Anisotropic Wave Behavior in Isotropic Material With Orthogonal Surface Perturbation

Khaleda Akter

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ANISOTROPIC WAVE BEHAVIOR IN ISOTROPIC MATERIAL WITH ORTHOGONAL SURFACE PERTURBATION

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DEDICATION

This work is dedicated to my parents, my husband and my beloved kids.
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I would like to express my deep gratitude and respect to my advisor Dr. Sourav Banerjee, associate professor and director of iMAPS lab, University of South Carolina for his guidance, invaluable suggestions, constructive criticisms and sharing his knowledge throughout this work. It has been a great privilege and honor for me to work with him. I would like to express my gratitude to my committee member Dr. Victor Giurgiutiu for his guidance and commenting on my work.

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ABSTRACT

Elastic wave propagation in anisotropic media is of great interest in various branches of engineering and applied sciences. In this study, an anisotropic wave propagation behavior in isotropic material with orthogonal surface perturbation is presented. The conventional method of estimating dispersion equations for isotropic material is to apply Helmholtz decomposition on the potential functions for Rayleigh-Lamb wave and Shear Horizontal (SH) waves. However, the presence of isotropic material with orthogonal surface perturbations in two coordinate directions significantly affect the wave propagation behavior due to its direction dependency, and hence, the Helmholtz decomposition of the potential functions cannot be applied to derive the dispersion equations. In this study, a generalized analytical expression for the Rayleigh-Lamb wave propagation in flat plate and a corrugated plate with orthogonal surface perturbation in two coordinate directions is developed by assuming the three potential functions introduced by Buchwald (1961) for anisotropic material. By setting the perturbation height to zero, the dispersion equations are solved using a logical root-finding algorithm for the flat plate and compared them with the results reported in literature. To validate the wave propagation behaviors in isotropic materials with orthogonal surface perturbated corrugated structure, a time domain simulation is performed by the Finite Element Method using a chirp signal to excite the corrugated plate. Finally, the displacements of the particles are obtained in multiple time steps and analyzed for wave propagation pattern at various points on the corrugated structure.
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LIST OF SYMBOLS

$\mathbb{C}_{ij}$  Constitutive Property Matrix of an Element

$\rho$  Density

$\varepsilon$  Permutation Symbol in vector algebra

$E$  Young’s Modulus of the materials used

$\nu$  Poisson’s Ratio

$n_j$  Direction Cosines in cartesian coordinate system

$\sigma_{ij}$  Stress developed due to applied force

$\varepsilon_{ij}$  Strain

$\Theta_l$  Potential Functions proposed by Buckwald

$\psi$  Vector Potentials in Helmholtz decomposition

$\phi$  Scaler Potentials in Helmholtz decomposition

$\mu, \lambda$  Lamé Constants for guided wave propagation

$\omega$  Angular Frequency
LIST OF ABBREVIATIONS

FEM .................................................................................................................... Finite Element Method

PML .................................................................................................................... Perfectly Matched Layer

SH ..................................................................................................................... Shear Horizontal

SV ..................................................................................................................... Shear Vertical
CHAPTER 1

INTRODUCTION

The general theory of wave propagation in both isotropic and anisotropic solid materials are well known for many years with significant contributions from scientists and researchers all around the world [1-3]. Elastic wave propagation in anisotropic media is of great interest in various branches of engineering and applied sciences. In addition, wave propagation in inhomogeneous or anisotropic materials is one of the classical problems in the theory of acoustics, electromagnetic, elastic, and seismic waves. In this context, design of energy absorbing structures made of anisotropic materials requires analysis based on anisotropic wave propagation. While dispersion equations for guided waves in structures made of isotropic material can be established analytically, similar development with anisotropic materials is challenging and complex in nature. Researchers have been investigating for decades to develop efficient energy absorbing structures for a wide range of engineering applications such as aerospace, automobile, civil, nuclear reactors etc. One of the broader objectives is to protect the structures from repetitive or high impact loading (i.e. crash loading) by isolating or trapping wave energies which would otherwise damage the structure. Many aerospace structures such as satellites, space shuttles, rocket components utilize energy absorbers to mitigate unwanted vibrations. Various types and geometric architecture energy absorbers have been developed in last decade, such as sandwich structures [4-6], column structure [7-9], plates [10, 11], and honeycomb structures [12, 13] etc. In general, traditional square and
circular tubes [14, 15] and conventional lightweight honeycomb structures are being utilized profoundly as energy absorbers for number of practical advantages such as ease of manufacture, configurable structural geometry, simple machinability and low costs [16]. The role of energy-absorbent materials and their periodicity also has significant impact in constructing energy absorbent structures [17-19]. Foams [20], structures with collapsible mechanisms [16, 21], bistable structures [22, 23] and periodic structures [21] are examples of such materials and structures. While impact mitigation is one of the primary methods of decreasing the damaging effects of the shock waves produced by the impacts on these structures, the energy absorption characteristics of these conventional structures are poor, especially in force efficiency terms [24]. To improve the energy absorption efficiency, researchers have introduced metastructures with artificial surface perturbations. Sinusoidally corrugated wave guide was shown to stop certain frequencies [25, 26] and could led to an effective energy absorber. Artificially created wave guides with surface perturbation are effective in stopping certain frequency bands. Similar orthogonal surface perturbations or corrugations to the structures should create wider stop bands in all directions of wave propagation through them, when perturbations or corrugation parameters are chosen appropriately. Investigation of wave propagation behavior in energy absorbent wave guide with orthogonal surface perturbations is an aspiring topic of research.

1.1 Background and Literature Review

Although corrugated or orthogonal surface perturbated structures are being proposed and investigated by the researchers in recent years, such geometric structures are widely found in biological structures, for example, woodpecker’s head, exhibiting
energy absorbing mechanisms in multiple parts of their bodies. It has been reported that a woodpecker’s micro-structured corrugated mesh skull has a mechanism which helps absorbing the shockwaves at 1000 times of gravitational acceleration and does not result in damage to their brain [27, 28]. The bones in human skull are stitched together to form a corrugated structure which also helps blocking certain lower-frequency stress waves. The biological corrugated architecture of such structures is, therefore, important to mitigate impact and reduce damage. Inspired from these biological instances, researchers have developed various types of corrugated waveguides, and studied the stopbands of sinusoidally corrugated plates [29, 30]. Multiple groups of researchers studied elastic wave propagation for stopbands and passbands in sinusoidally corrugated plate with single and doubly corrugated surface for Rayleigh-Lamb wave modes and homogeneous and inhomogeneous plane wave modes [31, 32]. Leduc et al. [33] studied Lamb wave propagation in a periodic grating plate with triangular grooves. Hou and Assouar [34] revised three-dimensional plane wave expansion method to investigate Lamb wave in a plate with two-dimensional phononic crystal layer coated on uniform substrate. Banerjee and Kundu [25] studied the symmetric and antisymmetric Rayleigh–Lamb modes in orthogonal surface perturbated sinusoidally corrugated isotropic plates by considering the P-wave and S-wave potentials. They observed passbands corresponding to both modes, which exhibited cut-on resonance along the direction of wave propagation. Zhang et al. [35] conducted Lamb wave propagation in a homogenous plate with periodic tapered surface to study the effect of geometry on the band gap using finite element method. Asfar et al. [36] developed an analytical model to obtain the mode conversion in periodically corrugated plates using perturbation method of multiple scales.
1.2 Motivation

After reviewing the literatures on different types of orthogonal surface perturbated waveguides, it is found that an in-depth study has been performed to investigate wave propagation behavior for guided waves in these structures. However, a generalized expression for Rayleigh-Lamb and Shear Horizontal (SH) waves is still in developing stage to understand the dispersion behavior in planar and corrugated waveguides. In 2006, Banerjee and Kundu [25] studied the stopbands in corrugated waveguide. However, using the same expression, these stopbands did not disappear when the corrugation depth was made zero which converts the corrugated wave guide as a planar waveguide. Recently, Tavaf and Banerjee [26] developed a generalized expression for planar and corrugated waveguide based on Helmholtz decomposition of two potential functions. In their study, a 3D symmetric element is analyzed with sinusoidal corrugation only in one coordinate direction. However, simultaneous addition of orthogonal perturbations in more than one coordinate direction cannot be solved using the potential functions derived by classical Helmholtz decomposition. This is because of the anisotropic wave behavior induced by the orthogonal surface perturbation of the wave guide. Therefore, a different set of potential functions is needed to understand the nature of wave propagation in these structures. Hence, the development of a generalized solution for a planar and orthogonal surface perturbated wave guide that explains the wave propagation behavior is the motivation of the current study.

1.3 Research Goal

Investigation and development of dispersion equations to understand the wave propagation behavior for Rayleigh-Lamb waves (symmetric and anti-symmetric) and
shear horizontal (SH) waves in isotropic material with orthogonal surface perturbations in two coordinate directions.

1.4 Solution Approach

The conventional method of determining the dispersion equations for Rayleigh-Lamb wave propagation is to use Navier’s equation for isotropic solids. Assuming suitable potential functions, the stress and displacement equations for Rayleigh-Lamb wave propagation are derived for both pressure wave (P-wave) and shear waves (SV-waves). However, it is very difficult to solve two and three-dimensional problems directly from Navier’s equation. Therefore, Stokes-Helmholtz decomposition is applied on the displacement field to transform the Navier’s equation of motion into simple wave equations. While applying Stokes-Helmholtz decomposition, two potential functions are assumed for P-waves and SV-waves. However, for the case of a 3D waveguide with orthogonal surface perturbations in two coordinate directions, the wave propagation behaviors are significantly affected by the geometry of the surface because of the omnidirectional interaction of the wavefront with the direction dependent corrugations. Directional dependency of the wave propagation leads to the anisotropic consideration of the dispersion equations. This consideration is applicable for the geometric structures made of isotropic material. Despite the presence of isotropic material properties of the waveguide with orthogonal surface perturbations in two coordinate directions, the dispersion behavior is direction dependent and, hence, the Helmholtz decomposition of the potential functions cannot be applied to derive the displacement equations in three principal directions. Therefore, it is required to assume an anisotropic wave propagation
behavior and the objective of the current study is to investigate the wave behavior in isotropic material with anisotropic surface perturbations.

It has been well established that the guided wave propagation in generalized anisotropic media is by far the most complex problem in Rayleigh–Lamb wave propagation. One of the prominent reasons of this complexity is that the fundamental bulk wave modes in anisotropic media are all coupled. While the SH wave is decoupled from the P and SV waves in isotropic plates, the anisotropic plate media behaves completely in a different way. This is one of the fundamental reasons that the Helmholtz decomposition cannot be applied to anisotropic media and, hence, the potentials for P and SV waves cannot be uniquely separated. In this study, three potential functions, such as $\Theta_1(x_j)e^{-i\omega t}$, $\Theta_2(x_j)e^{-i\omega t}$ and $\Theta_3(x_j)e^{-i\omega t}$, proposed by Buchwald in 1961 will be utilized to describe the three displacement functions in anisotropic media [37].

1.5 Research Objectives

A. Development of analytical dispersion equation for 3D waveguide with orthogonal surface perturbations in two coordinate directions.

B. Numerical verification of wave propagation behavior in a 3D wave guide described above.
CHAPTER 2
DEVELOPMENT OF ANALYTICAL DISPERSION EQUATION FOR 2D CORRUGATED STRUCTURES

In this study, a generalized mathematical expression has been developed for Rayleigh–Lamb and SH wave propagation in media with and without corrugation. In case of isotropic media, the equations for Rayleigh–Lamb and SH wave propagation in corrugated plates are developed analytically using scalar and vector potential functions as derived by the Helmholtz decomposition. However, researchers have derived solution of dispersion equations for orthogonal surface perturbation in one coordinate direction as shown in Figure 2-1a. Introduction of such corrugation in two coordinate directions, namely, $x_1$ and $x_2$ directions (Figure 2-1b), adds complexity in the structure and the wave propagation analysis in these structures requires all three vector potentials of Helmholtz decompositions. Therefore, it is hypothesized that by introducing a 2D corrugations, the dispersion equation can be derived by assuming it as an anisotropic media. By solving the dispersion equations, the dispersion curves for the corrugated waveguides for propagating and evanescent waves could be obtained.

