, the Jacobi symbol is

$$\left(\frac{a}{b}\right) = \prod_{p \mid b} \left(\frac{a}{p}\right)^{v_p(b)}. \tag{2.3.2}$$

To extend (2.3.2) to odd $b < 0$, for all $x \neq 0$ in $\mathbb{Z}$ we define

$$\operatorname{sign}(x) = \frac{x}{|x|} = (-1)^{\varepsilon(x)}, \text{ where } \varepsilon(x) = \frac{\operatorname{sign}(x) - 1}{2}.$$

Then for all $a \neq 0$ in $\mathbb{Z}$ and for all odd $b \in \mathbb{Z}$, following ([20], p. 52), we set

$$\left(\frac{a}{b}\right) = \left(\frac{a}{|b|}\right)(-1)^{\varepsilon(a)\varepsilon(b)}. \tag{2.3.3}$$

For such a and b with $ab \neq 0$, we note that $(-1)^{\varepsilon(a)\varepsilon(b)}$ agrees with the Hilbert symbol $(a,b)_\infty$. The law of quadratic reciprocity for odd a and b with $\gcd(a,b) = 1$ is

$$\left(\frac{a}{b}\right)\left(\frac{b}{a}\right) = (-1)^{\left(\frac{a-1}{2}\right)\left(\frac{b-1}{2}\right)}(-1)^{\varepsilon(a)\varepsilon(b)}. \tag{2.3.4}$$

For odd a, we also have

$$\left(\frac{-1}{a}\right) = (-1)^{\frac{|a|-1}{2}}(-1)^{\varepsilon(a)} = (-1)^{\frac{a-1}{2}}. \tag{2.3.5}$$

Following ([23], p. 50 and 51), we make the following further definitions:

Definition 2.3.1. *Let a, $b \in \mathbb{Z}$ with $a \neq 0$, b odd, and* $\gcd(a,b) = 1$.

We define

$$\left(\frac{a}{b}\right)^* = \left(\frac{a}{|b|}\right), \quad \left(\frac{0}{\pm 1}\right)^* = 1$$

and

$$\left(\frac{a}{b}\right)_* = \left(\frac{a}{b}\right), \quad \left(\frac{0}{1}\right)_* = 1, \quad \left(\frac{0}{-1}\right)_* = -1.$$

We observe that $\left(\frac{\cdot}{\cdot}\right)_*$ and $\left(\frac{\cdot}{\cdot}\right)^*$ are multiplicative in their upper and lower arguments.

We now give the definition of ν_η.

Definition 2.3.2. *For all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, *we define*

$$\nu_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{cases} \left(\frac{d}{c}\right)^* \exp\left\{\frac{2\pi i}{24}\Big((a+d)c - bd(c^2-1) - 3c\Big)\right\}, & \text{if } c \text{ is odd} \\ \left(\frac{c}{d}\right)_* \exp\left\{\frac{2\pi i}{24}\Big((a+d)c - bd(c^2-1) + 3d - 3 - 3cd\Big)\right\}, & \text{if } c \text{ is even.} \end{cases}$$

The following theorem states that $\eta(z)$ is a cusp form of weight $1/2$ on $\mathrm{SL}_2(\mathbb{Z})$ with respect to the multiplier system ν_η.

Proposition 2.3.3. *[23, Theorem 2, p. 51]. For all* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$*, we have*

$$\eta\left(\frac{az+b}{cz+d}\right) = \nu_\eta \begin{pmatrix} a & b \\ c & d \end{pmatrix} (cz+d)^{1/2}\eta(z).$$

Since ν_η is a 24^{th} root of unity, we have that

$$\eta\left(\frac{az+b}{cz+d}\right)^{24} = (cz+d)^{12}\eta(z)^{24}.$$

Therefore,

$$\eta(z)^{24} = q\prod_{n=1}^{\infty}(1-q^n)^{24} = q - 24q^2 + 252q^3 - 1472q^4 + \cdots \in S_{12}(\mathrm{SL}_2(\mathbb{Z})).$$

Since $\dim_{\mathbb{C}} S_{12}(\mathrm{SL}_2(\mathbb{Z})) = 1$, we have that $\eta(z)^{24} = \Delta$. By Proposition 1.0.2, we also have that

$$\frac{\eta(pz)^p}{\eta(z)}, \frac{\eta(z)^p}{\eta(pz)} \in M_{\frac{p-1}{2}}\left(\Gamma_0(p), \left(\frac{\cdot}{p}\right)\right). \tag{2.3.6}$$

2.4 Operators on Modular Forms

Let k and N be natural numbers and χ be a Dirichlet character modulo N. There are many important operators which act on modular forms. Some operators we define by how they act on $f(z)$ and some by how they act on q-expansions. First, we define the *slash operator of weight k*. For all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Q}^+)$, we have

$$f(z) \mid_k \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad-bc)^{k/2}(cz+d)^{-k} f\left(\frac{az+b}{cz+d}\right). \tag{2.4.1}$$

It follows that the first condition of Definition 1.0.1 can be rewritten in terms of the slash operator as $f(z) \mid_k \gamma = \chi(d)f(z)$ for all $\gamma \in \Gamma_0(N)$.

For $m \geq 1$, The *Hecke operator* $T_m = T_{m,k,\chi}$ is defined by

$$\sum_{n=0}^{\infty} a(n)q^n \mid T_m := \sum_{n=0}^{\infty} \sum_{d|\gcd(m,n)} d^{k-1}\chi(d)a\left(\frac{mn}{d^2}\right)q^n.$$

When ℓ is prime, the Hecke operator T_ℓ is

$$\sum_{n=0}^{\infty} a(n)q^n \mid T_\ell = \sum_{n=0}^{\infty} \left(a(\ell n) + \chi(\ell)\ell^{k-1}a\left(\frac{n}{\ell}\right)\right)q^n, \tag{2.4.2}$$

where $a\left(\frac{n}{\ell}\right) = 0$ for $\ell \nmid n$. The operator T_ℓ is expressible in terms of the slash operator (2.4.1) by

$$f(z)|T_\ell = \ell^{k/2-1}\left(\sum_{j=0}^{\ell-1} f(z)\Big|_k \begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} + \chi(\ell)f(z)\Big|_k \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}\right). \tag{2.4.3}$$

Let $M \geq 1$. We define the U_M operator and V_M operator on q-expansions as

$$f(z) \mid U_M = \sum_{n=0}^{\infty} a(n)q^n \mid U_M = \sum_{\substack{n=0 \\ M|n}}^{\infty} a(n)q^{n/M} = \sum_{n=0}^{\infty} a(Mn)q^n$$

and

$$f(z) \mid V_M = \sum_{n=0}^{\infty} a(n)q^n \mid V_M = \sum_{n=0}^{\infty} a(n)q^{Mn}.$$

Let ε be a Dirichlet character modulo M. Then the *twisting operator* is defined by

$$f(z) \otimes \varepsilon = \sum_{n=0}^{\infty} a(n)q^n \otimes \varepsilon = \sum_{n=0}^{\infty} \varepsilon(n)a(n)q^n.$$

The last operators on modular forms we discuss are the derivative operators. Recalling that $q = e^{2\pi iz}$, we define $\theta = q\frac{d}{dq} = \frac{1}{2\pi i}\frac{d}{dz}$, which gives

$$\theta(f(z)) = \theta\left(\sum_{n=0}^{\infty} a(n)q^n\right) = \sum_{n=0}^{\infty} na(n)q^n.$$

Note that the derivative operator θ fails to preserve modularity; however, it maps modular forms to quasi-modular forms. In fact, θ fails to preserve modularity for the same reason as $E_2(z)$. It follows that we can use $E_2(z)$ to adjust the derivative operator and preserve the modularity of the function. This is called the *Serre derivative*, denoted ϑ, and is defined by

$$\vartheta_k(f(z)) = \theta(f(z)) - \frac{k}{12}E_2(z)f(z).$$

All of the operators mentioned above except for the ordinary derivative operator θ preserve modularity. However, the new modular form may live in a space with different level, weight, or character. The following theorem gives the target space for each operator. Note that the level of the target space for U_M and $\otimes\, \varepsilon$ can be made more precise depending on the prime factorization of M and N.

Theorem 2.4.1. *Let k, N, χ, M, ℓ, and ε be defined as above. Then*

1. $T_\ell : M_k(\Gamma_0(N), \chi) \to M_k(\Gamma_0(N), \chi)$,

2. $U_M : M_k(\Gamma_0(N), \chi) \to \begin{cases} M_k(\Gamma_0(NM), \chi), & \text{if } M \nmid N, \\ M_k(\Gamma_0(N), \chi), & \text{if } M \mid N. \end{cases}$

3. $V_M : M_k(\Gamma_0(N), \chi) \to M_k(\Gamma_0(NM), \chi)$.

4. $\otimes\, \varepsilon : M_k(\Gamma_0(N), \chi) \to M_k(\Gamma_0(NM^2), \chi\varepsilon^2)$.

5. $\vartheta_k : M_k(\Gamma_0(N), \chi) \to M_{k+2}(\Gamma_0(N), \chi)$.

2.5 Modular Forms modulo p

Let p be prime and let $\mathbb{Z}_{(p)}$ denote the ring consisting of the p-integral elements of $\mathbb{Q}$, that is,

$$\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \in \mathbb{Q} : \gcd(a, b) = 1, p \nmid b \right\} \subseteq \mathbb{Q}.$$

Also, let $\mathbb{F}_p = \mathbb{Z}/p\mathbb{Z}$ be the finite field with p elements. Then the homomorphism $\mathbb{Z}_{(p)} \to \mathbb{F}_p$ defined by $a/b \mapsto ab^{-1} \pmod p$ extends to formal power series $\mathbb{Z}_{(p)}[\![q]\!] \to \mathbb{F}_p[\![q]\!]$ by reducing the coefficients of $f(z) = \sum a(n)q^n$ modulo p. Let $M_k^{(p)}(\Gamma_0(N), \chi) = M_k(\Gamma_0(N), \chi) \cap \mathbb{Z}_{(p)}[\![q]\!]$. We denote the *set of modular forms with p-integral coefficients reduced modulo p* by the image of $M_k^{(p)}(\Gamma_0(N), \chi)$ under the reduction modulo p homomorphism:

$$\widetilde{M}_k^{(p)}(\Gamma_0(N), \chi) := \operatorname{Im}\{M_k^{(p)}(\Gamma_0(N), \chi) \to \mathbb{F}_p[\![q]\!]\}.$$

Note that $\widetilde{M}_k^{(p)}(\Gamma_0(N),\chi)$ is a finite dimensional $\mathbb{F}_p$-vector space with dimension bounded above by $\frac{kN}{12}\prod_{p\mid N}\left(1+\frac{1}{p}\right)$. Lemma 2.1.1 implies that if $f(z)=\sum a(n)q^n$ and $g(z)=\sum b(n)q^n$ are in $M_k^{(p)}(\Gamma_0(N),\chi)$ and $a(n)\equiv b(n) \pmod p$ for all n less than or equal to the bound given in (2.1.2), then $f(z)\equiv g(z) \pmod p$ and $a(n)\equiv b(n) \pmod p$ for all n.

Let $k\geq 2$ and let $E_k(z)$ denote the Eisenstein series of weight k. The Von Staudt-Claussen Theorem on Bernoulli numbers implies that $E_k(z)\equiv 1 \pmod p$ for every p with $p-1\mid k$. Therefore, we have that $E_{p-1}\equiv 1 \pmod p$. Hence,

$$\widetilde{M}_k^{(p)}(\Gamma_0(N),\chi)\subseteq \widetilde{M}_{k+p-1}^{(p)}(\Gamma_0(N),\chi). \tag{2.5.1}$$

The Kummer congruences for Bernoulli numbers imply that $E_{p+1}(z)\equiv E_2(z) \pmod p$. Therefore, even though $E_2(z)$ is not a modular form, we have that $E_2(z)$ is a modular form modulo p for all primes p. In particular, for all primes p,

$$E_2(z)\in \widetilde{M}_{p+1}^{(p)}(\Gamma_0(N),\chi).$$

2.6 Operators on Modular Forms modulo p

Note that the operators from Section 2.4 preserve p-integrality and can be considered as operators on $\widetilde{M}_k^{(p)}(\Gamma_0(N),\chi)$. Below, we list the operators which can be refined modulo p.

1. $U_p:\widetilde{M}_k^{(p)}(\Gamma_0(N),\chi)\to \widetilde{M}_k^{(p)}(\Gamma_0(N),\chi)$

   $$f\mid U_p\equiv f\mid T_p \pmod p,$$

   for $k\geq 2$.

2. $V_p:\widetilde{M}_k^{(p)}(\Gamma_0(N),\chi)\to \widetilde{M}_{pk}^{(p)}(\Gamma_0(N),\chi)$

   $$f\mid V_p=\sum_{n=0}^{\infty}a(n)q^{pn}\equiv\sum_{n=0}^{\infty}a(n)^pq^{pn}\equiv\left(\sum_{n=0}^{\infty}a(n)q^n\right)^p\equiv f^p \pmod p,$$

   by Fermat's Little Theorem.

3. $\theta : \widetilde{M}_k^{(p)}(\Gamma_0(N), \chi) \to \widetilde{M}_{k+p+1}^{(p)}(\Gamma_0(N), \chi)$

$$\theta(f) = \vartheta_k(f) - \frac{k}{12}E_2(z)f(z) \equiv \vartheta_k(f)E_{p-1}(z) - \frac{k}{12}E_{p+1}(z)f(z) \pmod{p}.$$

4. $\otimes \left(\frac{\cdot}{p}\right) : \widetilde{M}_k^{(p)}(\Gamma_0(N), \chi) \to \widetilde{M}_{k+\frac{p^2-1}{2}}^{(p)}(\Gamma_0(N), \chi)$

$$f \otimes \left(\frac{\cdot}{p}\right) = \sum_{n=0}^{\infty} \left(\frac{n}{p}\right) a(n)q^n \equiv \sum_{n=0}^{\infty} n^{\frac{p-1}{2}} a(n)q^n \equiv \theta^{\frac{p-1}{2}}(f) \pmod{p},$$

by Euler's Criterion.

Note that the each of the above operators considered modulo p preserves the level of the space.

Chapter 3

Congruences for p-Core Partitions

3.1 The Partition Function

Let $n \in \mathbb{N}$. A *partition of* n is a representation of n as a sum of natural numbers in nonincreasing order. The partition function $p(n)$ gives the total number of partitions of the natural number n. For example $p(4) = 5$ since

$$\begin{aligned} 4 &= 4 \\ 4 &= 3+1 \\ 4 &= 2+2 \\ 4 &= 2+1+1 \\ 4 &= 1+1+1+1 \end{aligned}$$

are all the partitions of 4. Ramanujan discovered that the partition function satisfies some interesting congruences. In [27, 28], he proved that for every $n \in \mathbb{N}$,

$$p(5n+4) \equiv 0 \pmod{5}, \tag{3.1.1}$$

$$p(7n+5) \equiv 0 \pmod{7}, \tag{3.1.2}$$

$$p(11n+6) \equiv 0 \pmod{11}. \tag{3.1.3}$$

Ramanujan made two conjectures about further congruences of $p(n)$. First, he claimed

> *It appears there are no equally simple properties for any moduli involving primes other than these.*

In other words, he claimed that the only congruences of the form

$$p(\ell n + \beta_\ell) \equiv 0 \pmod{\ell}, \tag{3.1.4}$$

where ℓ is a prime, are (3.1.1) - (3.1.3). Second, he conjectured that congruences of this form also exist for powers of the primes 5, 7, and 11. The second conjecture was proved for $\ell = 5, 7$ by Watson [29] in 1938 and for $\ell = 11$ by Atkin [5] in 1967. They proved that for all $j \in \mathbb{N}$,

$$\begin{aligned} p(5^j n + 4) &\equiv 0 \pmod{5^j}, \\ p(7^j n + 5) &\equiv 0 \pmod{7^{\lfloor j/2 \rfloor + 1}}, \\ p(11^j n + 6) &\equiv 0 \pmod{11^j}. \end{aligned}$$

In 2003, Ahlgren and Boylan [3] proved the first conjecture. Even though there are no congruences of the form (3.1.4) for primes other than $5, 7$, and 11, there are still many congruences that the partition function satisfies for the other primes and even for most composite numbers. In 2001, Ahlgren and Ono [4] proved that there are congruences of the form

$$p(An + B) \equiv 0 \pmod{M}$$

for every natural number M relatively prime to 6. For example, when $M = 13$, then

$$p(11^3 \cdot 13n + 237) \equiv 0 \pmod{13}.$$

3.2 The t-Core Partition Function

We now examine the t-core partition function, a variation of the partition function. Consider the partition $10 = 5 + 3 + 2$. This partition can be represented as a Ferrer's diagram by

$$\begin{array}{ccccc} \bullet & \bullet & \bullet & \bullet & \bullet \\ \bullet & \bullet & \bullet & & \\ \bullet & \bullet & & & \end{array}$$

where the i^{th} row represents the i^{th} summand in the partition. The hook length of a node in the diagram is the number of nodes located directly below it and directly to its right and also the node itself. Note the same partition of 10, now with the hook length of each node included.

$$\begin{array}{ccccc} {}^{7}\bullet & {}^{6}\bullet & {}^{4}\bullet & {}^{2}\bullet & {}^{1}\bullet \\ {}^{4}\bullet & {}^{3}\bullet & {}^{1}\bullet & & \\ {}^{2}\bullet & {}^{1}\bullet & & & \end{array}$$

A partition of n is t-core if and only if none of the hook lengths associated to its Ferrer's diagram is a multiple of t. So the partition of 10 above is a 5-core partition and is a t-core partition for $t \geq 8$. Let $t \geq 2$ and let $a_t(n)$ denote the number of t-core partitions of n. In [15], Garvan, Kim, and Stanton give the generating function for $a_t(n)$ as

$$\sum_{m=0}^{\infty} a_t(m) q^m = \prod_{n=1}^{\infty} \frac{(1-q^{tn})^t}{1-q^n}. \tag{3.2.1}$$

Several congruence properties have been found for t-core partitions, especially when t is prime. In 2003, Garvan proved the following theorem.

Theorem 3.2.1 ([14])**.** *Let* $p = 5, 7, 11, 13, 17, 19, 23$ *and* $\delta_p = \frac{p^2-1}{24}$. *Then for each prime divisor* ℓ *of* $\frac{p-1}{2}$,

$$a_p(n - \delta_p) \equiv 0 \pmod{\ell},$$

whenever $\left(\frac{n}{p}\right) = \epsilon_p$ *and* $n \not\equiv 0 \pmod{\ell}$, *where* $\epsilon_5 = \epsilon_7 = \epsilon_{17} = \epsilon_{19} = \epsilon_{23} = -1$ *and* $\epsilon_{11} = \epsilon_{13} = 1$. *Furthermore,*

$$a_{17}(n - 12) \equiv 0 \pmod{8}$$

whenever $\left(\frac{n}{17}\right) = -1$ *and* $n \not\equiv 0 \pmod{2}$, *and*

$$a_{17}(n - 12) \equiv 0 \pmod{2}$$

whenever $\left(\frac{n}{17}\right) = -1$ *and* $n \not\equiv 0 \pmod{4}$.

In 2011, Radu and Sellers in their study of broken k-diamond partitions proved more Garvan type congruences for $a_p(n)$ when $5 \le p \le 23$.

Theorem 3.2.2 ([26])**.** *If* $p = 5, 11, 13, 19$, *then*

$$a_p(n - \delta_p) \equiv 0 \pmod{2}$$

whenver n *is odd and* $\left(\frac{n}{p}\right) = \epsilon_p$, *where* $\epsilon_5 = \epsilon_{11} = -1$ *and* $\epsilon_{13} = \epsilon_{19} = 1$. *If* $p = 7, 17, 23$, *then*

$$a_p(n - \delta_p) \equiv 0 \pmod{8}$$

whenever n *is odd and* $\left(\frac{n}{p}\right) = \epsilon_p$, *where* $\epsilon_7 = 1$ *and* $\epsilon_{17} = \epsilon_{23} = -1$.

Note that Garvan's theorem covers the case when $p = 5, 13$, and 17. In 2013, Chen proved the existence of fourteen more Garvan type congruences for $a_p(n)$ when $5 \le p \le 47$.

Theorem 3.2.3 ([8])**.** *Let* $\left(\frac{n}{m}\right)$ *be the Jacobi symbol. Then*

$$\begin{aligned}
a_5(n-1) &\equiv 0 \pmod{3} && \text{if } \left(\frac{n}{15}\right) = -1,\\
a_7(n-2) &\equiv 0 \pmod{5} && \text{if } \left(\frac{n}{5}\right) = 1 \text{ and } \left(\frac{n}{7}\right) = -1,\\
a_{11}(n-5) &\equiv 0 \pmod{3} && \text{if } \left(\frac{n}{33}\right) = 1.\\
a_{13}(n-7) &\equiv 0 \pmod{11} && \text{if } \left(\frac{n}{13}\right) = -1 \text{ and } \left(\frac{n}{11}\right) = 1,\\
a_{17}(n-12) &\equiv 0 \pmod{3} && \text{if } \left(\frac{n}{51}\right) = -1,\\
a_{17}(n-12) &\equiv 0 \pmod{5} && \text{if } \left(\frac{n}{17}\right) = 1 \text{ and } \left(\frac{n}{5}\right) = 1,\\
a_{23}(n-22) &\equiv 0 \pmod{3} && \text{if } \left(\frac{n}{69}\right) = 1,\\
a_{23}(n-22) &\equiv 0 \pmod{7} && \text{if } \left(\frac{n}{23}\right) = -1 \text{ and } \left(\frac{n}{7}\right) = -1,\\
a_{29}(n-35) &\equiv 0 \pmod{3} && \text{if } \left(\frac{n}{29}\right) = 1 \text{ and } \left(\frac{n}{3}\right) = 1,\\
a_{37}(n-57) &\equiv 0 \pmod{5} && \text{if } \left(\frac{n}{37}\right) = 1 \text{ and } \left(\frac{n}{5}\right) = 1,
\end{aligned}$$

$$
\begin{aligned}
a_{37}(n-57) &\equiv 0 \pmod 7 \qquad \textit{if } \left(\frac{n}{37}\right) = 1 \textit{ and } \left(\frac{n}{7}\right) = -1,\\
a_{41}(n-70) &\equiv 0 \pmod 3 \qquad \textit{if } \left(\frac{n}{41}\right) = 1 \textit{ and } \left(\frac{n}{3}\right) = 1,\\
a_{47}(n-92) &\equiv 0 \pmod 3 \qquad \textit{if } \left(\frac{n}{47}\right) = -1 \textit{ and } \left(\frac{n}{3}\right) = 1,\\
a_{47}(n-92) &\equiv 0 \pmod 5 \qquad \textit{if } \left(\frac{n}{47}\right) = 1 \textit{ and } \left(\frac{n}{5}\right) = 1.
\end{aligned}
$$

Not only do some of Chen's congruences apply to $a_p(n)$ for $p > 23$, but the modulus ℓ is not a divisor of $\frac{p-1}{2}$. In Section 3.3, we use the theory of modular forms to prove not only more Garvan type congruences, but the existence of infinite families of these congruences for $a_p(n)$ with $5 \leq p \leq 23$. In particular, we prove the following theorem.

Theorem 3.2.4. *Let $t \in \mathbb{Z}$ with $t \geq 0$ and let $\left(\frac{n}{m}\right)$ be the Jacobi symbol. Then we have the following congruences.*

$$
\begin{aligned}
a_5(5^t \cdot n - 1) &\equiv 0 \pmod 2, \qquad \textit{if } 2 \nmid n \textit{ and } \left(\frac{n}{5}\right) = -1\\
a_5(5^t \cdot n - 1) &\equiv 0 \pmod 3, \qquad \textit{if } \left(\frac{n}{15}\right) = -1\\
a_7(7^{2t} \cdot n - 2) &\equiv 0 \pmod 2, \qquad \textit{if } 2 \nmid n \textit{ and } \left(\frac{n}{7}\right) = 1\\
a_7(7^t \cdot n - 2) &\equiv 0 \pmod 3, \qquad \textit{if } 3 \nmid n \textit{ and } \left(\frac{n}{7}\right) = -1\\
a_7(7^t \cdot n - 2) &\equiv 0 \pmod 5, \qquad \textit{if } \left(\frac{n}{5}\right) = 1 \textit{ and } \left(\frac{n}{7}\right) = -1\\
a_{11}(11^t \cdot n - 5) &\equiv 0 \pmod 2, \qquad \textit{if } 2 \nmid n \textit{ and } \left(\frac{n}{22}\right) = -1\\
a_{11}(11^{2t} \cdot n - 5) &\equiv 0 \pmod 3, \qquad \textit{if } \left(\frac{n}{3}\right) = -1 \textit{ and } \left(\frac{n}{11}\right) = -1\\
a_{11}(11^{2t+1} \cdot n - 5) &\equiv 0 \pmod 3, \qquad \textit{if } \left(\frac{n}{3}\right) = 1 \textit{ and } \left(\frac{n}{11}\right) = -1\\
a_{11}(11^{6t} \cdot n - 5) &\equiv 0 \pmod 3, \qquad \textit{if } \left(\frac{n}{3}\right) = 1 \textit{ and } \left(\frac{n}{11}\right) = 1\\
a_{11}(11^{6t+3} \cdot n - 5) &\equiv 0 \pmod 3, \qquad \textit{if } \left(\frac{n}{3}\right) = -1 \textit{ and } \left(\frac{n}{11}\right) = 1\\
a_{13}(13^t \cdot n - 7) &\equiv 0 \pmod 2, \qquad \textit{if } 2 \nmid n \textit{ and } \left(\frac{n}{13}\right) = 1\\
a_{17}(17^{2t} \cdot n - 12) &\equiv 0 \pmod 2, \qquad \textit{if } 2 \nmid n \textit{ and } \left(\frac{n}{17}\right) = -1
\end{aligned}
$$

$$a_{19}(19^{2t}\cdot n-15)\equiv 0 \pmod 2, \qquad \textit{if } 2\nmid n \textit{ and } \left(\frac{n}{19}\right)=1$$

$$a_{23}(23^{2t}\cdot n-22)\equiv 0 \pmod 2, \qquad \textit{if } 2\nmid n \textit{ and } \left(\frac{n}{23}\right)=-1$$

3.3 Proof of Theorem 3.2.4

Proposition 1.0.2 and (3.2.1) give

$$f(z)=\frac{\eta(pz)^p}{\eta(z)}=\sum_{n=0}^{\infty}a_p(n-\delta_p)q^p\in M_{\frac{p-1}{2}}\left(\Gamma_0(p),\left(\frac{\cdot}{p}\right)\right),$$

for primes $p\geq 5$ and $\delta_p=\frac{p^2-1}{2}$. In order to prove each congruence in Theorem 3.2.4, we look for spaces which not only contain $f(z)$, but also contain eta-products with nice properties, that is, eta-products of the form $\eta(z)^{\ell^k}\eta(pz)^b$, where ℓ is the modulus of the congruence and $b,k\in\mathbb{N}$. We then use the operators given in Section 2.6 to construct a new form $g(z)$ which has support on specific congruence classes modulo p. Once we have found the appropriate spaces and forms, for every $t\geq 0$ we find linear combinations of the eta-products which are congruent to $g(z)\mid U_{p^t}$ modulo ℓ. We do not have to construct infinitely many linear combinations since for each of the congruences listed in Theorem 3.2.4, there exists $s\in\mathbb{N}$ such that $g(z)\mid U_{p^s}\equiv g(z) \pmod{\ell}$. To conclude the theorem, we analyze the support of the linear combination of eta-products modulo ℓ.

In the remainder of this section, we work out two of the results listed in the theorem in detail. Then, we briefly discuss how to prove the remaining congruences. We use the following notation throughout the examples. For each choice of p and ℓ, let $f(z)=\frac{\eta(pz)^p}{\eta(z)}$ and let $g_r(z)$ be the section of $f(z)$ with support on exponents congruent to $r \pmod{\ell}$, that is, if $f(z)=\sum a(n)q^n$, then $g_r(z)=\sum a(\ell n+r)q^{\ell n+r}$.

3.3.1 11-core partitions modulo 3

Let $f(z)=\frac{\eta(11z)^{11}}{\eta(z)}=\sum a_{11}(n-5)q^n\in M_5\left(\Gamma_0(11),\left(\frac{\cdot}{11}\right)\right)$. Recall from Section 2.6 that $\theta(f)\in\widetilde{M}_9^{(3)}\left(\Gamma_0(11),\left(\frac{\cdot}{11}\right)\right)$ and $(f\mid T_3)^3\in\widetilde{M}_{15}^{(3)}\left(\Gamma_0(11),\left(\frac{\cdot}{11}\right)\right)$. From (2.5.1),

we know that

$$\widetilde{M}_5^{(3)}\left(\Gamma_0(11),\left(\frac{\cdot}{11}\right)\right)\subseteq \widetilde{M}_9^{(3)}\left(\Gamma_0(11),\left(\frac{\cdot}{11}\right)\right)\subseteq \widetilde{M}_{15}^{(3)}\left(\Gamma_0(11),\left(\frac{\cdot}{11}\right)\right).$$

Let $g_1(z)=2(f+\theta(f)-(f\mid T_3)^3)$ and $g_2(z)=2(f-\theta(f)-(f\mid T_3)^3)$. We have that $g_1(z),g_2(z)\in \widetilde{M}_{15}^{(3)}\left(\Gamma_0(11),\left(\frac{\cdot}{11}\right)\right)$. Note that

$$g_1(z)\equiv 2\left(f+f\otimes\left(\frac{\cdot}{3}\right)-f\mid U_3\mid V_3\right)\equiv\sum_{n=0}^{\infty}a_{11}((3n+1)-5)q^{3n+1}\pmod 3.$$

Similarly, we have that $g_2(z)\equiv\sum a_{11}((3n+2)-5)q^{3n+2}\pmod 3$.

For any modular form $f(z)$, let $\overline{f(z)}$ be the series representation of $f(z)$ with each coefficient reduced modulo 3. We first analyze the support of $\overline{g_2(z)}$. Let $h(z)=\eta(z)^9\eta(11z)^{21}\in M_{15}\left(\Gamma_0(11),\left(\frac{\cdot}{11}\right)\right)$. Lemma 2.1.1 gives that $\overline{g_2(z)}=\overline{h(z)\mid T_{17}}$. Note that $\overline{h(z)\mid T_{17}\mid U_{11}}=\overline{h(z)\mid T_{19}}$ and $\overline{h(z)\mid T_{19}\mid U_{11}}=\overline{h(z)\mid T_{17}}$. Therefore, for all $t\geq 0$, we have that

$$\sum_{n=0}^{\infty}a_{11}(11^{2t}(3n+2)-5)q^{3n+2}=g_2(z)\mid U_{11^{2t}}\equiv h(z)\mid T_{17}\pmod 3,$$

$$\sum_{n=0}^{\infty}a_{11}(11^{2t+1}(3n+1)-5)q^{3n+1}=g_2(z)\mid U_{11^{2t+1}}\equiv h(z)\mid T_{19}\pmod 3.$$

Thus, in order to determine the support of $\overline{g_2(z)\mid U_{11^t}}$, we study the support of $\overline{h(z)\mid T_{17}}$ and $\overline{h(z)\mid T_{19}}$. Note that

$$\begin{aligned} h(z)&\equiv \eta(9z)\eta(11z)^{21}\pmod 3\\ &\equiv q^{10}\left(\prod_{n=1}^{\infty}\left(1-q^{11n}\right)\right)^{21}\left(1+\sum_{n=1}^{\infty}(-1)^n\left(q^{9n(3n-1)/2}+q^{9n(3n+1)/2}\right)\right)\pmod 3\end{aligned}$$

by (2.3.1). The support of the infinite product is restricted to multiples of 11 and the support of the infinite series is restricted modulo 11 to the congruence classes $0,1,3,7,8,9$. Therefore, we have that the support of $\overline{h(z)}$ is restricted modulo 11 to $0,2,6,7,8,10$. From (2.4.2), we see that T_ℓ changes the support by multiplying and dividing by ℓ. It follows that the supports of $\overline{h(z)\mid T_{17}}$ and $\overline{h(z)\mid T_{19}}$ are restricted modulo 11 to $0,1,3,4,5,9$, the quadratic residues modulo 11. Therefore, we have

that the support of $\overline{g_2(z) \mid U_{11^t}}$ is also restricted modulo 11 to the squares. Since the series $\overline{g_2(z) \mid U_{11^{2t}}}$ runs through all positive integers congruent to 2 modulo 3, we have that if $\left(\frac{n}{3}\right) = -1$ and $\left(\frac{n}{11}\right) = -1$, then

$$a_{11}(11^{2t} \cdot n - 5) \equiv 0 \pmod{3}.$$

Similarly, since the series $\overline{g_2(z) \mid U_{11^{2t+1}}}$ runs through all positive integers congruent to 1 modulo 3, we have that if $\left(\frac{n}{3}\right) = 1$ and $\left(\frac{n}{11}\right) = -1$, then

$$a_{11}(11^{2t+1} \cdot n - 5) \equiv 0 \pmod{3}.$$

To prove the remaining congruences for 11-core partitions modulo 3, we analyze $g_1(z)$. Let $h(z) = \eta(z)^3\eta(11z)^{15} \in \widetilde{M}_9\left(\Gamma_0(11), \left(\frac{\cdot}{11}\right)\right)$. By Lemma 2.1.1, we have that $\overline{g_1(z)} = \overline{g_1(z) \mid U_{11^6}} = \overline{2h(z)}$ and

$$\begin{aligned}
g_1(z) \mid U_{11^{6t}} &\equiv 2h(z) \pmod{3},\\
g_1(z) \mid U_{11^{6t+1}} &\equiv h(z) \mid T_5 + h(z) \mid T_{29} \pmod{3},\\
g_1(z) \mid U_{11^{6t+2}} &\equiv 2h(z) + 2h(z) \mid T_7 \pmod{3},\\
g_1(z) \mid U_{11^{6t+3}} &\equiv h(z) \mid T_5 \pmod{3},\\
g_1 \mid U_{11^{6t+4}} &\equiv 2h(z) + h(z) \mid T_7 \pmod{3},\\
g_1(z) \mid U_{11^{6t+5}} &\equiv h(z) \mid T_5 + 2h(z) \mid T_{29} \pmod{3}.
\end{aligned}$$

Using (2.3.1), we find that the supports of $\overline{h(z)}$ and $\overline{h(z) \mid T_5}$ are restricted modulo 11 to $0, 2, 6, 7, 8, 10$, zero and the non squares modulo 11, while the supports of $\overline{h(z) \mid T_7}$ and $\overline{h(z) \mid T_{29}}$ are restricted modulo 11 to $0, 1, 3, 4, 5, 9$, the squares modulo 11. It follows that the only congruences which can be obtained are

$$\begin{aligned}
a_{11}(11^{6t} \cdot n - 5) &\equiv 0 \pmod{3}, && \text{if } \left(\frac{n}{3}\right) = 1 \text{ and } \left(\frac{n}{11}\right) = 1,\\
a_{11}(11^{6t+3} \cdot n - 5) &\equiv 0 \pmod{3}, && \text{if } \left(\frac{n}{3}\right) = -1 \text{ and } \left(\frac{n}{11}\right) = 1.
\end{aligned}$$

3.3.2 7-core congruence modulo 5

Let $f(z) = \frac{\eta(7z)^7}{\eta(z)} = \sum a_7(n-2)q^n \in M_3\left(\Gamma_0(7), \left(\frac{\cdot}{7}\right)\right)$. Let χ be the Dirichlet character modulo 5 with $\chi(2) = i$. Let

$$g_{1,4}(z) := f \otimes \chi + f \otimes \overline{\chi} \equiv 2\sum_{n=0}^{\infty} a_7((5n+1)-2)q^{5n+1} + 3\sum_{n=0}^{\infty} a_7((5n+4)-2)q^{5n+4} \pmod 5$$

and

$$g_{2,3}(z) := f \otimes \chi - f \otimes \overline{\chi} \equiv 2i\sum_{n=0}^{\infty} a_7((5n+2)-2)q^{5n+2} + 3i\sum_{n=0}^{\infty} a_7((5n+3)-2)q^{5n+3} \pmod 5.$$

Let $h(z) = \eta(5z)\eta(7z)^{13}$. Note that $g_{1,4}(z), g_{2,3}(z), h(z) \in \widetilde{M}_7^{(5)}\left(\Gamma_0(7\cdot 25), \left(\frac{\cdot}{35}\right)\right)$. Let $A(s,r) := \sum a_7(7^s(5n+r)-2)q^{5n+r}$. By Lemma 2.1.1, we have that

$$\begin{aligned}
2A(4t,1) + 3A(4t,4) &= g_{1,4}(z) \mid U_{7^{4t}} \equiv h(z) \pmod 5,\\
2A(4t+1,1) + 3A(4t+1,4) &= 4ig_{2,3}(z) \mid U_{7^{4t+1}} \equiv 2h(z) \mid T_{11} \pmod 5,\\
2A(4t+2,1) + 3A(4t+2,4) &= 4g_{1,4}(z) \mid U_{7^{4t+2}} \equiv 4h(z) \mid T_{11} + 3h(z) \pmod 5,\\
2A(4t+3,1) + 3A(4t+3,4) &= ig_{2,3}(z) \mid U_{7^{4t+3}} \equiv 4h(z) \mid T_{11} + h(z) \pmod 5.
\end{aligned}$$

For any modular form $f(z)$, let $\overline{f(z)}$ be the series representation of $f(z)$ with each coefficient reduced modulo 5. By (2.3.1), we have that the support of $\overline{h(z)}$ is restricted modulo 7 to $0, 1, 2, 4$, the squares modulo 7. Since 11 is a square modulo 7, we have that the support of $\overline{h(z) \mid T_{11}}$ is also restricted modulo 7 to $0, 1, 2, 4$. Therefore, we have that for all $t \geq 0$,

$$a_7(7^t \cdot n - 2) \equiv 0 \pmod 5, \qquad \text{if } \left(\frac{n}{5}\right) = 1 \text{ and } \left(\frac{n}{7}\right) = -1.$$

3.3.3 Remaining congruences

The remaining congruences in the theorem can be proved using a similar strategy to the two examples given above. For each of the congruences, we keep the notation for $f(z)$ and $g_r(z)$ and give a quick sketch of the proof. We will also make use of $\Delta = \eta(z)^{24} \in S_{12}(\mathrm{SL}_2(\mathbb{Z}))$.

