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Trimming Complexes

Keller VandeBogert

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by

Keller VandeBogert

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Accepted by:

Matthew Ballard, Major Professor

Andrew Kustin, Major Professor

Matthew Boylan, Committee Member

Maria Girardi, Committee Member

Ralph Howard, Committee Member

Joshua Tebbs, Committee Member

Tracey Weldon, Interim Vice Provost and Dean of the Graduate School

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Abstract

We produce a family of complexes called trimming complexes and explore applications. We first study ideals defining type 2 compressed rings with socle minimally generated in degrees s and $2s-1$ for $s > 2$. We prove that all such ideals arise as trimmings of grade 3 Gorenstein ideals and show that trimming complexes yield an explicit free resolution. In particular, we give bounds on parameters arising in the Tor-algebra classification and construct explicit ideals attaining all intermediate values for every s. This partially answers a question of realizability of Tor-algebra structures posed by Avramov. Next, we study how trimming complexes can be used to deduce the Betti table for the minimal free resolution of the ideal generated by subsets of a generating set for an arbitrary ideal I. In particular, explicit Betti tables are computed for an infinite class of determinantal facet ideals; previously, Betti numbers for anything more than the linear strand had not been computed explicitly. Next, we study certain classes of equigenerated monomial ideals with the property that the so-called complementary ideal has no linear relations on the generators. We then use iterated trimming complexes to deduce Betti numbers for such ideals. Furthermore, using a result on splitting mapping cones by Miller and Rahmati, we construct the minimal free resolutions for all ideals under consideration explicitly and conclude with questions about extra structure on these complexes. Finally, we consider the iterated trimming complex associated to data yielding a complex of length 3. We compute an explicit algebra structure in this complex in terms of the algebra structures of the associated input data. Moreover, it is shown that many of these products become trivial after descending to homology. We apply these results to the problem of realizability for Tor-algebras of grade 3 perfect ideals, and show that under mild hypotheses, the process of “trimming” an ideal preserves Tor-algebra class. In particular, we construct new classes of ideals in arbitrary regular local rings defining rings realizing Tor-algebra classes $G(r)$ and $H(p, q)$ for a prescribed set of homological data.

Table of Contents

Chapter 1

Introduction

Let $(R, \mathfrak{m}, k)$ be a regular local ring with maximal ideal $\mathfrak{m}$. A result of Buchsbaum and Eisenbud (see [11]) established that any quotient R/I of R with projective dimension 3 admits the structure of an associative commutative differental graded (DG) algebra. Later, a complete classification of the multiplicative structure of the Tor algebra $\operatorname{Tor}_\bullet^R(R/I, k)$ for such quotients was established by Weyman in [43] and Avramov, Kustin, and Miller in [4].

One parametrized family arising from the aforementioned classification of Tor algebras is the class $G(r)$, where r is a parameter arising from the rank of the induced map

$$\delta : \operatorname{Tor}_2^R(R/I, k) \to \operatorname{Hom}_k(\operatorname{Tor}_1^R(R/I, k), \operatorname{Tor}_3^R(R/I, k)).$$

If $I \subset R$ is such that R/I is Gorenstein, then it is shown by Avramov and Golod in [3] that the Koszul homology algebra of R/I is a Poincaré duality algebra. Indeed, an equivalent characterization of the Tor algebra class $G(r)$ is that that there exists a subalgebra of the Tor algebra minimally exhibiting Poincaré duality, in the sense that there does not exist any nontrivial multiplication outside of this subalgebra (see Definition 3.4.5 for a precise statement). It can be shown that if R/I is Gorenstein (and not a complete intersection) of codimension 3, then R/I has Tor algebra class $G(\mu(I))$, where $\mu(I)$ denotes the minimal number of generators of I. Avramov conjectured in [2] that quotients of Tor algebra class G are necessarily Gorenstein rings.

The technique of "trimming" a Gorenstein ideal is used by Christensen, Veliche, and Weyman (see [16]) to produce codimension 3 non-Gorenstein rings with Tor algebra class G. If $(R, \mathfrak{m})$ is a regular local ring and $I = (\phi_1, \dots, \phi_n) \subseteq R$ is an $\mathfrak{m}$-primary ideal with R/I of codimension 3, then an example of this trimming process is the formation of the ideal $(\phi_1, \dots, \phi_{n-1}) + \mathfrak{m}\phi_n$.

The classification of perfect codimension 3 ideals has seen significant progress recently, starting with the paper [45] (extending the work started in [43]), which links this structure theory to the representation theory of Kac-Moody Lie algebras. Resolutions of a given format (sequence of Betti numbers) have an associated graph, and it is conjectured in [15] that an ideal is in the linkage class of a complete intersection if and only if this associated graph is a Dynkin diagram.

In [2, Question 3.8], Avramov poses a question of realizability; that is, which Tor algebra classes of codimension 3 local rings can actually occur? Using techniques of linkage, this question is explored in [17], refining the classification provided in [4] and showing that every grade 3 perfect ideal in a regular local ring is in the linkage class of either a complete intersection or an ideal defining a Golod ring.

The subject of this thesis is a generalization of the resolution introduced in [39, Theorem 5.4] and the exploration of its utility in a large expanse of topics relating to homological invariants of modules and ideals, including questions of realizability for Tor-algebra structures.

We first examine grade 3 homogeneous ideals $I \subset R := k[x, y, z]$ (with all variables having degree 1, and k being a field of arbitrary characteristic) defining an Artinian compressed ring with socle $\operatorname{Soc}(R/I) = k(-s)^\ell \oplus k(-2s+1)$ for some $\ell \geqslant 1$. The values s and $2s-1$ are interesting because they provide a boundary case for socle degrees; more precisely, it is not possible to have a ring with socle $k(-s_1)^\ell \oplus k(-s_2)$, where $s_2 \geqslant 2s_1$. In particular, we prove that all such ideals arise as *iterated* trimmings of a Gorenstein ideal (see Proposition 3.3.9) and are hence resolved by a so-called iterated trimming complex.

We then specialize to the case where $\ell = 1$. We use the resolution of Theorem 2.1.5 to produce general resolution for trimmed Gorenstein ideals that is minimal in some generic cases (see Proposition 3.2.4 for the relevant parameter space and the corresponding open subset). Even in the cases where this resolution is not minimal, there is valuable information to be gained from the relatively simple differentials involved. We give sharp bounds for the graded Betti numbers of ideals defining compressed rings with socle $k(-s) \oplus k(-2s+1)$. Furthermore, we produce a family of ideals attaining all possible intermediate Betti numbers. This family is also used to show that for any integers r and s with $s \geqslant 3$ and $s \leqslant r \leqslant 2s-1$, there exists a grade 3 ideal I defining an Artinian compressed ring with $\mathrm{Soc}(R/I) = k(-s) \oplus k(-2s+1)$ of Tor algebra class $G(r)$ (see Corollary 3.5.9), which partially answers Avramov's question of realizability mentioned above.

The homogeneous minimal free resolution of ideals generated by all minors of a given size of some matrix is well understood (see, for instance, [9]). It is less well understood what the minimal resolution/Betti table of the ideal generated by *subsets* of these minors must be. Certain classes of subsets have applications in algebraic statistics, including the adjacent 2-minors of an arbitrary matrix and arbitrary subsets of a $2 \times n$ matrix are considered (see [29], [31] for the former case). The latter case has been studied by Herzog et al (see [28]); in particular, such ideals are always radical, and the primary decomposition and Gröbner basis are known.

In [23], so-called determinantal facet ideals are studied. Every maximal minor has an associated indexing set consisting of the columns of the associated submatrix, and a collection of minors can then be indexed by the facets of a certain simplicial complex Δ on the vertex set $\{1, \dots, n\}$, for some n. Properties of the determinantal facet ideal may be deduced from properties of Δ. A study of the homological properties of these ideals is conducted in [30]; in particular, the Betti numbers of the linear strand of the minimal free resolution of these ideals is computed in terms of the f-vector of the associated clique complex.

In this thesis, we consider a subset of the cases addressed in [30] (indeed, determinantal facet ideals in general seem be *very* complicated, with essentially no well-known structure outside of the binomial edge ideal case); however, we compute Betti numbers *explicitly* in all degrees, instead of just the linear strand, and our formulas do not depend on any combinatorial machinery. We also deduce that the ideals under consideration are never linearly presented and hence never have linear resolutions.

Next, we study equigenerated monomial ideals; that is, ideals generated in a single degree. A naïve method of obtaining such ideals is to start with the ideal generated by all monomials of degree d, $(x_1, \dots, x_n)^d \subset k[x_1, \dots, x_n]$, and then delete some of the generators. The graded minimal free resolution of $(x_1, \dots, x_n)^d$ is well known (see Proposition 5.1.3), and so one would only need machinery for which the Betti numbers after deleting generators could be deduced. This machinery is provided by iterated trimming complexes.

Finally, we show that if the complexes associated to the input data of Setup 2.1.1 are length 3 DG-algebras, then the product on the resulting iterated trimming complex of Theorem 2.2.4 may be computed in terms of the products on the aforementioned complexes. The proof of this fact is a long and rather tedious computation; moreover, in full generality, the products have certain components that are only defined implicitly. In the case that the complexes involved admit additional module structures over one another, these products may be made more explicit (see Proposition 6.2.4). However, after descending to homology, many of these products either vanish completely or become considerably more simple. This fact is made explicit in the corollaries at the end of this section, and will be taken advantage of in Section 6.3. We focus on ideals defining rings of Tor-algebra G and H, and show that under very mild assumptions, trimming an ideal preserves these Tor-algebra classes. This allows us to construct novel examples of rings of class $G(r)$ and $H(p,q)$ obtained as quotients of arbitrary regular local rings $(R, \mathfrak{m}, k)$ of dimension 3, and we further add to the realizability question posed by Avramov (see Corollary 6.4.8 and 6.4.14).

Chapter 2

Construction Of Trimming Complexes

This chapter is the basis for the rest of the thesis. Sections 2.1 and 2.2 introduce the main machinery: the trimming complex and iterated trimming complex. We prove that these complexes are resolutions that are not necessarily minimal. However, due to the simple nature of the differentials, one can deduce the ranks appearing in the minimal free resolution of the ideal of interest. The construction itself is relatively simple, and can essentially be described as the comparison map of a certain induced morphism of complexes. The idea of using mapping cones to produce resolutions of sums of ideals has been known for some time; using a mapping cone to *delete* generators is a much newer idea.

2.1 Trimming Complexes

In this section, we introduce the notion of trimming complexes and show that, in fact, these complexes are resolutions. We begin by defining the quotient rings we aim to resolve and setting up the notation that we will use throughout the section.

Setup 2.1.1. *Let $R = k[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field k. Let $I \subseteq R$ be a homogeneous ideal and $(F_\bullet, d_\bullet)$ denote a homogeneous free resolution of R/I.*

Write $F_1 = F_1' \oplus Re_0$, where e_0 generates a free direct summand of F_1. Using the isomorphism

$$\mathrm{Hom}_R(F_2, F_1) = \mathrm{Hom}_R(F_2, F_1') \oplus \mathrm{Hom}_R(F_2, Re_0)$$

write $d_2 = d_2' + d_0$*, where* $d_2' \in \mathrm{Hom}_R(F_2, F_1')$*,* $d_0 \in \mathrm{Hom}_R(F_2, Re_0)$*. Let* $\mathfrak{a}$ *denote any homogeneous ideal with*

$$d_0(F_2) \subseteq \mathfrak{a}e_0,$$

and $(G_\bullet, m_\bullet)$ *be a homogeneous free resolution of* $R/\mathfrak{a}$*.*

Use the notation $K' := \mathrm{im}(d_1|_{F_1'} : F_1' \to R)$*,* $K_0 := \mathrm{im}(d_1|_{Re_0} : Re_0 \to R)$*, and let* $J := K' + \mathfrak{a} \cdot K_0$*.*

Our goal is to construct a resolution of the quotient ring R/J as in Setup 2.1.1. Observe that the length of $G_\bullet$ does not have to equal the length of $F_\bullet$.

Proposition 2.1.2. *Adopt notation and hypotheses as in Setup 2.1.1. Then*

$$(K' : K_0) \subseteq \mathfrak{a}.$$

Proof. Let $r \in R$ with $rK_0 \subseteq K'$. By definition there exists $e' \in F_1'$ such that

$$d_1(e' + re_0) = 0.$$

By exactness of $F_\bullet$, there exists $f \in F_2$ with $d_2(f) = e' + re_0$. Employing the decomposition $d_2 = d_2' + d_0$, we find

$$d_0(f) - re_0 = e' - d_2'(f) \in F_1' \cap Re_0 = 0$$

whence $d_0(f) = re_0$. By selection of $\mathfrak{a}$, we conclude $r \in \mathfrak{a}$. $\square$

Proposition 2.1.3. *Adopt notation and hypotheses as in Setup 2.1.1. Then there exists a map* $q_1 : F_2 \to G_1$ *such that the following diagram commutes:*

$$\begin{tikzcd} & F_2 \arrow[dl, "q_1"'] \arrow[d, "d_0'"] \\ G_1 \arrow[r, "m_1"'] & \mathfrak{a}, \end{tikzcd}$$

where $d_0' : F_2 \to R$ *is the composition*

$$F_2 \xrightarrow{d_0} Re_0 \longrightarrow R\ ,$$

and where the second map sends $e_0 \mapsto 1$*.*

Proof. This follows directly from the fact that F_2 is projective. □

Proposition 2.1.4. *Adopt notation and hypotheses as in Setup 2.1.1. Then there exist maps $q_k : F_{k+1} \to G_k$ for all $k \geqslant 2$ such that the following diagram commutes:*

$$\begin{tikzcd}
F_{k+1} \arrow[r, "d_{k+1}"] \arrow[d, "q_k"'] & F_k \arrow[d, "q_{k-1}"] \\
G_k \arrow[r, "m_k"'] & G_{k-1}
\end{tikzcd}$$

Proof. We build the q_k inductively. For $k = 2$, observe that

$$m_1 \circ q_1 \circ d_3 = d_0' \circ d_3 = 0,$$

so there exists $q_2 : F_3 \to G_2$ making the desired diagram commute. For $k > 2$, we assume that q_{k-1} has already been constructed. Then

$$m_{k-1} \circ q_{k-1} \circ d_{k+1} = q_{k-2} \circ d_k \circ d_{k+1} = 0,$$

so the desired map q_k exists. □

Theorem 2.1.5. *Adopt notation and hypotheses as in Setup 2.1.1. Then the mapping cone of the morphism of complexes*

$$\begin{tikzcd}
\cdots \arrow[r, "d_{k+1}"] & F_k \arrow[r, "d_k"] \arrow[d, "q_{k-1}"] & \cdots \arrow[r, "d_3"] & F_2 \arrow[r, "d_2'"] \arrow[d, "q_1"] & F_1' \arrow[d, "d_1"] \\
\cdots \arrow[r, "m_k"] & G_{k-1} \arrow[r, "m_{k-1}"] & \cdots \arrow[r, "m_2"] & G_1 \arrow[r, "-m_1(-)\cdot d_1(e_0)"] & R
\end{tikzcd} \tag{2.1.1}$$

is acyclic and is a free resolution of R/J.

Proof. We first verify that the maps given in the statement of Theorem 2.1.5 form a morphism of complexes. To this end, it suffices only to show that the first square commutes. Let $f \in F_2$; moving counterclockwise around the first square, we see

$$\begin{aligned} f &\mapsto -m_1(q_1(f)) \cdot d_1(e_0) \\ &= -d_0'(f) \cdot d_1(e_0) \\ &= -d_1(d_0'(f)e_0) \\ &= -d_1(d_0(f)) = d_1(d_2'(f)). \end{aligned}$$

Thus we have a well defined morphism of complexes. Let $q_\bullet$ denote the collection of vertical maps in 2.1.1, $F'_\bullet$ the top row of 2.1.1, and $G'_\bullet$ the bottom row of 2.1.1. There is a short exact sequence of complexes:

$$0 \to G'_\bullet \to \mathrm{Cone}(q_\bullet) \to F'_\bullet[-1] \to 0,$$

which induces the standard long exact sequence in homology. Using this long exact sequence of homology, $\mathrm{Cone}(q_\bullet)$ will be a resolution of R/J if:

1. The complex $F'_\bullet$ is a resolution of $K'/(K' \cap K_0)$.

2. The complex $G'_\bullet$ is a resolution of $R/(\mathfrak{a}K_0)$.

3. The induced map on 0th homology

$$\frac{K'}{K' \cap K_0} \to \frac{R}{\mathfrak{a}K_0}$$

   is an injection.

To prove (1), observe that the top row of 2.1.1 appears as the bottom row in the short exact sequence of complexes

$$
\begin{tikzcd}
 & & 0 \arrow[d] & 0 \arrow[d] & \\
 & 0 \arrow[r] & Re_0 \arrow[r, "d_1"] \arrow[d] & K_0 \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & F_2 \arrow[r, "d_2"] \arrow[d, equal] & F_1 \arrow[r, "d_1"] \arrow[d] & I \arrow[r] \arrow[d] & 0 \\
\cdots \arrow[r] & F_2 \arrow[r, "d_2'"] \arrow[d] & F_1' \arrow[r, "d_1"] \arrow[d] & \frac{K'}{K' \cap K_0} \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
\tag{2.1.2}
$$

The top row of 2.1.2 is exact since R is a domain. The middle row is exact since $F_\bullet$ is a resolution of R/I, so the bottom row must also be exact. Notice that the rightmost column is exact since $I/K_0 = (K' + K_0)/K_0 \cong K'/(K' \cap K_0)$.

Similarly, (2) holds because R is a domain. More precisely, if $g \in G$ and $m_1(g) \cdot d_1(e_0) = 0$, then $m_1(g) = 0$. Since $G_\bullet$ is exact by assumption, $g \in \operatorname{im}(d_2)$.

Lastly, to prove (3), simply observe that $K' \cap \mathfrak{a}K_0 \subseteq K' \cap K_0$. $\square$

Definition 2.1.6. The *trimming complex* associated to the data of Setup 2.1.1 is the resolution of Theorem 2.1.5.

Remark 2.1.7. Notice that in Definition 2.1.6, the associated trimming complex depends on a chosen generating set for I, not just the ideal itself.

In general, the trimming complex associated to the data of Setup 2.1.1 need not be minimal. However, the following Corollary allows us to deduce the (graded) Betti numbers even for a nonminimal resolution.

Corollary 2.1.8. *Adopt notation and hypotheses of Setup 2.1.1. Assume furthermore that the resolutions $F_\bullet$ and $G_\bullet$ are minimal. Then for $i \geqslant 2$,*

$$\dim_k \operatorname{Tor}_i^R(R/J, k) = \operatorname{rank} F_i + \operatorname{rank} G_i - \operatorname{rank}(q_{i-1} \otimes k) - \operatorname{rank}(q_i \otimes k),$$

and

$$\mu(J) = \mu(I) + \mu(\mathfrak{a}) - 1 - \operatorname{rank}(q_1 \otimes k).$$

Proof. Resolve R/J by the mapping cone of the diagram in Theorem 2.1.5, and let ℓ_i denote the ith differential. Then for $i \geqslant 2$,

$$\dim_k \operatorname{Tor}_i^R(R/J, k) = \dim_k \operatorname{Ker}(\ell_i \otimes k)/\operatorname{im}(\ell_{i+1} \otimes k).$$

Since the resolutions $F_\bullet$ and $G_\bullet$ are minimal by assumption,

$$\operatorname{rank}(\operatorname{im}(\ell_{i+1}) = \operatorname{rank}(q_i \otimes k), \text{ and}$$

$$\operatorname{rank}(\operatorname{Ker}(\ell_i \otimes k)) = \operatorname{rank} F_i + \operatorname{rank} G_i - \operatorname{rank}(q_{i-1} \otimes k).$$

For the latter claim, observe that $\ell_1 \otimes k = 0$, so

$$\dim_k \operatorname{Tor}_1(R/J, k) = \operatorname{rank} F_1' + \operatorname{rank} G_1 - \operatorname{rank}(q_1 \otimes k).$$

Since $\operatorname{rank} F_1' = \mu(I) - 1$ and $\operatorname{rank} G_1 = \mu(\mathfrak{a})$, the claim follows after recalling $\dim_k \operatorname{Tor}_1(R/J, k) = \mu(J)$. $\square$

Remark 2.1.9. Observe that in the setting of Corollary 2.1.8, if the resolutions $F_\bullet$ and $G_\bullet$ are also graded, then we may restrict the equalities to homogeneous pieces to find the graded Betti numbers as well.

2.2 Iterated Trimming Complexes

In this section, we consider an iterated version of the data of Setup 2.1.1, and construct a similar resolution. We conclude this section with a concrete example illustrating the construction.

Setup 2.2.1. *Let $R = k[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field k. Let $I \subseteq R$ be a homogeneous ideal and $(F_\bullet, d_\bullet)$ denote a homogeneous free resolution of R/I.*

Write $F_1 = F_1' \oplus \left(\bigoplus_{i=1}^t Re_0^i\right)$*, where, for each* $i = 1, \ldots, t$*,* e_0^i *generates a free direct summand of* F_1*. Using the isomorphism*

$$\operatorname{Hom}_R(F_2, F_1) = \operatorname{Hom}_R(F_2, F_1') \oplus \left(\bigoplus_{i=1}^t \operatorname{Hom}_R(F_2, Re_0^i)\right)$$

write $d_2 = d_2' + d_0^1 + \cdots + d_0^t$*, where* $d_2' \in \operatorname{Hom}_R(F_2, F_1')$ *and* $d_0^i \in \operatorname{Hom}_R(F_2, Re_0^i)$*.*

For each $i = 1, \ldots, t$*, let* $\mathfrak{a}_i$ *denote any homogeneous ideal with*

$$d_0^i(F_2) \subseteq \mathfrak{a}_i e_0^i,$$

and $(G_\bullet^i, m_\bullet^i)$ *be a homogeneous free resolution of* $R/\mathfrak{a}_i$*.*

Use the notation $K' := \operatorname{im}(d_1|_{F_1'} : F_1' \to R)$*,* $K_0^i := \operatorname{im}(d_1|_{Re_0^i} : Re_0^i \to R)$*, and let* $J := K' + \mathfrak{a}_1 \cdot K_0^1 + \cdots + \mathfrak{a}_t \cdot K_0^t$*.*

The next few Propositions are directly analogous to those of the previous section; the proofs are omitted since they are identical.

Proposition 2.2.2. *Adopt notation and hypotheses of Setup 2.2.1. Then for each* $i = 1, \ldots, t$ *there exist maps* $q_1^i : F_2 \to G_1^i$ *such that the following diagram commutes:*

$$\begin{tikzcd} & F_2 \arrow[dl, "q_1^i"'] \arrow[d, "{d_0^i}'"] \\ G_1^i \arrow[r, "m_1^i"'] & \mathfrak{a}_i, \end{tikzcd}$$

where ${d_0^i}' : F_2 \to R$ *is the composition*

$$F_2 \xrightarrow{d_0^i} Re_0^i \longrightarrow R\ ,$$

and where the second map sends $e_0^i \mapsto 1$*.*

Proposition 2.2.3. *Adopt notation and hypotheses as in Setup 2.2.1. Then for each* $i = 1, \ldots, t$ *there exist maps* $q_k^i : F_{k+1} \to G_k^i$ *for all* $k \geqslant 2$ *such that the following diagram commutes:*

$$\begin{tikzcd} F_{k+1} \arrow[r, "d_{k+1}"] \arrow[d, "q_k^i"'] & F_k \arrow[d, "q_{k-1}^i"] \\ G_k^i \arrow[r, "m_k^i"'] & G_{k-1}^i \end{tikzcd}$$

Theorem 2.2.4. *Adopt notation and hypotheses as in Setup 2.2.1. Then the mapping cone of the morphism of complexes*

$$\begin{tikzcd}[ampersand replacement=\&]
\cdots \arrow[r, "d_{k+1}"] \& F_k \arrow[r, "d_k"] \arrow[d, "\begin{pmatrix} q_{k-1}^1 \\ \vdots \\ q_{k-1}^t \end{pmatrix}"] \& \cdots \arrow[r, "d_3"] \& F_2 \arrow[r, "d_2'"] \arrow[d, "\begin{pmatrix} q_1^1 \\ \vdots \\ q_1^t \end{pmatrix}"] \& F_1' \arrow[d, "d_1"] \\
\cdots \arrow[r, "\bigoplus m_k^i"] \& \bigoplus_{i=1}^t G_{k-1}^i \arrow[r, "\bigoplus m_{k-1}^i"] \& \cdots \arrow[r, "\bigoplus m_2^i"] \& \bigoplus_{i=1}^t G_1^i \arrow[r, "-\sum_{i=1}^t m_1^i(-)\cdot d_1(e_0^i)"] \& R
\end{tikzcd} \tag{2.2.1}$$

is a free resolution of R/J.

The proof of Theorem 2.2.4 follows from iterating the construction of Theorem 2.1.5; however, there is some careful bookkeeping needed to deduce that the mapping cone of 2.2.1 can be obtained by iterating the mapping cone construction of Theorem 2.1.5.

Proof of Theorem 2.2.4. Adopt notation and hypotheses of Setup 2.2.1. Let $(F_\bullet^1, d_\bullet^1)$ denote the complex of Theorem 2.1.5 applied to the direct summand Re_0^1 of F_1; that is, the mapping cone of:

$$\begin{tikzcd}[ampersand replacement=\&]
\cdots \arrow[r, "d_{k+1}^1"] \& F_k \arrow[r, "d_k^1"] \arrow[d, "q_{k-1}^1"] \& \cdots \arrow[r, "d_3^1"] \& F_2 \arrow[r, "d_2^{1\prime}"] \arrow[d, "q_1^1"] \& F_1^{1\prime} \arrow[d, "d_1^1"] \\
\cdots \arrow[r, "m_k^1"] \& G_{k-1}^1 \arrow[r, "m_{k-1}^1"] \& \cdots \arrow[r, "m_2^1"] \& G_1^1 \arrow[r, "-m_1^1(-)\cdot d_1(e_0)"] \& R,
\end{tikzcd} \tag{2.2.2}$$

where $F_1^{1\prime} = F_1' \oplus \left(\bigoplus_{i=2}^t Re_0^i \right)$ and $d_2^{1\prime} = d_2' + d_0^2 + \cdots + d_0^t$. Proceed by induction on t. Observe that Theorem 2.1.5 is the base case $t = 1$. Let $t > 1$ and recall the notation of Setup 2.2.1. We may write

$$d_2^1 = \begin{pmatrix} d_2' & 0 \\ -q_1^1 & m_2^1 \end{pmatrix} + \begin{pmatrix} d_0^2 & 0 \\ 0 & 0 \end{pmatrix} + \cdots + \begin{pmatrix} d_0^t & 0 \\ 0 & 0 \end{pmatrix}$$

where for each $i = 2, \dots, t$,

$$\begin{pmatrix} d_0^i & 0 \\ 0 & 0 \end{pmatrix} : F_2^1 \to Re_0^i.$$

This means we are in the situation of Setup 2.2.1, only instead trimming $t-1$ generators from the ideal $K' + \mathfrak{a}_1 K_0^1 + K_0^2 + \cdots + K_0^t$. Observe that the maps $\begin{pmatrix} q_j^i & 0 \end{pmatrix} : F_{j+1}^1 = F_{j+1} \oplus G_{j+1}^1 \to G_j^i$ make the diagram of Proposition 2.2.3 commute. By induction, the mapping cone of

$$\begin{tikzcd}
\cdots \arrow[r, "d_{k+1}^1"] & F_k^1 \arrow[r, "d_k^1"] \arrow[d, "\begin{pmatrix} q_{k-1}^2 & 0 \\ \vdots & \\ q_{k-1}^t & 0 \end{pmatrix}"] & \cdots \arrow[r, "d_3^1"] & F_2^1 \arrow[rr, "\begin{pmatrix} d_2' & 0 \\ -q_1^1 & m_2^1 \end{pmatrix}"] \arrow[d, "\begin{pmatrix} q_1^2 & 0 \\ \vdots & \\ q_1^t & 0 \end{pmatrix}"] & & F_1' \arrow[d, "d_1 - m_1^1(-)\cdot d_1(e_0^1)"] \\
\cdots \arrow[r, "\bigoplus m_k^i"] & \bigoplus_{i=2}^t G_{k-1}^i \arrow[r, "\bigoplus m_{k-1}^i"] & \cdots \arrow[r, "\bigoplus m_2^i"] & \bigoplus_{i=2}^t G_1^i \arrow[rr, "-\sum_{i=2}^t m_1^i(-)\cdot d_1(e_0^i)"] & & R
\end{tikzcd}$$

forms a resolution of $K' + \mathfrak{a}_1 K_0^1 + \left(\mathfrak{a}_2 K_0^2 + \cdots + \mathfrak{a}_t K_0^t\right)$ (recall that the top row forms a resolution of $K' + (K_0^2 + \cdots + K_0^t) + \mathfrak{a}_1 K_0^1$ by Theorem 2.1.5). The differentials of this mapping cone are the same as the differentials induced by the mapping cone of diagram 2.2.1 as in the statement of Theorem 2.2.4. $\square$

Definition 2.2.5. The *iterated trimming complex* associated to the data of Setup 2.2.1 is the complex of Theorem 2.2.4.

As an immediate consequence, one obtains the following result (the proof of which is identical to that of Corollary 2.1.8):

Corollary 2.2.6. *Adopt notation and hypotheses of Setup 2.2.1. Assume furthermore that the complexes $F_\bullet$ and $G_\bullet$ are minimal. Then for $i \geqslant 2$,*

$$\dim_k \operatorname{Tor}_i^R(R/J, k) = \operatorname{rank} F_i + \sum_{j=1}^t \operatorname{rank} G_i^j - \operatorname{rank}\left(\begin{pmatrix} q_i^1 \\ \vdots \\ q_i^t \end{pmatrix} \otimes k\right) - \operatorname{rank}\left(\begin{pmatrix} q_{i-1}^1 \\ \vdots \\ q_{i-1}^t \end{pmatrix} \otimes k\right),$$

and

$$\mu(J) = \mu(I) - t + \sum_{j=1}^{t} \mu(\mathfrak{a}_j) - \operatorname{rank}\left(\begin{pmatrix} q_1^1 \\ \vdots \\ q_1^t \end{pmatrix} \otimes k \right). \qquad \square$$

We conclude this chapter with an example illustrating the construction of iterated trimming complexes.

Example 2.2.7. Let $R = k[x,y,z]$,

$$X = \begin{pmatrix} 0 & 0 & 0 & -x^2 & -z^2 \\ 0 & 0 & -x^2 & -z^2 & -y^2 \\ 0 & x^2 & 0 & -y^2 & 0 \\ x^2 & z^2 & y^2 & 0 & 0 \\ z^2 & y^2 & 0 & 0 & 0 \end{pmatrix},$$

and $I = \operatorname{Pf}(X)$, the ideal of submaximal pfaffians of X. Let $F_\bullet$ denote the complex

$$0 \longrightarrow R \xrightarrow{d_1^*} R^n \xrightarrow{X} R^n \xrightarrow{d_1} R,$$

with

$$d_1 = \begin{pmatrix} y^4 & -y^2z^2 & -x^2y^2 + z^4 & -x^2z^2 & x^4 \end{pmatrix}.$$

This is a minimal free resolution of R/I (see [11]). In the notation of Setup 2.2.1, let

$$K' := (-x^2y^2 + z^4, -x^2z^2, x^4), \ K_0^1 := (y^4), \ K_0^2 := (-y^2z^2),$$

and $\mathfrak{a}_1 = \mathfrak{a}_2 := (x,y,z)$. Let $G_\bullet^1 = G_\bullet^2$ denote the Koszul complex:

$$0 \longrightarrow R \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} -y & -z & 0 \\ x & 0 & -z \\ 0 & x & y \end{pmatrix}} R^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} R\,.$$

Then, one computes:

$$q_1^1 = \begin{pmatrix} 0 & 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -z \end{pmatrix} : R^5 \to R^3,$$

$$q_2^1 = \begin{pmatrix} 0 \\ -x^3 z \\ 0 \end{pmatrix} : R \to R^3,$$

$$q_1^2 = \begin{pmatrix} 0 & 0 & -x & 0 & 0 \\ 0 & 0 & 0 & 0 & -y \\ 0 & 0 & 0 & -z & 0 \end{pmatrix} : R^5 \to R^3,$$

$$q_2^2 = \begin{pmatrix} -x^3 y \\ x\, z^3 \\ 0 \end{pmatrix} : R \to R^3.$$

Then, the mapping cone of Theorem 2.2.4 forms a resolution of $R/(K'+\mathfrak{a}_1 K_0^1+\mathfrak{a}_2 K_0^2)$.

In particular, we deduce that this mapping cone is a minimal free resolution and hence the above quotient ring has Betti table

	0	1	2	3
total:	1	9	11	3
0:	1	.	.	.
1:	.	.	.	.
2:	.	.	.	.
3:	.	3	.	.
4:	.	6	11	2
5:	.	.	.	.
6:	.	.	.	.
7:	.	.	.	1.

Chapter 3

Ideals Defining Compressed Rings

The Hilbert scheme of points associated to projective space $\mathbb{P}^n$ is a scheme whose points parametrize 0-dimensional closed subvarieties by their corresponding *Hilbert polynomial*. Given such data, it is natural to consider the behavior of the varieties associated to closed points with *maximal* Hilbert polynomial. Taking global sections of such varieties, we obtain an Artinian k-algebra whose Hilbert polynomial is as large as possible and hence totally determined by its socle degrees; such a k-algebra is called *compressed*. Compressed k-algebras arise generically, meaning that they appear with sufficient ubiquity as to warrant close study. In this chapter, we examine grade 3 homogeneous ideals $I \subset R := k[x, y, z]$ (with all variables having degree 1, and k being a field of arbitrary characteristic) defining an Artinian compressed ring with socle $\operatorname{Soc}(R/I) = k(-s)^\ell \oplus k(-2s + 1)$ for some $s \geqslant 3$, $\ell > 0$.

The chapter is organized as follows. Sections 3.1 and 3.2 consist of preliminary material and notation. Section 3.3 proves the previously mentioned fact that any grade 3 ideal $I \subset k[x, y, z]$ defining an Artinian compressed ring with $\operatorname{Soc}(R/I) = k(s)^\ell \oplus k(-2s + 1)$ is obtained as the iterated trimming of some grade 3 Gorenstein ideal. In particular, all such ideals may be resolved by an iterated trimming complex. For sufficiently generic ideals, this resolution is minimal.

We explore the initial consequences of the explicit resolution provided by iterated trimming complexes. In the standard graded case, we find a remarkably simple criterion to deduce whether the trimmed generating set of a Gorenstein ideal is a minimal generating set. In particular, questions about minimal generators are translated into counting degrees of the entries of the presenting matrix of a Gorenstein ideal.

Section 3.4 delves into some more nontrivial consequences of the tools developed beforehand. In [16], all possible Tor algebra structures of trimmed Gorenstein ideals are enumerated. As a consequence, all possible Tor algebra structures for the ideals of interest may be deduced; in particular, these ideals are either class $G(r)$ or Golod.