Equations and geometry of 2D corrugated plate (Figure 2-1b) can be assumed as:

$$x_3^+ = h + \varepsilon \cos \left( \frac{2\pi x_1}{D_1} \right) \cos \left( \frac{2\pi x_2}{D_2} \right)$$

$$x_3^- = -h - \varepsilon \cos \left( \frac{2\pi x_1}{D_1} \right) \cos \left( \frac{2\pi x_2}{D_2} \right)$$
Where, $h$ is half of the average thickness of the plate, which is $d$. The corrugation depth is $\varepsilon$ whereas $e$ is the corrugation coefficient, which is to be multiplied with $h$ to control the corrugation depth. The wavelengths of the periodic waveguide are $D_1$ and $D_2$ in $x_1$ and $x_2$ directions, respectively.

![Figure 2.1: Sinusoidal corrugated structure with corrugation in (a) one dimension (b) two dimensions. (c) Projection of (b) through $x_1$-$x_3$ plane.](image)

2.1 Underlying Theory

The fundamental equation of motion or the Navier’s equation for isotropic solids can be written as:

$$(\lambda + 2\mu)u_{j,jl} - \mu\varepsilon_{ijk}\varepsilon_{kpq}u_{q,pj} + f_i = \rho\ddot{u}_i$$

where $\lambda$ and $\mu$ are the Lamé constants, $\varepsilon$ is the permutation symbol, and $u_i$ is the displacement field. To derive the stress and displacement equations for Rayleigh–Lamb wave propagation, both compressional waves (P-waves) and shear waves (SV-waves) are considered. Similarly, horizontal part of the shear waves (SH – waves) requires stress and displacement equations. By applying the Helmholtz decomposition, displacements fields are derived from scalar and vector potentials for P-waves and shear waves (both SV and SH waves). However, geometry of the 2D sinusoidally corrugated plate has wave propagation pattern which is dependent on the direction of the coordinate system.
Therefore, although the material properties of the structure are isotropic, a hypothesis that
the wave propagation in 2D corrugated structure will follow anisotropic behavior is
assumed in this study.

For decades, researchers have been trying to solve guided wave propagation in
generalized anisotropic media which is by far the most complex problem in wave
propagation. One of the reasons for these complexities is the underlying concepts in
fundamental bulk wave modes. In case of isotropic plate, the SH wave is decoupled from
the P and SV wave waves. Thus, the Helmholtz decomposition can be derived by
separately assuming scalar and vector potentials for P+SV wave and SH waves.
However, bulk wave modes in anisotropic media are all coupled and the potentials \( \phi \) and
\( \psi \) cannot be uniquely separated [38]. Therefore, to solve the wave propagation in
anisotropic media, in 1961, Buchwald proposed three potential functions, say
\( \Theta_1(x_j)e^{-i\omega t}, \Theta_2(x_j)e^{-i\omega t} \) and \( \Theta_3(x_j)e^{-i\omega t} \). While the potential functions in isotropic
media can be directly assumed from the equations of motion, potential functions that
Buchwald proposed are unknown for anisotropic media. As proposed by Buchwald, these
potentials are assumed to describe the displacement functions. A proper displacement
functions need to be found out by satisfying the fundamental equations of motion without
forcing function so that they can be utilized to calculate stress functions where boundary
conditions can be applied. By using Buchwald potential function, one can write the
displacement functions as follows:

\[
\begin{align*}
    u_1 &= \left( \frac{\partial \Theta_1}{\partial x_1} \right) e^{-i\omega t} \\
    u_2 &= \left( \frac{\partial \Theta_2}{\partial x_2} + \frac{\partial \Theta_3}{\partial x_3} \right) e^{-i\omega t} \\
    u_3 &= \left( \frac{\partial \Theta_2}{\partial x_3} - \frac{\partial \Theta_3}{\partial x_2} \right) e^{-i\omega t}
\end{align*}
\]

\[1\]
However, to calculate these displacement functions, the potential functions need to be evaluated. Banerjee and Lecky [39] presented a detailed derivation of the potential functions based on the work proposed by Mal et al. [40]. According to their derivations, the potential functions can be expressed as follows:

\[
\begin{bmatrix}
\Theta_1 \\
\Theta_2 \\
\Theta_3
\end{bmatrix} = 
\begin{bmatrix}
K_{11} & K_{12} & 0 \\
K_{21} & K_{22} & 0 \\
0 & 0 & 1
\end{bmatrix} 
\begin{bmatrix}
C_u e^{i\xi_1 x_3} + C_d e^{-i\xi_1 x_3} \\
C_u e^{i\xi_2 x_3} + C_d e^{-i\xi_2 x_3} \\
C_u e^{i\xi_3 x_3} + C_d e^{-i\xi_3 x_3}
\end{bmatrix} e^{i(k_1 x_1 + k_2 x_2)} \quad \text{.......................... (2)}
\]

Where,

\[
K_{11} = \frac{C_{12} + C_{55}}{\rho} - \left( \frac{(C_{11} C_{22} + C_{55}^2 - (C_{12} + C_{55})^2) k_1^2 - \omega^2 \rho (C_{22} + C_{55})}{2C_{22} C_{55}} \right)
\]

\[
\left[ \frac{(C_{11} C_{22} + C_{55}^2 - (C_{12} + C_{55})^2) k_1^2 - \omega^2 \rho (C_{22} + C_{55})}{2C_{22} C_{55}} \right] ^2
\]

\[
- \frac{(C_{11} k_1^2 - \omega^2)(C_{55} k_1^2 - \omega^2)}{\frac{C_{22} C_{55}}{\rho^2}}
\]

\[
K_{12} = \frac{C_{12} + C_{55}}{\rho} - \left( \frac{(C_{11} C_{22} + C_{55}^2 - (C_{12} + C_{55})^2) k_1^2 - \omega^2 \rho (C_{22} + C_{55})}{2C_{22} C_{55}} \right)
\]

\[
\left[ \frac{(C_{11} C_{22} + C_{55}^2 - (C_{12} + C_{55})^2) k_1^2 - \omega^2 \rho (C_{22} + C_{55})}{2C_{22} C_{55}} \right] ^2
\]

\[
+ \frac{(C_{11} k_1^2 - \omega^2)(C_{55} k_1^2 - \omega^2)}{\frac{C_{22} C_{55}}{\rho^2}}
\]
\[
K_{21} = \omega^2 - \frac{c_{22}^2 k_1^2}{\rho} - \frac{c_{55}}{\rho} - \left( \frac{c_{11} c_{22} + c_{55}^2 - (c_{12} + c_{55})^2}{2 c_{22} c_{55}} k_1^2 - \omega^2 \rho (c_{22} + c_{55}) \right)
\]

\[
- \left[ \frac{\left( c_{11} c_{22} + c_{55}^2 - (c_{12} + c_{55})^2 \right) k_1^2 - \omega^2 \rho (c_{22} + c_{55})}{2 c_{22} c_{55}} \right]^2
\]

\[
+ \left[ \frac{\left( c_{11} k_1^2 - \omega^2 \left( \frac{c_{55}}{\rho} k_1^2 - \omega^2 \right) \right)}{c_{22} c_{55}} \right]^{-1}
\]

\[
K_{22} = \omega^2 - \frac{c_{22}^2 k_1^2}{\rho}
\]

\[
- \frac{c_{55}}{\rho} - \left( \frac{c_{11} c_{22} + c_{55}^2 - (c_{12} + c_{55})^2}{2 c_{22} c_{55}} k_1^2 - \omega^2 \rho (c_{22} + c_{55}) \right)
\]

\[
+ \left[ \frac{\left( c_{11} c_{22} + c_{55}^2 - (c_{12} + c_{55})^2 \right) k_1^2 - \omega^2 \rho (c_{22} + c_{55})}{2 c_{22} c_{55}} \right]^2
\]

\[
- \left[ \frac{\left( c_{11} k_1^2 - \omega^2 \left( \frac{c_{55}}{\rho} k_1^2 - \omega^2 \right) \right)}{c_{22} c_{55}} \right]^{-1}
\]
\[ \xi_1^2 = -k_2^2 \]

\[
\begin{aligned}
- \left( \frac{(C_{11}C_{22} + C_{55}^2 - (C_{12} + C_{55})^2)k_1^2 - \omega^2 \rho(C_{22} + C_{55})}{2C_{22}C_{55}} \right) \\
+ \left[ \frac{(C_{11}C_{22} + C_{55}^2 - (C_{12} + C_{55})^2)k_1^2 - \omega^2 \rho(C_{22} + C_{55})}{2C_{22}C_{55}} \right]^2 \\
- \frac{\left( \frac{C_{11}k_1^2}{\rho} - \omega^2 \right) \left( \frac{C_{55}k_1^2}{\rho} - \omega^2 \right)}{\frac{C_{22}C_{55}}{\rho^2}}
\end{aligned}
\]

\[ \xi_2^2 = -k_2^2 + \left[ \frac{(C_{11}C_{22} + C_{55}^2 - (C_{12} + C_{55})^2)k_1^2 - \omega^2 \rho(C_{22} + C_{55})}{2C_{22}C_{55}} \right]^2 \\
+ \left[ \frac{(C_{11}C_{22} + C_{55}^2 - (C_{12} + C_{55})^2)k_1^2 - \omega^2 \rho(C_{22} + C_{55})}{2C_{22}C_{55}} \right]^2 \\
- \frac{\left( \frac{C_{11}k_1^2}{\rho} - \omega^2 \right) \left( \frac{C_{55}k_1^2}{\rho} - \omega^2 \right)}{\frac{C_{22}C_{55}}{\rho^2}}
\]

\[ \xi_3^2 = -k_2^2 + \rho(\omega^2 - C_{55}/\rho k_1^2)/C_{44} \]

In short, \( K_{11}, K_{12}, K_{21} \) and \( K_{22} \), and \( \xi_1, \xi_2 \) and \( \xi_3 \) can be expressed as follows:

\[ K_{11} = \Lambda_3 P_+ \]

\[ K_{12} = \Lambda_3 P_- \]

\[ K_{21} = \omega^2 - \Lambda_2 k_1^2 - \Lambda_5 P_+ \]

\[ K_{22} = \omega^2 - \Lambda_2 k_1^2 - \Lambda_5 P_- \]

\[ P_{+, -} = -\left( \frac{\beta}{2\alpha} \right) \pm \sqrt{\left( \frac{\beta}{2\alpha} \right)^2 - \frac{\gamma}{\alpha}} \]

\[ \alpha, \beta, \gamma \]
\[ \alpha = \Lambda_2 \Lambda_5 = \frac{C_{22}}{\rho} \frac{C_{55}}{\rho} \]
\[ \beta = (\Lambda_1 \Lambda_2 + \Lambda_2^2 - \Lambda_3^2)k_1^2 - \omega^2(\Lambda_2 + \Lambda_5) \]
\[ \gamma = (\Lambda_1 k_1^2 - \omega^2)(\Lambda_5 k_1^2 - \omega^2) \]
\[ \xi_1^2 = -k_2^2 + P_+ \]
\[ \xi_2^2 = -k_2^2 + P_- \]
\[ \xi_3^2 = -k_2^2 + (\omega^2 - \Lambda_5 k_1^2)/\Lambda_4 \]
\[ \Lambda_1 = \frac{C_{11}}{\rho} \]
\[ \Lambda_2 = \frac{C_{22}}{\rho} \]
\[ \Lambda_3 = \frac{(C_{12} + C_{55})}{\rho} \]
\[ \Lambda_4 = \frac{C_{44}}{\rho} \]
\[ \Lambda_5 = \frac{C_{55}}{\rho} \]

Equation (2) can be written in following form,
\[ \Theta_1 = \left[ K_{11} \left( C_u e^{i\xi_1 x_3} + C_d e^{-i\xi_1 x_3} \right) + K_{12} \left( C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3} \right) \right] e^{i(k_1 x_1 + k_2 x_2)} \]
\[ \Theta_2 = \left[ K_{21} \left( C_u e^{i\xi_1 x_3} + C_d e^{-i\xi_1 x_3} \right) + K_{22} \left( C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3} \right) \right] e^{i(k_1 x_1 + k_2 x_2)} \]
\[ \Theta_3 = \left[ C_u^3 e^{i\xi_3 x_3} + C_d^3 e^{-i\xi_3 x_3} \right] e^{i(k_1 x_1 + k_2 x_2)} \]

2.2 Application of Bloch theorem and derivation of displacement functions

Derivation of displacement functions (Eq. 1) from the potential functions (Eq. 3) thus satisfies the fundamental equations of motion. To obtain the equations of motion for 2D corrugated structure, it is assumed that a plane harmonic monochromatic wave propagates through the structure as described in Figure (2-1b). The propagation of wave is such that the time-harmonic part of the potentials is \( e^{-i\omega t} \). Since the structure in Figure 2-1b consisting of 2D periodically corrugated media, the Bloch Theorem can be utilized
to obtain its solution. Thus, applying Bloch theorem in reciprocal wave-number space in assuming displacement field while propagating wave through the phononic crystals, it can be assumed:

\[ u_i(x_i) = Ae^{i(k \cdot x + G \cdot x)} \]

Where,

\[ k = k_i \hat{e}_i \] while \( k_1 \) and \( k_2 \) are fundamental component of the wavenumber along the \( x_1 \) and \( x_2 \) directions

\[ G = G_i \hat{e}_i \] while \( G_1 = \frac{2\pi}{D_1} \) and \( G_2 = \frac{2\pi m}{D_2} \)

\( D_1 \) and \( D_2 \) are the periodicity of the structure or the length of a single wave in the periodic structure along \( x_1 \) and \( x_2 \) directions.