$\boldsymbol{p=5,\ell=2:}$ Let $h(z)=\eta(z)^4\eta(5z)^4\in M_4(\Gamma_0(5))$. Then $g_1(z)\equiv h(z)\pmod 2$ and $h(z)\mid U_5\equiv h(z)\pmod 2$.

$\boldsymbol{p=5,\ell=3:}$ Let $h_1(z)=\eta(z)^9\eta(5z)^3, h_2(z)=\eta(z)^3\eta(5z)^9\in M_6\left(\Gamma_0(5),\left(\frac{\cdot}{5}\right)\right)$. Note that $g_1(z)\equiv h_1(z)\pmod 3$ and $g_2(z)\equiv h_2(z)\pmod 3$. We also have that $h_1(z)\mid U_5\equiv 2h_2(z)\pmod 3$ and $h_2(z)\mid U_5\equiv 2h_1(z)\pmod 3$.

$\boldsymbol{p=7,\ell=2:}$ Let $h(z)=\eta(z)^2\eta(7z)^{10}\in M_6(\Gamma_0(7))$. Then $g_1(z)\equiv h(z)\pmod 2$ and $h(z)\mid U_{7^2}\equiv h(z)\pmod 2$.

$\boldsymbol{p=7,\ell=3:}$ Let $h_1(z)=\eta(z)^{27}\eta(7z)^3, h_2(z)=\eta(z)^3\eta(7z)^{27}\in M_{15}\left(\Gamma_0(7),\left(\frac{\cdot}{7}\right)\right)$. Note that $g_1(z)\equiv h_1(z)\mid T_{11}\pmod 3$ and $g_2(z)\equiv h_1(z)+2h_2(z)\pmod 3$. Also, $h_1(z)\mid U_7\equiv h_2(z)\pmod 3$ and $h_2(z)\mid U_7\equiv h_1(z)\pmod 3$.

$\boldsymbol{p=11,\ell=2:}$ Let $h(z)=\eta(z)^2\eta(11z)^2\in M_2(\Gamma_0(11))$. Note $g_1(z)+g_1(z)\mid U_{11}\equiv h(z)\pmod 2$ and $h(z)\mid U_{11}\equiv h(z)\pmod 2$.

$\boldsymbol{p=13,\ell=2:}$ Note that $g_1(z)+g_1(z)\mid U_{13}\equiv\Delta+\Delta\mid U_{13}\pmod 2$. Also, $\Delta\mid U_{13^2}\equiv\Delta\pmod 2$.

$\boldsymbol{p=17,\ell=2:}$ Let $h_1(z)=\eta(z)^2\eta(17z)^{38}, h_2(z)=\eta(z)^8\eta(17z)^{32}\in M_{20}(\Gamma_0(17))$. Note that $g_1(z)\equiv h_1(z)\mid T_3+h_1(z)\mid T_{29}+h_2(z)\mid T_{23}\pmod 2$. Also, $h_1(z)\mid U_{17^2}\equiv h_1(z)\pmod 2$ and $h_2(z)\mid U_{17^2}\equiv h_2(z)\pmod 2$.

$\boldsymbol{p=19,\ell=2:}$ Let $h(z)=\eta(z)^2\eta(19z)^{34}\in M_{18}(\Gamma_0(19))$. Note that $g_1(z)\equiv h(z)+h(z)\mid T_7\pmod 2$ and $h(z)\mid U_{19^2}\equiv h(z)\pmod 2$.

$\boldsymbol{p=23,\ell=2:}$ Let $h(z)=\eta(z)^2\eta(23z)^{26}\in M_{14}(\Gamma_0(23))$. Note that $g_1(z)\equiv h(z)+\Delta(z)\mid U_{23}\pmod 2$. Also, $h(z)\mid U_{23^2}\equiv h(z)\pmod 2$ and $\Delta(z)\mid U_{23^2}\equiv\Delta(z)\pmod 2$.

Chapter 4

Spaces of Modular Forms with Eta-Multiplier

4.1 Notation and Statement of Theorem 4.1.8

For our next result, we study the action of Hecke operators on subspaces of modular forms which arise as multiples of modular eta-quotients. More precisely, let $w \in \mathbb{Z}$, let $N \geq 1$, let χ be a Dirichlet character modulo N, let $f(z)$ be a holomorphic level N eta-quotient with integer weight, let D be the least integer for which $f(Dz)$ is a modular form, and let $\ell \geq 5$ be prime. Our main result, Theorem 4.1.8, completely describes how the Hecke operators T_ℓ permute subspaces of modular forms of the type

$$\{f(Dz)F(Dz) : F(z) \in M_w(\Gamma_0(N), \chi)\},$$

where $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$ and $f(z)$ is "minimal" in the sense that it has order of vanish in $[0, 1)$ at all cusps. We restrict to these levels since they are precisely the levels which satisfy Propositions 4.1.1 and 4.1.6 below. It may be possible to adapt our methods to prove analogous results for other levels.

4.1.1 General Results

We describe a framework for explaining results of the type in Theorem 1.0.4 and Corollary 1.0.5. This framework requires preliminary definitions, facts, and notation. The following fact partly motivates our focus on eta-quotients with level $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$. It is a consequence of the valence formula and (1.0.3).

Proposition 4.1.1. *Let $N \geq 1$ and suppose, for all $d \mid N$, that there exists a holomorphic integer weight level N eta-quotient f_d which has $ord_{\frac{1}{d}}(f_d) = 1$ and for*

all $\delta \mid N$ with $\delta \neq d$ has $ord_{\frac{1}{\delta}}(f_d) = 0$. Then we have $N \in \{1,2,3,4,5,6,8,9\}$. Furthermore, the forms f_d for these levels are as follows.

Table 4.1 Eta-quotients with order conditions

N	d	f_d	N	d	f_d	N	d	f_d
1	1	$\eta(z)^{24}$	5	1	$\frac{\eta(z)^5}{\eta(5z)}$	8	1	$\frac{\eta(z)^4}{\eta(2z)^2}$
2	1	$\frac{\eta(z)^{16}}{\eta(2z)^8}$	5	5	$\frac{\eta(5z)^5}{\eta(z)}$	8	2	$\frac{\eta(2z)^{10}}{\eta(z)^4\eta(4z)^4}$
2	2	$\frac{\eta(2z)^{16}}{\eta(z)^8}$	6	1	$\frac{\eta(z)^6\eta(6z)}{\eta(2z)^3\eta(3z)^2}$	8	4	$\frac{\eta(4z)^{10}}{\eta(2z)^4\eta(8z)^4}$
3	1	$\frac{\eta(z)^9}{\eta(3z)^3}$	6	2	$\frac{\eta(2z)^6\eta(3z)}{\eta(z)^3\eta(6z)^2}$	8	8	$\frac{\eta(8z)^4}{\eta(4z)^2}$
3	3	$\frac{\eta(3z)^9}{\eta(z)^3}$	6	3	$\frac{\eta(3z)^6\eta(2z)}{\eta(6z)^3\eta(z)^2}$	9	1	$\frac{\eta(z)^3}{\eta(3z)}$
4	1	$\frac{\eta(z)^8}{\eta(2z)^4}$	6	6	$\frac{\eta(6z)^6\eta(z)}{\eta(3z)^3\eta(2z)^2}$	9	3	$\frac{\eta(3z)^{10}}{\eta(z)^3\eta(9z)^3}$
4	2	$\frac{\eta(2z)^{20}}{\eta(z)^8\eta(4z)^8}$				9	9	$\frac{\eta(9z)^3}{\eta(3z)}$
4	4	$\frac{\eta(4z)^8}{\eta(2z)^4}$						

For every eta-quotient $f(z)$, Proposition 1.0.2 implies that there is an integer $D \geq 1$ such that $f(Dz)$ is modular. We highlight the least such D in the following definition.

Definition 4.1.2. *The denominator of an eta-quotient $f(z)$ with level $N \geq 1$ is the least integer $D \geq 1$ such that for all $d \mid N$, we have*

$$D \cdot ord_{\frac{1}{d}}(f) \in \mathbb{Z}. \tag{4.1.1}$$

Equivalently, (1.0.3) implies that $D \geq 1$ is smallest such that for all $d \mid N$, we have

$$24D \cdot \mathrm{ord}_{\frac{1}{d}}(f) = \frac{DN}{\gcd\left(d, \frac{N}{d}\right) d} \sum_{\delta \mid N} \frac{\gcd(d,\delta)^2 r_\delta}{\delta} \equiv 0 \pmod{24}.$$

Remark 1. We also have

$$D = \frac{24}{\gcd\left(24, 24\,\mathrm{ord}_0(f), 24\,\mathrm{ord}_\infty(f)\right)}. \tag{4.1.2}$$

It follows that $D \mid 24$ and that $D = 1$ if and only if for all $d \mid N$, we have $\mathrm{ord}_{\frac{1}{d}}(f) \in \mathbb{Z}$.

Remark 2. If we suppose that $D' \geq 1$ has $D' \cdot \operatorname{ord}_{\frac{1}{d}}(f) \in \mathbb{Z}$ for all $d \mid N$, then we have $D \mid D'$.

Remark 3. If $f(z)$ is an eta-quotient with level N, integer weight k, s as in Proposition 1.0.2, and denominator D, then we have

$$f(Dz) \in M_k^! \left(\Gamma_0(D^2N), \left(\frac{(-1)^k s}{\cdot}\right)\right).$$

Remark 4. Our definition of denominator differs from the definition in [22], where the author defines the denominator to be the least $D \geq 1$ for which $D \cdot \operatorname{ord}_{\frac{1}{N}}(f) = D \cdot \operatorname{ord}_{\infty}(f) \in \mathbb{Z}$.

We next focus on holomorphic eta-quotients whose order of vanish at cusps is minimal in the following precise sense.

Definition 4.1.3. *An eta-quotient $f(z)$ of level $N \geq 1$ is minimal if and only if for all $d \mid N$, we have $0 \leq ord_{\frac{1}{d}}(f) < 1$.*

Equivalently, an eta-quotient is minimal if and only if for all $d \mid N$, the integer $24 \cdot \operatorname{ord}_{\frac{1}{d}}(f)$ lies in $[0, 23]$. There are finitely many minimal eta-quotients of a given level. Tables containing all minimal eta-quotients for levels $1, 2, 3, 4, 5, 6, 8$, and 9 are given in Appendix B.

We next observe, for $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$, that a holomorphic integer weight level N eta-quotient is the product of a minimal integer weight level N eta-quotient and a modular form.

Proposition 4.1.4. *Let $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$, and let $f(z)$ be a holomorphic integer weight level N eta-quotient. Then there exists $k \geq 0$, a Dirichlet character χ modulo N, and a minimal level N eta-quotient $g(z)$ with integer weight such that*

$$f(z) \in g(z)M_k(\Gamma_0(N), \chi) = \{g(z)h(z) : h(z) \in M_k(\Gamma_0(N), \chi)\}.$$

Proof. For each N, the proof follows by using the level N modular forms in Table 4.1. To illustrate, we consider $N = 2$. A holomorphic integer weight level 2 eta-quotient $f(z) = \eta(z)^{r_1}\eta(2z)^{r_2}$ has

$$\mathrm{ord}_0(f) = \frac{2r_1 + r_2}{24}, \quad \mathrm{ord}_\infty(f) = \frac{r_1 + 2r_2}{24}.$$

We let $a = \lfloor \mathrm{ord}_0(f) \rfloor$, $b = \lfloor \mathrm{ord}_\infty(f) \rfloor$. Then we have $f(z) = g(z)h(z)$ with

$$h(z) = \left(\frac{\eta(z)^{16}}{\eta(2z)^8}\right)^a \left(\frac{\eta(2z)^{16}}{\eta(z)^8}\right)^b = \eta(z)^{16a-8b}\eta(2z)^{16b-8a} \in M_{4(a+b)}(\Gamma_0(2))$$

and

$$g(z) = \eta(z)^{r_1-16a+8b}\eta(2z)^{r_2+8a-16b}$$

minimal. $\square$

For convenience in stating our main results, we adopt the following notation for eta-quotients.

Definition 4.1.5. *Let $N \geq 1$, let $d(N)$ denote the number of positive divisors of N, and let $\boldsymbol{v} = \langle r_\delta : \delta \mid N \rangle = \langle r_1 = 1, \ldots, r_\delta, \ldots, r_N = N \rangle \in \mathbb{Z}^{d(N)}$. Then we have*

$$f_{N,\boldsymbol{v}}(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}.$$

Therefore, a vector $\mathbf{v} \in \mathbb{Z}^{d(N)}$, indexed by positive divisors of N, uniquely specifies a level N eta-quotient. When $\mathbf{v} = \mathbf{0}$ is the zero vector, we have $f_{N,\mathbf{0}} = 1$. Furthermore, we remark that $f_{N,\mathbf{v}} = f_{N,\mathbf{v}'}$, and hence that $\mathbf{v} = \mathbf{v}'$, if and only if for all $d \mid N$, we have $\mathrm{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}}) = \mathrm{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}'})$.

Let $\overline{x}$ denote the least non-negative residue of x modulo 24, for all $x \in \mathbb{Z}$.

Proposition 4.1.6. *Let $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$, let $f_{N,\boldsymbol{v}}(z)$ be a minimal holomorphic integer weight level N eta-quotient with weight $k \geq 1$ and denominator D, and let $\ell \neq 2, 3$ be prime. Then there exists a unique $\boldsymbol{v}' = \langle r'_\delta : \delta \mid N \rangle \in \mathbb{Z}^{d(N)}$ such that for all $d \mid N$, we have*

$$24 \,\mathrm{ord}_{\frac{1}{d}}(f_{N,\boldsymbol{v}'}) = \overline{\ell \cdot 24 \,\mathrm{ord}_{\frac{1}{d}}(f_{N,\boldsymbol{v}})}. \tag{4.1.3}$$

Furthermore, $f_{N,\mathbf{v}'}(z)$ is a level N eta-quotient which is minimal, holomorphic, has integer weight $k' \geq 1$, and has denominator D.

Proof. For all $d \mid N$, we let $a_d = \lfloor \ell \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}}) \rfloor$. For all N in the statement of the proposition, the $d(N) \times d(N)$ system in the variables $r'_1 \ldots, r'_\delta, \ldots, r'_N$ has a unique solution given in Table 4.2 below. Furthermore, the table shows that the unique

Table 4.2 Solutions for r'_δ

N	d	$\operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}})$	δ	r'_δ
1	1	$\frac{r_1}{24}$	1	$\ell r_1 - 24a_1$
$p=2,3,5$	1	$\frac{pr_1+r_p}{24}$	1	$\ell r_1 - \frac{24}{p^2-1}(pa_1 - a_p)$
	p	$\frac{r_1+pr_p}{24}$	p	$\ell r_p - \frac{24}{p^2-1}(-a_1 + pa_p)$
$p^2=4,9$	1	$\frac{p^2r_1+pr_p+r_{p^2}}{24}$	1	$\ell r_1 - \frac{24}{p^2-1}(a_1 - a_p)$
	p	$\frac{r_1+pr_p+r_{p^2}}{24}$	p	$\ell r_p - \frac{24}{p(p^2-1)}(-a_1 + (p^2+1)a_p - a_{p^2})$
	p^2	$\frac{r_1+pr_p+p^2r_{p^2}}{24}$	p^2	$\ell r_{p^2} - \frac{24}{p^2-1}(-a_p + a_{p^2})$
6	1	$\frac{6r_1+3r_2+2r_3+r_6}{24}$	1	$\ell r_1 - (6a_1 - 3a_2 - 2a_3 + a_6)$
	2	$\frac{3r_1+6r_2+r_3+2r_6}{24}$	2	$\ell r_2 - (-3a_1 + 6a_2 + a_3 - 2a_6)$
	3	$\frac{2r_1+r_2+6r_3+3r_6}{24}$	3	$\ell r_3 - (-2a_1 + a_2 + 6a_3 - 3a_6)$
	6	$\frac{r_1+2r_2+3r_3+6r_6}{24}$	6	$\ell r_6 - (a_1 - 2a_2 - 3a_3 + 6a_6)$
8	1	$\frac{8r_1+4r_2+2r_4+r_8}{24}$	1	$\ell r_1 - 4(a_1 - a_2)$
	2	$\frac{2r_1+4r_2+2r_4+r_8}{24}$	2	$\ell r_2 - 2(-a_1 + 5a_2 - 2a_4)$
	4	$\frac{r_1+2r_2+4r_4+2r_8}{24}$	4	$\ell r_4 - 2(-2a_2 + 5a_4 - a_8)$
	8	$\frac{r_1+2r_2+4r_4+8r_8}{24}$	8	$\ell r_8 - 4(-a_4 + a_8)$

solution $\mathbf{v}'$ lies in $\mathbb{Z}^{d(N)}$. We recall that the eta-quotient $f_{N,\mathbf{v}}$ has weight $k = \frac{1}{2}\sum_{\delta|N} r_\delta \in \mathbb{Z}$. Then the eta-quotient $f_{N,\mathbf{v}'}$ determined by $\mathbf{v}'$ has weight $k' = \frac{1}{2}\sum_{\delta|N} r'_\delta \in \mathbb{Z}$, as in Table 4.3.

Table 4.3 Weight of $f_{N,\mathbf{v}'}$

N	k'
1	$k\ell - 12a_1$
$p=2,3,5$	$k\ell - \frac{12}{p+1}(a_1 + a_p)$
$p^2=4,9$	$k\ell - \frac{12}{p(p+1)}(a_1 + (p-1)a_p + a_{p^2})$
6	$k\ell - (a_1 + a_2 + a_3 + a_6)$
8	$k\ell - (a_1 + a_2 + a_4 + a_8)$

We also observe that $f_{N,\mathbf{v}'}$ is holomorphic and minimal since for all $d \mid N$, we have

$$0 \leq 24 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}'}) = \overline{\ell \cdot 24 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}})} \leq 23.$$

It remains to show that $f_{N,\mathbf{v}'}$ has denominator D. We suppose that it has denominator D'. For all $d \mid N$, we observe that

$$D \cdot 24 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}'}) \equiv 24\ell \cdot D \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}}) \equiv 0 \pmod{24},$$

where the first congruence follows by multiplying (4.1.3) by D and the second follows from (4.1.1). We conclude that $D \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}'}) \in \mathbb{Z}$, and hence, that $D' \mid D$. Similarly, we have

$$D'\ell \cdot 24 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}}) \equiv 24 \cdot D' \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}'}) \equiv 0 \pmod{24}$$

which gives $D'\ell \cdot 24 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}}) \in \mathbb{Z}$. Therefore, we have $D \mid D'\ell$; since $\ell \geq 5$ and $D \mid 24$, we have $D \mid D'$. □

Definition 4.1.7. *Let $w \in \mathbb{Z}$, let $N \geq 1$, let χ be a Dirichlet character modulo N, and let $f_{N,\boldsymbol{v}}$ be a holomorphic level N eta-quotient with integer weight $k \geq 1$ and denominator D. Then we define*

$$\mathcal{A}_{N,\boldsymbol{v},w,\chi} = \{f_{N,\boldsymbol{v}}(Dz)F(Dz) : F(z) \in M_w(\Gamma_0(N),\chi)\}.$$

Remarks. When $\mathbf{v} = \mathbf{0}$, we note that $\mathcal{A}_{N,\mathbf{0},w,\chi} = M_w(\Gamma_0(N),\chi)$. We also note that both $w < 0$ and $\chi(-1) \neq (-1)^w$ imply that $M_w(\Gamma_0(N),\chi) = \{0\}$, in which case we have $\mathcal{A}_{N,\mathbf{v},w,\chi} = \{0\}$. When $w = 0$, we have $M_0(\Gamma_0(N),\chi) = \mathbb{C}$, in which case we have $A_{N,\mathbf{v},w,\chi} = \mathbb{C}f_{N,\mathbf{v}}$.

We now turn to our main theorem, which precisely describes how the Hecke operators map subspaces of the type in Definition 4.1.7.

Theorem 4.1.8. *Let $w \in \mathbb{Z}$, let $N \in \{1,2,3,4,5,6,8,9\}$, let χ be a Dirichlet character modulo N, let $f_{N,\boldsymbol{v}}$ be a minimal holomorphic level N eta-quotient with integer*

weight $k \geq 1$, and let $\ell \geq 5$ be prime. Then, with $\boldsymbol{v}'$ and k' as in Proposition 4.1.6, and with

$$\chi'(\cdot) = \begin{cases} 1_N, & N \in \{1,2,4,5\}, \\ \left(\frac{\cdot}{3}\right)^{k+k'}, & N \in \{3,6,9\}, \\ \left(\frac{-1}{\cdot}\right)^{k+k'}, & N = 8, \end{cases} \tag{4.1.4}$$

we have

$$T_\ell : \mathcal{A}_{N,\boldsymbol{v},w,\chi} \to \mathcal{A}_{N,\boldsymbol{v}',w+k-k',\chi\chi'}.$$

From Theorem 4.1.8, we deduce corollaries on how Hecke operators with index coprime to 6 map subspaces $\mathcal{A}_{N,\mathbf{v},w,\chi}$. Let $M \geq 1$, let $j \geq 0$, and let ψ be a Dirichlet character modulo M. For primes ℓ and $m \geq 2$, we recall that the Hecke operator with index ℓ^m on $M_j(\Gamma_0(M), \psi)$ is

$$T_{\ell^m} = T_\ell \circ T_{\ell^{m-1}} - \psi(\ell)\ell^{j-1} T_{\ell^{m-2}}. \tag{4.1.5}$$

Corollary 4.1.9. *Keeping the notation from Theorem 4.1.8, for all $m \geq 0$, we have*

$$T_{\ell^m} : \mathcal{A}_{N,\boldsymbol{v},w,\chi} \to \begin{cases} \mathcal{A}_{N,\boldsymbol{v},w,\chi}, & m \text{ even} \\ \mathcal{A}_{N,\boldsymbol{v}',w+k-k',\chi\chi'}, & m \text{ odd.} \end{cases}$$

Proof. The proof follows by induction on m. When $m = 0$, the Hecke operator T_{ℓ^m} is the identity. Theorem 4.1.8 covers the case $m = 1$. When $m \geq 2$ is odd, the induction hypothesis together with (4.1.5) implies the result. When $m \geq 2$ is even, the induction hypothesis together with (4.1.5) implies that $T_{\ell^{m-1}} : \mathcal{A}_{N,\mathbf{v},w,\chi} \to \mathcal{A}_{N,\mathbf{v}',w+k-k',\chi\chi'}$ and that $T_{\ell^{m-2}}$ stabilizes $\mathcal{A}_{N,\mathbf{v},w,\chi}$. From Theorem 4.1.8, we have

$$T_\ell : \mathcal{A}_{N,\mathbf{v}',w+k-k',\chi\chi'} \to \mathcal{A}_{N,\mathbf{v}'',w+k-k'+(k'-k''),\chi\chi'\chi''},$$

where $\mathbf{v}'' = \langle r''_\delta : \delta \mid N \rangle \in \mathbb{Z}^{d(N)}$ is the unique solution to the $d(N) \times d(N)$ system defined, for all $d \mid N$, by

$$24 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}''}) = \overline{\ell \cdot 24 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}'})} \tag{4.1.6}$$

as in Proposition 4.1.6; $k'' = \frac{1}{2}\sum_{\delta|N} r''_\delta$; and χ'' is as in the theorem. In view of (4.1.5), it suffices to show that $\mathbf{v}'' = \mathbf{v}$. To see this, we combine (4.1.3) and (4.1.6) to obtain, for all $d \mid N$,

$$24\operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}''}) = \overline{24\ell\operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}'})} = \overline{24\ell^2\operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}})} = 24\operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}}).$$

The third equality holds since $\ell \geq 5$ and $f_{N,\mathbf{v}}$ is minimal. Since an eta-quotient is uniquely determined by its orders at cusps $\frac{1}{\delta}$ for all $\delta \mid N$, we conclude that $f_{N,\mathbf{v}''} = f_{N,\mathbf{v}}$ and that $\mathbf{v}'' = \mathbf{v}$. It follows that $k = k''$ and that $\chi'\chi'' = 1_N$. □

To recall the definition of the Hecke operator T_n for all $n \geq 1$, we suppose that n has prime factorization $n = \prod \ell_i^{e_i}$. The Hecke operator with index n on $M_j(\Gamma_0(M), \psi)$ is

$$T_n = \prod T_{\ell_i^{e_i}}. \tag{4.1.7}$$

Corollary 4.1.10. *Let w, N, χ, and $f_{N,\boldsymbol{v}}$ be as in Theorem 4.1.8, and suppose that $f_{N,\boldsymbol{v}}$ has denominator D. Suppose also that m, $n \geq 1$ have* $\gcd(mn, 6) = 1$ *and $m \equiv n$* (mod D)*. Then there exists $w' \in \mathbb{Z}$, a Dirichlet character ε modulo N, and $f_{N,\boldsymbol{u}}$, a minimal holomorphic level N eta-quotient with integer weight and denominator D such that*

$$T_m, T_n : \mathcal{A}_{N,\boldsymbol{v},w,\chi} \to \mathcal{A}_{N,\boldsymbol{u},w',\varepsilon}.$$

In particular, the space to which T_n maps $\mathcal{A}_{N,\boldsymbol{v},w,\chi}$ depends only on n modulo D, the denominator of $f_{N,\boldsymbol{v}}$.

Proof. Let m_0 and n_0 be the square-free parts of m and n, respectively. We note that $D \mid 24$ and $\gcd(m, 6) = 1$ imply, for all primes $\ell \mid m$, that $\ell^2 \equiv 1 \pmod{D}$. It follows that $m \equiv m_0 \pmod{D}$. Similarly, we see that $n \equiv n_0 \pmod{D}$. Since $m \equiv n$ (mod D), we conclude that $m_0 \equiv n_0 \pmod{D}$.

Next, we observe that (4.1.7) and Theorem 4.1.8 imply that $T_{m_0} : \mathcal{A}_{N,\mathbf{v},w,\chi} \to \mathcal{A}_{N,\mathbf{v}_m,w_m,\chi_m}$, where $\mathbf{v}_m = \langle r_\delta^{(m)} : \delta \mid N\rangle \in \mathbb{Z}^{d(N)}$ is the unique solution to the

$d(N) \times d(N)$ system defined for all $d \mid N$ by $24\operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}_m}) = \overline{24m_0 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}})}$, $w_m = \frac{1}{2}\sum_{\delta|N} r_\delta^{(m)}$, and χ_m is as in Theorem 4.1.8. We note, by Corollary 4.1.9, that $T_m : \mathcal{A}_{N,\mathbf{v},w,\chi} \to \mathcal{A}_{N,\mathbf{v}_m,w_m,\chi_m}$ also. Similarly, we find that T_n, $T_{n_0} : \mathcal{A}_{N,\mathbf{v},w,\chi} \to \mathcal{A}_{N,\mathbf{v}_n,w_n,\chi_n}$, where $\mathbf{v}_n = \langle r_\delta^{(n)} : \delta \mid N\rangle \in \mathbb{Z}^{d(N)}$ is the unique solution to the system defined for all $d \mid N$ by $24\operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}_n}) = \overline{24n_0 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}})}$, $w_n = \frac{1}{2}\sum_{\delta|N} r_\delta^{(n)}$, and χ_n is as in Theorem 4.1.8.

To conclude, it suffices to show that $\mathbf{v}_m = \mathbf{v}_n$. To see this, we observe that $m_0 \equiv n_0 \pmod{D}$ implies, for all $d \mid N$, that

$$24\operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}_m}) = \overline{24m_0 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}})} = \overline{24n_0 \operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}})} = 24\operatorname{ord}_{\frac{1}{d}}(f_{N,\mathbf{v}_n}).$$

Hence, we have $\mathbf{v}_m = \mathbf{v}_n$, $w_m = w_n$, $\chi_m = \chi_n$, and

$$T_m, T_n : \mathcal{A}_{N,\mathbf{v},w,\chi} \to \mathcal{A}_{N,\mathbf{v}_m,w_m,\chi_m}.$$

□

Corollary 4.1.10 allows us to identify Hecke operators which preserve subspaces of the type $\mathcal{A}_{N,\chi,w,\chi}$.

Corollary 4.1.11. *Let w, N, χ, and $f_{N,\boldsymbol{v}}$ be as in Theorem 4.1.8, and suppose that $f_{N,\boldsymbol{v}}$ has denominator D.*

1. *Suppose that $n \geq 1$ has $\gcd(n, 6) = 1$ and $n \equiv 1 \pmod{D}$. Then we have*

$$T_n : \mathcal{A}_{N,\boldsymbol{v},w,\chi} \to \mathcal{A}_{N,\boldsymbol{v},w,\chi}.$$

2. *Let $\ell \geq 5$ be prime. Then we have $T_\ell^2 = T_\ell \circ T_\ell : \mathcal{A}_{N,\boldsymbol{v},w,\chi} \to \mathcal{A}_{N,\boldsymbol{v},w,\chi}$.*

Proof. Part one follows from Corollary 4.1.10 since T_1 is the identity. Since $\ell \geq 5$ and $D \mid 24$, we have $\ell^2 \equiv 1 \pmod{D}$. Therefore, part one of the corollary and (4.1.5) imply part two of the corollary. □

Remark. When $N = q$ is prime, Newman [24] classified level N eta-quotients $f_{N,\mathbf{v}}$ which are eigenforms of T_p with $p \equiv 1 \pmod{D}$, where D is the denominator of $f_{N,\mathbf{v}}$. Setting $w = 0$ in part one of the corollary, we immediately recover all of the eigenforms in Newman's list (121 out of 147) which are minimal. The remaining 26 in Newman's list all have level $N \leq 23$ and either order 1 at 0 and/or at infinity. Moreover, Theorem 4.1.8 and Corollary 4.1.10 allow us to quickly identify minimal eta-quotients with level $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$ which are eigenforms for T_n for n in residue classes $r \pmod{24}$ with $\gcd(r, 6) = 1$.

4.2 Extended Example

We start by finding all minimal eta-quotients of level 2. Let $f(z) = \eta(z)^{r_1}\eta(2z)^{r_2}$ be a minimal holomorphic eta-quotient of integer weight $k = \frac{1}{2}(r_1 + r_2)$. When $N = 2$, (1.0.3) gives that

$$\mathrm{ord}_\infty(f(z)) = \frac{r_1 + 2r_2}{24} \quad \text{and} \quad \mathrm{ord}_0(f(z)) = \frac{2r_1 + r_2}{24}.$$

Therefore, the comments following Definition 4.1.3 imply that

$$0 \leq r_1 + 2r_2 \leq 23 \tag{4.2.1}$$

$$0 \leq 2r_1 + r_2 \leq 23 \tag{4.2.2}$$

Adding (4.2.1) and (4.2.2) and dividing by six yields $k \in [1, 7]$. Replacing r_2 with $2k - r_1$ and solving for r_1 in (4.2.1) yields $r_1 \in [4k - 23, 4k]$. Making the same substitution in (4.2.2) and solving for r_1 yields $r_1 \in [-2k, 23 - 2k]$. Thus,

$$\max\{4k - 23, -2k\} \leq r_1 \leq \min\{23 - 2k, 4k\} \tag{4.2.3}$$

To identify all minimal eta-quotients, first fix $k \in [1, 7]$ and then let r_1 run through each of the integers in (4.2.3). Once r_1 and $r_2 = 2k - r_1$ have been determined, the denominator of the form can be calculated using Definition 4.1.1 or using (4.1.2).

Table 4.4 contains every pair (r_1, r_2) such that $f(z)$ is a minimal eta-quotient of level 2 and weight 1 along with its order of vanish at the cusps and its denominator which is calculated using (4.1.2).

Table 4.4 Minimal holomorphic level 2 eta-quotients of weight 1

r_1	r_2	$24\mathrm{ord}_\infty(f)$	$24\mathrm{ord}_0(f)$	D
-2	4	6	0	4
-1	3	5	1	24
0	2	4	2	12
1	1	3	3	8
2	0	2	4	12
3	-1	1	5	24
4	-2	0	6	4

Recall from Definition 4.1.5 that $f_{2,\langle r_1,r_2\rangle}(z) = \eta(z)^{r_1}\eta(2z)^{r_2}$. Consider the following minimal eta-quotients of level 2 and denominator 24.

Table 4.5 Minimal holomorphic level 2 eta-quotients with denominator 24

f	k	$24\mathrm{ord}_\infty(f)$	$24\mathrm{ord}_0(f)$
$f_{2,\langle 11,-5\rangle}$	3	1	17
$f_{2,\langle 7,-1\rangle}$	3	5	13
$f_{2,\langle 13,-3\rangle}$	5	7	23
$f_{2,\langle 9,1\rangle}$	5	11	19
$f_{2,\langle -1,7\rangle}$	3	13	5
$f_{2,\langle -5,11\rangle}$	3	17	1
$f_{2,\langle 1,9\rangle}$	5	19	11
$f_{2,\langle -3,13\rangle}$	5	23	7

Note that for each f in Table 4.5 and for each prime $\ell \geq 5$, there exists an f' in Table 4.5 such that $\overline{24\ell\,\mathrm{ord}_\infty(f)} = 24\,\mathrm{ord}_\infty(f')$ and $\overline{24\ell\,\mathrm{ord}_0(f)} = 24\,\mathrm{ord}_0(f')$. Therefore, for each $f_{2,\mathbf{v}}$ in Table 4.5, $f_{2,\mathbf{v}'}$ is also in the table.

Recall Definition 4.1.7 and let $\mathcal{A}_{r_1,r_2,w,\chi} := \mathcal{A}_{2,\langle r_1,r_2,\rangle,w,\chi}$. Note that Theorem 4.1.8 gives the following relationships among the spaces associated to the eta-quotients $f_{N,\mathbf{v}}$ from Table 4.5,

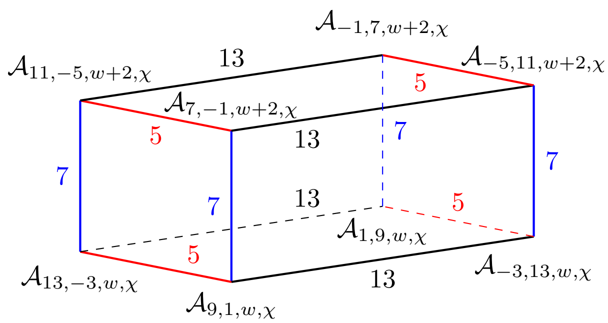

Figure 4.1 Diagram depicting the action of T_ℓ on $\mathcal{A}_{r_1,r_2,w,\chi}$

where two spaces are connected by the prime ℓ if T_ℓ maps one space into the other. For example,

$$T_{13} : \mathcal{A}_{11,-5,w+2,\chi} \to \mathcal{A}_{-1,7,w+2,\chi}$$

and vice versa. Since the set $\{5, 7, 13\}$ generates $\mathbb{Z}/24\mathbb{Z}^\times$, the diagram gives where T_ℓ maps the space $\mathcal{A}_{r_1,r_2,w,\chi}$ for every prime $\ell \geq 5$. Since $71 \equiv 23 \equiv 5 * 7 * 13 \pmod{24}$, we have that

$$T_{71} : \mathcal{A}_{11,-5,w+2,\chi} \to \mathcal{A}_{-3,13,w,\chi}.$$

Since $M_{-2}(\Gamma_0(2), \chi) = \{0\}$, we have $T_{71} : \mathcal{A}_{11,-5,0,\chi} \to \{0\}$, so $f_{2,\langle 11,-5\rangle}(24z) \mid T_{71} = 0$. Furthermore, we have $f_{2,\langle 11,-5\rangle}(z) \mid T_\ell = 0$ for all $\ell \equiv 23 \pmod{24}$. Since $101 \equiv 5 \pmod{24}$, we have that

$$T_{101} : \mathcal{A}_{11,-5,w+2,\chi} \to \mathcal{A}_{7,-1,w+2,\chi}.$$

Taking $w = -2$, we obtain $f_{2,\langle 11,-5\rangle}(24z) \mid T_{101} = a_{101} f_{2,\langle 7,-1\rangle}(24z)$, where $a_{101} \in \mathbb{C}$. Moreover, we have that $f_{2,\langle 11,-5\rangle}(24z) \mid T_\ell = a_\ell f_{2,\langle 7,-1\rangle}(24z)$ for all primes $\ell \equiv 5 \pmod{24}$ and for some $a_\ell \in \mathbb{C}$. For each prime $\ell \geq 5$, we use Figure 4.1 to determine $f_{2,\langle 11,-5\rangle}(24z) \mid T_\ell$ and list the result in Table 4.6. We also list $f_{2\langle 13,-3\rangle}(24z) \mid T_\ell$ for each prime $\ell \geq 5$. For $r \in \mathbb{Z}/24\mathbb{Z}^*$, let $f_r(z)$ denote the form in Table 4.5 with $24 \operatorname{ord}_\infty(f) = r$ and let $g_r(z) = f_r(24z)$. Then $f_{2,\langle 11,-5\rangle}(24z) = g_1(z)$ and $f_{2,\langle 13,-3\rangle}(24z) = g_7(z)$. Let $E_2(z)$ be the Eisenstein series of weight 2, which is a quasi-modular form and define $E_{2,2}(z) := 2E_2(48z) - E_2(24z) \in M_2(\Gamma_0(2 \cdot 24))$.