Section 3.5 specializes to the case that the ideal defines a compressed ring of type 2. Combining given by [16] the bounds on the minimal number of generators, we show that all such ideals are class $G(r)$ for some $s \leqslant r \leqslant 2s-1$. Furthermore, using information from the free resolution provided by trimming complexes, we show that every such r value between s and $2s-1$ may be achieved.

3.1 Compressed Rings and Inverse Systems

Definition 3.1.1. Let A be a local Artinian k-algebra, where k is a field and $\mathfrak{m}$ denotes the maximal ideal. The top socle degree is the maximum s with $\mathfrak{m}^s \neq 0$ and the socle polynomial of A is the formal polynomial $\sum_{i=0}^s c_i z^i$, where

$$c_i = \dim_k \frac{\operatorname{Soc}(A) \cap \mathfrak{m}^i}{\operatorname{Soc}(A) \cap \mathfrak{m}^{i+1}}.$$

An Artinian k-algebra is *standard graded* if it is generated as an algebra in degree 1.

Definition 3.1.2. A standard graded Artinian k-algebra A with embedding dimension e, top socle degree s, and socle polynomial $\sum_{i=0}^s c_i z^i$ is *compressed* if

$$\dim_k \mathfrak{m}^i/\mathfrak{m}^{i+1} = \min\left\{ \binom{e-1+i}{i}, \sum_{\ell=0}^{s} c_\ell \binom{e-1+\ell-i}{\ell-i} \right\}$$

for $i = 0, \ldots, s$.

Setup 3.1.3. *Let $n \geqslant 1$ be an integer and k denote a field of arbitrary characteristic. Let V be a vector space of dimension n over k. Give the symmetric algebra $S(V) =: R$ and divided power algebra $D(V^*)$ the standard grading (that is, $S_1(V) = V$, $D_1(V^*) =$*

V^*). The notation $S_i := S_i(V)$ denotes the degree i component of the symmetric algebra on V. Similarly, the notation $D_i := D_i(V^*)$ denotes the degree i component of the divided power algebra on V^*.

Given a homogeneous $I \subseteq S(V)$ defining an Artinian ring, there is an associated inverse system $0 :_{D(V^*)} I$. Similarly, for any finitely generated graded submodule $N \subseteq D(V^*)$ there is a corresponding homogeneous ideal $0 :_{S(V)} N$ defining an Artinian ring.

If I is a homogeneous ideal with associated inverse system minimally generated by elements $\phi_1, \ldots, \phi_k$ with $\deg \phi_i = s_i$, then there are induced vector space homomorphisms

$$\Phi_i : S_i \to \bigoplus_{j=1}^{k} D_{s_j - i}$$

sending $f \mapsto (f \cdot \phi_1, \ldots, f \cdot \phi_k)$.

Observation 3.1.4. Let $I \subseteq S(V)$ be a homogeneous ideal with associated inverse system minimally generated by elements $\phi_1, \ldots, \phi_k$ with $\deg \phi_i = s_i$. If the induced maps Φ_i of Setup 3.1.3 have maximal rank for all i, then the ring R/I is compressed, as in Definition 3.1.2.

Proof. By definition, $I_i = \operatorname{Ker} \Phi_i$; by the rank-nullity theorem,

$$\begin{aligned} \dim_k (R/I)_i &= \dim_k \operatorname{im} \Phi_i \\ &= \min \left\{ \dim_k S_i, \dim_k \bigoplus_{j=1}^{\ell} D_{s_j - i} \right\} \end{aligned}$$

where the latter equality follows by the assumption that Φ_i has maximal rank. $\square$

Definition 3.1.5. Let $I \subseteq S(V)$ be a homogeneous ideal with associated inverse system minimally generated by elements $\phi_1, \ldots, \phi_k$ with $\deg \phi_i = s_i$. Let m denote the first integer for which Φ_m is a surjection. Then m is called the *tipping point* of I; this is well defined since the rank of the domain and codomain of each Φ_i is increasing/decreasing in i, respectively (and the codomain is eventually 0).

Proposition 3.1.6 ([34], Lemma 1.13)**.** *Let ϕ be a homogeneous element of $D(V^*)$ of degree s. Then the tipping point of the ideal $0 :_{S(V)} \phi$ is $\lceil s/2 \rceil$. In addition, the induced maps Φ_i satisfy the following properties for every integer i.*

(a) $\operatorname{Hom}_k(\Phi_i, k) = \Phi_{s-i}$

(b) Φ_i is surjective if and only if Φ_{s-i} is injective.

Definition 3.1.7. Adopt Setup 3.1.3 with $R = S(V)$ and let $\psi : V \to R$. The Koszul complex $K_\bullet$ on ψ is the complex obtained by setting

$$K_i := \bigwedge^i V \otimes R(-i)$$

with differential

$$\delta_i : \bigwedge^i V \otimes R(-i) \to \bigwedge^{i-1} V \otimes R(-i+1)$$

defined as multiplication by $\psi \in V^*$ (where $\bigwedge^\bullet V$ is given the standard module structure over $\bigwedge^\bullet V^*$).

The following can be found as Proposition 2.5 of [6]:

Proposition 3.1.8. *Let I be a homogeneous ideal in $R := S(V)$ of initial degree t, and set $A = R/I$. Then*

$$\operatorname{Tor}_i^R(A, k)_{i+t-1} \cong \operatorname{Ker}(\pi) \cap \operatorname{Ker}(\delta_{i-1}), \quad i = 2, \dots, n$$

where δ_{i-1} is the Koszul differential $\bigwedge^{i-1} V \otimes R_t \to \bigwedge^{i-2} V \otimes R_{t+1}$ and π is the quotient map $\bigwedge^{i-1} V \otimes R_t \to \bigwedge^{i-1} V \otimes A_t$.

Remark 3.1.9. Adopt notation and hypotheses of Setup 3.1.3. Let $\psi : V \to R$ be such that $\operatorname{im} \psi = R_+$. Observe that $\operatorname{Ker}(\pi) \cap \operatorname{Ker}(\delta_{i-1})$ as in Proposition 3.1.8 is precisely

$\mathrm{Ker}(\pi) \cap \mathrm{im}(\delta_i)$ by exactness of the Koszul complex. The latter set may be described as the kernel of the composition of k-vector space homomorphisms

$$\bigwedge^i V \otimes S_t(V) \xrightarrow{\delta_i} \bigwedge^{i-1} V \otimes S_{t+1}(V)$$

$$\xrightarrow{1\otimes\Phi_{t+1}} \bigwedge^{i-1} V \otimes D_{c-t-1}(V^*).$$

Denote the above composition of k-vector space homomorphisms by

$$\Theta_i(\phi) : \bigwedge^i V \otimes S_t(V) \to \bigwedge^{i-1} V \otimes D_{c-t-1}(V^*).$$

Proposition 3.1.10. *Adopt Setup 3.1.3. Let $A = R/I$ where $I = (0 :_R \phi)$, $\phi \in D_c(V^*)$, and let t denote the initial degree of I, $n = \mathrm{pd}_R R/I$. Then,*

$$\dim_k \mathrm{Tor}_i^R(A,k)_{i+t-1} = \binom{t-1+i-1}{i-1}\binom{t-1+n}{n-i} - \mathrm{rank}\,\Theta_i(\phi)$$

for all $i = 2, \dots, n$.

Proof. First observe that the minimal homogeneous free resolution of

$$\mathrm{im}(\delta_i : \bigwedge^{i+1} V \otimes R_{t-1} \to \bigwedge^i V \otimes R_t)$$

is obtained by truncating the Koszul complex:

$$0 \to \bigwedge^{i+t} V \otimes R_0 \to \cdots \to \bigwedge^{i+1} V \otimes R_{t-1} \to \mathrm{im}(\delta_i) \to 0$$

whence

$$\begin{aligned}\dim \mathrm{im}(\delta_i) &= \sum_{j=1}^{t} (-1)^{j+1} \dim \Big(\bigwedge^{i+j} V \otimes R_{t-j} \Big) \\ &= \sum_{j=1}^{t} (-1)^{j+1} \binom{n+1}{i+j} \cdot \binom{n+t-j}{t-j}.\end{aligned}$$

By Lemma 1.2 of [12], this sum is equal to $\binom{i+t-1}{i} \cdot \binom{n+t}{i+t}$. Similarly, by construction $\dim \mathrm{Ker}(\Theta_i(\phi)) = \dim \Big(\mathrm{Ker}(\pi) \cap \mathrm{im}(\delta_i) \Big)$. By exactness of the Koszul complex, $\mathrm{im}(\delta_i) = \mathrm{Ker}(\delta_{i-1})$; combining this with Proposition 3.1.8:

$$\dim \mathrm{Tor}_i^R(A,k)_{i+t-1} = \dim \mathrm{Ker}(\Theta_i(\phi)).$$

By the rank-nullity theorem,

$$\dim \operatorname{Tor}_i^R(A,k)_{i+t-1} = \dim \operatorname{im}(\delta_i) - \operatorname{rank}_k \Theta_i(\phi)$$

$\square$

Corollary 3.1.11. *Adopt notation and hypotheses as in Setup 3.1.3. Then there is a nonempty open set U in the Grassmannian parametrizing all 1-dimensional subspaces of $D_c(V^*)$ such that the Betti numbers of the k-algebra $A = R/(0 :_{S(V)} \phi)$ are the same for all $[\phi]$ in U (where $[\phi]$ denotes the class of the subspace spanned by $\phi \in D_c(V^*)$).*

Proof. Take the open subset U to be the set of all 1-dimensional subspaces $[\phi]$ of $D_c(V^*)$ such that $\Theta_i(\phi)$ has maximal rank for each $i = 2, \ldots, n$.

We may identify the Grassmannian $\operatorname{Gr}(1, D_c(V^*))$ with the projective space $\mathbb{P}^{\binom{c+2}{2}-1}$, so it suffices to show that the complement of U is the zero set of homogeneous polynomials in the variables $p_1, \ldots, p_{\binom{c+2}{2}}$, where $[\phi] = [p_1 : \cdots : p_{\binom{c+2}{2}}]$.

Let ϵ_β denote any standard basis element of $\bigwedge^i V$, so $\beta = (\beta_1, \ldots, \beta_i)$ with $\beta_1 < \cdots < \beta_i$. Let $m \in S_t(V)$ be any degree t monomial. We compute

$$\Theta_i(\phi)(m) = \sum_{j=1}^{i} (-1)^{i+1} \epsilon_{\beta \setminus \beta_j} \otimes (\psi(\epsilon_{\beta_j})m) \cdot \phi,$$

implying that the matrix representation of $\Theta_i(\phi)$ has entries of the form $\pm p_\ell$, for $\ell = 1, \ldots, \binom{c+2}{2}$, where the basis chosen for $\bigwedge^{i-1} V \otimes D_{c-t-1}(V^*)$ consists of the tensor products of the standard basis for $\bigwedge^{i-1} V$ and the monomial basis for $D_{c-t-1}(V^*)$.

The complement of U is the union of the zero sets of the determinant of the above matrix representation for each $i = 2, \ldots, n$, which is a homogeneous polynomial in the p_ℓ. As a finite union of closed sets, this set is closed. Thus U is an open set, and by Proposition 3.1.10, any $[\phi] \in U$ gives rise to an ideal $(0 :_R \phi)$ whose Betti numbers are independent of the choice of ϕ. $\square$

Definition 3.2.1. A standard graded Artinian algebra is *level* if its socle is concentrated entirely in a single degree.

Proposition 3.2.2 ([6], Proposition 3.6)**.** *Let R/I be a standard graded compressed level Artinian algebra of embedding dimension r, socle degree c, socle dimension m, and assume I has initial degree t. Then*

$$\dim_k \operatorname{Tor}_i^R(R/I,k)_{t+i-1} - \dim_k \operatorname{Tor}_{i-1}^R(R/I,k)_{t+i-1} = \binom{t-1+i-1}{i-1}\cdot\binom{t-1+r}{r-i} - m\binom{c-t+r-i}{r-i}\cdot\binom{c-t+r}{i-1}$$

for $i = 1, \ldots, r-1$.

Proposition 3.2.3. *Let $R = k[x,y,z]$ be standard graded and I a homogeneous grade 3 Gorenstein ideal with R/I compressed and* $\operatorname{Soc}(R/I) = k(-2s+1)$ *for some integer s. Then R/I has Betti table of the form*

	0	1	2	3
0	1	$\cdot$	$\cdot$	$\cdot$
$s-1$	$\cdot$	$s+1$	b	$\cdot$
s	$\cdot$	b	$s+1$	$\cdot$
$2s-1$	$\cdot$	$\cdot$	$\cdot$	1

where b is some integer. Moreover, $b \leqslant s$.

Proof. Employ Proposition 3.2.2, where $r = 3$, $c = 2s-1$, $m = 1$, and $t = s$ ($= \lceil (2s-1)/2 \rceil$; see Proposition 3.1.6). Using the notation

$$T_i := \operatorname{Tor}_i^R(R/I,k),$$

we obtain

$$\dim(T_1)_s = s+1$$

$$\dim(T_2)_{s+1} - \dim(T_1)_{s+1} = 0$$

$$\dim(T_2)_{s+2} = s+1.$$

Thus define $b := \dim_k(T_2)_{s+1}$. The final claim that $b \leqslant s$ follows from the fact that the Betti table has the following decomposition into standard pure Betti diagrams:

$$\frac{b}{2(s+1)^2-2}\cdot \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & s+2 & \cdot & \cdot & \cdot \\ s-1 & \cdot & 2(s+1)^2 & 2(s+1)^2-2 & \cdot \\ s & \cdot & \cdot & \cdot & \cdot \\ 2s-1 & \cdot & \cdot & \cdot & s \end{array}$$

$$+\frac{(s+1)^2-1-(s+1)b}{(s+1)^2-1}\cdot \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & 1 & \cdot & \cdot & \cdot \\ s-1 & \cdot & s+1 & \cdot & \cdot \\ s & \cdot & \cdot & s+1 & \cdot \\ 2s-1 & \cdot & \cdot & \cdot & 1 \end{array}$$

$$+\frac{b}{2(s+1)^2-2}\cdot \begin{array}{c|cccc} & 0 & 1 & 2 & 3 \\ \hline 0 & s & \cdot & \cdot & \cdot \\ s-1 & \cdot & \cdot & \cdot & \cdot \\ s & \cdot & 2(s+1)^2-2 & 2(s+1)^2 & \cdot \\ 2s-1 & \cdot & \cdot & \cdot & s+2 \end{array}$$

If $b \geqslant s+1$, then the middle coefficient of the above decomposition becomes negative, which is a contradiction to results of Boij-Söderberg theory (see, for instance, [7, Theorem 2]). □

In the following, recall that the notation $[\phi] \in \mathrm{Gr}(1, D_c(V^*))$ means the class of the subspace spanned by the element $\phi \in D_c(V^*)$.

Proposition 3.2.4. *Let $R = S(V)$ be standard graded, where V is a 3-dimensional vector space over a field k. If s is even, then there is a nonempty open set U in the Grassmannian parametrizing all 1-dimensional subspaces of $D_{2s-1}(V^*)$ such that for* *all* $[\phi] \in U$*,* $I := (0 :_{S(V)} \phi)$ *has Betti table*

	0	1	2	3
0	1	·	·	·
$s-1$	·	$s+1$	·	·
s	·	·	$s+1$	·
$2s-1$	·	·	·	1

If s *is odd, then there is a nonempty open set* U *in the Grassmannian parametrizing all* 1*-dimensional subspaces of* $D_{2s-1}(V^*)$ *such that for all* $[\phi] \in U$*,* $I := (0 :_{S(V)} \phi)$ *has Betti table*

	0	1	2	3
0	1	·	·	·
$s-1$	·	$s+1$	1	·
s	·	1	$s+1$	·
$2s-1$	·	·	·	1

Proof. The goal is to find minimal values for b, where b is as in Proposition 3.2.3, since b is minimized precisely when the rank of $\Theta_i(\phi)$ is maximized by Proposition 3.1.10. To this end, we exhibit an explicit I for each s attaining the Betti table as in the statement and argue that no smaller values of b can be obtained. The matrices used below are those from Proposition 6.2 of [11] with minor alterations; in our case, some of the entries are squared.

Choosing a basis for V, we may view $S(V)$ as the standard graded polynomial ring $k[x, y, z]$. Assume first that s is even. Consider the $(s+1) \times (s+1)$ alternating matrix

$$\begin{pmatrix} 0 & x^2 & 0 & \cdots & 0 & z^2 \\ -x^2 & 0 & y^2 & \cdots & z^2 & 0 \\ 0 & -y^2 & 0 & \cdots & & 0 \\ \vdots & & & & & \vdots \\ -z^2 & 0 & \cdots & & & 0 \end{pmatrix}$$

To see the pattern more clearly, the first two matrices are

$$H_1^{ev} = \begin{pmatrix} 0 & x^2 & z^2 \\ -x^2 & 0 & y^2 \\ -z^2 & -y^2 & 0 \end{pmatrix}, \quad H_2^{ev} = \begin{pmatrix} 0 & x^2 & 0 & 0 & z^2 \\ -x^2 & 0 & y^2 & z^2 & 0 \\ 0 & -y^2 & 0 & x^2 & 0 \\ 0 & -z^2 & -x^2 & 0 & y^2 \\ -z^2 & 0 & 0 & -y^2 & 0 \end{pmatrix}$$

The ideal generated by the $s \times s$ Pfaffians has grade 3 according to section 6 of [11] (a much more explicit generating set is exhibited in Proposition 7.6 of [21]), and hence has minimal free resolution

$$0 \to R(-2s+1) \to R(-s-2)^{s+1} \to R(-s)^{s+1} \to R.$$

The above gives an ideal for which $b = 0$, and this is clearly the smallest possible.

Similarly, if s is odd, consider the following $(s+2) \times (s+2)$ matrix:

$$\begin{pmatrix} 0 & x^2 & 0 & \cdots & 0 & z \\ -x^2 & 0 & y^2 & \cdots & z^2 & 0 \\ 0 & -y^2 & 0 & \cdots & & 0 \\ \vdots & & & & & \vdots \\ -z & 0 & \cdots & & & 0 \end{pmatrix}$$

The first two matrices in this case are

$$H_1^{odd} = \begin{pmatrix} 0 & x^2 & z \\ -x^2 & 0 & y \\ -z & -y & 0 \end{pmatrix}, \quad H_2^{odd} = \begin{pmatrix} 0 & x^2 & 0 & 0 & z \\ -x^2 & 0 & y^2 & z^2 & 0 \\ 0 & -y^2 & 0 & x^2 & 0 \\ 0 & -z^2 & -x^2 & 0 & y \\ -z & 0 & 0 & -y & 0 \end{pmatrix}$$

Again, the ideal generated by the submaximal Pfaffians is grade 3 Gorenstein with $b = 1$. Moreover, no smaller value of b can be achieved since otherwise the ideal would have an even number of minimal generators, which is impossible by work of Watanabe in [42] or Corollary 2.2 of [11]. □

3.3 Ideals Defining Rings with Socle $k(-s)^\ell \oplus k(-2s+1)$

Setup 3.3.1. *Let k be a field and let $R = k[x,y,z]$ be a standard graded polynomial ring over a field k. Let $I \subset R$ be a grade 3 homogeneous ideal defining a compressed ring with $\operatorname{Soc}(R/I) = k(-s)^\ell \oplus k(-2s+1)$, where $s \geqslant 3$.*

Write $I = I_1 \cap I_2 \cap \cdots \cap I_\ell \cap I_t$ for $I_1, \dots, I_\ell$ homogeneous grade 3 Gorenstein ideals defining rings with socle degrees s and I_t a homogeneous grade 3 Gorenstein ideal defining a ring with socle degree $2s-1$. The notation R_+ will denote the irrelevant ideal ($R_{>0}$).

Let $F_\bullet$ denote the minimal free resolution of R/I_t and let $G_\bullet$ denote the Koszul complex induced by the map $U = Re_x \oplus Re_y \oplus Re_z \to R$ sending $e_x \mapsto x,\ e_y \mapsto y,\ e_z \mapsto z$.

Theorem 3.3.2. *Adopt notation and hypotheses of Setup 3.3.1. Then the tipping point of I is equal to s. In particular, $\ell \leqslant s+1$.*

The proof of Theorem 3.3.2 will follow easily after a series of Lemmas.

Lemma 3.3.3. *Adopt notation and hypotheses of Setup 3.3.1. Then the tipping point of I is $\geqslant s$.*

Proof. Recall the notation of Setup 3.1.3; we may view R as $S(V)$ for some 3-dimensional vector space over k. By counting initial degrees, we eliminate all possibilities except for the case that $\Phi_{s-1} : S_{s-1} \to D_1^{\oplus \ell} \oplus D_s$ is an isomorphism and I has initial degree s. Counting ranks, this implies $\ell s = 1 - s \leqslant 0$, which is a clear contradiction. □

Lemma 3.3.4. *Adopt notation and hypotheses of Setup 3.3.1. Then I_t defines a compressed ring.*

Proof. Recall the notation of Setup 3.1.3; we may view R as $S(V)$ for some 3-dimensional vector space over k. Let $\phi_i \in D_s$ denote the inverse system for each

I_i and $\phi_t \in D_{2s-1}$ denote the inverse system for I_t. By Lemma 3.3.3, the maps Φ_i for $i \geqslant s$ are surjective; Proposition 3.1.6 guarantees that the map $f \mapsto f \cdot \phi_t$ is surjective for $f \in S_s$.

For $i > s$, the maps $\Phi_i : S_i \to D_{2s-1-i}$ are identically the maps $f \mapsto f \cdot \phi_t$ for $f \in S_i$. By assumption, these are surjections; by Proposition 3.1.6, I_t defines a compressed ring. □

Lemma 3.3.5. *Adopt notation and hypotheses of Setup 3.3.1. Then the tipping point of I is $\leqslant s+1$.*

Proof. Suppose for sake of contradiction that the tipping point is $\geqslant s+2$. Recall the notation of Setup 3.1.3; we may view R as $S(V)$ for some 3-dimensional vector space over k. Let $\phi_i \in D_s$ denote the inverse system for each I_i and $\phi_t \in D_{2s-1}$ denote the inverse system for I_t.

By Proposition 3.1.6, I_t has tipping point s. If I has tipping point $\geqslant s+2$, then $\Phi_{s+1} : S_{s+1} \to D_{s-2}$ is injective; this is impossible by counting ranks. □

Lemma 3.3.6. *Adopt notation and hypotheses of Setup 3.3.1. Let $\phi_1, \ldots, \phi_{s+1}, \psi_1, \ldots, \psi_b$ denote a minimal generating set for I_t, where $\deg \phi_i = s$, $\deg \psi_i = s+1$. Then the ideal*

$$(\phi_1, \ldots, \phi_{s+1-\ell}, \psi_1, \ldots, \psi_b) + R_+\phi_{s+2-\ell} + \cdots + R_+\phi_{s+1}$$

defines a ring of type $i+1$. In particular, $(I_t)_{\geqslant s+1}$ defines a ring of type $s+2$.

Remark 3.3.7. By Proposition 3.2.3 combined with Lemma 3.3.4, I_t is minimally generated by homogeneous forms $\phi_1, \ldots, \phi_{s+1}$, $\psi_1, \ldots, \psi_b$, where $\deg \phi_i = s$, $\deg \psi_i = s+1$ and $b < s+1$, so it makes sense to choose a generating set as in the statement of Lemma 3.3.6.

Proof. This is a consequence of Corollary 2.2.6. Let $J := (\phi_1, \ldots, \phi_{s+1-\ell}, \psi_1, \ldots, \psi_b) + R_+\phi_{s+2-\ell} + \cdots + R_+\phi_{s+1}$. In the notation of Theorem 2.2.4, notice that $G^j_\bullet = K_\bullet$, the

Koszul complex resolving R_+, for all $j = 1, \ldots, i$. Counting degrees on the diagram of Proposition 2.2.3, we find

$$\deg q_2^j \geqslant s - 1 > 0$$

so that $q_2^j \otimes k = 0$, for all $j = 1, \ldots, i$. Since $q_3^j = 0$ for each $j = 1, \ldots, i$, Corollary 2.2.6 implies that

$$\begin{aligned}\dim_k \operatorname{Tor}_3^R(R/J, k) &= \operatorname{rank} F_3 + \sum_{j=1}^{i} \operatorname{rank} K_3 \\ &= i + 1\end{aligned}$$

□

Proof of Theorem 3.3.2. By Lemmas 3.3.5 and 3.3.3, we only need to check that the tipping point cannot equal $s+1$. Assume that I has tipping point $= s+1$. This implies that the map

$$\Phi_s : S_s \to D_0^{\oplus \ell} \oplus D_{s-1}$$

is an injection. Counting ranks, $\ell + \binom{s+1}{2} \geqslant \binom{s+2}{2}$. If we have equality, then Φ_s is an isomorphism, implying that the tipping point is $\leqslant s$. Thus there is strict inequality, and $\ell \geqslant s+2$.

This implies I has type $\geqslant s+3$ and initial degree $s+1$. However, Lemma 3.3.4 forces I_t to be compressed. Counting ranks in each homogeneous component and using the definition of compressed, we must have $I = (I_t)_{\geqslant s+1}$. By Lemma 3.3.6, I has type $s+2$; this contradiction yields the result. □

Corollary 3.3.8. *Adopt notation and hypotheses of Setup 3.3.1. Then for each $i \leqslant \ell$, the ideal*

$$I_i \cap \cdots \cap I_\ell \cap I_t$$

defines a compressed ring with socle $k(-s)^{\ell-i} \oplus k(-2s+1)$.

Proof. Recall the notation of Setup 3.1.3; we may view R as $S(V)$ for some 3-dimensional vector space over k. Let $\phi_i \in D_s$ denote the inverse system for each I_i and $\phi_t \in D_{2s-1}$ denote the inverse system for I_t.

For $j < s$, the map $f \mapsto f \cdot \phi_t$ is injective, since I_t defines a compressed ring by Lemma 3.3.4. Similarly, for $j \geqslant s$, the map $\Phi_j : S_j \to D_j^{\oplus \ell} \oplus D_{2s-1-j}$ associated to the ideal I is surjective. The map Φ_j' associated to the ideal $I_i \cap \cdots \cap I_\ell \cap I_t$ is the composition of Φ_j with the canonical projection $D_j^{\oplus \ell} \oplus D_{2s-1-j} \to D_j^{\oplus \ell - i} \oplus D_{2s-1-j}$. As a composition of surjections, Φ_j' is a surjection for $j \geqslant s$. $\square$

Proposition 3.3.9. *Adopt notation and hypotheses of Setup 3.3.1. Then there exists a minimal generating set*

$$\phi_1, \ldots, \phi_{s+1}, \ \psi_1, \ldots, \psi_b$$

for I_t such that

$$I = (\phi_1, \ldots, \phi_{s+1-\ell}, \psi_1, \ldots, \psi_b) + R_+ \phi_{s+2-\ell} + \cdots + R_+ \phi_{s+1},$$

where $\deg \phi_i = s$, $\deg \psi_i = s+1$, *and* $b < s+1$.

Proof. Observe that, by definition of compressed,

$$\dim_k (I)_s = s + 1 - \ell.$$

Choose a basis $\phi_1, \ldots, \phi_{s+1-\ell}$ for I_s; notice that $\dim_k (I_t)_s = s+1$, so we may extend this set to a basis

$$\phi_1, \ldots, \phi_{s+1}$$

for $(I_t)_s$. Since $I_{\geqslant s+1} = (I_t)_{\geqslant s+1}$, there exist elements $\psi_1, \ldots, \psi_b \in (I_t)_{s+1}$ such that

$$I_{s+1} = (R_+(\phi_1, \ldots, \phi_{s+1}))_{s+1} + \operatorname{Span}_k \{\psi_1, \ldots, \psi_b\}.$$

In particular, the assumption that I defines a compressed ring forces every minimal generating set to be concentrated in two consecutive degrees. This immediately yields that

$$I = (\phi_1, \ldots, \phi_{s+1-\ell}, \psi_1, \ldots, \psi_b) + R_+ \phi_{s+2-\ell} + \cdots + R_+ \phi_{s+1}.$$

$\square$

3.4 Tor-Algebra Structure

Definition 3.4.1. Let $(R, \mathfrak{m}, k)$ be a regular local ring with $I \subset \mathfrak{m}^2$ and ideal such that $\operatorname{pd}_R(R/I) = 3$. Let $T_\bullet := \operatorname{Tor}_\bullet^R(R/I, k)$. Then R/I has Tor algebra class $G(r)$ if, for $m = \mu(I)$ and $t = \operatorname{type}(R/I)$, there exist bases for T_1, T_2, and T_3

$$e_1, \ldots, e_m, \quad f_1, \ldots, f_{m+t-1}, \quad g_1, \ldots, g_t,$$

respectively, such that the only nonzero products are given by

$$e_i f_i = g_1 = f_i e_i, \quad 1 \leqslant i \leqslant r.$$

Such a Tor algebra structure has

$$T_1 \cdot T_1 = 0, \quad \operatorname{rank}_k(T_1 \cdot T_2) = 1, \quad \operatorname{rank}_k(T_2 \to \operatorname{Hom}_k(T_1, T_3)) = r,$$

where $r \geqslant 2$.

Theorem 3.4.2 ([16], Theorem 2.4, Homogeneous version)**.** *Let $R = k[x, y, z]$ with the standard grading and let $I \subseteq R_+^2$ be an R_+-primary homogeneous Gorenstein ideal minimally generated by elements $\phi_1, \ldots, \phi_{2m+1}$. Then the ideal*

$$J = (\phi_1, \ldots, \phi_{2m}) + R_+ \phi_{2m+1}$$

is a homogeneous R_+-primary ideal and defines a ring of type 2. Moreover,

(a) If $m = 1$, then $\mu(J) = 5$ and R/J is class B.

(b) If $m = 2$, then one of the following holds:

- *$\mu(J) = 4$ and R/J is class $H(3, 2)$*
- *$\mu(J) = 5$ and R/J is class B*
- *$\mu(J) \in \{6, 7\}$ and R/J is class $G(r)$ with $\mu(J) - 2 \geqslant r \geqslant \mu(J) - 3$*

(c) If $m \geqslant 3$, then R/J is class $G(r)$ with $\mu(J) - 2 \geqslant r \geqslant \mu(J) - 3$.

The proof of Theorem 3.4.3 is essentially that of [16, Theorem 2.4].

Theorem 3.4.3. *Adopt the notation and hypotheses of Setup 3.3.1. Then the rank of the induced map*

$$\delta_I : \operatorname{Tor}_2^R(R/I, k) \to \operatorname{Hom}_k(\operatorname{Tor}_1^R(R/I, k), \operatorname{Tor}_3^R(R/I, k))$$

is at least $\mu(I) - 3\ell$.

Proof. Throughout the proof, use the notation

$$T_i^A := \operatorname{Tor}_i^R(A, k),$$

where A is any R-module. Notice that by Proposition 3.3.9,

$$I_t/I \cong k^\ell.$$

Considering the long exact sequence of Tor associated to the short exact sequence

$$0 \longrightarrow \frac{I_t}{I} \xrightarrow{\iota} \frac{R}{I} \xrightarrow{p} \frac{R}{I_t} \longrightarrow 0\ ,$$

one counts ranks to find that

$$\operatorname{rank} \operatorname{Tor}_2^R(p, k) = \mu(I) - 2\ell, \quad \operatorname{rank} \operatorname{Hom}_k(\operatorname{Tor}_1^R(p, k), T_3^{R/I_t}) = \mu(I_t) - \ell.$$

Consider the following commutative diagram:

$$\begin{tikzcd}
T_2^{R/I_t} \arrow[r, "\delta_{I_t}"] & \operatorname{Hom}_k(T_1^{R/I_t}, T_3^{R/I_t}) \arrow[d, "\operatorname{Hom}_k(\operatorname{Tor}_1^R(p,k), T_3^{R/I_t})"] \\
T_2^{R/I} \arrow[u, "\operatorname{Tor}_2(p,k)"] \arrow[r, "\varepsilon"] \arrow[d, equal] & \operatorname{Hom}_k(T_1^{R/I}, T_3^{R/I}) \\
T_2^{R/I} \arrow[r, "\delta_I"] & \operatorname{Hom}_k(T_1^{R/I}, T_3^{R/I}) \arrow[u, "\operatorname{Hom}_k(T_1^{R/I}, \operatorname{Tor}_3^R(p,k))"']
\end{tikzcd}\ .$$

Since I_t is Gorenstein, δ_{I_t} is an isomorphism. This implies that $\operatorname{rank} \delta_I \geqslant \operatorname{rank} \varepsilon \geqslant (\mu(I) - 2\ell) - \ell$, which yields the result. □

Corollary 3.4.4. *Adopt the notation and hypotheses of Setup 3.3.1. If $\ell \leqslant s-1$, then R/I has Tor algebra class $G(r)$ for some $r \geqslant \mu(I) - 3\ell$.*

Proof. It suffices to show that, in the notation of the proof of Theorem 3.4.3, the map δ_I has rank at least 2 and $T_1 \cdot T_1 = 0$. Since $s \geqslant 3$, a degree count shows $T_1 \cdot T_1 = 0$. By Corollary 2.2.6,

$$\mu(I) = \mu(I_t) + 2\ell - \operatorname{rank}\left(\begin{pmatrix} q_1^1 \\ \vdots \\ q_1^m \end{pmatrix} \otimes k \right).$$

Since $\mu(I_t) \geqslant s+1$ and $\operatorname{rank}\left(\begin{pmatrix} q_1^1 \\ \vdots \\ q_1^m \end{pmatrix} \otimes k \right) \geqslant 0$, we deduce that

$$\operatorname{rank}(\delta_I) \geqslant s + 1 - \ell \geqslant 2.$$

□

Proposition 3.4.5. *Adopt the notation and hypotheses of Setup 3.3.1. If $\ell \leqslant s-1$, then R/I has Tor algebra class $G(\mu(I) - 3\ell)$.*

Proof. In the notation of the proof of Theorem 3.4.3, it suffices to show that

$$\operatorname{rank} \delta_I \leqslant \mu(I) - 3\ell.$$

and that $T_1 \cdot T_1 = 0$. Since $s \geqslant 3$, a degree count immediately yields that $T_1 \cdot T_1 = 0$. Observe that $\dim_k (T_1^{R/I})_s = s + 1 - \ell$, so that

$$\dim_k (T_1^{R/I})_{s+1} = \mu(I) - s - 1 + \ell.$$

Moreover, counting ranks on the degree $s+1$ homogeneous strand of the long exact sequence of Tor associated to the short exact sequence

$$0 \to \frac{I_t}{I} \to \frac{R}{I} \to \frac{R}{I_t} \to 0,$$

we obtain

$$\dim_k(T_1^{R/I})_{s+1} - \dim_k(T_2^{R/I})_{s+1} = 3\ell$$

$$\implies \dim_k(T_2^{R/I})_{s+1} = \mu(I) - s - 1 - 2\ell.$$

Similarly, a rank count on the degree $s+2$ strand yields

$$\dim_k(T_2^{R/I})_{s+2} = s + 4\ell.$$

By counting degrees, the only nontrivial products can occur between $(T_1^{R/I})_s,\ (T_2^{R/I})_{s+2}$ and $(T_1^{R/I})_{s+1},\ (T_2^{R/I})_{s+1}$; this implies that

$$\begin{aligned}\operatorname{rank}\delta_I &\leqslant \dim_k(T_1^{R/I})_s + \dim_k(T_2^{R/I})_{s+1}\\ &= s + 1 - \ell + \mu(I) - s - 1 - 2\ell\\ &= \mu(I) - 3\ell.\end{aligned}$$

Combining this with Corollary 3.4.4, we find that R/I must be class $G(\mu(I)-3\ell)$. □

3.5 Realizability in The Type 2 Case

Adopt Setup 3.3.1 with $\ell = 1$. The ideal I_t has Betti table arising from Proposition 3.2.3 for some integer $b < s+1$ by Corollary 3.3.8. We may fit I into the short exact sequence

$$0 \to I_t/I \to R/I \to R/I_t \to 0$$

whence upon counting ranks on the graded strands of the long exact sequence of Tor, we deduce that $\dim_k \operatorname{Tor}_1^R(R/I,k)_{s+1} \leqslant b+3$. Since $b \leqslant s$, $\dim_k \operatorname{Tor}_1^R(R/I,k)_{s+1} \leqslant s+3$. Furthermore:

Proposition 3.5.1. *Let I be as in Setup 3.1.3 with $\ell = 1$. Then* $\dim_k \operatorname{Tor}_1^R(R/I,k)_{s+1} \leqslant$ *$s+2$.*

Proof. By counting ranks on the long exact sequence of Tor induced by the short exact sequence

$$0 \to I_t/I \to R/I \to R/I_t \to 0,$$

we must have $\dim_k \operatorname{Tor}_1^R(R/I_t, k)_{s+1} = s$, which is the maximum possible. By Proposition 3.3.9, I may be written

$$I = (\phi_1, \ldots, \phi_s, \psi_1, \ldots, \psi_s) + R_+\phi_{s+1}$$

where $\phi_1, \ldots, \phi_{s+1}, \psi_1, \ldots, \psi_s$ is a minimal generating set for I_t. Assume for sake of contradiction that $\dim_k \operatorname{Tor}_1^R(R/I, k)_{s+1} = s+3$; this means that the above generating set for I is minimal. Thus the resolution of Theorem 2.1.5 is a minimal free resolution for R/I.