\( n \) and \( m \) are positive and negative integers in real number domain.

Application of Bloch theorem modifies the potential functions described by equation (3). Thus, the three potential functions can be derived for all the upgoing and down going compressional and shear waves and can be expressed by equations (4). It can be noted that the time harmonic part of the monochromatic wave \( e^{-i\omega t} \) has been excluded from the potential equations.

\[ \Theta_1 = \sum_{m=-\infty}^{m=+\infty} \sum_{n=+\infty}^{n=-\infty} \left[ K_{11} (C_{1u} e^{i\xi_1 x_3} + C_{1d} e^{-i\xi_1 x_3}) + K_{12} (C_{2u} e^{i\xi_2 x_3} + C_{2d} e^{-i\xi_2 x_3}) \right] e^{i(k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \] .................................................. 4(a)

\[ \Theta_2 = \sum_{m=-\infty}^{m=+\infty} \sum_{n=+\infty}^{n=-\infty} \left[ K_{21} (C_{1u} e^{i\xi_1 x_3} + C_{1d} e^{-i\xi_1 x_3}) + K_{22} (C_{2u} e^{i\xi_2 x_3} + C_{2d} e^{-i\xi_2 x_3}) \right] e^{i(k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \] .................................................. 4(b)

\[ \Theta_3 = \sum_{m=-\infty}^{m=+\infty} \sum_{n=+\infty}^{n=-\infty} \left[ C_{3u} e^{i\xi_3 x_3} + C_{3d} e^{-i\xi_3 x_3} \right] e^{i(k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \] .......................... 4(c)
Displacement field equations are thus can be calculated by using the formulation mentioned in equation (1) which becomes,

\[ u_1 = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} i \left( k_1 + \frac{2\pi}{D_1} \right) \left[ K_{11} (C_u e^{i\xi_1 x_3} + C_1 d e^{-i\xi_1 x_3}) + K_{12} (C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) \right] e^{i(k_1 x_1 + 2\pi m x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \] ........................................ 5(a)

\[ u_2 = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} i \left[ \left( k_2 + \frac{2\pi m}{D_2} \right) K_{21} (C_u e^{i\xi_1 x_3} + C_1 d e^{-i\xi_1 x_3}) + \left( k_2 + \frac{2\pi m}{D_2} \right) \right] K_{22} (C_u e^{i\xi_2 x_3} + C_2 d e^{-i\xi_2 x_3}) + \xi_3 (C_3 u e^{i\xi_3 x_3} - C_3 d e^{-i\xi_3 x_3}) \] ........................................ 5(b)

\[ u_3 = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} i \left[ \xi_1 K_{21} (C_u e^{i\xi_1 x_3} - C_1 d e^{-i\xi_1 x_3}) + \xi_2 K_{22} (C_u e^{i\xi_2 x_3} - C_2 d e^{-i\xi_2 x_3}) - (k_2 + \frac{2\pi m}{D_2}) (C_3 u e^{i\xi_3 x_3} + C_3 d e^{-i\xi_3 x_3}) \right] e^{i(k_1 x_1 + 2\pi m x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} ....... 5(c)

2.3 Derivation of strain equations

The strain-displacement equations for a linear elastic medium can be written as:

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \quad i,j = 1,2,3 \] .............................................. (6)

Using these relationships, strains are calculated, and the final expressions are stated below:

\[ \varepsilon_{11} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1} \right) = \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} - \left( k_1 + \frac{2\pi n}{D_1} \right) \left[ K_{11} (C_u e^{i\xi_1 x_3} + C_1 d e^{-i\xi_1 x_3}) + K_{12} (C_2 u e^{i\xi_2 x_3} + C_2 d e^{-i\xi_2 x_3}) \right] e^{i(k_1 x_1 + 2\pi n x_1 + k_2 x_2 + \frac{2\pi n}{D_2} x_2)} \]
\[
\varepsilon_{22} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2} \right) \\
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} - \left( k_2 + \frac{2\pi m}{D_2} \right) \left( k_2 + \frac{2\pi m}{D_2} \right) K_{21} \left( C_1 u e^{i\xi_1 x_3} + C_1 d e^{-i\xi_1 x_3} \right) \\
+ \left( k_2 + \frac{2\pi m}{D_2} \right) K_{22} \left( C_1 u e^{i\xi_2 x_3} + C_1 d e^{-i\xi_2 x_3} \right) \\
+ \xi_3 \left( C_1 u e^{i\xi_3 x_3} - C_1 d e^{-i\xi_3 x_3} \right) \right] e^{i \left( k_1 x_1 + \frac{2\pi m}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2 \right)} \\
\varepsilon_{33} = \frac{1}{2} \left( \frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3} \right) \\
= \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} - \left[ \xi_2^2 K_{21} \left( C_1 u e^{i\xi_1 x_3} + C_1 d e^{-i\xi_1 x_3} \right) + \xi_2^2 K_{22} \left( C_1 u e^{i\xi_2 x_3} + C_1 d e^{-i\xi_2 x_3} \right) \\
- \xi_3 \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_1 u e^{i\xi_3 x_3} - C_1 d e^{-i\xi_3 x_3} \right) \right] e^{i \left( k_1 x_1 + \frac{2\pi m}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2 \right)} \\
\varepsilon_{12} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \right) \\
= -\frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ \left( k_1 + \frac{2\pi m}{D_1} \right) \left( k_2 + \frac{2\pi m}{D_2} \right) \left( k_1 + \frac{2\pi m}{D_1} \right) \left( k_2 + \frac{2\pi m}{D_2} \right) \right] \left[ K_{11} \left( C_1 u e^{i\xi_1 x_3} + C_1 d e^{-i\xi_1 x_3} \right) \\
+ K_{12} \left( C_1 u e^{i\xi_2 x_3} + C_1 d e^{-i\xi_2 x_3} \right) \\
+ \xi_3 \left( C_1 u e^{i\xi_3 x_3} - C_1 d e^{-i\xi_3 x_3} \right) \right] e^{i \left( k_1 x_1 + \frac{2\pi m}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2 \right)} \\
\text{16}
\]
\[ \varepsilon_{13} = \frac{1}{2} \left( \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} \right) \]

\[ = - \frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ k_1 + \frac{2\pi n}{D_1} \right] \left[ \xi_1 K_{11} \left( C_u^1 e^{i\xi_1 x_3} - C_d^1 e^{-i\xi_1 x_3} \right) + \xi_2 K_{12} \left( C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3} \right) \right] \]

\[ + \left( k_1 + \frac{2\pi m}{D_2} \right) \left[ \xi_1 K_{21} \left( C_u^1 e^{i\xi_1 x_3} - C_d^1 e^{-i\xi_1 x_3} \right) + \xi_2 K_{22} \left( C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3} \right) \right] \]

\[ - \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u^3 e^{i\xi_3 x_3} + C_d^3 e^{-i\xi_3 x_3} \right) e^{i(k_1 x_1 + \frac{2\pi m}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \]

\[ \varepsilon_{23} = \frac{1}{2} \left( \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \right) \]

\[ = - \frac{1}{2} \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ \xi_1 \left( k_2 + \frac{2\pi m}{D_2} \right) K_{21} \left( C_u^1 e^{i\xi_1 x_3} - C_d^1 e^{-i\xi_1 x_3} \right) + \xi_2 \left( k_2 + \frac{2\pi m}{D_2} \right) K_{22} \left( C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3} \right) \right] \]

\[ + \xi_2 \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u^3 e^{i\xi_3 x_3} + C_d^3 e^{-i\xi_3 x_3} \right) \]

\[ + \left( k_2 + \frac{2\pi m}{D_2} \right) \left[ \xi_1 K_{21} \left( C_u^1 e^{i\xi_1 x_3} - C_d^1 e^{-i\xi_1 x_3} \right) + \xi_2 K_{22} \left( C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3} \right) \right] \]

\[ - \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u^3 e^{i\xi_3 x_3} + C_d^3 e^{-i\xi_3 x_3} \right) e^{i(k_1 x_1 + \frac{2\pi m}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \]
\[ \varepsilon_{31} = \frac{1}{2} \left( \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \right) \]

\[ = -\frac{1}{2} \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \left( k_1 + \frac{2\pi n}{D_1} \right) \left\{ \xi_1 K_{21} \left( C_u e^{i\xi_1 x_3} - C_d e^{-i\xi_1 x_3} \right) 
+ \xi_2 K_{22} \left( C_u e^{i\xi_2 x_3} - C_d e^{-i\xi_2 x_3} \right) 
- \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u e^{i\xi_3 x_3} + C_d e^{-i\xi_3 x_3} \right) \right\} 
+ \left( k_1 + \frac{2\pi n}{D_1} \right) \left\{ \xi_1 K_{11} \left( C_u e^{i\xi_1 x_3} - C_d e^{-i\xi_1 x_3} \right) 
+ \xi_2 K_{12} \left( C_u e^{i\xi_2 x_3} - C_d e^{-i\xi_2 x_3} \right) \right\} e^{i(k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \]

\[ \varepsilon_{32} = \frac{1}{2} \left( \frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3} \right) \]

\[ = -\frac{1}{2} \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \left( k_2 + \frac{2\pi m}{D_2} \right) \left\{ \xi_1 K_{21} \left( C_u e^{i\xi_1 x_3} - C_d e^{-i\xi_1 x_3} \right) 
+ \xi_2 K_{22} \left( C_u e^{i\xi_2 x_3} - C_d e^{-i\xi_2 x_3} \right) 
- \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u e^{i\xi_3 x_3} + C_d e^{-i\xi_3 x_3} \right) \right\} 
+ \left[ \xi_1 \left( k_2 + \frac{2\pi m}{D_2} \right) K_{21} \left( C_u e^{i\xi_1 x_3} - C_d e^{-i\xi_1 x_3} \right) 
+ \xi_2 \left( k_2 + \frac{2\pi m}{D_2} \right) K_{22} \left( C_u e^{i\xi_2 x_3} - C_d e^{-i\xi_2 x_3} \right) 
+ \xi_3 ^2 \left( C_u e^{i\xi_3 x_3} + C_d e^{-i\xi_3 x_3} \right) \right\} e^{i(k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \]
\( \varepsilon_{kk} = \varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33} \)

\[
= - \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \left( k_1 + \frac{2\pi n}{D_1} \right)^2 \left[ K_{11} \left( C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3} \right) + K_{12} \left( C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3} \right) \right] \\
+ \left( k_2 + \frac{2\pi m}{D_2} \right) \left[ \left( k_2 + \frac{2\pi m}{D_2} \right) K_{21} \left( C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3} \right) + K_{22} \left( C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3} \right) \right] \\
+ \left[ \xi_1^2 K_{21} \left( C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3} \right) + \xi_2^2 K_{22} \left( C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3} \right) \right] \\
- \xi_3 \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u^3 e^{i\xi_3 x_3} - C_d^3 e^{-i\xi_3 x_3} \right) \right] e^{i \left( k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2 \right)}
\]