Table 4.6 Image of $f_{2,\langle 11,-5\rangle}(24z) \mid T_\ell$ and $f_{2,\langle 13,-3\rangle}(24z) \mid T_\ell$ for $\ell \geq 5$ prime

$\ell \pmod{24}$	$g_1(z) \mid T_\ell$	$g_7(z) \mid T_\ell$
1	$a_\ell g_1(z)$	$b_\ell g_7(z)$
5	$a_\ell g_5(z)$	$b_\ell g_{11}(z)$
7	0	$b_\ell E_{2,2}(z) g_1(z)$
11	0	$b_\ell E_{2,2}(z) g_5(z)$
13	$a_\ell g_{13}(z)$	$b_\ell g_{19}(z)$
17	$a_\ell g_{17}(z)$	$b_\ell g_{23}(z)$
19	0	$b_\ell E_{2,2}(z) g_{13}(z)$
23	0	$b_\ell E_{2,2}(z) g_{17}(z)$

We now demonstrate how to use Table 4.6 to complete forms as done by Gordon and his coauthors in [17], [18], and [19]. In particular, we will complete the forms $g_1(z)$ and $g_7(z)$ into Hecke eigenforms. Using Table 4.6, we see that for all prime $\ell \equiv 1, 5, 13, 17 \pmod{24}$, we have $g_1(z) \mid T_\ell = a g_{\overline{\ell}}(z)$, for some $a \in \mathbb{C}$. Similarly, for all $r \in \{1, 5, 13, 17\}$ and $\ell \equiv 1, 5, 13, 17 \pmod{24}$, we have for some $a \in \mathbb{C}$ that $g_r(z) \mid T_\ell = a g_{\overline{r\ell}}(z)$. Consider the expansions of $g_r(z)$ for $r \in \{1, 5, 13, 17\}$.

$$g_1(z) = \sum_{n=0}^{\infty} a_1(24n+1) q^{24n+1} = q - 11q^{5^2} + 49q^{49} - 110q^{73} + \cdots - 407q^{13^2} + \cdots - 33q^{17^2} + \cdots$$

$$g_5(z) = \sum_{n=0}^{\infty} a_5(24n+5) q^{24n+5} = q^5 - 7q^{29} + 15q^{53} + \cdots - 64q^{13(17)} + \cdots - 110q^{5(73)} + \cdots$$

$$g_{13}(z) = \sum_{n=0}^{\infty} a_{13}(24n+13) q^{24n+13} = q^{13} + q^{37} - 5q^{61} - 4q^{5(17)} + \cdots - 110q^{15(73)} + \cdots$$

$$g_{17}(z) = \sum_{n=0}^{\infty} a_{17}(24n+17) q^{24n+17} = q^{17} + 5q^{41} + 9q^{5(13)} + 10q^{89} + \cdots - 110q^{17(73)} + \cdots$$

We focus on the action of T_ℓ on the above forms for the primes $\ell = 5, 13, 17$, and 73, one prime from each congruence class modulo 24 for which the action of T_ℓ is nonzero. Using (2.4.2), we obtain

Table 4.7 Values used in the completion of $f_{2,\langle 11,-5\rangle}(24z)$

$f(z)$	$f(z) \mid T_{73}$	$f(z) \mid T_5$	$f(z) \mid T_{13}$	$f(z) \mid T_{17}$
$g_1(z)$	$-110g_1(z)$	$-36g_5(z)$	$-576g_{13}(z)$	$256g_{17}(z)$
$g_5(z)$	$-110g_5(z)$	$g_1(z)$	$-64g_{17}(z)$	$-64g_{13}(z)$
$g_{13}(z)$	$-110g_{13}(z)$	$-4g_{17}(z)$	$g_1(z)$	$-4g_5(z)$
$g_{17}(z)$	$-110g_{17}(z)$	$9g_{13}(z)$	$9g_5(z)$	$g_1(z)$

Let $A, B, C \in \mathbb{C}$ and let

$$f(z) = g_1(z) + Ag_5(z) + Bg_{13}(z) + Cg_{17}(z) = \sum_{n=0}^{\infty} a(n)q^n.$$

We determine choices of A, B, C so that $f(z)$ is a Hecke eigenform, that is, for all primes ℓ, $f(z) \mid T_\ell = a(\ell)f(z)$. By Figure 4.1, we see that for all primes $\ell \equiv 7, 11, 19, 23 \pmod{24}$ we have $f(z) \mid T_\ell = 0 = a(\ell)f(z)$. By Table 4.7, we have that $f(z) \mid T_{73} = -110f(z) = a(73)f(z)$. If $f(z) \mid T_5 = a(5)f(z)$, then

$$A(g_1(z) + Ag_5(z) + Bg_{13}(z) + Cg_{17}(z)) = Ag_1(z) - 36g_5(z) + 9Cg_{13}(z) - 4Bg_{17}(z).$$

It follows that $A^2 = -36$ and so $A = \pm 6i$. Similarly, if $f(z) \mid T_{13} = a(13)f(z)$ and $f(z) \mid T_{17} = a(17)f(z)$, then we get that $B = \mp 24i$ and $C = \pm 16$. Therefore, we have that

$$f(z) = \frac{\eta(24z)^{11}}{\eta(48z)^5} \pm 6i\frac{\eta(24z)^7}{\eta(48z)} \mp 24i\frac{\eta(48z)^7}{\eta(24z)} \pm 16\frac{\eta(48z)^{11}}{\eta(24z)^5}$$

is a Hecke eigenform.

We now complete $f_{2,\langle 13,-3\rangle}(24z) = g_7(z)$. Consider the following q-expansions of forms which are in $M_5\left(\Gamma_0(2 \cdot 24^2), \left(\frac{-2}{\cdot}\right)\right)$.

$$\begin{aligned}
F_1(z) := E_{2,2}(z)g_1(z) &= q + 13q^{25} - 191q^{49} + 898q^{73} - 2366q^{97} + \cdots \\
F_5(z) := E_{2,2}(z)g_5(z) &= q^5 + 17q^{29} - 129q^{53} + 288q^{77} - 321q^{101} + \cdots \\
F_7(z) := g_7(z) &= q^7 - 13q^{31} + 68q^{55} - 169q^{79} + 139q^{103} + \cdots \\
F_{11}(z) := g_{11}(z) &= q^{11} - 9q^{35} + 26q^{59} - 3q^{83} - 118q^{107} + \cdots \\
F_{13}(z) := E_{2,2}(z)g_{13}(z) &= q^{13} + 25q^{37} + 43q^{61} - 4q^{85} - 91q^{109} + \cdots \\
F_{17}(z) := E_{2,2}(z)g_{17}(z) &= q^{17} + 29q^{41} + 153q^{65} + 442q^{89} + 974q^{113} + \cdots \\
F_{19}(z) := g_{19}(z) &= q^{19} - q^{43} - 10q^{67} + 9q^{91} + 36q^{115} + \cdots \\
F_{23}(z) := g_{23}(z) &= q^{23} + 3q^{47} - 4q^{71} - 17q^{95} - q^{119} + \cdots
\end{aligned}$$

We construct a table similar to Table 4.7 which depicts how the Hecke operator permutes these forms. For sake of space, we simply give the coefficients $a_{r,\ell} \in \mathbb{C}$ so that $F_r(z) \mid T_\ell = a_{r,\ell}F_{\overline{r\ell}}(z)$.

Table 4.8 Values used in the completion of $f_{2,\langle 13,-3\rangle}(24z)$

$r \backslash \ell$	73	5	7	11	13	17	19	23
1	898	−612	−2592	19584	−9792	256	313344	−553552
5	898	1	288	288	−64	−64	−18432	−18432
7	898	68	1	68	−1088	−256	−1088	−256
11	898	−9	−9	1	−576	64	64	−576
13	898	−4	−288	1152	1	−4	−288	1152
17	898	153	2592	4896	153	1	4896	2592
19	898	36	9	4	9	4	1	36
23	898	−17	−1	17	17	−1	−17	1

Let $a_5, a_7, a_{11}, a_{13}, a_{17}, a_{19}, a_{23} \in \mathbb{C}$ and let $f(z) = F_1(z) + \sum a_i F_i(z)$. In order for $f(z)$ to be a Hecke-eigenform, we need that $f(z) \mid T_\ell = af(z)$ for every prime $\ell \geq 5$, where $a \in \mathbb{C}$. Using Table 4.8, we see that $f(z) \mid T_{73} = 898 f(z)$ no matter the choice of the a_i. In order for $f(z) \mid T_5 = af(z)$, we see by Table 4.8 that $a_5 = \pm\sqrt{-612} = \pm 6i\sqrt{17}$. We do this for all primes ℓ in the table and obtain that

$$\begin{aligned} f(z) \;=\; & E_{2,2}(z)\frac{\eta(24z)^{11}}{\eta(48z)^5} + 6i\sqrt{17}E_{2,2}(z)\frac{\eta(24z)^7}{\eta(48z)} - 36i\sqrt{2}\frac{\eta(24z)^{13}}{\eta(48z)^3} - 24\sqrt{34}\eta(24z)^9\eta(48z) \\ & -24i\sqrt{17}E_{2,2}(z)\frac{\eta(48z)^7}{\eta(24z)} + 16E_{2,2}(z)\frac{\eta(48z)^{11}}{\eta(24z)^5} - 96\sqrt{34}\eta(24z)\eta(48z)^9 + 576i\sqrt{2}\frac{\eta(48z)^{13}}{\eta(24z)^3} \end{aligned}$$

is a Hecke eigenform.

4.3 Proof of Theorem 4.1.8

4.3.1 The setting of Theorem 4.1.8.

We let $N \geq 1$, $\ell \geq 5$ be prime, and $\mathbf{v} = \langle r_\delta : \delta \mid N\rangle \in \mathbb{Z}^{d(N)}$, where $d(N)$ is the number of positive divisors of N. We define

$$f(z) := f_{N,\mathbf{v}}(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta} \tag{4.3.1}$$

to be a minimal holomorphic eta-quotient with weight k and denominator D as in Definitions 4.1.1, 4.1.3, and 4.1.5. Let w be a non-negative integer, χ be a Dirichlet character modulo N, and $F(z) \in M_w(\Gamma_0(N), \chi)$. Proposition 4.1.6 guarantees the

existence of unique $\mathbf{v}' = \langle r'_\delta : \delta \mid N\} \in \mathbb{Z}^{d(N)}$ such that for all $\delta \mid N$, we have

$$24 \operatorname{ord}_{\frac{1}{\delta}}(f_{N,\mathbf{v}'}) = \overline{\ell \cdot 24 \operatorname{ord}_{\frac{1}{\delta}}(f_{N,\mathbf{v}})}. \tag{4.3.2}$$

Furthermore, the proposition implies that

$$f^\dagger(z) := f_{N,\mathbf{v}'}(z) = \prod_{\delta \mid N} \eta(\delta z)^{r'_\delta} \tag{4.3.3}$$

is minimal with weight $k' = \frac{1}{2} \sum_{\delta \mid N} r'_\delta$ and denominator D.

4.3.2 A reduction.

With $f(z)$ as in (4.3.1) and $f^\dagger(z)$ as in (4.3.3), our proof of Theorem 4.1.8 requires careful study of

$$H(z) := \frac{(f(Dz)F(Dz)) \mid T_\ell}{f^\dagger(Dz)} \quad \text{and} \quad G(z) := H\left(\frac{z}{D}\right). \tag{4.3.4}$$

With χ' as in (4.1.4), we will prove that $G(z) \in M_{w+k-k'}(\Gamma_0(N), \chi\chi')$. In the notation of Definition 4.1.7, it then follows from (4.3.4) that T_ℓ maps $f(Dz)F(Dz) \in \mathcal{A}_{N,\mathbf{v},w,\chi}$ to $f^\dagger(Dz)G(Dz) \in \mathcal{A}_{N,\mathbf{v}',w+k-k',\chi\chi'}$. This is precisely the statement of Theorem 4.1.8.

To prove that $G(z) \in M_{w+k-k'}(\Gamma_0(N), \chi\chi')$, we first observe that $G(z)$ is holomorphic on $\mathfrak{h}$ since $f^\dagger$, as an eta-quotient, is holomorphic and non-vanishing on $\mathfrak{h}$. We next prove that $G(z)$ is holomorphic at the cusps. Let $\mathfrak{s}$ be a cusp of $\Gamma_0(N)$. The comments following (1.0.3) imply that it suffices to verify that $\operatorname{ord}_\mathfrak{s}(G(z)) \geq 0$, or equivalently, $\operatorname{ord}_\mathfrak{s}(G(Dz)) = \operatorname{ord}_\mathfrak{s}(H(z)) \geq 0$. We have

$$\operatorname{ord}_\mathfrak{s}(H(z)) = \operatorname{ord}_\mathfrak{s}((f(Dz)F(Dz)) \mid T_\ell) - \operatorname{ord}_\mathfrak{s}(f^\dagger(Dz)). \tag{4.3.5}$$

From Theorem 49 in [16], we have that

$$\operatorname{ord}_\mathfrak{s}((f(Dz)F(Dz)) \mid T_\ell) \geq \min\left\{\ell \operatorname{ord}_{\ell\cdot\mathfrak{s}}(f(Dz)F(Dz)), \frac{\operatorname{ord}_{\ell^{-1}\cdot\mathfrak{s}}(f(Dz)F(Dz)) + nD}{\ell}\right\},$$

where $a \cdot \mathfrak{s}$ is the cusp resulting from the action of $\mathbb{Z}/D^2N\mathbb{Z}$ on the cusps of $\Gamma_0(D^2N)$ and n is the least nonnegative integer such that $n \equiv -D^{-1} \operatorname{ord}_{\ell^{-1}\cdot\mathfrak{s}}(f(Dz)F(Dz)) \pmod{\ell}$. Note that

$$\begin{aligned}
\frac{\operatorname{ord}_{\ell^{-1}\cdot\mathfrak{s}}(f(Dz)F(Dz)) + nD}{\ell} &\equiv \ell^{-1}(\operatorname{ord}_{\ell^{-1}\cdot\mathfrak{s}}(f(Dz)F(Dz)) + nD) \pmod{D} \\
&\equiv \ell \operatorname{ord}_{\ell^{-1}\cdot\mathfrak{s}}(f(Dz)F(Dz)) \pmod{D}.
\end{aligned}$$

For this section only, let $\overline{a}$ be the least nonnegative residue of a modulo D. Therefore, we have that

$$\operatorname{ord}_{\mathfrak{s}}((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \operatorname{ord}_{a\cdot s}(f(Dz)F(Dz))},$$

where $a \in \{\ell, \ell^{-1}\}$. Since $F(z)$ is a modular form, we know that $\operatorname{ord}_{\mathfrak{s}}(F(Dz))$ is divisible by D for all cusps $\mathfrak{s}$. Therefore, we have that

$$\overline{\ell \operatorname{ord}_{a\cdot\mathfrak{s}}(f(Dz)F(Dz))} = \overline{\ell \operatorname{ord}_{a\cdot\mathfrak{s}}(f(Dz)) + \ell \operatorname{ord}_{a\cdot\mathfrak{s}}(F(Dz))} = \overline{\ell \operatorname{ord}_{a\cdot\mathfrak{s}}(f(Dz))}.$$

Hence,

$$\operatorname{ord}_{\mathfrak{s}}((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \operatorname{ord}_{a\cdot\mathfrak{s}}(f(Dz))}. \tag{4.3.6}$$

For the cusps $\mathfrak{s} \in \{0, \infty\}$, we have that $a \cdot \mathfrak{s} = \mathfrak{s}$. Therefore,

$$\operatorname{ord}_0((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \operatorname{ord}_0(f(Dz))}$$

and

$$\operatorname{ord}_\infty((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \operatorname{ord}_\infty(f(Dz))}.$$

We now consider the remaining cusps. For $N = 4$, the only remaining cusp can be represented by $\frac{1}{2}$. Since the action $a \cdot \frac{1}{2} = \frac{1}{2}$ for all odd a, we have that

$$\operatorname{ord}_{1/2}((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \operatorname{ord}_{1/2}(f(Dz))}.$$

When $N = 6$, there are two remaining cusps in $\Gamma_0(N)$. These can be represented by $\frac{1}{2}$ and $\frac{1}{3}$. Again we have that $a \cdot \frac{1}{2} = \frac{1}{2}$ for all odd a. Therefore, we have

$$\operatorname{ord}_{1/2}((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \operatorname{ord}_{1/2}(f(Dz))}.$$

For the cusp $\frac{1}{3}$, note that $a \cdot \frac{1}{3}$ is either the cusp $\frac{1}{3}$ or the cusp $\frac{2}{3}$, which are inequivalent cusp in $\Gamma_0(6D^2)$ for $D \in \{3, 6, 12, 24\}$. However, since the order of vanish at these cusps is the same for an eta-quotient, we still conclude that

$$\mathrm{ord}_{1/3}((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \, \mathrm{ord}_{1/3}(f(Dz))}.$$

For $N = 8$, we have $a \cdot \frac{1}{2} = \frac{1}{2}$ and $a \cdot \frac{1}{4}$ is either $\frac{1}{4}$ or $\frac{3}{4}$. It follows that

$$\mathrm{ord}_{1/2}((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \, \mathrm{ord}_{1/2}(f(Dz))}$$

and

$$\mathrm{ord}_{1/4}((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \, \mathrm{ord}_{1/4}(f(Dz))}.$$

Lastly, for $N = 9$, we have that $a \cdot \left\{\frac{1}{3}, \frac{2}{3}\right\} = \left\{\frac{1}{3}, \frac{2}{3}\right\}$, so if $\mathfrak{s} \in \left\{\frac{1}{3}, \frac{2}{3}\right\}$, then

$$\mathrm{ord}_{\mathfrak{s}}((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \, \mathrm{ord}_{\mathfrak{s}}(f(Dz))}$$

Therefore, for every $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$ and for every cusp $\mathfrak{s}$ we have that

$$\mathrm{ord}_{\mathfrak{s}}((f(Dz)F(Dz)) \mid T_\ell) \geq \overline{\ell \, \mathrm{ord}_{\mathfrak{s}}(f(Dz))}.$$

By construction of $f^\dagger$, we have that for all cusps $\mathfrak{s}$,

$$\mathrm{ord}_{\mathfrak{s}}(f^\dagger(Dz)) = \overline{\ell \, \mathrm{ord}_{\mathfrak{s}}(f(Dz))}.$$

Therefore by (4.3.5), we obtain $\mathrm{ord}_{\mathfrak{s}}(H(z)) \geq 0$. It follows that $G(z)$ is holomorphic at the cusps of $\Gamma_0(N)$.

We devote the rest of this chapter to proving that $G(z)$ satisfies the appropriate transformation law. The following proposition asserts that it suffices to prove a corresponding transformation law for $H(z)$.

Proposition 4.3.1. *We keep the notation of this section, and we let $t = w + k - k'$. We suppose, for all $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, that*

$$H(z)\Big|_t \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \chi\chi'(d)H(z). \tag{4.3.7}$$

Then we have

$$G(z)\Big|_t \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi\chi'(d)G(z). \tag{4.3.8}$$

Proof. We apply $\Big|_t \begin{pmatrix} \frac{1}{D} & 0 \\ 0 & 1 \end{pmatrix}$ to (4.3.7) to obtain

$$H(z)\Big|_t \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \chi\chi'(d)H(z)\Big|_t \begin{pmatrix} \frac{1}{D} & 0 \\ 0 & 1 \end{pmatrix}.$$

We then use (2.4.1) and (4.3.4) to get (4.3.8). □

In view of Proposition 4.3.1, we will prove (4.3.7).

4.3.3 Some important lemmas.

The proof of the transformation law (4.3.7) requires several lemmas. The first, whose proof is left for Section 4.4, concerns factorizations of matrices in $\mathrm{GL}_2(\mathbb{Q})$. We recall from the first remark following Definition 4.1.1 that $D \mid 24$. Since $\ell \geq 5$ is prime, we observe that $\ell \nmid D$.

Lemma 4.3.2. *Let $\ell \geq 5$ be prime, let D, $N \geq 1$, and let* $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$.

1. *Suppose that $0 \leq j \leq \ell - 1$ has $a + Dcj \not\equiv 0 \pmod{\ell}$. Then we have*

$$\begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a + Dcj & x \\ c\ell & y \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & \ell \end{pmatrix},$$

where $z \in \mathbb{Z}$ has

$$z \equiv (b + Ddj)(a + Dcj)^{-1}D^{-1} \pmod{\ell}$$

and

$$x\ell = b + D(dj - (a + Dcj)z), \quad y = d - Dcz.$$

2. Suppose that $0 \le t \le \ell - 1$ has $a + Dct \equiv 0 \pmod{\ell}$. Then we have

$$\begin{pmatrix} 1 & t \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u & b + Ddt \\ c & d\ell \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix},$$

where

$$u\ell = a + Dct.$$

3. Suppose that $\ell \nmid c$. Then we have

$$\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a\ell & m \\ c & n \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & \ell \end{pmatrix},$$

where $w \in \mathbb{Z}$ has

$$w \equiv d(Dc)^{-1} \pmod{\ell}$$

and

$$m = b - Daw, \quad nl = d - Dcw.$$

4. Suppose that $\ell \mid c$. Then we have

$$\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b\ell \\ c/\ell & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}.$$

We also require the following elementary number theory lemma. When ℓ is prime, we let $\mathbb{F}_\ell = \mathbb{Z}/\ell\mathbb{Z}$.

Lemma 4.3.3. *Let $\ell \ge 5$ be prime, let $D, N \ge 1$, and let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Then the map*

$$\phi : \mathbb{F}_\ell \setminus \{-a(Dc)^{-1} \pmod{\ell}\} \to \mathbb{F}_\ell \setminus \{d(Dc)^{-1} \pmod{\ell}\}$$

defined by

$$j \pmod{\ell} \longmapsto (b + Ddj)(a + Dcj)^{-1}D^{-1} \pmod{\ell}$$

is a bijection.

Proof. To see that ϕ is well-defined, we suppose on the contrary that there exists $j \in \mathbb{F}_\ell \setminus \{-a(Dc)^{-1} \pmod{\ell}\}$ such that $(b+Ddj)(a+Dcj)^{-1}D^{-1} \equiv d(Dc)^{-1} \pmod{\ell}$. Simplification yields $ad - bc \equiv 0 \pmod{\ell}$, which contradicts $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. We next suppose that $j_1,\ j_2 \in \mathbb{F}_\ell \setminus \{-a(Dc)^{-1} \pmod{\ell}\}$ have $(b + Ddj_1)(a + Dcj_1)^{-1} \equiv (b+Ddj_2)(a+Dcj_2)^{-1} \pmod{\ell}$. Using $ad-bc = 1$, we conclude that $j_1 \equiv j_2 \pmod{\ell}$ and hence, that ϕ is injective. $\square$

The following lemma plays a fundamental role in the computations in this paper.

Lemma 4.3.4. *Let $N \geq 1$, let $\begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \Gamma_0(N)$, and let $f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$ be a minimal holomorphic level N eta-quotient with denominator D and weight k as in* (4.3.1)*. Then we have*

$$f(Dz)\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} m & n\delta \\ p/\delta & q \end{pmatrix}^{r_\delta} f(Dz).$$

Proof. We first remark that it suffices to prove that

$$\eta(D\delta z)\Big|_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \nu_\eta \begin{pmatrix} m & n\delta \\ p/\delta & q \end{pmatrix} \eta(D\delta z). \tag{4.3.9}$$

Since $\begin{pmatrix} m & n \\ p & q \end{pmatrix} \in \Gamma_0(N)$, we observe, for all $\delta \mid N$, that $\begin{pmatrix} m & n\delta \\ p/\delta & q \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. Using (2.4.1), we find that

$$\eta(D\delta z)\Big|_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} = D^{-\frac{1}{4}} \eta(\delta z) \tag{4.3.10}$$

and

$$\eta(\delta z)\Big|_{\frac{1}{2}} \begin{pmatrix} m & n \\ p & q \end{pmatrix} = \nu_\eta \begin{pmatrix} m & n\delta \\ p/\delta & q \end{pmatrix} \eta(\delta z). \tag{4.3.11}$$

To conclude, we use (4.6.7) and (4.6.8) to compute

$$\eta(D\delta z)\Big|_{\frac{1}{2}} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = D^{-\frac{1}{4}} \eta(\delta z)\Big|_{\frac{1}{2}} \begin{pmatrix} m & n \\ p & q \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}$$

$$= D^{-\frac{1}{4}} \nu_\eta \begin{pmatrix} m & n\delta \\ p/\delta & q \end{pmatrix} \eta(\delta z)\Big|_{\frac{1}{2}} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \nu_\eta \begin{pmatrix} m & n\delta \\ p/\delta & q \end{pmatrix} \eta(D\delta z).$$

□

We defer proofs of the next two lemmas to later sections. The first of these constitutes the bulk of the proof of Theorem 4.1.8. Its proof is in Section 4.5.

Lemma 4.3.5. *Let $N \geq 1$, let $\ell \geq 5$ be prime, let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, let $f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$ be a minimal holomorphic level N eta-quotient with denominator D and weight k as in* (4.3.1)*, and let $s = \prod_{\delta \mid N} \delta^{r_\delta}$.*

1. *Suppose that $0 \leq j \leq \ell - 1$ has $a + Dcj \not\equiv 0 \pmod{\ell}$. With x and y as in part 1 of Lemma 4.3.2, we have*

$$f(Dz)\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a + Dcj & x \\ c\ell & y \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta} f(Dz).$$

2. *Suppose that $0 \leq t \leq \ell - 1$ has $a + Dct \equiv 0 \pmod{\ell}$. With u as in part 2 of Lemma 4.3.2, we have*

$$f(Dz)\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u & b + Ddt \\ c & d\ell \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \left(\frac{(-1)^k s}{\ell} \right) \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta} f(Dz).$$

3. *Suppose that $\ell \nmid c$. With m and n as in part 3 of Lemma 4.3.2, we have*

$$f(Dz)\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a\ell & m \\ c & n \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \left(\frac{(-1)^k s}{\ell} \right) \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta} f(Dz).$$

4. Suppose that $\ell \mid c$. Then we have

$$f(Dz)\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b\ell \\ c/\ell & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta} f(Dz).$$

The proof of the following lemma is the content of Section 4.6.

Lemma 4.3.6. *With k' and r'_δ as in (4.3.3), and with the hypotheses in Lemma 4.3.5, we have*

$$\prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta - r'_\delta} = \begin{cases} 1_N & \text{if } N \in \{1,2,4,5\}, \\ \left(\frac{d}{3}\right)^{k+k'} & \text{if } N \in \{3,6,9\}, \\ \left(\frac{-1}{d}\right)^{k+k'} & \text{if } N = 8, \end{cases}$$

where 1_N is the trivial character modulo N.

Hence, the product of eta-multipliers in the lemma gives the Dirichlet character χ' as in (4.1.4).

4.3.4 Proof of Theorem 4.1.8.

We keep the notation of Sections 4.3.1, 4.3.2, and 4.3.3: We let $N \geq 1$, we let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and we let $\ell \geq 5$ be prime. We let $f(z) = \prod_{\delta \mid N} \eta(\delta z)^{r_\delta}$ and $f^\dagger(z) = \prod_{\delta \mid N} \eta(\delta z)^{r'_\delta}$ be minimal holomorphic level N eta-quotients with denominator D and weights k and k' as in (4.3.1) and (4.3.3), and we let $F(z) \in M_w(\Gamma_0(N), \chi)$, where $w \geq 0$ is an integer. We recall from (4.3.4) that

$$H(z) = \frac{(f(Dz)F(Dz)) \mid T_\ell}{f^\dagger(Dz)}.$$

In this section we apply the lemmas of Section 4.3.3 to prove, with χ' as in (4.1.4), that

$$H(z)\Big|_{w+k-k'} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \chi\chi'(d) H(z),$$

as in (4.3.7) of Proposition 4.3.1. We compute the slash operation on the numerator and denominator of $H(z)$ separately. For the denominator, Lemma 4.3.4 yields

$$f^{\dagger}(Dz)\Big|_{k'}\begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}D&0\\0&1\end{pmatrix}=\prod_{\delta\mid N}\nu_\eta\begin{pmatrix}a&b\delta\\c/\delta&d\end{pmatrix}^{r'_\delta}f^{\dagger}(Dz). \tag{4.3.12}$$

For the numerator, we consider cases depending on whether $\ell \mid c$. When $\ell \nmid c$, we also have $\ell \nmid N$. We compute

$$\begin{aligned}
&((f(Dz)F(Dz)) \mid T_\ell)\Big|_{w+k}\begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}D&0\\0&1\end{pmatrix}\\
&=\ell^{\frac{k}{2}-1}\left(\sum_{j=0}^{\ell-1}(f(Dz)F(Dz))\Big|_{w+k}\begin{pmatrix}1&j\\0&\ell\end{pmatrix}\begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}D&0\\0&1\end{pmatrix}\right)\\
&\quad+\ell^{\frac{k}{2}-1}\left(\left(\frac{(-1)^k s}{\ell}\right)\chi(\ell)(f(Dz)F(Dz))\Big|_{w+k}\begin{pmatrix}\ell&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}a&b\\c&d\end{pmatrix}\begin{pmatrix}D&0\\0&1\end{pmatrix}\right)\\
&=\ell^{\frac{k}{2}-1}\sum_{\substack{j=0\\ j\not\equiv -a(Dc)^{-1}(\text{mod }\ell)}}^{\ell-1}(f(Dz)F(Dz))\Big|_{w+k}\begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}a+Dcj&x\\c\ell&y\end{pmatrix}\begin{pmatrix}D&0\\0&1\end{pmatrix}\begin{pmatrix}1&z\\0&\ell\end{pmatrix}\\
&\quad+\ell^{\frac{k}{2}-1}\left(\frac{(-1)^k s}{\ell}\right)\chi(\ell)(f(Dz)F(Dz))\Big|_{w+k}\begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}a\ell&m\\c&n\end{pmatrix}\begin{pmatrix}D&0\\0&1\end{pmatrix}\begin{pmatrix}1&w\\0&\ell\end{pmatrix}\\
&\quad+\ell^{\frac{k}{2}-1}(f(Dz)F(Dz))\Big|_{w+k}\begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}u&b+Ddt\\c&d\ell\end{pmatrix}\begin{pmatrix}D&0\\0&1\end{pmatrix}\begin{pmatrix}\ell&0\\0&1\end{pmatrix}\\
&=\chi(d)\prod_{\delta\mid N}\nu_\eta\begin{pmatrix}a&b\delta\\c/\delta&d\end{pmatrix}^{\ell r_\delta}\left(\ell^{\frac{k}{2}-1}\sum_{\substack{j=0\\ j\not\equiv -a(Dc)^{-1}(\text{mod }\ell)}}^{\ell-1}(f(Dz)F(Dz))\Big|_{w+k}\begin{pmatrix}1&z\\0&\ell\end{pmatrix}\right)\\
&\quad+\left(\frac{(-1)^k s}{\ell}\right)^2\chi(\ell)^2\chi(d)\prod_{\delta\mid N}\nu_\eta\begin{pmatrix}a&b\delta\\c/\delta&d\end{pmatrix}^{\ell r_\delta}\left(\ell^{\frac{k}{2}-1}(f(Dz)F(Dz))\Big|_{w+k}\begin{pmatrix}1&w\\0&\ell\end{pmatrix}\right)\\
&\quad+\left(\frac{(-1)^k s}{\ell}\right)\chi(\ell)\chi(d)\prod_{\delta\mid N}\nu_\eta\begin{pmatrix}a&b\delta\\c/\delta&d\end{pmatrix}^{\ell r_\delta}\left(\ell^{\frac{k}{2}-1}(f(Dz)F(Dz))\Big|_{w+k}\begin{pmatrix}\ell&0\\0&1\end{pmatrix}\right)\\
&=\chi(d)\prod_{\delta\mid N}\nu_\eta\begin{pmatrix}a&b\delta\\c/\delta&d\end{pmatrix}^{\ell r_\delta}(f(Dz)F(Dz)) \mid T_\ell.
\end{aligned}$$

The first equality in the computation follows from (2.4.3); the second equality follows from parts (1), (2), and (3) of Lemma 4.3.2. The third equality results from parts

(1), (2), and (3) of Lemma 4.3.5, and the fourth equality results from another application of (2.4.3) together with Lemma 4.3.3. The transformation of $F(Dz)$ in the computation holds since $F(z) \in M_w(\Gamma_0(N), \chi)$.

When $\ell \mid c$, we note, for all $0 \le j \le \ell - 1$, that $a + Dcj \equiv a \not\equiv 0 \pmod{\ell}$. When $\ell \nmid N$, we compute

$$\begin{aligned}
&((f(Dz)F(Dz)) \mid T_\ell)\Big|_{w+k} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \\
&\qquad = \ell^{\frac{k}{2}-1} \left(\sum_{j=0}^{\ell-1} (f(Dz)F(Dz))\Big|_{w+k} \begin{pmatrix} 1 & j \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&\qquad\qquad + \ell^{\frac{k}{2}-1} \left(\left(\frac{(-1)^k s}{\ell} \right) \chi(\ell) (f(Dz)F(Dz))\Big|_{w+k} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&\qquad = \ell^{\frac{k}{2}-1} \sum_{j=0}^{\ell-1} (f(Dz)F(Dz))\Big|_{w+k} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a + Dcj & x \\ c\ell & y \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & z \\ 0 & \ell \end{pmatrix} \\
&\qquad\qquad + \ell^{\frac{k}{2}-1} \left(\frac{(-1)^k s}{\ell} \right) \chi(\ell) (f(Dz)F(Dz))\Big|_{w+k} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b\ell \\ c/\ell & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \\
&\qquad = \chi(d) \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta} \left(\ell^{\frac{k}{2}-1} \sum_{j=0}^{\ell-1} (f(Dz)F(Dz))\Big|_{w+k} \begin{pmatrix} 1 & z \\ 0 & \ell \end{pmatrix} \right) \\
&\qquad\qquad + \left(\frac{(-1)^k s}{\ell} \right) \chi(\ell)\chi(d) \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta} \left(\ell^{\frac{k}{2}-1} (f(Dz)F(Dz))\Big|_{w+k} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \right) \\
&\qquad = \chi(d) \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta} (f(Dz)F(Dz)) \mid T_\ell.
\end{aligned}$$

Since $a + Dcj \not\equiv 0 \pmod{\ell}$ for all j, the computation differs from the one in the previous section. The second equality follows from parts (1) and (4) of Lemma 4.3.2, the third equality follows from parts (1) and (4) of Lemma 4.3.5, and we do not require Lemma 4.3.3. When $\ell \mid N$, the Hecke operator T_ℓ reduces to the U_ℓ operator; in the computation above, the terms with $\chi(\ell)$ vanish and the result continues to hold.

To conclude the proof of Theorem 4.1.8, we use (4.3.12), the computations above and Lemma 4.3.6 to compute

$$\left(\frac{(f(Dz)F(Dz)) \mid T_\ell}{f^\dagger(Dz)}\right)\Big|_{w+k-k'}\begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}D & 0\\ 0 & 1\end{pmatrix}$$

$$= \chi(d)\prod_{\delta\mid N}\nu_\eta\begin{pmatrix}a & b\delta\\ c/\delta & d\end{pmatrix}^{\ell r_\delta - r'_\delta}\frac{(f(Dz)F(Dz)) \mid T_\ell}{f^\dagger(Dz)}$$

$$= \chi\chi'(d)\,\frac{(f(Dz)F(Dz)) \mid T_\ell}{f^\dagger(Dz)}.$$

Proposition 4.3.1 now implies Theorem 4.1.8.