Adopt the notation of Theorem 2.1.5, where $F_\bullet$ resolves R/I_t and $G_\bullet$ resolves k. Let us examine the map $q_1 : F_2 \to G_1$. By counting degrees, one finds that $q_1(F_2) \not\subset R_+G_1$ or that q_1 is identically the 0 map. Either case is a contradiction, so that $\operatorname{rank}_k(q_1 \otimes k) \geqslant 1$. $\square$

We now exhibit a class of ideals defining compressed rings with top socle degree $2s-1$ and $\dim_k \operatorname{Tor}_1^R(R/I, k)_{s+1} = s+2$, showing that the inequality of 3.5.1 is sharp. To do this, we first need some notation.

Definition 3.5.2. Let U_m^j (for $j \leqslant m$) denote the $m \times m$ matrix with entries from the polynomial ring $R = k[x, y, z]$ defined by:

$$U^j_{i,m-i} = x^2, \quad U^j_{i,m-i+1} = z^2, \quad U^{ev}_{i,m-i+2} = y^2 \text{ for } i < j$$

$$U^j_{i,m-i} = x, \quad U^j_{i,m-i+1} = z, \quad U^{ev}_{i,m-i+2} = y \text{ for } i \geqslant j$$

and all other entries are defined to be 0. Define $d_m^j := \det(U_m^j)$.

To see the pattern, we have:

$$U_2^1 = \begin{pmatrix} x^2 & z^2 \\ z & y \end{pmatrix},\ U_3^1 = \begin{pmatrix} 0 & x^2 & z^2 \\ x^2 & z^2 & y^2 \\ z & y & 0 \end{pmatrix},\ U_3^2 = \begin{pmatrix} 0 & x^2 & z^2 \\ x & z & y \\ z & y & 0 \end{pmatrix}$$

Definition 3.5.3. Define V_m^j (for $j < m$) to be the $(2m+1)\times(2m+1)$ skew symmetric matrix

$$V_m^j := \begin{pmatrix} O & O_{x^2} & (U_m^j)^T \\ -(O_{x^2})^T & 0 & {}^{y^2}O \\ -U_m^j & -({}^{y^2}O)^T & O \end{pmatrix}$$

and if $j = m$, then V_m^m is the skew symmetric matrix

$$V_m^j := \begin{pmatrix} O & O_{x^2} & (U_m^m)^T \\ -(O_{x^2})^T & 0 & {}^{y}O \\ -U_m^m & -({}^{y}O)^T & O \end{pmatrix}$$

Observe that the ideal of pfaffians $\mathrm{Pf}(V_m^j)$ is a grade 3 Gorenstein ideal with graded Betti table

	0	1	2	3
0	1	·	·	·
$2m-j-1$	·	$2m+1-j$	j	·
$2m-j$	·	j	$2m+1-j$	·
$4m-2j-1$	·	·	·	1

In particular, for any integer s, $\mathrm{Pf}(V_s^s)$ has Betti table

	0	1	2	3
0	1	·	·	·
$s-1$	·	$s+1$	s	·
s	·	s	$s+1$	·
$2s-1$	·	·	·	1

3.5.4. Given an $n \times n$ alternating matrix M, the notation $\mathrm{Pf}(M)$ will denote the ideal of submaximal pfaffians of the matrix M. Similarly,

$$(\mathrm{Pf}(M) \backslash \mathrm{Pf}_i(M))$$

is shorthand for the ideal

$$(\mathrm{Pf}_j(M) \mid 1 \leqslant j \leqslant n,\ j \neq i),$$

where $\mathrm{Pf}_j(M)$ denotes the pfaffian of the matrix obtained by deleting the jth row and column of M.

Setup 3.5.5. *In the notation of Definition 3.5.3, the following is a minimal free resolution of $Pf(V_m^j)$:*

$$0 \longrightarrow R \xrightarrow{d_1^*} F_1^* \xrightarrow{V_m^j} F_1 \xrightarrow{d_1} R$$

where d_1 is the row vector whose ith entry is the ith signed submaximal pfaffian of V_m^j. Observe that $F_1 = \bigoplus_{i=1}^{2m+1} Re_i$.

Let $G_\bullet$ be the Koszul complex on $U = Re_x \oplus Re_y \oplus Re_z$ with map $X : e_x \mapsto x$, $e_y \mapsto y$, and $e_z \mapsto z$.

Proposition 3.5.6. *Let $s \geqslant 1$ be an integer and $R = k[x, y, z]$, where k is any field. The ideal*

$$I := (Pf(V_s^s) \backslash Pf_{s+1}(V_s^s)) + R_+ Pf_{s+1}(V_s^s)$$

is minimally generated by $2s+2$ elements and defines a compressed ring with $\mathrm{Soc}(R/I) \cong k(-s) \oplus k(-2s+1)$ and Betti table

	0	1	2	3
0	1	·	·	·
$s-1$	·	s	$s-1$	·
s	·	$s+2$	$s+4$	1
$2s-1$	·	·	·	1

Proof. Adopt notation of Theorem 2.1.5 and Setup 3.5.5. Let $F_\bullet$ denote the graded minimal free resolution of $R/\mathrm{Pf}(V_s^s)$ as in Setup 3.5.5. Write $F_1 = F_1' \oplus Re_{s+1}$. By definition of the matrix V_s^s, the induced map $d_0 : F_1^* \to Re_{s+1}$ sends $e_s^* \mapsto -x^2 e_{s+1}$, $e_{s+2}^* \mapsto ye_{s+1}$ and all other basis vectors to 0.

Let $q_1 : F_1^* \to U$ be the map sending $e_s^* \mapsto -xe_x$, $e_{s+2}^* \mapsto e_y$, and all other basis vectors to 0. Then the following diagram commutes:

$$\begin{tikzcd} & F_1^* \arrow[dl, "q_1"'] \arrow[d, "d_0'"] \\ U \arrow[r, "X"] & \operatorname{im} X \end{tikzcd}$$

In particular, $\mathrm{rank}_k(q_1 \otimes k) = 1$. We do not have to compute the map $q_2 : F_3 \to \bigwedge^2 U$, since a degree count tells us that $q_2 \otimes k = 0$. Employing Corollary 2.1.8 yields the result. $\square$

Corollary 3.5.7. *Adopt Setup 3.3.1 with $\ell = 1$. Then I has Tor algebra class $G(r)$ for some $s \leqslant r \leqslant 2s-1$.*

Proof. Observe that $s+3 \leqslant \mu(I) \leqslant 2s+2$, where the upper bound follows from Proposition 3.5.1. The lower bound follows from Corollary 2.1.8, since

$$\mu(I) = \mu(I_t) + 2 - \mathrm{rank}_k(q_1 \otimes k),$$

and $\mu(I_t) \geqslant s+1$. A degree count shows that if I_t has Betti table given by Proposition 3.2.3, then $\mathrm{rank}_k(q_1 \otimes k) \leqslant \min\{b, 3\}$; this immediately yields the lower bound. Combining these bounds with Proposition 3.4.5, the result follows. $\square$

A natural question arising from Corollary 3.5.7 is whether or not every possible r value may be obtained for a given $s \geqslant 3$, where I is obtained from Setup 3.1.3. The next proposition will allow us to answer in the affirmative:

Proposition 3.5.8. *Let $s \geqslant 3$ be an integer and $R = k[x, y, z]$, where k is any field.*

1. *For $1 \leqslant i < s/2$, the ideal*

$$I := (Pf(V_{s-i}^{s-2i}) \backslash Pf_{s-i+1}(V_{s-i}^{s-2i})) + R_+ Pf_{s-i+1}(V_{s-i}^{s-2i})$$

   has Tor algebra class $G(2s-2i)$.

2. *For $1 \leqslant i < s/2$, the ideal*

$$I := (Pf(V_{s-i}^{s-2i}) \backslash Pf_{i+1}(V_{s-i}^{s-2i})) + R_+ Pf_{i+1}(V_{s-i}^{s-2i})$$

   has Tor algebra class $G(2s-2i-1)$.

Proof. Adopt notation and hypotheses as in Setup 3.5.5. In view of Corollary 2.1.8 and Proposition 3.4.5, it suffices to compute the rank of the map $q_1 : F_1^* \to U$ as in Theorem 2.1.5 to find the minimal number of generators.

We compute the map q_1 explicitly in each case. Write $F_1 = F_1' \oplus Re_{s-i+1}$, we see (recalling that $i > 0$) that the map $d_0 : F_1^* \to U$ is defined by sending $e_{s-i}^* \mapsto -x^2 e_{s-i+1},\ e_{s-i+2} \mapsto y^2 e_{s-i+2}$ and all other basis vectors to 0.

Take $q_1 : F_1^* \to U$ to be the map sending $e_{s-i}^* \mapsto -xe_x$, $e_{s-i+2}^* \mapsto ye_y$, and all other basis vectors map to 0. Clearly $q_1 \otimes k = 0$, whence the resolution of Theorem 2.1.5 is minimal. In particular, $\mu(I) = 2s + 3 - 2i$.

For the second case, retain much of the notation as above. Decompose $F_1 = F_1' \oplus Re_{i+1}$ and observe that $d_0 : F_1^* \to Re_{i+1}$ is the map sending $e_{2s-i}^* \mapsto x^2 e_{i+1},\ e_{2s-i+1}^* \mapsto z^2 e_{i+1},\ e_{2s-i+2}^* \mapsto ye_{i+1}$, and all other basis vectors to 0.

Take $q_1 : F_1^* \to U$ to be the map sending $e_{2s-i}^* \mapsto xe_x$, $e_{2s-i+1}^* \mapsto ze_z$, $e_{2s-i+2}^* \mapsto e_y$, and all other basis vectors to 0. In this case $\operatorname{rank}_k(q_1 \otimes k) = 1$, whence $\mu(I) = 2s + 2 - 2i$. □

Corollary 3.5.9. *Let $R = k[x, y, z]$ with the standard grading, where k is any field. Given any $s \geqslant 3$ and any r with $s \leqslant r \leqslant 2s - 1$, there exists an ideal I with* $\operatorname{Soc}(R/I) = k(-s) \oplus k(-2s+1)$ *and defining an Artinian compressed ring of Tor algebra class $G(r)$.*

Proof. Assume first that r is even and $s \leqslant r < 2s-1$. Employ Proposition 3.5.8 on the ideal

$$(\mathrm{Pf}(V^{r-s}_{r/2})\backslash \mathrm{Pf}_{r/2+1}(V^{r-s}_{r/2})) + R_{+}\mathrm{Pf}_{r/2+1}(V^{r-s}_{r/2})$$

Assume now that r is odd, with $s \leqslant r \leqslant 2s-1$. If $r = 2s-1$, use the ideal from Proposition 3.5.6. If $r < 2s-1$, apply Proposition 3.5.8 to the ideal

$$(\mathrm{Pf}(V^{r+1-s}_{(r+1)/2})\backslash \mathrm{Pf}_{s-(r+1)/2+1}(V^{r+1-s}_{(r+1)/2})) + R_{+}\mathrm{Pf}_{s-(r+1)/2+1}(V^{r+1-s}_{(r+1)/2})$$

□

Chapter 4

Applications to Determinantal Facet Ideals

Let $S = k[x_{ij} \mid 1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m]$ where k is any field, and M be a generic $n \times m$ matrix of indeterminates. The study of the ideal generated by all minors of a given size of M has a long history, and such ideals are well understood (see, for instance, [9]). In a similar vein, one can instead consider the ideal generated by *some* of the minors of a given size of M; these are known as *determinantal facet ideals* and were introduced by Ene, Herzog, Hibi, and Mohammadi in [23]. The study of determinantal facet ideals turns out to be much more subtle and has seen comparably less attention, even though such ideals arise naturally in algebraic statistics (see [18] and [29]). In [30], the linear strand of determinantal facet ideals is constructed in terms of a *generalized* Eagon-Northcott complex. In particular, the linear Betti numbers of such ideals may be computed in terms of the f-vector of an associated simplicial complex.

Determinantal facet ideals for the case $n = 2$ were originally introduced as binomial edge ideals independently by Ohtani [36] and Herzog, et. al. [29]; this generalized work of Diaconis, Eisenbud, and Sturmfels in [18]. To study binomial edge ideals, one can associate each column of M with a vertex of a graph G, and one can associate a minor of M involving two columns i and j with an edge (i, j) in the graph. For example, the ideal generated by all maximal minors of a $2 \times m$ matrix corresponds to a complete graph on m vertices. The relationship between homological invariants of ideals generated by some maximal minors of M and combinatorial invariants of the associated graph G has been widely studied; see the survey paper [32] for a compilation of such results. determinantal facet ideals naturally extend this idea by instead associating a pure simplicial complex Δ on m vertices to the ideal J_Δ, where each $(n-1)$-dimensional facet of Δ corresponds to a maximal minor in the set of generators of J_Δ. Mohammadi and Rauh further generalized this notion to that of a determinantal hypergraph ideal, which associates a minor to each hyperedge of a graph, allowing for an ideal that is generated by minors of different sizes.

In this chapter, we use trimming complexes to compute Betti tables of certain ideals of pfaffians and the previously mentioned determinantal facet ideals. The chapter is organized as follows. In Section 4.1, we show how to use the complex of Section 2.1 to resolve ideals generated by certain subsets of a minimal generating set of an arbitrary ideal I. As applications, we compute the Betti tables of the ideals obtained by removing a single generator from the ideal of submaximal pfaffians (see Proposition 4.1.4) and from the ideal of maximal minors of a generic $n \times m$ matrix M (see Theorem 4.1.15). In Section 4.2 we use the iterated trimming complex of Section 2.2 to compute the Betti tables of ideals obtained by removing certain additional generators from the generating set of the ideal of maximal minors of a generic $n \times m$ matrix M. As an application, we are able to deduce pieces of the f-vector of the simplicial complex associated to certain classes of uniform clutters.

4.1 Betti Tables for Ideals Obtained by Removing a Generator from Generic Submaximal Pfaffian Ideals and Ideals of Maximal Minors

In this section, we demonstrate how to use trimming complexes to compute the Betti table of the ideal generating by removing a single generator from a given generating set of an ideal I.

Setup 4.1.1. *Let $R = k[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field k, with $R_+ := R_{>0}$. Let $I \subseteq R$ be a homogeneous R_+-primary ideal and $(F_\bullet, d_\bullet)$ denote a homogeneous free resolution of R/I.*

Write $F_1 = F_1' \oplus Re_0$, where e_0 generates a free direct summand of F_1. Using the isomorphism

$$\operatorname{Hom}_R(F_2, F_1) = \operatorname{Hom}_R(F_2, F_1') \oplus \operatorname{Hom}_R(F_2, Re_0)$$

write $d_2 = d_2' + d_0$, where $d_2' \in \operatorname{Hom}_R(F_2, F_1')$, $d_0 \in \operatorname{Hom}_R(F_2, Re_0)$. Let $\mathfrak{a}$ denote a homogeneous ideal with

$$d_0(F_2) = \mathfrak{a}e_0,$$

and $(G_\bullet, m_\bullet)$ be a homogeneous free resolution of $R/\mathfrak{a}$.

Use the notation $K' := \operatorname{im}(d_1|_{F_1'} : F_1' \to R)$, $K_0 := \operatorname{im}(d_1|_{Re_0} : Re_0 \to R)$, and let $J := K' + \mathfrak{a} \cdot K_0$.

Proposition 4.1.2. *Adopt notation and hypotheses as in Setup 4.1.1. Then the resolution of Theorem 2.1.5 resolves K'.*

Proof. It will be shown that $\mathfrak{a} = K' : K_0$. Observe that $K' : K_0 \subseteq \mathfrak{a}$ by Proposition 2.1.2. Let $r \in \mathfrak{a}$; by assumption, there exists $f \in F_2$ such that $d_0(f) = re_0$. Since $F_\bullet$ is a complex, $d_1(re_0) = -d_1(d_2'(f))$, so that $rK_0 \subseteq K'$. This yields that $\mathfrak{a} = K' : K_0$. In particular, we find that $\mathfrak{a}K_0 \subset K'$. The resolution of Theorem 2.1.5 resolves $K' + \mathfrak{a}K_0 = K'$, so the result follows. $\square$

Notation 4.1.3. *Given a skew symmetric matrix $X \in M_n(R)$, where R is some commutative ring, the notation $Pf_j(X)$ will denote the pfaffian of the matrix obtained by removing the jth row and column from X.*

Proposition 4.1.4. *Let $R = k[x_{ij} \mid 1 \leqslant i < j \leqslant n]$ and let X denote a generic $n \times n$ skew symmetric matrix, with $n \geqslant 7$ odd. Given $1 \leqslant i \leqslant n$, the ideal*

$$J := (Pf_j(X) \mid i \neq j)$$

has Betti table

	0	1	2	3	$\cdots$	k	$\cdots$	$n-1$
0	1	$\cdot$	$\cdot$		$\cdots$	$\cdot$	$\cdots$	$\cdot$
$\vdots$								
$(n-3)/2$	$\cdot$	$n-1$	1	$\cdot$	$\cdots$		$\cdots$	$\cdot$
$\vdots$								
$(n-1)/2$	$\cdot$	$\cdot$	$\binom{n-1}{2}$	$\binom{n-1}{3}$	$\cdots$	$\binom{n-1}{k}$	$\cdots$	1
$\vdots$								
$n-3$	$\cdot$	$\cdot$	$\cdot$	1	$\cdots$	$\cdot$	$\cdots$	$\cdot$

In the case where $n = 5$, the Betti table is

	0	1	2	3	4
0	1	$\cdot$	$\cdot$	$\cdot$	$\cdot$
1	$\cdot$	4	1	$\cdot$	$\cdot$
2	$\cdot$	$\cdot$	6	5	1

Proof. In view of Corollary 2.1.8, it suffices to compute the ranks of the maps $q_i \otimes k$ for all appropriate i. Let $F_\bullet$ denote the minimal free resolution of the ideal of submaximal pfaffians of X. Observe that $F_\bullet$ is of the form

$$0 \longrightarrow R \xrightarrow{d_1^*} R^n \xrightarrow{X} R^n \xrightarrow{d_1} R,$$

where $d_1 = (\mathrm{Pf}_1(X), -\mathrm{Pf}_2(X), \ldots, (-1)^{n+1}\mathrm{Pf}_n(X))$ (see, for instance, [11]). Fix an integer $1 \leqslant \ell \leqslant n$ and let $K' := (\mathrm{Pf}_i(X) \mid i \neq \ell)$, $K_0 := (\mathrm{Pf}_\ell(X))$. Observe that ℓth row of X generates the ideal

$$\begin{cases} (x_{12}, \ldots, x_{1n}) \quad \text{if } \ell = 1, \\ (x_{1\ell}, \ldots, x_{\ell-1,\ell}, x_{\ell,\ell+1}, \ldots, x_{\ell,n}) \quad \text{if } 1 < \ell < n, \\ (x_{1n}, \ldots, x_{n-1,n}) \quad \text{if } \ell = n. \end{cases}$$

Notice that this ideal is a complete intersection on $n-1$ generators; in the notation of Setup 4.1.1, the ideal $\mathfrak{a}$ is this complete intersection (so that $\mathfrak{a}K_0 \subseteq K'$). Let $G_\bullet$ denote the Koszul complex resolving $\mathfrak{a}$.

Observe that for $i \geqslant 3$,

$$q_i : F_{i+1} = 0 \to G_i,$$

so $q_i \otimes k = 0$ for $i \geqslant 3$. By counting degrees, one finds $q_2 \otimes k = 0$. Finally, the map q_1 is simply the projection

$$q_1 : F_2 \cong R^n \to G_1 \cong R^{n-1}$$

onto the appropriate summands; this map has $\operatorname{rank}(q_1 \otimes k) = n-1$. Combining this information with Corollary 2.1.8 and Remark 2.1.9, for $i \geqslant 4$,

$$\dim_k \operatorname{Tor}_i^R(R/J, k) = \binom{n-1}{i}.$$

For $i = 3$ and $n \geqslant 7$,

$$\begin{aligned}
\dim_k \operatorname{Tor}_3^R(R/J)_{(n+5)/2} &= \operatorname{rank} G_3 \\
&= \binom{n-1}{3} \\
\dim_k \operatorname{Tor}_3^R(R/J)_n &= \operatorname{rank} F_3 \\
&= 1.
\end{aligned}$$

For $i = 3$ and $n = 5$, observe that $n = (n+5)/2$, so

$$\begin{aligned}
\dim_k \operatorname{Tor}_3^R(R/J) &= \operatorname{rank} F_3 + \operatorname{rank} G_3 \\
&= 1 + \binom{n-1}{3} = 5.
\end{aligned}$$

Finally, for $i = 2$ and $n \geqslant 5$,

$$\dim_k \operatorname{Tor}_2^R(R/J)_{(n+1)/2} = \operatorname{rank} F_2 - \operatorname{rank}(q_1 \otimes k)$$

$$
\begin{aligned}
&= n - (n-1) = 1 \\
\dim_k \operatorname{Tor}_2^R(R/J)_{(n+3)/2} &= \operatorname{rank} G_2 \\
&= \binom{n-1}{2}.
\end{aligned}
$$

□

Observe the difference between the Betti table of Proposition 4.1.4 and the classical case of the ideal generated by all submaximal pfaffians of a generic skew symmetric matrix. In the latter case, this ideal is always a grade 3 Gorenstein ideal (in particular, the projective dimension is 3). After removing a generator, one sees that the projective dimension can become arbitrarily large based on the size of the matrix X.

Next, we want to compute the graded Betti numbers when removing a generator from an ideal of maximal minors of a generic $n \times m$ matrix. This case requires more work since the q_ℓ maps of Proposition 2.1.4 must be computed explicitly in order to compute the ranks. For convenience, we recall the definition of the Eagon-Northcott complex.

Notation 4.1.5. *Let V be a k-vector space, where k is any field. The notation $\bigwedge^i V$ denotes the ith exterior power of V and $D_i(V)$ denotes the ith divided power of V (see [20, Section A2.4] for the definition of $D_i(V)$).*

Definition 4.1.6. Let $\phi : F \to G$ be a homomorphism of free modules of ranks f and g, respectively, with $f \geqslant g$. Let c_ϕ be the image of ϕ under the isomorphism $\operatorname{Hom}_R(F, G) \xrightarrow{\cong} F^* \otimes G$. The *Eagon-Northcott complex* is the complex

$$
0 \to D_{f-g}(G^*) \otimes \bigwedge^{f} F \to D_{f-g-1}(G^*) \otimes \bigwedge^{f-1} F \to \cdots \to G^* \otimes \bigwedge^{g+1} F \to \bigwedge^{g} F \to \bigwedge^{g} G
$$

with differentials in homological degree $\geqslant 2$ induced by multiplication by the element $c_\phi \in F^* \otimes G$, and the map $\bigwedge^g F \to \bigwedge^g G$ is $\bigwedge^g \phi$.

Setup 4.1.7. *Let $R = k[x_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m]$ and $M = (x_{ij})_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m}$ denote a generic $n \times m$ matrix, where $n \leqslant m$. View M as a homomorphism $M : F \to G$ of free modules F and G of rank m and n, respectively.*

Let f_i, $i = 1, \ldots, m$, g_j, $j = 1, \ldots, n$ denote the standard bases with respect to which M has the above matrix representation. Write

$$\bigwedge^n F = F' \oplus Rf_\sigma$$

for some free module F', where $\sigma = (\sigma_1 < \cdots < \sigma_n)$ is a fixed index set, and the notation f_σ denotes $f_{\sigma_1} \wedge \cdots \wedge f_{\sigma_n}$. Recall that the Eagon-Northcott complex of Definition 4.1.6 resolves the quotient ring defined by $I_n(M)$, the ideal of $n \times n$ minors of M.

We will consider the submodule of $\bigwedge^{n+\ell} F$ generated by all elements of the form $f_{\sigma,\tau}$, where $\tau = (\tau_1 < \cdots < \tau_\ell)$ and $\sigma \cap \tau = \varnothing$. The notation $f_{\sigma,\tau}$ denotes the element $f_\sigma \wedge f_\tau$. If $\tau = (\tau_1 < \cdots < \tau_n)$, let Δ_τ denote the determinant of the matrix formed by columns $\tau_1, \ldots, \tau_n$ of M. Then, in the notation of Setup 4.1.1,

$$K' = (\Delta_\tau \mid \tau \neq \sigma)$$

and $K_0 = (\Delta_\sigma)$.

Observe that the Eagon-Northcott differential $d_2 : G^* \otimes \bigwedge^{n+1} F \to \bigwedge^n F$ induces a homomorphism $d_0 : G^* \otimes \bigwedge^{n+1} F \to Rf_\sigma$ by sending

$$g_i^* \otimes f_{\{j\},\sigma} \mapsto x_{ij} f_\sigma,$$

and all other basis elements to 0. In the notation of Setup 4.1.1,

$$\mathfrak{a} = (x_{ij} \mid i = 1, \ldots, n,\ j \notin \sigma).$$

This means $\mathfrak{a}$ is a complete intersection generated by $n(m-n)$ elements, and hence is resolved by the Koszul complex. Moreover, $\mathfrak{a}K_0 \subseteq K'$. Let

$$U = \bigoplus_{\substack{1 \leqslant i \leqslant n \\ j \notin \sigma}} Re_{ij}$$

with differential induced by the homomorphism $m_1 : U \to R$ *sending* $e_{ij} \mapsto x_{ij}$*. If* $L = (i, j)$ *is a* 2*-tuple, then the notation* e_L *will denote* e_{ij}*.*

The proof of the following Proposition is a straightforward computation.

Proposition 4.1.8. *Adopt notation and hypotheses of Setup 4.1.7. Define* $q_1 : G^* \otimes \bigwedge^{n+1} F \to U$ *by sending* $g_i^* \otimes f_{\{j\},\sigma} \mapsto e_{ij}$ *and all other basis elements to* 0*. Then the following diagram commutes:*

$$\begin{tikzcd} & G^* \otimes \bigwedge^{n+1} F \arrow[dl, "q_1"'] \arrow[d, "d_0'"] \\ U \arrow[r, "m_1"'] & \mathfrak{a}, \end{tikzcd}$$

where $d_0' : G^* \otimes \bigwedge^{n+1} F \to R$ *is the composition*

$$G^* \otimes \bigwedge^{n+1} F \xrightarrow{d_0} Rf_\sigma \longrightarrow R\ ,$$

and where the second map sends $f_\sigma \mapsto 1$*.*

We will need the following definition before introducing the q_i maps for $i \geqslant 2$.

Definition 4.1.9. Let $\tau = (\tau_1, \ldots, \tau_\ell)$ be an indexing set of length ℓ with $\tau_1 < \cdots < \tau_\ell$. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$, with $\alpha_i \geqslant 0$ for each i. Define $\mathcal{L}_{\alpha,\tau}$ to be the subset of size ℓ subsets of the cartesian product

$$\{i \mid \alpha_i \neq 0\} \times \tau,$$

where $\{(r_1, \tau_1), \ldots, (r_\ell, \tau_\ell)\} \in \mathcal{L}_{\alpha,\tau}$ if $|\{i \mid r_i = j\}| = \alpha_j$.

Observe that $L_{\alpha,\tau}$ is empty unless $\alpha_1 + \cdots + \alpha_n = \ell$.

Example 4.1.10. One easily computes:

$$\mathcal{L}_{(2,0,1),(1,2,3)} = \{\{(3,1),(1,2),(1,3)\},\{(1,1),(3,2),(1,3)\},\{(1,1),(1,2),(3,3)\}\}$$

$$\mathcal{L}_{(2,0,2),(1,2,3,4)} = \{\{(3,1)\,,\ (3,2)\,,(1,3)\,,\ (1,4)\}\,,\ \{(3,1)\,,\ (1,2)\,,\ (3,3)\,,(1,4)\}\,,$$

$$\{(1,1)\,,\,(3,2)\,,\,(3,3)\,,(1,4)\}\,,\,\{(3,1)\,,\,(1,2)\,,\,(1,3)\,,\,(3,4)\}\,,$$
$$\{(1,1)\,,\,(3,2)\,,\,(1,3)\,,\,(3,4)\}\,,\,\{(1,1)\,,\,(1,2)\,,\,(3,3)\,,\,(3,4)\}\}$$

Lemma 4.1.11. *Let $\tau = (\tau_1, \ldots, \tau_\ell)$ be an indexing set of length ℓ with $\tau_1 < \cdots < \tau_\ell$. Let $\alpha = (\alpha_1, \cdots, \alpha_n)$, with $\alpha_i \geqslant 0$ for each i. Use the notation $\alpha^i := (\alpha_1, \ldots, \alpha_i - 1, \ldots, \alpha_n)$. Then any $L' \in \mathcal{L}_{\alpha^i, \tau\setminus\tau_k}$ is contained in a unique element $L \in \mathcal{L}_{\alpha,\tau}$.*

Proof. Given L', take $L := L' \cup (i, j_k)$, ordered appropriately. Assume that $L' \subseteq L''$ for some other $L'' \in \mathcal{L}_{\alpha,\tau}$. It is easy to see that $L''\setminus L' = (a, \sigma_k)$, where a is some integer. However, since α^i differs by α by 1 in the ith spot, $a = i$, whence $L = L''$ and L is unique. □

Lemma 4.1.12. *Adopt notation and hypotheses of Setup 4.1.7. Define*

$$q_\ell : D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \to \bigwedge^{\ell} U, \quad \ell \geqslant 2,$$

by sending

$$g_1^{*(\alpha_1)} \cdots g_n^{*(\alpha_n)} \otimes f_{\tau, I\sigma} \mapsto \sum_{L \in \mathcal{L}_{\alpha,\tau}} e_{L_1} \wedge \cdots \wedge e_{L_\ell},$$

where $\mathcal{L}_{\alpha,\tau}$ is defined in Definition 4.1.9. All other basis elements are sent to 0. Then the following diagram commutes:

$$\begin{tikzcd}
D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \arrow[r, "d_\ell"] \arrow[d, "q_\ell"'] & D_{\ell-1}(G^*) \otimes \bigwedge^{n+\ell-1} F \arrow[d, "q_{\ell-1}"] \\
\bigwedge^\ell U \arrow[r, "m_\ell"'] & \bigwedge^{\ell-1} U.
\end{tikzcd}$$

Proof. We first compute the image of the element

$$g_1^{*(\alpha_1)} \cdots g_n^{*(\alpha_n)} \otimes f_{\tau,\sigma}$$

going clockwise about the diagram. We obtain:

$$g_1^{*(\alpha_1)} \cdots g_n^{*(\alpha_n)} \otimes f_{\tau,\sigma} \mapsto \sum_{\substack{\{i \mid \alpha_i \neq 0\} \\ 1 \leqslant j \leqslant \ell}} (-1)^{j+1} x_{i\tau_j} g_1^{*(\alpha_1)} \cdots g_i^{*(\alpha_i - 1)} \cdots g_n^{*(\alpha_n)} \otimes f_{\tau\setminus\tau_j,\sigma}$$

$$
\begin{aligned}
&+ \sum_{\substack{\{i|\alpha_i\neq 0\}\\ 1\leqslant j\leqslant n}} (-1)^{m-n+j+1} x_{i\sigma_j} g_1^{*(\alpha_1)}\cdots g_i^{*(\alpha_i-1)}\cdots g_n^{*(\alpha_n)}\otimes f_{J,\sigma\setminus\sigma_j}\\
&\mapsto \sum_{\substack{\{i|\alpha_i\neq 0\}\\ 1\leqslant j\leqslant \ell}} \sum_{L\in\mathcal{L}_{\alpha^i,\tau\setminus\tau_j}} (-1)^{j+1} x_{i\tau_j} e_{L_1}\wedge\cdots\wedge e_{L_{\ell-1}}
\end{aligned}
$$

where in the above, denote $\alpha^i := (\alpha_1,\dots,\alpha_i-1,\dots,\alpha_n)$ and L_i the ith entry of $L\in\mathcal{L}_{\alpha,\tau\setminus\tau_j}$. According to Lemma 4.1.11,

$$
\begin{aligned}
&\sum_{\substack{\{i|\alpha_i\neq 0\}\\ 1\leqslant j\leqslant \ell}} \sum_{L\in\mathcal{L}_{\alpha^i,\tau\setminus\tau_j}} (-1)^{j+1} x_{i\tau_j} e_{L_1}\wedge\cdots\wedge e_{L_{\ell-1}}\\
&= \sum_{1\leqslant j\leqslant \ell} \sum_{L\in\mathcal{L}_{\alpha,\tau}} (-1)^{j+1} x_{L_j} e_{L_1}\wedge\cdots\wedge\widehat{e_{L_j}}\wedge\cdots\wedge e_{L_\ell}.
\end{aligned}
$$

Moving in the counterclockwise direction, we obtain:

$$
\begin{aligned}
g_1^{*(\alpha_1)}\cdots g_n^{*(\alpha_n)}\otimes f_{\tau,\sigma} &\mapsto \sum_{L\in\mathcal{L}_{\alpha,\tau}} e_{L_1}\wedge\cdots\wedge e_{L_\ell}\\
&\mapsto \sum_{L\in\mathcal{L}_{\alpha,\tau}} \sum_{1\leqslant j\leqslant \ell} (-1)^{j+1} x_{L_j} e_{L_1}\wedge\cdots\wedge\widehat{e_{L_j}}\wedge\cdots\wedge e_{L_\ell}
\end{aligned}
$$

□

Lemma 4.1.13. *Adopt notation and hypotheses of Setup 4.1.7. Then the maps*

$$
q_\ell : D_\ell(G^*)\otimes \bigwedge^{n+\ell} F \to \bigwedge^{\ell} U
$$

have $\operatorname{rank}(q_\ell\otimes k) = \binom{n+\ell-1}{\ell}\cdot\binom{m-n}{\ell}$ *for all* $\ell = 1,\cdots,m-n+1$ *and* $\operatorname{rank}(q_\ell\otimes k) = 0$ *for all* $\ell = m-n+2,\dots,n(m-n)$.