2.4 Derivation of stress equations

The stress-strain relation for a linear elastic material can be written as

\( \sigma_{ij} = 2\mu \varepsilon_{ij} + \lambda \varepsilon_{kk} \delta_{ij} \)

In this equation, the stresses and the strains in three co-ordinate directions are expressed where \( \delta_{ij} \) is known as Kronecker delta. Using equation (7), stresses can be calculated by using the strains derived above. The final form of the stress equations in all three directions are calculated and shown below:
\[ \sigma_{11} = 2\mu \varepsilon_{11} + \lambda \varepsilon_{kk} \delta_{11} \]

\[ = - \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} 2\mu \left(k_1 + \frac{2\pi n}{D_1}\right)^2 \left[K_{11}(C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) \right. \]

\[ + K_{12}(C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) + \lambda \left(k_1 + \frac{2\pi n}{D_1}\right)^2 \left[K_{11}(C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) \right. \]

\[ + K_{12}(C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) \]

\[ \left. + \left(k_2 + \frac{2\pi m}{D_2}\right) \left(k_2 + \frac{2\pi m}{D_2}\right) K_{21}(C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) \right] \]

\[ + \left(k_2 + \frac{2\pi m}{D_2}\right) K_{22}(C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) \]

\[ + \xi_3(C_u^3 e^{i\xi_3 x_3} - C_d^3 e^{-i\xi_3 x_3}) \]

\[ + \left[\xi_1^2 K_{21}(C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) + \xi_2^2 K_{22}(C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) \right. \]

\[ - \xi_3 \left(k_2 + \frac{2\pi m}{D_2}\right) (C_u^3 e^{i\xi_3 x_3} \]

\[ \left. - C_d^3 e^{-i\xi_3 x_3}\right] e^{i(k_1 x_1 + \frac{2\pi m}{D_1} x_1 + k_2 x_2 + \frac{2\pi n}{D_2} x_2)} \]
\[ \sigma_{12} = 2\mu\varepsilon_{12} + \lambda\varepsilon_{kk}\delta_{12} \]
\[ = - \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \mu \left( k_1 + \frac{2\pi n}{D_1} \right) \left( k_2 + \frac{2\pi m}{D_2} \right) \left[ K_{11}(C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) \right. \]
\[ + K_{12}(C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) \left. \right] \]
\[ + \left( k_1 + \frac{2\pi n}{D_1} \right) \left( k_2 + \frac{2\pi m}{D_2} \right) K_{21}(C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) \]
\[ + \left( k_2 + \frac{2\pi m}{D_2} \right) K_{22}(C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) \]
\[ + \xi_3 (C_u^3 e^{i\xi_3 x_3} - C_d^3 e^{-i\xi_3 x_3}) \left] e^{i(k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \right. \]
\[ \sigma_{13} = 2\mu\varepsilon_{13} + \lambda\varepsilon_{kk}\delta_{13} \]
\[ = - \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \mu \left[ \xi_1 K_{11}(C_u^1 e^{i\xi_1 x_3} - C_d^1 e^{-i\xi_1 x_3}) \right. \]
\[ + \xi_2 K_{12}(C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3}) \left. \right] \]
\[ + \left( k_1 + \frac{2\pi n}{D_1} \right) \left[ \xi_1 K_{21}(C_u^1 e^{i\xi_1 x_3} - C_d^1 e^{-i\xi_1 x_3}) \right. \]
\[ + \xi_2 K_{22}(C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3}) \left. \right] \]
\[ - \left( k_2 + \frac{2\pi m}{D_2} \right) \left(C_u^3 e^{i\xi_3 x_3} \right. \]
\[ + C_d^3 e^{-i\xi_3 x_3} \left. \right) \left] e^{i(k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \right. \]
\[
\sigma_{22} = 2\mu \varepsilon_{22} + \lambda \varepsilon_{kk} \delta_{22}
\]

\[
= - \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ 2\mu \left( k_2 + \frac{2\pi m}{D_2} \right) \left( k_2 + \frac{2\pi n}{D_2} \right) K_{21} \left( C_u e^{i\xi_1 x_3} + C_d e^{-i\xi_1 x_3} \right) 
\right. 

+ \left. \left( k_2 + \frac{2\pi m}{D_2} \right) K_{22} \left( C_u e^{i\xi_2 x_3} + C_d e^{-i\xi_2 x_3} \right) \right]

+ \xi_3 \left( C_u e^{i\xi_3 x_3} - C_d e^{-i\xi_3 x_3} \right)

+ \lambda \left[ \left( k_1 + \frac{2\pi n}{D_1} \right)^2 \left( k_1 + \frac{2\pi m}{D_1} \right) K_{11} \left( C_u e^{i\xi_1 x_3} + C_d e^{-i\xi_1 x_3} \right) 
\right. 

+ \left. K_{12} \left( C_u e^{i\xi_2 x_3} + C_d e^{-i\xi_2 x_3} \right) \right]

+ \left( k_2 + \frac{2\pi m}{D_2} \right) \left( k_2 + \frac{2\pi n}{D_2} \right) K_{21} \left( C_u e^{i\xi_1 x_3} + C_d e^{-i\xi_1 x_3} \right)

+ \left( k_2 + \frac{2\pi m}{D_2} \right) K_{22} \left( C_u e^{i\xi_2 x_3} + C_d e^{-i\xi_2 x_3} \right)

+ \xi_3 \left( C_u e^{i\xi_3 x_3} - C_d e^{-i\xi_3 x_3} \right)

+ \left[ \xi_1^2 K_{21} \left( C_u e^{i\xi_1 x_3} + C_d e^{-i\xi_1 x_3} \right) + \xi_2^2 K_{22} \left( C_u e^{i\xi_2 x_3} + C_d e^{-i\xi_2 x_3} \right) 
\right. 

- \xi_3 \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u e^{i\xi_3 x_3} 
\right. 

+ \left. C_d e^{-i\xi_3 x_3} \right) \left( e^{i(k_1 x_1 + \frac{2\pi m}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \right) \right]
\[ \sigma_{23} = 2\mu\varepsilon_{23} + \lambda\varepsilon_{kk}\delta_{23} \]

\[ = -\sum_{m=+\infty}^{m=-\infty} \sum_{n=+\infty}^{n=-\infty} \mu \left[ \xi_1 \left( k_2 + \frac{2\pi m}{D_2} \right) K_{21} \left( C_u e^{i\xi_1 x_3} - C_d e^{-i\xi_1 x_3} \right) \right. \]

\[ + \xi_2 \left( k_2 + \frac{2\pi m}{D_2} \right) K_{22} \left( C_u e^{i\xi_2 x_3} - C_d e^{-i\xi_2 x_3} \right) \]

\[ + \xi_3 \left( C_u^3 e^{i\xi_3 x_3} + C_d^3 e^{-i\xi_3 x_3} \right) \]

\[ + \left( k_2 + \frac{2\pi m}{D_2} \right) \left[ \xi_2 K_{21} \left( C_u^1 e^{i\xi_2 x_3} - C_d^1 e^{-i\xi_2 x_3} \right) \right. \]

\[ + \xi_2 K_{22} \left( C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3} \right) \]

\[ - \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u^3 e^{i\xi_3 x_3} \right. \]

\[ + \left. C_d^3 e^{-i\xi_3 x_3} \right) \right] e^{i \left( k_1 x_1 + \frac{2\pi n x_1 + k_2 x_2 + 2\pi}{D_2} x_2 \right)} \]

It can be noted that \( \sigma_{21} \) is equivalent to \( \sigma_{12} \) can be expressed with the expression derived for \( \sigma_{12} \). The last row of the stress matrix is derived in similar manner and the final derived expressions are shown below.
\[ \sigma_{31} = 2\mu \varepsilon_{31} + \lambda \varepsilon_{kk} \delta_{31} \]

\[ = -\mu \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \left[ (k_1 + \frac{2\pi n}{D_1}) \left[ \xi_1 K_{21} \left( C_u e^{i\xi_1 x_3} - C_d e^{-i\xi_1 x_3} \right) \\
+ \xi_2 K_{22} \left( C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3} \right) \\
- \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u^3 e^{i\xi_3 x_3} + C_d^3 e^{-i\xi_3 x_3} \right) \right] \\
+ \left( k_1 + \frac{2\pi n}{D_1} \right) \left[ \xi_1 K_{11} \left( C_u e^{i\xi_1 x_3} - C_d e^{-i\xi_1 x_3} \right) \\
+ \xi_2 K_{12} \left( C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3} \right) \right] \right] e^{i(k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \]

\[ \sigma_{32} = 2\mu \varepsilon_{32} + \lambda \varepsilon_{kk} \delta_{32} \]

\[ = -\mu \sum_{m=-\infty}^{m=+\infty} \sum_{n=-\infty}^{n=+\infty} \left[ (k_2 + \frac{2\pi m}{D_2}) \left[ \xi_1 K_{21} \left( C_u e^{i\xi_1 x_3} - C_d e^{-i\xi_1 x_3} \right) \\
+ \xi_2 K_{22} \left( C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3} \right) \\
- \left( k_2 + \frac{2\pi m}{D_2} \right) \left( C_u^3 e^{i\xi_3 x_3} + C_d^3 e^{-i\xi_3 x_3} \right) \right] \\
+ \left[ \xi_1 \left( k_2 + \frac{2\pi m}{D_2} \right) K_{21} \left( C_u e^{i\xi_1 x_3} - C_d e^{-i\xi_1 x_3} \right) \\
+ \xi_2 \left( k_2 + \frac{2\pi m}{D_2} \right) K_{22} \left( C_u^2 e^{i\xi_2 x_3} - C_d^2 e^{-i\xi_2 x_3} \right) \\
+ \xi_3 \left( C_u^3 e^{i\xi_3 x_3} + C_d^3 e^{-i\xi_3 x_3} \right) \right] e^{i(k_1 x_1 + \frac{2\pi n}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \]
\[ \sigma_{33} = 2\mu \varepsilon_{33} + \lambda \varepsilon_{kk} \delta_{33} \]

\[
= - \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left[ 2\mu \left( \xi_1^2 K_{21} (C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) + \xi_2^2 K_{22} (C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) \right) 
- \xi_3 (k_2 + \frac{2\pi m}{D_2}) (C_u^3 e^{i\xi_3 x_3} - C_d^3 e^{-i\xi_3 x_3}) \right] 
+ \lambda \left[ (k_1 + \frac{2\pi n}{D_1})^2 K_{11} (C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) + K_{12} (C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) \right] 
+ \left( k_2 + \frac{2\pi m}{D_2} \right) \left( k_2 + \frac{2\pi m}{D_2} \right) K_{21} (C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) 
+ \left( k_2 + \frac{2\pi m}{D_2} \right) K_{22} (C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) + \xi_3 (C_u^3 e^{i\xi_3 x_3} - C_d^3 e^{-i\xi_3 x_3}) \right] 
+ \xi_1^2 K_{21} (C_u^1 e^{i\xi_1 x_3} + C_d^1 e^{-i\xi_1 x_3}) + \xi_2^2 K_{22} (C_u^2 e^{i\xi_2 x_3} + C_d^2 e^{-i\xi_2 x_3}) 
- \xi_3 (k_2 + \frac{2\pi m}{D_2}) (C_u^3 e^{i\xi_3 x_3} - C_d^3 e^{-i\xi_3 x_3}) \right] \left[ e^{i(k_1 x_1 + \frac{2\pi m}{D_1} x_1 + k_2 x_2 + \frac{2\pi m}{D_2} x_2)} \right] 
\]

2.5 Implementing Traction Free Boundary Conditions

An effective approach to analyze the stress field is the assumption of traction free boundary in a solid continuum medium. This boundary condition can be applied to the upper \( (x_3^+) \) and lower \( (x_3^-) \) surfaces of the 2D corrugated structure as shown in Figure (2-1b) to derive the wave dispersion equation. Under the traction free boundary, the Cauchy traction equations can be written as,
\[ T_i = \sigma_{ij} n_j = 0 \] \hspace{10cm} (8)

Where, \( \sigma_{ij} \) is the stress function and \( n_j \) is the direction cosines.