4.4 Proof of Lemma 4.3.2

In this section, we prove the factorizations of the matrices given in Lemma 4.3.2. For part (1), let $j \in \{0, \ldots, \ell-1\}$ with $a + Dcj \not\equiv 0 \pmod{\ell}$. Since $D \mid 24$ and $\ell \not\equiv 2, 3$, we know $D(a+Dcj) \not\equiv 0 \pmod{\ell}$. Choose z to be the least nonnegative integer with $z \equiv (b+Ddj)(D(a+Dcj))^{-1} \pmod{\ell}$. It follows that $Dz(a+Dcj) \equiv b+Ddj \pmod{\ell}$. Let x and y be integers such that $x\ell = b + Ddj - Dz(a+Dcj)$ and $y = d - Dcz$. Then

$$\begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}a+Dcj & x\\ c\ell & y\end{pmatrix}\begin{pmatrix}D & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}1 & z\\ 0 & \ell\end{pmatrix} = \begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}a+Dcj & x\\ c\ell & y\end{pmatrix}\begin{pmatrix}D & Dz\\ 0 & \ell\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}D(a+Dcj) & Dz(a+Dcj)+x\ell\\ Dc\ell & D\ell cz + y\ell\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}D(a+Dcj) & Dz(a+Dcj)+(b+Ddj-Dz(a+Dcj))\\ Dc\ell & D\ell cz + \ell(d-Dcz)\end{pmatrix}$$

$$= \begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}D(a+Dcj) & b+Ddj\\ Dc\ell & d\ell\end{pmatrix}$$

$$= \begin{pmatrix}D(a+Dcj) & b+Ddj\\ D^2c\ell & Dd\ell\end{pmatrix}$$

$$= \begin{pmatrix}1 & Dj\\ 0 & D\ell\end{pmatrix}\begin{pmatrix}Da & b\\ Dc & d\end{pmatrix}$$

$$= \begin{pmatrix}1 & j\\ 0 & \ell\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & D\end{pmatrix}\begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}D & 0\\ 0 & 1\end{pmatrix}$$

For part (2), let $t \in \{0, \ldots, \ell-1\}$ with $a + Dct \equiv 0 \pmod{\ell}$ and let u be the integer such that $u\ell = a + Dct$. Then we have

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u & b+Ddt \\ c & d\ell \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u & b+Ddt \\ c & d\ell \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} u\ell & b+Ddt \\ c\ell & d\ell \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a+Dct & b+Ddt \\ c\ell & d\ell \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} a+Dct & b+Ddt \\ Dc\ell & Dd\ell \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & Dt \\ 0 & D\ell \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & t \\ 0 & \ell \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

For part (3), assume $\ell \nmid c$. Therefore, $\ell \nmid Dc$ since $\ell \geq 5$. Let w be the least nonnegative integer with $w \equiv d(Dc)^{-1} \pmod{\ell}$. It follows that $Dcw \equiv d \pmod{\ell}$. Let m and n be integers such that $m = b - Daw$ and $n\ell = d - Dcw$. Then

$$\begin{aligned}
\begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a\ell & m \\ c & n \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & w \\ 0 & \ell \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a\ell & m \\ c & n \end{pmatrix} \begin{pmatrix} D & Dw \\ 0 & \ell \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} Da\ell & Daw\ell + m\ell \\ Dc & Dcw + n\ell \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} Da\ell & Daw\ell + (b - Daw)\ell \\ Dc & Dcw + (d - Dcw) \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} Da\ell & b\ell \\ Dc & d \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

Assume $\ell \mid c$. Part (4) follows from the fact that diagonal matrices commute with one another and

$$\begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} \ell a & \ell b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b\ell \\ c/\ell & d \end{pmatrix} \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix}.$$

4.5 Proof of Lemma 4.3.5

We suppose that $0 \leq j \leq \ell - 1$ has $a + Dcj \not\equiv 0 \pmod{\ell}$, and we let x, y, and z be as as in the first part of Lemma 4.3.2. In particular, we recall that

$$x\ell = b + D(dj - (a + Dcj)z), \quad y = d - Dcz. \tag{4.5.1}$$

From Lemma 4.3.4, we have

$$f(Dz)\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a + Dcj & x \\ c\ell & y \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a + Dcj & x\delta \\ c\ell/\delta & y \end{pmatrix} f(Dz).$$

Therefore, it suffices to show that

$$\prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a + Dcj & x\delta \\ c\ell/\delta & y \end{pmatrix} = \prod_{\delta \mid N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta}. \tag{4.5.2}$$

We also require

$$A = (jy - az)(d - Dcz) - bcz. \tag{4.5.3}$$

The following proposition gives a useful expression for xy modulo 24.

Proposition 4.5.1. *In the notation above, we have*

$$xy \equiv AD\ell + bd\ell \pmod{24}.$$

Proof. Equation 4.5.1 implies that $x\ell = b + D(jy - az)$. Since $\ell \geq 5$ is prime, (4.5.1) and (4.5.3) give

$$xy \equiv \ell y(x\ell) \equiv \ell(d - Dcz)(b + D(jy - az)) \equiv AD\ell + bd\ell \pmod{24}.$$

□

In view of (4.5.2) and Definition 2.3.2, we consider c/δ even and c/δ odd separately.

The case of c/δ even.

We note that d is odd since $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, and hence, that $y = d - Dcz$ is odd.

Definition 2.3.2 gives the following expression for $\nu_\eta \begin{pmatrix} a + Dcj & x\delta \\ c\ell/\delta & y \end{pmatrix}$:

$$\left(\frac{c\ell/\delta}{y}\right)_* e\left((a + Dcj + y)\frac{c\ell}{\delta} - xy\delta\left(\left(\frac{c\ell}{\delta}\right)^2 - 1\right) + 3y - 3 - 3y\frac{c\ell}{\delta}\right). \tag{4.5.4}$$

We now proceed to simplify (4.5.4). Since $\ell \geq 5$ is prime, a computation using Proposition 4.5.1 and (4.5.1) shows that

$$\begin{aligned}
&(a + Dcj + y)\frac{c\ell}{\delta} - xy\delta\left(\left(\frac{c\ell}{\delta}\right)^2 - 1\right) + 3y - 3 - 3y\frac{c\ell}{\delta} \\
&\quad \equiv \ell\left((a + d)\frac{c}{\delta} - bd\delta\left(\frac{c^2}{\delta^2} - 1\right) + 3d - 3 - 3\frac{cd}{\delta}\right) && (4.5.5) \\
&\quad + \frac{Dc^2\ell}{\delta}(j - z) - AD\ell\delta\left(\frac{c^2}{\delta^2} - 1\right) + 3Dcz\left(\frac{c\ell}{\delta} - 1\right) - 3(d - 1)(\ell - 1) \pmod{24}.
\end{aligned}$$

We use (2.3.4), (2.3.5), and Definition 2.3.1 to see that

$$\left(\frac{c\ell/\delta}{y}\right)_* = \left(\frac{c/\delta}{dy}\right)_* \left(\frac{c/\delta}{d}\right)_* \left(\frac{-1}{\ell}\right)^{\frac{y-1}{2}} \left(\frac{y}{\ell}\right). \tag{4.5.6}$$

We also use (2.3.5) and (4.5.1) to compute

$$\begin{aligned}
e(-3(d - 1)(\ell - 1)) &= e(-3(y + Dcz - 1)(\ell - 1)) \\
&= e(-3(y - 1)(\ell - 1) - 3Dcz(\ell - 1)) \\
&= e\left(-12\left(\frac{y - 1}{2}\right)\left(\frac{\ell - 1}{2}\right)\right) e(-3Dcz(\ell - 1)) \\
&= \left(\frac{-1}{\ell}\right)^{\frac{y-1}{2}} e(-3Dcz(\ell - 1)). && (4.5.7)
\end{aligned}$$

Substituting (4.5.5), (4.5.6), and (4.5.7) in (4.5.4) and using Definition 2.3.2 yields

$$\begin{aligned}
\nu_\eta \begin{pmatrix} a + Dcj & x\delta \\ c\ell/\delta & y \end{pmatrix} &= \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell} \left(\frac{y}{\ell}\right) e\left(\frac{Dc^2\ell}{\delta}(j - z) - AD\ell\delta\left(\frac{c^2}{\delta^2} - 1\right)\right) \\
&\quad \times \left(\frac{c/\delta}{dy}\right)_* e\left(3Dcz\ell\left(\frac{c}{\delta} - 1\right)\right). && (4.5.8)
\end{aligned}$$

The case of c/δ odd.

In this case, Definition 2.3.2 implies that

$$\nu_\eta \begin{pmatrix} a+Dcj & x\delta \\ c\ell/\delta & y \end{pmatrix} = \left(\frac{y}{c\ell/\delta}\right)^* e\left((a+Dcj+y)\frac{c\ell}{\delta} - xy\delta\left(\left(\frac{c\ell}{\delta}\right)^2 - 1\right) - 3\frac{c\ell}{\delta}\right) \tag{4.5.9}$$

To simplify (4.5.9), we note that $\ell \geq 5$ is prime and apply Proposition 4.5.1 and (4.5.1) to obtain

$$\begin{aligned} &(a+Dcj+y)\frac{c\ell}{\delta} - xy\delta\left(\left(\frac{c\ell}{\delta}\right)^2 - 1\right) - 3\frac{c\ell}{\delta} \\ &= \ell\left((a+d)\frac{c}{\delta} - bd\delta\left(\frac{c^2}{\delta^2} - 1\right) - 3\frac{c}{\delta}\right) + \frac{Dc^2\ell}{\delta}(j-z) - AD\ell\delta\left(\frac{c^2}{\delta^2} - 1\right) \pmod{24}. \end{aligned} \tag{4.5.10}$$

We use Definition 2.3.1 and (4.5.1) to get

$$\left(\frac{y}{c\ell/\delta}\right)^* = \left(\frac{y}{\ell}\right)\left(\frac{d}{c/\delta}\right)^*. \tag{4.5.11}$$

Inserting (4.5.10) and (4.5.11) in (4.5.9) and using Definition 2.3.2 gives

$$\nu_\eta \begin{pmatrix} a+Dcj & x\delta \\ c\ell/\delta & y \end{pmatrix} = \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^\ell \left(\frac{y}{\ell}\right) e\left(\frac{Dc^2\ell}{\delta}(j-z) - AD\ell\delta\left(\frac{c^2}{\delta^2} - 1\right)\right). \tag{4.5.12}$$

Consolidating the two cases.

Equations (4.5.8) and (4.5.12) show, whether c/δ is even or odd, that

$$\begin{aligned} \prod_{\delta|N} \nu_\eta \begin{pmatrix} a+Dcj & x\delta \\ c\ell/\delta & y \end{pmatrix} &= \prod_{\delta|N} \left(\nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^\ell \left(\frac{y}{\ell}\right) e\left(\frac{Dc^2\ell}{\delta}(j-z) - AD\ell\delta\left(\frac{c^2}{\delta^2} - 1\right)\right)\right)^{r_\delta} \\ &\times \prod_{\substack{\delta|N \\ c/\delta \text{ even}}} \left(\left(\frac{c/\delta}{dy}\right)_* e\left(3Dcz\ell\left(\frac{c}{\delta} - 1\right)\right)\right)^{r_\delta} \end{aligned} \tag{4.5.13}$$

In order to further simplify the first product, we observe from Definition 4.1.1 that

$$D\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24}, \quad DN\sum_{\delta|N} \frac{r_\delta}{\delta} \equiv 0 \pmod{24}.$$

These facts imply that

$$\prod_{\delta|N} e\left(\frac{Dc^2\ell}{\delta}(j-z)\right)^{r_\delta} = e\left(\frac{c^2\ell(j-z)}{N} DN \sum_{\delta|N} \frac{r_\delta}{\delta}\right) = 1 \tag{4.5.14}$$

and that

$$\prod_{\delta|N} e\left(-AD\ell\delta\left(\frac{c^2}{\delta^2}-1\right)\right)^{r_\delta} = \prod_{\delta|N} e\left(-\frac{A\ell c^2}{N} DN \sum_{\delta|N} \frac{r_\delta}{\delta} + A\ell D \sum_{\delta|N} \delta r_\delta\right) = 1. \tag{4.5.15}$$

We also recall that the eta-quotient $f(z)$ has integer weight $k = \sum_{\delta|N} r_\delta \equiv 0 \pmod 2$, which we use to deduce that

$$\prod_{\delta|N} \left(\frac{y}{\ell}\right)^{r_\delta} = \left(\frac{y}{\ell}\right)^{\sum_{\delta|N} r_\delta} = 1. \tag{4.5.16}$$

Using (4.5.14), (4.5.15), and (4.5.16) in (4.5.13), gives us

$$\prod_{\delta|N} \nu_\eta \begin{pmatrix} a+Dcj & x\delta \\ c\ell/\delta & y \end{pmatrix} = \prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta} \prod_{\substack{\delta|N \\ c/\delta \text{ even}}} \left(\left(\frac{c/\delta}{dy}\right)_* e\left(3Dcz\ell\left(\frac{c}{\delta}-1\right)\right)\right)^{r_\delta}. \tag{4.5.17}$$

Conclusion of part one of Lemma 4.3.5.

In view of (4.5.17), to conclude the proof of the first part of Lemma 4.3.5, it suffices to prove the following proposition.

Proposition 4.5.2. *In the notation of this section, we have*

$$\prod_{\substack{\delta|N \\ c/\delta \text{ even}}} \left(\left(\frac{c/\delta}{dy}\right)_* e\left(3Dcz\ell\left(\frac{c}{\ell}-1\right)\right)\right)^{r_\delta} = 1. \tag{4.5.18}$$

Before proceeding, we remark that c/δ even and $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ imply that c is even and that d and $y = d - Dcz$ are odd. We also define $v = v_2(c/\delta) \geq 0$ and $c' \in \mathbb{Z}$ by

$$c = 2^v c'. \tag{4.5.19}$$

Hence, c'/δ is the odd part of c/δ. The following lemma simplifies the $\left(\frac{c/\delta}{dy}\right)_*$ factors in (4.5.18), thereby providing an equivalent condition sufficient for the proof of Part 1 of Lemma 4.3.5.

Lemma 4.5.3. *In the notation of this section, the expression* (4.5.18) *in Proposition 4.5.2 is equal to*

$$\prod_{\substack{\delta|N \\ c/\delta \text{ even}}} \left(\frac{2}{|dy|}\right)^{vr_\delta} e\left(3Dczr_\delta\left(\ell\left(\frac{c}{\delta}-1\right)+d\left(\frac{c'}{\delta}-1\right)\right)\right). \tag{4.5.20}$$

Proof. Definition 2.3.1 and (4.5.19) give

$$\left(\frac{c/\delta}{dy}\right)_* = \left(\frac{2}{|dy|}\right)^v \left(\frac{c'/\delta}{|dy|}\right)(-1)^{\varepsilon(c)\varepsilon(dy)}. \tag{4.5.21}$$

From (4.5.1), we have

$$\left(\frac{dy}{|c'|/\delta}\right) = \left(\frac{d^2 - Dcdz}{|c'|/\delta}\right) = \left(\frac{d}{|c'|/\delta}\right)^2 = 1. \tag{4.5.22}$$

We use (2.3.4), (4.5.1), and (4.5.22) to find that

$$\begin{aligned}
\left(\frac{c'/\delta}{|dy|}\right) &= \left(\frac{c'/\delta}{|dy|}\right)\left(\frac{dy}{|c'|/\delta}\right) = (-1)^{\left(\frac{c'/\delta-1}{2}\right)\left(\frac{d^2-Dcdz-1}{2}\right)}(-1)^{\varepsilon(c)\varepsilon(dy)} \\
&= (-1)^{\left(\frac{c'/\delta-1}{2}\right)\left(-\frac{Dcdz}{2}\right)}(-1)^{\varepsilon(c)\varepsilon(dy)} = e\left(12\left(\frac{c'/\delta-1}{2}\right)\left(-\frac{Dcdz}{2}\right)\right)(-1)^{\varepsilon(c)\varepsilon(dy)} \\
&= e\left(3Dcdz\left(\frac{c'}{\delta}-1\right)\right)(-1)^{\varepsilon(c)\varepsilon(dy)}.
\end{aligned} \tag{4.5.23}$$

The lemma follows from inserting (4.5.21) and (4.5.23) in (4.5.18). □

To conclude the proof of the first part of Lemma 4.3.5, we devote the rest of this section to proving that the expression (4.5.20) in Lemma 4.5.3 is equal to one. We require integers X and Y defined by

$$X := \sum_{\delta|N} \delta r_\delta = \sum_{j=0}^{v_2(N)} \sum_{\substack{\delta|N \\ v_2(\delta)=j}} \delta r_\delta, \tag{4.5.24}$$

$$Y := \sum_{\delta|N} \frac{Nr_\delta}{\delta} = \sum_{j=0}^{v_2(N)} \sum_{\substack{\delta|N \\ v_2(\delta)=j}} \frac{Nr_\delta}{\delta}. \tag{4.5.25}$$

Definition 4.1.1 implies that $X \equiv Y \equiv 0 \pmod{24/D}$. We also observe from (4.3.1) that the integer weight k has

$$2k = \sum_{\delta|N} r_\delta = \sum_{j=0}^{v_2(N)} \sum_{\substack{\delta|N \\ v_2(\delta)=j}} r_\delta \equiv 0 \pmod{2}. \tag{4.5.26}$$

Our arguments separate into cases depending on $v_2(Dc)$.

Case of $\boldsymbol{v_2(Dc) \geq 3}$. From (4.5.1), we have $|dy| = |d(d - Dcz)| \equiv d^2 \equiv 1 \pmod{8}$. It follows that $\left(\frac{2}{|dy|}\right) = 1$ in (4.5.20). Hence, since

$$3Dczr_\delta \left(\ell\left(\frac{c}{\delta} - 1\right) + d\left(\frac{c'}{\delta} - 1\right)\right) \equiv 0 \pmod{24},$$

the full expression (4.5.20) reduces to 1, as desired.

Case of $\boldsymbol{v_2(Dc) = 2}$. If c is odd, then the product (4.5.20) is empty and there is nothing to prove. Therefore, we suppose that c is even. We have $(v_2(c), v_2(D)) \in \{(1,1),(2,0)\}$. Since $N \mid c$, we have $v_2(N) \leq v_2(c) \leq 2$. Using (4.5.24) and observing that $v_2(D) \in \{0,1\}$, we obtain

$$A := \sum_{\substack{\delta|N \\ v_2(\delta)=0}} r_\delta \equiv \sum_{j=0}^{v_2(N)} \sum_{\substack{\delta|N \\ v_2(\delta)=j}} \delta r_\delta \equiv X \equiv 0 \pmod{2}. \tag{4.5.27}$$

Similarly, when $v_2(N) = 2$, we find from (4.5.25) and $v_2(D) \in \{0,1\}$ that

$$\sum_{\substack{\delta|N \\ v_2(\delta)=2}} r_\delta \equiv \sum_{j=0}^{2} \sum_{\substack{\delta|N \\ v_2(\delta)=j}} \frac{Nr_\delta}{\delta} \equiv Y \equiv 0 \pmod{2}. \tag{4.5.28}$$

Therefore, when $v_2(N) \in \{1,2\}$, we use (4.5.26), (4.5.27), and (4.5.28) to see that

$$B := \sum_{\substack{\delta|N \\ v_2(\delta)=1}} r_\delta \equiv 0 \pmod{2}. \tag{4.5.29}$$

We now use (4.5.27) and (4.5.29) to simplify (4.5.20). Since $v_2(c) \leq 2$, we observe that the factors in (4.5.20) have $v_2(\delta) \in \{0,1\}$. We also recall from (4.5.19) that

$v = v_2(c/\delta)$. Therefore, we compute

$$\begin{aligned}\prod_{\substack{\delta \mid N \\ c/\delta \text{ even}}} \left(\frac{2}{|dy|}\right)^{vr_\delta} &= \prod_{\substack{\delta \mid N \\ v_2(\delta)=0}} \left(\frac{2^{v_2(c)}}{|dy|}\right)^{r_\delta} \prod_{\substack{\delta \mid N \\ v_2(\delta)=1}} \left(\frac{2^{v_2(c)-1}}{|dy|}\right)^{r_\delta} \\ &= \left(\frac{2^{v_2(c)}}{|dy|}\right)^{A} \left(\frac{2^{v_2(c)-1}}{|dy|}\right)^{B} = 1,\end{aligned} \tag{4.5.30}$$

where the final equality follows from (4.5.27) and (4.5.29). Next, when c/δ is even, we recall that c'/δ and d are odd. Hence, since ℓ is odd and $Dc \equiv 0 \pmod 4$, we find that there exists $m \in \mathbb{Z}$ such that we can simplify the argument of the exponential in (4.5.20) as

$$3Dczr_\delta \left(\ell\left(\frac{c}{\delta} - 1\right) + d\left(\frac{c'}{\delta} - 1\right)\right) \equiv 12mzr_\delta \pmod{24}. \tag{4.5.31}$$

We then use (4.5.27), (4.5.29), and (4.5.31) to see that

$$\begin{aligned}\prod_{\substack{\delta \mid N \\ c/\delta \text{ even}}} e\left(3Dczr_\delta \left(\ell\left(\frac{c}{\delta} - 1\right) + d\left(\frac{c'}{\delta} - 1\right)\right)\right) &= \prod_{\substack{\delta \mid N \\ c/\delta \text{ even}}} e(12mzr_\delta) = e\left(12mz \sum_{\substack{\delta \mid N \\ c/\delta \text{ even}}} r_\delta\right) \\ &= e(12mz(A+B)) = 1.\end{aligned} \tag{4.5.32}$$

To conclude, we use (4.5.30) and (4.5.32) along with (4.5.27) and (4.5.28) to reduce (4.5.20) to the value 1.

Case of $v_2(Dc) = 1$. As in the previous case, it suffices to suppose that c is even. Then we have $v_2(D) = 0$, $v_2(N) \leq v_2(c) = 1$, $c = 2c'$, and the factors in (4.5.20) have $v_2(\delta) = 0$. Equation (4.5.27) from the previous case continues to hold, and we use it to compute

$$\prod_{\substack{\delta \mid N \\ c/\delta \text{ even}}} \left(\frac{2}{|dy|}\right)^{vr} = \left(\frac{2}{|dy|}\right)^{A} = 1.$$

For the remaining part of (4.5.20), we compute

$$\begin{aligned}
\prod_{\substack{\delta|N\\ c/\delta\ \text{even}}} & e\left(3Dczr_\delta\left(\ell\left(\frac{c}{\delta}-1\right)+d\left(\frac{c'}{\delta}-1\right)\right)\right)\\
&= e\left(6z\sum_{\substack{\delta|N\\ c/\delta\ \text{even}}} r_\delta\left(\ell\left(\frac{c}{\delta}-1\right)+d\left(\frac{c'}{\delta}-1\right)\right)\right)\\
&= e\left(6z(c'd+c\ell)\sum_{\substack{\delta|N\\ v_2(\delta)=0}}\frac{r_\delta}{\delta}-6z(\ell+d)A\right)\\
&= e\left(6z(c'd+c\ell)\sum_{\substack{\delta|N\\ v_2(\delta)=0}}\frac{r_\delta}{\delta}\right),
\end{aligned} \tag{4.5.33}$$

where the third equality holds by (4.5.27) since ℓ and d are odd. In view of (4.5.33), it suffices to show that

$$(c'd+c\ell)\sum_{\substack{\delta|N\\ v_2(\delta)=0}}\frac{r_\delta}{\delta}\equiv 0 \pmod 4. \tag{4.5.34}$$

We consider cases depending on $v_2(N)\in\{0,1\}$.

Case of $v_2(N)=0$. Since $N\mid c$ and $c=2c'$, we have $N\mid c'$. We compute

$$(c'd+c\ell)\sum_{\substack{\delta|N\\ v_2(\delta)=0}}\frac{r_\delta}{\delta}=\left(\frac{c'd+c\ell}{N}\right)\sum_{\delta|N}\frac{Nr_\delta}{\delta}\equiv 0 \pmod 8,$$

where the congruence follows from (4.5.25) since $v_2(D)=0$, thereby establishing (4.5.34).

Case of $v_2(N)=1$. Observing that $v_2(D)=0$, we see from (4.5.24) that $\sum\limits_{\delta|N}\delta r_\delta\equiv 0 \pmod 8$. Since $v_2(N)=1$, it follows that

$$\sum_{\substack{\delta|N\\ v_2(\delta)=0}}\delta r_\delta\equiv-\sum_{\substack{\delta|N\\ v_2(\delta)=1}}\delta r_\delta \pmod 8. \tag{4.5.35}$$

Hence, we have

$$\sum_{\substack{\delta|N\\ v_2(\delta)=0}} r_\delta \equiv \sum_{\substack{\delta|N\\ v_2(\delta)=0}} \delta r_\delta \equiv \sum_{\substack{\delta|N\\ v_2(\delta)=1}} \delta r_\delta \equiv 0 \pmod 2. \tag{4.5.36}$$

Using $v_2(N) = 1$, (4.5.26), and (4.5.36), we obtain

$$0 \equiv 2k \equiv \sum_{\substack{\delta|N\\ v_2(\delta)=0}} r_\delta + \sum_{\substack{\delta|N\\ v_2(\delta)=1}} r_\delta \equiv \sum_{\substack{\delta|N\\ v_2(\delta)=1}} r_\delta \pmod 2. \tag{4.5.37}$$

Next, we claim that

$$\sum_{\substack{\delta|N\\ v_2(\delta)=0}} \frac{N}{\delta} r_\delta \equiv 0 \pmod 8. \tag{4.5.38}$$

To see this, we note from (4.5.35) that

$$\sum_{\substack{\delta|N\\ v_2(\delta)=0}} \frac{N}{\delta} r_\delta \equiv \sum_{\substack{\delta|N\\ v_2(\delta)=0}} N\delta r_\delta \equiv - \sum_{\substack{\delta|N\\ v_2(\delta)=1}} N\delta r_\delta \pmod 8, \tag{4.5.39}$$

where we used $\delta^2 \equiv 1 \pmod 8$ when δ is odd for the first congruence. Now, since $v_2(N) = 1$, when $v_2(\delta) = 1$, there exists odd c_δ with $-N\delta = 4c_\delta$. We use this observation and (4.5.37) to conclude the computation in (4.5.39) and prove (4.5.38):

$$\sum_{\substack{\delta|N\\ v_2(\delta)=0}} \frac{N}{\delta} r_\delta \equiv 4 \sum_{\substack{\delta|N\\ v_2(\delta)=1}} c_\delta r_\delta \equiv 0 \pmod 8. \tag{4.5.40}$$

Recalling that $N \mid c$, $v_2(N) = 1$, and $2c' = c$, we compute

$$(c'd + c\ell) \sum_{\substack{\delta|N\\ v_2(\delta)=0}} \frac{r_\delta}{\delta} \equiv \frac{c}{N}\,\ell \sum_{\substack{\delta|N\\ v_2(\delta)=0}} \frac{N}{\delta} r_\delta + \frac{2c'}{N}\,d \sum_{\substack{\delta|N\\ v_2(\delta)=0}} \frac{N r_\delta}{2\delta} \equiv \frac{cd}{N}\,\frac{1}{2} \sum_{\substack{\delta|N\\ v_2(\delta)=0}} \frac{N r_\delta}{\delta} \equiv 0 \pmod 4,$$

which proves (4.5.34). Specifically, we used (4.5.39) for the second and third congruences. Therefore, when $v_2(Dc) = 1$, we find that (4.5.20) is 1. This concludes the proof of Proposition 4.5.2, and with it, the proof of the first part of Lemma 4.3.5.

4.5.1 The remaining parts of Lemma 4.3.5.

The proofs of the remaining parts of Lemma 4.3.5 have structures similar to the proof of the first part. Therefore, we only sketch the arguments. We require

$$D\sum_{\delta|N}\delta r_\delta \equiv 0 \pmod{24}, \quad DN\sum_{\delta|N}\frac{r_\delta}{\delta} \equiv 0 \pmod{24}, \tag{4.5.41}$$

which follow from Definition 4.1.1.

Part two of Lemma 4.3.5.

We suppose that $0 \leq t \leq \ell - 1$ and $u \in \mathbb{Z}$ have $u\ell = a + Dct$ as in part two of Lemma 4.3.2. From Lemma 4.3.4, we have

$$f(Dz)\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix}\begin{pmatrix} u & b+Ddt \\ c & d\ell \end{pmatrix}\begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \prod_{\delta|N}\nu_\eta\begin{pmatrix} u & (b+Ddt)\delta \\ c/\delta & d\ell \end{pmatrix}^{r_\delta} f(Dz).$$

Hence, it suffices to prove that

$$\prod_{\delta|N}\nu_\eta\begin{pmatrix} u & (b+Ddt)\delta \\ c/\delta & d\ell \end{pmatrix}^{r_\delta} = \left(\frac{(-1)^k s}{\ell}\right)\prod_{\delta|N}\nu_\eta\begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta}, \tag{4.5.42}$$

where $s = \prod_{\delta|N}\delta^{r_\delta}$. Using Definition 2.3.2 and arguing as we did above, we find that $\prod_{\delta|N}\nu_\eta\begin{pmatrix} u & (b+Ddt)\delta \\ c/\delta & d\ell \end{pmatrix}^{r_\delta}$ is

$$\prod_{\delta|N}\left(\left(\frac{c/\delta}{\ell}\right)e\left(Dt\ell d^2\delta - Dt\ell(d^2-1)\frac{c^2}{\delta} + 3(\ell-1)\right)\nu_\eta\begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell}\right)^{r_\delta}. \tag{4.5.43}$$

From (4.5.41), it follows that

$$\begin{aligned}\prod_{\delta|N}e\left(Dt\ell d^2\delta - Dt\ell(d^2-1)\frac{c^2}{\delta}\right)^{r_\delta} &= e\left(t\ell d^2\cdot\sum_{\delta|N}\delta r_\delta\right) \\ &\times e\left(-t\ell(d^2-1)\frac{c^2}{N}\cdot DN\sum_{\delta|N}\frac{r_\delta}{\delta}\right) = 1. \end{aligned}\tag{4.5.44}$$

Since $s=\prod_{\delta|N}\delta^{r_\delta}$ and $2k=\sum_{\delta|N} r_\delta$, we also have

$$\prod_{\delta|N}\left(\left(\frac{c/\delta}{\ell}\right)e(3(\ell-1))\right)^{r_\delta}=\left(\frac{(-1)^k s}{\ell}\right). \tag{4.5.45}$$

To conclude, we substitute (4.5.44) and (4.5.45) in (4.5.43) to obtain (4.5.42).

Part three of Lemma 4.3.5.

We suppose that $\ell\nmid c$, and we let w, m, and n be as in the third part of Lemma 4.3.2. Specifically, we have

$$w\equiv d(Dc)^{-1}\pmod{\ell},\quad m=b-Daw,\quad n\ell=d-Dcw.$$

For convenience, we set

$$B:=-bcw\ell-aw\ell(d-Dcw).$$

Lemma 4.3.4 gives

$$f(Dz)\Big|_k\begin{pmatrix}1&0\\0&D\end{pmatrix}\begin{pmatrix}a\ell&m\\c&n\end{pmatrix}\begin{pmatrix}D&0\\0&1\end{pmatrix}=\prod_{\delta|N}\nu_\eta\begin{pmatrix}a\ell&m\delta\\c/\delta&n\end{pmatrix}^{r_\delta}f(Dz).$$

We will prove that

$$\prod_{\delta|N}\nu_\eta\begin{pmatrix}a\ell&m\delta\\c/\delta&n\end{pmatrix}^{r_\delta}=\left(\frac{(-1)^k s}{\ell}\right)\prod_{\delta|N}\nu_\eta\begin{pmatrix}a&b\delta\\c/\delta&d\end{pmatrix}^{\ell r_\delta}, \tag{4.5.46}$$

where $s=\prod_{\delta|N}\delta^{r_\delta}$. We use Definition 2.3.2 and simplify as we did above to see that

$$\begin{aligned}\prod_{\delta|N}\nu_\eta\begin{pmatrix}a\ell&m\delta\\c/\delta&n\end{pmatrix}^{r_\delta}&=\prod_{\delta|N}\left(\left(\frac{c/\delta}{\ell}\right)e\left(-D(w\ell+B)\frac{c^2}{\delta}+BD\delta+3(\ell-1)\right)\nu_\eta\begin{pmatrix}a&b\delta\\c/\delta&d\end{pmatrix}^{\ell}\right)^{r_\delta}\\&\times\prod_{\substack{\delta|N\\c/\delta\text{ even}}}\left(\left(\frac{c/\delta}{dn\ell}\right)_* e\left(3Dcw\ell\left(\frac{c}{\delta}-1\right)\right)\right)^{r_\delta}.\end{aligned} \tag{4.5.47}$$

With $n\ell$ in place of y and w in place of z in Proposition 4.5.2, we obtain

$$\prod_{\substack{\delta|N\\c/\delta\text{ even}}}\left(\left(\frac{c/\delta}{dn\ell}\right)_* e\left(3Dcw\ell\left(\frac{c}{\delta}-1\right)\right)\right)^{r_\delta}=1. \tag{4.5.48}$$

Applying (4.5.41) yields

$$\prod_{\delta|N} e\left(-D(w\ell+B)\frac{c^2}{\delta}+BD\delta\right)^{r_\delta} = e\left(-(w\ell+B)\frac{c^2}{N}\cdot DN\sum_{\delta|N}\frac{r_\delta}{\delta}\right) \times e\left(B\cdot D\sum_{\delta|N}\delta r_\delta\right) = 1. \tag{4.5.49}$$

Noting that (4.5.45) continues to hold, we substitute it, (4.5.48), and (4.5.49) in (4.5.47) to get (4.5.46).

Part four of Lemma 4.3.5.

We suppose that $\ell \mid c$. Lemma 4.3.4 implies that

$$f(Dz)\Big|_k \begin{pmatrix} 1 & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b\ell \\ c/\ell & d \end{pmatrix} \begin{pmatrix} D & 0 \\ 0 & 1 \end{pmatrix} = \prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\ell\delta \\ c/\ell\delta & d \end{pmatrix}^{r_\delta} f(Dz).$$

As in the proofs of the previous parts, we simplify the expressions for the eta-multiplier from Definition 2.3.2 and use the fact that $2k = \sum_{\delta|N} r_\delta$ to get

$$\prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\ell\delta \\ c/\ell\delta & d \end{pmatrix}^{r_\delta} = \prod_{\delta|N}\left(\left(\frac{d}{\ell}\right)\nu_\eta\begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell}\right)^{r_\delta} = \prod_{\delta|N}\nu_\eta\begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta}.$$

4.6 Proof of Lemma 4.3.6

We let $N \in \{1,2,3,4,5,6,8,9\}$, we let $\ell \geq 5$ be prime, and we let $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$. Proposition 4.1.6 implies, for all $d \mid N$, that

$$\frac{N}{\gcd(d^2,N)}\sum_{\delta|N}\frac{\gcd(d,\delta)^2}{\delta}(\ell r_\delta - r'_\delta) \equiv 0 \pmod{24}. \tag{4.6.1}$$

When $d = 1$ and $d = N$ in (4.6.1), we obtain

$$N\sum_{\delta|N}\frac{\ell r_\delta - r'_\delta}{\delta} \equiv 0 \pmod{24}, \quad \sum_{\delta|N}\delta(\ell r_\delta - r'_\delta) \equiv 0 \pmod{24}, \tag{4.6.2}$$

respectively.

4.6.1 A preliminary simplification

Toward a proof of Lemma 4.3.6, we simplify the product of eta-multipliers in the lemma depending on the parity of the matrix entry c.

Proposition 4.6.1. *In the notation of Lemma 4.3.6, we have*

$$\prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta - r'_\delta} = \begin{cases} \prod\limits_{\delta|N} \left(\frac{d}{\delta}\right)^{\ell r_\delta - r'_\delta}, & c \text{ is odd}, \\ \left(\frac{-1}{d}\right)^{k+k'} \prod\limits_{\delta|N} \left(\frac{\delta}{d}\right)^{\ell r_\delta - r'_\delta}, & c \text{ is even}. \end{cases}$$

Proof. Definition 2.3.2 gives

$$\begin{aligned} \prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta - r'_\delta} &= \prod_{\delta|N} \left((a+d)\frac{c}{\delta} - bd\delta\left(\frac{c^2}{\delta^2} - 1\right) - \frac{3c}{\delta}\right)^{\ell r_\delta - r'_\delta} \prod_{\substack{\delta|N \\ c/\delta \text{ odd}}} \left(\frac{d}{|c/\delta|}\right)^{\ell r_\delta - r'_\delta} \\ &\times \prod_{\substack{\delta|N \\ c/\delta \text{ even}}} \left(\left(\frac{c/\delta}{|d|}\right)(-1)^{\varepsilon(c)\varepsilon(d)} e\left(-3(d-1)\left(\frac{c}{\delta} - 1\right)\right)\right)^{\ell r_\delta - r'_\delta}. \end{aligned} \tag{4.6.3}$$

Using (4.6.2), we find that the first product in (4.6.3) is

$$\begin{aligned} \prod_{\delta|N} \left((a+d)\frac{c}{\delta} - bd\delta\left(\frac{c^2}{\delta^2} - 1\right) - \frac{3c}{\delta}\right)^{\ell r_\delta - r'_\delta} &= e\left((a+d-bdc-3)\frac{c}{N}\left(N\sum_{\delta|N} \frac{\ell r_\delta - r_{\delta'}}{\delta}\right)\right) \\ &\times e\left(bd\sum_{\delta|N} \delta(\ell r_\delta - r_{\delta'})\right) = 1, \end{aligned}$$

which allows us to simplify (4.6.3) as

$$\begin{aligned} \prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta - r'_\delta} &= \prod_{\substack{\delta|N \\ c/\delta \text{ odd}}} \left(\frac{d}{|c/\delta|}\right)^{\ell r_\delta - r'_\delta} \\ &\times \prod_{\substack{\delta|N \\ c/\delta \text{ even}}} \left(\left(\frac{c/\delta}{|d|}\right)(-1)^{\varepsilon(c)\varepsilon(d)} e\left(-3(d-1)\left(\frac{c}{\delta} - 1\right)\right)\right)^{\ell r_\delta - r'_\delta}. \end{aligned} \tag{4.6.4}$$

We now simplify (4.6.4) according to the parity of c.