Remark 4.1.14. If we use the convention that $\binom{r}{s} = 0$ for $s > r$, then the above says that $\operatorname{rank}(q_\ell\otimes k) = \binom{n+\ell-1}{\ell}\cdot\binom{m-n}{\ell}$ for all $\ell \geqslant 1$.

Proof. First observe that since the Eagon-Northcott complex is 0 in homological degrees $\geqslant m-n+2$, it is immediate that $q_\ell = 0$ for $\ell \geqslant m-n+2$. For the first claim, this follows from the fact that for $\alpha \neq \alpha'$,

$$\mathcal{L}_{\alpha,\tau} \cap \mathcal{L}_{\alpha',\tau} = \varnothing,$$

which implies that the image of each element $g_1^{*(\alpha_1)} \cdots g_n^{*(\alpha_n)} \otimes f_{\tau,\sigma} \in D_\ell(G^*) \otimes \bigwedge^{n+\ell} F$ under q_ℓ has maximal rank (recall that these are the only elements with nonzero image). This rank is computed by counting all such basis elements; it is clear that there are $\binom{m-n}{\ell}$ possible elements of the form $f_{\tau,\sigma}$, since σ is a fixed index set of length n. The rank of $D_\ell(G^*)$ is $\binom{n+\ell-1}{\ell}$, therefore we conclude that the rank of each q_ℓ is

$$\binom{n+\ell-1}{\ell}\binom{m-n}{\ell}$$

□

Theorem 4.1.15. *Adopt notation and hypotheses of Setup 4.1.7. If $\tau = (\tau_1 < \cdots < \tau_n)$, let Δ_τ denote the determinant of the matrix formed by columns $\tau_1, \ldots, \tau_n$ of M. Then the ideal*

$$K' := (\Delta_\tau \mid \tau \neq \sigma)$$

has Betti table

	0	1	$\cdots$	ℓ	$\cdots$	$n(m-n)-1$	$n(m-n)$
0	1	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$	$\cdot$
$\vdots$							
$n-1$	$\cdot$	$\binom{m}{n}-1$	$\cdots$	$\binom{n+\ell-2}{\ell-1}\left(\binom{m}{n+\ell-1}-\binom{m-n}{\ell-1}\right)$	$\cdots$	$\cdot$	$\cdot$
n	$\cdot$	$\cdot$	$\cdots$	$\binom{n(m-n)}{\ell}-\binom{n+\ell-1}{\ell}\binom{m-n}{\ell}$	$\cdots$	$n(m-n)$	1

Proof. We employ Corollary 2.1.8 and Remark 2.1.9. By selection, $\mathfrak{a}K_0 \subseteq K'$. Let $E_\bullet$ denote the Eagon-Northcott complex as in Setup 4.1.7. For $\ell \geqslant 1$,

$$\operatorname{rank} E_\ell = \binom{n+\ell-2}{\ell-1}\binom{m}{n+\ell-1}.$$

Similarly, let $K_\bullet$ denote the Koszul complex resolving $\mathfrak{a}$ as in Setup 4.1.7. Then

$$\operatorname{rank} K_\ell = \binom{n(m-n)}{\ell}.$$

Combining the information above with that of Lemma 4.1.13, Corollary 2.1.8, and Remark 2.1.9, we have:

$$\begin{aligned}
\dim_k \operatorname{Tor}_\ell^R(R/K', k)_{n+\ell} &= \operatorname{rank} E_\ell - \operatorname{rank}_k(q_{\ell-1} \otimes k) \\
&= \binom{n+\ell-2}{\ell-1}\binom{m}{n+\ell-1} - \binom{n+\ell-2}{\ell-1}\binom{m-n}{\ell-1}, \\
\dim_k \operatorname{Tor}_\ell^R(R/K', k)_{n+\ell+1} &= \operatorname{rank} K_\ell - \operatorname{rank}_k(q_\ell \otimes k) \\
&= \binom{n(m-n)}{\ell} - \binom{n+\ell-1}{\ell}\binom{m-n}{\ell}.
\end{aligned}$$

This concludes the proof. $\square$

4.2 Betti Tables for a Class of Determinantal Facet Ideals

In this section we consider the case for removing multiple generators from the ideal of maximal minors of a generic $n \times m$ matrix M. Such ideals belong to the class of ideals called determinantal facet ideals, which were studied in [23] and [30]. Graded Betti numbers for these ideals appearing in higher degrees have not been previously computed, even in simple cases. In Theorem 4.2.6, the graded Betti numbers of an infinite class of determinantal facet ideals defining quotient rings of regularity $n+1$ are computed explicitly in all degrees. In [30] the linear strand for such ideals is computed in terms of the f-vector of some associated simplicial complex. We use the linear strand of Theorem 4.2.6 to deduce the f-vector of the simplicial complex associated to an n-uniform clutter obtained by removing pairwise disjoint subsets from all n-subsets of $[m]$ (see Corollary 4.2.14).

Setup 4.2.1. *Let $R = k[x_{ij} \mid 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m]$ and $M = (x_{ij})_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ denote a generic $n \times m$ matrix, where $n \leqslant m$. View M as a homomorphism $M : F \to G$ of free modules F and G of rank m and n, respectively.*

Fix indexing sets $\sigma_j = (\sigma_{j1} < \cdots < \sigma_{jn})$ for $j = 1, \ldots, r$ pairwise disjoint; that is, $\sigma_i \cap \sigma_j = \varnothing$ for $i \neq j$ (this intersection is taken as sets).

Let f_i, for $i = 1, \ldots, m$, and g_j, for $j = 1, \ldots, n$ denote the standard bases with respect to which M has the above matrix representation. Write

$$\bigwedge^n F = F' \oplus Rf_{\sigma_1} \oplus \cdots \oplus Rf_{\sigma_r}$$

for some free module F', where the notation f_{σ_j} denotes $f_{\sigma_{j1}} \wedge \cdots \wedge f_{\sigma_{jn}}$. Recall that the Eagon-Northcott complex of Definition 4.1.6 resolves the quotient ring defined by $I_n(M)$. If $\tau = (\tau_1 < \cdots < \tau_n)$, let Δ_τ denote the determinant of the matrix formed by columns $\tau_1, \ldots, \tau_n$ of M. Then, in the iterated version of Setup 4.1.1,

$$K' = (\Delta_\tau \mid \tau \neq \sigma_j,\ j = 1, \ldots, r)$$

and $K_0^j = (\Delta_{\sigma_j})$.

Observe that the Eagon-Northcott differential $d_2 : G^* \otimes \bigwedge^{n+1} F \to \bigwedge^m F$ induces homomorphisms $d_0^\ell : G^* \otimes \bigwedge^{n+1} F \to Rf_{\sigma_j}$ by sending

$$g_i^* \otimes f_{\{\ell\},\sigma_j} \mapsto x_{i\ell} f_{\sigma_j},$$

and all other basis elements to 0. In the notation of Setup 4.1.1, this means we are considering the family of ideals

$$\mathfrak{a}_j = (x_{i\ell} \mid i = 1, \ldots, n,\ \ell \notin \sigma_j).$$

For each $j = 1, \ldots, r$, $\mathfrak{a}_j$ is a complete intersection generated by $n(m-n)$ elements, and hence is resolved by the Koszul complex. Let

$$U_j = \bigoplus_{\substack{1\leqslant i\leqslant n \\ \ell\notin\sigma_j}} Re_{i\ell}$$

with differential induced by the homomorphism $m_1^j : U_j \to R$ sending $e_{i\ell} \mapsto x_{i\ell}$. If $L = (i,j)$ is a 2-tuple, then the notation e_L will denote e_{ij}.

Remark 4.2.2. The assumption $\sigma_i \cap \sigma_j = \varnothing$ for $i \neq j$ implies that, if we define $K' := (\Delta_\tau \mid \tau \neq \sigma_j,\ j = 1, \dots, r)$, each Δ_{σ_j} satisfies $\mathfrak{a}_j \Delta_{\sigma_j} \subset K'$. This is significant since it allows us to use the resolution of Theorem 2.2.4. If the indexing sets were not pairwise disjoint, then we would have to apply Theorem 2.1.5 iteratively and compute the minimal presenting matrix at each step explicitly.

Another way of saying that $\sigma_i \cap \sigma_j = \varnothing$ is that the Alexander dual $\Delta^\vee$ of Δ is totally disconnected.

Proposition 4.2.3. *Adopt notation and hypotheses of Setup 4.2.1. Define $q_1^j : G^* \otimes \bigwedge^{n+1} F \to U_j$ by sending $g_i^* \otimes f_{\{\ell\},\sigma_j} \mapsto e_{i\ell}$ and all other basis elements to 0. Then the following diagram commutes:*

$$\begin{tikzcd}
 & G^* \otimes \bigwedge^{n+1} F \arrow[dl, "q_1^j"'] \arrow[d, "d_0'"] \\
U_j \arrow[r, "m_1^j"'] & \mathfrak{a}_j,
\end{tikzcd}$$

where $d_0' : G^* \otimes \bigwedge^{n+1} F \to R$ is the composition

$$G^* \otimes \bigwedge^{n+1} F \xrightarrow{d_0} Rf_{\sigma_j} \longrightarrow R\,,$$

and where the second map sends $f_{\sigma_j} \mapsto 1$.

Proposition 4.2.4. *Adopt notation and hypotheses of Setup 4.2.1. Define*

$$q_\ell^j : D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \to \bigwedge^{\ell} U_j, \quad \ell \geqslant 2,$$

by sending

$$g_1^{*(\alpha_1)} \cdots g_n^{*(\alpha_n)} \otimes f_{\tau,\sigma_j} \mapsto (-1)^n \sum_{L \in \mathcal{L}_{\alpha,\tau}} e_{L_1} \wedge \cdots \wedge e_{L_\ell},$$

where $\mathcal{L}_{\alpha,\tau}$ is defined in Definition 4.1.9. All other basis elements are sent to 0*. Then the following diagram commutes:*

$$\begin{tikzcd} D_\ell(G^*) \otimes \bigwedge^{n+\ell} F \arrow[r, "d_\ell"] \arrow[d, "q_\ell^j"'] & D_{\ell-1}(G^*) \otimes \bigwedge^{n+\ell-1} F \arrow[d, "q_{\ell-1}^j"] \\ \bigwedge^\ell U_j \arrow[r, "m_\ell^j"'] & \bigwedge^{\ell-1} U_j \end{tikzcd},$$

where m_ℓ^j is the standard Koszul differential induced by the map m_1^j as in Setup 4.2.1.

Lemma 4.2.5. *Adopt notation and hypotheses of Setup 4.2.1. Then*

$$\operatorname{rank}_k\left(\begin{pmatrix} q_\ell^1 \\ \vdots \\ q_\ell^r \end{pmatrix} \otimes k\right) = \binom{n+\ell-1}{\ell} \cdot \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \binom{m-in}{\ell-(i-1)n}$$

Proof. For convenience, use the notation

$$\operatorname{rk}_\ell := \operatorname{rank}_k\left(\begin{pmatrix} q_\ell^1 \\ \vdots \\ q_\ell^r \end{pmatrix} \otimes k\right).$$

As already noted, $\mathcal{L}_{\alpha',\tau} \cap \mathcal{L}_{\alpha,\tau} = \varnothing$ for $\alpha \neq \alpha'$, so $\operatorname{rk}_\ell \leqslant r\binom{n+\ell-1}{\ell}\binom{m-n}{\ell}$. We want to count all elements

$$g_1^{*(\alpha_1)} \cdots g_n^{*(\alpha_n)} \otimes f_\tau$$

such that there exists $1 \leqslant j \leqslant r$ with

$$0 \neq q_\ell^j(g_1^{*(\alpha_1)} \cdots g_n^{*(\alpha_n)} \otimes f_\tau),$$

taking into account the fact that some elements will have nonzero image under multiple q_ℓ^j. Thus, we count all elements

$$g_1^{*(\alpha_1)} \cdots g_n^{*(\alpha_n)} \otimes f_\tau$$

such that the image under at least i distinct q_ℓ^j is nonzero, then apply the inclusion exclusion principle.

It is easy to see that this set is obtained by choosing all indexing sets τ with $|\tau| = n + \ell$ such that $\tau = \sigma_{j_1} \cup \cdots \cup \sigma_{j_i} \cup \tau'$ for some i and τ' with $\tau' \cap \sigma_{j_s} = \varnothing$ for each $s = 1, \dots, i$. Fixing i, there are $\binom{r}{i}$ unique choices for the union $\sigma_{j_1} \cup \cdots \cup \sigma_{j_i}$. For the indexing set τ', there are $m - in$ total choices of indices after removing all elements of the union $\sigma_{j_1} \cup \cdots \cup \sigma_{j_i}$, and we are choosing $\ell + n - in = \ell - (i-1)n$ elements. Using the inclusion-exclusion principle, the total number of indexing sets τ as above is

$$\sum_{i=2}^{r} (-1)^i \binom{r}{i} \binom{m - in}{\ell - (i-1)n}.$$

Multiplying by rank $D_\ell(G)$ and subtracting from $r\binom{n+\ell-1}{\ell}\binom{m-n}{\ell}$, we obtain the result.

□

Theorem 4.2.6. *Adopt notation and hypotheses as in Setup 4.2.1. Define*

$$rk_\ell := \binom{n+\ell-1}{\ell} \cdot \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \binom{m - in}{\ell - (i-1)n}.$$

If $\tau = (\tau_1 < \cdots < \tau_n)$, let Δ_τ denote the determinant of the matrix formed by columns $\tau_1, \dots, \tau_n$ of M. Then the ideal

$$K' := (\Delta_\tau \mid \tau \neq \sigma_j, \ j = 1, \dots, r)$$

has Betti table

	0	1	$\cdots$	ℓ	$\cdots$	$n(m-n)-1$	$n(m-n)$
0	1	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$	$\cdot$
$\vdots$							
$n-1$	$\cdot$	$\binom{m}{n} - r$	$\cdots$	$\binom{n+\ell-2}{\ell-1}\binom{m}{n+\ell-1} - rk_{\ell-1}$	$\cdots$	$\cdot$	$\cdot$
n	$\cdot$	$\cdot$	$\cdots$	$r \cdot \binom{n(m-n)}{\ell} - rk_\ell$	$\cdots$	$r \cdot n(m-n)$	r

Proof. Let $E_\bullet$ denote the Eagon-Northcott complex as in Setup 4.2.1. For $\ell \geqslant 1$,

$$\operatorname{rank} E_\ell = \binom{n+\ell-2}{\ell-1}\binom{m}{n+\ell-1}.$$

Similarly, let $K^j_\bullet$ denote the Koszul complex resolving $\mathfrak{a}_j$ as in Setup 4.2.1. Then for each $j = 1, \ldots, r$,

$$\operatorname{rank} K^j_\ell = \binom{n(m-n)}{\ell}.$$

Combining the information above with that of Lemma 4.2.5, Corollary 2.2.6, and the iterated version of Remark 2.1.9, we have:

$$\begin{aligned}
\dim_k \operatorname{Tor}^R_\ell(R/K', k)_{n+\ell} &= \operatorname{rank} E_\ell - \operatorname{rank}_k \left(\begin{pmatrix} q^1_{\ell-1} \\ \vdots \\ q^r_{\ell-1} \end{pmatrix} \otimes k \right) \\
&= \binom{n+\ell-2}{\ell-1}\binom{m}{n+\ell-1} - \operatorname{rk}_{\ell-1}, \\
\dim_k \operatorname{Tor}^R_\ell(R/K', k)_{n+\ell+1} &= \sum_{j=1}^{r} \operatorname{rank} K^j_\ell - \operatorname{rank}_k \left(\begin{pmatrix} q^1_{\ell} \\ \vdots \\ q^r_{\ell} \end{pmatrix} \otimes k \right) \\
&= r \cdot \binom{n(m-n)}{\ell} - \operatorname{rk}_\ell.
\end{aligned}$$

$\square$

The following definitions assume familiarity of the reader with the language of simplicial complexes. For an introduction, see, for instance, Chapter 5 of [8]. Given a pure $(n-1)$-dimensional simplicial complex Δ, the determinantal facet ideal J_Δ associated to Δ is generated by all maximal minors $\det(M_\tau)$, where $\tau = (\tau_1 < \cdots < \tau_n) \in \Delta$ is a facet of Δ.

Definition 4.2.7. Let Δ be a simplicial complex. The f-vector $(f_0(\Delta), \ldots, f_{\dim \Delta}(\Delta))$ is the sequence of integers with

$$f_i(\Delta) = |\{\sigma \in \Delta \mid \dim \sigma = i\}|.$$

Definition 4.2.8. A *clutter* C on the vertex set $[n] := \{1, \dots, n\}$ is a collection of subsets of $[n]$ such that no element of C is contained in another. Any element of C is called a *circuit*. If all circuits of C have the same cardinality m, then C is called an *m-uniform* clutter.

If C is an m-uniform clutter, then a *clique* of C is a subset σ of $[n]$ such that each m-subset τ of σ is a circuit of C.

Definition 4.2.9. Let M be a generic $n \times m$ matrix, with $n \leqslant m$. Given an n-uniform clutter C on the vertex set $[m]$, associate to each circuit $\tau = \{j_1, \dots, j_n\}$ with $j_1 < \cdots < j_n$ the determinant $\det(M_\tau)$ of the submatrix formed by columns $j_1, \dots, j_n$ of M.

The ideal $J_C := \{\det(M_\tau) \mid \tau \in C\}$ is called the *determinantal facet ideal* associated to C.

Similarly, define the *clique complex* $\Delta(C)$ as the associated simplicial complex whose facets are the circuits of C.

The following definition is introduced in [30].

Definition 4.2.10. Let $\phi : F \to G$ be a homomorphism of free modules of rank m and n, respectively. Let $f_1, \dots, f_m$ and $g_1, \dots, g_n$ denote bases of F and G, respectively. Let Δ be a simplicial complex on the vertex set $[m]$. Then the *generalized Eagon-Northcott complex* $\mathcal{C}_\bullet(\Delta; \phi)$ associated to Δ is the subcomplex

$$0 \to \mathcal{C}_{m-n+1} \to \cdots \to \mathcal{C}_1 \to \mathcal{C}_0$$

of the Eagon-Northcott complex with $\mathcal{C}_0 = \bigwedge^n G$ and $\mathcal{C}_\ell \subseteq D_{\ell-1}(G^*) \otimes \bigwedge^{n+\ell-1} F$ for $\ell \geqslant 1$ the submodule generated by all elements $g_1^{*(\alpha_1)} \cdots g_n^{*(\alpha_n)} \otimes f_\sigma$, where $\sigma \in \Delta$ and $\dim \sigma = n + \ell - 2$.

Remark 4.2.11. Notice that by definition of the f-vector in Definition 4.2.7 combined with Definition 4.2.10, for $\ell \geqslant 1$,

$$\operatorname{rank} C_\ell(\Delta; \phi) = \binom{n+\ell-2}{\ell-1} f_{n+\ell-2}(\Delta).$$

Definition 4.2.12. Let $F_\bullet$ be a minimal graded complex of free R-modules. The *linear strand* $F_\bullet^{\mathrm{lin}}$ of $F_\bullet$ is the complex with $F_i^{\mathrm{lin}} =$ degree i part of F_i, and differentials induced by the differentials of $F_\bullet$.

The following result illustrates the connection between the complex of Definition 4.2.10 and resolutions of determinantal facet ideals.

Theorem 4.2.13. *[30, Theorem 4.1] Let $\phi : F \to G$ be a homomorphism of free modules of rank m and n, respectively. Let C be an n-uniform clutter on the vertex set $[m]$ with associated simplicial complex $\Delta(C)$. Let J_C denote determinantal facet ideal associated to C, with minimal free resolution $\mathcal{F}_\bullet$. Then,*

$$\mathcal{F}_i^{lin} = \mathcal{C}_i(\Delta(C); \phi),$$

where $\mathcal{F}_\bullet^{lin}$ denotes the linear strand of the complex $\mathcal{F}_\bullet$.

Corollary 4.2.14. *Let C denote the n-uniform clutter on the vertex set $[m]$ obtained by removing r pairwise disjoint elements from all n-subsets of $[m]$. Then for $\ell \geqslant 1$,*

$$f_{n+\ell-2}(\Delta(C)) = \binom{m}{n+\ell-1} - \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \binom{m-in}{\ell-(i-1)n}$$

Proof. Let $\phi : F \to G$ be a generic homomorphism of free modules of rank m and n, respectively, and let J_C denote the determinantal facet ideal associated to C with minimal free resolution $\mathcal{F}_\bullet$. By Theorem 4.2.6 with $\ell \geqslant 1$,

$$\operatorname{rank} \mathcal{F}_\ell^{\mathrm{lin}} = \binom{n+\ell-2}{\ell-1} \left(\binom{m}{n+\ell-1} - \sum_{i=1}^{r} (-1)^{i+1} \binom{r}{i} \binom{m-in}{\ell-(i-1)n} \right).$$

Combining this with Theorem 4.2.13 and Remark 4.2.11, the result follows. $\square$

Chapter 5

Minimal Free Resolutions for Certain Classes of Monomial Ideals

Let $(R, \mathfrak{m}, k)$ denote a local ring. The computation of minimal free resolutions of arbitrary ideals $I \subseteq R$ is a problem that remains open, even in relatively simple cases. In this chapter, we consider instead the class of monomial ideals; that is, ideals minimally generated by monomials. Such ideals seem to exist at the intersection of commutative algebra and combinatorics, and are hence the subject of a large body of research.

In [38], Taylor constructed what is now called the Taylor resolution. This complex, aside from being a free resolution for any monomial ideal I, also possesses many other desirable properties. For instance, it always admits the structure of an associative differential graded (DG) algebra, and is cellular (see [5]). In general, however, this resolution is highly nonminimal.

Kaplansky posed the problem of describing the minimal free resolution of a monomial ideal in a polynomial ring. In general, this has turned out to be a difficult problem. A large class of ideals for which an explicit minimal free resolution can be constructed is for so-called Borel-fixed ideals. This resolution was constructed in [22] and is now called the Eliahou-Kervaire resolution. This resolution, similar to the Taylor resolution, admits the structure of a DG algebra (see [37]) and is cellular (see [33]). Likewise, a squarefree analogue of the Eliahou-Kervaire resolution is considered in [1], for which many of the properties of the standard Eliahou-Kervaire resolution remain valid.

Monomial ideals are a class of ideals for which combinatorial techniques have also proved very effective for the computation of such minimal free resolutions. One can reduce the study of arbitrary monomial ideals to the study of squarefree monomial ideals via polarization; once this reduction is made, there is a standard one-to-one correspondence between squarefree monomial ideals $I \subseteq k[x_1, \ldots, x_n]$ and simplicial complexes Δ on n vertices. This perspective was introduced in [35] and is used to deduce homological information of a monomial ideal I based on the combinatorial data of Δ. An excellent survey of this perspective, along with a collection of the literature on the topic, may be found in [27].

Even more recently, the problem of a general minimal free resolution for all monomial ideals has been attacked in [19]. This fascinating construction relies heavily on extensive combinatorial machinery; as a result of its generality, the complex itself is not simple to construct, but has the advantage of being described almost entirely in a combinatorial fashion.

In this chapter, we restrict ourselves to the case of equigenerated monomial ideals; that is, ideals generated in a single degree. A naïve method of obtaining such ideals is to start with the ideal generated by all monomials of degree d, $(x_1, \ldots, x_n)^d \subset k[x_1, \ldots, x_n]$, and then delete some of the generators. The graded minimal free resolution of $(x_1, \ldots, x_n)^d$ is well known (see Proposition 5.1.3), and so one would only need machinery for which the Betti numbers after deleting generators could be deduced. This machinery is provided by iterated trimming complexes.

This chapter is organized as follows. Section 5.1 introduces necessary background, conventions, and definitions. We then introduce background of Schur and Specht modules and two standard resolutions for both powers of complete intersections and the ideal generated by all squarefree monomials of a given degree in some polynomial ring. In Section 5.2 we build so-called q_i-maps for the aforementioned complexes to be used in the construction of trimming complexes.

In Sections 5.3 and 5.4, we compute explicit Betti tables for certain classes of equigenerated monomial ideals. In particular, we produce a large class of (squarefree) equigenerated monomial ideals with linear resolution. The definition of these ideals is phrased in terms of its so-called *complementary ideal* (see Definition 5.3.1. More precisely, we impose the condition that there are *no* linear syzygies on the complementary ideal; in this case, certain maps associated to the complexes introduced in Section 5.1 become much simpler.

Finally, in Section 5.5, we use a result of Miller and Rahmati (see [34]) about splitting mapping cones to compute explicit minimal free resolutions for the ideals of Section 5.4 (see Theorem 5.5.8). In the case where these complexes are linear, the minimal free resolution is even simpler to describe: it is constructed as the kernel of a certain morphism of complexes (see Corollary 5.5.9 and Theorem 5.5.12).

5.1 L-Complexes And Resolutions of Squarefree Monomials

The material up until Proposition 5.1.3, along with proofs, can be found in [10] or Section 2 of [21]. This first setup will be needed for the construction of the L-complexes of Buchsbaum and Eisenbud.

Setup 5.1.1. *Let F denote a free R-module of rank n, and $S = S(F)$ the symmetric algebra on F with the standard grading. Define a complex*

$$\cdots \longrightarrow \bigwedge^{a+1} F \otimes_R S_{b-1} \xrightarrow{\kappa_{a+1,b-1}} \bigwedge^a F \otimes_R S_b \xrightarrow{\kappa_{a,b}} \cdots$$

where the maps $\kappa_{a,b}$ are defined as the composition

$$\begin{aligned} \bigwedge^a F \otimes_R S_b &\to \bigwedge^{a-1} F \otimes_R F \otimes_R S_b \\ &\to \bigwedge^{a-1} F \otimes_R S_{b+1} \end{aligned}$$

where the first map is comultiplication in the exterior algebra and the second map is the standard module action (where we identify $F = S_1(F)$). Define

$$L_b^a(F) := \operatorname{Ker} \kappa_{a,b}.$$

Let $\psi : F \to R$ be a morphism of R-modules with $\operatorname{im}(\psi)$ an ideal of grade n. Let $Kos^{\psi} : \bigwedge^i F \to \bigwedge^{i-1} F$ denote the standard Koszul differential; that is, the composition

$$\begin{aligned} \bigwedge^{i} F &\to F \otimes_R \bigwedge^{i-1} F \quad \textit{(comultiplication)} \\ &\to \bigwedge^{i-1} F \quad \textit{(module action)} \end{aligned}$$

Definition 5.1.2. Adopt notation and hypotheses of Setup 5.1.1. Define the complex

$$L(\psi, b): \; 0 \longrightarrow L_b^{n-1} \xrightarrow{\operatorname{Kos}^{\psi} \otimes 1} \cdots \xrightarrow{\operatorname{Kos}^{\psi} \otimes 1} L_b^0 \xrightarrow{S_b(\psi)} R \longrightarrow 0$$

where $\operatorname{Kos}^{\psi} \otimes 1 : L_b^a(F) \to L_b^{a-1}$ is induced by making the following diagram commute:

$$\begin{tikzcd} \bigwedge^a F \otimes S_b(F) \arrow[r, "\operatorname{Kos}^{\psi}\otimes 1"] & \bigwedge^{a-1} F \otimes S_b(F) \\ L_b^a(F) \arrow[u] \arrow[r, "\operatorname{Kos}^{\psi}\otimes 1"] & L_b^{a-1}(F) \arrow[u] \end{tikzcd}$$

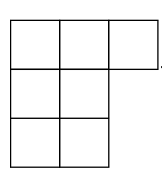
Proposition 5.1.3. *Let $\psi : F \to R$ be a map from a free module F of rank n such that the image $\operatorname{im}(\psi)$ is a grade n ideal. Then the complex $L(\psi, b)$ of Definition 5.1.2 is a minimal free resolution of $R/\operatorname{im}(\psi)^b$*

We also have (see Proposition 2.5(c) of [10])

$$\operatorname{rank}_R L_b^a(F) = \binom{n+b-1}{a+b}\binom{a+b-1}{a}.$$

Moreover, using the notation and language of Chapter 2 of [44], $L_b^a(F)$ is the Schur module $L_{(a+1,1^{b-1})}(F)$. This allows us to identify a standard basis for such modules.

Notation 5.1.4. *We use the English convention for partition diagrams. That is, the partition $(3, 2, 2)$ corresponds to the diagram*



A Young tableau is standard if it is strictly increasing in both the columns and rows. It is semistandard if it is strictly increasing in the columns and nondecreasing in the rows.

Proposition 5.1.5. *Adopt notation and hypotheses as in Setup 5.1.1. Then a basis for $L_b^a(F)$ is represented by all Young tableaux of the form*

$$\begin{array}{|c|c|c|c|}\hline i_0 & j_1 & \cdots & j_{b-1} \\ \hline i_1 & \multicolumn{3}{c}{} \\ \cline{1-1} \vdots & \multicolumn{3}{c}{} \\ \cline{1-1} i_a & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

with $i_0 < \cdots < i_a$ and $i_0 \leqslant j_1 \leqslant \cdots \leqslant j_{b-1}$.

Proof. See Proposition 2.1.4 of [44] for a more general statement. □

Remark 5.1.6. Adopt notation and hypotheses of Setup 5.1.1. Let F have basis $f_1, \ldots, f_n$. In the statement of Proposition 5.1.5, we think of the tableau as representing the element

$$\kappa_{a+1,b-1}(f_{i_1} \wedge \cdots \wedge f_{i_{a+1}} \otimes f_{j_1} \cdots f_{j_{b-1}}) \in \bigwedge^a F \otimes S_b(F).$$

We will often write $f_{i_1} \wedge \cdots \wedge f_{i_{a+1}} \otimes f_{j_1} \cdots f_{j_{b-1}} \in L_b^a(F)$, with the understanding that we are identifying $L_b^a(F)$ with the cokernel of $\kappa_{a+2,b-2} : \bigwedge^{a+2} F \otimes S_{b-2}(F) \to \bigwedge^{a+1} F \otimes S_{b-1}(F)$.

Next, we give a brief introduction of Specht modules and define the complex constructed by Galetto in [25]. The construction of Specht modules used here may be considered the dual construction, as in 7.4 of [24]. Instead of the more standard presentation using row tabloids, the Specht modules here are constructed as the quotient of all column tabloids by the so-called straightening relations.

Definition 5.1.7. Let λ be a partition and k a field. A column tabloid $[T]$ is an equivalence class of a tableau T modulo alternating columns.

Let M^λ denote the formal span of all column tabloids of shape λ. Define the map $\pi_{j,k} : M^\lambda \to M^\lambda$ by sending $[T] \mapsto \sum[S]$, where the sum is over all tableau S obtained from T by exchanging the top k elements of the $(j+1)$st column with the k elements in the jth column of T, while preserving the vertical order of each set of k elements.

Let $\mu = \lambda^t$ denote the transpose partition. Then the maps $\pi_{j,k}$ are defined for $1 \leqslant j \leqslant \lambda_1 - 1$, $1 \leqslant k \leqslant \mu_{j+1}$. Define the submodule $Q^\lambda \subset M^\lambda$ to be the subspace spanned by all elements of the form

$$[T] - \pi_{j,k}([T]),$$

where $j,\ k$ vary as above.

Then, with notation as above, define the Specht module S^λ to be the quotient M^λ / Q^λ.

Definition 5.1.8. Let $d \leqslant n$ be integers. Define

$$\begin{aligned} U_i^{d,n} &= \operatorname{Ind}_{S_{d+i}\times S_{n-d-i}}^{S_n}\left(S^{(d,1^i)} \otimes S^{(n-d-i)}\right) \\ &= \bigoplus_\sigma \sigma\left(S^{(d,1^i)} \otimes S^{(n-d-i)}\right) \end{aligned}$$

where the direct sum is taken over all coset representatives for $S_{d+i} \times S_{n-d-i}$.

Definition 5.1.9. Let $R = k[x_1, \ldots, x_n]$ where k is a field. Let $1 \leqslant d \leqslant n$ and $1 \leqslant i \leqslant n - d + 1$. Define

$$F_i^{d,n} := U_{i-1}^{d,n} \otimes_k R(-d - i + 1),$$

where $U_i^{d,n}$ is as in Definition 5.1.8. Given any Tableau T, define the differential

$$\partial_i^{d,n}([T]) := \sum_{j=0}^{i} (-1)^{i-j} x_{a_j} [T \backslash a_j],$$

where

$$T := \begin{array}{|c|c|c|c|} \hline a_1 & b_1 & \cdots & b_{d-1} \\ \hline a_2 \\ \cline{1-1} \vdots \\ \cline{1-1} a_i \\ \cline{1-1} \end{array},$$

and $i > 1$. When $i = 1$, define

$$\partial_1^{d,n}\left(\begin{array}{|c|c|c|c|}\hline a_1 & b_1 & \cdots & b_{d-1} \\ \hline \end{array}\right) = x_{a_1} x_{b_1} \cdots x_{b_{d-1}}.$$

Let $F_\bullet^{d,n}$ denote the complex

$$0 \longrightarrow F_{n-d+1}^{d,n} \xrightarrow{\partial_{n-d+1}^{d,n}} \cdots \xrightarrow{\partial_2^{d,n}} F_1^{d,n} \xrightarrow{\partial_1^{d,n}} R\,.$$

Theorem 5.1.10 ([25], Theorem 4.11)**.** *Let n and d be integers with $1 \leqslant d \leqslant n$. Then the complex $F_\bullet^{d,n}$ of Definition 5.1.9 is a S_n-equivariant minimal free resolution of quotient ring defined by the ideal generated by all squarefree monomials of degree d in R.*

Notation 5.1.11. *Adopt notation as in Definition 5.1.9. To the tabloid $[T]$ we will associate a formal basis element*

$$[T] \leftrightarrow f_{a_1} \wedge f_{a_2} \wedge \cdots \wedge f_{a_i} \otimes f_{b_1} \cdot f_{b_2} \cdots f_{b_{d-1}},$$

where the notation is meant to mimic the notation used for the modules L_d^i. This should cause no confusion, since the straightening relations/tabloid properties are directly compatible with the straightening relations for L_d^i and the exterior/symmetric algebra relations.

5.2 q_i Maps for Certain Schur and Specht Modules

In this section, we construct the maps of Proposition 2.2.3 in the case where the relevant modules are Schur and Specht modules, and they are being mapped to a Koszul complex. These maps are essential for the rest of the chapter, as they are the building blocks employed for the iterated trimming complex construction. At the end of this section, we also take the opportunity to compute certain colon ideals; these colons will be used in later sections in order to count rank and deduce higher strands appearing in the minimal free resolutions of the ideals of interest.