**Determination of direction cosines:** A direction cosine is the cosine angle between the normal unit vector and its projection along the coordinate axis. If \( V = V_{x_1} e_{x_1} + V_{x_2} e_{x_2} + V_{x_3} e_{x_3} \) is a vector quantity where \( e_{x_1}, e_{x_2} \) and \( e_{x_3} \) are the standard basis vector, then the direction cosines are

\[
\begin{align*}
    n_1 &= \cos \alpha = \frac{V \cdot e_{x_1}}{|V|} = \frac{V_{x_1}}{\sqrt{V_{x_1}^2 + V_{x_2}^2 + V_{x_3}^2}} \\
    n_2 &= \cos \beta = \frac{V \cdot e_{x_2}}{|V|} = \frac{V_{x_2}}{\sqrt{V_{x_1}^2 + V_{x_2}^2 + V_{x_3}^2}} \\
    n_3 &= \cos \gamma = \frac{V \cdot e_{x_3}}{|V|} = \frac{V_{x_3}}{\sqrt{V_{x_1}^2 + V_{x_2}^2 + V_{x_3}^2}}
\end{align*}
\]

Implementing these concepts in upper and lower side of the corrugated plate, one can get,

\[
\begin{align*}
    V_{x_1} &= \frac{\partial x_i}{\partial x_1} = \pm \frac{2 \pi \varepsilon}{D_1} \sin \left( \frac{2 \pi x_1}{D_1} \right) \cos \left( \frac{2 \pi x_2}{D_2} \right) \\
    V_{x_2} &= \frac{\partial x_i}{\partial x_2} = \pm \frac{2 \pi \varepsilon}{D_2} \cos \left( \frac{2 \pi x_1}{D_1} \right) \sin \left( \frac{2 \pi x_2}{D_2} \right) \\
    V_{x_3} &= \pm 1
\end{align*}
\]

Therefore, the direction cosines for the corrugated plate can be written as,

\[
\begin{align*}
    n_1^\pm &= \frac{\pm \frac{2 \pi \varepsilon}{D_1} \sin \left( \frac{2 \pi x_1}{D_1} \right) \cos \left( \frac{2 \pi x_2}{D_2} \right)}{\sqrt{\left[ \frac{2 \pi \varepsilon}{D_1} \sin \left( \frac{2 \pi x_1}{D_1} \right) \cos \left( \frac{2 \pi x_2}{D_2} \right) \right]^2 + \left[ \frac{2 \pi \varepsilon}{D_2} \cos \left( \frac{2 \pi x_1}{D_1} \right) \sin \left( \frac{2 \pi x_2}{D_2} \right) \right]^2} + 1
\end{align*}
\]
Once the direction cosines are derived, the Cauchy traction equations can be written as:

\[ T_1 = \sigma_{11}n_1^+ + \sigma_{12}n_2^+ + \sigma_{13}n_3^+ = 0 \]
\[ T_2 = \sigma_{21}n_1^+ + \sigma_{22}n_2^+ + \sigma_{23}n_3^+ = 0 \]
\[ T_3 = \sigma_{31}n_1^+ + \sigma_{32}n_2^+ + \sigma_{33}n_3^+ = 0 \]

Dividing by Eq. (9) by \( n_3^+ \),

\[ \sigma_{11} \frac{n_1^+}{n_3^+} + \sigma_{12} \frac{n_2^+}{n_3^+} + \sigma_{13} = 0 \]
\[ \sigma_{21} \frac{n_1^+}{n_3^+} + \sigma_{22} \frac{n_2^+}{n_3^+} + \sigma_{23} = 0 \]
\[ \sigma_{31} \frac{n_1^+}{n_3^+} + \sigma_{32} \frac{n_2^+}{n_3^+} + \sigma_{33} = 0 \]

Due to the Bloch expansion of the wave function, orthogonality principles are applied to equation (9) and (10). By applying orthogonality to the guided wave modes, the stress functions above are multiplied with normalized eigen functions which is

\[ e^{-i(k_1x_1 + \frac{2\pi n}{D_1}x_1 + k_2x_2 + \frac{2\pi m}{D_2}x_2)} \]

and thus boundary conditions are integrated over the respective periods of the corrugation along \( x_1 \) and \( x_2 \) axis. This facilitates the imposition of conditions of orthogonality on the boundary conditions and the resultant equations are transferred to the wave number frequency domain. Therefore, the traction equations for
the first Bloch mode, \( n = 0 \) and \( m = 0 \), with one period of corrugation in stress components on the upper and lower surfaces can be expressed by following three equations. Note that, in traction equations, the wavenumber, \( k_1 \), along \( x_1 \)-axis and, \( k_2 \), along \( x_2 \)-axis are the fundamental wave number of the corrugated waveguide where, \( k_1 \) and, \( k_2 \) are the components of wavevector \( k \). However, according to Bloch theorem, if \( k \) is a solution of the system with \( n = 0 \) and \( m = 0 \), then all other wavenumbers that follow the rule \( k_1 + \frac{2\pi}{D_1} \) and \( k_2 + \frac{2\pi m}{D_2} \) are also the solutions of the system for all other values of \( n (-\infty < n < \infty) \) and \( m (-\infty < m < \infty) \). Therefore, while imposing Bloch theorem, equation (9) can be written as,

\[
T_1 = \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} \int_{-\frac{D_1}{2}}^{\frac{D_1}{2}} \left[ \begin{array}{c}
2\mu k^2 [K_{11}(C_{1u0}e^{i\xi_1 x_3} + C_{1d0}e^{-i\xi_1 x_3}) \\
+ K_{12}(C_{2u0}e^{i\xi_2 x_3} + C_{2d0}e^{-i\xi_2 x_3}) \\
+ \lambda [k_1^2 K_{11}(C_{1u0}e^{i\xi_1 x_3} + C_{1d0}e^{-i\xi_1 x_3}) \\
+ K_{12}(C_{2u0}e^{i\xi_2 x_3} + C_{2d0}e^{-i\xi_2 x_3})] \\
+ k_2 [k_2 K_{21}(C_{1u0}e^{i\xi_1 x_3} + C_{1d0}e^{-i\xi_1 x_3}) \\
+ k_2 K_{22}(C_{2u0}e^{i\xi_2 x_3} + C_{2d0}e^{-i\xi_2 x_3}) + \xi_3 (C_{3u0}e^{i\xi_3 x_3} - C_{3d0}e^{-i\xi_3 x_3})] \\
+ [\xi_2^2 K_{21}(C_{1u0}e^{i\xi_1 x_3} + C_{1d0}e^{-i\xi_1 x_3}) + \xi_2^2 K_{22}(C_{2u0}e^{i\xi_2 x_3} + C_{2d0}e^{-i\xi_2 x_3}) \\
- \xi_3 k_2(C_{3u0}e^{i\xi_3 x_3} - C_{3d0}e^{-i\xi_3 x_3})]] \right] 2\pi \frac{D_1}{D_1} \sin \left(\frac{2\pi x_1}{D_1}\right) \cos \left(\frac{2\pi x_2}{D_2}\right) \\
+ \mu [k_1 k_2 K_{11}(C_{1u0}e^{i\xi_1 x_3} + C_{1d0}e^{-i\xi_1 x_3}) \\
+ K_{12}(C_{2u0}e^{i\xi_2 x_3} + C_{2d0}e^{-i\xi_2 x_3})] \\
+ k_2 (k_2 K_{21}(C_{u} e^{i\xi_3 x_3} + C_{d} e^{-i\xi_3 x_3}) + k_2 K_{22}(C_{u} e^{i\xi_2 x_3} + C_{d} e^{-i\xi_2 x_3}) \\
+ \xi_3 (C_{3u0}e^{i\xi_3 x_3} - C_{3d0}e^{-i\xi_3 x_3})] \right] \frac{2\pi \varepsilon}{D_2} \cos \left(\frac{2\pi x_1}{D_1}\right) \sin \left(\frac{2\pi x_2}{D_2}\right) \\
- \mu [k_1 k_2 K_{11}(C_{1u0}e^{i\xi_1 x_3} - C_{1d0}e^{-i\xi_1 x_3}) + \xi_2 K_{12}(C_{2u0}e^{i\xi_2 x_3} - C_{2d0}e^{-i\xi_2 x_3})] \\
+ k_2 (k_2 K_{21}(C_{u} e^{i\xi_3 x_3} - C_{d} e^{-i\xi_3 x_3}) + k_2 K_{22}(C_{u} e^{i\xi_2 x_3} - C_{d} e^{-i\xi_2 x_3}) \\
- k_2(C_{3u0}e^{i\xi_3 x_3} + C_{3d0}e^{-i\xi_3 x_3})] \, dx_1 \, dx_2
\]
\[ T_2 = \int_{D_2} \int_{D_2} \mu \left[ k_1 k_2 \left[ K_{11} \left( C_{1u0} e^{i\xi_1 x_3} + C_{1d0} e^{-i\xi_1 x_3} \right) \right. \right. \\
+ K_{12} \left( C_{2u0} e^{i\xi_2 x_3} + C_{2d0} e^{-i\xi_2 x_3} \right) ] + k_1 \left[ k_2 K_{21} \left( C_{1u0} e^{i\xi_1 x_3} + C_{1d0} e^{-i\xi_1 x_3} \right) \right. \\
+ k_2 K_{22} \left( C_{2u0} e^{i\xi_2 x_3} + C_{2d0} e^{-i\xi_2 x_3} \right) \left. \right] \left. \right] \frac{2\pi \varepsilon}{D_1} \sin \left( \frac{2\pi x_1}{D_1} \right) \cos \left( \frac{2\pi x_2}{D_2} \right) \\
+ \left[ 2\mu k_2 k_2 K_{21} \left( C_{1u0} e^{i\xi_1 x_3} + C_{1d0} e^{-i\xi_1 x_3} \right) \right. \\
\left. + k_2 K_{22} \left( C_{2u0} e^{i\xi_2 x_3} + C_{2d0} e^{-i\xi_2 x_3} \right) + \xi_3 \left( C_{3u0} e^{i\xi_3 x_3} - C_{3d0} e^{-i\xi_3 x_3} \right) \right] \left. \right] \frac{2\pi \varepsilon}{D_1} \sin \left( \frac{2\pi x_1}{D_1} \right) \sin \left( \frac{2\pi x_2}{D_2} \right) \\
- \left[ 2\mu k_2 k_2 K_{21} \left( C_{1u0} e^{i\xi_1 x_3} + C_{1d0} e^{-i\xi_1 x_3} \right) \right. \\
\left. + k_2 K_{22} \left( C_{2u0} e^{i\xi_2 x_3} + C_{2d0} e^{-i\xi_2 x_3} \right) + \xi_3 \left( C_{3u0} e^{i\xi_3 x_3} - C_{3d0} e^{-i\xi_3 x_3} \right) \right] \left. \right] \frac{2\pi \varepsilon}{D_2} \cos \left( \frac{2\pi x_1}{D_1} \right) \sin \left( \frac{2\pi x_2}{D_2} \right) \\
- \left[ \xi_1 K_{21} \left( C_{1u0} e^{i\xi_1 x_3} - C_{1d0} e^{-i\xi_1 x_3} \right) \right. \\
\left. + \xi_2 K_{22} \left( C_{2u0} e^{i\xi_2 x_3} - C_{2d0} e^{-i\xi_2 x_3} \right) + \xi_3 \left( C_{3u0} e^{i\xi_3 x_3} + C_{3d0} e^{-i\xi_3 x_3} \right) \right] \left. \right] \frac{2\pi \varepsilon}{D_2} \cos \left( \frac{2\pi x_1}{D_1} \right) \sin \left( \frac{2\pi x_2}{D_2} \right) \\
- k_2 \left[ \xi_1 K_{21} \left( C_{1u0} e^{i\xi_1 x_3} - C_{1d0} e^{-i\xi_1 x_3} \right) \right. \\
\left. + \xi_2 K_{22} \left( C_{2u0} e^{i\xi_2 x_3} - C_{2d0} e^{-i\xi_2 x_3} \right) \right] \right] dx_1 dx_2 \]
\[ T_3 = \int_{-D_z/2}^{D_z/2} \int_{-D_z/2}^{D_z/2} \left[ \mu \left[ k_1 \left[ \xi_1 K_{21} \left( C_{1u0} e^{i \xi_1 x_3} - C_{1d0} e^{-i \xi_1 x_3} \right) \right] 
\right. \\
+ \xi_2 K_{22} \left( C_{2u0} e^{i \xi_2 x_3} - C_{2d0} e^{-i \xi_2 x_3} \right) - k_2 \left( C_{3u0} e^{i \xi_3 x_3} + C_{3d0} e^{-i \xi_3 x_3} \right) \right] \\
+ k_1 \left[ \xi_1 K_{11} \left( C_{1u0} e^{i \xi_1 x_3} - C_{1d0} e^{-i \xi_1 x_3} \right) \right] \right] \frac{2 \pi e}{D_1} \sin \left( \frac{2 \pi x_1}{D_1} \right) \cos \left( \frac{2 \pi x_2}{D_2} \right) \\
+ \mu \left[ k_2 \left[ \xi_1 K_{21} \left( C_{1u0} e^{i \xi_1 x_3} - C_{1d0} e^{-i \xi_1 x_3} \right) \right] 
\right. \\
+ \xi_2 K_{22} \left( C_{2u0} e^{i \xi_2 x_3} - C_{2d0} e^{-i \xi_2 x_3} \right) - k_2 \left( C_{3u0} e^{i \xi_3 x_3} + C_{3d0} e^{-i \xi_3 x_3} \right) \right] \\
+ \xi_2 K_{22} \left( C_{2u0} e^{i \xi_2 x_3} - C_{2d0} e^{-i \xi_2 x_3} \right) - k_2 \left( C_{3u0} e^{i \xi_3 x_3} + C_{3d0} e^{-i \xi_3 x_3} \right) \right] \frac{2 \pi e}{D_2} \cos \left( \frac{2 \pi x_1}{D_1} \right) \sin \left( \frac{2 \pi x_2}{D_2} \right) \\
- \left[ 2 \mu \left[ \xi_1 K_{21} \left( C_{1u0} e^{i \xi_1 x_3} + C_{1d0} e^{-i \xi_1 x_3} \right) \right] 
\right. \\
+ \xi_2 K_{22} \left( C_{2u0} e^{i \xi_2 x_3} + C_{2d0} e^{-i \xi_2 x_3} \right) - k_2 \left( C_{3u0} e^{i \xi_3 x_3} + C_{3d0} e^{-i \xi_3 x_3} \right) \right] \frac{2 \pi e}{D_2} \cos \left( \frac{2 \pi x_1}{D_1} \right) \sin \left( \frac{2 \pi x_2}{D_2} \right) \\
\right] dx_1 dx_2 \]