Case of c odd. When c is odd, we have δ and c/δ odd for all $\delta \mid N$. Since the weights k and k' are integers, we have that

$$\sum_{\delta|N}(\ell r_\delta - r'_\delta) = 2k\ell - 2k' \equiv 0 \pmod 2. \tag{4.6.5}$$

It follows from (4.6.5) that (4.6.4) becomes

$$\begin{aligned}\prod_{\delta|N}\begin{pmatrix} a & b\delta \\ c/\delta & d\end{pmatrix}^{\ell r_\delta - r'_\delta} &= \prod_{\substack{\delta|N \\ c/\delta \text{ odd}}}\left(\frac{d}{|c/\delta|}\right)^{\ell r_\delta - r'_\delta} = \prod_{\delta|N}\left(\frac{d}{|c|}\right)^{\ell r_\delta - r'_\delta}\prod_{\delta|N}\left(\frac{d}{\delta}\right)^{\ell r_\delta - r'_\delta} \\ &= \left(\frac{d}{|c|}\right)^{\sum_{\delta|N}(\ell r_\delta - r'_\delta)}\prod_{\delta|N}\left(\frac{d}{\delta}\right)^{\ell r_\delta - r'_\delta} = \prod_{\delta|N}\left(\frac{d}{\delta}\right)^{\ell r_\delta - r'_\delta}.\end{aligned}$$

Case of c even. We first suppose that c/δ is odd, in which case (2.3.4) gives

$$\begin{aligned}\prod_{\substack{\delta|N \\ c/\delta \text{ odd}}}\left(\frac{d}{|c/\delta|}\right)^{\ell r_\delta - r'_\delta} &= \prod_{\substack{\delta|N \\ c/\delta \text{ odd}}}\left(\left(\frac{c/\delta}{|d|}\right)(-1)^{\frac{d-1}{2}\cdot\frac{c/\delta-1}{2}}(-1)^{\varepsilon(c)\varepsilon(d)}\right)^{\ell r_\delta - r'_\delta} \\ &= \prod_{\substack{\delta|N \\ c/\delta \text{ odd}}}\left(\left(\frac{c/\delta}{|d|}\right)e\left(-3(d-1)\left(\frac{c}{\delta}-1\right)\right)(-1)^{\varepsilon(c)\varepsilon(d)}\right)^{\ell r_\delta - r'_\delta}\end{aligned}$$

Hence, when c is even, (4.6.4) is

$$\prod_{\delta|N}\begin{pmatrix} a & b\delta \\ c/\delta & d\end{pmatrix}^{\ell r_\delta - r'_\delta} = \prod_{\delta|N}\left(\left(\frac{c/\delta}{|d|}\right)e\left(-3(d-1)\left(\frac{c}{\delta}-1\right)\right)(-1)^{\varepsilon(c)\varepsilon(d)}\right)^{\ell r_\delta - r'_\delta} \tag{4.6.6}$$

We achieve further simplification of (4.6.6) through the following computations. Using (2.3.3) and (4.6.5), we find that

$$\begin{aligned}\prod_{\delta|N}\left(\left(\frac{c/\delta}{|d|}\right)(-1)^{\varepsilon(c)\varepsilon(d)}\right)^{\ell r_\delta - r'_\delta} &= \prod_{\delta|N}\left(\left(\frac{c}{|d|}\right)(-1)^{\varepsilon(c)\varepsilon(d)}\right)^{\ell r_\delta - r'_\delta}\prod_{\delta|N}\left(\frac{\delta}{|d|}\right)^{\ell r_\delta - r'_\delta} \\ &= \left(\left(\frac{c}{|d|}\right)(-1)^{\varepsilon(c)\varepsilon(d)}\right)^{\sum_{\delta|N}(\ell r_\delta - r'_\delta)}\prod_{\delta|N}\left(\frac{\delta}{|d|}\right)^{\ell r_\delta - r'_\delta} = \prod_{\delta|N}\left(\frac{\delta}{d}\right)^{\ell r_\delta - r'_\delta}.\end{aligned} \tag{4.6.7}$$

We also compute

$$\begin{aligned}\prod_{\delta|N} e\left(-3(d-1)\left(\frac{c}{\delta}-1\right)\right)^{\ell r_\delta - r'_\delta} &= e\left(-3(d-1)\sum_{\delta|N}\left(\frac{c}{\delta}-1\right)(\ell r_\delta - r'_\delta)\right)\\ &= e\left(-3(d-1)\frac{c}{N}\cdot N\sum_{\delta|N}\frac{\ell r_\delta - r'_\delta}{\delta} + 3(d-1)\sum_{\delta|N}(\ell r_\delta - r'_\delta)\right)\\ &= e\left(3(d-1)\sum_{\delta|N}(\ell r_\delta - r'_\delta)\right) = e(6(d-1)(k\ell - k'))\\ &= e\left(12\left(\frac{d-1}{2}\right)(k+k')\right) = (-1)^{\frac{d-1}{2}(k+k')} = \left(\frac{-1}{d}\right)^{k+k'}, \quad (4.6.8)\end{aligned}$$

where the third equality holds by (4.6.2), the fourth by (4.6.5), and the sixth by (2.3.5). We substitute (4.6.7) and (4.6.8) in (4.6.6) to conclude the proposition when c is even. $\square$

The proof of Lemma 4.3.6 now proceeds by studying the value of

$$\prod_{\delta|N}\nu_\eta\begin{pmatrix} a & b\delta \\ c/\delta & d\end{pmatrix}^{\ell r_\delta - r'_\delta} = \begin{cases}\prod_{\delta|N}\left(\frac{d}{\delta}\right)^{\ell r_\delta - r'_\delta}, & c \text{ is odd},\\ \left(\frac{-1}{d}\right)^{k+k'}\prod_{\delta|N}\left(\frac{\delta}{d}\right)^{\ell r_\delta - r'_\delta}, & c \text{ is even}.\end{cases}$$

in Proposition 4.6.1 for levels $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$. Our study requires formulas for r'_δ found in Table 4.2 and for k' found in Table 4.3. The formulas express r'_δ and k' in terms of $a_d = \lfloor \ell \operatorname{ord}_{\frac{1}{d}}(f)\rfloor$ for $d \mid N$. We group together N which require similar arguments.

$\boldsymbol{N = 1}$. By (4.6.2), we have $\ell r_1 - r'_1 \equiv 0 \pmod{24}$. Therefore, we have

$$\prod_{\delta|N}\nu_\eta\begin{pmatrix} a & b\delta \\ c/\delta & d\end{pmatrix}^{\ell r_\delta - r'_\delta} = \nu_\eta\begin{pmatrix} a & b \\ c & d\end{pmatrix}^{\ell r_1 - r'_1} = 1.$$

$\boldsymbol{N = 2, 4, 8}$. For these N, we have c even and d odd. We summarize here the relevant information from Table 4.2 and Table 4.3.

Table 4.9 Values from Table 4.2 and Table 4.3 for $N = 2, 4, 8$

N	$k+k'$	$\ell r_2 - r_2'$	$\ell r_8 - r_8'$
2	$k(\ell+1) - 4(a_1+a_2)$	$8(-a_1+a_2)$	–
4	$k(\ell+1) - 2(a_1+a_2+a_4)$	$4(-a_1+5a_2+a_4)$	–
8	$k(\ell+1) - (a_1+a_2+a_4+a_8)$	$2(-2a_2+5a_4-a_8)$	$2(-a_1+5a_2-2a_4)$

When $N \in \{2, 4\}$, we have $k + k' \equiv \ell r_2 - r_2' \equiv 0 \pmod 2$. In these cases, it follows from Proposition 4.6.1 that

$$\prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta - r_\delta'} = \left(\frac{-1}{d}\right)^{k+k'} \left(\frac{2}{d}\right)^{\ell r_2 - r_2'} = 1.$$

When $N = 8$, we have $\ell r_2 - r_2' \equiv \ell r_8 - r_8' \equiv 0 \pmod 2$. It follows in this case that

$$\prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta - r_\delta'} = \left(\frac{-1}{d}\right)^{k+k'} \left(\frac{2}{d}\right)^{\ell r_2 - r_2'} \left(\frac{8}{d}\right)^{\ell r_8 - r_8'} = \left(\frac{-1}{d}\right)^{k+k'}.$$

$\boldsymbol{N = 3, 9}$. We record the relevant information from Table 4.2 and Table 4.3:

Table 4.10 Values from Table 4.2 and Table 4.3 for $N = 3, 9$

N	$k+k'$	$\ell r_3 - r_3'$
3	$k(\ell+1) - 3(a_1+a_3)$	$3(-a_1+3a_3)$
9	$k(\ell+1) - (a_1+2a_3+a_9)$	$-a_1+10a_3-a_9$

In both instances, we see that $\ell r_3 - r_3' \equiv k + k' \pmod 2$. Hence, in both instances, Proposition 4.6.1 implies that

$$\prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta - r_\delta'} = \begin{cases} \left(\frac{d}{3}\right)^{k+k'}, & c \text{ odd}, \\ \left(\left(\frac{-1}{d}\right)\left(\frac{3}{d}\right)\right)^{k+k'} = \left(\frac{d}{3}\right)^{k+k'}, & c \text{ even}, \end{cases}$$

where the equality for c even holds by (2.3.4) and (2.3.5).

$\boldsymbol{N = 5}$. Table 4.3 gives that

$$k + k' = k(\ell+1) - 2(a_1 + a_5) \equiv 0 \pmod 2.$$

Proposition 4.6.1 gives

$$\prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta - r'_\delta} = \begin{cases} \left(\frac{d}{5}\right)^{\ell r_5 - r'_5}, & c \text{ odd}, \\ \left(\frac{-1}{d}\right)^{k+k'} \left(\frac{5}{d}\right)^{\ell r_5 - r'_5} = \left(\frac{d}{5}\right)^{\ell r_5 - r'_5}, & c \text{ even}, \end{cases}$$

where the equality for c even follows from (2.3.4). Hence, it remains to show, for all primes ℓ and for all minimal eta-quotients of level 5, that $\ell r_5 - r'_5 \equiv 0 \pmod 2$. There are 11 minimal eta-quotients $f_{5,\mathbf{v}}(z)$ with level 5. These form are given in Table **??**. We verify, for all primes $\ell \geq 5$ and all minimal eta-quotients of level 5, that $\ell r_5 - r'_5 \equiv 0 \pmod 2$.

***N* = 6**. The corresponding formulas in Table 4.2 and Table 4.3 are

$$\begin{aligned} k + k' &= k(\ell+1) - (a_1 + a_2 + a_3 + a_6) \\ \ell r_2 - r'_2 &= -3a_1 + 6a_2 + a_3 - 2a_6, \\ \ell r_3 - r'_3 &= -2a_1 + a_2 + 6a_3 - 3a_6, \\ \ell r_6 - r'_6 &= a_1 - 2a_2 - 3a_3 + 6a_6. \end{aligned}$$

We observe that

$$(\ell r_2 - r'_2) + (\ell r_6 + r'_6) \equiv 0 \pmod 2, \quad (\ell r_3 - r'_3) + (\ell r_6 + r'_6) \equiv k + k' \pmod 2. \tag{4.6.9}$$

Since c is even in this case, Proposition 4.6.1 now gives

$$\begin{aligned} \prod_{\delta|N} \nu_\eta \begin{pmatrix} a & b\delta \\ c/\delta & d \end{pmatrix}^{\ell r_\delta - r'_\delta} &= \left(\frac{-1}{d}\right)^{k+k'} \left(\frac{2}{d}\right)^{\ell r_2 - r'_2} \left(\frac{3}{d}\right)^{\ell r_3 - r'_3} \left(\frac{6}{d}\right)^{\ell r_6 - r'_6} \\ &= \left(\left(\frac{-1}{d}\right)\left(\frac{3}{d}\right)\right)^{k+k'} = \left(\frac{d}{3}\right)^{k+k'}, \end{aligned}$$

where the second equality holds by (4.6.9) and the third holds by (2.3.4).

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Appendix A

Maple Code For Computations

The following Maple code contains several functions which were helpful in both verifying and proving our main results. In addition to our code, we made use of several of the functions created by Frank Garvan in his two packages **qseries** and **ETA**. Most importantly, we use *gp2etaprod* to create eta-quotients and *etaprodtoqseries* to obtain the q-expansions for these eta-quotients. These packages and several others can be downloaded from Garvan's website at

`https://qseries.org/fgarvan/research.html`.

Also, instructions on how to use **qseries** and the other packages can be found on his website and in [12].

Our code is used to generate Eisenstein series and to apply various operators to the q-expansions of modular forms. In particular, we give code to calculate the action of the Hecke operator T_n, the U_m operator, the V_m operator, the θ^k operator, the Serre derivative, and the twisting operator on q-expansions. For each of these operators, we constructed a function which acts on eta-quotients and a separate function which acts on any q-expansion. The reason for handling eta-quotients separately is that the code runs much faster for these modular forms. In the following sections, we include the code for the operators on q-expansions. Then, we list code which is useful for generating Eisenstein series and code for analyzing the coefficients in a certain arithmetic progressions of a given q-series.

A.1 Hecke Operator

```
HECKE[HeckeSeriesWithChar] := proc (Series, wt, char, primes, N)
## Computes the first N coefficients of Series | Tp1 | Tp2 | ... | Tpn,
    where Series is a modular form of weight wt and character (char/*).
    Also, primes = [p1, p2, ..., pn].
  local poly, terms:
  poly := convert(Series,polynom):
  a := Array(PolynomialTools[CoefficientList](poly,q)):
  terms := ArrayTools[Size](a)[2]:
  for k from 1 to ArrayTools[Size](primes)[2] do:
    p := primes[k]:
    b := Array([seq(0,k=1..terms/p+2)]):
    for j from 0 to terms/p-1 do:
      if evalb(j mod p = 0) then
        b[j+1] := a[p*j+1] + NumberTheory[LegendreSymbol](char,p) * p^(
            wt-1) * a[j/p+1]:
      else
        b[j+1] := a[p*j+1]:
      fi:
    od:
    a := b:
    terms := terms/p:
  od:
  b := convert(b, list):
  RETURN(series(PolynomialTools[FromCoefficientList](b,q),q, min(floor(
      terms),N))):
end:
```

A.2 U_m Operator

```
HECKE[uOperatorSeries] := proc(Series, m, N)
## Computes the first N coefficients of Series | U_m
  local g, a, b, j, M, terms, i:
  g := convert(Series, polynom):
  a := Array(PolynomialTools[CoefficientList](g, q)):
  terms := ArrayTools[Size](a)[2]:
  M := min((terms-1)/m - 1,N):
  b := Array([seq(0, i = 1 .. M+2)]):
  for j from 0 to M do:
    b[j + 1] := a[j*m + 1]:
  od:
  b := convert(b, list):
  RETURN(series(PolynomialTools[FromCoefficientList](b, q), q, N+1)):
end:
```

A.3 V_m Operator

```
HECKE[vOperatorSeries] := proc(Series, m, N)
## Computes the first N coefficients of series | V_m
  local g, a, b, j, i, terms, M:
  g := convert(Series, polynom):
  a := Array(PolynomialTools[CoefficientList](g, q)):
  terms := ArrayTools[Size](a)[2]:
  M := min(m*terms,N):
  b := Array([seq(0, i = 1 .. M + m)]):
  for j from 0 to M/m do:
    b[m*j + 1] := a[j+1]:
  od:
  b := convert(b, list):
  RETURN(series(PolynomialTools[FromCoefficientList](b, q), q, N+1)):
end:
```

A.4 Twisting Operator

```
HECKE[twistingOperatorSeries] := proc(Series, chi, N)
## Computes the first N coefficients of Series twisted by chi
  local g, a, b, j, terms, M, i:
  g := convert(Series, polynom):
  a := Array(PolynomialTools[CoefficientList](g, q)):
  terms := ArrayTools[Size](a)[2]:
  M := min(terms, N):
  b := Array([seq(0, i = 1 .. M + 2)]):
  for j from 1 to M+1 do:
    b[j] := a[j]*chi(j-1):
  od:
  b := convert(b, list):
  RETURN(series(PolynomialTools[FromCoefficientList](b, q), q, N+1)):
end:
```

A.5 θ Operator

```
HECKE[thetaOperatorSeries] := proc(Series, N)
## Computes the first N coefficients of theta(Series)
  local g, a, b, j, terms, M, i:
  g := convert(Series, polynom):
  a := Array(PolynomialTools[CoefficientList](g, q)):
  terms := ArrayTools[Size](a)[2]:
  M := min(terms, N):
  b := Array([seq(0, i = 1 .. M + 2)]):
  for j from 1 to M do:
    b[j] := a[j]*(j-1):
  od:
  b := convert(b, list):
  RETURN(series(PolynomialTools[FromCoefficientList](b, q), q, N+1)):
end:
```

A.6 Serre Derivative

```
HECKE[SerreDerivativeSeries] := proc(Series, k, N)
## Computes the first N coefficients of the Serre Derivative of Series
   of weight k
  RETURN(series(HECKE[thetaOperatorSeries](Series,N) − k/12*HECKE[
      Eisenstein](2,N+1)*Series,q,N+1):
end:
```

A.7 Eisenstein Series

```
HECKE[divisor]:=proc(num, power)
## Computes the sum of divisors of num raised to power
   local total:
   total := 0:
   for i in NumberTheory[Divisors](num) do:
     total := total + i^power;
   od:
  RETURN(total):
end:
```

```
HECKE[Eisenstein] := proc(k, N)
## Computes the first N terms of the Eisenstein series of weight k
   local poly:
   poly := 1:
   for i from 1 to N+1 do:
     poly := poly − 2*k/bernoulli(k) * divisor(i, k−1) * q^i:
   od:
  RETURN(series(poly, q,N+1));
end:
```

```
HECKE[Eisenstein2] := proc(k, r, N)
## Computes the first N terms of the Eisenstein series of weight k, with
    argument z replaced by rz
  local poly:
  poly := 1:
  for i from 1 to N+1 do:
    poly := poly - 2*k/bernoulli(k) * divisor(i, k-1) * q^(r*i):
  od:
  RETURN(series(poly, q,N+1));
end:
```

A.8 Coefficients in Progressions

```
HECKE[SeriesProgression] := proc(Series, k, n, N)
## Returns the first N coefficients of Series in the arithmetic
    progression k modulo n
  local g, h, j, i:
  g := convert(Series, polynom):
  g := Array(PolynomialTools[CoefficientList](g, q)):
  h := Array([seq(0, j = 1 .. N)]):
  for i from 0 to N/n - 1 do:
    h[i*n + k + 1] := g[i*n + k + 1]:
  od:
  h := convert(h, list):
  RETURN(series(PolynomialTools[FromCoefficientList](h, q), q, N + 1)):
end:
```

Appendix B

Minimal Eta-Quotients

Let $N \in \{1, 2, 3, 4, 5, 6, 8, 9\}$. The following tables contain every minimal eta-quotient of level N. There are 11 forms of level 1, 95 forms of level 2, 35 forms of level 3, 383 forms of level 4, 11 forms of level 5, 287 forms of level 6, 383 forms of level 8, and 35 forms of level 9. The forms are grouped with their corresponding forms given by Theorem 4.1.8. For each f in the table, let $v_{\mathfrak{s}} = 24\,\mathrm{ord}_{\mathfrak{s}}(f)$,

B.1 Minimal Eta-Quotients of Level 1

Table B.1 Minimal eta-quotients of level 1

f	D	k	v_∞
$\eta(z)^2$	12	1	2
$\eta(z)^{10}$	12	5	10
$\eta(z)^{14}$	12	7	14
$\eta(z)^{22}$	12	11	22
$\eta(z)^4$	6	2	4
$\eta(z)^{20}$	6	10	20
$\eta(z)^6$	4	3	6
$\eta(z)^{18}$	4	9	18
$\eta(z)^8$	3	4	8
$\eta(z)^{16}$	3	8	16
$\eta(z)^{12}$	2	6	12

B.2 Minimal Eta-Quotients of Level 2

Table B.2 Minimal eta-quotients of level 2

f	D	k	v_∞	v_0	f	D	k	v_∞	v_0
$\frac{\eta(z)^{15}}{\eta(2z)^7}$	24	4	1	23	$\frac{\eta(z)^{11}}{\eta(2z)^5}$	24	3	1	17
$\frac{\eta(z)^{11}}{\eta(2z)^3}$	24	4	5	19	$\frac{\eta(z)^7}{\eta(2z)}$	24	3	5	13
$\frac{\eta(z)^9}{\eta(2z)}$	24	4	7	17	$\frac{\eta(z)^{13}}{\eta(2z)^3}$	24	5	7	23
$\eta(z)^5\eta(2z)^3$	24	4	11	13	$\eta(z)^9\eta(2z)$	24	5	11	19
$\eta(z)^3\eta(2z)^5$	24	4	13	11	$\frac{\eta(2z)^7}{\eta(z)}$	24	3	13	5
$\frac{\eta(2z)^9}{\eta(z)}$	24	4	17	7	$\frac{\eta(2z)^{11}}{\eta(z)^5}$	24	3	17	1
$\frac{\eta(2z)^{11}}{\eta(z)^3}$	24	4	19	5	$\eta(z)\eta(2z)^9$	24	5	19	11
$\frac{\eta(2z)^{15}}{\eta(z)^7}$	24	4	23	1	$\frac{\eta(2z)^{13}}{\eta(z)^3}$	24	5	23	7
$\frac{\eta(z)^7}{\eta(2z)^3}$	24	2	1	11	$\frac{\eta(z)^3}{\eta(2z)}$	24	1	1	5
$\eta(z)^3\eta(2z)$	24	2	5	7	$\frac{\eta(2z)^3}{\eta(z)}$	24	1	5	1
$\eta(z)\eta(2z)^3$	24	2	7	5	$\eta(z)^5\eta(2z)$	24	3	7	11
$\frac{\eta(2z)^7}{\eta(z)^3}$	24	2	11	1	$\eta(z)\eta(2z)^5$	24	3	11	7
$\eta(z)^{11}\eta(2z)$	24	6	13	23	$\eta(z)^7\eta(2z)^3$	24	5	13	17
$\eta(z)^7\eta(2z)^5$	24	6	17	19	$\eta(z)^3\eta(2z)^7$	24	5	17	13
$\eta(z)^5\eta(2z)^7$	24	6	19	17	$\eta(z)^9\eta(2z)^5$	24	7	19	23
$\eta(z)\eta(2z)^{11}$	24	6	23	13	$\eta(z)^5\eta(2z)^9$	24	7	23	19
$\frac{\eta(z)^{14}}{\eta(2z)^6}$	12	4	2	22	$\frac{\eta(z)^6}{\eta(2z)^2}$	12	2	2	10
$\eta(z)^6\eta(2z)^2$	12	4	10	14	$\frac{\eta(2z)^6}{\eta(z)^2}$	12	2	10	2
$\eta(z)^2\eta(2z)^6$	12	4	14	10	$\eta(z)^{10}\eta(2z)^2$	12	6	14	22
$\frac{\eta(2z)^{14}}{\eta(z)^6}$	12	4	22	2	$\eta(z)^2\eta(2z)^{10}$	12	6	22	14
$\frac{\eta(z)^{10}}{\eta(2z)^4}$	12	3	2	16	$\frac{\eta(2z)^{10}}{\eta(z)^4}$	12	3	16	2
$\eta(z)^2\eta(2z)^4$	12	3	10	8	$\eta(z)^4\eta(2z)^2$	12	3	8	10
$\eta(z)^6\eta(2z)^4$	12	5	14	16	$\eta(z)^4\eta(2z)^6$	12	5	16	14
$\frac{\eta(2z)^{12}}{\eta(z)^2}$	12	5	22	8	$\frac{\eta(z)^{12}}{\eta(2z)^2}$	12	5	8	22
$\eta(z)^2$	12	1	2	4	$\eta(2z)^2$	12	1	4	2
$\eta(z)^{10}$	12	5	10	20	$\eta(2z)^{10}$	12	5	20	10
$\frac{\eta(2z)^8}{\eta(z)^2}$	12	3	14	4	$\frac{\eta(z)^8}{\eta(2z)^2}$	12	3	4	14
$\eta(z)^6\eta(2z)^8$	12	7	22	20	$\eta(z)^8\eta(2z)^6$	12	7	20	22

Table B.2 cont. Minimal eta-quotients of level 2

f	D	k	v_∞	v_0	f	D	k	v_∞	v_0
$\frac{\eta(z)^{13}}{\eta(2z)^5}$	8	4	3	21	$\frac{\eta(z)^9}{\eta(2z)^3}$	8	3	3	15
$\eta(z)^7\eta(2z)$	8	4	9	15	$\frac{\eta(z)^{11}}{\eta(2z)}$	8	5	9	21
$\eta(z)\eta(2z)^7$	8	4	15	9	$\frac{\eta(2z)^9}{\eta(z)^3}$	8	3	15	3
$\frac{\eta(2z)^{13}}{\eta(z)^5}$	8	4	21	3	$\frac{\eta(2z)^{11}}{\eta(z)}$	8	5	21	9
$\frac{\eta(z)^5}{\eta(2z)}$	8	2	3	9	$\eta(z)\eta(2z)$	8	1	3	3
$\frac{\eta(2z)^5}{\eta(z)}$	8	2	9	3	$\eta(z)^3\eta(2z)^3$	8	3	9	9
$\eta(z)^9\eta(2z)^3$	8	6	15	21	$\eta(z)^5\eta(2z)^5$	8	5	15	15
$\eta(z)^3\eta(2z)^9$	8	6	21	15	$\eta(z)^7\eta(2z)^7$	8	7	21	21
$\frac{\eta(z)^{12}}{\eta(2z)^4}$	6	4	4	20	$\eta(z)^4$	6	2	4	8
$\frac{\eta(2z)^{12}}{\eta(z)^4}$	6	4	20	4	$\eta(z)^4\eta(2z)^8$	6	6	20	16
$\eta(2z)^4$	6	2	8	4	$\frac{\eta(z)^{10}}{\eta(2z)^2}$	4	4	6	18
$\eta(z)^8\eta(2z)^4$	6	6	16	20	$\frac{\eta(2z)^{10}}{\eta(z)^2}$	4	4	18	6
$\frac{\eta(z)^4}{\eta(2z)^2}$	4	1	0	6	$\frac{\eta(2z)^4}{\eta(z)^2}$	4	1	6	0
$\frac{\eta(z)^{12}}{\eta(2z)^6}$	4	3	0	18	$\frac{\eta(2z)^{12}}{\eta(z)^6}$	4	3	18	0
$\eta(z)^6$	4	3	6	12	$\eta(2z)^6$	4	3	12	6
$\eta(z)^2\eta(2z)^8$	4	5	18	12	$\eta(z)^8\eta(2z)^2$	4	5	12	18
$\eta(z)^2\eta(2z)^2$	4	2	6	6	$\eta(z)^8$	3	4	8	16
$\eta(z)^6\eta(2z)^6$	4	6	18	18	$\eta(2z)^8$	3	4	16	8
$\eta(z)^4\eta(2z)^4$	2	4	12	12	$\frac{\eta(z)^8}{\eta(2z)^4}$	2	2	0	12
$\frac{\eta(2z)^8}{\eta(z)^4}$	2	2	12	0					

B.3 Minimal Eta-Quotients of Level 3

Table B.3 Minimal eta-quotients of level 3

f	D	k	v_∞	v_0	f	D	k	v_∞	v_0
$\frac{\eta(z)^8}{\eta(3z)^2}$	12	3	2	22	$\frac{\eta(z)^5}{\eta(3z)}$	12	2	2	14
$\eta(z)^4\eta(3z)^2$	12	3	10	14	$\eta(z)^7\eta(3z)$	12	4	10	22
$\eta(z)^2\eta(3z)^4$	12	3	14	10	$\frac{\eta(3z)^5}{\eta(z)}$	12	2	14	2
$\frac{\eta(3z)^8}{\eta(z)^2}$	12	3	22	2	$\eta(z)\eta(3z)^7$	12	4	22	10
$\eta(z)^2$	12	1	2	6	$\eta(3z)^2$	12	1	6	2
$\eta(z)\eta(3z)^3$	12	2	10	6	$\eta(z)^3\eta(3z)$	12	2	6	10
$\eta(z)^5\eta(3z)^3$	12	4	14	18	$\eta(z)^3\eta(3z)^5$	12	4	18	14
$\eta(z)^4\eta(3z)^6$	12	5	22	18	$\eta(z)^6\eta(3z)^4$	12	5	18	22
$\frac{\eta(z)^7}{\eta(3z)}$	6	3	4	20	$\eta(z)^4$	6	2	4	12
$\frac{\eta(3z)^7}{\eta(z)}$	6	3	20	4	$\eta(z)^2\eta(3z)^6$	6	4	20	12
$\eta(3z)^4$	6	2	12	4	$\eta(z)\eta(3z)$	6	1	4	4
$\eta(z)^6\eta(3z)^2$	6	4	12	20	$\eta(z)^5\eta(3z)^5$	6	5	20	20
$\eta(z)^6$	4	3	6	18	$\eta(z)^5\eta(3z)$	3	3	8	16
$\eta(3z)^6$	4	3	18	6	$\eta(z)\eta(3z)^5$	3	3	16	8
$\eta(z)^2\eta(3z)^2$	3	2	8	8	$\frac{\eta(z)^3}{\eta(3z)}$	3	1	0	8
$\eta(z)^4\eta(3z)^4$	3	4	16	16	$\frac{\eta(z)^6}{\eta(3z)^2}$	3	2	0	16
$\frac{\eta(3z)^3}{\eta(z)}$	3	1	8	0	$\eta(z)^3\eta(3z)^3$	2	3	12	12
$\frac{\eta(3z)^6}{\eta(z)^2}$	3	2	16	0					

B.4 Minimal Eta-Quotients of Level 4

Table B.4 Minimal eta-quotients of level 4

f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0
$\frac{\eta(z)^3\eta(2z)^7}{\eta(4z)^4}$	24	3	1	13	22	$\frac{\eta(2z)^7\eta(4z)^3}{\eta(z)^4}$	24	3	22	13	1
$\frac{\eta(2z)^{11}}{\eta(z)\eta(4z)^4}$	24	3	5	17	14	$\frac{\eta(2z)^{11}}{\eta(z)^4\eta(4z)}$	24	3	14	17	5
$\frac{\eta(2z)^{13}}{\eta(z)^3\eta(4z)^4}$	24	3	7	19	10	$\frac{\eta(2z)^{13}}{\eta(z)^4\eta(4z)^3}$	24	3	10	19	7
$\frac{\eta(2z)^{17}}{\eta(z)^7\eta(4z)^4}$	24	3	11	23	2	$\frac{\eta(2z)^{17}}{\eta(z)^4\eta(4z)^7}$	24	3	2	23	11
$\frac{\eta(z)^7\eta(4z)^4}{\eta(2z)^5}$	24	3	13	1	22	$\frac{\eta(z)^4\eta(4z)^7}{\eta(2z)^5}$	24	3	22	1	13
$\frac{\eta(z)^3\eta(4z)^4}{\eta(2z)}$	24	3	17	5	14	$\frac{\eta(z)^4\eta(4z)^3}{\eta(2z)}$	24	3	14	5	17
$\eta(z)\eta(2z)\eta(4z)^4$	24	3	19	7	10	$\eta(z)^4\eta(2z)\eta(4z)$	24	3	10	7	19
$\frac{\eta(2z)^5\eta(4z)^4}{\eta(z)^3}$	24	3	23	11	2	$\frac{\eta(z)^4\eta(2z)^5}{\eta(4z)^3}$	24	3	2	11	23
$\frac{\eta(2z)^{13}}{\eta(z)\eta(4z)^6}$	24	3	1	19	16	$\frac{\eta(2z)^{13}}{\eta(z)^6\eta(4z)}$	24	3	16	19	1
$\frac{\eta(2z)^{17}}{\eta(z)^5\eta(4z)^6}$	24	3	5	23	8	$\frac{\eta(2z)^{17}}{\eta(z)^6\eta(4z)^5}$	24	3	8	23	5
$\frac{\eta(z)\eta(2z)^7}{\eta(4z)^2}$	24	3	7	13	16	$\frac{\eta(2z)^7\eta(4z)}{\eta(z)^2}$	24	3	16	13	7
$\frac{\eta(2z)^{11}}{\eta(z)^3\eta(4z)^2}$	24	3	11	17	8	$\frac{\eta(2z)^{11}}{\eta(z)^2\eta(4z)^3}$	24	3	8	17	11
$\eta(z)^3\eta(2z)\eta(4z)^2$	24	3	13	7	16	$\eta(z)^2\eta(2z)\eta(4z)^3$	24	3	16	7	13
$\frac{\eta(2z)^5\eta(4z)^2}{\eta(z)}$	24	3	17	11	8	$\frac{\eta(z)^2\eta(2z)^5}{\eta(4z)}$	24	3	8	11	17
$\frac{\eta(z)^5\eta(4z)^6}{\eta(2z)^5}$	24	3	19	1	16	$\frac{\eta(z)^6\eta(4z)^5}{\eta(2z)^5}$	24	3	16	1	19
$\frac{\eta(z)\eta(4z)^6}{\eta(2z)}$	24	3	23	5	8	$\frac{\eta(z)^6\eta(4z)}{\eta(2z)}$	24	3	8	5	23
$\frac{\eta(2z)^{16}}{\eta(z)^3\eta(4z)^7}$	24	3	1	22	13	$\frac{\eta(z)\eta(2z)^{10}}{\eta(4z)^5}$	24	3	1	16	19
$\frac{\eta(z)\eta(2z)^8}{\eta(4z)^3}$	24	3	5	14	17	$\frac{\eta(z)^5\eta(2z)^2}{\eta(4z)}$	24	3	5	8	23
$\frac{\eta(z)^3\eta(2z)^4}{\eta(4z)}$	24	3	7	10	19	$\frac{\eta(2z)^{10}}{\eta(z)\eta(4z)^3}$	24	3	7	16	13
$\frac{\eta(z)^7\eta(4z)^3}{\eta(2z)^4}$	24	3	11	2	23	$\eta(z)^3\eta(2z)^2\eta(4z)$	24	3	11	8	17
$\frac{\eta(2z)^{16}}{\eta(z)^7\eta(4z)^3}$	24	3	13	22	1	$\frac{\eta(2z)^{10}}{\eta(z)^3\eta(4z)}$	24	3	13	16	7
$\frac{\eta(2z)^8\eta(4z)}{\eta(z)^3}$	24	3	17	14	5	$\eta(z)\eta(2z)^2\eta(4z)^3$	24	3	17	8	11
$\frac{\eta(2z)^4\eta(4z)^3}{\eta(z)}$	24	3	19	10	7	$\frac{\eta(2z)^{10}\eta(4z)}{\eta(z)^5}$	24	3	19	16	1
$\frac{\eta(z)^3\eta(4z)^7}{\eta(2z)^4}$	24	3	23	2	11	$\frac{\eta(2z)^2\eta(4z)^5}{\eta(z)}$	24	3	23	8	5