Notation 5.2.1. *Let R be a commutative ring. Let F be a free R-module of rank n with basis $f_1, \ldots, f_n$ and let ℓ, b be integers. Fix indexing sets $J = (j_1, \ldots, j_\ell)$ with $j_1 < \cdots < j_\ell$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \geqslant 0$ for each $i = 1, \ldots, n$, $|\alpha| = b$.*

The notation f_J denotes $f_{j_1} \wedge \cdots \wedge f_{j_\ell} \in \bigwedge^\ell F$, the notation f^J denotes $f_{j_1} \cdots f_{j_\ell} \in S_\ell(F)$, and the notation f^α denotes $f_1^{\alpha_1} \cdots f_n^{\alpha_n} \in S_b(F)$.

Definition 5.2.2. Let R be a commutative ring. Let F be a free R-module of rank n with basis $f_1, \ldots, f_n$ and let ℓ, b be integers. Fix indexing sets $J = (j_1, \ldots, j_\ell)$ with $j_1 < \cdots < j_\ell$ and $\alpha = (\alpha_1, \ldots, \alpha_n)$ with $\alpha_i \geqslant 0$ for each $i = 1, \ldots, n$, $|\alpha| = b$. Define the maps $\phi_i^{J,\alpha} : \bigwedge^i F \otimes S_b(F) \to \bigwedge^i F$ via

$$\phi_i^{J,\alpha}(f_I \otimes f^\beta) = \begin{cases} f_I & \text{if } I \subseteq J \text{ and } \beta = \alpha \\ 0 & \text{otherwise} \end{cases}$$

Observation 5.2.3. Adopt notation and hypotheses as in Definition 5.2.2. Let $\psi : F \to R$ be a homomorphism of R-modules, and $\operatorname{Kos}^\psi : \bigwedge^i F \to \bigwedge^{i-1} F$ the induced Koszul differential. Then the following diagram commutes:

$$\begin{tikzcd} \bigwedge^i F \otimes S_b(F) \arrow[r, "\operatorname{Kos}^\psi \otimes 1"] \arrow[d, "\phi_i^{J,\alpha}"] & \bigwedge^{i-1} F \otimes S_b(F) \arrow[d, "\phi_{i-1}^{J,\alpha}"] \\ \bigwedge^i F \arrow[r, "\operatorname{Kos}^\psi"] & \bigwedge^{i-1} F. \end{tikzcd}$$

Moreover, for all $i \geqslant 1$, $\phi_i^{J,\alpha}$ induces the commutative diagram

$$\begin{tikzcd} L_b^i(F) \arrow[r, "\operatorname{Kos}^\psi \otimes 1"] \arrow[d, "\phi_i^{J,\alpha}"] & L_b^{i-1}(F) \arrow[d, "\phi_{i-1}^{J,\alpha}"] \\ \bigwedge^i F \arrow[r, "\operatorname{Kos}^\psi"] & \bigwedge^{i-1} F, \end{tikzcd}$$

where $L_b^i(F)$ is as in Setup 5.1.1. More precisely, this map is realized as:

$$\phi_i^{J,\alpha}(\kappa_{i+1,b-1}(f_I \otimes f^\beta)) = \begin{cases} \operatorname{sgn}(i) f_{I \setminus i} & \text{if } i \in I,\ \beta + \epsilon_i = \alpha \\ 0 & \text{otherwise} \end{cases}$$

Definition 5.2.4. Let R be a commutative ring. Let F be a free R-module of rank m with basis $f_1, \ldots, f_m$ and let ℓ, d be integers. Fix indexing sets $J = (j_1, \ldots, j_\ell)$ with $j_1 < \cdots < j_\ell$ and $I = (i_1, \ldots, i_d)$ with $i_1 < \cdots < i_d$. Let $\psi : F \to R$ be an R-module homomorphism and

$$T := \begin{array}{|c|c|c|c|}\hline a_0 & b_1 & \cdots & b_{d-1} \\ \hline a_1 & \multicolumn{3}{c}{} \\ \cline{1-1} \vdots & \multicolumn{3}{c}{} \\ \cline{1-1} a_\ell & \multicolumn{3}{c}{} \\ \cline{1-1} \end{array}$$

a standard tableau with $a_0 < \cdots < a_\ell$ and $b_1 < \cdots < b_{d-1}$. Define maps

$$\psi_\ell^{J,I} : S^{(d,1^\ell)} \to \bigwedge^\ell F$$

on the equivalence class of the column tabloid $[T] \in S^{(d,1^\ell)}$ by setting

$$\psi_\ell^{J,I}\Big([T]\Big) := \begin{cases} \operatorname{sgn}(a_i) f_{\{a_0,\ldots,\widehat{a_i},\ldots,a_\ell\}} & \text{if } I = \{b_1, \ldots, b_{d-1}\} \cup \{a_i\} \text{ for some } 0 \leqslant i \leqslant \ell \\ & \text{and } \{a_0, \ldots, \widehat{a_i}, \ldots, a_\ell\} \subseteq J, \\ 0 & \text{otherwise.} \end{cases}$$

Observe that this is well defined since the above definition is compatible with the shuffling relations on $S^{(d,1^\ell)}$. Moreover, extending by linearity, this induces a map

$$\phi_\ell^{J,I} : F_\ell^{d,n} \to \bigwedge^\ell U$$

making the following diagram commute:

$$\begin{tikzcd} F_\ell^{d,n} \arrow[r, "\partial_\ell^{d,n}"] \arrow[d, "\phi_\ell^{J,\alpha}"] & F_{\ell-1}^{d,n} \arrow[d, "\phi_{\ell-1}^{J,\alpha}"] \\ \bigwedge^\ell F \arrow[r, "\mathrm{Kos}^\psi"] & \bigwedge^{\ell-1} F, \end{tikzcd}$$

where $F_\ell^{d,n}$ and $\partial_\ell^{d,n}$ are as in Definition 5.1.9 and Kos^ψ denotes the induced Koszul differential.

Proposition 5.2.5. *Adopt notation and hypotheses as in Setup 2.2.1, and assume that* $d_0^i(F_2) = \mathfrak{a}_i e_0^i$*. Then the ideals* $\mathfrak{a}_i \subseteq R$ *do not depend on the choice of differential* d_2*.*

Proof. Assume for simplicity that $m = 1$. Then we will prove a slightly stronger statement; namely, $\mathfrak{a}_1 = (K' : K_0^1)$. The containment $\mathfrak{a}_1 \subseteq (K' : K_0^1)$ is trivial, so let $r \in (K' : K_0^1)$. Assume rank $F_1' = f'$ and let $e_1, \ldots, e_{f'}$ denote a basis for F_1'.

By definition, there exist elements $r_i \in R$ such that

$$r_1 d_1(e_1) + \cdots + r_{f'} d_1(e_{f'}) = r d_1(e_0^1),$$

$$\implies d_1(r_1 e_1 + \cdots + r_{f'} e_{f'} - r e_0^1) = 0.$$

However, by the assumption on $\mathfrak{a}_1$, this implies $r \in \mathfrak{a}_1$ as desired. $\square$

Notation 5.2.6. *Let* $R = k[x_1, \ldots, x_n]$*, where* k *is any field. If* $\alpha = (\alpha_1, \ldots, \alpha_n)$*, then the notation* x^α *denotes* $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$*. Given such an* α*, define* $|\alpha| := \alpha_1 + \cdots + \alpha_n$*. If* $J = \{j_1 < \cdots < j_n\}$*, then the notation* x^J *will denote* $x_{j_1} \cdots x_{j_n}$*. Given such a* J*, define* $|J| = n$*, the cardinality of* J*.*

The notation ϵ_i *will denote the vector with a* 1 *in the* i*th entry and* 0*'s elsewhere.*

The following Propositions are immediate.

Proposition 5.2.7. *Let* $R = k[x_1, \ldots, x_n]$ *where* k *is any field and let* $\alpha = (\alpha_1, \ldots, \alpha_n)$ *be an exponent vector with* $|\alpha| = d$*. If* $K' := (x^\beta \mid |\beta| = d,\ \beta \neq \alpha)$*, then*

$$(K' : x^\alpha) = \begin{cases} (x_1, \ldots, \widehat{x_i}, \ldots, x_n) & \text{if } \alpha = d\epsilon_i \text{ for some } 1 \leqslant i \leqslant n \\ (x_1, \ldots, x_n) & \text{otherwise.} \end{cases}$$

Proposition 5.2.8. *Let* $R = k[x_1, \ldots, x_n]$ *where* k *is any field and let* $J = \{j_1 < \cdots < j_d\}$*. If* $K' := (x^I \mid |I| = d,\ I \neq J)$*, then*

$$(K' : x^J) = (x_i \mid i \notin J).$$

5.3 q_i Maps for the Complexes $L(\psi, b)$ and $F_\bullet^{n,m}$

We can now use the maps constructed in Section 5.2 to find the Betti tables for resolving certain subsets of the standard generating sets for powers of the maximal ideal and all squarefree monomials of a given degree. Our first goal is to compute the ranks of the maps $\phi_\ell^{J,\alpha}$ of Definition 5.2.2 and $\psi_\ell^{J,I}$ of Definition 5.1.7. We begin with some definitions and notation related to monomial ideals which will be in play for the rest of the chapter.

Definition 5.3.1. Let $R = k[x_1, \dots, x_n]$ be a standard graded polynomial ring over a field k. Let K denote an equigenerated monomial ideal with generators in degree d. Define

$$G(K) := \text{unique minimal generating set of } K \text{ consisting of monic monomials.}$$

Given a monomial ideal K, define the (squarefree) complementary ideal $\overline{K}$ to be the ideal with minimal generating set:

$$G(\overline{K}) = \begin{cases} \{\text{degree } d \text{ squarefree monomials}\} \backslash G(K) & \text{if } K \text{ squarefree,} \\ \{\text{degree } d \text{ monomials}\} \backslash G(K) & \text{otherwise.} \end{cases}$$

The following setup will be used for constructing the Betti table/minimal free resolution when the monomial ideals of interest are not squarefree.

Setup 5.3.2. *Let $R = k[x_1, \dots, x_n]$ where k is a field and let $F = \bigoplus_{i=1}^n Re_i$ be a free module of rank n with map $\psi : F \to R$ sending $e_i \mapsto x_i$. Let $d \geqslant 1$ denote any integer and $L(\psi, d)$ the complex of Definition 5.1.2. Fix an exponent vector $\alpha = (\alpha_1, \dots, \alpha_n)$ with $|\alpha| = d$. Let*

$$U = \begin{cases} \bigoplus_{j \neq i} Re_j & \text{if } \alpha = d\epsilon_i \\ F & \text{otherwise,} \end{cases}$$

with map $\psi : U \to R$ defined by sending $e_j \mapsto x_j$.

Let $\phi_\ell^{I,\alpha} : L_d^\ell(F) \to \bigwedge^\ell U$ for $1 \leqslant \ell \leqslant n$ be the maps of Definition 5.2.2, where

$$I = \begin{cases} [n]\backslash\{i\} & \text{if } \alpha = d\epsilon_i \\ [n] & \text{otherwise.} \end{cases}$$

The following notation will be convenient in many of the ensuing computations:

Notation 5.3.3. *Adopt notation and hypotheses of Setup 5.3.2. Let* $\operatorname{Supp}(\alpha) = \{i \mid \alpha_i > 0\}$ *and define*

$$n_\alpha := |\operatorname{Supp}(\alpha)|$$

Proposition 5.3.4. *Adopt notation and hypotheses of Setup 5.3.2 with $\alpha = d\epsilon_i$ for some $1 \leqslant i \leqslant n$. The maps $\phi_\ell^{I,\alpha} : L_d^\ell(F) \to \bigwedge^\ell U$ are surjective for all $1 \leqslant \ell \leqslant n-1$. In particular,*

$$\operatorname{rank}(\phi_\ell^{I,\alpha} \otimes k) = \binom{n-1}{\ell}$$

Proof. Let $J \subset I$ with $J = (j_1, \dots, j_\ell)$. It suffices to show that e_J is in the image of $\phi_\ell^{I,d\epsilon_i}$ for any choice of J. Order the set $J \cup \{i\}$ so that

$$j_1 < \cdots < j_k < i < j_{k+1} < \cdots < j_\ell.$$

This is possible since $i \notin J$ by construction of the free module U. Then, by definition,

$$\phi_\ell^{I,d\epsilon_i}(e_{J\cup\{i\}} \otimes e_i^{d-1}) = \operatorname{sgn}(i) e_J.$$

□

Corollary 5.3.5. *Adopt notation and hypotheses of Setup 5.3.2 with $\alpha = d\epsilon_i$ for some $1 \leqslant i \leqslant n$. Let K' be an equigenerated momomial ideal with $\overline{K'} = (x_i^d)$. Then, R/K' has Betti table*

	0	1	$\cdots$	ℓ	$\cdots$	n
0	1	.	$\cdots$	.	$\cdots$	.
$\vdots$	.	.	$\cdots$	.	$\cdots$	.
$d-1$	.	$\binom{n+d-1}{d} - 1$	$\cdots$	$\binom{n+d-1}{\ell+d}\binom{d+\ell-2}{\ell-1} - \binom{n-1}{\ell-1}$	$\cdots$	$\binom{n+d-2}{n-1} - 1$

In particular, R/K' has projective dimension n with linear resolution and defines a ring of type $\binom{n+d-2}{n-1} - 1$.

Proposition 5.3.6. *Adopt notation and hypotheses of Setup 5.3.2. Then the maps $\phi_\ell^{I,\alpha} : L_d^\ell(F) \to \bigwedge^\ell U$ are such that*

$$\operatorname{rank}(\phi_\ell^{I,\alpha} \otimes k) = \binom{n}{\ell} - \binom{n-n_\alpha}{\ell - n_\alpha},$$

for all $1 \leqslant \ell \leqslant n$.

Proof. We shall enumerate a subset of bases whose images under $\phi_\ell^{I,\alpha}$ form a linearly independent set, then show that the image of any other standard basis element lies in the image spanned by this set. Counting the size of this set will then yield the rank.

To this end, enumerate the set $\{i \mid \alpha_i > 0\} = \{k_1, \dots, k_{n_\alpha}\}$, where $k_1 < \cdots < k_{n_\alpha}$. Consider the set S consisting of all standard basis elements of the form

$$e_{\{k_1,\dots,k_s\} \cup J'} \otimes e^{\alpha - \epsilon_{k_s}},$$

in $L_\ell^d(F)$ with $s \leqslant n_\alpha$, $|J'| = \ell - s + 1$. By definition,

$$\phi_\ell^{I,\alpha}(e_{\{k_1,\dots,k_s\} \cup J'} \otimes e^{\alpha - \epsilon_{k_s}}) = \begin{cases} \operatorname{sgn}(k_s) e_{J'} & \text{if } s = 1 \\ \operatorname{sgn}(k_s) e_{\{k_1,\dots,k_{s-1}\} \cup J'} & \text{otherwise.} \end{cases}$$

The collection of all basis elements as above, where $1 \leqslant s \leqslant n_\alpha$, is evidently a linearly independent set since it is an irredundant subset of a basis for $\bigwedge^\ell U$.

Let $1 \leqslant r \leqslant n_\alpha$ and consider any standard basis element of the form $e_{\{k_r\} \cup J'} \otimes e^{\alpha - \epsilon_{k_r}}$. Let $t := \min\{s \mid k_s \notin J\}$. Assume first that $t > 1$; by definition of t, $\{k_1, \dots, k_{t-1}\} \subseteq J'$, so we may write $J' = \{k_1, \dots, k_{t-1}\} \cup J''$ for some J''. Then,

$$\phi_\ell^{I,\alpha}(\operatorname{sgn}(k_r) e_{\{k_r\} \cup J'} \otimes e^{\alpha - \epsilon_{k_r}}) = \phi_\ell^{I,\alpha}(-\operatorname{sgn}(k_t) e_{\{k_1,\dots,k_t\} \cup J''} \otimes e^{\alpha - \epsilon_{k_t}}),$$

and the element on the right is the image of an element of S. Likewise, if $t = 1$, then

$$\phi_\ell^{I,\alpha}(\operatorname{sgn}(k_r) e_{\{k_r\} \cup J'} \otimes e^{\alpha - \epsilon_{k_r}}) = \phi_\ell^{I,\alpha}(-\operatorname{sgn}(k_1) e_{\{k_1\} \cup J'} \otimes e^{\alpha - \epsilon_{k_1}}),$$

and again the element on the right is the image of an element of S. Thus, counting the cardinality of S, we see that this is counting all possible indexing sets J' with $|J'| = \ell - s + 1$ and $J' \cap \{k_1, \ldots, k_s\} = \varnothing$, for $1 \leqslant s \leqslant n_\alpha$. It is a trivial counting exercise to see

$$|S| = \sum_{i=1}^{n_\alpha} \binom{n-i}{\ell - i + 1} = \sum_{i=1}^{n_\alpha} \binom{n-i}{n-\ell-1},$$

and one can moreover check that

$$\sum_{i=1}^{n_\alpha} \binom{n-i}{n-\ell-1} = \binom{n}{\ell} - \binom{n-n_\alpha}{\ell - n_\alpha}.$$

□

The following is an immediate result of Proposition 5.3.6 combined with Corollary 2.2.6.

Corollary 5.3.7. *Adopt notation and hypotheses as in Setup 5.3.2. Let K' be an equigenerated momomial ideal with $\overline{K'} = (x^\alpha)$. Then, R/K' has Betti table*

	0	1	$\cdots$	ℓ	$\cdots$	n
0	1	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$d-1$	$\cdot$	$\binom{n+d-1}{d} - 1$	$\cdots$	$\binom{n+d-1}{\ell+d}\binom{d+\ell-2}{\ell-1} - \binom{n}{\ell-1} + \binom{n-n_\alpha}{\ell-1-n_\alpha}$	$\cdots$	$\binom{n+d-2}{n-1} - n_\alpha$
d	$\cdot$	$\cdot$	$\cdots$	$\binom{n-n_\alpha}{\ell-n_\alpha}$	$\cdots$	1

The following result in the case of an Artinian ideal is a statement about the non-cyclicity of the associated inverse system; this behavior is highly dependent on the chosen generating set. For instance, choosing instead the generating set to be the maximal minors of the associated Sylvester matrix for $(x_1, \ldots, x_n)^2$, it is not hard to see that removing the generator $x_1 x_n$ will yield a grade n Gorenstein ideal for all $n \geqslant 2$.

Corollary 5.3.8. *Adopt notation and hypotheses of Setup 5.3.2. Let K' be an equigenerated momomial ideal generated in degree $d \geqslant 2$ with $\overline{K'} = (x^\alpha)$. Then, R/K' is Gorenstein if and only if $n = d = 2$, in which case $K' = (x_1^2, x_2^2)$.*

Proof. By Gorenstein duality, it is immediate that if K' is Gorenstein, then $d = 2$. This implies that for any choice of α, $n_\alpha \leqslant 2$. Moreover, using the Betti table of Corollary 5.3.7, K' defines a ring of type $n - n_\alpha + 1 \geqslant n - 1$, whence $n = 2$. $\square$

Next, we adopt the following setup. This setup is the squarefree analog of Setup 5.3.2, and will instead be used to compute the Betti table/minimal free resolution when the ideals of interest are squarefree.

Setup 5.3.9. *Let $R = k[x_1, \dots, x_n]$ where k is a field and let $F_\bullet^{d,n}$ denote the complex of Definition 5.1.9. Fix an indexing set $I = (i_1, \dots, i_d)$ and let $U = \bigoplus_{j \notin I} Re_j$ with map $\psi : U \to R$ defined by sending $e_j \mapsto x_j$.*

Let $\psi_\ell^{I,I^c} : F_\ell^{d,n} \to \bigwedge^\ell U$ for $1 \leqslant \ell \leqslant n - d$ be the maps of Definition 5.2.4, where $I^c = [n] \backslash I$.

Proposition 5.3.10. *Adopt notation and hypotheses as in Setup 5.3.9. The maps $\psi_\ell^{I,I^c} : F_\ell^{d,n} \to \bigwedge^\ell U$ are surjective for all $1 \leqslant \ell \leqslant n - d$. In particular,*

$$\operatorname{rank}(\psi_\ell^{I,I^c} \otimes k) = \binom{n-d}{\ell}$$

Proof. Let $J \subset I^c$ be any indexing set with $J = (j_1, \dots, j_\ell)$. It suffices to show that the basis element $e_J \in \bigwedge^\ell U$ is in the image of ψ_ℓ^{I,I^c}.

Order the set $J \cup \{i_1\}$, so that

$$j_1 < \cdots < j_k < i_1 < j_{k+1} < \cdots < j_\ell$$

for some $k < \ell$. Then, observe that the hook tableau with $J \cup \{i_1\}$ ordered appropriately in the first column and $(i_2, \dots, i_d)$ along the first row has image $\operatorname{sgn}(i_1)e_J$. $\square$

Corollary 5.3.11. *Adopt notation and hypotheses as in Setup 5.3.9. Let K' be a squarefree equigenerated momomial ideal with $\overline{K'} = (x^I)$. Then, R/K'*

	0	1	$\cdots$	ℓ	$\cdots$	$n-d+1$
0	1	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$d-1$	$\cdot$	$\binom{n}{d}-1$	$\cdots$	$\binom{d+\ell-2}{\ell-1}\binom{n}{d+\ell-1}-\binom{n-d}{\ell-1}$	$\cdots$	$\binom{n-1}{n-d}-1$

In particular, R/K' has projective dimension $n-d+1$ with linear resolution and defines a ring of type $\binom{n-1}{n-d}-1$.

5.4 Betti Tables for Certain Classes of Equigenerated Monomial Ideals

This section is an iterated version of Section 5.3; that is, we consider the *iterated* trimming complex associated to the maps constructed in the previous section. It turns out that under sufficient hypotheses, these maps stay well-behvaed when removing multiple generators at a time. The following setup is similar to Setup 5.3.2, but with more data to keep track of:

Setup 5.4.1. *Let $R = k[x_1, \ldots, x_n]$ where k is a field and let $F = \bigoplus_{i=1}^n Re_i$ be a free module of rank n with map $\psi : F \to R$ sending $e_i \mapsto x_i$. Let $d \geqslant 1$ denote any integer and $L(\psi, d)$ the complex of Definition 5.1.2. Fix exponent vectors $\alpha^s = (\alpha_1^s, \ldots, \alpha_n^s)$ with $|\alpha^s| = d$ for $1 \leqslant s \leqslant r$. Assume that for all $s \neq t$, $\deg lcm(x^{\alpha^s}, x^{\alpha^t}) \geqslant d+2$. Let*

$$U_s = \begin{cases} \bigoplus_{j\neq i} Re_j & \text{if } \alpha^s = d\epsilon_i \\ F & \text{otherwise,} \end{cases}$$

with map $\psi : U_s \to R$ induced by sending $e_j \mapsto x_j$.

Let $\phi_\ell^{I_s,\alpha^s} : L_d^\ell(F) \to \bigwedge^\ell U$ for $1 \leqslant \ell \leqslant n$ be the maps of Definition 5.2.2, where

$$I_s = \begin{cases} [n]\backslash\{i\} & \text{if } \alpha^s = d\epsilon_i \\ [n] & \text{otherwise.} \end{cases}$$

Observation 5.4.2. Adopt notation and hypotheses as in Setup 5.4.1. Let K' be an equigenerated momomial ideal with $\overline{K'} = (x^{\alpha^1}, \dots, x^{\alpha^r})$ and let

$$\mathfrak{a}_s := \begin{cases} (x_1, \dots, \widehat{x_i}, \dots, x_n) & \text{if } \alpha^s = d\epsilon_i \\ (x_1, \dots, x_n) & \text{otherwise} \end{cases}.$$

Then $\mathfrak{a}_s x^{\alpha^s} \subseteq K'$ for all $1 \leqslant s \leqslant r$.

Proof. Suppose for sake of contradiction that the containment $\mathfrak{a}_t x^{\alpha^t} \not\subseteq K'$ for some $1 \leqslant t \leqslant r$. Let

$$K_t := (x^\beta \mid |\beta| = d,\ \beta \neq \alpha^t\},$$

and observe that $(K_t : x^{\alpha^t}) = \mathfrak{a}_t$ by Proposition 5.2.7. This means that for some $s \neq t$, $x_i \cdot x^{\alpha^s} = x_j \cdot x^{\alpha^t}$, contradicting the LCM hypothesis on each α^s. □

Remark 5.4.3. In the notation of the statement of Observation 5.4.2, this is saying that the construction of Theorem 2.2.4 applied to the ideals $\mathfrak{a}_s$, for $1 \leqslant s \leqslant r$, yields a resolution of R/K'.

The following Proposition makes precise the previously mentioned fact that the maps of Definition 5.2.2 are "well-behaved" when removing multiple generators.

Proposition 5.4.4. *Adopt notation and hypotheses as in Setup 5.4.1. Enumerate the set* $\operatorname{Supp}(\alpha) = \{k_1^s, \dots, k_{n_{\alpha^s}}^s\}$ *with* $k_1^s < \dots < k_{n_{\alpha^s}}^s$*. Then for all* $t \neq s$ *and* $p \leqslant n_{\alpha^s}$*,*

$$\phi^{I^t, \alpha^t}(e_{\{k_1^s, \dots, k_p^s\} \cup J'} \otimes e^{\alpha^s - \epsilon_{k_p^s}}) = 0.$$

Proof. Suppose for sake of contradiction that there exists some $t \neq s$ and $1 \leqslant p \leqslant n_{\alpha^s}$ such that

$$\phi^{I^t, \alpha^t}(e_{\{k_1^s, \dots, k_p^s\} \cup J'} \otimes e^{\alpha^s - \epsilon_{k_p^s}}) \neq 0.$$

This is possible if and only if there exists $q \in \{k_1^s, \dots, k_p^s\} \cup J'$ such that $\alpha^t = \alpha^s - \epsilon_{k_p^s} + \epsilon_q$. This implies that $\alpha^t - \alpha^s = \epsilon_q - \epsilon_{k_p^s}$, which is a clear contradiction to the LCM hypothesis on each α^s. □

Corollary 5.4.5. *Adopt notation and hypotheses as in Setup 5.4.1. Then,*

$$\operatorname{rank}\left(\begin{pmatrix} \phi_\ell^{I_1,\alpha^1} \\ \phi_\ell^{I_2,\alpha^2} \\ \vdots \\ \phi_\ell^{I_r,\alpha^r} \end{pmatrix} \otimes k\right) = \sum_{s=1}^r \operatorname{rank}(\phi_\ell^{I_s,\alpha^s} \otimes k).$$

Corollary 5.4.6. *Adopt notation and hypotheses as in Setup 5.4.1. Define $rk_\ell := \sum_{s=1}^r \operatorname{rank}(\phi_\ell^{I_s,\alpha^s} \otimes k)$. Let K' be an equigenerated momomial ideal with $\overline{K'} = (x^{\alpha^1}, \ldots, x^{\alpha^r})$. Then R/K' has Betti table*

	0	1	$\cdots$	ℓ	$\cdots$	n
0	1	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$d-1$	$\cdot$	$\binom{n+d-1}{d} - r$	$\cdots$	$\binom{n+d-1}{\ell+d}\binom{d+\ell-2}{\ell-1} - rk_{\ell-1}$	$\cdots$	$\binom{n+d-2}{n-1} - \sum_{s=1}^r n_{\alpha^s}$
d	$\cdot$	$\cdot$	$\cdots$	$\sum_{s=1}^r \operatorname{rank} \bigwedge^\ell U_s - rk_\ell$	$\cdots$	r

As a special case of the above, we can compute the Betti table of an equigenerated monomial ideal whose complementary ideal consists only of pure powers.

Corollary 5.4.7. *Adopt notation and hypotheses as in Setup 5.4.1 and let $B = \{k_1 < \cdots < k_r\}$. Let K' be an equigenerated momomial ideal with $\overline{K'} = (x_{k_1}^d, \ldots, x_{k_r}^d)$. Then R/K' has Betti table*

	0	1	$\cdots$	ℓ	$\cdots$	$n-d+1$
0	1	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$\cdots$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$d-1$	$\cdot$	$\binom{n+d-1}{d} - r$	$\cdots$	$\binom{n+d-1}{\ell+d}\binom{d+\ell-2}{\ell-1} - r\binom{n-1}{\ell-1}$	$\cdots$	$\binom{n+d-2}{n-1} - r$

In particular, R/K' has projective dimension n with linear resolution and defines a ring of type $\binom{n+d-2}{n-1} - r$.

The rest of this section is just the squarefree analog of the first half of this section. It turns out that the squarefree case is, in some sense, much simpler than the non-squarefree case. We will see that these ideals *always* have a linear minimal free resolution. We will first need to adopt the following setup, which the reader should take as the squarefree analog of Setup 5.4.1.

Setup 5.4.8. *Let $R = k[x_1, \dots, x_n]$ where k is a field and let $F_\bullet^{d,n}$ denote the complex of Definition 5.1.9. Fix indexing sets $I_j = (i_{j1}, \dots, i_{jd})$ for $1 \leqslant j \leqslant r$ with the property that $|I_j \cap I_i| \leqslant d-2$ for all $i \neq j$. Let $U_j = \bigoplus_{\ell \notin I_j} Re_\ell$ with map $\psi : U_j \to R$ defined by sending $e_\ell \mapsto x_\ell$.*

Let $\psi_\ell^{I_j, I_j^c} : F_\ell^{d,n} \to \bigwedge^\ell U_j$ for $1 \leqslant \ell \leqslant n-d$ be the maps of Definition 5.2.4, where $I_j^c = [n] \backslash I_j$.

Observe that the proof of the following is essentially identical to that of Observation 5.4.2, where we employ Proposition 5.2.8 instead.

Observation 5.4.9. Adopt notation and hypotheses as in Setup 5.4.8. Let K' be a squarefree equigenerated momomial ideal with $\overline{K'} = (x^{I_1}, \dots, x^{I_r})$ and let

$$\mathfrak{a}_s := (x_j \mid j \notin I_s) \quad (1 \leqslant s \leqslant r).$$

Then $\mathfrak{a}_s x^{I_s} \subseteq K'$ for all $1 \leqslant s \leqslant r$.

In a similar manner, the proof of the following Proposition is essentially identical to that of Proposition 5.4.4.

Proposition 5.4.10. *Adopt notation and hypotheses as in Setup 5.4.8. Then for all $t \neq s$ and $p \leqslant d$,*

$$\psi^{I_t, I_t^c}(e_{\{i_{1s}, \dots, i_{ps}\} \cup J'} \otimes e^{I_s - \epsilon_{i_{ps}}}) = 0.$$

Corollary 5.4.11. *Adopt notation and hypotheses as in Setup 5.4.8. Then,*

$$\operatorname{rank}\left(\begin{pmatrix} \psi_\ell^{I_1, I_1^c} \\ \psi_\ell^{I_2, I_2^c} \\ \vdots \\ \psi_\ell^{I_r, I_r^c} \end{pmatrix} \otimes k\right) = \sum_{s=1}^r \operatorname{rank}(\psi_\ell^{I_s, I_s^c} \otimes k).$$

Corollary 5.4.12. *Adopt notation and hypotheses as in Setup 5.4.8. Let K' be a squarefree equigenerated momomial ideal with $\overline{K'} = (x^{I_1}, \dots, x^{I_r})$. Then R/K' has Betti table*

	0	1	$\cdots$	ℓ	$\cdots$	$n-d+1$
0	1	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$\vdots$	$\cdot$	$\cdot$	$\cdots$	$\cdot$	$\cdots$	$\cdot$
$d-1$	$\cdot$	$\binom{n}{d}-r$	$\cdots$	$\binom{d+\ell-2}{\ell-1}\binom{n}{d+\ell-1}-r\binom{n-d}{\ell-1}$	$\cdots$	$\binom{n-1}{n-d}-r$

In particular, R/K' has projective dimension $n-d+1$ with linear resolution and defines a ring of type $\binom{n-1}{n-d}-r$.

5.5 Explicit Minimal Free Resolutions

In this section we produce the explicit minimal free resolutions of all of the ideals considered in Section 5.4. In particular, for the cases where the resolutions were linear, these resolutions may be obtained by simply taking the kernel of the morphisms of complexes constructed in the previous sections. The proofs of these results are based on the following more general theorem, which describes how to extract "minimal" summands of mapping cones of complexes when the associated morphism of complexes is split. This first result is a specialized version of a result by Miller and Rahmati (see [34, Proposition 2.1])

Theorem 5.5.1. *Consider the morphism of complexes*

$$\begin{tikzcd} \cdots \arrow[r, "d_{k+1}"] & F_k \arrow[r, "d_k"] \arrow[d, "q_k"] & \cdots \arrow[r, "d_2"] & F_1 \arrow[r, "d_1"] \arrow[d, "q_1"] & F_0 \arrow[d, "d_0"] \\ \cdots \arrow[r, "m_{k+1}"] & G_k \arrow[r, "m_k"] & \cdots \arrow[r, "m_2"] & G_1 \arrow[r, "m_1"] & R. \end{tikzcd} \tag{5.5.1}$$

For each $k > 0$, let

$$A_k := \operatorname{Ker} q_k, \ C_k := \operatorname{Coker} q_k, \ \textit{and } B_n := \operatorname{im} q_k,$$

and assume that the short exact sequences

$$0 \to A_k \to F_k \to B_k \to 0, \text{ and}$$

$$0 \to B_k \to G_k \to C_k \to 0$$

are split, with $C_1 = 0$. Then the mapping cone of 5.5.1 is the direct sum of a split exact complex and the following complex:

$$\cdots \to \begin{matrix} A_{k-1} \\ \oplus \\ C_k \end{matrix} \xrightarrow{\ell_k} \begin{matrix} A_{k-2} \\ \oplus \\ C_{k-1} \end{matrix} \xrightarrow{\ell_{k-1}} \cdots \xrightarrow{\ell_3} \begin{matrix} A_1 \\ \oplus \\ C_2 \end{matrix} \xrightarrow{\ell_2} F_0 \xrightarrow{d_0} R,$$

where

$$\ell_k := \begin{pmatrix} d_{k-1} & \Theta_k \\ 0 & -m_k \end{pmatrix}, \quad (k \geqslant 3),$$

$$\ell_2 := \begin{pmatrix} d_1 & \Theta_2 \end{pmatrix},$$

and $\Theta_k : C_k \to A_{k-2}$ is the composition

$$\begin{aligned} C_k &\xrightarrow{\text{inclusion}} G_k \\ &\xrightarrow{m_k} G_{k-1} \\ &\xrightarrow{\text{projection}} B_{k-1} \\ &\xrightarrow{\text{inclusion}} F_{k-1} \\ &\xrightarrow{d_{k-1}} F_{k-2} \\ &\xrightarrow{\text{projection}} A_{k-2} \end{aligned}$$

Remark 5.5.2. In the statement of Theorem 5.5.1, it is understood that the differentials d_k and m_k appearing in the matrix form of ℓ_k are the maps induced by restricting to the subcomplex/quotient complex $A_\bullet$ and $C_\bullet$, respectively.

In the next few definitions/results, we will be constructing the constituent building blocks of the minimal free resolution of the ideals of interest in Theorem 5.5.8.