2.6 Integration of traction equations and derivation of dispersion equation

The above three equations have integration terms where the integrations need to be estimated first along \( x_1 \) axis for one period of corrugation \( D_1 \) and then along \( x_2 \) axis for another period of corrugation \( D_2 \). While evaluating the integrations, the stress terms with \( \frac{n_1^+}{n_3^+} \) or \( \frac{n_2^+}{n_3^+} \) automatically become zero. With the remaining terms, one can get the traction equations as follows:
\[ T_1 = \int_{-D_2}^{D_2} \int_{-D_1}^{D_1} \left[ -\mu k_1 \xi_1 (K_{11} + K_{21}) C_{1u0} e^{i\xi_1 x_3} + \mu k_1 \xi_1 (K_{11} + K_{21}) C_{1d0} e^{-i\xi_1 x_3} - \mu k_1 \xi_2 (K_{12} + K_{22}) C_{2u0} e^{i\xi_2 x_3} + \mu k_1 \xi_2 (K_{12} + K_{22}) C_{2d0} e^{-i\xi_2 x_3} + \mu k_1 k_2 C_{3u0} e^{i\xi_3 x_3} + \mu k_1 k_2 C_{3d0} e^{-i\xi_3 x_3} \right] dx_1 dx_2 \]

\[ T_2 = \int_{-D_2}^{D_2} \int_{-D_1}^{D_1} \left[ -2\mu k_2 \xi_1 K_{21} C_{1u0} e^{i\xi_1 x_3} + 2\mu k_2 \xi_1 K_{21} C_{1d0} e^{-i\xi_1 x_3} - 2\mu k_2 \xi_2 K_{22} C_{2u0} e^{i\xi_2 x_3} + 2\mu k_2 \xi_2 K_{22} C_{2d0} e^{-i\xi_2 x_3} + \mu (k_2^2 - \xi_3^2) C_{3u0} e^{i\xi_3 x_3} + \mu (k_2^2 - \xi_3^2) C_{3d0} e^{-i\xi_3 x_3} \right] dx_1 dx_2 \]

\[ T_3 = \int_{-D_2}^{D_2} \int_{-D_1}^{D_1} \left[ -(2\mu \xi_1^2 K_{21} + \lambda k_1^2 K_{11} + \lambda k_2^2 K_{21} + \lambda \xi_1^2 K_{21}) C_{1u0} e^{i\xi_1 x_3} - (2\mu \xi_1^2 K_{21} + \lambda k_1^2 K_{11} + \lambda k_2^2 K_{21} + \lambda \xi_1^2 K_{21}) C_{1d0} e^{-i\xi_1 x_3} - (2\mu \xi_2^2 K_{22} + \lambda k_2^2 K_{12} + \lambda k_2^2 K_{22} + \lambda \xi_2^2 K_{22}) C_{2u0} e^{i\xi_2 x_3} - (2\mu \xi_2^2 K_{22} + \lambda k_2^2 K_{12} + \lambda k_2^2 K_{22} + \lambda \xi_2^2 K_{22}) C_{2d0} e^{-i\xi_2 x_3} + 2\mu k_2 \xi_3 C_{3u0} e^{i\xi_3 x_3} - 2\mu k_2 \xi_3 C_{3d0} e^{-i\xi_3 x_3} \right] dx_1 dx_2 \]

After integrating the traction equations for upper and lower surfaces with respect to \( x_1 \), we can get following equations,

From \( T_1^{x_3^+} \):

\[
\int_{D_2}^{D_2} -\mu D_1 k_1 (K_{11} + K_{21}) \xi_1 J_0 (z \xi_1) e^{ih\xi_1} C_{1u0} \ dx_2 + \int_{D_2}^{D_2} \mu D_1 k_1 (K_{11} + K_{21}) \xi_1 J_0 (z \xi_1) e^{ih\xi_1} C_{1d0} \ dx_2 = \int_{D_2}^{D_2} -\mu D_1 k_1 (K_{12} + K_{22}) \xi_2 J_0 (z \xi_2) e^{ih\xi_2} C_{2u0} \ dx_2 + \int_{D_2}^{D_2} \mu D_1 k_1 (K_{12} + K_{22}) \xi_2 J_0 (z \xi_2) e^{ih\xi_2} C_{2d0} \ dx_2
\]

\[
\int_{D_2}^{D_2} -\mu D_1 k_1 (K_{11} + K_{21}) \xi_1 J_0 (z \xi_1) e^{ih\xi_1} C_{1u0} \ dx_2 + \int_{D_2}^{D_2} \mu D_1 k_1 (K_{11} + K_{21}) \xi_1 J_0 (z \xi_1) e^{ih\xi_1} C_{1d0} \ dx_2 + \int_{D_2}^{D_2} -\mu D_1 k_1 (K_{12} + K_{22}) \xi_2 J_0 (z \xi_2) e^{ih\xi_2} C_{2u0} \ dx_2 + \int_{D_2}^{D_2} \mu D_1 k_1 (K_{12} + K_{22}) \xi_2 J_0 (z \xi_2) e^{ih\xi_2} C_{2d0} \ dx_2 + \int_{D_2}^{D_2} -\mu D_1 k_2 \xi_3 J_0 (z \xi_3) e^{ih\xi_3} C_{3u0} \ dx_2 + \int_{D_2}^{D_2} \mu D_1 k_2 \xi_3 J_0 (z \xi_3) e^{ih\xi_3} C_{3d0} \ dx_2 = 0
\]
From $T_{1}^{x_{3}^{-}}$:

$$
\int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} -\mu D_{1} k_{1} (K_{11} + K_{21}) \xi_{1} J_{0} (z \xi_{1}) e^{-ih\xi_{1}} C_{1u0} \, dx_{2}
\quad + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} \mu D_{1} k_{1} (K_{11} + K_{21}) \xi_{1} J_{0} (z \xi_{1}) e^{ih\xi_{1}} C_{1d0} \, dx_{2}
\quad + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} -\mu D_{1} k_{1} (K_{12} + K_{22}) \xi_{2} J_{0} (z \xi_{2}) e^{-ih\xi_{2}} C_{2u0} \, dx_{2}
\quad + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} \mu D_{1} k_{1} (K_{12} + K_{22}) \xi_{2} J_{0} (z \xi_{2}) e^{ih\xi_{2}} C_{2d0} \, dx_{2}
\quad + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} \mu D_{1} k_{2} J_{0} (z \xi_{3}) e^{-ih\xi_{3}} C_{3u0} \, dx_{2}
\quad + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} \mu D_{1} k_{2} J_{0} (z \xi_{3}) e^{ih\xi_{3}} C_{3d0} \, dx_{2} = 0
$$

From $T_{2}^{x_{3}^{+}}$:

$$
\int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} -2\mu D_{1} k_{2} K_{21} \xi_{1} J_{0} (z \xi_{1}) e^{ih\xi_{1}} C_{1u0} \, dx_{2} + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} \frac{2\mu D_{1} k_{2} K_{21} \xi_{1} J_{0} (z \xi_{1}) e^{-ih\xi_{1}} C_{1d0}}{2} \, dx_{2}
\quad + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} -2\mu D_{1} k_{2} K_{22} \xi_{2} J_{0} (z \xi_{2}) e^{ih\xi_{2}} C_{2u0} \, dx_{2}
\quad + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} \frac{2\mu D_{1} k_{2} K_{22} \xi_{2} J_{0} (z \xi_{2}) e^{-ih\xi_{2}} C_{2d0}}{2} \, dx_{2}
\quad + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} \mu D_{1} (k_{2}^{2} - \xi_{3}^{2}) J_{0} (z \xi_{3}) e^{ih\xi_{3}} C_{3u0} \, dx_{2}
\quad + \int_{-\frac{D_{2}}{2}}^{\frac{D_{2}}{2}} \frac{\mu D_{1} (k_{2}^{2} - \xi_{3}^{2}) J_{0} (z \xi_{3}) e^{-ih\xi_{3}} C_{3d0}}{2} \, dx_{2} = 0
$$
From $T_{x^3}^+\downarrow$:

\[
\int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} -2\mu D_1 k_2 K_{21} \xi_1 J_0(z \xi_1) e^{-i \theta \xi_1} C_{1u0} \, dx_2 + \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 K_{21} \xi_1 J_0(z \xi_1) e^{i \theta \xi_1} C_{1d0} \, dx_2
\]

\[
+ \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} -2\mu D_1 k_2 K_{22} \xi_2 J_0(z \xi_2) e^{-i \theta \xi_2} C_{2u0} \, dx_2
\]

\[
+ \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 K_{22} \xi_2 J_0(z \xi_2) e^{i \theta \xi_2} C_{2d0} \, dx_2
\]

\[
+ \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} \mu D_1 (k_2^2 - \xi_3^2) J_0(z \xi_3) e^{-i \theta \xi_3} C_{3u0} \, dx_2
\]

\[
+ \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} \mu D_1 (k_2^2 - \xi_3^2) J_0(z \xi_3) e^{i \theta \xi_3} C_{3d0} \, dx_2 = 0
\]