Table B.4 cont. Minimal eta-quotients of level 4

f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0
$\frac{\eta(z)^7}{\eta(2z)^3}$	24	2	1	1	22	$\frac{\eta(4z)^7}{\eta(2z)^3}$	24	2	22	1	1
$\eta(z)^3\eta(2z)$	24	2	5	5	14	$\eta(2z)\eta(4z)^3$	24	2	14	5	5
$\eta(z)\eta(2z)^3$	24	2	7	7	10	$\eta(2z)^3\eta(4z)$	24	2	10	7	7
$\frac{\eta(2z)^7}{\eta(z)^3}$	24	2	11	11	2	$\frac{\eta(2z)^7}{\eta(4z)^3}$	24	2	2	11	11
$\eta(z)^3\eta(2z)^5$	24	4	13	13	22	$\eta(2z)^5\eta(4z)^3$	24	4	22	13	13
$\frac{\eta(2z)^9}{\eta(z)}$	24	4	17	17	14	$\frac{\eta(2z)^9}{\eta(4z)}$	24	4	14	17	17
$\frac{\eta(2z)^{11}}{\eta(z)^3}$	24	4	19	19	10	$\frac{\eta(2z)^{11}}{\eta(4z)^3}$	24	4	10	19	19
$\frac{\eta(2z)^{15}}{\eta(z)^7}$	24	4	23	23	2	$\frac{\eta(2z)^{15}}{\eta(4z)^7}$	24	4	2	23	23
$\frac{\eta(z)^3\eta(2z)^3}{\eta(4z)^2}$	24	2	1	7	16	$\frac{\eta(2z)^3\eta(4z)^3}{\eta(z)^2}$	24	2	16	7	1
$\frac{\eta(2z)^7}{\eta(z)\eta(4z)^2}$	24	2	5	11	8	$\frac{\eta(2z)^7}{\eta(z)^2\eta(4z)}$	24	2	8	11	5
$\frac{\eta(z)^5\eta(4z)^2}{\eta(2z)^3}$	24	2	7	1	16	$\frac{\eta(z)^2\eta(4z)^5}{\eta(2z)^3}$	24	2	16	1	7
$\eta(z)\eta(2z)\eta(4z)^2$	24	2	11	5	8	$\eta(z)^2\eta(2z)\eta(4z)$	24	2	8	5	11
$\frac{\eta(2z)^{11}}{\eta(z)\eta(4z)^2}$	24	4	13	19	16	$\frac{\eta(2z)^{11}}{\eta(z)^2\eta(4z)}$	24	4	16	19	13
$\frac{\eta(2z)^{15}}{\eta(z)^5\eta(4z)^2}$	24	4	17	23	8	$\frac{\eta(2z)^{15}}{\eta(z)^2\eta(4z)^5}$	24	4	8	23	17
$\eta(z)\eta(2z)^5\eta(4z)^2$	24	4	19	13	16	$\eta(z)^2\eta(2z)^5\eta(4z)$	24	4	16	13	19
$\frac{\eta(2z)^9\eta(4z)^2}{\eta(z)^3}$	24	4	23	17	8	$\frac{\eta(z)^2\eta(2z)^9}{\eta(4z)^3}$	24	4	8	17	23
$\frac{\eta(2z)^9}{\eta(z)\eta(4z)^4}$	24	2	1	13	10	$\frac{\eta(2z)^9}{\eta(z)^4\eta(4z)}$	24	2	10	13	1
$\frac{\eta(2z)^{13}}{\eta(z)^5\eta(4z)^4}$	24	2	5	17	2	$\frac{\eta(2z)^{13}}{\eta(z)^4\eta(4z)^5}$	24	2	2	17	5
$\frac{\eta(z)\eta(2z)^{11}}{\eta(4z)^4}$	24	4	7	19	22	$\frac{\eta(2z)^{11}\eta(4z)}{\eta(z)^4}$	24	4	22	19	7
$\frac{\eta(2z)^{15}}{\eta(z)^3\eta(4z)^4}$	24	4	11	23	14	$\frac{\eta(2z)^{15}}{\eta(z)^4\eta(4z)^3}$	24	4	14	23	11
$\frac{\eta(z)^3\eta(4z)^4}{\eta(2z)^3}$	24	2	13	1	10	$\frac{\eta(z)^4\eta(4z)^3}{\eta(2z)^3}$	24	2	10	1	13
$\frac{\eta(2z)\eta(4z)^4}{\eta(z)}$	24	2	17	5	2	$\frac{\eta(z)^4\eta(2z)}{\eta(4z)}$	24	2	2	5	17
$\frac{\eta(z)^5\eta(4z)^4}{\eta(2z)}$	24	4	19	7	22	$\frac{\eta(z)^4\eta(4z)^5}{\eta(2z)}$	24	4	22	7	19
$\eta(z)\eta(2z)^3\eta(4z)^4$	24	4	23	11	14	$\eta(z)^4\eta(2z)^3\eta(4z)$	24	4	14	11	23
$\frac{\eta(2z)^{15}}{\eta(z)^5\eta(4z)^6}$	24	2	1	19	4	$\frac{\eta(2z)^{15}}{\eta(z)^6\eta(4z)^5}$	24	2	4	19	1
$\frac{\eta(2z)^{15}}{\eta(z)\eta(4z)^6}$	24	4	5	23	20	$\frac{\eta(2z)^{15}}{\eta(z)^6\eta(4z)}$	24	4	20	23	5
$\frac{\eta(2z)^9}{\eta(z)^3\eta(4z)^2}$	24	2	7	13	4	$\frac{\eta(2z)^9}{\eta(z)^2\eta(4z)^3}$	24	2	4	13	7
$\frac{\eta(z)\eta(2z)^9}{\eta(4z)^2}$	24	4	11	17	20	$\frac{\eta(2z)^9\eta(4z)}{\eta(z)^2}$	24	4	20	17	11
$\frac{\eta(2z)^3\eta(4z)^2}{\eta(z)}$	24	2	13	7	4	$\frac{\eta(z)^2\eta(2z)^3}{\eta(4z)}$	24	2	4	7	13
$\eta(z)^3\eta(2z)^3\eta(4z)^2$	24	4	17	11	20	$\eta(z)^2\eta(2z)^3\eta(4z)^3$	24	4	20	11	17
$\frac{\eta(z)\eta(4z)^6}{\eta(2z)^3}$	24	2	19	1	4	$\frac{\eta(z)^6\eta(4z)}{\eta(2z)^3}$	24	2	4	1	19
$\frac{\eta(z)^5\eta(4z)^6}{\eta(2z)^3}$	24	4	23	5	20	$\frac{\eta(z)^6\eta(4z)^5}{\eta(2z)^3}$	24	4	20	5	23

Table B.4 cont. Minimal eta-quotients of level 4

f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0
$\frac{\eta(z)\eta(2z)^6}{\eta(4z)^3}$	24	2	1	10	13	$\frac{\eta(2z)^{18}}{\eta(z)^7\eta(4z)^7}$	24	2	1	22	1
$\frac{\eta(z)^5\eta(4z)}{\eta(2z)^2}$	24	2	5	2	17	$\frac{\eta(2z)^{10}}{\eta(z)^3\eta(4z)^3}$	24	2	5	14	5
$\frac{\eta(2z)^{14}}{\eta(z)\eta(4z)^5}$	24	4	7	22	19	$\frac{\eta(2z)^6}{\eta(z)\eta(4z)}$	24	2	7	10	7
$\frac{\eta(z)^3\eta(2z)^6}{\eta(4z)}$	24	4	11	14	23	$\frac{\eta(z)^3\eta(4z)^3}{\eta(2z)^2}$	24	2	11	2	11
$\frac{\eta(2z)^6\eta(4z)}{\eta(z)^3}$	24	2	13	10	1	$\frac{\eta(2z)^{14}}{\eta(z)^3\eta(4z)^3}$	24	4	13	22	13
$\frac{\eta(z)\eta(4z)^5}{\eta(2z)^2}$	24	2	17	2	5	$\eta(z)\eta(2z)^6\eta(4z)$	24	4	17	14	17
$\frac{\eta(2z)^{14}}{\eta(z)^5\eta(4z)}$	24	4	19	22	7	$\eta(z)^3\eta(2z)^2\eta(4z)^3$	24	4	19	10	19
$\frac{\eta(2z)^6\eta(4z)^3}{\eta(z)}$	24	4	23	14	11	$\frac{\eta(z)^7\eta(4z)^7}{\eta(2z)^6}$	24	4	23	2	23
$\frac{\eta(z)^5}{\eta(4z)}$	24	2	1	4	19	$\frac{\eta(2z)^{12}}{\eta(z)^3\eta(4z)^5}$	24	2	1	16	7
$\frac{\eta(z)\eta(2z)^{12}}{\eta(4z)^5}$	24	4	5	20	23	$\frac{\eta(z)\eta(2z)^4}{\eta(4z)}$	24	2	5	8	11
$\eta(z)^3\eta(4z)$	24	2	7	4	13	$\frac{\eta(2z)^{12}}{\eta(z)^5\eta(4z)^3}$	24	2	7	16	1
$\frac{\eta(2z)^{12}}{\eta(z)\eta(4z)^3}$	24	4	11	20	17	$\frac{\eta(2z)^4\eta(4z)}{\eta(z)}$	24	2	11	8	5
$\eta(z)\eta(4z)^3$	24	2	13	4	7	$\frac{\eta(z)\eta(2z)^8}{\eta(4z)}$	24	4	13	16	19
$\frac{\eta(2z)^{12}}{\eta(z)^3\eta(4z)}$	24	4	17	20	11	$\eta(z)^5\eta(4z)^3$	24	4	17	8	23
$\frac{\eta(4z)^5}{\eta(z)}$	24	2	19	4	1	$\frac{\eta(2z)^8\eta(4z)}{\eta(z)}$	24	4	19	16	13
$\frac{\eta(2z)^{12}\eta(4z)}{\eta(z)^5}$	24	4	23	20	5	$\eta(z)^3\eta(4z)^5$	24	4	23	8	17
$\frac{\eta(z)^3}{\eta(2z)}$	24	1	1	1	10	$\frac{\eta(4z)^3}{\eta(2z)}$	24	1	10	1	1
$\frac{\eta(2z)^3}{\eta(z)}$	24	1	5	5	2	$\frac{\eta(2z)^3}{\eta(4z)}$	24	1	2	5	5
$\eta(z)^5\eta(2z)$	24	3	7	7	22	$\eta(2z)\eta(4z)^5$	24	3	22	7	7
$\eta(z)\eta(2z)^5$	24	3	11	11	14	$\eta(2z)^5\eta(4z)$	24	3	14	11	11
$\frac{\eta(2z)^7}{\eta(z)}$	24	3	13	13	10	$\frac{\eta(2z)^7}{\eta(4z)}$	24	3	10	13	13
$\frac{\eta(2z)^{11}}{\eta(z)^5}$	24	3	17	17	2	$\frac{\eta(2z)^{11}}{\eta(4z)^5}$	24	3	2	17	17
$\eta(z)\eta(2z)^9$	24	5	19	19	22	$\eta(2z)^9\eta(4z)$	24	5	22	19	19
$\frac{\eta(2z)^{13}}{\eta(z)^3}$	24	5	23	23	14	$\frac{\eta(2z)^{13}}{\eta(4z)^3}$	24	5	14	23	23
$\frac{\eta(2z)^5}{\eta(z)\eta(4z)^2}$	24	1	1	7	4	$\frac{\eta(2z)^5}{\eta(z)^2\eta(4z)}$	24	1	4	7	1
$\frac{\eta(z)^3\eta(2z)^5}{\eta(4z)^2}$	24	3	5	11	20	$\frac{\eta(2z)^5\eta(4z)^3}{\eta(z)^2}$	24	3	20	11	5
$\frac{\eta(z)\eta(4z)^2}{\eta(2z)}$	24	1	7	1	4	$\frac{\eta(z)^2\eta(4z)}{\eta(2z)}$	24	1	4	1	7
$\frac{\eta(z)^5\eta(4z)^2}{\eta(2z)}$	24	3	11	5	20	$\frac{\eta(z)^2\eta(4z)^5}{\eta(2z)}$	24	3	20	5	11
$\frac{\eta(2z)^{13}}{\eta(z)^5\eta(4z)^2}$	24	3	13	19	4	$\frac{\eta(2z)^{13}}{\eta(z)^2\eta(4z)^5}$	24	3	4	19	13
$\frac{\eta(2z)^{13}}{\eta(z)\eta(4z)^2}$	24	5	17	23	20	$\frac{\eta(2z)^{13}}{\eta(z)^2\eta(4z)}$	24	5	20	23	17
$\frac{\eta(2z)^7\eta(4z)^2}{\eta(z)^3}$	24	3	19	13	4	$\frac{\eta(z)^2\eta(2z)^7}{\eta(4z)^3}$	24	3	4	13	19
$\eta(z)\eta(2z)^7\eta(4z)^2$	24	5	23	17	20	$\eta(z)^2\eta(2z)^7\eta(4z)$	24	5	20	17	23

Table B.4 cont. Minimal eta-quotients of level 4

f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0
$\frac{\eta(2z)^8}{\eta(z)^3\eta(4z)^3}$	24	1	1	10	1	$\frac{\eta(z)\eta(2z)^2}{\eta(4z)}$	24	1	1	4	7
$\eta(z)\eta(4z)$	24	1	5	2	5	$\frac{\eta(2z)^{14}}{\eta(z)^3\eta(4z)^5}$	24	3	5	20	11
$\frac{\eta(2z)^{16}}{\eta(z)^5\eta(4z)^5}$	24	3	7	22	7	$\frac{\eta(2z)^2\eta(4z)}{\eta(z)}$	24	1	7	4	1
$\frac{\eta(2z)^8}{\eta(z)\eta(4z)}$	24	3	11	14	11	$\frac{\eta(2z)^{14}}{\eta(z)^5\eta(4z)^3}$	24	3	11	20	5
$\eta(z)\eta(2z)^4\eta(4z)$	24	3	13	10	13	$\frac{\eta(z)^5\eta(4z)^3}{\eta(2z)^2}$	24	3	13	4	19
$\frac{\eta(z)^5\eta(4z)^5}{\eta(2z)^4}$	24	3	17	2	17	$\frac{\eta(z)\eta(2z)^{10}}{\eta(4z)}$	24	5	17	20	23
$\frac{\eta(2z)^{12}}{\eta(z)\eta(4z)}$	24	5	19	22	19	$\frac{\eta(z)^3\eta(4z)^5}{\eta(2z)^2}$	24	3	19	4	13
$\eta(z)^3\eta(2z)^4\eta(4z)^3$	24	5	23	14	23	$\frac{\eta(2z)^{10}\eta(4z)}{\eta(z)}$	24	5	23	20	17
$\frac{\eta(z)^2\eta(2z)^8}{\eta(4z)^4}$	12	3	2	14	20	$\frac{\eta(2z)^8\eta(4z)^2}{\eta(z)^4}$	12	3	20	14	2
$\frac{\eta(2z)^{16}}{\eta(z)^6\eta(4z)^4}$	12	3	10	22	4	$\frac{\eta(2z)^{16}}{\eta(z)^4\eta(4z)^6}$	12	3	4	22	10
$\frac{\eta(z)^6\eta(4z)^4}{\eta(2z)^4}$	12	3	14	2	20	$\frac{\eta(z)^4\eta(4z)^6}{\eta(2z)^4}$	12	3	20	2	14
$\frac{\eta(2z)^4\eta(4z)^4}{\eta(z)^2}$	12	3	22	10	4	$\frac{\eta(z)^4\eta(2z)^4}{\eta(4z)^2}$	12	3	4	10	22
$\frac{\eta(2z)^{14}}{\eta(z)^2\eta(4z)^6}$	12	3	2	20	14	$\frac{\eta(z)^2\eta(2z)^4}{\eta(4z)^2}$	12	2	2	8	14
$\frac{\eta(z)^6\eta(4z)^2}{\eta(2z)^2}$	12	3	10	4	22	$\frac{\eta(z)^2\eta(2z)^8}{\eta(4z)^2}$	12	4	10	16	22
$\frac{\eta(2z)^{14}}{\eta(z)^6\eta(4z)^2}$	12	3	14	20	2	$\frac{\eta(2z)^4\eta(4z)^2}{\eta(z)^2}$	12	2	14	8	2
$\frac{\eta(z)^2\eta(4z)^6}{\eta(2z)^2}$	12	3	22	4	10	$\frac{\eta(2z)^8\eta(4z)^2}{\eta(z)^2}$	12	4	22	16	10
$\frac{\eta(z)^6}{\eta(2z)^2}$	12	2	2	2	20	$\frac{\eta(4z)^6}{\eta(2z)^2}$	12	2	20	2	2
$\frac{\eta(2z)^6}{\eta(z)^2}$	12	2	10	10	4	$\frac{\eta(2z)^6}{\eta(4z)^2}$	12	2	4	10	10
$\eta(z)^2\eta(2z)^6$	12	4	14	14	20	$\eta(2z)^6\eta(4z)^2$	12	4	20	14	14
$\frac{\eta(2z)^{14}}{\eta(z)^6}$	12	4	22	22	4	$\frac{\eta(2z)^{14}}{\eta(4z)^6}$	12	4	4	22	22
$\frac{\eta(2z)^{10}}{\eta(z)^2\eta(4z)^4}$	12	2	2	14	8	$\frac{\eta(2z)^{10}}{\eta(z)^4\eta(4z)^2}$	12	2	8	14	2
$\frac{\eta(2z)^{14}}{\eta(z)^2\eta(4z)^4}$	12	4	10	22	16	$\frac{\eta(2z)^{14}}{\eta(z)^4\eta(4z)^2}$	12	4	16	22	10
$\frac{\eta(z)^2\eta(4z)^4}{\eta(2z)^2}$	12	2	14	2	8	$\frac{\eta(z)^4\eta(4z)^2}{\eta(2z)^2}$	12	2	8	2	14
$\eta(z)^2\eta(2z)^2\eta(4z)^4$	12	4	12	10	16	$\eta(z)^4\eta(2z)^2\eta(4z)^2$	12	4	16	10	12
$\frac{\eta(2z)^{16}}{\eta(z)^6\eta(4z)^6}$	12	2	2	20	2	$\frac{\eta(2z)^6}{\eta(z)^2\eta(4z)^2}$	12	1	2	8	2
$\eta(z)^2\eta(4z)^2$	12	2	10	4	10	$\frac{\eta(2z)^{10}}{\eta(z)^2\eta(4z)^2}$	12	3	10	16	10
$\frac{\eta(2z)^{12}}{\eta(z)^2\eta(4z)^2}$	12	4	14	20	14	$\eta(z)^2\eta(2z)^2\eta(4z)^2$	12	3	14	8	14
$\frac{\eta(z)^6\eta(4z)^6}{\eta(2z)^4}$	12	4	22	4	22	$\eta(z)^2\eta(2z)^6\eta(4z)^2$	12	5	22	16	22
$\eta(z)^2$	12	1	2	2	8	$\eta(4z)^2$	12	1	8	2	2
$\eta(z)^2\eta(2z)^4$	12	3	10	10	16	$\eta(2z)^4\eta(4z)^2$	12	3	16	10	10
$\frac{\eta(2z)^8}{\eta(z)^2}$	12	3	14	14	8	$\frac{\eta(2z)^8}{\eta(4z)^2}$	12	3	8	14	14
$\frac{\eta(2z)^{12}}{\eta(z)^2}$	12	5	22	22	16	$\frac{\eta(2z)^{12}}{\eta(4z)^2}$	12	5	16	22	22

Table B.4 cont. Minimal eta-quotients of level 4

f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0
$\frac{\eta(z)\eta(2z)^9}{\eta(4z)^4}$	8	3	3	15	18	$\frac{\eta(2z)^9\eta(4z)}{\eta(z)^4}$	8	3	18	15	3
$\frac{\eta(2z)^{14}}{\eta(z)^5\eta(4z)^4}$	8	3	9	21	6	$\frac{\eta(2z)^{14}}{\eta(z)^4\eta(4z)^5}$	8	3	6	21	9
$\frac{\eta(z)^5\eta(4z)^4}{\eta(2z)^3}$	8	3	15	3	18	$\frac{\eta(z)^4\eta(4z)^5}{\eta(2z)^3}$	8	3	18	3	15
$\frac{\eta(2z)^3\eta(4z)^4}{\eta(z)}$	8	3	21	9	6	$\frac{\eta(z)^4\eta(2z)^3}{\eta(4z)}$	8	3	6	9	21
$\frac{\eta(2z)^{15}}{\eta(z)^3\eta(4z)^6}$	8	3	3	21	12	$\frac{\eta(2z)^{15}}{\eta(z)^6\eta(4z)^3}$	8	3	12	21	3
$\frac{\eta(2z)^9}{\eta(z)\eta(4z)^2}$	8	3	9	15	12	$\frac{\eta(2z)^9}{\eta(z)^2\eta(4z)}$	8	3	12	15	9
$\eta(z)\eta(2z)^3\eta(4z)^2$	8	3	15	9	12	$\eta(z)^2\eta(2z)^3\eta(4z)$	8	3	12	9	15
$\frac{\eta(z)^3\eta(4z)^6}{\eta(2z)^3}$	8	3	21	3	12	$\frac{\eta(z)^6\eta(4z)^3}{\eta(2z)^3}$	8	3	12	3	21
$\frac{\eta(z)^3\eta(2z)^6}{\eta(4z)^3}$	8	3	3	12	21	$\frac{\eta(2z)^{12}}{\eta(z)\eta(4z)^5}$	8	3	3	18	15
$\frac{\eta(z)\eta(2z)^6}{\eta(4z)}$	8	3	9	12	15	$\eta(z)^5\eta(4z)$	8	3	9	6	21
$\frac{\eta(2z)^6\eta(4z)}{\eta(z)}$	8	3	15	12	9	$\frac{\eta(2z)^{12}}{\eta(z)^5\eta(4z)}$	8	3	15	18	3
$\frac{\eta(2z)^6\eta(4z)^3}{\eta(z)^3}$	8	3	21	12	3	$\eta(z)\eta(4z)^5$	8	3	21	6	9
$\frac{\eta(z)^6}{\eta(2z)\eta(4z)}$	8	2	0	3	21	$\frac{\eta(4z)^6}{\eta(z)\eta(2z)}$	8	2	21	3	0
$\frac{\eta(z)^2\eta(2z)^5}{\eta(4z)^3}$	8	2	0	9	15	$\frac{\eta(2z)^5\eta(4z)^2}{\eta(z)^3}$	8	2	15	9	0
$\frac{\eta(2z)^{11}}{\eta(z)^2\eta(4z)^5}$	8	2	0	15	9	$\frac{\eta(2z)^{11}}{\eta(z)^5\eta(4z)^2}$	8	2	9	15	0
$\frac{\eta(2z)^{17}}{\eta(z)^6\eta(4z)^7}$	8	2	0	21	3	$\frac{\eta(2z)^{17}}{\eta(z)^7\eta(4z)^6}$	8	2	3	21	0
$\frac{\eta(z)^7\eta(4z)}{\eta(2z)^4}$	8	2	3	0	21	$\frac{\eta(2z)^{14}}{\eta(z)^5\eta(4z)^5}$	8	2	3	18	3
$\frac{\eta(z)^5\eta(4z)^3}{\eta(2z)^4}$	8	2	9	0	15	$\eta(z)\eta(2z)^2\eta(4z)$	8	2	9	6	9
$\frac{\eta(z)^3\eta(4z)^5}{\eta(2z)^4}$	8	2	15	0	9	$\frac{\eta(2z)^{10}}{\eta(z)\eta(4z)}$	8	4	15	18	15
$\frac{\eta(z)\eta(4z)^7}{\eta(2z)^4}$	8	2	21	0	3	$\frac{\eta(z)^5\eta(4z)^5}{\eta(2z)^2}$	8	4	21	6	21
$\frac{\eta(z)^5}{\eta(2z)}$	8	2	3	3	18	$\frac{\eta(4z)^5}{\eta(2z)}$	8	2	18	3	3
$\frac{\eta(2z)^5}{\eta(z)}$	8	2	9	9	6	$\frac{\eta(2z)^5}{\eta(4z)}$	8	2	6	9	9
$\eta(z)\eta(2z)^7$	8	4	15	15	18	$\eta(2z)^7\eta(4z)$	8	4	18	15	15
$\frac{\eta(2z)^{13}}{\eta(z)^5}$	8	4	21	21	6	$\frac{\eta(2z)^{13}}{\eta(4z)^5}$	8	4	6	21	21
$\frac{\eta(z)\eta(2z)^5}{\eta(4z)^2}$	8	2	3	9	12	$\frac{\eta(2z)^5\eta(4z)}{\eta(z)^2}$	8	2	12	9	3
$\frac{\eta(z)^3\eta(4z)^2}{\eta(2z)}$	8	2	9	3	12	$\frac{\eta(z)^2\eta(4z)^3}{\eta(2z)}$	8	2	12	3	9
$\frac{\eta(2z)^{13}}{\eta(z)^3\eta(4z)^2}$	8	4	15	21	12	$\frac{\eta(2z)^{13}}{\eta(z)^2\eta(4z)^3}$	8	4	12	21	15
$\frac{\eta(2z)^7\eta(4z)^2}{\eta(z)}$	8	4	21	15	12	$\frac{\eta(z)^2\eta(2z)^7}{\eta(4z)}$	8	4	12	15	21
$\frac{\eta(2z)^8}{\eta(z)\eta(4z)^3}$	8	2	3	12	9	$\frac{\eta(z)^3\eta(2z)^2}{\eta(4z)}$	8	2	3	6	15
$\frac{\eta(2z)^8}{\eta(z)^3\eta(4z)}$	8	2	9	12	3	$\frac{\eta(z)\eta(2z)^{10}}{\eta(4z)^3}$	8	4	9	18	21
$\eta(z)^3\eta(2z)^4\eta(4z)$	8	4	15	12	21	$\frac{\eta(2z)^2\eta(4z)^3}{\eta(z)}$	8	2	15	6	3
$\eta(z)\eta(2z)^4\eta(4z)^3$	8	4	21	12	15	$\frac{\eta(2z)^{10}\eta(4z)}{\eta(z)^3}$	8	4	21	18	9

Table B.4 cont. Minimal eta-quotients of level 4

f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0
$\frac{\eta(2z)^{11}}{\eta(z)^3\eta(4z)^4}$	8	2	3	15	6	$\frac{\eta(2z)^{11}}{\eta(z)^4\eta(4z)^3}$	8	2	6	15	3
$\frac{\eta(2z)^{13}}{\eta(z)\eta(4z)^4}$	8	4	9	21	18	$\frac{\eta(2z)^{13}}{\eta(z)^4\eta(4z)}$	8	4	18	21	9
$\frac{\eta(z)\eta(4z)^4}{\eta(2z)}$	8	2	15	3	6	$\frac{\eta(z)^4\eta(4z)}{\eta(2z)}$	8	2	6	3	15
$\eta(z)^3\eta(2z)\eta(4z)^4$	8	4	21	9	18	$\eta(z)^4\eta(2z)\eta(4z)^3$	8	4	18	9	21
$\frac{\eta(z)^2\eta(2z)}{\eta(4z)}$	8	1	0	3	9	$\frac{\eta(2z)\eta(4z)^2}{\eta(z)}$	8	1	9	3	0
$\frac{\eta(2z)^7}{\eta(z)^2\eta(4z)^3}$	8	1	0	9	3	$\frac{\eta(2z)^7}{\eta(z)^3\eta(4z)^2}$	8	1	3	9	0
$\frac{\eta(z)^2\eta(2z)^9}{\eta(4z)^5}$	8	3	0	15	21	$\frac{\eta(2z)^9\eta(4z)^2}{\eta(z)^5}$	8	3	21	15	0
$\frac{\eta(2z)^{15}}{\eta(z)^2\eta(4z)^7}$	8	3	0	21	15	$\frac{\eta(2z)^{15}}{\eta(z)^7\eta(4z)^2}$	8	3	15	21	0
$\frac{\eta(z)^3\eta(4z)}{\eta(2z)^2}$	8	1	3	0	9	$\frac{\eta(2z)^4}{\eta(z)\eta(4z)}$	8	1	3	6	3
$\frac{\eta(z)\eta(4z)^3}{\eta(2z)^2}$	8	1	9	0	3	$\frac{\eta(2z)^{12}}{\eta(z)^3\eta(4z)^3}$	8	3	9	18	9
$\frac{\eta(z)^7\eta(4z)^5}{\eta(2z)^6}$	8	3	15	0	21	$\eta(z)^3\eta(4z)^3$	8	3	15	6	15
$\frac{\eta(z)^5\eta(4z)^7}{\eta(2z)^6}$	8	3	21	0	15	$\eta(z)\eta(2z)^8\eta(4z)$	8	5	21	18	21
$\eta(z)\eta(2z)$	8	1	3	3	6	$\eta(2z)\eta(4z)$	8	1	6	3	3
$\eta(z)^3\eta(2z)^3$	8	3	9	9	18	$\eta(2z)^3\eta(4z)^3$	8	3	18	9	9
$\frac{\eta(2z)^9}{\eta(z)^3}$	8	3	15	15	6	$\frac{\eta(2z)^9}{\eta(4z)^3}$	8	3	6	15	15
$\frac{\eta(2z)^{11}}{\eta(z)}$	8	5	21	21	18	$\frac{\eta(2z)^{11}}{\eta(4z)}$	8	5	18	21	21
$\frac{\eta(2z)^{10}}{\eta(z)^4}$	6	3	16	16	4	$\frac{\eta(2z)^{10}}{\eta(4z)^4}$	6	3	4	16	16
$\eta(z)^4\eta(2z)^2$	6	3	8	8	20	$\eta(2z)^2\eta(4z)^4$	6	3	20	8	8
$\frac{\eta(z)^4\eta(4z)^4}{\eta(2z)^2}$	6	3	16	4	16	$\frac{\eta(2z)^{12}}{\eta(z)^4\eta(4z)^4}$	6	2	4	16	4
$\frac{\eta(2z)^{14}}{\eta(z)^4\eta(4z)^4}$	6	3	8	20	8	$\eta(z)^4\eta(4z)^4$	6	4	20	8	20
$\eta(z)^4$	6	2	4	4	16	$\eta(4z)^4$	6	2	16	4	4
$\frac{\eta(2z)^{12}}{\eta(z)^4}$	6	4	20	20	8	$\frac{\eta(2z)^{12}}{\eta(4z)^4}$	6	4	8	20	20
$\eta(2z)^2$	6	1	4	4	4	$\frac{\eta(z)^6\eta(4z)^2}{\eta(2z)^4}$	4	2	6	0	18
$\eta(2z)^{10}$	6	5	20	20	20	$\frac{\eta(z)^2\eta(4z)^6}{\eta(2z)^4}$	4	2	18	0	6
$\frac{\eta(z)^4\eta(2z)^2}{\eta(4z)^2}$	4	2	0	6	18	$\frac{\eta(2z)^2\eta(4z)^4}{\eta(z)^2}$	4	2	18	6	0
$\frac{\eta(2z)^{14}}{\eta(z)^4\eta(4z)^6}$	4	2	0	18	6	$\frac{\eta(2z)^{14}}{\eta(z)^6\eta(4z)^4}$	4	2	6	18	0
$\eta(z)^4\eta(4z)^2$	4	3	12	6	18	$\eta(z)^2\eta(4z)^4$	4	3	18	6	12
$\frac{\eta(2z)^{12}}{\eta(z)^4\eta(4z)^2}$	4	3	12	18	6	$\frac{\eta(2z)^{12}}{\eta(z)^2\eta(4z)^4}$	4	3	6	18	12
$\frac{\eta(z)^2\eta(2z)^6}{\eta(4z)^2}$	4	3	6	12	18	$\frac{\eta(z)^2\eta(4z)^2}{\eta(2z)^2}$	4	1	6	0	6
$\frac{\eta(2z)^6\eta(4z)^2}{\eta(z)^2}$	4	3	18	12	6	$\frac{\eta(z)^6\eta(4z)^6}{\eta(2z)^6}$	4	3	18	0	18
$\frac{\eta(2z)^4}{\eta(z)^2}$	4	1	6	6	0	$\frac{\eta(2z)^4}{\eta(4z)^2}$	4	1	0	6	6
$\frac{\eta(2z)^{12}}{\eta(z)^6}$	4	3	18	18	0	$\frac{\eta(2z)^{12}}{\eta(4z)^6}$	4	3	0	18	18

Table B.4 cont.   Minimal eta-quotients of level 4

f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	v_0
$\eta(z)^2\eta(2z)^2$	4	2	6	6	12	$\eta(2z)^2\eta(4z)^2$	4	2	12	6	6
$\frac{\eta(2z)^{10}}{\eta(z)^2}$	4	4	18	18	12	$\frac{\eta(2z)^{10}}{\eta(4z)^2}$	4	4	12	18	18
$\frac{\eta(2z)^8}{\eta(z)^2\eta(4z)^2}$	4	2	6	12	6	$\eta(2z)^4$	3	2	8	8	8
$\eta(z)^2\eta(2z)^4\eta(4z)^2$	4	4	18	12	18	$\eta(2z)^8$	3	4	16	16	16
$\frac{\eta(z)^4}{\eta(2z)^2}$	2	1	0	0	12	$\frac{\eta(4z)^4}{\eta(2z)^2}$	2	1	12	0	0
$\frac{\eta(2z)^{10}}{\eta(z)^4\eta(4z)^4}$	2	1	0	12	0	$\frac{\eta(z)^4\eta(4z)^4}{\eta(2z)^4}$	2	2	12	0	12
$\frac{\eta(2z)^8}{\eta(z)^4}$	2	2	12	12	0	$\frac{\eta(2z)^8}{\eta(4z)^4}$	2	2	0	12	12
$\eta(2z)^6$	2	3	12	12	12						

B.5 Minimal Eta-Quotients of Level 5

Table B.5 Minimal eta-quotients of level 5

f	D	k	v_∞	v_0
$\eta(z)^2$	12	1	2	10
$\eta(5z)^2$	12	1	10	2
$\eta(z)^4\eta(5z)^2$	12	3	14	22
$\eta(z)^2\eta(5z)^4$	12	3	22	14
$\eta(z)^3\eta(5z)$	3	2	8	16
$\eta(z)\eta(5z)^3$	3	2	16	8
$\eta(z)^4$	6	2	4	20
$\eta(5z)^4$	6	2	20	4
$\eta(z)\eta(5z)$	4	1	6	6
$\eta(z)^3\eta(5z)^3$	4	3	18	18
$\eta(z)^2\eta(5z)^2$	2	2	12	12

B.6 Minimal Eta-Quotients of Level 6

Table B.6 Minimal eta-quotients of level 6

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0
$\frac{\eta(z)^3\eta(2z)^2}{\eta(6z)}$	24	2	1	5	19	23	$\frac{\eta(z)\eta(2z)^3\eta(3z)^2}{\eta(6z)^2}$	24	2	1	11	19	17
$\frac{\eta(z)^2\eta(2z)^3}{\eta(3z)}$	24	2	5	1	23	19	$\frac{\eta(2z)^4\eta(3z)}{\eta(6z)}$	24	2	5	7	23	13
$\eta(z)^2\eta(2z)\eta(3z)$	24	2	7	11	13	17	$\frac{\eta(z)^4\eta(6z)}{\eta(3z)}$	24	2	7	5	13	23
$\eta(z)\eta(2z)^2\eta(6z)$	24	2	11	7	17	13	$\frac{\eta(z)^3\eta(2z)\eta(6z)^2}{\eta(3z)^2}$	24	2	11	1	17	19
$\eta(z)\eta(3z)^2\eta(6z)$	24	2	13	17	7	11	$\frac{\eta(2z)\eta(3z)^4}{\eta(z)}$	24	2	13	23	7	5
$\eta(2z)\eta(3z)\eta(6z)^2$	24	2	17	13	11	7	$\frac{\eta(2z)^2\eta(3z)^3\eta(6z)}{\eta(z)^2}$	24	2	17	19	11	1
$\frac{\eta(3z)^3\eta(6z)^2}{\eta(2z)}$	24	2	19	23	1	5	$\frac{\eta(z)^2\eta(3z)\eta(6z)^3}{\eta(2z)^2}$	24	2	19	17	1	11
$\frac{\eta(3z)^2\eta(6z)^3}{\eta(z)}$	24	2	23	19	5	1	$\frac{\eta(z)\eta(6z)^4}{\eta(2z)}$	24	2	23	13	5	7
$\frac{\eta(z)^2\eta(2z)\eta(3z)^3}{\eta(6z)^2}$	24	2	1	17	11	19	$\frac{\eta(2z)^4\eta(3z)^4}{\eta(z)\eta(6z)^3}$	24	2	1	17	19	11
$\frac{\eta(z)^4\eta(3z)}{\eta(2z)}$	24	2	5	13	7	23	$\frac{\eta(2z)^5\eta(3z)^3}{\eta(z)^2\eta(6z)^2}$	24	2	5	13	23	7
$\frac{\eta(z)\eta(3z)^4}{\eta(6z)}$	24	2	7	23	5	13	$\frac{\eta(2z)^3\eta(3z)^5}{\eta(z)^2\eta(6z)^2}$	24	2	7	23	13	5
$\frac{\eta(z)^3\eta(3z)^2\eta(6z)}{\eta(2z)^2}$	24	2	11	19	1	17	$\frac{\eta(2z)^4\eta(3z)^4}{\eta(z)^3\eta(6z)}$	24	2	11	19	17	1
$\frac{\eta(2z)^4\eta(6z)}{\eta(z)}$	24	2	13	5	23	7	$\frac{\eta(z)^5\eta(6z)^3}{\eta(2z)^2\eta(3z)^2}$	24	2	13	5	7	23
$\frac{\eta(z)\eta(2z)^2\eta(6z)^3}{\eta(3z)^2}$	24	2	17	1	19	11	$\frac{\eta(z)^4\eta(6z)^4}{\eta(2z)\eta(3z)^3}$	24	2	17	1	11	19
$\frac{\eta(2z)^3\eta(3z)\eta(6z)^2}{\eta(z)^2}$	24	2	19	11	17	1	$\frac{\eta(z)^4\eta(6z)^4}{\eta(2z)^3\eta(3z)}$	24	2	19	11	1	17
$\frac{\eta(2z)\eta(6z)^4}{\eta(3z)}$	24	2	23	7	13	5	$\frac{\eta(z)^3\eta(6z)^5}{\eta(2z)^2\eta(3z)^2}$	24	2	23	7	5	13
$\frac{\eta(2z)^2\eta(3z)^5}{\eta(6z)^3}$	24	2	1	23	11	13	$\frac{\eta(2z)^5\eta(3z)^6}{\eta(z)^3\eta(6z)^4}$	24	2	1	23	19	5
$\frac{\eta(z)^2\eta(3z)^3}{\eta(6z)}$	24	2	5	19	7	17	$\frac{\eta(2z)^6\eta(3z)^5}{\eta(z)^4\eta(6z)^3}$	24	2	5	19	23	1
$\frac{\eta(z)^3\eta(3z)^2}{\eta(2z)}$	24	2	7	17	5	19	$\frac{\eta(2z)^2\eta(3z)^3}{\eta(6z)}$	24	2	7	17	13	11
$\frac{\eta(z)^5\eta(6z)^2}{\eta(2z)^3}$	24	2	11	13	1	23	$\frac{\eta(2z)^3\eta(3z)^2}{\eta(z)}$	24	2	11	13	17	7
$\frac{\eta(2z)^5\eta(3z)^2}{\eta(z)^3}$	24	2	13	11	23	1	$\frac{\eta(z)^3\eta(6z)^2}{\eta(2z)}$	24	2	13	11	7	17
$\frac{\eta(2z)^3\eta(6z)^2}{\eta(z)}$	24	2	17	7	19	5	$\frac{\eta(z)^2\eta(6z)^3}{\eta(3z)}$	24	2	17	7	11	13
$\frac{\eta(2z)^2\eta(6z)^3}{\eta(3z)}$	24	2	19	5	17	7	$\frac{\eta(z)^6\eta(6z)^5}{\eta(2z)^4\eta(3z)^3}$	24	2	19	5	1	23
$\frac{\eta(z)^2\eta(6z)^5}{\eta(3z)^3}$	24	2	23	1	13	11	$\frac{\eta(z)^5\eta(6z)^6}{\eta(2z)^3\eta(3z)^4}$	24	2	23	1	5	19