Definition 5.5.3. Adopt notation and hypotheses as in Setup 5.4.1, and let $B = \{\alpha^1, \dots, \alpha^r\}$. For each s, write $\operatorname{Supp}(\alpha^s) = \{k_1^s < \cdots < k_{n_{\alpha^s}}^s\}$. For each $i > 0$, define the free submodule $L_d^{i,B}(F) \subseteq L_d^i(F)$ to be generated by the following collection of basis elements, denoted $\mathcal{S}$ (all terms appearing are assumed to be standard basis elements as in Remark 5.1.6):

$$\begin{cases} e_J \otimes e^{\beta} & \text{if } \beta \neq \alpha^s - \epsilon_{k_i^s} \text{ for some } i, \\ e_J \otimes e^{\alpha - \epsilon_{k_p^s}} & \text{if } k_p^s \notin J, \\ \operatorname{sgn}(k_p^s) e_{J \cup \{k_p^s\}} \otimes e^{\alpha - \epsilon_{k_p^s}} + \operatorname{sgn}(k_t^s) e_{\{k_1^s, \dots, k_t^s\} \cup J'} \otimes e^{\alpha - \epsilon_{k_t^s}} & \text{if } J = \{k_1^s, \dots, k_{t-1}^s\} \cup J' \\ & \text{for some } J', \ k_t^s \notin J \end{cases}$$

for all $1 \leqslant s \leqslant r$, $1 \leqslant p \leqslant n_{\alpha^s}$, where $J = (j_0 < \cdots < j_i)$.

Remark 5.5.4. In the case that $\alpha^s = d\epsilon_{i_s}$ for some indicies $i_1 < \cdots < i_r$, the submodules $L_d^{i,B}$ as in Definition 5.5.3 are obtained by simply deleting all standard basis elements of the form

$$e_{\{i_s\} \cup J} \otimes e_{i_s}^{d-1} \qquad (i_s \notin J).$$

Observation 5.5.5. Let $L_d^{i,B}(F)$ denote the submodule of Definition 5.5.3. Then the Koszul differential induces a map

$$L_d^{i,B}(F) \to L_d^{i-1,B}(F).$$

Moreover, if rk_i is as in the statement of Corollary 5.4.6, then

$$\begin{aligned} \operatorname{rank} L_d^{i,B}(F) &= \operatorname{rank} L_d^i(F) - \mathrm{rk}_i \\ &= \binom{n+d-1}{i+d}\binom{d+i-1}{i} - \mathrm{rk}_i. \end{aligned}$$

Proof. The first observation is clear by noticing that $\mathcal{S}$ as in Definition 5.5.3 generates Ker $\begin{pmatrix} \phi_i^{I_1,\alpha^1} \\ \phi_i^{I_2,\alpha^2} \\ \vdots \\ \phi_i^{I_r,\alpha^r} \end{pmatrix}$, where each $\phi_i^{I_s,\alpha^s}$ is as in Definition 5.2.2. The fact that this generates the kernel follows by the proof of Proposition 5.3.6. For the rank count, observe that the count for each omitted basis element is precisely the count done in the proof of Proposition 5.3.6. Indeed, the basis elements omitted are precisely the elements whose images form a basis for the image of the q_i^s maps, for each $1 \leqslant s \leqslant r$. □

Definition 5.5.6. Let $\alpha := (\alpha_1, \ldots, \alpha_n)$ be an exponent vector. Define $\operatorname{Supp}(\alpha) := \{i \mid \alpha_i > 0\}$. If $n_\alpha > 1$, define $K_\bullet^{\alpha^c}$ to be the complex induced by the map

$$\begin{aligned} \psi : K_1^\alpha := \bigoplus_{i \notin \operatorname{Supp}(\alpha)} Re_i &\to R \\ e_i &\mapsto x_i. \end{aligned}$$

If $n_\alpha = 1$, then $K_\bullet^\alpha$ is defined to be the 0 complex.

It turns out that the following result tells us that the top linear strand of the minimal free resolution quotient defined by the ideals of Theorem 5.5.8 will always be a direct sum of shifted Koszul complexes.

Proposition 5.5.7. *Adopt notation and hypotheses as in Setup 5.3.2. Then there is an isomorphism of complexes*

$$\Phi_\bullet : \operatorname{Coker} \phi_\bullet^{I,\alpha} \to K_\bullet^\alpha[-n_\alpha]$$

Proof. If $n_\alpha = 1$, then the claim is true. Assume that $n_\alpha > 1$. Observe that $\operatorname{Coker} \phi_i^{I,\alpha}$ is free on all basis elements of the form

$$\{e_{\operatorname{Supp}(\alpha) \cup J} \mid \operatorname{Supp}(\alpha) \cap J = \varnothing, \ |J| = i - n_\alpha\}.$$

Consider the map

$$\Phi_i : \operatorname{Coker} \phi_i^\alpha \to K_{i-n_\alpha}^\alpha$$

$$e_{\{k_1,\ldots,k_{n_\alpha}\}\cup J} \mapsto \operatorname{sgn}(J) e_J.$$

This map is clearly an isomorphism, whence it remains to show that $\Phi_\bullet$ is a morphism of complexes. For $i \geqslant 1$, consider the diagram

$$\begin{tikzcd} \operatorname{Coker} \phi_i^\alpha \arrow[r, "d_i"] \arrow[d, "\Phi_i"] & \operatorname{Coker} \phi_{i-1}^\alpha \arrow[d, "\Phi_{i-1}"'] \\ K_{i-n_\alpha}^\alpha \arrow[r, "\mathrm{Kos}"] & K_{i-1-n_\alpha}^\alpha \end{tikzcd} \tag{5.5.1}$$

Going clockwise around 5.5.1, one has:

$$\begin{aligned} e_{\{k_1,\ldots,k_{n_\alpha}\}\cup J} \stackrel{d_i}{\longmapsto} & \sum_{i=1}^{n_\alpha} \operatorname{sgn}(k_i) x_{k_i} e_{\{k_1,\ldots,\widehat{k_i},\ldots k_{n_\alpha}\}\cup J} \\ & + \sum_{j\in J} \operatorname{sgn}(j \in \operatorname{Supp}(\alpha) \cup J) x_j e_{\{k_1,\ldots,k_{n_\alpha}\}\cup J\backslash j} \\ & = \sum_{j\in J} \operatorname{sgn}(j \in \operatorname{Supp}(\alpha) \cup J) x_j e_{\{k_1,\ldots,k_{n_\alpha}\}\cup J\backslash j} \\ & \xmapsto{\Phi_{i-1}} \sum_{j\in J} \operatorname{sgn}(j \in \operatorname{Supp}(\alpha) \cup J) \operatorname{sgn}(J\backslash j \subseteq \operatorname{Supp}(\alpha)) x_j e_{J\backslash j}. \end{aligned}$$

where the equality in the penultimate line follows by noticing that $\operatorname{im} \phi_i^{I,\alpha}$ is free on basis elements of the form

$$\{e_J \mid \operatorname{Supp}(\alpha) \not\subset J,\ |J| = i\}.$$

Moving counterclockwise around 5.5.1:

$$\begin{aligned} e_{\{k_1,\ldots,k_{n_\alpha}\}\cup J} & \stackrel{\Phi_i}{\longmapsto} \operatorname{sgn}(J) e_J \\ & \stackrel{\mathrm{Kos}}{\longmapsto} \sum_{j\in J} \operatorname{sgn}(J \subset \operatorname{Supp}(\alpha)) \operatorname{sgn}(j \in J) x_j e_{J\backslash j}. \end{aligned}$$

To conclude, observe that

$$\operatorname{sgn}(j \in \operatorname{Supp}(\alpha) \cup J) \operatorname{sgn}(J\backslash j \subseteq \operatorname{Supp}(\alpha)) = \operatorname{sgn}(J \subset \operatorname{Supp}(\alpha)) \operatorname{sgn}(j \in J).$$

$\square$

Combining these building blocks with Theorem 5.5.1, one obtains:

Theorem 5.5.8. *Adopt notation and hypotheses as in Setup 5.4.1. Let K' be an equigenerated momomial ideal with $\overline{K'} = (x^{\alpha^1}, \dots, x^{\alpha^r})$. Then the minimal free resolution of R/K' is given by the complex*

$$F_i := L_d^{i-1,B} \oplus \Big(\bigoplus_{j=1}^{|B|} K^{\alpha^j}_{i-n_{\alpha^j}} \Big) \quad (i > 0),$$
$$F_0 = R,$$

with differentials

$$\ell_i := \begin{pmatrix} Kos^{\psi} \otimes 1 & \Theta_i \\ 0 & -\bigoplus_{j=1}^{|B|} Kos^{\psi} \end{pmatrix}, \quad (i > 2),$$
$$\ell_2 := \begin{pmatrix} Kos^{\psi} \otimes 1 & \Theta_i \end{pmatrix},$$
$$\ell_1 := S(\psi)|_{L_d^{0,B}}$$

where Θ_p restricted to each direct summand $K^{\alpha^{s},c}_{p-n_{\alpha^s}}$ is the map:

$$\begin{aligned} \Theta_i : K^{\alpha^{j},c}_{p-n_{\alpha^s}} &\to L_d^{p-2,B} \\ e_J^{\alpha^j} &\mapsto \operatorname{sgn}(J) \sum_{i=1}^{n_{\alpha^s}} \operatorname{sgn}(k_i^s) x^2_{k_i^s} e_{\{k_1^s, \dots, \widehat{k_i^s}, \dots, k^s_{n_{\alpha^s}}\} \cup J} \otimes e^{\alpha^s - \epsilon_{k_i^s}} \\ &+ \operatorname{sgn}(J) \sum_{i<j} x_{k_i^s} x_{k_j^s} \Big(\operatorname{sgn}(k_j^s) e_{\{k_1^s, \dots, \widehat{k_j^s}, \dots, k^s_{n_{\alpha^s}}\} \cup J} \otimes e^{\alpha^s - \epsilon_{k_i^s}} \\ &\qquad + \operatorname{sgn}(k_i^s) e_{\{k_1^s, \dots, \widehat{k_i^s}, \dots, k^s_{n_{\alpha^s}}\} \cup J} \otimes e^{\alpha^s - \epsilon_{k_j^s}} \Big) \end{aligned}$$

Proof. Using Theorem 5.5.1, the proof comes down to computing the map Θ_i explicitly. For ease of notation/computation, assume that $|B| = 1$. It will be understood that in the case $|B| > 1$, this computation yields the restriction of Θ_p to each direct summand.

Let $p \geqslant 2$; one computes the image of an arbitrary $e_J \in K^{\alpha^c}_{p-n_\alpha}$ under Θ_p:

$$e_J \xmapsto{\text{Prop } 5.5.7} \operatorname{sgn}(J) e_{\{k_1, \dots, k_{n_\alpha}\} \cup J}$$

$$
\begin{aligned}
&\xmapsto{\mathrm{Kos}^{\psi}} \operatorname{sgn}(J)\sum_{j\in J}\operatorname{sgn}(j)x_j e_{\{k_1,\dots,k_t\}\cup J}\\
&+\operatorname{sgn}(J)\sum_{i=1}^{n_\alpha}\operatorname{sgn}(k_i)x_{k_i}e_{\{k_1,\dots,\widehat{k_i},\dots,k_{n_\alpha}\}\cup J}\\
&\xmapsto{\text{projection}} \operatorname{sgn}(J)\sum_{i=1}^{n_\alpha}\operatorname{sgn}(k_i)x_{k_i}e_{\{k_1,\dots,\widehat{k_i},\dots,k_{n_\alpha}\}\cup J}\\
&\xmapsto{\text{inclusion}} \operatorname{sgn}(J)\sum_{i=1}^{n_\alpha}x_{k_i}e_{\{k_1,\dots,k_{n_\alpha}\}\cup J}\otimes e^{\alpha-\epsilon_{k_i}}\\
&\xmapsto{\mathrm{Kos}^{\psi}\otimes 1} \operatorname{sgn}(J)\sum_{i=1}^{n_\alpha}\operatorname{sgn}(j\in J)x_{k_i}x_j e_{\{k_1,\dots,k_{n_\alpha}\}\cup J\backslash j}\otimes e^{\alpha-\epsilon_{k_i}}\\
&+\operatorname{sgn}(J)\sum_{i,j=1}^{n_\alpha}\operatorname{sgn}(k_j)x_{k_i}x_{k_j}e_{\{k_1,\dots,\widehat{k_j},\dots,k_{n_\alpha}\}\cup J}\otimes e^{\alpha-\epsilon_{k_i}}\\
&\xmapsto{\text{projection}} \operatorname{sgn}(J)\sum_{i,j=1}^{n_\alpha}\operatorname{sgn}(k_j)x_{k_i}x_{k_j}e_{\{k_1,\dots,\widehat{k_j},\dots,k_{n_\alpha}\}\cup J}\otimes e^{\alpha-\epsilon_{k_i}}\\
&=\operatorname{sgn}(J)\sum_{i=1}^{n_\alpha}\operatorname{sgn}(k_i)x_{k_i}^2 e_{\{k_1,\dots,\widehat{k_i},\dots,k_{n_\alpha}\}\cup J}\otimes e^{\alpha-\epsilon_{k_i}}\\
&+\operatorname{sgn}(J)\sum_{i<j}x_{k_i}x_{k_j}\Big(\operatorname{sgn}(k_j)e_{\{k_1,\dots,\widehat{k_j},\dots,k_{n_\alpha}\}\cup J}\otimes e^{\alpha-\epsilon_{k_i}}\\
&\qquad\qquad+\operatorname{sgn}(k_i)e_{\{k_1,\dots,\widehat{k_i},\dots,k_{n_\alpha}\}\cup J}\otimes e^{\alpha-\epsilon_{k_j}}\Big).
\end{aligned}
$$

The final equality of the above is written in terms of the basis elements of $L_d^{p-2,B}$ □

As a Corollary, we obtain the previously mentioned fact that the minimal free resolution in the case that the complementary ideal consists of pure powers is obtained by simply restricting to the subcomplex $A_\bullet$ (with notation as in Theorem 5.5.1).

Corollary 5.5.9. *Adopt notation and hypotheses as in Setup 5.4.1, with $\alpha_s = d\epsilon_{k_s}$ for $1 \leqslant s \leqslant r \leqslant n$, where $B = \{d\epsilon_{k_1} < \cdots < d\epsilon_{k_r}\}$. Let K' be an equigenerated momomial ideal with $\overline{K'} = (x_{k_1}^d, \dots, x_{k_r}^d)$. Then the minimal free resolution of R/K' is given by the complex*

$$
L^B(\psi,d):\ 0\longrightarrow L_d^{n-1,B}\xrightarrow{Kos^{\psi}\otimes 1}\cdots\xrightarrow{Kos^{\psi}\otimes 1}L_d^{0,B}\xrightarrow{S_d(\psi)}R\longrightarrow 0.
$$

Proof. This follows immediately by Theorem 5.5.1, since $C_k = 0$ for all k by Proposition 5.3.4. $\square$

Next, we define the necessary building blocks in the squarefree case. This resolution will be much simpler to describe, since we have already seen that these ideals have linear minimal free resolutions. This implies that, as in Corollary 5.5.9, the minimal free resolution is obtained by taking the kernel of an appropriate morphism of complexes.

Definition 5.5.10. Adopt notation and hypotheses as in Setup 5.4.8, with $i = \{I_1, \dots, I_r\}$. For each i, define the free submodule $F_i^{d,n,i} \subseteq F_i^{d,n}$ to be generated by the following collections of basis elements, denoted $\mathcal{T}$ (all terms appearing are assumed to be standard tableau with strictly increasing columns and rows; recall that Notation 5.1.11 is in play here):

$$\begin{cases} f_J \otimes f^L & \text{if } J \neq I_s \backslash \{i_{ps}\} \text{ for any } p, \ s, \\ f_J \otimes f^{I_s - \epsilon_{i_{ps}}} & \text{if } i_{ps} \notin J \text{ for any } p, \ s, \\ \operatorname{sgn}(i_{ps}) f_{J \cup \{i_{ps}\}} \otimes f^{I_s - \epsilon_{i_{ps}}} + \operatorname{sgn}(i_{1s}) f_{J \cup i_{1s}} \otimes f^{I_s - \epsilon_{i_{1s}}} & \text{where } I_s \cap J = \varnothing, \end{cases}$$

where $J = (j_0 < \ldots < j_{i-1})$, $1 \leqslant s \leqslant r$, and $1 < p \leqslant d$.

Observation 5.5.11. Adopt notation and hypotheses as in Setup 5.4.8, with $i = \{I_1, \dots, I_r\}$. Let $F_i^{d,n,i}$ denote the submodule of Definition 5.5.10. Then the differential $\partial_i^{d,n} : F_i^{d,n} \to F_{i-1}^{d,n}$ induces a differential

$$\partial_i^{d,n} : F_i^{d,n,i} \to F_{i-1}^{d,n,i}.$$

Moreover,

$$\begin{aligned}
\operatorname{rank} F_i^{d,n,i} &= \operatorname{rank} F_i^{d,n} - r \cdot \binom{n-d}{i-1} \\
&= \binom{n}{d+i-1}\binom{d+i-2}{i-1} - r\binom{n-d}{i-1}.
\end{aligned}$$

Proof. The first claim follows after noting that $F_i^{d,n,\mathcal{I}}$ generates $\operatorname{Ker}\begin{pmatrix} \psi_\ell^{I_1,I_1^c} \\ \psi_\ell^{I_2,I_2^c} \\ \vdots \\ \psi_\ell^{I_r,I_r^c} \end{pmatrix}$, where each $\psi_\ell^{I_s,I_s^c}$ is as in Definition 5.2.4. For the second claim, fix an indexing set $I_s = (i_{1s} < \cdots < i_{ds})$. The module $F_i^{d,n,\mathcal{I}}$ omits precisely all standard basis elements of the form

$$\begin{array}{|c|c|c|c|}
\hline
\vdots & i_{2s} & \cdots & i_{ds} \\
\hline
\multicolumn{1}{|c|}{J'} \\
\cline{1-1}
\multicolumn{1}{|c|}{\vdots} \\
\cline{1-1}
\end{array},$$

where $J' = (j'_0 < \cdots < j'_{i-1})$ and $J' \cap I_s = \{i_{1s}\}$; there are $\binom{n-d}{i-1}$ such choices for J' and r choices of s, so the result follows. □

Theorem 5.5.12. *Adopt notation and hypotheses as in Setup 5.4.8, with $i = \{I_1, \dots, I_r\}$. Let K' be a squarefree equigenerated momomial ideal with $\overline{K'} = (x^{I_1}, \dots, x^{I_r})$. Then the minimal free resolution of R/K' is given by the complex*

$$F_\bullet^{d,n,i} : \ 0 \longrightarrow F_{n-d+1}^{d,n,i} \xrightarrow{\partial_{n-d}^{d,n}} \cdots \xrightarrow{\partial_1^{d,n}} F_1^{d,n,i} \longrightarrow R \longrightarrow 0.$$

Proof. This follows immediately by combining Theorem 5.5.1 with Proposition 5.3.10.

□

We conclude with some questions about additional structure on the complexes above. Firstly, it is well known the the L-complexes admit the structure of an associative DG-algebra. Likewise, the complex constructed by Galetto will also admit the structure of an associative DG-algebra, since the squarefree Elihou-Kervaire complex admits such a structure by work of Peeva (see [37]). One is then tempted to ask:

Question 5.5.13. Do the complexes of Theorem 5.5.8 or Theorem 5.5.12 admit the structure of an associative DG-algebra?

Similarly, it is well known that the (squarefree) Eliahou-Kervaire resolution is cellular by work of Mermin (see [33]). Since one can reformat the above constructions of this section in terms of taking kernels/cokernels of the Eliahou-Kervaire resolution, we also pose:

Question 5.5.14. Are the complexes of Theorem 5.5.8 or Theorem 5.5.12 cellular?

Chapter 6

DG-Algebra Structure on Length 3 Trimming Complexes and Applications to Tor-Algebras

Let $(R, \mathfrak{m}, k)$ be a regular local ring with maximal ideal $\mathfrak{m}$ and residue field k with R/I a quotient ring of projective dimension 3. Recall that a complete classification of the multiplicative structure of the Tor algebra $\mathrm{Tor}_\bullet^R(R/I, k)$ for such quotients was established by Weyman in [43] and Avramov, Kustin, and Miller in [4].

Absent from this classification was a complete description of which Tor-algebra structures actually arise as the Tor-algebra of some quotient R/I with some prescribed homological data. More precisely, let R/I have a length 3 DG-algebra minimal free resolution:

$$F_\bullet : \quad 0 \to F_3 \to F_2 \to F_1 \to R,$$

with $m = \mathrm{rank}(F_1)$, $n = \mathrm{rank}(F_3)$. Let $\bar{\cdot} := \cdot \otimes k$, and define

$$p = \mathrm{rank}(\overline{F_1}^2), \quad q = \mathrm{rank}(\overline{F_1} \cdot \overline{F_2}),$$

$$r = \mathrm{rank}\left(\overline{F_2} \to \mathrm{Hom}_k(\overline{F_1}, \overline{F_3}) \right).$$

Then, Avramov posed the question (see [2, Question 3.8]):

Which tuples (m, n, p, q, r) are realized by some quotient ring R/I?

This is often referred to as the *realizability question*, and Avramov gives bounds on the possible tuples that can occur in this same paper along with some conjectures on the tuples associated to certain Tor-algebra classes. One such conjecture was related to the Tor-algebra class G, where Avramov had posed:

Question 6.0.1. If R/I is class $G(r)$ for some $r \geqslant 2$, then is R/I Gorenstein?

The counterexamples to Question 6.0.1 were originally constructed by Christensen and Veliche in [14] and produced on a much larger scale by Christensen, Veliche, and Weyman using a remarkably simple construction. Given an $\mathfrak{m}$-primary ideal $I = (\phi_1, \ldots, \phi_n) \subseteq R$, one can "trim" the ideal I by, for instance, forming the ideal $(\phi_1, \ldots, \phi_{n-1}) + \mathfrak{m}\phi_n$. It turns out that this will yield an ideal that defines a *non-Gorenstein* quotient ring which is also of class $G(r)$ (see [16]).

More generally, computational evidence suggests that the process of trimming an ideal tends to preserve Tor-algebra class. In this chapter, we set out to answer why this is true. In practice, there are two ways of computing multiplication in the Tor-algebra $\operatorname{Tor}_\bullet(R/I, k)$ for a given ideal I. First, let $K_\bullet$ denote the Koszul complex resolving $R/\mathfrak{m}$. Then, one can descend to the homology algebra $H_\bullet(K_\bullet \otimes R/I)$, with multiplication induced by the exterior algebra $K_\bullet$. Alternatively, one can produce an explicit DG-algebra free resolution $F_\bullet$ of R/I, tensor with $R/\mathfrak{m}$, and descend to homology with product induced by the algebra structure on $F_\bullet$.

We take the latter approach in this chapter. Luckily, an explicit free resolution of trimmed ideals is constructed in [41]. More generally, we construct an explicit algebra structure on arbitrary *iterated trimming complexes* of length 3 (see Theorem 6.2.3). In the case that the ambient ring is local, we are then able to show that the possible nontrivial multiplications in the Tor-algebra are rather restricted (see Corollary 6.2.7).

This algebra structure is then applied to the previously mentioned case where the ideal I is obtained by trimming. We focus on ideals defining rings of Tor-algebra G and H, and show that under very mild assumptions, trimming an ideal preserves these Tor-algebra classes. This allows us to construct novel examples of rings of class $G(r)$ and $H(p, q)$ obtained as quotients of arbitrary regular local rings $(R, \mathfrak{m}, k)$ of dimension 3, and we further add to the realizability question posed by Avramov (see Corollary 6.4.8 and 6.4.14).

This chapter is organized as follows. In Section 6.1, we set the stage with conventions and notation to be used throughout the rest of the chapter along with some background. In Section 6.2, an explicit product on the length 3 iterated trimming complex is constructed. In the case that the complexes involved further admit the structure of DG-modules over each other, then this product may be made even more explicit (see Proposition 6.2.4). As corollaries, we find that only a subset of the products on the iterated trimming complex are nontrivial after descending to homology.

In Section 6.3, we focus on the case of trimming an ideal (in the sense of Christensen, Veliche, and Weyman [16]). Assuming that certain products on the minimal free resolution of an ideal sit in sufficiently high powers of the maximal ideal, we show that trimming an ideal will either preserve the Tor-algebra class or yield a Golod ring (see Lemma 6.3.10 and 6.3.13). In the case that the minimal presenting matrix for these quotient rings has entries in $\mathfrak{m}^2$, the restrictions become even tighter and we can say *precisely* which Tor-algebra class these new ideals will occupy (see Corollary 6.3.11 and 6.3.14).

Finally, in Section 6.4, we begin to construct explicit quotient rings realizing tuples of the form (m, n, p, q, r). In particular, we construct an infinite class of new examples of class $G(r)$, and can say in general that there are rings of arbitrarily large type with Tor-algebra class $G(r)$, for any $r \geqslant 2$ (this was previously known in the case that the ambient ring was $k[x, y, z]$, see [39]). Likewise, we construct an infinite class of rings of Tor-algebra class $H(p, q)$ that are not hyperplane sections, which, combined with the process of linkage, can be used to conclusively show the existence of rings realizing many of the tuples falling within the bounds imposed by Christensen, Veliche, and Weyman in [17].

6.1 Background, Notation, and Conventions

In this section, we first introduce some of the notation and conventions that will be in play throughout the chapter. We will introduce iterated trimming complexes (see Definition 2.2.5), the algebra structure on which is the main subject of Section 6.2. We also discuss the realizability question posed originally by Avramov, and discuss the progress on this question due to Christensen, Veliche, and Weyman in [17].

Throughout the chapter, all complexes will be assumed to have nontrivial terms appearing only in nonnegative homological degrees.

Notation 6.1.1. *The notation* $(F_\bullet, d_\bullet)$ *will denote a complex* $F_\bullet$ *with differentials* $d_\bullet$*. When no confusion may occur,* F *or* $F_\bullet$ *may be written instead.*

Given a complex $F_\bullet$ *as above, elements of* F_n *will often be denoted* f_n*, without specifying that* $f_n \in F_n$*.*

Definition 6.1.2. A *differential graded algebra* $(F_\bullet, d_\bullet)$ (DG-algebra) over a commutative Noetherian ring R is a complex of finitely generated free R-modules with differential d and with a unitary, associative multiplication $F \otimes_R F \to F$ satisfying

(a) $F_i \cdot F_j \subseteq F_{i+j}$,

(b) $d_{i+j}(x_i x_j) = d_i(x_i)x_j + (-1)^i x_i d_j(x_j)$,

(c) $x_i x_j = (-1)^{ij} x_j x_i$, and

(d) $x_i^2 = 0$ if i is odd,

where $x_k \in F_k$.

The following setup will be used for the rest of this section, and will be used to discuss results related to Question 6.1.4.

Setup 6.1.3. *Let $(R, \mathfrak{m}, k)$ denote a local ring. Let R/I have a length 3 DG-algebra minimal free resolution:*

$$F_\bullet : \qquad 0 \to F_3 \to F_2 \to F_1 \to R,$$

with $m = \operatorname{rank}(F_1)$, $n = \operatorname{rank}(F_3)$. Let $\bar{\cdot} := \cdot \otimes k$, and define

$$p = \operatorname{rank}(\overline{F_1}^2), \quad q = \operatorname{rank}(\overline{F_1} \cdot \overline{F_2}),$$

$$r = \operatorname{rank}\left(\overline{F_2} \to \operatorname{Hom}_k(\overline{F_1}, \overline{F_3})\right).$$

Question 6.1.4. Which tuples (m, n, p, q, r) are realized by the data of Setup 6.1.3 for some quotient ring R/I?

For the definition of the Tor-algebra class $H(p, q)$ appearing in the following Theorem, see Theorem 6.3.1.

Theorem 6.1.5 ([17], Theorem 1.1)**.** *Adopt notation and hypotheses as in Setup 6.1.3. Let $Q = R/I$ be a Cohen-Macaulay local ring of codimension 3 and class $H(p, q)$. Then the following inequalities hold:*

$$p \leqslant m - 1 \quad \textit{and} \quad q \leqslant n.$$

Moreover, the following are equivalent:

(i) $p = n + 1$.

(ii) $q = m - 2$.

(iii) $p = m - 1$ *and* $q = n$.

If the conditions $(i) - (iii)$ are not satisfied, then there are inequalities

$$p \leqslant n - 1 \quad \textit{and} \quad q \leqslant m - 4$$

with

$$p = n - 1 \iff q \equiv_2 m - 2 \quad \textit{and} \quad q = m - 4 \iff p \equiv_2 n - 1.$$

Christensen, Veliche, and Weyman further conjecture that within the bounds supplied by Theorem 6.1.5, there are ideals defining rings of class $H(p,q)$ realizing the tuples (m,n,p,q) (that is, these bounds are sharp; see the conjectures of 7.4 in [17]).

Next, recall that an ideal $J \subseteq R$ is *directly linked* to the ideal I if there exists a grade 3 complete intersection $\mathfrak{a} \subseteq I$ such that $J = (\mathfrak{a} : I)$. Two ideals I and J are *linked* if there exists a sequence of direct links connecting I and J. In [17], it is carefully studied how the Tor-algebra of an ideal relates to that of any directly linked ideal. In particular, one has the following Proposition.

Proposition 6.1.6 ([17]). *Adopt notation and hypotheses as in Setup 6.1.3. Assume $I \subseteq R$ is a grade 3 perfect ideal.*

(a) If I defines a ring of Tor-algebra class $G(r)$, then it is directly linked to a grade 3 perfect ideal J defining a ring of Tor-algebra class $H(p',q')$ realizing the tuple

$$(n+3, m-3, p', q'),$$

where $p' \geqslant \min\{r, 3\}$.

(b) If I defines a ring of Tor-algebra class $H(p,q)$ with $p \geqslant 2$, $q \geqslant 3$, and $m-3 \geqslant p$, then it is directly linked to a grade 3 perfect ideal J of class $H(q,p)$ realizing the tuple

$$(n+3, m-3, q, p).$$

(c) If I defines a ring of Tor-algebra class $H(p,q)$ with $p \geqslant 3$, $q \geqslant 3$, and $m-2 \geqslant p \geqslant 1$, then it is directly linked to a grade 3 perfect ideal J of class $H(q,p-1)$ realizing the tuple

$$(n+2, m-3, q, p-1).$$

(d) If I defines a ring of Tor-algebra class $H(p,q)$ with $p \geqslant 4$, $q \geqslant 3$, and $m-1 \geqslant p \geqslant 1$, then it is directly linked to a grade 3 perfect ideal J of class $H(q,p-2)$

realizing the tuple

$$(n+1, m-3, q, p-2),$$

where $m \geqslant 5$ or $n \geqslant 3$.

Combining Proposition 6.1.6 with the explicit examples given in Section 6.4 will allow us to combine the processes of linkage and trimming to compute explicit examples realizing the tuples falling within the bounds of Theorem 6.1.5.

6.2 Algebra Structure on Length 3 Iterated Trimming Complexes

In this section, we show that if the complexes associated to the input data of Setup 2.1.1 are length 3 DG-algebras, then the product on the resulting iterated trimming complex of Theorem 2.2.4 may be computed in terms of the products on the aforementioned complexes. The proof of this fact is a long and rather tedious computation; moreover, in full generality, the products have certain components that are only defined implicitly. In the case that the complexes involved admit additional module structures over one another, these products may be made more explicit (see Proposition 6.2.4). However, after descending to homology, many of these products either vanish completely or become considerably more simple. This fact is made explicit in the corollaries at the end of this section, and will be taken advantage of in Section 6.3.

The following is essentially the proof of Proposition 1.3 of [11]; for convenience, the proof is reproduced here.

Proposition 6.2.1. *Let $(F_\bullet, d_\bullet)$ denote a length 3 resolution of a cyclic module M admitting a product satisfying axioms $(a)-(d)$ of Definition 6.1.2. Then, the product is associative.*

Proof. Since $F_\bullet$ has length 3, the only nontrivial triple product can occur between 3 elements $e_1,\ e_2$, and e_3 of homological degree 1. One computes:

$$\begin{aligned}
d_3\Big((e_1 \cdot e_2) \cdot e_3 - e_1 \cdot (e_2 \cdot e_3)\Big) &= d_2(e_1 \cdot e_2) \cdot e_3 + d_1(e_3) e_1 \cdot e_2 \\
&- d_1(e_1) e_2 \cdot e_3 + e_1 \cdot d_2(e_2 \cdot e_3) \\
&= d_1(e_1) e_2 \cdot e_3 - d_1(e_2) e_1 \cdot e_3 + d_1(e_3) e_1 \cdot e_2 \\
&- d_1(e_1) e_2 \cdot e_3 + d_1(e_2) e_1 \cdot e_3 - d_1(e_3) e_1 \cdot e_2 \\
&= 0.
\end{aligned}$$

Since d_3 is injective, the result follows. □

Notation 6.2.2. *Given a DG-algebra $F_\bullet$, the notation $\cdot_F$ denotes the product on $F_\bullet$. Given two free modules F and G, elements of the direct sum $F \oplus G$ will be denoted $f + g \in F \oplus G$.*

Theorem 6.2.3. *Adopt notation and hypotheses as in Setup 2.1.1, and assume that the complexes $F_\bullet$ and $G^i_\bullet$ $(1 \leqslant i \leqslant t)$ are length 3 DG-algebras. Then the length 3 iterated trimming complex of Theorem 2.2.4 admits the structure of an associative DG-algebra with a product of the form:*

$$\begin{aligned}
&F_1' \otimes F_1' \to F_2 \oplus \Big(\bigoplus_{i=1}^t G_2^i\Big) \\
&f_1 \cdot_T f_1' := f_1 \cdot_F f_1' + \sum_{i=1}^t g_2^i, \ \textit{where } m_2^i(g_2^i) = q_1^i(f_1 \cdot_F f_1'), \\
&F_1' \otimes G_1^i \to F_2 \oplus \Big(\bigoplus_{j=1}^t G_2^j\Big) \\
&f_1 \cdot_T g_1^i := m_1^i(g_1^i) e_0^i \cdot_F f_1 + \sum_{j=1}^t g_2^j, \\
&\quad \textit{where } m_2^i(g_2^i) = d_1(f_1) g_1^i + m_1^i(g_1^i) q_1^i(e_0^i \cdot_F f_1), \\
&\qquad m_2^j(g_2^j) = m_1^i(g_1^i) q_1^j(e_0^i \cdot_F f_1) \ \textit{for } j \neq i
\end{aligned}$$

$$
\begin{aligned}
&G_1^i \otimes G_1^i \to F_2 \oplus \Big(\bigoplus_{j=1}^{t} G_2^j \Big) \\
&\quad g_1^i \cdot_T {g'}_1^i := -g_1^i \cdot_{G^i} {g'}_1^i d_1(e_0^i), \\
&G_1^i \otimes G_1^j \to F_2 \oplus \Big(\bigoplus_{k=1}^{t} G_2^k \Big) \qquad i \neq j \\
&\quad g_1^i \cdot_T g_1^j := m_1^i(g_1^i) m_1^j(g_1^j) e_0^i \cdot_F e_0^j + \sum_{k=1}^{t} g_2^k, \\
&\quad \textit{where } m_2^i(g_2^i) = m_1^i(g_1^i) m_1^j(g_1^j) q_1^i(e_0^i \cdot_F e_0^j) + m_1^j(g_1^j) d_1(e_0^j) g_1^i, \\
&\qquad\quad m_2^j(g_2^j) = m_1^i(g_1^i) m_1^j(g_1^j) q_1^j(e_0^i \cdot_F e_0^j) - m_1^i(g_1^i) d_1(e_0^i) g_1^j, \\
&\qquad\quad m_1^k(g_2^k) = m_1^i(g_1^i) m_1^j(g_1^j) q_1^k(e_0^i \cdot_F e_0^j) \textit{ for } k \neq i,\ j, \\
&F_1' \otimes F_2 \to F_3 \oplus \Big(\bigoplus_{i=1}^{t} G_3^i \Big) \\
&\quad f_1 \cdot_T f_2 := f_1 \cdot_F f_2 + \sum_{i=1}^{t} g_3^i, \textit{for some } g_3^i \in G_3^i, \\
&F_1' \otimes G_2^i \to F_3 \oplus \Big(\bigoplus_{j=1}^{t} G_3^j \Big) \\
&\quad f_1 \cdot_T g_2^i := \sum_{j=1}^{t} g_3^j, \quad \textit{for some } g_3^j \in G_3^j \\
&G_1^i \otimes G_2^i \to F_3 \oplus \Big(\bigoplus_{j=1}^{t} G_3^j \Big) \\
&\quad g_1^i \cdot_T g_2^i = -g_1^i \cdot_{G^i} g_2^i d_1(e_0^i), \\
&G_1^i \otimes G_2^j \to F_3 \oplus \Big(\bigoplus_{k=1}^{t} G_3^k \Big) \\
&\quad g_1^i \cdot_T g_2^j := \sum_{k=1}^{t} g_3^k,\ i \neq j,\ \textit{for some } g_3^k \in G_3^k, \\
&G_1^i \otimes F_2 \to F_3 \oplus \Big(\bigoplus_{j=1}^{t} G_3^j \Big) \\
&\quad g_1^i \cdot_T f_2 := -m_1^i(g_1^i) e_0^i \cdot_F f_2 + \sum_{j=1}^{t} g_3^j,\ \textit{for some } g_3^j \in G_3^j.
\end{aligned}
$$

Proof. Observe that, by Proposition 6.2.1, it suffices to show that the contended products satisfy axiom (b) of Definition 6.1.2. The proof thus becomes a straightforward verification of this identity, and will be split accordingly into all of the cases. For convenience, let $\pi^i : F_1 \to R$ denote the composition

$$F_1 \xrightarrow{\text{projection}} Re_0^i \to R,$$

where the second map sends $e_0^i \mapsto 1$. Observe that $m_1^i \circ q_1^i = \pi^i \circ d_2 =: {d'_0}^i$. Likewise, let $p : F_1 \to F'_1$ denote the natural projection. Observe that $d'_2 = p \circ d_2$.