From $T_{x^3}^+\downarrow$:

\[
\int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} -D_1 (\lambda k_1^2 K_{11} + K_{21} (\lambda k_2^2 + (\lambda + 2\mu) \xi_1^2)) J_0(z \xi_1) e^{i \theta \xi_1} C_{1u0} \, dx_2
\]

\[
+ \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} -D_1 (\lambda k_1^2 K_{11} + K_{21} (\lambda k_2^2 + (\lambda + 2\mu) \xi_1^2)) J_0(z \xi_1) e^{-i \theta \xi_1} C_{1d0} \, dx_2
\]

\[
+ \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} -D_1 (\lambda k_1^2 K_{12} + K_{22} (\lambda k_2^2 + (\lambda + 2\mu) \xi_2^2)) J_0(z \xi_2) e^{i \theta \xi_2} C_{2u0} \, dx_2
\]

\[
+ \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} -D_1 (\lambda k_1^2 K_{12} + K_{22} (\lambda k_2^2 + (\lambda + 2\mu) \xi_2^2)) J_0(z \xi_2) e^{-i \theta \xi_2} C_{2d0} \, dx_2
\]

\[
+ \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 \xi_3 J_0(z \xi_3) e^{i \theta \xi_3} C_{3u0} \, dx_2
\]

\[
+ \int_{-\frac{D_2}{2}}^{\frac{D_2}{2}} -2\mu D_1 k_2 \xi_3 J_0(z \xi_3) e^{-i \theta \xi_3} C_{3d0} \, dx_2 = 0
\]
From $T_{3}^{x_{3}}$:

\[
\int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} -D_1(\lambda k_1^2 K_{11} + K_{21}(\lambda k_2^2 + (\lambda + 2\mu)\xi_1^2))J_0(z\xi_1)e^{-i\hbar z_1^2}C_{1u0} \, dx_2
\]

\[
+ \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} D_1(\lambda k_1^2 K_{12} - K_{22}(\lambda k_2^2 + (\lambda + 2\mu)\xi_2^2))J_0(z\xi_2) e^{i\hbar z_2^2}C_{1d0} \, dx_2
\]

\[
+ \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} D_1(\lambda k_1^2 K_{12} + K_{22}(\lambda k_2^2 + (\lambda + 2\mu)\xi_2^2))J_0(z\xi_2) e^{-i\hbar z_2^2}C_{2u0} \, dx_2
\]

\[
+ \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} -2\mu D_1 k_2\xi_3 J_0(z\xi_3) e^{-i\hbar z_3^2}C_{3u0} \, dx_2
\]

\[
+ \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} 2\mu D_1 k_2\xi_3 J_0(z\xi_3) e^{i\hbar z_3^2}C_{3d0} \, dx_2 = 0
\]

The above six equations with six unknown amplitudes can be arranged in a matrix form as follows:

\[
\begin{bmatrix}
\tau_{1 x_{3}}^{+} \\
\tau_{1 x_{3}}^{-} \\
\tau_{2 x_{3}}^{+} \\
\tau_{2 x_{3}}^{-} \\
\tau_{3 x_{3}}^{+} \\
\tau_{3 x_{3}}^{-}
\end{bmatrix} =
\begin{bmatrix}
C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\
C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\
C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36} \\
C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46} \\
C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56} \\
C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}
\end{bmatrix}
\begin{bmatrix}
C_{1u0} \\
C_{1d0} \\
C_{2u0} \\
C_{2d0} \\
C_{3u0} \\
C_{3d0}
\end{bmatrix} = 0
\]

(11)

The generalized coefficients $C_{ij}$ where $i, j = 1, 2, 3, 4, 5, 6$ in the matrix equation (11) for a corrugated wave guide can be expressed as:

\[
C_{11} = -\int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} \mu D_1 k_1(K_{11} + K_{21})\xi_1 J_0(z\xi_1) e^{-i\hbar z_1^2} \, dx_2
\]

\[
C_{12} = \int_{-\frac{d_2}{2}}^{\frac{d_2}{2}} \mu D_1 k_1(K_{11} + K_{21})\xi_1 J_0(z\xi_1) e^{-i\hbar z_1^2} \, dx_2
\]
\[ C_{13} = - \int \frac{D_2}{2} \mu D_1 k_1 (K_{12} + K_{22}) \xi_2 J_0 (z \xi_2) e^{i \theta \xi_2} \ dx_2 \]

\[ C_{14} = \int \frac{D_2}{2} \mu D_1 k_1 (K_{12} + K_{22}) \xi_2 J_0 (z \xi_2) e^{-i \theta \xi_2} \ dx_2 \]

\[ C_{15} = \int \frac{D_2}{2} \mu D_1 k_2 J_0 (z \xi_3) e^{i \theta \xi_3} \ dx_2 \]

\[ C_{16} = \int \frac{D_2}{2} \mu D_1 k_2 J_0 (z \xi_3) e^{-i \theta \xi_3} \ dx_2 \]

\[ C_{21} = - \int \frac{D_2}{2} \mu D_1 k_1 (K_{11} + K_{21}) \xi_1 J_0 (z \xi_1) e^{-i \theta \xi_1} \ dx_2 \]

\[ C_{22} = \int \frac{D_2}{2} \mu D_1 k_1 (K_{11} + K_{21}) \xi_1 J_0 (z \xi_1) e^{i \theta \xi_1} \ dx_2 \]

\[ C_{23} = - \int \frac{D_2}{2} \mu D_1 k_1 (K_{12} + K_{22}) \xi_2 J_0 (z \xi_2) e^{-i \theta \xi_2} \ dx_2 \]

\[ C_{24} = \int \frac{D_2}{2} \mu D_1 k_1 (K_{12} + K_{22}) \xi_2 J_0 (z \xi_2) e^{i \theta \xi_2} \ dx_2 \]

\[ C_{25} = \int \frac{D_2}{2} \mu D_1 k_2 J_0 (z \xi_3) e^{-i \theta \xi_3} \ dx_2 \]

\[ C_{26} = \int \frac{D_2}{2} \mu D_1 k_2 J_0 (z \xi_3) e^{i \theta \xi_3} \ dx_2 \]

\[ C_{31} = - \int \frac{D_2}{2} 2 \mu D_1 k_2 K_{21} \xi_1 J_0 (z \xi_1) e^{i \theta \xi_1} \ dx_2 \]
\[ C_{32} = \int_{\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 K_{21} \xi_1 J_0(z \xi_1) e^{-ih\xi_1} \, dx_2 \]

\[ C_{33} = -\int_{\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 K_{22} \xi_2 J_0(z \xi_2) e^{ih\xi_2} \, dx_2 \]

\[ C_{34} = \int_{\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 K_{22} \xi_2 J_0(z \xi_2) e^{-ih\xi_2} C_d \, dx_2 \]

\[ C_{35} = \int_{\frac{D_2}{2}}^{\frac{D_2}{2}} \mu D_1 (k_2^2 - \xi_3^2) J_0(z \xi_3) e^{ih\xi_3} \, dx_2 \]

\[ C_{36} = \int_{\frac{D_2}{2}}^{\frac{D_2}{2}} \mu D_1 (k_2^2 - \xi_3^2) J_0(z \xi_3) e^{-ih\xi_3} \, dx_2 \]

\[ C_{41} = -\int_{\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 K_{21} \xi_1 J_0(z \xi_1) e^{-ih\xi_1} \, dx_2 \]

\[ C_{42} = \int_{\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 K_{21} \xi_1 J_0(z \xi_1) e^{ih\xi_1} \, dx_2 \]

\[ C_{43} = -\int_{\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 K_{22} \xi_2 J_0(z \xi_2) e^{-ih\xi_2} \, dx_2 \]

\[ C_{44} = \int_{\frac{D_2}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 K_{22} \xi_2 J_0(z \xi_2) e^{ih\xi_2} C_d \, dx_2 \]

\[ C_{45} = \int_{\frac{D_2}{2}}^{\frac{D_2}{2}} \mu D_1 (k_2^2 - \xi_3^2) J_0(z \xi_3) e^{-ih\xi_3} \, dx_2 \]

\[ C_{46} = \int_{\frac{D_2}{2}}^{\frac{D_2}{2}} \mu D_1 (k_2^2 - \xi_3^2) J_0(z \xi_3) e^{ih\xi_3} \, dx_2 \]
\[ C_{51} = -\int_{\frac{D_1}{2}}^{\frac{D_2}{2}} D_1(\lambda k^2_1 K_{11} + K_{21}(\lambda k^2_2 + (\lambda + 2\mu)\xi^2_1))J_0(z,\xi) e^{ih\xi_1} dx_2 \]

\[ C_{52} = -\int_{\frac{D_1}{2}}^{\frac{D_2}{2}} D_1(\lambda k^2_1 K_{11} + K_{21}(\lambda k^2_2 + (\lambda + 2\mu)\xi^2_1))J_0(z,\xi) e^{-ih\xi_1} dx_2 \]

\[ C_{53} = -\int_{\frac{D_1}{2}}^{\frac{D_2}{2}} D_1(\lambda k^2_1 K_{12} + K_{22}(\lambda k^2_2 + (\lambda + 2\mu)\xi^2_2))J_0(z,\xi) e^{ih\xi_2} dx_2 \]

\[ C_{54} = -\int_{\frac{D_1}{2}}^{\frac{D_2}{2}} D_1(\lambda k^2_1 K_{12} + K_{22}(\lambda k^2_2 + (\lambda + 2\mu)\xi^2_2))J_0(z,\xi) e^{-ih\xi_2} dx_2 \]

\[ C_{55} = \int_{\frac{D_1}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 \xi_3 J_0(z,\xi) e^{ih\xi_3} dx_2 \]

\[ C_{56} = -\int_{\frac{D_1}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 \xi_3 J_0(z,\xi) e^{-ih\xi_3} dx_2 \]

\[ C_{61} = -\int_{\frac{D_1}{2}}^{\frac{D_2}{2}} D_1(\lambda k^2_1 K_{11} + K_{21}(\lambda k^2_2 + (\lambda + 2\mu)\xi^2_1))J_0(z,\xi) e^{-ih\xi_1} dx_2 \]

\[ C_{62} = \int_{\frac{D_1}{2}}^{\frac{D_2}{2}} D_1(-\lambda k^2_1 K_{11} + K_{21}(-\lambda k^2_2 + (\lambda + 2\mu)\xi^2_1))J_0(z,\xi) e^{ih\xi_1} dx_2 \]

\[ C_{63} = -\int_{\frac{D_1}{2}}^{\frac{D_2}{2}} D_1(\lambda k^2_1 K_{12} + K_{22}(\lambda k^2_2 + (\lambda + 2\mu)\xi^2_2))J_0(z,\xi) e^{-ih\xi_2} dx_2 \]

\[ C_{64} = \int_{\frac{D_1}{2}}^{\frac{D_2}{2}} D_1(-\lambda k^2_1 K_{12} - K_{22}(\lambda k^2_2 + (\lambda + 2\mu)\xi^2_2))J_0(z,\xi) e^{ih\xi_2} dx_2 \]

\[ C_{65} = \int_{\frac{D_1}{2}}^{\frac{D_2}{2}} 2\mu D_1 k_2 \xi_3 J_0(z,\xi) e^{-ih\xi_3} dx_2 \]
\[ C_{66} = - \int_{-D_2}^{D_1} \frac{2\mu D_1 k_2 \xi_3 J_0(z\xi_3)}{D_2} e^{i\theta\xi_3} \, dx_2 \]

The expressions of \( C_{ij} \) involves zeroth order Bessel function and the integrands can be determined numerically along \( x_2 \)-axis. It can be noted that the generalized coefficients \( C_{ij} \) in equation (11) are function of geometric parameters and material properties of waveguide. Therefore, the determinant of matrix \( C_{ij} \) must be zero to obtain the dispersion solutions. In this study, the validity of the solutions was first determined by comparing the solution with a planar wave guide. To obtain Lamb wave dispersion curve for the planar waveguide, the value of \( \varepsilon \) is set to zero and thus 2D corrugated structure becomes a flat plate. The equations for the dispersion relationships in the frequency-wavenumber domain are determined by plugging in geometric parameters and material properties using a root finding algorithm where the determinant of matrix \( C_{ij} \) is evaluated for a set of frequency and wavenumber values. Whenever the determinant becomes zero, the corresponding frequency and wavenumber set is recorded. Since the wavenumber can be of real and imaginary in nature, the wavenumber domain thus contains extended solutions of the dispersion relationship. The root finding algorithm can be translated into any programming language, and in this study MATLAB code was developed to implement this procedure. The procedure, as described in Figure 2-2, is performed individually for a certain assumed frequency with a range of real and imaginary wavenumbers. It can be noted that the root-finding algorithm runs an identical procedure for real and imaginary wavenumbers. With specific geometric parameters and material properties, a range of frequency values are assumed. At each of the frequency values, the determinant of the matrix \( C_{ij} \) is calculated for all the real wavenumbers in the intended
problem domain. This determinant can be of real or imaginary in nature. Either a determinant is real or imaginary, a bisection method is employed to find the wavenumber with a tolerance between two successive iteration is $10^{-6}$.