Table B.6 cont. Minimal eta-quotients of level 6

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0
$\frac{\eta(z)^3\eta(3z)^4}{\eta(2z)\eta(6z)^2}$	24	2	1	23	3	21	$\frac{\eta(2z)^4\eta(6z)^3}{\eta(z)^2\eta(3z)}$	24	2	21	3	23	1
$\frac{\eta(2z)^3\eta(3z)^4}{\eta(z)\eta(6z)^2}$	24	2	5	19	15	9	$\frac{\eta(2z)^4\eta(3z)^3}{\eta(z)^2\eta(6z)}$	24	2	9	15	19	5
$\frac{\eta(2z)^5\eta(3z)^4}{\eta(z)^3\eta(6z)^2}$	24	2	7	17	21	3	$\frac{\eta(2z)^4\eta(3z)^5}{\eta(z)^2\eta(6z)^3}$	24	2	3	21	17	7
$\eta(z)^2\eta(3z)\eta(6z)$	24	2	11	13	9	15	$\eta(z)\eta(2z)\eta(6z)^2$	24	2	15	9	13	11
$\eta(2z)^2\eta(3z)\eta(6z)$	24	2	13	11	15	9	$\eta(z)\eta(2z)\eta(3z)^2$	24	2	9	15	11	13
$\frac{\eta(z)^5\eta(6z)^4}{\eta(2z)^3\eta(3z)^2}$	24	2	17	7	3	21	$\frac{\eta(z)^4\eta(6z)^5}{\eta(2z)^2\eta(3z)^3}$	24	2	21	3	7	17
$\frac{\eta(z)^3\eta(6z)^4}{\eta(2z)\eta(3z)^2}$	24	2	19	5	9	15	$\frac{\eta(z)^4\eta(6z)^3}{\eta(2z)^2\eta(3z)}$	24	2	15	9	5	19
$\frac{\eta(2z)^3\eta(6z)^4}{\eta(z)\eta(3z)^2}$	24	2	23	1	21	3	$\frac{\eta(z)^4\eta(3z)^3}{\eta(2z)^2\eta(6z)}$	24	2	3	21	1	23
$\frac{\eta(2z)^3\eta(3z)^3}{\eta(z)^2\eta(6z)^2}$	24	1	1	11	11	1	$\frac{\eta(2z)^2\eta(3z)}{\eta(6z)}$	24	1	1	5	11	7
$\eta(2z)\eta(3z)$	24	1	5	7	7	5	$\frac{\eta(z)^2\eta(6z)}{\eta(3z)}$	24	1	5	1	7	11
$\eta(z)\eta(6z)$	24	1	7	5	5	7	$\frac{\eta(2z)\eta(3z)^2}{\eta(z)}$	24	1	7	11	5	1
$\frac{\eta(z)^3\eta(6z)^3}{\eta(2z)^2\eta(3z)^2}$	24	1	11	1	1	11	$\frac{\eta(z)\eta(6z)^2}{\eta(2z)}$	24	1	11	7	1	5
$\frac{\eta(2z)^4\eta(3z)^4}{\eta(z)\eta(6z)}$	24	3	13	23	23	13	$\eta(z)\eta(2z)^3\eta(3z)^2$	24	3	13	17	23	19
$\eta(z)\eta(2z)^2\eta(3z)^2\eta(6z)$	24	3	17	19	19	17	$\eta(z)^3\eta(2z)\eta(6z)^2$	24	3	17	13	19	23
$\eta(z)^2\eta(2z)\eta(3z)\eta(6z)^2$	24	3	19	17	17	19	$\eta(2z)^2\eta(3z)^3\eta(6z)$	24	3	19	23	17	13
$\frac{\eta(z)^4\eta(6z)^4}{\eta(2z)\eta(3z)}$	24	3	23	13	13	23	$\eta(z)^2\eta(3z)\eta(6z)^3$	24	3	23	19	13	17
$\frac{\eta(z)^3}{\eta(2z)}$	24	1	1	5	3	15	$\frac{\eta(6z)^3}{\eta(3z)}$	24	1	15	3	5	1
$\frac{\eta(2z)^3}{\eta(z)}$	24	1	5	1	15	3	$\frac{\eta(3z)^3}{\eta(6z)}$	24	1	3	15	1	5
$\frac{\eta(2z)^4\eta(3z)^2}{\eta(z)\eta(6z)}$	24	2	7	11	21	9	$\frac{\eta(2z)^2\eta(3z)^4}{\eta(z)\eta(6z)}$	24	2	9	21	11	7
$\frac{\eta(z)^4\eta(6z)^2}{\eta(2z)\eta(3z)}$	24	2	11	7	9	21	$\frac{\eta(z)^2\eta(6z)^4}{\eta(2z)\eta(3z)}$	24	2	21	9	7	11
$\frac{\eta(2z)^3\eta(3z)^3}{\eta(z)^2}$	24	2	13	17	15	3	$\frac{\eta(2z)^3\eta(3z)^3}{\eta(6z)^2}$	24	2	3	15	17	13
$\frac{\eta(z)^3\eta(6z)^3}{\eta(2z)^2}$	24	2	17	13	3	15	$\frac{\eta(z)^3\eta(6z)^3}{\eta(3z)^2}$	24	2	15	3	13	17
$\frac{\eta(z)^3\eta(3z)^2\eta(6z)^2}{\eta(2z)}$	24	3	19	23	9	21	$\frac{\eta(z)^2\eta(2z)^2\eta(6z)^3}{\eta(3z)}$	24	3	21	9	23	19
$\frac{\eta(2z)^3\eta(3z)^2\eta(6z)^2}{\eta(z)}$	24	3	23	19	21	9	$\frac{\eta(z)^2\eta(2z)^2\eta(3z)^3}{\eta(6z)}$	24	3	9	21	19	23
$\frac{\eta(z)\eta(3z)^2}{\eta(6z)}$	24	1	1	11	3	9	$\frac{\eta(2z)^2\eta(6z)}{\eta(z)}$	24	1	9	3	11	1
$\eta(z)^3\eta(2z)$	24	2	5	7	15	21	$\eta(3z)\eta(6z)^3$	24	2	21	15	7	5
$\eta(z)\eta(2z)^3$	24	2	7	5	21	15	$\eta(3z)^3\eta(6z)$	24	2	15	21	5	7
$\frac{\eta(2z)\eta(6z)^2}{\eta(3z)}$	24	1	11	1	9	3	$\frac{\eta(z)^2\eta(3z)}{\eta(2z)}$	24	1	3	9	1	11
$\eta(z)^2\eta(2z)\eta(3z)^3$	24	3	13	23	15	21	$\eta(2z)^3\eta(3z)\eta(6z)^2$	24	3	21	15	23	13
$\frac{\eta(z)\eta(3z)^2\eta(6z)^2}{\eta(2z)}$	24	2	17	19	3	9	$\frac{\eta(z)^2\eta(2z)^2\eta(6z)}{\eta(3z)}$	24	2	9	3	19	17
$\frac{\eta(2z)\eta(3z)^2\eta(6z)^2}{\eta(z)}$	24	2	19	17	9	3	$\frac{\eta(z)^2\eta(2z)^2\eta(3z)}{\eta(6z)}$	24	2	3	9	17	19
$\eta(z)\eta(2z)^2\eta(6z)^3$	24	3	23	13	21	15	$\eta(z)^3\eta(3z)^2\eta(6z)$	24	3	15	21	13	23

Table B.6 cont. Minimal eta-quotients of level 6

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0
$\frac{\eta(2z)\eta(3z)^4}{\eta(z)\eta(6z)^2}$	24	1	1	17	3	3	$\frac{\eta(2z)^4\eta(3z)}{\eta(z)^2\eta(6z)}$	24	1	3	3	17	1
$\frac{\eta(z)\eta(2z)^2\eta(3z)^2}{\eta(6z)}$	24	2	5	13	15	15	$\frac{\eta(2z)^2\eta(3z)^2\eta(6z)}{\eta(z)}$	24	2	15	15	13	5
$\frac{\eta(z)\eta(2z)^3\eta(3z)^4}{\eta(6z)^2}$	24	3	7	23	21	21	$\frac{\eta(2z)^4\eta(3z)^3\eta(6z)}{\eta(z)^2}$	24	3	21	21	23	7
$\eta(2z)\eta(3z)^3$	24	2	11	19	9	9	$\eta(2z)^3\eta(3z)$	24	2	9	9	19	11
$\frac{\eta(z)^2\eta(2z)\eta(6z)^2}{\eta(3z)}$	24	2	13	5	15	15	$\frac{\eta(z)^2\eta(3z)\eta(6z)^2}{\eta(2z)}$	24	2	15	15	5	13
$\frac{\eta(z)\eta(6z)^4}{\eta(2z)\eta(3z)^2}$	24	1	17	1	3	3	$\frac{\eta(z)^4\eta(6z)}{\eta(2z)^2\eta(3z)}$	24	1	3	3	1	17
$\eta(z)\eta(6z)^3$	24	2	19	11	9	9	$\eta(z)^3\eta(6z)$	24	2	9	9	11	19
$\frac{\eta(z)^3\eta(2z)\eta(6z)^4}{\eta(3z)^2}$	24	3	23	7	21	21	$\frac{\eta(z)^4\eta(3z)\eta(6z)^3}{\eta(2z)^2}$	24	3	21	21	7	23
$\frac{\eta(2z)^6\eta(3z)^6}{\eta(z)^4\eta(6z)^4}$	12	2	2	22	22	2	$\frac{\eta(z)^3\eta(2z)\eta(3z)}{\eta(6z)}$	12	2	2	10	14	22
$\eta(2z)^2\eta(3z)^2$	12	2	10	14	14	10	$\frac{\eta(z)\eta(2z)^3\eta(6z)}{\eta(3z)}$	12	2	10	2	22	14
$\eta(z)^2\eta(6z)^2$	12	2	14	10	10	14	$\frac{\eta(z)\eta(3z)^3\eta(6z)}{\eta(2z)}$	12	2	14	22	2	10
$\frac{\eta(z)^6\eta(6z)^6}{\eta(2z)^4\eta(3z)^4}$	12	2	22	2	2	22	$\frac{\eta(2z)\eta(3z)\eta(6z)^3}{\eta(z)}$	12	2	22	14	10	2
$\frac{\eta(2z)^4\eta(3z)^2}{\eta(6z)^2}$	12	2	2	10	22	14	$\frac{\eta(2z)^3\eta(3z)^5}{\eta(z)\eta(6z)^3}$	12	2	2	22	14	10
$\frac{\eta(z)^4\eta(6z)^2}{\eta(3z)^2}$	12	2	10	2	14	22	$\frac{\eta(2z)^5\eta(3z)^3}{\eta(z)^3\eta(6z)}$	12	2	10	14	22	2
$\frac{\eta(2z)^2\eta(3z)^4}{\eta(z)^2}$	12	2	14	22	10	2	$\frac{\eta(z)^5\eta(6z)^3}{\eta(2z)^3\eta(3z)}$	12	2	14	10	2	22
$\frac{\eta(z)^2\eta(6z)^4}{\eta(2z)^2}$	12	2	22	14	2	10	$\frac{\eta(z)^3\eta(6z)^5}{\eta(2z)\eta(3z)^3}$	12	2	22	2	10	14
$\frac{\eta(z)^2\eta(2z)^3}{\eta(6z)}$	12	2	2	4	22	20	$\frac{\eta(3z)^3\eta(6z)^2}{\eta(z)}$	12	2	20	22	4	2
$\frac{\eta(2z)^3\eta(3z)^4}{\eta(z)^2\eta(6z)}$	12	2	10	20	14	4	$\frac{\eta(2z)^4\eta(3z)^3}{\eta(z)\eta(6z)^2}$	12	2	4	14	20	10
$\frac{\eta(z)^4\eta(6z)^3}{\eta(2z)\eta(3z)^2}$	12	2	14	4	10	20	$\frac{\eta(z)^3\eta(6z)^4}{\eta(2z)^2\eta(3z)}$	12	2	20	10	4	14
$\frac{\eta(3z)^2\eta(6z)^3}{\eta(2z)}$	12	2	22	20	2	4	$\frac{\eta(z)^3\eta(2z)^2}{\eta(3z)}$	12	2	4	2	20	22
$\frac{\eta(2z)^5\eta(3z)^4}{\eta(z)^2\eta(6z)^3}$	12	2	2	16	22	8	$\frac{\eta(2z)^4\eta(3z)^5}{\eta(z)^3\eta(6z)^2}$	12	2	8	22	16	2
$\eta(z)^2\eta(2z)\eta(6z)$	12	2	10	8	14	16	$\eta(z)\eta(3z)\eta(6z)^2$	12	2	16	14	8	10
$\eta(2z)\eta(3z)^2\eta(6z)$	12	2	14	16	10	8	$\eta(z)\eta(2z)^2\eta(3z)$	12	2	8	10	16	14
$\frac{\eta(z)^4\eta(6z)^5}{\eta(2z)^3\eta(3z)^2}$	12	2	22	8	2	16	$\frac{\eta(z)^5\eta(6z)^4}{\eta(2z)^2\eta(3z)^3}$	12	2	22	8	2	16
$\frac{\eta(z)\eta(2z)^2\eta(3z)^3}{\eta(6z)^2}$	12	2	2	16	14	16	$\frac{\eta(2z)^3\eta(3z)^2\eta(6z)}{\eta(z)^2}$	12	2	16	14	16	2
$\frac{\eta(2z)^4\eta(3z)}{\eta(z)}$	12	2	10	8	22	8	$\frac{\eta(2z)\eta(3z)^4}{\eta(6z)}$	12	2	8	22	8	10
$\frac{\eta(z)^3\eta(3z)\eta(6z)^2}{\eta(2z)^2}$	12	2	14	16	2	16	$\frac{\eta(z)^2\eta(2z)\eta(6z)^3}{\eta(3z)^2}$	12	2	16	2	16	14
$\frac{\eta(z)\eta(6z)^4}{\eta(3z)}$	12	2	22	8	10	8	$\frac{\eta(z)^4\eta(6z)}{\eta(2z)}$	12	2	8	10	8	22
$\frac{\eta(z)^2\eta(3z)^4}{\eta(6z)^2}$	12	2	2	22	6	18	$\frac{\eta(2z)^4\eta(6z)^2}{\eta(z)^2}$	12	2	18	6	22	2
$\frac{\eta(z)^3\eta(3z)\eta(6z)}{\eta(2z)}$	12	2	10	14	6	18	$\frac{\eta(z)\eta(2z)\eta(6z)^3}{\eta(3z)}$	12	2	18	6	14	10
$\frac{\eta(2z)^3\eta(3z)\eta(6z)}{\eta(z)}$	12	2	14	10	18	6	$\frac{\eta(z)\eta(2z)\eta(3z)^3}{\eta(6z)}$	12	2	6	18	10	14
$\frac{\eta(2z)^2\eta(6z)^4}{\eta(3z)^2}$	12	2	22	2	18	6	$\frac{\eta(z)^4\eta(3z)^2}{\eta(2z)^2}$	12	2	6	18	2	22

Table B.6 cont. Minimal eta-quotients of level 6

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0
$\frac{\eta(2z)^2\eta(3z)^4}{\eta(z)^2\eta(6z)^2}$	12	1	2	16	6	0	$\frac{\eta(2z)^4\eta(3z)^2}{\eta(z)^2\eta(6z)^2}$	12	1	0	6	16	2
$\frac{\eta(2z)\eta(3z)\eta(6z)}{\eta(z)}$	12	1	10	8	6	0	$\frac{\eta(z)\eta(2z)\eta(3z)}{\eta(6z)}$	12	1	0	6	8	10
$\frac{\eta(2z)^4\eta(3z)^3}{\eta(z)^3}$	12	2	14	16	18	0	$\frac{\eta(2z)^3\eta(3z)^4}{\eta(6z)^3}$	12	2	0	18	16	14
$\frac{\eta(2z)^3\eta(6z)^3}{\eta(z)^2}$	12	2	22	8	18	0	$\frac{\eta(z)^3\eta(3z)^3}{\eta(6z)^2}$	12	2	0	18	8	22
$\frac{\eta(z)^2\eta(6z)^4}{\eta(2z)^2\eta(3z)^2}$	12	1	16	2	0	6	$\frac{\eta(z)^4\eta(6z)^2}{\eta(2z)^2\eta(3z)^2}$	12	1	6	0	2	16
$\frac{\eta(z)\eta(3z)\eta(6z)}{\eta(2z)}$	12	1	8	10	0	6	$\frac{\eta(z)\eta(2z)\eta(6z)}{\eta(3z)}$	12	1	6	0	10	8
$\frac{\eta(z)^4\eta(6z)^3}{\eta(2z)^3}$	12	2	16	14	0	18	$\frac{\eta(z)^3\eta(6z)^4}{\eta(3z)^3}$	12	2	18	0	14	16
$\frac{\eta(z)^3\eta(3z)^3}{\eta(2z)^2}$	12	2	8	22	0	18	$\frac{\eta(2z)^3\eta(6z)^3}{\eta(3z)^2}$	12	2	18	0	22	8
$\frac{\eta(2z)\eta(3z)^2}{\eta(6z)}$	12	1	2	10	6	6	$\frac{\eta(2z)^2\eta(3z)}{\eta(z)}$	12	1	6	6	10	2
$\frac{\eta(z)\eta(6z)^2}{\eta(3z)}$	12	1	10	2	6	6	$\frac{\eta(z)^2\eta(6z)}{\eta(2z)}$	12	1	6	6	2	10
$\eta(z)\eta(2z)^2\eta(3z)^3$	12	3	14	22	18	18	$\eta(2z)^3\eta(3z)^2\eta(6z)$	12	3	18	18	22	14
$\eta(z)^2\eta(2z)\eta(6z)^3$	12	3	22	14	18	18	$\eta(z)^3\eta(3z)\eta(6z)^2$	12	3	18	18	14	22
$\frac{\eta(2z)^3\eta(3z)}{\eta(z)\eta(6z)}$	12	1	2	4	14	4	$\frac{\eta(2z)\eta(3z)^3}{\eta(z)\eta(6z)}$	12	1	4	14	4	2
$\frac{\eta(z)\eta(2z)^3\eta(3z)^3}{\eta(6z)}$	12	3	10	20	22	20	$\frac{\eta(2z)^3\eta(3z)^3\eta(6z)}{\eta(z)}$	12	3	20	22	20	10
$\frac{\eta(z)\eta(6z)^3}{\eta(2z)\eta(3z)}$	12	1	14	4	2	4	$\frac{\eta(z)^3\eta(6z)}{\eta(2z)\eta(3z)}$	12	1	4	2	4	14
$\frac{\eta(z)^3\eta(3z)\eta(6z)^3}{\eta(2z)}$	12	3	22	20	10	20	$\frac{\eta(z)^3\eta(2z)\eta(6z)^3}{\eta(3z)}$	12	3	20	10	20	22
$\eta(z)^2$	12	1	2	4	6	12	$\eta(6z)^2$	12	1	12	6	4	2
$\eta(z)\eta(3z)^3$	12	2	10	20	6	12	$\eta(2z)^3\eta(6z)$	12	2	12	6	20	10
$\frac{\eta(z)\eta(2z)^2\eta(6z)^2}{\eta(3z)}$	12	2	14	4	18	12	$\frac{\eta(z)^2\eta(3z)^2\eta(6z)}{\eta(2z)}$	12	2	12	18	4	14
$\eta(2z)^2\eta(3z)^2\eta(6z)^2$	12	3	22	20	18	12	$\eta(z)^2\eta(2z)^2\eta(3z)^2$	12	3	12	18	20	22
$\eta(2z)^2$	12	1	4	2	12	6	$\eta(3z)^2$	12	1	6	12	2	4
$\eta(2z)\eta(6z)^3$	12	2	20	10	12	6	$\eta(z)^3\eta(3z)$	12	2	6	12	10	20
$\frac{\eta(z)^2\eta(2z)\eta(3z)^2}{\eta(6z)}$	12	2	4	14	12	18	$\frac{\eta(2z)^2\eta(3z)\eta(6z)^2}{\eta(z)}$	12	2	18	12	14	4
$\eta(z)^2\eta(3z)^2\eta(6z)^2$	12	3	20	22	12	18	$\eta(z)^2\eta(2z)^2\eta(6z)^2$	12	3	18	12	22	20
$\frac{\eta(z)^3\eta(3z)^2}{\eta(6z)}$	8	2	3	15	9	21	$\frac{\eta(z)\eta(2z)\eta(3z)^4}{\eta(6z)^2}$	8	2	3	21	9	15
$\frac{\eta(z)^2\eta(3z)^3}{\eta(2z)}$	8	2	9	21	3	15	$\frac{\eta(z)^4\eta(3z)\eta(6z)}{\eta(2z)^2}$	8	2	9	15	3	21
$\frac{\eta(2z)^3\eta(6z)^2}{\eta(3z)}$	8	2	15	3	21	9	$\frac{\eta(2z)^4\eta(3z)\eta(6z)}{\eta(z)^2}$	8	2	15	9	21	3
$\frac{\eta(2z)^2\eta(6z)^3}{\eta(z)}$	8	2	21	9	15	3	$\frac{\eta(z)\eta(2z)\eta(6z)^4}{\eta(3z)^2}$	8	2	21	3	15	9
$\frac{\eta(2z)^2\eta(3z)^2}{\eta(z)\eta(6z)}$	8	1	3	9	9	3	$\eta(z)\eta(2z)$	8	1	3	3	9	9
$\frac{\eta(z)^2\eta(6z)^2}{\eta(2z)\eta(3z)}$	8	1	9	3	9	3	$\eta(3z)\eta(6z)$	8	1	9	9	3	3
$\eta(2z)^3\eta(3z)^3$	8	3	15	21	21	15	$\eta(z)^2\eta(2z)^2\eta(3z)\eta(6z)$	8	3	15	15	21	21
$\eta(z)^3\eta(6z)^3$	8	3	21	15	15	21	$\eta(z)\eta(2z)\eta(3z)^2\eta(6z)^2$	8	3	21	21	15	15

Table B.6 cont.   Minimal eta-quotients of level 6

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{3}}$	v_0
$\frac{\eta(2z)^5\eta(3z)^5}{\eta(z)^3\eta(6z)^3}$	6	2	4	20	20	4	$\frac{\eta(z)^3\eta(3z)^3}{\eta(2z)\eta(6z)}$	6	2	4	20	4	20
$\frac{\eta(z)^5\eta(6z)^5}{\eta(2z)^3\eta(3z)^3}$	6	2	20	4	4	20	$\frac{\eta(2z)^3\eta(6z)^3}{\eta(z)\eta(3z)}$	6	2	20	4	20	4
$\frac{\eta(z)\eta(2z)^3\eta(3z)}{\eta(6z)}$	6	2	4	8	20	16	$\frac{\eta(2z)\eta(3z)^3\eta(6z)}{\eta(z)}$	6	2	16	20	8	4
$\frac{\eta(z)\eta(3z)\eta(6z)^3}{\eta(2z)}$	6	2	20	16	4	8	$\frac{\eta(z)^3\eta(2z)\eta(6z)}{\eta(3z)}$	6	2	8	4	16	20
$\frac{\eta(2z)^2\eta(3z)^4}{\eta(6z)^2}$	6	2	4	20	12	12	$\frac{\eta(2z)^4\eta(3z)^2}{\eta(z)^2}$	6	2	12	12	20	4
$\frac{\eta(z)^2\eta(6z)^4}{\eta(3z)^2}$	6	2	20	4	12	12	$\frac{\eta(z)^4\eta(6z)^2}{\eta(2z)^2}$	6	2	12	12	4	20
$\frac{\eta(2z)^3\eta(3z)^2}{\eta(z)^2\eta(6z)}$	6	1	4	8	12	0	$\frac{\eta(2z)^2\eta(3z)^3}{\eta(z)\eta(6z)^2}$	6	1	0	12	8	4
$\frac{\eta(2z)^2\eta(3z)^2\eta(6z)^2}{\eta(z)^2}$	6	2	20	16	12	0	$\frac{\eta(z)^2\eta(2z)^2\eta(3z)^2}{\eta(6z)^2}$	6	2	0	12	16	20
$\frac{\eta(z)^3\eta(6z)^2}{\eta(2z)^2\eta(3z)}$	6	1	8	4	0	12	$\frac{\eta(z)^2\eta(6z)^3}{\eta(2z)\eta(3z)^2}$	6	1	12	0	4	8
$\frac{\eta(z)^2\eta(3z)^2\eta(6z)^2}{\eta(2z)^2}$	6	2	16	20	0	12	$\frac{\eta(z)^2\eta(2z)^2\eta(6z)^2}{\eta(3z)^2}$	6	2	12	0	20	16
$\eta(z)\eta(3z)$	6	1	4	8	4	8	$\eta(2z)\eta(6z)$	6	1	8	4	8	4
$\eta(z)\eta(2z)^2\eta(3z)\eta(6z)^2$	6	3	20	16	20	16	$\eta(z)^2\eta(2z)\eta(3z)^2\eta(6z)$	6	3	16	20	16	20
$\frac{\eta(2z)^4}{\eta(z)^2}$	4	1	6	0	18	0	$\frac{\eta(3z)^4}{\eta(6z)^2}$	4	1	0	18	0	6
$\frac{\eta(6z)^4}{\eta(3z)^2}$	4	1	18	0	6	0	$\frac{\eta(z)^4}{\eta(2z)^2}$	4	1	0	6	0	18
$\frac{\eta(2z)^4\eta(3z)^4}{\eta(z)^2\eta(6z)^2}$	4	2	6	18	18	6	$\eta(z)^2\eta(2z)^2$	4	2	6	6	18	18
$\frac{\eta(z)^4\eta(6z)^4}{\eta(2z)^2\eta(3z)^2}$	4	2	18	6	6	18	$\eta(3z)^2\eta(6z)^2$	4	2	18	18	6	6
$\frac{\eta(2z)^3\eta(3z)^2}{\eta(6z)}$	4	2	6	12	18	12	$\frac{\eta(2z)^2\eta(3z)^3}{\eta(z)}$	4	2	12	18	12	6
$\frac{\eta(z)^2\eta(6z)^3}{\eta(2z)}$	4	2	18	12	6	12	$\frac{\eta(z)^3\eta(6z)^2}{\eta(3z)}$	4	2	12	6	12	18
$\frac{\eta(3z)^3}{\eta(z)}$	3	1	8	16	0	0	$\frac{\eta(2z)^3}{\eta(6z)}$	3	1	0	0	16	8
$\frac{\eta(6z)^3}{\eta(2z)}$	3	1	16	8	0	0	$\frac{\eta(z)^3}{\eta(3z)}$	3	1	0	0	8	16
$\frac{\eta(2z)^3\eta(3z)^3}{\eta(z)\eta(6z)}$	3	2	8	16	16	8	$\eta(z)^2\eta(3z)^2$	3	2	8	16	8	16
$\frac{\eta(z)^3\eta(6z)^3}{\eta(2z)\eta(3z)}$	3	2	16	8	8	16	$\eta(2z)^2\eta(6z)^2$	3	2	16	8	16	8
$\frac{\eta(2z)^2\eta(6z)^2}{\eta(z)\eta(3z)}$	2	1	12	0	12	0	$\frac{\eta(z)^2\eta(3z)^2}{\eta(2z)\eta(6z)}$	2	1	0	12	0	12
$\eta(z)\eta(2z)\eta(3z)\eta(6z)$	2	2	12	12	12	12							

B.7 Minimal Eta-Quotients of Level 8

Table B.7 Minimal eta-quotients of level 8

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0
$\frac{\eta(z)\eta(2z)^4}{\eta(8z)}$	24	2	1	7	17	23	$\frac{\eta(2z)^7}{\eta(z)\eta(4z)\eta(8z)}$	24	2	1	7	23	17
$\frac{\eta(z)\eta(2z)^2\eta(4z)^2}{\eta(8z)}$	24	2	5	11	13	19	$\frac{\eta(2z)^5\eta(4z)}{\eta(z)\eta(8z)}$	24	2	5	11	19	13
$\frac{\eta(2z)^8\eta(8z)}{\eta(z)\eta(4z)^4}$	24	2	7	1	23	17	$\frac{\eta(z)\eta(2z)^5\eta(8z)}{\eta(4z)^3}$	24	2	7	1	17	23
$\frac{\eta(2z)^6\eta(8z)}{\eta(z)\eta(4z)^2}$	24	2	11	5	19	13	$\frac{\eta(z)\eta(2z)^3\eta(8z)}{\eta(4z)}$	24	2	11	5	13	19
$\frac{\eta(z)\eta(4z)^6}{\eta(2z)^2\eta(8z)}$	24	2	13	19	5	11	$\frac{\eta(2z)\eta(4z)^5}{\eta(z)\eta(8z)}$	24	2	13	19	11	5
$\frac{\eta(z)\eta(4z)^8}{\eta(2z)^4\eta(8z)}$	24	2	17	23	1	7	$\frac{\eta(4z)^7}{\eta(z)\eta(2z)\eta(8z)}$	24	2	17	23	7	1
$\frac{\eta(2z)^2\eta(4z)^2\eta(8z)}{\eta(z)}$	24	2	19	13	11	5	$\frac{\eta(z)\eta(4z)^3\eta(8z)}{\eta(2z)}$	24	2	19	13	5	11
$\frac{\eta(4z)^4\eta(8z)}{\eta(z)}$	24	2	23	27	7	1	$\frac{\eta(z)\eta(4z)^5\eta(8z)}{\eta(2z)^3}$	24	2	23	17	1	7
$\frac{\eta(z)^3\eta(4z)^7}{\eta(2z)^3\eta(8z)^3}$	24	2	1	19	5	23	$\frac{\eta(2z)^6\eta(4z)^4}{\eta(z)^3\eta(8z)^3}$	24	2	1	19	23	5
$\frac{\eta(z)^3\eta(4z)^9}{\eta(2z)^5\eta(8z)^3}$	24	2	5	23	1	19	$\frac{\eta(2z)^4\eta(4z)^6}{\eta(z)^3\eta(8z)^3}$	24	2	5	23	19	1
$\frac{\eta(z)\eta(2z)\eta(4z)^3}{\eta(8z)}$	24	2	7	13	11	17	$\frac{\eta(2z)^4\eta(4z)^2}{\eta(z)\eta(8z)}$	24	2	7	13	17	11
$\frac{\eta(z)\eta(4z)^5}{\eta(2z)\eta(8z)}$	24	2	11	17	7	13	$\frac{\eta(2z)^2\eta(4z)^4}{\eta(z)\eta(8z)}$	24	2	11	17	13	7
$\frac{\eta(2z)^5\eta(8z)}{\eta(z)\eta(4z)}$	24	2	13	7	17	11	$\eta(z)\eta(2z)^2\eta(8z)$	24	2	13	7	11	17
$\frac{\eta(2z)^3\eta(4z)\eta(8z)}{\eta(z)}$	24	2	17	11	13	7	$\eta(z)\eta(4z)^2\eta(8z)$	24	2	17	11	7	13
$\frac{\eta(2z)^9\eta(8z)^3}{\eta(z)^3\eta(4z)^5}$	24	2	19	1	23	5	$\frac{\eta(z)^3\eta(8z)^3}{\eta(4z)^2}$	24	2	19	1	5	23
$\frac{\eta(2z)^7\eta(8z)^3}{\eta(z)^3\eta(4z)^3}$	24	2	23	5	19	1	$\frac{\eta(z)^3\eta(8z)^3}{\eta(2z)^2}$	24	2	23	5	1	19
$\frac{\eta(z)\eta(4z)^6}{\eta(8z)^3}$	24	2	1	19	11	17	$\frac{\eta(2z)^3\eta(4z)^5}{\eta(z)\eta(8z)^3}$	24	2	1	19	17	11
$\frac{\eta(z)\eta(4z)^8}{\eta(2z)^2\eta(8z)^3}$	24	2	5	23	7	13	$\frac{\eta(2z)\eta(4z)^7}{\eta(z)\eta(8z)^3}$	24	2	5	23	13	7
$\frac{\eta(z)^3\eta(4z)^4}{\eta(2z)^2\eta(8z)}$	24	2	7	13	5	23	$\frac{\eta(2z)^7\eta(4z)}{\eta(z)^3\eta(8z)}$	24	2	7	13	23	5
$\frac{\eta(z)^3\eta(4z)^6}{\eta(2z)^4\eta(8z)}$	24	2	11	17	1	19	$\frac{\eta(2z)^5\eta(4z)^3}{\eta(z)^3\eta(8z)}$	24	2	11	17	19	1
$\frac{\eta(2z)^8\eta(8z)}{\eta(z)^3\eta(4z)^2}$	24	2	13	7	23	5	$\frac{\eta(z)^3\eta(4z)\eta(8z)}{\eta(2z)}$	24	2	13	7	5	23
$\frac{\eta(2z)^6\eta(8z)}{\eta(z)^3}$	24	2	17	11	19	1	$\frac{\eta(z)^3\eta(4z)^3\eta(8z)}{\eta(2z)^3}$	24	2	17	11	1	19
$\frac{\eta(2z)^6\eta(8z)^3}{\eta(z)\eta(4z)^4}$	24	2	19	1	17	11	$\frac{\eta(z)\eta(2z)^3\eta(8z)^3}{\eta(4z)^3}$	24	2	19	1	11	17
$\frac{\eta(2z)^4\eta(8z)^3}{\eta(z)\eta(4z)^2}$	24	2	23	5	13	7	$\frac{\eta(z)\eta(2z)\eta(8z)^3}{\eta(4z)}$	24	2	23	5	7	13

Table B.7 cont. Minimal eta-quotients of level 8

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0
$\frac{\eta(z)\eta(2z)^2\eta(4z)^3}{\eta(8z)^2}$	24	2	1	13	14	20	$\frac{\eta(2z)^3\eta(4z)^2\eta(8z)}{\eta(z)^2}$	24	2	20	14	13	1
$\frac{\eta(2z)^6\eta(4z)^3}{\eta(z)^3\eta(8z)^2}$	24	2	5	17	22	4	$\frac{\eta(2z)^3\eta(4z)^6}{\eta(z)^2\eta(8z)^3}$	24	2	4	22	17	5
$\frac{\eta(z)^3\eta(4z)^7}{\eta(2z)^4\eta(8z)^2}$	24	2	7	19	2	20	$\frac{\eta(2z)^7\eta(8z)^3}{\eta(z)^2\eta(4z)^4}$	24	2	20	2	19	7
$\frac{\eta(4z)^7}{\eta(z)\eta(8z)^2}$	24	2	11	23	10	4	$\frac{\eta(2z)^7}{\eta(z)^2\eta(8z)}$	24	2	4	10	23	11
$\frac{\eta(z)\eta(2z)^4\eta(8z)^2}{\eta(4z)^3}$	24	2	13	1	14	20	$\frac{\eta(z)^2\eta(4z)^4\eta(8z)}{\eta(2z)^3}$	24	2	20	14	1	13
$\frac{\eta(2z)^8\eta(8z)^2}{\eta(z)^3\eta(4z)^3}$	24	2	17	5	22	4	$\frac{\eta(z)^2\eta(4z)^8}{\eta(2z)^3\eta(8z)^3}$	24	2	4	22	5	17
$\frac{\eta(z)^3\eta(4z)\eta(8z)^2}{\eta(2z)^2}$	24	2	19	7	2	20	$\frac{\eta(z)^2\eta(2z)\eta(8z)^3}{\eta(4z)^2}$	24	2	20	2	7	19
$\frac{\eta(2z)^2\eta(4z)\eta(8z)^2}{\eta(z)}$	24	2	23	11	10	4	$\frac{\eta(z)^2\eta(2z)\eta(4z)^2}{\eta(8z)}$	24	2	4	10	11	23
$\frac{\eta(2z)^5\eta(4z)^2}{\eta(z)\eta(8z)^2}$	24	2	1	13	20	14	$\frac{\eta(2z)^2\eta(4z)^5}{\eta(z)^2\eta(8z)}$	24	2	14	20	13	1
$\frac{\eta(z)^3\eta(4z)^6}{\eta(2z)^3\eta(8z)^2}$	24	2	5	17	4	22	$\frac{\eta(2z)^6\eta(8z)^3}{\eta(z)^2\eta(4z)^3}$	24	2	22	4	17	5
$\frac{\eta(2z)^5\eta(4z)^4}{\eta(z)^3\eta(8z)^2}$	24	2	7	19	20	2	$\frac{\eta(2z)^4\eta(4z)^5}{\eta(z)^2\eta(8z)^3}$	24	2	2	20	19	7
$\frac{\eta(z)\eta(4z)^8}{\eta(2z)^3\eta(8z)^2}$	24	2	11	23	4	10	$\frac{\eta(2z)^8\eta(8z)}{\eta(z)^2\eta(4z)^3}$	24	2	10	4	23	11
$\frac{\eta(2z)^7\eta(8z)^2}{\eta(z)\eta(4z)^4}$	24	2	13	1	20	14	$\frac{\eta(z)^2\eta(4z)^7}{\eta(2z)^4\eta(8z)}$	24	2	14	20	1	13
$\frac{\eta(z)^3\eta(8z)^2}{\eta(2z)}$	24	2	17	5	4	22	$\frac{\eta(z)^2\eta(8z)^3}{\eta(4z)}$	24	2	22	4	5	17
$\frac{\eta(2z)^7\eta(8z)^2}{\eta(z)^3\eta(4z)^2}$	24	2	19	7	20	2	$\frac{\eta(z)^2\eta(4z)^7}{\eta(2z)^2\eta(8z)^3}$	24	2	2	20	7	19
$\frac{\eta(z)\eta(4z)^2\eta(8z)^2}{\eta(2z)}$	24	2	23	11	4	10	$\frac{\eta(z)^2\eta(2z)^2\eta(8z)}{\eta(4z)}$	24	2	10	4	11	23
$\frac{\eta(z)\eta(4z)^2}{\eta(8z)}$	24	1	1	7	5	11	$\frac{\eta(2z)^3\eta(4z)}{\eta(z)\eta(8z)}$	24	1	1	7	11	5
$\frac{\eta(z)\eta(4z)^4}{\eta(2z)^2\eta(8z)}$	24	1	5	11	1	7	$\frac{\eta(2z)\eta(4z)^3}{\eta(z)\eta(8z)}$	24	1	5	11	7	1
$\frac{\eta(2z)^4\eta(8z)}{\eta(z)\eta(4z)^2}$	24	1	7	1	11	5	$\frac{\eta(z)\eta(2z)\eta(8z)}{\eta(4z)}$	24	1	7	1	5	11
$\frac{\eta(2z)^2\eta(8z)}{\eta(z)}$	24	1	11	5	7	1	$\frac{\eta(z)\eta(4z)\eta(8z)}{\eta(2z)}$	24	1	11	5	1	7
$\frac{\eta(z)\eta(2z)^2\eta(4z)^4}{\eta(8z)}$	24	3	13	19	17	23	$\frac{\eta(2z)^5\eta(4z)^3}{\eta(z)\eta(8z)}$	24	3	13	19	23	17
$\frac{\eta(z)\eta(4z)^6}{\eta(8z)}$	24	3	17	23	13	19	$\frac{\eta(2z)^3\eta(4z)^5}{\eta(z)\eta(8z)}$	24	3	17	23	19	13
$\frac{\eta(2z)^6\eta(8z)}{\eta(z)}$	24	3	19	13	23	17	$\eta(z)\eta(2z)^3\eta(4z)\eta(8z)$	24	3	19	13	17	23
$\frac{\eta(2z)^4\eta(4z)^2\eta(8z)}{\eta(z)}$	24	3	23	17	19	13	$\eta(z)\eta(2z)\eta(4z)^3\eta(8z)$	24	3	23	17	13	19
$\frac{\eta(z)^3}{\eta(2z)}$	24	1	1	1	2	20	$\frac{\eta(8z)^3}{\eta(4z)}$	24	1	20	2	1	1
$\frac{\eta(2z)^3}{\eta(z)}$	24	1	5	5	10	4	$\frac{\eta(4z)^3}{\eta(8z)}$	24	1	4	10	5	5
$\eta(z)\eta(2z)^3$	24	2	7	7	14	20	$\eta(4z)^3\eta(8z)$	24	2	20	14	7	7
$\frac{\eta(2z)^7}{\eta(z)^3}$	24	2	11	11	22	4	$\frac{\eta(4z)^7}{\eta(8z)^3}$	24	2	4	22	11	11
$\frac{\eta(z)^3\eta(4z)^4}{\eta(2z)^3}$	24	2	13	13	2	20	$\frac{\eta(2z)^4\eta(8z)^3}{\eta(4z)^3}$	24	2	20	2	13	13
$\frac{\eta(2z)\eta(4z)^4}{\eta(z)}$	24	2	17	17	10	4	$\frac{\eta(2z)^4\eta(4z)}{\eta(8z)}$	24	2	4	10	17	17
$\eta(z)\eta(2z)\eta(4z)^4$	24	3	19	19	14	20	$\eta(2z)^4\eta(4z)\eta(8z)$	24	3	20	14	19	19
$\frac{\eta(2z)^5\eta(4z)^4}{\eta(z)^3}$	24	3	23	23	22	4	$\frac{\eta(2z)^4\eta(4z)^5}{\eta(8z)^3}$	24	3	4	22	23	23