Case 1: $F'_1 \otimes F'_1 \to F_2 \oplus \left(\bigoplus_{i=1}^t G_2^i \right)$. We first need to verify the existence of each g_2^i; by exactness of each $G_\bullet^i$, it suffices to show that $q_1^i(f_1 \cdot_F f'_1)$ is a cycle. One computes:

$$\begin{aligned} m_1^i \circ q_1^i(f_1 \cdot_F f'_1) &= \pi^i \Big(d_2(f_1 \cdot_F f'_1) \Big) \\ &= \pi^i \Big(d_1(f_1) f'_1 - d_1(f'_1) f_1 \Big) \\ &= 0, \text{ since } \pi^i(F'_1) = 0 \text{ for all } i. \end{aligned}$$

Thus the desired g_2^i exists for all i. It remains to verify the DG axiom:

$$\begin{aligned} \ell_2(f_1 \cdot_T f'_1) &= d'_2(f_1 \cdot_F f'_1) + \sum_{i=1}^t \Big(-q_1^i(f_1 \cdot_F f'_1) + m_2^i(g_2^i) \Big) \\ &= d_1(f_1) f'_1 - d_1(f'_1) f_1 \\ &= \ell_1(f_1) f'_1 - \ell_1(f'_1) f_1. \end{aligned}$$

Case 2: $F'_1 \otimes G_1^i \to F_2 \oplus \left(\bigoplus_{j=1}^t G_2^j \right)$. We first verify the existence of the desired g_2^j. One computes:

$$\begin{aligned} m_1^i \Big(d_1(f_1) g_1^i + m_1^i(g_1^i) q_1^i(e_0^i \cdot_F f_1) \Big) &= d_1(f_1) m_1^i(g_1^i) + m_1^i(g_1^i) \pi^i(d_2(e_0^i \cdot_F f_1)) \\ &= d_1(f_1) m_1^i(g_1^i) - m_1^i(g_1^i) d_1(f_1) \\ &= 0, \\ m_1^j \Big(m_1^i(g_1^i) q_1^j(e_0^i \cdot_F f_1) \Big) &= m_1^i(g_1^i) \pi^j(d_2(e_0^i \cdot_F f_1)) \end{aligned}$$

$$= 0.$$

It remains to verify the DG axioms:

$$\begin{aligned}
\ell_2(f_1 \cdot_T g_1^i) &= m_1^i(g_1^i)d_2'(e_0^i \cdot_F f_1) - \sum_{j=1}^t m_1^i(g_1^i)q_1^j(e_0^i \cdot_F f_1) + \sum_{j=1}^t m_2^i(g_2^i) \\
&= m_1^i(g_1^i)d_1(e_0)f_1 + d_1(f_1)g_1^i \\
&= \ell_1(f_1)g_1^i - \ell_1(g_1^i)f_1.
\end{aligned}$$

The above uses that $d_2' = p \circ d_2$, implying $d_2'(e_0^i \cdot_F f_1) = d_1(e_0^i)f_1$.

Case 3: $G_1^i \otimes G_1^i \to F_2 \oplus \left(\bigoplus_{j=1}^t G_2^j\right)$. One computes directly:

$$\begin{aligned}
\ell_2(g_1^i \cdot_T {g'}_1^i) &= m_2^i(-g_1^i \cdot_{G^i} {g'}_1^i d_1(e_0^i)) \\
&= -m_1(g_1^i){g'}_1^i d_1(e_0^i) + m_1({g'}_1^i)g_1^i d_1(e_0^i) \\
&= \ell_1(g_1^i){g'}_1^i - \ell_1({g'}_1^i)g_1^i
\end{aligned}$$

Case 4: $G_1^i \otimes G_1^j \to F_2 \oplus \left(\bigoplus_{k=1}^t G_2^k\right)$, $i \neq j$. We verify the existence of g_2^i; the proof of the existence of g_2^j is identical. One computes:

$$\begin{aligned}
&m_2^i\Big(m_1^i(g_1^i)m_1^j(g_1^j)q_1^i(e_0^i \cdot_F e_0^j) + m_1^j(g_1^j)d_1(e_0^j)g_1^i\Big) \\
=&m_1^i(g_1^i)m_1^j(g_1^j)\pi^i \circ d_2(e_0^i \cdot_F e_0^j) + m_1^j(g_1^j)d_1(e_0^j)m_1^i(g_1^i) \\
=&-m_1^i(g_1^i)m_1^j(g_1^j)d_1(e_0^j) + m_1^j(g_1^j)d_1(e_0^j)m_1^i(g_1^i) \\
=&0.
\end{aligned}$$

It remains to show the DG axiom:

$$\begin{aligned}
\ell_2(g_1^i \cdot_T g_1^j) = m_1^i(g_1^i)m_1^j(g_1^j)d_2'(e_0^i \cdot_F e_0^j) \\
+ \sum_{k=1}^t \Big(- m_1^i(g_1^i)m_1^j(g_1^j)q_1^i(e_0^i \cdot_F e_0^j) + m_2^k(g_2^k)\Big)
\end{aligned}$$

$$= m_1^j(g_1^j)d_1(e_0^j)g_1^i - m_1^i(g_1^i)d_1(e_0^i)g_1^j$$

$$= \ell_1(g_1^i)g_1^j - \ell_1(g_1^j)g_1^i.$$

In the above, notice that $d_2'(e_0^i \cdot_F e_0^j) = p(d_1(e_0^i)e_0^j - d_1(e_0^j)e_0^i) = 0$.

Case 5: $F_1' \otimes F_2 \to F_3 \oplus \left(\bigoplus_{i=1}^t G_3^i \right)$. Observe that

$$\begin{aligned} f_1 \cdot_T (d_2'(f_2) - \sum_{i=1}^t q_1^i(f_2)) &= f_1 \cdot_F d_2'(f_2) + \sum_{i=1}^t g_2^i \\ &\quad - \sum_{i=1}^t \left(m_1^i \circ q_1^i(f_2)e_0 \cdot_F f_1 + \sum_{j=1}^t g_2^{i,j} \right), \end{aligned}$$

$$\begin{aligned} \text{where} \qquad & m_2^i \left(g_2^i - \sum_{j=1}^t g_2^{j,i} \right) \\ &= q_1^i(f_1 \cdot_F d_2'(f_2)) - d_1(f_1)q_1^i(f_2) - \sum_{j=1}^t q_1^i(d_0^j(f_2) \cdot_F f_1) \\ &= q_1^i(f_1 \cdot_F d_2(f_2) - d_1(f_1)f_2) \\ &= - m_2^i \circ q_2^i(f_1 \cdot_F f_2), \text{ for each } i = 1, \ldots, t. \end{aligned}$$

This implies that $g_2^i - \sum_{j=1}^t g_2^{j,i} + q_2^i(f_1 \cdot_F f_2)$ is a cycle, so that there exist $g_3^i \in G_3^i$ such that

$$m_3^i(g_3^i) = g_2^i - \sum_{j=1}^t g_2^{j,i} + q_2^i(f_1 \cdot_F f_2).$$

Using this along with the fact that

$$\sum_{i=1}^t \left(g_2^i - \sum_{j=1}^t g_2^{i,j} \right) = \sum_{i=1}^t \left(g_2^i - \sum_{j=1}^t g_2^{j,i} \right),$$

one obtains:

$$f_1 \cdot_T (d_2'(f_2) - \sum_{i=1}^t q_1^i(f_2)) = f_1 \cdot_F d_2(f_2) - \sum_{i=1}^t q_2^i(f_1 \cdot_F f_2) + \sum_{i=1}^t g_3^i,$$

whence upon choosing

$$f_1 \cdot_T f_2 := f_1 \cdot_F f_2 + \sum_{i=1}^{t} g_3^i,$$

one immediately obtains

$$\begin{aligned}
\ell_3(f_1 \cdot_T f_2) &= d_1(f_1)f_2 - f_1 \cdot_F d_2(f_2) + \sum_{i=1}^{t} q_2^i(f_1 \cdot_F f_2) + \sum_{i=1}^{t} g_3^i \\
&= d_1(f_1)f_2 - f_1 \cdot_T (d_2'(f_2) - \sum_{i=1}^{t} q_1^i(f_2)) \\
&= \ell_1(f_1)f_2 - f_1 \cdot_T \ell_2(f_2).
\end{aligned}$$

Case 6: $F_1' \otimes G_2^i \to F_3 \oplus \left(\bigoplus_{j=1}^t G_3^j\right)$. One computes:

$$\ell_1(f_1)g_2^i - f_1 \cdot_T m_2^i(g_2^i) = d_1(f_1)g_2^i - \sum_{j=1}^{t} {g'}_2^j,$$

$$\text{where } m_2^j({g'}_2^j) = \begin{cases} d_1(f_1)m_2^i(g_2^i) & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

This implies that

$$\begin{cases} d_1(f_1)g_2^i - {g_2'}^i & \text{if } i = j, \text{ and} \\ {g_2'}^j & \text{otherwise} \end{cases}$$

are cycles. By exactness of each $G_\bullet^j$, there exist $g_3^j \in G_3^j$ such that

$$m_3^j(g_3^j) = \begin{cases} d_1(f_1)g_2^i - {g_2'}^i & \text{if } i = j, \\ -{g_2'}^j & \text{otherwise.} \end{cases} \tag{6.2.1}$$

Case 7: $G_1^i \otimes G_2^i \to F_3 \oplus \left(\bigoplus_{j=1}^t G_3^j\right)$. One computes:

$$\begin{aligned}
\ell_3(g_1^i \cdot_T g_2^i) &= m_3^i(-g_1^i \cdot_{G^i} g_2^i d_1(e_0^i)) \\
&= -m_1^i(g_1^i)g_2^i d_1(e_0^i) + g_1^i \cdot_{G^i} m_2^i(g_2^i) d_1(e_0^i)
\end{aligned}$$

$$= \ell_1(g_1^i)g_2^i - g_1^i \cdot_T \ell_2(g_2^i).$$

Case 8: $G_1^i \otimes G_2^j \to F_3 \oplus \left(\bigoplus_{k=1}^t G_3^k \right)$, $i \neq j$. One computes:

$$\ell_1(g_1^i)g_2^j - g_1^i \cdot_T \ell_2(g_2^j) = -m_1^i(g_1^i)d_1(e_0^i)g_2^j - \sum_{k=1}^t {g'}_2^k$$

$$\text{where } m_2^k({g'}_2^k) = \begin{cases} -m_1^i(g_1^i)d_1(e_0^i)m_2^j(g_2^j) & \text{if } k = j \\ 0 & \text{otherwise,} \end{cases}$$

whence

$$\begin{cases} {g'}_2^i + m_1^i(g_1^i)d_1(e_0^i)g_2^j & \text{if } k = j \\ {g'}_2^k & \text{otherwise,} \end{cases}$$

are cycles, implying there exists $g_3^k \in G_3^k$ such that

$$m_3^k(g_3^k) = \begin{cases} {g'}_2^i + m_1^i(g_1^i)d_1(e_0^i)g_2^j & \text{if } k = j \\ {g'}_2^k & \text{otherwise.} \end{cases} \tag{6.2.2}$$

Thus, one may define

$$g_1^i \cdot_T g_2^j := \sum_{k=1}^t g_3^k,$$

and this product will satisfy the Leibniz rule.

Case 9: $G_1^i \otimes F_2 \to F_3 \oplus \left(\bigoplus_{j=1}^t G_3^j \right)$. One computes:

$$\begin{aligned} &\ell_1(g_1^i)f_2 - g_1^i \cdot_T \ell_2(f_2) \\ &= -m_1^i(g_1^i)d_1(e_0^i)f_2 - g_1^i \cdot_T (d_2'(f_2) - \sum_{j=1}^t q_1^j(f_2)) \\ &= -m_1^i(g_1^i)d_1(e_0^i)f_2 + d_2'(f_2) \cdot_T g_1^i + \sum_{j=1}^t g_1^i \cdot_T q_1^j(f_2) \end{aligned}$$

$$= -m_1^i(g_1^i)d_1(e_0^i)f_2 + m_1^i(g_1^i)e_0^i \cdot_F d_2'(f_2) + \sum_{j=1}^t g_2^j - g_1^i \cdot_{G^i} q_1^i(f_2)d_1(e_0^i)$$
$$+ \sum_{j\neq i}^t \left(m_1^i(g_1^i)m_1^j(q_1^j(f_2))e_0^i \cdot_F e_0^j + \sum_{k=1}^t {g_2'}^{j,k} \right)$$
$$= -m_1^i(g_1^i)d_3(e_0 \cdot_F f_2) - g_1^i \cdot_{G^i} q_1^i(f_2)d_1(e_0^i) + g_2^i + \sum_{j\neq i}^t \left(g_2^j + \sum_{k=1}^t {g_2'}^{j,k} \right).$$

Observe that $\sum_{j\neq i}^t \left(g_2^j + \sum_{k=1}^t {g_2'}^{j,k} \right) = \sum_{j\neq i}^t \left(g_2^j + \sum_{k\neq i}^t {g_2'}^{k,j} \right) + \sum_{k\neq i}^t {g_2'}^{k,i}$, and

$$\begin{aligned}
m_2^j\Big(g_2^j + \sum_{k\neq i}^t {g_2'}^{k,j} \Big) &= m_1^i(g_1^i)q_1^j(e_0^i \cdot_F d_2'(f_2)) \\
&+ \sum_{k\neq i} m_1^i(g_1^i){d_0'}^k(f_2)q_1^j(e_0^i \cdot_F e_0^k) \\
&- m_1^i(g_1^i)d_1(e_0^i)q_1^j(f_2) \\
&= m_1(g_1^i)q_1^j(e_0^i \cdot_F d_2(f_2)) - m_1^i(g_1^i)d_1(e_0^i)q_1^j(f_2) \\
&= -m_1^i(g_1^i)m_2^j(q_2^j(e_0^i \cdot_F f_2)), \\
m_2^i\Big(- g_1^i \cdot q_1^i(f_2)d_1(e_0^i) + g_2^i + \sum_{k\neq i}^t {g_2'}^{k,i} \Big) &= -m_1^i(g_1^i)q_1^i(f_2)d_1(e_0^i) \\
&+ {d_0'}^i(f_2)d_1(e_0^i) + d_1(d_2'(f_2))g_1^i \\
&+ m_1^i(g_1^i)q_1^i(e_0^i \cdot_F d_2'(f_2)) \\
&+ \sum_{k\neq i} \left(m_1^i(g_1^i){d_0'}^k(f_2)q_1^i(e_0^i \cdot_F e_0^k) + {d_0'}^k(f_2)d_1(e_0^k)g_1^i \right) \\
&= d_1(d_2(f_2))g_1^i - m_1^i(g_1^i)q_1^i(f_2)d_1(e_0^i) \\
&+ m_1^i(g_1^i)q_1^i(e_0^i \cdot_F d_2(f_2)) \\
&= -m_1^i(g_1^i)m_2^i(q_2^i(e_0^i \cdot_F f_2)).
\end{aligned}$$

Thus,

$$g_2^j + \sum_{k\neq i}^t {g_2'}^{k,j} + m_1^i(g_1^i)q_2^j(e_0^i \cdot_F f_2), \text{ and} \tag{6.2.3}$$

$$- g_1^i \cdot q_1^i(f_2)d_1(e_0^i) + g_2^i + \sum_{k\neq i}^{t} {g_2'}^{k,i} + m_1^i(g_1^i)q_2^i(e_0^i \cdot_F f_2) \tag{6.2.4}$$

are both cycles, implying there exist $g_3^j \in G_3^j$ such that

$$m_3^j(g_3^j) = \begin{cases} -g_1^i \cdot q_1^i(f_2)d_1(e_0^i) + g_2^i + \sum_{k\neq i}^t {g_2'}^{k,i} + q_2^i(e_0^i \cdot_F f_2) & \text{if } i = j, \\ g_2^j + \sum_{k\neq i}^t {g_2'}^{k,j} + q_2^j(e_0^i \cdot_F f_2) & \text{otherwise.} \end{cases}$$

Defining

$$g_1^i \cdot_T f_2 := -m_1^i(g_1^i)e_0^i \cdot_F f_2 + \sum_{j=1}^t g_3^j,$$

one combines this with the first computation of this case to find:

$$\begin{aligned} \ell_3(g_1^i \cdot_T f_2) &= \ell_3(-m_1^i(g_1^i)e_0^i \cdot_F f_2 + \sum_{j=1}^t g_3^j) \\ &= -m_1^i(g_1^i)d_3(e_0^i \cdot_F f_2) + \sum_{j=1}^t -m_1^i(g_1^i)q_2^j(e_0^i \cdot_F f_2) + \sum_{j=1}^t m_3^j(g_3^j) \\ &= \ell_1(g_1^i)f_2 - g_1^i \cdot_T \ell_2(f_2). \end{aligned}$$

□

As previously mentioned, in the case that each $G_\bullet^i$ has an additional DG-module structure, some of the products of Theorem 6.2.3 may be made more explicit.

Proposition 6.2.4. *Adopt notation and hypotheses as in the statement of Theorem 6.2.3, and assume that $G_\bullet^i$ admits the structure of a DG-module over each $G_\bullet^j$ for all $1 \leqslant j \leqslant t$. Then, the following products can be extended to a DG-algebra structure on $T_\bullet$:*

$$F_1' \otimes G_1^i \to F_2 \oplus \left(\bigoplus_{j=1}^t G_2^j\right)$$

$$f_1 \cdot_T g_1^i := m_1^i(g_1^i)e_0^i \cdot_F f_1 + \sum_{j=1}^t g_1^i \cdot_{G^j} q_1^j(e_0^i \cdot_F f_1),$$

$$G_1^i \otimes G_1^j \to F_2 \oplus \left(\bigoplus_{k=1}^t G_2^k\right) \qquad i < j$$

$$g_1^i \cdot_T g_1^j := m_1^i(g_1^i)m_1^j(g_1^j)e_0^i \cdot_F e_0^j + \sum_{k=1}^t g_2^k,$$

$$\text{where } g_2^k = \begin{cases} m_1^j(g_1^j)g_1^i \cdot_{G^i} q_1^i(e_0^i \cdot_F e_0^j) & \text{if } k = i \\ m_1^i(g_1^i)g_1^j \cdot_{G^j} q_1^j(e_0^i \cdot_F e_0^j) & \text{if } k = j \\ m_1^i(g_1^i)g_1^j \cdot_{G^k} q_1^k(e_0^i \cdot_F e_0^j) & \text{otherwise,} \end{cases}$$

$$F_1' \otimes G_2^i \to F_3 \oplus \left(\bigoplus_{j=1}^t G_3^j\right)$$

$$f_1 \cdot_T g_2^i := -\sum_{j=1}^t g_2^i \cdot_{G^j} q_1^j(e_0^i \cdot_F f_1),$$

$$G_1^i \otimes G_2^j \to F_3 \oplus \left(\bigoplus_{k=1}^t G_3^k\right)$$

$$g_1^i \cdot_T g_2^j := \begin{cases} -\sum_{k \neq i} m_1^i(g_1^i)g_2^j \cdot_{G^k} q_1^k(e_0^i \cdot_F e_0^j) & \text{if } i < j \\ -m_1^i(g_1^i)g_2^j \cdot_{G^j} q_1^j(e_0^i \cdot_F e_0^j) & \text{if } j < i, \end{cases}$$

Remark 6.2.5. Observe that the assumption that $G_\bullet$ is a DG-module over each $G_\bullet^j$ is satisfied if $G_\bullet^i = G_\bullet^j$ for all $1 \leqslant i,\ j \leqslant t$.

Proof. In order to show that the products in the statement of the Proposition are well-defined and may be extended to a product on all of $T_\bullet$, one only needs to verify the identities in the statement and proof of Theorem 6.2.3. The verification is split into all 4 cases:

Case 1: $F_1' \otimes G_1^i \to F_2 \oplus \left(\bigoplus_{j=1}^t G_2^j\right)$. One has:

$$\begin{aligned} m_2^j(g_1^i \cdot_{G^j} q_1^j(e_0^i \cdot_F f_1)) &= m_1^i(g_1^i)q_1^j(e_0^i \cdot_F f_1) - g_1^i \cdot \pi^j(d_2(e_0 \cdot_F f_1)) \\ &= m_1^i(g_1^i)q_1^j(e_0^i \cdot_F f_1) + \begin{cases} d_1(f_1)g_1^i & \text{if } i = j \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Case 2: $G_1^i \otimes G_1^j \to F_2 \oplus \left(\bigoplus_{k=1}^t G_2^k \right)$. One computes:

$$\begin{aligned}
m_2^i(g_2^i) &= m_1^j(g_1^j)m_1^i(g_1^i)q_1^i(e_0^i \cdot_F e_0^j) - m_1^j(g_1^j)g_1^i \cdot \pi^i(d_2(e_0^i \cdot_F e_0^j)) \\
&= m_1^j(g_1^j)m_1^i(g_1^i)q_1^i(e_0^i \cdot_F e_0^j) + m_1^j(g_1^j)d_1(e_0^j)g_1^i, \\
m_2^j(g_2^j) &= m_1^i(g_1^i)m_1^j(g_1^j)q_1^j(e_0^i \cdot_F e_0^j) - m_1^i(g_1^i)g_1^j \cdot \pi^j(d_2(e_0^i \cdot_F e_0^j)) \\
&= m_1^i(g_1^i)m_1^j(g_1^j)q_1^j(e_0^i \cdot_F e_0^j) - m_1^i(g_1^i)d_1(e_0^i)g_1^j, \\
m_2^k(g_2^k) &= m_1^i(g_1^i)m_1^j(g_1^j)q_1^k(e_0^i \cdot_F e_0^j) - m_1^i(g_1^i)g_1^j \cdot \pi^k(d_2(e_0^i \cdot_F e_0^j)) \\
&= m_1^i(g_1^i)m_1^j(g_1^j)q_1^j(e_0^i \cdot_F e_0^j).
\end{aligned}$$

Case 3: $F_1' \otimes G_2^i \to F_3 \oplus \left(\bigoplus_{j=1}^t G_3^j \right)$. The identity 6.2.1 must be verified. One computes:

$$\begin{aligned}
m_3^j(-g_3^j) &= -m_2^i(g_2^i) \cdot_{G^j} q_1^j(e_0^i \cdot_F f_1) - g_2^i \cdot \pi^j(d_2(e_0^i \cdot_F f_1)) \\
&= d_1(f_1)g_2^i - m_2^i(g_2^i) \cdot_{G^j} q_1^j(e_0^i \cdot_F f_1) \quad \text{if } i = j, \\
&= -m_2^i(g_2^i) \cdot_{G^j} q_1^j(e_0^i \cdot_F f_1), \quad \text{otherwise.}
\end{aligned}$$

Case 4: $G_1^i \otimes G_2^j \to F_3 \oplus \left(\bigoplus_{k=1}^t G_3^k \right)$. The identities of 6.2.2 must be verified. One computes:

$$\begin{aligned}
m_3^k(m_1^i(g_1^i)g_2^j \cdot_{G^k} q_1^k(e_0^i \cdot_F e_0^j)) &= m_1^i(g_1^i)m_2^j(g_2^j) \cdot_{G^k} q_1^k(e_0^i \cdot_F e_0^j) \\
&\quad + \begin{cases} m_1^i(g_1^i)d_1(e_0^i)g_2^j & \text{if } k = j \\ 0 & \text{otherwise} \end{cases} \quad \text{if } i < j \\
m_3^j(m_1^i(g_1^i)g_2^j \cdot_{G^j} q_1^j(e_0^i \cdot_F e_0^j)) &= m_1^i(g_1^i)m_2^j(g_2^j) \cdot_{G^j} q_1^j(e_0^i \cdot_F e_0^j) \\
&\quad + m_1^i(g_1^i)d_1(e_0^j)g_2^j \\
&= g_1^i \cdot_T m_2^j(g_2^j) + m_1^i(g_1^i)d_1(e_0^j)g_2^j \quad \text{if } j < i.
\end{aligned}$$

This completes the proof. $\square$

Notation 6.2.6. *Let $(R, \mathfrak{m}, k)$ denote a regular local ring. Let $\bar{\cdot}$ denote the functor $\cdot \otimes_R k$.*

The following corollaries are immediate consequences of the statement and proof of Theorem 6.2.3.

Corollary 6.2.7. *Let $(R, \mathfrak{m}, k)$ denote a regular local ring. Assume that the complexes $F_\bullet$ and $G^i_\bullet$ (for $1 \leqslant i \leqslant t$) are minimal. Then the only possible nontrivial products in the algebra $\overline{T_\bullet}$ are*

$$\overline{F_1'} \cdot_T \overline{F_1'}, \quad \overline{F_1'} \cdot_T \overline{F_2}, \quad \overline{G_1^i} \cdot_T \overline{F_2}, \quad \textit{and} \quad \overline{F_1'} \cdot_T \overline{G_2^i}.$$

Corollary 6.2.8. *Let $(R, \mathfrak{m}, k)$ denote a regular local ring. Assume that the complexes $F_\bullet$ and $G^i_\bullet$ (for $1 \leqslant i \leqslant t$) are minimal. Then the map*

$$\begin{aligned}
&\overline{T_\bullet} \to \overline{F_\bullet} \\
&\overline{f_i} \mapsto \overline{f_i}, \qquad (f_i \in F_i), \\
&\overline{g_i^j} \mapsto 0, \qquad (g_i^j \in G_i^j),
\end{aligned}$$

is a homomorphism of k-algebras.

6.3 Consequences for Tor Algebra Structures

In this section, we take advantage of Corollary 6.2.7 and study how the process of trimming an ideal affects the Tor-algebra class. As in turns out, if the multiplication between certain homological degrees has coefficients appearing in sufficiently high powers of the maximal ideal, then the Tor-algebra class will be preserved. We also give explicit examples showing that if this assumption is not satisfied, then it is possible to obtain new nontrivial multiplication in the associated trimming complex.

We begin with the Tor-algebra classification provided by Avramov, Kustin, and Miller in [4].

Theorem 6.3.1 ([4], Theorem 2.1)**.** *There are nonnegative integers p, q, and r and bases $\{f_1^i\}$, $\{f_2^i\}$, and $\{f_3^i\}$ for $\operatorname{Tor}_1^R(R/I,k)$, $\operatorname{Tor}_2^R(R/I,k)$, and $\operatorname{Tor}_3^R(R/I,k)$, respectively, such that the multiplication in $\operatorname{Tor}_+^R(R/I,k)$ is given by one of the following:*

$$
\begin{aligned}
CI: &\qquad f_2^1 = f_1^2 f_1^3,\ f_2^2 = f_1^3 f_1^1,\ f_2^3 = f_1^1 f_1^2 \\
&\qquad f_1^i f_2^j = \delta_{ij} f_3^1 \text{ for } 1 \leqslant i,\ j \leqslant 3 \\
TE: &\qquad f_2^1 = f_1^2 f_1^3,\ f_2^2 = f_1^3 f_1^1,\ f_2^3 = f_1^1 f_1^2 \\
B: &\qquad f_1^1 f_1^2 = f_2^3,\ f_1^1 f_2^1 = f_3^1, f_1^2 f_2^2 = f_3^1 \\
G(r): &\qquad f_1^i f_2^i = f_3^1,\ 1 \leqslant i \leqslant r \\
H(p,q): &\qquad f_1^{p+1} f_1^i = f_2^i,\ 1 \leqslant i \leqslant p \\
&\qquad f_1^{p+1} f_2^{p+i} = f_3^i,\ 1 \leqslant i \leqslant q
\end{aligned}
$$

Remark 6.3.2. In terms of the tuples presented in the Realizability Question 6.1.4, the classes $G(r)$ and $H(p,q)$ have:

$$
\begin{aligned}
\mathrm{G}(r): &\qquad (m,n,0,1,r) \\
\mathrm{H}(p,q): &\qquad (m,n,p,q,q).
\end{aligned}
$$

For this reason, a tuple coming from a class $G(r)$ ring will often be shortened to (m,n,r), and a tuple coming from a class $H(p,q)$ ring will be shortened to (m,n,p,q). These tuples will be referred to as *the associated tuple.*

The following definition is introduced out of convenience for stating the results appearing later in this section.

Definition 6.3.3. A Tor-algebra is in *standard form* if the basis elements have been chosen such that the multiplication is given by one of the possibilities of Theorem 6.3.1. A DG-algebra free resolution $F_\bullet$ is in standard form if the multiplication descends to a Tor-algebra in standard form after applying $-\otimes_R k$.

Example 6.3.4. Let $R = k[x_{ij} \mid i < j]$ and $X = (x_{ij})$ be a generic $n \times n$ skew symmetric matrix, with n odd. Given an indexing set $I = (i_1 < \cdots < i_k)$, let

$$\mathrm{Pf}_I(X) := \text{pfaffian of } X \text{ with rows and columns from } I \text{ removed.}$$

Define

$$d_1 := (\mathrm{Pf}_1(X), -\mathrm{Pf}_2(X), \dots, (-1)^{i+1}\mathrm{Pf}_i(X), \dots, \mathrm{Pf}_n(X)),$$

and consider the complex

$$F_\bullet : \qquad 0 \longrightarrow R \xrightarrow{d_1^*} R^n \xrightarrow{X} R^n \xrightarrow{d_1} R \longrightarrow 0 .$$

Then $F_\bullet$ admits the structure of an associative DG-algebra with the following products:

$$f_1^i \cdot_F f_1^j = \sum_{k=1}^{n} (-1)^{i+j+k} \mathrm{Pf}_{ijk}(X) f_2^k,$$
$$f_1^i \cdot_F f_2^j = \delta_{ij} f_3^1.$$

If $n \geqslant 5$, then $F_\bullet$ is in standard form since it descends to a Tor-algebra of class $G(n)$ in standard form.

Notation 6.3.5. *If $A = A_3 \oplus A_2 \oplus A_1 \oplus k$ is a finite dimensional graded-commutative k-algebra, then*

$$A_2^\perp := \{a \in A_1 \mid a \cdot A_2 = 0\}.$$

The following Lemma is a coordinate free characterization for length 3 k-algebras realizing certain types of algebra classes.

Lemma 6.3.6 ([4], Lemma 2.3)**.** *Suppose $A = A_3 \oplus A_2 \oplus A_1 \oplus k$ is a finite dimensional graded-commutative k-algebra with $A_1^2 = 0$. Then:*

(a) A has form $H(0,0)$ if and only if $A_1 \cdot A_2 = 0$.

(b) A has form $H(0,q)$ for some $q \geqslant 1$ if and only if $\operatorname{codim} A_2^\perp = 1$ and $\dim A_1 \cdot A_2 = q$.

(c) A has form $G(r)$ for some $r \geqslant 2$ if and only if $\dim A_1 \cdot A_2 = 1$ *and* $\operatorname{codim} A_2^{\perp} = r$.

Notation 6.3.7. *Let $I = (\phi_1, \dots, \phi_n) \subseteq R$ be an $\mathfrak{m}$-primary ideal and $F_\bullet$ a DG-algebra free resolution of R/I in standard form. Given an indexing set $\sigma = \{1 \leqslant \sigma_1 < \cdots < \sigma_t \leqslant n\}$, define*

$$\operatorname{tm}_\sigma(I) := (\phi_i \mid i \notin \sigma) + \mathfrak{m}(\phi_j \mid j \in \sigma).$$

The transformation $I \mapsto \operatorname{tm}_\sigma(I)$ will be referred as trimming *the ideal I.*

The following setup will be in effect for the remainder of this section.

Setup 6.3.8. *Let $(R, \mathfrak{m}, k)$ denote a regular local ring of dimension 3. Let $I = (\phi_1, \dots, \phi_n) \subseteq R$ be an ideal and $\sigma = (1 \leqslant \sigma_1 < \cdots < \sigma_t \leqslant n)$ be an indexing set. Let $(F_\bullet, d_\bullet)$ and $(K_\bullet, m_\bullet)$ be minimal DG-algebra free resolutions and R/I and k, respectively. By Theorem 2.2.4, a free resolution of $R/\operatorname{tm}_\sigma(I)$ may be obtained as the mapping cone of a morphism of complexes of the form:*

$$\begin{tikzcd}
 & 0 \arrow[r] & F_3 \arrow[r, "d_3"] \arrow[d, "Q_2"] & F_2 \arrow[r, "d_2'"] \arrow[d, "Q_1"] & F_1' \arrow[d, "d_1"] \\
\bigoplus_{i=1}^t K_3 \arrow[rr, "\bigoplus_{i=1}^t m_3"] & & \bigoplus_{i=1}^t K_2 \arrow[r, "\bigoplus_{i=1}^t m_2"] & \bigoplus_{i=1}^t K_1 \arrow[r, "-\sum_{i=1}^t m_1(-) d_1(e_0^{\sigma_i})"] & R,
\end{tikzcd} \tag{6.3.1}$$

where

$$F_1' := \bigoplus_{j \notin \sigma} R e_0^j \quad \text{and} \quad d_2' : F_2 \xrightarrow{d_2} F_1 \xrightarrow{proj} F_1'.$$

Let $T_\bullet$ denote the mapping cone of 6.3.1.