Figure 2.2: Root finding procedure
CHAPTER 3

RESULTS AND DISCUSSIONS

3.1 Results

In this study Lamb wave propagation in flat and corrugated isotropic plate is investigated using an analytical and numerical method. For both of these plates following material properties of Aluminum are assumed.

- Young’s Modulus, $E$ 72.4 GPa
- Poisson’s Ratio, $\nu$ 0.33
- Density, $\rho$ 2780 kg/m$^3$

Verification of the proposed analytical approach is obtained by comparing the dispersion curves of a 3D planar wave guide with that of the corrugated wave guide by setting the value of $\varepsilon = 0$. Therefore, to achieve this comparison, first, the dispersion curves for the Rayleigh-Lamb wave modes in a planar wave guide are considered as reported in [38]. Next, the dispersion curves are determined by solving the proposed analytical approach (equation 11) while setting the value of $\varepsilon = 0$. The comparison of the dispersion curves are shown in Figure 3-1. To obtain a non-biased generalized solution, the axes of the wavenumber and frequency are made dimensionless. This ensures the generalizations of the guided Rayleigh-Lamb and SH wave propagation phenomena. Since the angular frequency and wavenumber in SI units are rad/sec and 1/m, the dimensionless frequency,
\[ \Omega = \frac{f_d}{C_p} \] and wavenumber, \( \bar{k} = kd \) are considered. For the guided SH waves, \( C_p \) in dimensionless frequency is replaced by \( C_s \). The wave velocities, \( C_p \) and \( C_s \), for Aluminum media are calculated as follows:

\[
\begin{align*}
\lambda &= \frac{E \nu}{(1 - 2\nu)(1 + \nu)} \\
\mu &= \frac{E}{2(1 + \nu)} \\
C_p &= \sqrt{\frac{1 + 2\mu}{\rho}} = 6212 \text{ m/s} \\
C_s &= \sqrt{\frac{\mu}{\rho}} = 3130 \text{ m/s}
\end{align*}
\]

Therefore, a value of \( \Omega = 1.5 \) indicates \( f \approx 3106 \text{ kHz} \) for Rayleigh-Lamb wave and \( f \approx 1565 \text{ kHz} \) for SH wave with a plate thickness of 3 mm (\( 2h = d = 3 \text{ mm} \)). The wavenumbers (\( k \)), which are the roots generated while solving equation 11, at each normalized frequency are calculated utilizing the material properties and geometric dimensions of the planar waveguide. The calculations and subsequent plots of the dispersion curves are performed by writing computer codes in MATLAB. Figure 3-1(a) and 3-1(b) show the comparison of dispersion curves of the P + SV waves. While Figure 1(a) shows the dispersion curves obtained by solving equation 11, Figure 3-1(b) shows the dispersion curves reported previously in the literature [38]. Similarly, the dispersion curves of SH waves are also compared in Figure 3-1(c) and 3-1(d) where the dispersion curves in Figure 3-1(c) are obtained by taking the part of the solution of equation 11. It can be noted that the potential functions assumed in equation 1 in Chapter 2 contain both P+SV and SH wave modes. Therefore, each of the solutions obtained by solving equation...
contains components of both types of P+SV and SH wave modes. In this regard, during the verification phase of the dispersion curves of the flat plate, SH wave modes are separated manually by separating the appropriate SH wave modes. In later cases, for corrugated plates, dispersion curves are presented exactly as obtained from the solutions of equation 11. In Figure 3-1, it can be seen that the dispersion curve obtained using the proposed analytical method for a planar waveguide are almost similar to those reported previously. The first two bands with real wave numbers in present study closely matched with that of the literature results. This verifies the applicability of the analytical solution obtained in this study.

Figure 3.1: Comparison of dispersion curves of Rayleigh-Lamb waves and SH waves. (a) and (c) results obtained in present study, (b) and (d) results reported in literature [38]
Next, the solutions are obtained by introducing and varying the corrugation in the geometry to understand the effect of the corrugation height on the guided waves. The dispersion curves are plotted for $\varepsilon = 0.1h$, $\varepsilon = 0.2h$, $\varepsilon = 0.3h$ and $\varepsilon = 0.5h$ while keeping a fixed periodicity ($D_1 = D_2 = d$) in both $x_1$ and $x_2$ directions. The results are shown in Figure 3-2. It can be seen that additional evanescent modes are generated due to the introduction of corrugation in the geometry. Moreover, additional modes are present in the real part of the dispersion curves. These additional modes in the dispersion curves are the direct result of increased reflections of the guided wave propagation through the orthogonal surface perturbated geometry. By comparing the dispersion curves of the flat wave guide and that of the corrugated wave guide, an anisotropic wave propagation behavior in corrugated structure is evident. To validate this claim, a pictorial presentation of wave propagation is discussed in next paragraphs.
Figure 3.2: Dispersion curves of the orthogonal surface perturbated geometry with varying corrugation height, (a) $0.1h$, (b) $0.2h$, (c) $0.1h$ and (d) $0.1h$

**Validation of Anisotropic wave propagation**

To validate the anisotropic wave propagation behavior in corrugated structure made of isotropic material, the time domain simulations are performed using the FEM model, and displacement of the particles are obtained in various time steps. The multiphysics software COMSOL is used to simulate the propagation of Rayleigh-Lamb wave in a corrugated wave guide. A corrugation amplitude, $\varepsilon$, of $0.5h$ and periodicity, $D_1 = D_2 = d$ are considered for the orthogonally surface perturbated wave guide. The overall dimension of this geometry is assumed as $40 \text{ mm} \times 40 \text{ mm} \times 3 \text{ mm}$ where a cylindrical through thickness hole with a diameter of $2.5 \text{ mm}$ is assume at the center of the wave guide. A cylindrical excitor with a diameter of $2.5 \text{ mm}$ and height of $2h$ is
placed at the center of the geometry to match with the cylindrical hole. A chirp signal with frequencies in the range of 100 kHz to 1000 kHz is used to radially excite the cylinder.

Figure 3.3 shows the geometry of the corrugated waveguide with cylindrical excitor. Stress free boundary conditions are imposed on the upper ($x_1$, $x_3$ plane) and lower ($x_1$, $x_3$ plane) surfaces of the waveguide. In addition, perfectly matched layer boundary conditions are applied around the entire geometry as shown in Figure 3.3. While setting up simulation parameters, the spatial and temporal discretizations are calculated such that the convergence of the solutions are obtained for wave propagations. The maximum element size for the spatial discretization ($\Delta x$) and maximum time step for temporal discretization ($\Delta t$) for achieving the convergence criteria are calculated from the following expressions:

\[ \Delta x = \frac{\lambda_{\text{min}}}{10} \]

\[ \Delta t = \frac{\Delta x}{\sqrt{3} C_{\text{max}}} \]

Where, $\lambda_{\text{min}}$ is the minimum wavelength of the Lamb wave modes and $C_{\text{max}}$ is the maximum phase velocity of the wave propagation modes [41]. A mesh size of 0.4 mm and a maximum time step of $2 \times 10^{-8}$ s are calculated to meet convergence criteria found in [41]. The linear triangular elements are chosen for finite element simulation, and a total of 5,898,108 elements are found based on mesh size.
A similar study is also performed with identical simulation parameters for a plate with corrugation amplitude, $\varepsilon$, of zero. By setting $\varepsilon = 0$, this geometry becomes a flat plate. To run the simulations, a high-performance ~50 core CPU with 128 Gigabyte RAM is utilized for efficient processing of the computations.

Figure 3.4 shows the comparison of displacement behavior of wave propagation in a flat plate and an orthogonal surface perturbed corrugated structure at various time steps. The total displacement behavior which has the displacement components in all three coordinate directions are plotted in Figure 3.4. The Subplots (a) to (e) show the normalized total displacements at various time steps for the flat plate, whereas the subplots (f) to (j) show the normalized total displacements in the corrugated plate. While the normalized total displacement ranges from zero to 1, this normalizations are
performed based on the maximum total displacements of the particles up to 4 \( \mu s \) for the both plates.

![Figure 3.4: Comparison of total displacements of wave propagation from 80 ns to 4 \( \mu s \), (a) to (e) for flat plate, and (f) to (j) corrugated plate.](image)

Clearly, the displacements are originated from the excitor placed at the center of the geometries and propagated towards the edge of the geometries as the time steps increased. At 80 ns, the wave propagation pattern is seen fully circular as the wave is generated around the excitor. As the time increases, guided wave is generated and propagated radially. Since the isotropic material properties are assumed to construct the both of the geometries, the conventional concept is that a circular wave propagation would be observed which is found for the flat plate. However, by looking at the wave propagation at the increased time steps such as from 1 \( \mu s \) to 4 \( \mu s \), it is clearly evident that the wave propagation behavior is non-circular for the corrugated plate which is the case for an anisotropic structure. Similarly, the displacement component only in \( x_3 \) direction also demonstrates the anisotropic wave propagation behavior in corrugated plate as compared to flat plate, which is shown in Figure 3.5. It can be noted that the wave
propagation velocity is affected by the corrugated geometric structure. The outer circular displacements in the flat plate (Figure 3.5d and 3.5e) are larger than the outer rectangular displacements in the corrugated plate. This indicates that the multiple reflections at the corrugations result in smaller growth in the outer particle displacements compared to the flat plate.

Figure 3.5: Comparison of displacement along $x_3$ direction of the wave propagation from 80 ns to 4 $\mu$s, (a) to (e) for flat plate, and (f) to (j) corrugated plate.

Therefore, despite the assumption of the isotropic material properties in the corrugated geometry, presence of the orthogonal surface perturbation influences the wave propagation and an anisotropic wave behavior is observed.
Further, for the corrugated structure, three time history signals are extracted at 30 μs from three points, A, B and C, located on a circle with a radius of 14 mm from the center of the excitation point. These three points are shown in Figure 3.6 (d). The individual time history signal of points A, B and C are shown in Figure 3.6 (a), (b) and (c) respectively. A superposition of these three signals are plotted in Figure 3.6 (e). From these superposition plot, it is evident that the displacement signals of point A and C are of exact match. However, the displacement signal of point B is somewhat different than that of point A and C. This indicates that despite the points A, B and C are on a circular path, the wave propagation is not circular. Therefore, an anisotropic wave behavior is found from orthogonal surface perturbated geometry made of isotropic material.
CHAPTER 4
CONCLUSIONS AND FUTURE WORKS

4.1 Conclusions

In summary, we present an anisotropic wave behavior in isotropic material with orthogonal surface perturbation. First, a generalized analytical expression for the Rayleigh-Lamb wave propagation in flat plate and a corrugated plate with orthogonal surface perturbation in two coordinate directions is developed. The Bloch theorem is applied to the three potential functions introduced by Buchwald (1961) for anisotropic material to derive the dispersion relation with the stress-free boundary conditions. Next, by setting the perturbation height to zero, the dispersion equations are solved using a logical root-finding algorithm for the flat plate and found excellent match with the dispersion curves reported in the literature. After this verification, the dispersion curves are determined by varying the perturbation depth and an anisotropic wave propagation behavior is observed. For validation of the anisotropic wave propagation of the Rayleigh-Lamb wave in isotropic material with orthogonal surface perturbations, a time domain simulation is obtained by the Finite Element Method based COMSOL Multiphysics software. A chirp signal is used to excite the corrugated plate and the displacement of the particles are obtained in various time steps. By investigating the displacements of the particles at various time steps, a non-circular wave propagation behavior is observed.
This ensures that the Rayleigh-Lamb wave propagation in isotropic material with orthogonal surface perturbations is anisotropic.

4.2 Future Work

This study can be extended further to investigate the generalization of the developed expression. Followings are the tasks that can be performed as future work:

1. Determination of the dispersion curves by simultaneously varying the periodicity in two coordinate directions and the perturbation depth.

2. Determination and comparison of the dispersion curves from the time domain analysis of the Finite Element Model for the Flat plate and the corrugated wave guide.

3. Extension of the generalized expressions isotropic materials with various other geometries such as cylindrical structure with orthogonal surface perturbations in multiple coordinate directions. The expression can also be developed and validated for anisotropic material systems.
REFERENCES