Table B.7 cont. Minimal eta-quotients of level 8

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0
$\frac{\eta(2z)^8}{\eta(z)^3\eta(4z)^3}$	24	1	1	1	20	2	$\frac{\eta(4z)^8}{\eta(2z)^3\eta(8z)^3}$	24	1	2	20	1	1
$\eta(z)\eta(4z)$	24	1	5	5	4	10	$\eta(2z)\eta(8z)$	24	1	10	4	5	5
$\frac{\eta(2z)^6}{\eta(z)\eta(4z)}$	24	2	7	7	20	14	$\frac{\eta(4z)^6}{\eta(2z)\eta(8z)}$	24	2	14	20	7	7
$\frac{\eta(z)^3\eta(4z)^3}{\eta(2z)^2}$	24	2	11	11	4	22	$\frac{\eta(2z)^3\eta(8z)^3}{\eta(4z)^2}$	24	2	22	4	11	11
$\frac{\eta(2z)^6\eta(4z)}{\eta(z)^3}$	24	2	13	13	20	2	$\frac{\eta(2z)\eta(4z)^6}{\eta(8z)^3}$	24	2	2	20	13	13
$\frac{\eta(z)\eta(4z)^5}{\eta(2z)^2}$	24	2	17	17	4	10	$\frac{\eta(2z)^5\eta(8z)}{\eta(4z)^2}$	24	2	10	4	17	17
$\frac{\eta(2z)^4\eta(4z)^3}{\eta(z)}$	24	3	19	19	20	14	$\frac{\eta(2z)^3\eta(4z)^4}{\eta(8z)}$	24	3	14	20	19	19
$\frac{\eta(z)^3\eta(4z)^7}{\eta(2z)^4}$	24	3	23	23	4	22	$\frac{\eta(2z)^7\eta(8z)^3}{\eta(4z)^4}$	24	3	22	4	23	23
$\frac{\eta(z)\eta(2z)^2}{\eta(4z)}$	24	1	1	1	8	14	$\frac{\eta(4z)^2\eta(8z)}{\eta(2z)}$	24	1	14	8	1	1
$\frac{\eta(z)\eta(2z)^4}{\eta(4z)}$	24	2	5	5	16	22	$\frac{\eta(4z)^4\eta(8z)}{\eta(2z)}$	24	2	22	16	5	5
$\frac{\eta(2z)^2\eta(4z)}{\eta(z)}$	24	1	7	7	8	2	$\frac{\eta(2z)\eta(4z)^2}{\eta(8z)}$	24	1	2	8	7	7
$\frac{\eta(2z)^4\eta(4z)}{\eta(z)}$	24	2	11	11	16	10	$\frac{\eta(2z)\eta(4z)^4}{\eta(8z)}$	24	2	10	16	11	11
$\eta(z)\eta(4z)^3$	24	2	13	13	8	14	$\eta(2z)^3\eta(8z)$	24	2	14	8	13	13
$\eta(z)\eta(2z)^2\eta(4z)^3$	24	3	17	17	16	22	$\eta(2z)^3\eta(4z)^2\eta(8z)$	24	3	22	16	17	17
$\frac{\eta(4z)^5}{\eta(z)}$	24	2	19	19	8	2	$\frac{\eta(2z)^5}{\eta(8z)}$	24	2	2	8	19	19
$\frac{\eta(2z)^2\eta(4z)^5}{\eta(z)}$	24	3	23	23	16	10	$\frac{\eta(2z)^5\eta(4z)^2}{\eta(8z)}$	24	3	10	16	23	23
$\frac{\eta(2z)^5}{\eta(z)\eta(4z)^2}$	24	1	1	1	14	8	$\frac{\eta(4z)^5}{\eta(2z)^2\eta(8z)}$	24	1	8	14	1	1
$\frac{\eta(2z)^7}{\eta(z)\eta(2z)^2}$	24	2	5	5	22	16	$\frac{\eta(4z)^7}{\eta(2z)^2\eta(8z)}$	24	2	16	22	5	5
$\frac{\eta(z)\eta(4z)^2}{\eta(2z)}$	24	1	7	7	2	8	$\frac{\eta(2z)^2\eta(8z)}{\eta(4z)}$	24	1	8	2	7	7
$\eta(z)\eta(2z)\eta(4z)^2$	24	2	11	11	10	16	$\eta(2z)^2\eta(4z)\eta(8z)$	24	2	16	10	11	11
$\frac{\eta(2z)^3\eta(4z)^2}{\eta(z)}$	24	2	13	13	14	8	$\frac{\eta(2z)^2\eta(4z)^3}{\eta(8z)}$	24	2	8	14	13	13
$\frac{\eta(2z)^5\eta(4z)^2}{\eta(z)}$	24	3	17	17	22	16	$\frac{\eta(2z)^2\eta(4z)^5}{\eta(8z)}$	24	3	16	22	17	17
$\frac{\eta(z)\eta(4z)^6}{\eta(2z)^3}$	24	2	19	19	2	8	$\frac{\eta(2z)^6\eta(8z)}{\eta(4z)^3}$	24	2	8	2	19	19
$\frac{\eta(z)\eta(4z)^6}{\eta(2z)}$	24	3	23	23	10	16	$\frac{\eta(2z)^6\eta(8z)}{\eta(4z)}$	24	3	16	10	23	23
$\frac{\eta(z)\eta(4z)^5}{\eta(2z)^2\eta(8z)^2}$	24	1	1	13	2	8	$\frac{\eta(2z)^5\eta(8z)}{\eta(z)^2\eta(4z)^2}$	24	1	8	2	13	1
$\frac{\eta(z)\eta(4z)^5}{\eta(8z)^2}$	24	2	5	17	10	16	$\frac{\eta(2z)^5\eta(8z)}{\eta(z)^2}$	24	2	16	10	17	5
$\frac{\eta(2z)^2\eta(4z)^5}{\eta(z)\eta(8z)^2}$	24	2	7	19	14	8	$\frac{\eta(2z)^5\eta(4z)^2}{\eta(z)^2\eta(8z)}$	24	2	8	14	19	7
$\frac{\eta(2z)^4\eta(4z)^5}{\eta(z)\eta(8z)^2}$	24	3	11	23	22	16	$\frac{\eta(2z)^5\eta(4z)^4}{\eta(z)^2\eta(8z)}$	24	3	16	22	23	11
$\frac{\eta(z)\eta(8z)^2}{\eta(4z)}$	24	1	13	1	2	8	$\frac{\eta(z)^2\eta(8z)}{\eta(2z)}$	24	1	8	2	1	13
$\frac{\eta(z)\eta(2z)^2\eta(8z)^2}{\eta(4z)}$	24	2	17	5	10	16	$\frac{\eta(z)^2\eta(4z)^2\eta(8z)}{\eta(2z)}$	24	2	16	10	5	17
$\frac{\eta(2z)^4\eta(8z)^2}{\eta(z)\eta(4z)}$	24	2	19	7	14	8	$\frac{\eta(z)^2\eta(4z)^4}{\eta(2z)\eta(8z)}$	24	2	8	14	7	19
$\frac{\eta(2z)^6\eta(8z)^2}{\eta(z)\eta(4z)}$	24	3	23	11	22	16	$\frac{\eta(z)^2\eta(4z)^6}{\eta(2z)\eta(8z)}$	24	3	16	22	11	23

Table B.7 cont. Minimal eta-quotients of level 8

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0
$\frac{\eta(2z)\eta(4z)^4}{\eta(z)\eta(8z)^2}$	24	1	1	13	8	2	$\frac{\eta(2z)^4\eta(4z)}{\eta(z)^2\eta(8z)}$	24	1	2	8	13	1
$\frac{\eta(2z)^3\eta(4z)^4}{\eta(z)\eta(8z)^2}$	24	2	5	17	16	10	$\frac{\eta(2z)^4\eta(4z)^3}{\eta(z)^2\eta(8z)}$	24	2	10	16	17	5
$\frac{\eta(z)\eta(4z)^6}{\eta(2z)\eta(8z)^2}$	24	2	7	19	8	14	$\frac{\eta(2z)^6\eta(8z)}{\eta(z)^2\eta(4z)}$	24	2	14	8	19	7
$\frac{\eta(z)\eta(2z)\eta(4z)^6}{\eta(8z)^2}$	24	3	11	23	16	22	$\frac{\eta(2z)^6\eta(4z)\eta(8z)}{\eta(2z)^2}$	24	3	22	16	23	11
$\frac{\eta(2z)^3\eta(8z)^2}{\eta(z)\eta(4z)^2}$	24	1	13	1	8	2	$\frac{\eta(z)^2\eta(4z)^3}{\eta(2z)^2\eta(8z)}$	24	1	2	8	1	13
$\frac{\eta(2z)^5\eta(8z)^2}{\eta(z)\eta(4z)^2}$	24	2	17	5	16	10	$\frac{\eta(z)^2\eta(4z)^5}{\eta(2z)^2\eta(8z)}$	24	2	10	16	5	17
$\eta(z)\eta(2z)\eta(8z)^2$	24	2	19	7	8	14	$\eta(z)^2\eta(4z)\eta(8z)$	24	2	14	8	7	19
$\eta(z)\eta(2z)^3\eta(8z)^2$	24	3	23	11	16	22	$\eta(z)^2\eta(4z)^3\eta(8z)$	24	3	22	16	11	23
$\frac{\eta(2z)^3\eta(4z)^3}{\eta(8z)^2}$	12	2	2	14	16	16	$\frac{\eta(2z)^3\eta(4z)^3}{\eta(z)^2}$	12	2	16	16	14	2
$\frac{\eta(4z)^7}{\eta(2z)\eta(8z)^2}$	12	2	10	22	8	8	$\frac{\eta(2z)^7}{\eta(z)^2\eta(4z)}$	12	2	8	8	22	10
$\frac{\eta(2z)^5\eta(8z)^2}{\eta(4z)^3}$	12	2	14	2	16	16	$\frac{\eta(z)^2\eta(4z)^5}{\eta(2z)^3}$	12	2	16	16	2	14
$\eta(2z)\eta(4z)\eta(8z)^2$	12	2	22	10	8	8	$\eta(z)^2\eta(2z)\eta(4z)$	12	2	8	8	10	22
$\frac{\eta(z)^2\eta(4z)^4}{\eta(8z)^2}$	12	2	2	14	10	22	$\frac{\eta(2z)^6\eta(4z)^2}{\eta(z)^2\eta(8z)^2}$	12	2	2	14	22	10
$\frac{\eta(z)^2\eta(4z)^8}{\eta(2z)^4\eta(8z)^2}$	12	2	10	22	2	14	$\frac{\eta(2z)^2\eta(4z)^6}{\eta(z)^2\eta(8z)^2}$	12	2	10	22	14	2
$\frac{\eta(2z)^8\eta(8z)^2}{\eta(z)^2\eta(4z)^4}$	12	2	14	2	22	10	$\frac{\eta(z)^2\eta(2z)^2\eta(8z)^2}{\eta(4z)^2}$	12	2	14	2	10	22
$\frac{\eta(2z)^4\eta(8z)^2}{\eta(z)^2}$	12	2	22	10	14	2	$\frac{\eta(z)^2\eta(4z)^2\eta(8z)^2}{\eta(2z)^2}$	12	2	22	10	2	14
$\frac{\eta(2z)^7}{\eta(4z)^3}$	12	2	2	2	22	22	$\frac{\eta(2z)^3}{\eta(4z)}$	12	1	2	2	10	10
$\eta(2z)^3\eta(4z)$	12	2	10	10	14	14	$\frac{\eta(4z)^3}{\eta(2z)}$	12	1	10	10	2	2
$\eta(2z)\eta(4z)^3$	12	2	14	14	10	10	$\eta(2z)^5\eta(4z)$	12	3	14	14	22	22
$\frac{\eta(4z)^7}{\eta(2z)^3}$	12	2	22	22	2	2	$\eta(2z)\eta(4z)^5$	12	3	22	22	14	14
$\frac{\eta(4z)^5}{\eta(2z)\eta(8z)^2}$	12	1	2	14	4	4	$\frac{\eta(2z)^5}{\eta(z)^2\eta(4z)}$	12	1	4	4	14	2
$\frac{\eta(2z)^3\eta(4z)^5}{\eta(8z)^2}$	12	3	10	22	20	20	$\frac{\eta(2z)^5\eta(4z)^3}{\eta(z)^2}$	12	3	20	20	22	10
$\frac{\eta(2z)\eta(8z)^2}{\eta(4z)}$	12	1	14	2	4	4	$\frac{\eta(z)^2\eta(4z)}{\eta(2z)}$	12	1	4	4	2	14
$\frac{\eta(2z)^5\eta(8z)^2}{\eta(4z)}$	12	3	22	10	20	20	$\frac{\eta(z)^2\eta(4z)^5}{\eta(2z)}$	12	3	20	20	10	22
$\eta(z)^2$	12	1	2	2	4	16	$\eta(8z)^2$	12	1	16	4	2	2
$\frac{\eta(2z)^6}{\eta(z)^2}$	12	2	10	10	20	8	$\frac{\eta(4z)^6}{\eta(8z)^2}$	12	2	8	20	10	10
$\frac{\eta(z)^2\eta(4z)^4}{\eta(2z)^2}$	12	2	14	14	4	16	$\frac{\eta(2z)^4\eta(8z)^2}{\eta(4z)^2}$	12	2	16	4	14	14
$\frac{\eta(2z)^4\eta(4z)^4}{\eta(z)^2}$	12	3	22	22	20	8	$\frac{\eta(2z)^4\eta(4z)^4}{\eta(8z)^2}$	12	3	8	20	22	22
$\frac{\eta(2z)^6}{\eta(z)^2\eta(4z)^2}$	12	1	2	2	16	4	$\frac{\eta(4z)^6}{\eta(2z)^2\eta(8z)^2}$	12	1	4	16	2	2
$\eta(z)^2\eta(4z)^2$	12	2	10	10	8	20	$\eta(2z)^2\eta(8z)^2$	12	2	20	8	10	10
$\frac{\eta(2z)^4\eta(4z)^2}{\eta(z)^2}$	12	2	14	14	16	4	$\frac{\eta(2z)^2\eta(4z)^4}{\eta(8z)^2}$	12	2	4	16	14	14
$\frac{\eta(z)^2\eta(4z)^6}{\eta(2z)^2}$	12	3	22	22	8	20	$\frac{\eta(2z)^6\eta(8z)^2}{\eta(4z)^2}$	12	3	20	8	22	22

Table B.7 cont. Minimal eta-quotients of level 8

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0
$\frac{\eta(z)\eta(2z)^3\eta(4z)}{\eta(8z)}$	8	2	3	9	15	21	$\frac{\eta(2z)^6}{\eta(z)\eta(8z)}$	8	2	3	9	21	15
$\frac{\eta(2z)^7\eta(8z)}{\eta(z)\eta(4z)^3}$	8	2	9	3	21	15	$\frac{\eta(z)\eta(2z)^4\eta(8z)}{\eta(4z)^2}$	8	2	9	3	15	21
$\frac{\eta(z)\eta(4z)^7}{\eta(2z)^3\eta(8z)}$	8	2	15	21	3	9	$\frac{\eta(4z)^6}{\eta(z)\eta(8z)}$	8	2	15	21	9	3
$\frac{\eta(2z)\eta(4z)^3\eta(8z)}{\eta(z)}$	8	2	21	15	9	3	$\frac{\eta(z)\eta(4z)^4\eta(8z)}{\eta(2z)^2}$	8	2	21	15	3	9
$\frac{\eta(z)\eta(4z)^7}{\eta(2z)\eta(8z)^3}$	8	2	3	21	9	15	$\frac{\eta(2z)^2\eta(4z)^6}{\eta(z)\eta(8z)^3}$	8	2	3	21	15	9
$\frac{\eta(z)^3\eta(4z)^5}{\eta(2z)^3\eta(8z)}$	8	2	9	15	3	21	$\frac{\eta(2z)^6\eta(4z)^2}{\eta(z)^3\eta(8z)}$	8	2	9	15	21	3
$\frac{\eta(2z)^7\eta(8z)}{\eta(z)^3\eta(4z)}$	8	2	15	9	21	3	$\frac{\eta(z)^3\eta(4z)^2\eta(8z)}{\eta(2z)^2}$	8	2	15	9	3	21
$\frac{\eta(2z)^5\eta(8z)^3}{\eta(z)\eta(4z)^3}$	8	2	21	3	15	9	$\frac{\eta(z)\eta(2z)^2\eta(8z)^3}{\eta(4z)^2}$	8	2	21	3	9	15
$\frac{\eta(2z)^5\eta(4z)^5}{\eta(z)^3\eta(8z)^3}$	8	2	3	21	21	3	$\frac{\eta(z)^3\eta(4z)^8}{\eta(2z)^4\eta(8z)^3}$	8	2	3	21	3	21
$\frac{\eta(2z)^3\eta(4z)^3}{\eta(z)\eta(8z)}$	8	2	9	15	15	9	$\frac{\eta(z)\eta(4z)^4}{\eta(8z)}$	8	2	9	15	9	15
$\eta(z)\eta(2z)\eta(4z)\eta(8z)$	8	2	15	9	9	15	$\frac{\eta(2z)^4\eta(8z)}{\eta(z)}$	8	2	15	9	15	9
$\frac{\eta(z)^3\eta(8z)^3}{\eta(2z)\eta(4z)}$	8	2	21	3	3	21	$\frac{\eta(2z)^8\eta(8z)^3}{\eta(z)^3\eta(4z)^4}$	8	2	21	3	21	3
$\frac{\eta(z)\eta(2z)\eta(4z)^4}{\eta(8z)^2}$	8	2	3	15	12	18	$\frac{\eta(2z)^4\eta(4z)\eta(8z)}{\eta(z)^2}$	8	2	18	12	15	3
$\frac{\eta(2z)\eta(4z)^6}{\eta(z)\eta(8z)^2}$	8	2	9	21	12	6	$\frac{\eta(2z)^6\eta(4z)}{\eta(z)^2\eta(8z)}$	8	2	6	12	21	9
$\frac{\eta(z)\eta(2z)^3\eta(8z)^2}{\eta(4z)^2}$	8	2	15	3	12	18	$\frac{\eta(z)^2\eta(4z)^3\eta(8z)}{\eta(2z)^2}$	8	2	18	12	3	15
$\frac{\eta(2z)^3\eta(8z)^2}{\eta(z)}$	8	2	21	9	12	6	$\frac{\eta(z)^2\eta(4z)^3}{\eta(8z)}$	8	2	6	12	9	21
$\frac{\eta(2z)^4\eta(4z)^3}{\eta(z)\eta(8z)^2}$	8	2	3	15	18	12	$\frac{\eta(2z)^3\eta(4z)^4}{\eta(z)^2\eta(8z)}$	8	2	12	18	15	3
$\frac{\eta(z)\eta(4z)^7}{\eta(2z)^2\eta(8z)^2}$	8	2	9	21	6	12	$\frac{\eta(2z)^7\eta(8z)}{\eta(z)^2\eta(4z)^2}$	8	2	12	6	21	9
$\frac{\eta(2z)^6\eta(8z)^2}{\eta(z)\eta(4z)^3}$	8	2	15	3	18	12	$\frac{\eta(z)^2\eta(4z)^6}{\eta(2z)^3\eta(8z)}$	8	2	12	18	3	15
$\eta(z)\eta(4z)\eta(8z)^2$	8	2	21	9	6	12	$\eta(z)^2\eta(2z)\eta(8z)$	8	2	12	6	9	21
$\frac{\eta(z)^3\eta(4z)}{\eta(2z)^2}$	8	1	3	3	0	18	$\frac{\eta(2z)\eta(8z)^3}{\eta(4z)^2}$	8	1	18	0	3	3
$\frac{\eta(z)\eta(4z)^3}{\eta(2z)^2}$	8	1	9	9	0	6	$\frac{\eta(2z)^3\eta(8z)}{\eta(4z)^2}$	8	1	6	0	9	9
$\frac{\eta(z)^3\eta(4z)^5}{\eta(2z)^4}$	8	2	15	15	0	18	$\frac{\eta(2z)^5\eta(8z)^3}{\eta(4z)^4}$	8	2	18	0	15	15
$\frac{\eta(z)\eta(4z)^7}{\eta(2z)^4}$	8	2	21	21	0	6	$\frac{\eta(2z)^7\eta(8z)}{\eta(4z)^4}$	8	2	6	0	21	21
$\frac{\eta(2z)^7}{\eta(z)^3\eta(4z)^2}$	8	1	3	3	18	0	$\frac{\eta(4z)^7}{\eta(2z)^2\eta(8z)^3}$	8	1	0	18	3	3
$\frac{\eta(2z)\eta(4z)^2}{\eta(z)}$	8	1	9	9	6	0	$\frac{\eta(2z)^2\eta(4z)}{\eta(8z)}$	8	1	0	6	9	9
$\frac{\eta(2z)^5\eta(4z)^2}{\eta(z)^3}$	8	2	15	15	18	0	$\frac{\eta(2z)^2\eta(4z)^5}{\eta(8z)^3}$	8	2	0	18	15	15
$\frac{\eta(4z)^6}{\eta(z)\eta(2z)}$	8	2	21	21	6	0	$\frac{\eta(2z)^6}{\eta(4z)\eta(8z)}$	8	2	0	6	21	21
$\frac{\eta(z)\eta(4z)^6}{\eta(2z)^3\eta(8z)^2}$	8	1	3	15	0	6	$\frac{\eta(2z)^6\eta(8z)}{\eta(z)^2\eta(4z)^3}$	8	1	6	0	15	3
$\frac{\eta(z)^3\eta(4z)^8}{\eta(2z)^5\eta(8z)^2}$	8	2	9	21	0	18	$\frac{\eta(2z)^8\eta(8z)^3}{\eta(z)^2\eta(4z)^5}$	8	2	18	0	21	9
$\frac{\eta(z)\eta(8z)^2}{\eta(2z)}$	8	1	15	3	0	6	$\frac{\eta(z)^2\eta(8z)}{\eta(4z)}$	8	1	6	0	3	15
$\frac{\eta(z)^3\eta(4z)^2\eta(8z)^2}{\eta(2z)^3}$	8	2	21	9	0	18	$\frac{\eta(z)^2\eta(2z)^2\eta(8z)^3}{\eta(4z)^3}$	8	2	18	0	9	21

Table B.7 cont. Minimal eta-quotients of level 8

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0
$\frac{\eta(4z)^5}{\eta(z)\eta(8z)^2}$	8	1	3	15	6	0	$\frac{\eta(2z)^5}{\eta(z)^2\eta(8z)}$	8	1	0	6	15	3
$\frac{\eta(2z)^4\eta(4z)^5}{\eta(z)^3\eta(8z)^2}$	8	2	9	21	18	0	$\frac{\eta(2z)^5\eta(4z)^4}{\eta(z)^2\eta(8z)^3}$	8	2	0	18	21	9
$\frac{\eta(2z)^2\eta(8z)^2}{\eta(z)\eta(4z)}$	8	1	15	3	6	0	$\frac{\eta(z)^2\eta(4z)^2}{\eta(2z)\eta(8z)}$	8	1	0	6	3	15
$\frac{\eta(2z)^6\eta(8z)^2}{\eta(z)^3\eta(4z)}$	8	2	21	9	18	0	$\frac{\eta(z)^2\eta(4z)^6}{\eta(2z)\eta(8z)^3}$	8	2	0	18	9	21
$\frac{\eta(2z)^2\eta(4z)^2}{\eta(z)\eta(8z)}$	8	1	3	9	9	3	$\frac{\eta(z)\eta(4z)^3}{\eta(2z)\eta(8z)}$	8	1	3	9	3	9
$\eta(z)\eta(8z)$	8	1	9	3	3	9	$\frac{\eta(2z)^3\eta(8z)}{\eta(z)\eta(4z)}$	8	1	9	3	9	3
$\frac{\eta(2z)^4\eta(4z)^4}{\eta(z)\eta(8z)}$	8	3	15	21	21	15	$\frac{\eta(z)\eta(2z)\eta(4z)^5}{\eta(8z)}$	8	3	15	21	15	21
$\eta(z)\eta(2z)^2\eta(4z)^2\eta(8z)$	8	3	21	15	15	21	$\frac{\eta(2z)^5\eta(4z)\eta(8z)}{\eta(z)}$	8	3	21	15	21	15
$\eta(z)\eta(2z)$	8	1	3	3	6	12	$\eta(4z)\eta(8z)$	8	1	12	6	3	3
$\frac{\eta(2z)^5}{\eta(z)}$	8	2	9	9	18	12	$\frac{\eta(4z)^5}{\eta(8z)}$	8	2	12	18	9	9
$\frac{\eta(z)\eta(4z)^4}{\eta(2z)}$	8	2	15	15	6	12	$\frac{\eta(2z)^4\eta(8z)}{\eta(4z)}$	8	2	12	6	15	15
$\frac{\eta(2z)^3\eta(4z)^4}{\eta(z)}$	8	3	21	21	18	12	$\frac{\eta(2z)^4\eta(4z)^3}{\eta(8z)}$	8	3	12	18	21	21
$\frac{\eta(2z)^4}{\eta(z)\eta(4z)}$	8	1	3	3	12	6	$\frac{\eta(4z)^4}{\eta(2z)\eta(8z)}$	8	1	6	12	3	3
$\eta(z)\eta(2z)^2\eta(4z)$	8	2	9	9	12	18	$\eta(2z)\eta(4z)^2\eta(8z)$	8	2	18	12	9	9
$\frac{\eta(2z)^2\eta(4z)^3}{\eta(z)}$	8	2	15	15	12	6	$\frac{\eta(2z)^3\eta(4z)^2}{\eta(8z)}$	8	2	6	12	15	15
$\eta(z)\eta(4z)^5$	8	3	21	21	12	18	$\eta(2z)^5\eta(8z)$	8	3	18	12	21	21
$\frac{\eta(z)^2\eta(4z)^5}{\eta(2z)\eta(8z)^2}$	6	2	4	16	8	20	$\frac{\eta(2z)^7\eta(8z)^2}{\eta(z)^2\eta(4z)^3}$	6	2	16	4	20	8
$\frac{\eta(2z)^5\eta(8z)^2}{\eta(z)^2\eta(4z)}$	6	2	20	8	16	4	$\frac{\eta(z)^2\eta(4z)^7}{\eta(2z)^3\eta(8z)^2}$	6	2	8	20	4	16
$\frac{\eta(2z)^5\eta(4z)^3}{\eta(z)^2\eta(8z)^2}$	6	2	4	16	20	8	$\frac{\eta(2z)^3\eta(4z)^5}{\eta(z)^2\eta(8z)^2}$	6	2	8	20	16	4
$\frac{\eta(z)^2\eta(4z)\eta(8z)^2}{\eta(2z)}$	6	2	20	8	4	16	$\frac{\eta(z)^2\eta(2z)\eta(8z)^2}{\eta(4z)}$	6	2	16	4	8	20
$\eta(2z)^2$	6	1	4	4	8	8	$\eta(4z)^2$	6	1	8	8	4	4
$\eta(2z)^2\eta(4z)^4$	6	3	20	20	16	16	$\eta(2z)^4\eta(4z)^2$	6	3	16	16	20	20
$\frac{\eta(2z)^6}{\eta(4z)^2}$	6	2	4	4	20	20	$\eta(2z)\eta(4z)$	4	1	6	6	6	6
$\frac{\eta(4z)^6}{\eta(2z)^2}$	6	2	20	20	4	4	$\eta(2z)^3\eta(4z)^3$	4	3	18	18	18	18
$\frac{\eta(4z)^7}{\eta(2z)^3\eta(8z)^2}$	4	1	6	18	0	0	$\frac{\eta(2z)^7}{\eta(z)^2\eta(4z)^3}$	4	1	0	0	18	6
$\frac{\eta(4z)\eta(8z)^2}{\eta(2z)}$	4	1	18	6	0	0	$\frac{\eta(z)^2\eta(2z)}{\eta(4z)}$	4	1	0	0	6	18
$\frac{\eta(z)^2\eta(4z)^2}{\eta(2z)^2}$	4	1	6	6	0	12	$\frac{\eta(2z)^2\eta(8z)^2}{\eta(4z)^2}$	4	1	12	0	6	6
$\frac{\eta(z)^2\eta(4z)^6}{\eta(2z)^4}$	4	2	18	18	0	12	$\frac{\eta(2z)^6\eta(8z)^2}{\eta(4z)^4}$	4	2	12	0	18	18
$\frac{\eta(2z)^4}{\eta(z)^2}$	4	1	6	6	12	0	$\frac{\eta(4z)^4}{\eta(8z)^2}$	4	1	0	12	6	6
$\frac{\eta(2z)^2\eta(4z)^4}{\eta(z)^2}$	4	2	18	18	12	0	$\frac{\eta(2z)^4\eta(4z)^2}{\eta(8z)^2}$	4	2	0	12	18	18
$\frac{\eta(2z)\eta(4z)^5}{\eta(8z)^2}$	4	2	6	18	12	12	$\frac{\eta(2z)^5\eta(4z)}{\eta(z)^2}$	4	2	12	12	18	6
$\frac{\eta(2z)^3\eta(8z)^2}{\eta(4z)}$	4	2	18	6	12	12	$\frac{\eta(z)^2\eta(4z)^3}{\eta(2z)}$	4	2	12	12	6	18

Table B.7 cont. Minimal eta-quotients of level 8

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0
$\frac{\eta(2z)^5}{\eta(4z)}$	4	2	6	6	18	18
$\frac{\eta(4z)^5}{\eta(2z)}$	4	2	18	18	6	6
$\frac{\eta(2z)^4\eta(4z)^4}{\eta(z)^2\eta(8z)^2}$	4	2	6	18	18	6
$\eta(z)^2\eta(8z)^2$	4	2	18	6	6	18
$\frac{\eta(2z)^4}{\eta(4z)^2}$	2	1	0	0	12	12
$\frac{\eta(z)^2\eta(4z)^5}{\eta(2z)^3\eta(8z)^2}$	2	1	0	12	0	12
$\frac{\eta(2z)^3\eta(4z)^3}{\eta(z)^2\eta(8z)^2}$	2	1	0	12	12	0
$\eta(2z)^2\eta(4z)^2$	2	2	12	12	12	12

f	D	k	v_∞	$v_{\frac{1}{2}}$	$v_{\frac{1}{4}}$	v_0
$\frac{\eta(z)^2\eta(4z)^6}{\eta(2z)^2\eta(8z)^2}$	4	2	6	18	6	18
$\frac{\eta(2z)^6\eta(8z)^2}{\eta(z)^2\eta(4z)^2}$	4	2	18	6	18	6
$\eta(2z)^4$	3	2	8	8	16	16
$\eta(4z)^4$	3	2	16	16	8	8
$\frac{\eta(4z)^4}{\eta(2z)^2}$	2	1	12	12	0	0
$\frac{\eta(2z)^5\eta(8z)^2}{\eta(z)^2\eta(4z)^3}$	2	1	12	0	12	0
$\frac{\eta(z)^2\eta(8z)^2}{\eta(2z)\eta(4z)}$	2	1	12	0	0	12

B.8 Minimal Eta-Quotients of Level 9

Table B.8 Minimal eta-quotients of level 9

f	D	k	v_∞	$v_{\frac{1}{3}}$	v_0	f	D	k	v_∞	$v_{\frac{1}{3}}$	v_0
$\frac{\eta(3z)^7}{\eta(z)\eta(9z)^2}$	12	2	2	18	10	$\frac{\eta(3z)^4}{\eta(z)\eta(9z)}$	12	1	2	10	2
$\frac{\eta(3z)^7}{\eta(z)^2\eta(9z)}$	12	2	10	18	2	$\eta(z)\eta(9z)$	12	1	10	2	10
$\eta(z)^2\eta(3z)\eta(9z)$	12	2	14	6	22	$\frac{\eta(3z)^8}{\eta(z)\eta(9z)}$	12	3	14	22	14
$\eta(z)\eta(3z)\eta(9z)^2$	12	2	22	6	14	$\eta(z)\eta(3z)^4\eta(9z)$	12	3	22	14	22
$\eta(z)^2$	12	1	2	2	18	$\eta(9z)^2$	12	1	18	2	2
$\eta(z)\eta(3z)^3$	12	2	10	10	18	$\eta(3z)^3\eta(9z)$	12	2	18	10	10
$\frac{\eta(3z)^5}{\eta(z)}$	12	2	14	14	6	$\frac{\eta(3z)^5}{\eta(9z)}$	12	2	6	14	14
$\frac{\eta(3z)^8}{\eta(z)^2}$	12	3	22	22	6	$\frac{\eta(3z)^8}{\eta(9z)^2}$	12	3	6	22	22
$\frac{\eta(3z)^8}{\eta(z)^2\eta(9z)^2}$	6	2	4	20	4	$\frac{\eta(z)\eta(3z)^4}{\eta(9z)}$	6	2	4	12	20
$\eta(z)^2\eta(9z)^2$	6	2	20	4	20	$\frac{\eta(3z)^4\eta(9z)}{\eta(z)}$	6	2	20	12	4
$\eta(z)\eta(3z)$	6	1	4	4	12	$\eta(3z)\eta(9z)$	6	1	12	4	4
$\frac{\eta(3z)^7}{\eta(z)}$	6	3	20	20	12	$\frac{\eta(3z)^7}{\eta(9z)}$	6	3	12	20	20
$\eta(3z)^2$	4	1	6	6	6	$\frac{\eta(z)^2\eta(9z)}{\eta(3z)}$	3	1	8	0	16
$\eta(3z)^6$	4	3	18	18	18	$\frac{\eta(z)\eta(9z)^2}{\eta(3z)}$	3	1	16	0	8
$\frac{\eta(3z)^6}{\eta(z)\eta(9z)}$	3	2	8	16	8	$\frac{\eta(3z)^3}{\eta(z)}$	3	1	8	8	0
$\eta(z)\eta(3z)^2\eta(9z)$	3	2	16	8	16	$\frac{\eta(3z)^6}{\eta(z)^2}$	3	2	16	16	0
$\frac{\eta(3z)^3}{\eta(9z)}$	3	1	0	8	8	$\eta(3z)^4$	2	2	12	12	12
$\frac{\eta(3z)^6}{\eta(9z)^2}$	3	2	0	16	16						