The following Lemma says that in the context of Setup 6.3.8, the possible non-trivial multiplications in the homology algebra are even more restricted than that of Corollary 6.2.7.

Lemma 6.3.9. *Adopt notation and hypotheses as in Setup 6.3.8 and assume that $I \subseteq \mathfrak{m}^2$. Then,*

$$\overline{F_1} \cdot_F \overline{F_1} \subseteq \operatorname{Ker}(Q_1 \otimes k)$$

$$\overline{F_1} \cdot_F \overline{F_2} \subseteq \operatorname{Ker}(Q_2 \otimes k)$$

In particular, the only possible nontrivial products in the algebra $\overline{T_\bullet}$ are given by

$$\overline{F_1'} \cdot_T \overline{F_1'} \quad \textit{and} \quad \overline{F_1'} \cdot_T \overline{F_2}.$$

Proof. One has:

$$\begin{aligned} & d_2(F_1 \cdot_F F_1) \subseteq \mathfrak{m}^2 F_1 \\ \Longrightarrow\ & m_1^i \circ q_1(F_1 \cdot_F F_1) = {d_0'}^i(F_1 \cdot_F F_1) \subseteq \mathfrak{m}^2 \\ \Longrightarrow\ & q_1^i(F_1 \cdot_F F_1) \subseteq \mathfrak{m} K_1 \text{ for all } i. \end{aligned}$$

Likewise,

$$\begin{aligned} q_1^i(d_3(F_1 \cdot_F F_2)) &\subseteq \mathfrak{m} \cdot q_1(F_1 \cdot_F F_1) \\ &\subseteq \mathfrak{m}^2 G_1^i \\ \Longrightarrow\ m_2(q_2^i(F_1 \cdot_F F_2)) &= q_1(d_3(F_1 \cdot_F F_2)) \subseteq \mathfrak{m}^2 G_1^i \\ \Longrightarrow\ q_2^i(F_1 \cdot_F F_2) &\subseteq \mathfrak{m} K_2 \text{ for all } i. \end{aligned}$$

The latter claim about the triviality of the products $\overline{K_1} \cdot_T \overline{F_2}$ and $\overline{F_1'} \cdot_T \overline{K_2}$ follows immediately from the definition of the products given in Theorem 6.2.3 along with the identities 6.2.1 and 6.2.3. □

The following lemma makes precise the previously mentioned fact that, under mild hypotheses, the Tor-algebra class $G(r)$ is preserved by trimming.

Lemma 6.3.10. *Adopt notation and hypotheses as in Setup 6.3.8 and assume that $F_\bullet$ is in standard form with $F_1 \cdot_F F_1 \subseteq \mathfrak{m}^2 F_2$. If R/I defines a ring of class $G(r)$, then for all indexing sets σ, the ideal $\operatorname{tm}_\sigma(I)$ defines either a Golod ring or a ring of Tor algebra class $G(r')$, for some $r - |\sigma \cap [r]| - \operatorname{rank}(Q_1 \otimes k) \leqslant r' \leqslant r - |\sigma \cap [r]|$.*

Proof. By Lemma 6.3.9, f_3^1 may be chosen as part of a basis for $\operatorname{Ker}(Q_2 \otimes k)$; the parameter r' arises from counting the rank of the induced map

$$\operatorname{Ker}(Q_1 \otimes k) \to \operatorname{Hom}_k(\overline{F_1'}, \overline{F_3}).$$

By definition of the product on $T_\bullet$, the assumption $F_1 \cdot_F F_1 \subseteq \mathfrak{m}^2 F_2$ implies that

$$\overline{f_1} \cdot_T \overline{f_1'} = \overline{f_1} \cdot_F \overline{f_1'} = 0 \text{ for all } f_1, f_1' \in F_1'.$$

Thus, the induced map $\overline{F_2} \to \operatorname{Hom}_k(\overline{F_1'}, \overline{F_3})$ has rank $r - |\sigma \cap [r]|$. But this map may be written as the composition

$$\operatorname{Ker}(Q_1 \otimes k) \hookrightarrow \overline{F_2} \to \operatorname{Hom}_k(\overline{F_1'}, \overline{F_3}),$$

whence one finds that $r - |\sigma \cap [r]| - \operatorname{rank}(Q_1 \otimes k) \leqslant r' \leqslant r - |\sigma \cap [r]|$. $\square$

Corollary 6.3.11. *Adopt notation and hypotheses as in the statement of Lemma 6.3.10. If $d_2(F_2) \subseteq \mathfrak{m}^2 F_1$, then $\operatorname{tm}_\sigma(I)$ defines a ring of Tor-algebra class $G(r - |\sigma \cap [r]|)$.*

Proof. The assumption $d_2(F_2) \subseteq \mathfrak{m}^2 F_1$ implies that $Q_1 \otimes k = 0$. $\square$

The following example shows that the assumption $F_1 \cdot_F F_1 \subseteq \mathfrak{m}^2 F_2$ in Lemma 6.3.10 is necessary.

Example 6.3.12. Let $R = k[x_1, x_2, x_3]$ and

$$X := \begin{pmatrix} 0 & 0 & 0 & x_1 & x_2 \\ 0 & 0 & x_1 & x_2 & x_3 \\ 0 & -x_1 & 0 & x_3 & 0 \\ -x_1 & -x_2 & -x_3 & 0 & 0 \\ -x_2 & -x_3 & 0 & 0 & 0 \end{pmatrix}.$$

Let $I = \operatorname{Pf}(X) = (x_3^2,\, -x_2x_3,\, x_2^2 - x_1x_3,\, -x_1x_2,\, x_1^2)$, the ideal of 4×4 pfaffians of X. The ring R/I has Tor-algebra class $G(5)$ and the multiplication satisfies $F_1 \cdot F_1 \subseteq \mathfrak{m} F_2$

and $F_1 \cdot F_1 \not\subseteq \mathfrak{m}^2 F_2$. It may be shown using Macaulay2 [26] that $\mathrm{tm}_1(I)$ defines a ring of Tor-algebra class B and $\mathrm{tm}_2(I)$ defines a ring of Tor-algebra class $H(3,2)$. Both of these Tor-algebras have nontrivial multiplication of elements in homological degree 1, which shows that the multiplication on $\overline{T_\bullet}$ cannot possibly agree with the multiplication on $\overline{F_\bullet}$.

It turns out that trimming also tends to preserve the Tor-algebra class $H(p,q)$:

Lemma 6.3.13. *Adopt notation and hypotheses as in Setup 6.3.8 and assume that $F_\bullet$ is in standard form of class $H(p,q)$ with the property that*

$$f_1^i \cdot_F f_1^j \in \mathfrak{m}^2 F_2 \textit{ for all } i,j \neq p+1.$$

Then,

(i) if $p+1 \in \sigma$, $\mathrm{tm}_\sigma(I)$ defines a Golod ring, and

(ii) if $p+1 \notin \sigma$, then $\mathrm{tm}_\sigma(I)$ defines either a Golod ring or a ring of class $H(p - |\sigma \cap [p]|, q')$, where $q - \mathrm{rank}(Q_1 \otimes k) \leqslant q' \leqslant q$.

Proof. In an identical manner to the proof of Lemma 6.3.10, the assumption $f_1^i \cdot_F f_1^j \in \mathfrak{m}^2 F_2$ implies

$$\overline{f_1^i} \cdot_T \overline{f_1^j} = \overline{f_1^i} \cdot_F \overline{f_1^j} = 0 \quad \text{for all } i,j \neq p+1.$$

Case 1: $p+1 \in \sigma$. Since $f_1^{p+1} \notin F_1'$, it follows that $\overline{F_1'} \cdot_T \overline{F_1'} = 0$. For identical reasons, $\overline{F_1'} \cdot_T \overline{F_2} = 0$. Thus, the multiplication in the Tor-algebra is trivial, so R/I is a Golod ring.

Case 2: $p+1 \notin \sigma$. By Lemma 6.3.9, $\overline{f_2^1}, \dots, \overline{f_2^p}$ may be chosen as part of a basis of $\mathrm{Ker}(Q_1 \otimes k)$. This immediately implies that the only nontrivial products in $\overline{T_\bullet}$ are of the form

$$\overline{f_1^i} \cdot_T \overline{f_1^{p+1}} = \overline{f_2^i} \text{ for } 1 \leqslant i \leqslant p, \ i \notin \sigma.$$

Thus, $\dim_k \overline{F_1'} \cdot_T \overline{F_1'} = p - |\sigma \cap [p]|$. Likewise, $\overline{f_3^{p+1}}, \ldots, \overline{f_3^{p+q}}$ may be chosen as part of a basis for $\operatorname{Ker}(Q_2 \otimes k)$. Moreover, the rank of the induced map

$$\operatorname{Ker}(Q_1 \otimes k) \to \operatorname{Hom}_k(\overline{F_1'}, \overline{F_3})$$

is at least $q - \operatorname{rank}(Q_1 \otimes k)$, and it is evidently at most q. □

Corollary 6.3.14. *Adopt notation and hypotheses as in the statement of Lemma 6.3.13. If $d_2(F_2) \subseteq \mathfrak{m}^2 F_2$, then $\operatorname{tm}_\sigma(I)$ is either Golod or defines a ring of Tor-algebra class $H(p - |\sigma \cap [p]|, q)$.*

Again, the assumption that $f_1^i \cdot_F f_1^j \in \mathfrak{m}^2 F_2$ for $i, j \neq p+1$ in Lemma 6.3.13 is necessary, as the following example shows.

Example 6.3.15. Let $R = k[x_1, x_2, x_3]$ and $I = (x_2^2 - x_1x_3, \ -x_1x_2, \ x_1^2, \ x_3^2)$. The ring R/I has Tor-algebra class $H(3,2)$, and the multiplication on the minimal free resolution $F_\bullet$ of R/I satisfies $f_1^i \cdot f_1^j \in \mathfrak{m}F_2$, $f_1^i \cdot f_2^j \notin \mathfrak{m}^2 F_2$, where $i, \ j \neq 5$. However, it can be shown using Macaulay2 [26] that $\operatorname{tm}_2(I)$ has Tor-algebra class TE.

6.4 Examples

In this section, we employ the theory developed in Section 6.3 for the construction of explicit examples of rings realizing Tor-algebra classes $G(r)$ and $H(p,q)$ for a given set of parameters (m,n,p,q,r) (as in Setup 6.1.3 and Question 6.1.4). These examples are constructed in an arbitrary regular local ring $(R, \mathfrak{m}, k)$; in particular, we will construct explicit novel examples of ideals defining rings of Tor-algebra class $G(r)$ (and arbitrarily large type). One can combine the examples of this section with the results of Proposition 6.1.6 and Section 6.3 to obtain an even larger class of tuples.

We begin by first adopting the following simple setup.

Setup 6.4.1. *Let $(R, \mathfrak{m}, k)$ denote a regular local ring of dimension 3 (or a standard graded polynomial ring over a field). Let $\mathfrak{m} = (x_1, x_2, x_3)$, where x_1, x_2, x_3 is a regular* *sequence. Let $K_1 := Re_1 \oplus Re_2 \oplus Re_3$ and $K_\bullet$ denote the Koszul complex induced by the map sending $e_i \mapsto x_i$.*

The matrices appearing in the following two definitions were inspired by matrices constructed in [16] and further generalized in [39]. Here, we extend this definition to arbitrary local rings for the construction of our examples.

Definition 6.4.2. Adopt notation and hypotheses as in Setup 6.4.1. Let U_m^j (for $j \leqslant m$) denote the $m \times m$ matrix with entries from R defined by:

$$(U_m^j)_{i,m-i} = x_1^2, \quad (U_m^j)_{i,m-i+1} = x_3^2, \quad (U_m^j)_{i,m-i+2} = x_2^2 \text{ for } i \leqslant m-j$$

$$(U_m^j)_{i,m-i} = x_1, \quad (U_m^j)_{i,m-i+1} = x_3, \quad (U_m^j)_{i,m-i+2} = x_2 \text{ for } i > m-j$$

and all other entries are defined to be 0.

To see the pattern, observe that:

$$U_2^1 = \begin{pmatrix} x_1^2 & x_3^2 \\ x_3 & x_2 \end{pmatrix}, \; U_3^1 = \begin{pmatrix} 0 & x_1^2 & x_3^2 \\ x_1^2 & x_3^2 & x_2^2 \\ x_3 & x_2 & 0 \end{pmatrix}, \; U_3^2 = \begin{pmatrix} 0 & x_1^2 & x_3^2 \\ x_1 & x_3 & x_2 \\ x_3 & x_2 & 0 \end{pmatrix}$$

Definition 6.4.3. Define V_m^j (for $j < m$) to be the $(2m+1) \times (2m+1)$ skew symmetric matrix

$$V_m^j := \begin{pmatrix} O & O_{x_1^2} & (U_m^j)^T \\ -(O_{x_1^2})^T & 0 & {}^{x_2^2}O \\ -U_m^j & -({}^{x_2^2}O)^T & O \end{pmatrix}.$$

If $j = m$, then V_m^m is the skew symmetric matrix

$$V_m^m := \begin{pmatrix} O & O_{x_1^2} & (U_m^m)^T \\ -(O_{x_1^2})^T & 0 & {}^{x_2}O \\ -U_m^m & -({}^{x_2}O)^T & O \end{pmatrix}.$$

Lastly, if $j = m+1$, then V_m^{m+1} is the skew symmetric matrix

$$V_m^{m+1} := \begin{pmatrix} O & O_{x_1} & (U_m^m)^T \\ -(O_{x_1})^T & 0 & {}^{x_2}O \\ -U_m^m & -({}^{x_2}O)^T & O \end{pmatrix}.$$

Definition 6.4.4. Let $m \geqslant 2$ be an integer. Define the ideal I_m^j (for $0 \leqslant j \leqslant m+1$) by

$$I_m^j := \mathrm{Pf}(V_m^j),$$

where $\mathrm{Pf}(V_m^j)$ denotes the ideal of $2m \times 2m$ pfaffians of V_m^j.

Setup 6.4.5. *Adopt notation and hypotheses as in Setup 6.4.1. Define*

$$d_1 := (Pf_1(V_m^j), -Pf_2(V_m^j), \ldots, (-1)^{i+1}Pf_i(V_m^j), \ldots, Pf_n(V_m^j)),$$

(for $m \geqslant 2$ and $0 \leqslant j \leqslant m+1$) and consider the complex

$$F_\bullet : \qquad 0 \longrightarrow R \xrightarrow{d_1^*} R^n \xrightarrow{V_m^j} R^n \xrightarrow{d_1} R \longrightarrow 0\,.$$

Recall that $F_\bullet$ is a minimal free resolution of R/I_m^j in standard form of class $G(2m+1)$ with product as in Example 6.3.4.

Proposition 6.4.6. *Adopt notation and hypotheses as in Setup 6.4.5. Define q_1^i : $F_2 \to K_1$ by sending:*

$$f_2^{2m+3-i} \mapsto \begin{cases} e_2 & \text{if } 1 < i \leqslant j+1 \leqslant m+1 \\ x_2e_2 & \text{if } j+1 < i \leqslant m+1 \\ -x_2e_2 & \text{if } m+1 < i \leqslant 2m+1-j \\ -e_2 & \text{if } 2m+1-j < i \leqslant 2m+1, \end{cases}$$

$$f_2^{2m+2-i} \mapsto \begin{cases} e_3 & \textit{if } 1 \leqslant i \leqslant j,\ i < m+1 \\ x_3e_3 & \textit{if } j < i < m+1 \\ 0 & \textit{if } i = m+1 \\ -x_3e_3 & \textit{if } m+1 < i \leqslant 2m+1-j \\ -e_3 & \textit{if } 2m+1-j < i \leqslant 2m+1, \end{cases}$$

$$f_2^{2m+1-i} \mapsto \begin{cases} e_1 & \textit{if } 1 \leqslant i \leqslant j-1 \\ x_1e_1 & \textit{if } j-1 < i < m+1 \\ -x_1e_1 & \textit{if } i = m+1,\ j < m+1 \\ -e_1 & \textit{if } i = m+1,\ j = m+1 \\ -x_1e_1 & \textit{if } m+1 < i \leqslant 2m+1-j \\ -e_1 & \textit{if } 2m+1-j < i < 2m+1, \end{cases}$$

and all other basis elements are sent to 0. *Then the following diagram commutes:*

$$\begin{tikzcd} F_2 \arrow[d, "q_1^i"'] \arrow[dr, "{d'_0}^i"] & \\ K_1 \arrow[r, "m_1"] & \mathfrak{m}. \end{tikzcd}$$

Notation 6.4.7. *Let a and b be positive integers with $a < b$. The notation $[a]$ will denote the set $\{1, 2, \ldots, a-1, a\}$ and the notation $[a, b]$ will denote the set $\{a, a+1, \ldots, b-1, b\}$.*

Corollary 6.4.8. *Adopt notation and hypotheses as in Setup 6.4.5. Let $\sigma = (1 \leqslant \sigma_1 < \cdots < \sigma_t \leqslant 2m+1)$ denote an indexing and assume:*

(a) $m \geqslant 3$, *or*

(b) $m = 2$ *and* $j = 0$.

Then:

1. *The ideal* $\mathrm{tm}_i(I_m^j)$ *defines either a Golod ring or a ring of Tor algebra class* $G(2m+1-t-\mathrm{rank}(q_1^i \otimes k))$.

2. *If* $t \leqslant 2m+1-j$, *then the ideal* $\mathrm{tm}_{[t]}(I_m^j)$ *defines either a Golod ring or a ring of Tor-algebra class* $G(2m+1-t-\min\{1+t,j\})$.

3. *More generally, the ideal* $\mathrm{tm}_\sigma(I_m^j)$ *defines either a Golod ring or a ring of Tor-algebra class*

$$G(2m+1-t-\mathrm{rank}(Q_1 \otimes k) + |\sigma \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\}|)$$

Proof. The assumptions (a) and (b) ensure that $F_1 \cdot F_1 \subseteq \mathfrak{m}^2 F_2$, so that by Lemma 6.3.10, $\mathrm{tm}_\sigma(I_m^j)$ defines either a Golod ring or a ring of class $G(r')$, where $2m - \mathrm{rank}(q_1^i \otimes k) \leqslant r' \leqslant 2m$. By construction, $\mathrm{Ker}(Q_1 \otimes k)$ for each i has basis given by

$$\{\overline{f_2^j} \in \overline{F_2} \mid \overline{Q_1(f_2^j)} = 0\}.$$

Case 1: By the above, one immediately has that $\mathrm{rank}\Big(\mathrm{Ker}(Q_1 \otimes k) \to \mathrm{Hom}_k(\overline{F_1'}, \overline{F_3})\Big) = 2m - \mathrm{rank}(q_1^i \otimes k)$. This is because (by Proposition 6.4.6)

$$i \notin \{j \mid \overline{Q_1(f_2^j)} \neq 0\}.$$

Case 2: Using Proposition 6.4.6, one finds that $\mathrm{rank}(Q_1 \otimes k) = \min\{1+r, j\}$. Moreover, since $t < 2m+1-j$,

$$[t] \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\} = \varnothing,$$

whence $\mathrm{rank}\Big(\mathrm{Ker}(Q_1 \otimes k) \to \mathrm{Hom}_k(\overline{F_1'}, \overline{F_3})\Big) \leqslant 2m+1-t-\min\{1+t,j\}$. By Lemma 6.3.10, one has equality.

Case 3: Define $S := \sigma \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\}$. If $j \in S$, then in $T_\bullet$, the direct summand Rf_1^j has been omitted from F_1. Thus, removal of the direct summand generated by f_2^j has no effect on the rank of the induced map

$$\delta : \mathrm{Ker}(Q_1 \otimes k) \to \mathrm{Hom}_k(\overline{F_1'}, \overline{F_3}).$$

By inclusion-exclusion, this implies that δ has rank

$$2m + 1 - t - \mathrm{rank}(Q_1 \otimes k) + |S|.$$

$\square$

Remark 6.4.9. Let $S := \sigma \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\}$. Then, in terms of the associated tuple (see Remark 6.3.2), the transformation $I_m^j \mapsto \mathrm{tm}_\sigma(I_m^j)$ transforms the tuple $(2m+1, 1, 2m+1)$ as so:

$$\begin{aligned} I_m^j &\mapsto \mathrm{tm}_\sigma(I_m^j) \\ &(2m+1, 1, 2m+1) \\ \mapsto &(2m + 2t + 1 - \mathrm{rank}(Q_1 \otimes k), 1 + t, 2m + 1 - t - \mathrm{rank}(Q_1 \otimes k) + |S|) \end{aligned}$$

Corollary 6.4.8 immediately allows us to fill in a large class of tuples:

Corollary 6.4.10. *Adopt notation and hypotheses as in Setup 6.3.8. Let (m, n, r) be a tuple of positive integers satisfying either:*

1. $m - r = 3(n-1)$, $n \geqslant 2$, *and* $n + r \geqslant 6$,
2. $m - r = 3(n-1) - 2$, $n \geqslant 3$, *and* $n + r \geqslant 6$,
3. $m - r = 3(n-2)$, $n \geqslant 4$, *and* $n + r \geqslant 7$.

Then there exists an ideal J defining a ring of Tor-algebra class $G(r)$ realizing this tuple.

Proof. **Case 1(a):** $n + r$ is even. Write $n + r = 2k + 2$ for some integer $k \geqslant 2$. Consider the ideal $\mathrm{tm}_{[n-1]}(I_k^0)$. By Corollary 6.4.8, this has the effect of transforming the associated tuple in the following way:

$$(2k+1, 1, 2k+1) \mapsto (2k + 1 + 2(n-1), 1 + n - 1, 2k + 1 - (n-1))$$

$$= (n + r + 2n - 3, n, n + r - 1 - n + 1)$$

$$= (m, n, r).$$

Case 1(b): $n + r$ is odd. Write $n + r = 2k + 1$ for some integer $k \geqslant 3$. Consider the ideal $\mathrm{tm}_{[n-1]}(I_k^1)$. By Corollary 6.4.8, this has the effect of transforming the associated tuple in the following way:

$$(2k + 1, 1, 2k + 1) \mapsto (2k + 1 + 2(n - 1) - 1, 1 + n - 1, 2k + 1 - (n - 1) - 1)$$

$$= (n + r + 2n - 3, n, n + r - n + 1 - 1)$$

$$= (m, n, r).$$

Case 2(a): $n + r$ is even. Write $n + r = 2k$ for some $k \geqslant 3$. Consider the ideal

$$\begin{cases} \mathrm{tm}_{1,2k+1}(I_k^2) & \text{if } n = 3, \\ \mathrm{tm}_{\{1,2k+1\} \cup [3,n-1]}(I_k^2) & \text{if } n \geqslant 4. \end{cases}$$

By Proposition 6.4.6, $\operatorname{rank}(Q_1 \otimes k) = 4$ and

$$\sigma \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\} = \{1, 2k + 1\}$$

so $|\sigma \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\}| = 2$. By Corollary 6.4.8, this has the effect of transforming the associated tuple as so:

$$(2k + 1, 1, 2k + 1) \mapsto (2k + 1 + 2(n - 1) - 4, n, 2k + 1 - (n - 1) - 4 + 2)$$

$$= (n + r + 2n - 5, n, n + r - 1 - (n - 1))$$

$$= (m, n, r).$$

Case 2(b): $n + r$ is odd. Write $n + r = 2k + 1$ for some $k \geqslant 3$. Consider the ideal

$$\begin{cases} \mathrm{tm}_{1,2k+1}(I_k^1) & \text{if } n = 3, \\ \mathrm{tm}_{\{1,2k_1\} \cup [3,n-1]}(I_k^1) & \text{if } n \geqslant 4. \end{cases}$$

By Proposition 6.4.6, $\operatorname{rank}(Q_1 \otimes k) = 3$, and exactly as in Case $2(a)$,

$$|\sigma \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\}| = |\{1, 2k+1\}| = 2.$$

By Corollary 6.4.8, this has the effect of transforming the associated tuple as so:

$$\begin{aligned}(2k+1, 1, 2k+1) &\mapsto (2k+1+2(n-1)-3, n, 2k+1-(n-1)-3+2)\\ &= (n+r+2n-5, n, n+r-1-(n-1))\\ &= (m, n, r).\end{aligned}$$

Case 3(a): $n+r$ is odd. Write $n+r = 2k+1$ for some $k \geqslant 3$. Consider the ideal $\operatorname{tm}_{[n-2],2k+1}(I_k^2)$. By Proposition 6.4.6, $\operatorname{rank}(Q_1 \otimes k) = 4$, and (recalling that $n \geqslant 4$)

$$\sigma \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\} = \{1, 2, 2k+1\},$$

so $|\sigma \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\}| = 3$. By Corollary 6.4.8, this has the effect of transforming the associated tuple as so:

$$\begin{aligned}(2k+1, 1, 2k+1) &\mapsto (2k+1+2(n-1)-4, n, 2k+1-(n-1)-4+3)\\ &= (n+r+2n-6, n, n+r-1-(n-1))\\ &= (m, n, r).\end{aligned}$$

Case 3(b): $n+r$ is even. Write $n+r = 2k+2$ for some $k \geqslant 3$. Consider the ideal $\operatorname{tm}_{[n-2],2k+1}(I_k^1)$. By Proposition 6.4.6, $\operatorname{rank}(Q_1 \otimes k) = 3$, and (recalling that $n \geqslant 4$)

$$|\sigma \cap \{j \mid \overline{Q_1(f_2^j)} \neq 0\}| = |\{1, 2, 2k+1\}| = 3$$

By Corollary 6.4.8, this has the effect of transforming the associated tuple as so:

$$\begin{aligned}(2k+1, 1, 2k+1) &\mapsto (2k+1+2(n-1)-3, n, 2k+1-(n-1)-3+3)\\ &= (n+r+2n-6, n, n+r-1-(n-1))\\ &= (m, n, r).\end{aligned}$$

□

Example 6.4.11. Corollary 6.4.10 is far from being an exhaustive list of the possible tuples (m,n,r). For example, let $R = k[x_1,x_2,x_3]$ and consider the ideal

$$J := (x_1^2x_3, x_1^2x_2 - x_3^3, x_2^2x_3^2, x_1x_2^2x_3, x_2^4, x_1x_2^3, x_1^4).$$

One may verify using the TorAlgebras package in Macaulay2 that J defines a ring of Tor-algebra class $G(2)$ and realizes the tuple $(7,2,2)$, which does not fall into any of the cases of Corollary 6.4.10.

As of yet, there is no standardized method for producing non-Gorenstein rings of Tor-algebra class $G(r)$ en masse besides trimming; because of this, the realizable classes covered by Corollary 6.4.10 are bound to be rather restricted. Next, we consider rings of class $H(p,q)$.

Setup 6.4.12. *Adopt notation and hypotheses as in Setup 6.4.1. Let X_p denote the $p \times (p-1)$ matrix*

$$X_p := \begin{pmatrix} x_1 & 0 & 0 & \cdots & 0 \\ x_2 & x_1 & 0 & \cdots & 0 \\ x_3 & x_2 & x_1 & \cdots & 0 \\ 0 & x_3 & x_2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & x_1 \\ 0 & 0 & \cdots & \cdots & x_2 \end{pmatrix}$$

and define

$$J_p := I_{p-1}(X_p) + (x_3^{p-1}) = (\Delta_1, \dots, \Delta_p, x_3^{p-1}).$$

Let $H_\bullet$ denote the Hilbert-Burch resolution of $R/I_{p-1}(X_p)$ and $G_\bullet := 0 \to R \xrightarrow{x_3^{p-1}} R$. The minimal free resolution of J_p may be obtained as the tensor product $F_\bullet := (H \otimes G)_\bullet$:

$$F_\bullet : \quad 0 \longrightarrow H_2 \otimes G_1 \longrightarrow \big(H_1 \otimes G_1\big) \oplus H_2 \longrightarrow H_1 \oplus G_1 \longrightarrow R.$$

The following multiplication makes $F_\bullet$ into an algebra resolution in standard form of Tor-algebra class $H(p, p-1)$:

$$h_1 \cdot_F h_1' = h_1 \cdot_H h_1',$$
$$h_1 \cdot_F g_1 = h_1 \otimes g_1,$$
$$h_2 \cdot_F g_1 = h_2 \otimes g_1,$$
$$\textit{where } h_1,\ h_1' \in H_1, h_2 \in H_2, g_1 \in G_1.$$

In an identical manner, let $F_\bullet'$ be a minimal algebra resolution of $J_p' := I_{p-1}(X_p') + (x_2^{2n-2})$ in standard from of Tor-algebra class $H(p, p-1)$, where

$$X_p' := \begin{pmatrix} x_1^2 & 0 & 0 & \cdots & 0 \\ x_2^2 & x_1^2 & 0 & \cdots & 0 \\ x_3^2 & x_2^2 & x_1^2 & \cdots & 0 \\ 0 & x_3^2 & x_2^2 & \cdots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & x_1^2 \\ 0 & 0 & \cdots & \cdots & x_2^2 \end{pmatrix}.$$

Proposition 6.4.13. *Adopt notation and hypotheses as in Setup 6.4.12. Assume $H_1 = \bigoplus_{i=1}^p Rh_1^i$, where $h_1^i \mapsto \Delta_i$. Define $q_1^i : F_2 \to K_1$ for $i < p$ by sending:*

$$h_2^{i-2} \mapsto e_3 \quad (i > 2),$$
$$h_2^{i-1} \mapsto e_2 \quad (i > 1),$$
$$h_2^{i} \mapsto e_1 \quad (i < p),$$
$$h_1^i \otimes g_1 \mapsto -x_3^{p-2} e_3,$$

and all other basis elements to 0*. If* $i = p+1$*, write each* $\Delta_j = x_1\Delta_{1,j} + x_2\Delta_{2,j} + x_3\Delta_{3,j}$*. Define* $q_1^{p+1} : F_2 \to K_1$ *by sending:*

$$h_1^j \otimes g_1 \mapsto \Delta_{1,j}e_1 + \Delta_{2,j}e_2 + \Delta_{3,j}e_3, \quad (1 \leqslant j \leqslant p)$$

and all other basis elements to 0*. Then the following diagram commutes:*

$$\begin{tikzcd} F_2 \arrow[d, "q_1^i"'] \arrow[dr, "{d'_0}^i"] & \\ K_1 \arrow[r, "m_1"] & \mathfrak{m}. \end{tikzcd}$$

Corollary 6.4.14. *Adopt notation and hypotheses as in Setup 6.4.12 with* $p \geqslant 4$*. Then*

1. *if* $p+1 \notin \sigma$*, the ideal* $\mathrm{tm}_\sigma(J_p)$ *defines either a Golod ring or a ring of Tor-algebra class* $H(p-t, p-1-\mathrm{rank}(Q_1 \otimes k)$*. In particular, if* $\sigma = [t]$ *for some* $t < p$*, the ideal* $\mathrm{tm}_{[t]}(J_p)$ *defines a ring of Tor-algebra class* $H(p-t, p-1-t)$*.*

2. *if* $p+1 \in \sigma$*, the ideal* $\mathrm{tm}_\sigma(I)$ *defines a Golod ring.*

3. *if* $p+1 \notin \sigma$*, the ideal* $\mathrm{tm}_\sigma(J'_p)$ *defines a ring of Tor-algebra class* $H(p-t, p-1)$*.*

4. *if* $p+1 \in \sigma$*, the ideal* $\mathrm{tm}_\sigma(J'_p)$ *defines a Golod ring.*

Proof. As in the proof of Corollary 6.4.8, one has

$$\mathrm{Ker}(Q_1 \otimes k) = \mathrm{Span}_k\{\overline{f_2^j} \in \overline{F_2} \mid \overline{Q_1(f_2^j)} = 0\}.$$

Moreover, the assumption that $p \geqslant 4$ implies that the hypotheses of Lemma 6.3.13 are satisfied.

Case 1: Since $\mathrm{Ker}(Q_1 \otimes k)$ is obtained by simply deleting basis elements with nonzero image under $Q_1 \otimes k$, it follows that $\mathrm{rank}(\mathrm{Ker}(Q_1 \otimes k) \to \mathrm{Hom}_k(\overline{F'_1}, \overline{F_3}) \leqslant r - \mathrm{rank}(Q_1 \otimes k)$. By Lemma 6.3.13, the result follows.

Case 2: This is simply case (i) of Lemma 6.3.13.

Cases 3 and 4: Observe that $F'_\bullet$ has the property that $d_2(F_2) \subseteq \mathfrak{m}^2 F_1$. Thus the conclusion follows by Corollary 6.3.11. $\square$

Remark 6.4.15. In terms of the tuples of the associated tuple (see Remark 6.3.2), the transformation $J_p \mapsto \mathrm{tm}_\sigma(J_p)$ transforms the tuple $(p+1, p-1, p, p-1)$ as so:

$$\begin{aligned} J_p &\mapsto \mathrm{tm}_\sigma(J_p) \\ (p+1, p-1, p, p-1) &\mapsto (p+1+2t-\mathrm{rank}(Q_1 \otimes k), p-1+t, p-t, p-1-\mathrm{rank}(Q_1 \otimes k)), \\ J'_p &\mapsto \mathrm{tm}_\sigma(J'_p) \\ (p+1, p-1, p, p-1) &\mapsto (p+1+2t, p-1+t, p-t, p-1). \end{aligned}$$

We conclude with some discussion on the problem of realizability. The results of Section 6.3 are stated for arbitrary rings of a given Tor-algebra class. However, one must start with a ring of a given Tor-algebra class and then apply the trimming process to obtain a new ideal with some new set of parameters. The only simple candidates for “initial" ideals of Tor-algebra class $G(r)$ and $H(p,q)$ are grade 3 Gorenstein ideals and grade 3 hyperplane sections, respectively. Even though using a combination of linkage and trimming can obtain *many* of the tuples falling within the bounds of Theorem 6.1.5, one is tempted to ask:

Question 6.4.16. Are there other “canonical" sources of rings of Tor-algebra class $G(r)$ and $H(p,q)$, distinct from grade 3 Gorenstein ideals or hyperplane sections?

Enlarging the set of starting ideals from which one can begin the process of linkage/trimming would immediately allow one to add to the question of realizability. As it turns out, rings of Tor-algebra class $G(r)$ arise generically when working in a polynomial ring. The examples arising in [40] are already obtained by trimming a Gorenstein ideal, but it is shown more generally in [13] that generically obtained rings of type 2 are of class $G(r)$ under appropriate hypotheses. To the author's knowledge, there are fewer results of this flavor for rings of Tor-algebra class $H(p, q)$, even though these rings seem to be ubiquitous.

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