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Widely Digitally Stable Numbers and Irreducibility Criteria For Polynomials With Prime Values

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by

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Dedication

For the friends, family, teachers, and mentors that always believed in me, but especially to my wife, Christian, for being by my side every step of the way.

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Finally, I thank God for the gifts and talents that have enabled me to pursue my degree. I could have never made it this far without His helping hand. He continues to surprise me as He orchestrates my life and things come together in ways I could never have imagined.

Abstract

This dissertation considers three different topics. In the first part of the dissertation, we show for an integer $b > 2$ that if a polynomial $f(x)$ with non-negative integer coefficients is such that $f(b)$ is prime, then there are explicit bounds $M_1(b)$, $M_2(b)$, and $M_3(b)$ such that if the coefficients of $f(x)$ are each $\leq M_1(b)$, then $f(x)$ is irreducible; if the coefficients of $f(x)$ are each $\leq M_2(b)$ and $f(x)$ is reducible, then it is divisible by the shifted cyclotomic polynomial $\Phi_3(x-b)$ for $3 \leq b \leq 5$, and divisible by $\Phi_4(x-b)$ for $b > 5$; and if the coefficients of $f(x)$ are each $\leq M_3(b)$ and $f(x)$ is reducible, then it is divisible by at least one of $\Phi_3(x-b)$ and $\Phi_4(x-b)$. Furthermore, if $b > 69$ and the coefficients of $f(x)$ are each $\leq M_4(b)$, then $f(x)$ is either irreducible or divisible by at least one of $\Phi_3(x-b)$, $\Phi_4(x-b)$, and $\Phi_6(x-b)$.

In the second part of the dissertation, we show that there are only finitely many values of t such that the truncated binomial polynomial of degree 6,

$$q_{6,t}(x) = \sum_{j=0}^{6} \binom{t}{j} x^j.$$

has Galois group $PGL_2(5)$, a transitive subgroup of S_6 isomorphic to S_5. When the Galois group of the truncated binomial of degree 6 is not $PGL_2(5)$, it has been shown to be S_6. Additionally, we show that the truncated binomial of degree 6 is irreducible for all values of t.

In the third part of the dissertation, we show that there are infinitely many composite numbers, N, with the property that inserting a digit between any two digits in base 10 of N, including between any two of the infinitely many leading zeros and to the right of N, always results in a composite number. We show that the same result holds for bases $b \in \{2, 3, \cdots, 8, 9, 11, 31\}$.

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Chapter 1

Introduction

In this dissertation, we explore two main topics pertaining to number theory; polynomial irreducibility and digit insertions. This introduction is therefore split into three sections, two of which pertain to polynomial irreducibility and one on digit insertions. The proofs that follow in the dissertation are split into three chapters based on the first three sections in the introduction. A fourth section of the introduction lists several results which the author helped to establish.

1.1 Irreducibility criteria for polynomials with prime values

If $d_n d_{n-1} \dots d_1 d_0$ is the decimal representation of a prime, then a result of A. Cohn [28] asserts that

$$f(x) = d_n x^n + d_{n-1} x^{n-1} + \cdots + d_1 x + d_0$$

is irreducible over the integers. If we generalize this setting and view $f(x)$ as a polynomial with non-negative integer coefficients with $f(10)$ prime, a few questions arise. Does the irreducibility of $f(x)$ depend on the coefficients being less than 10? Can we find a degree n so that if $f(x)$ has degree less than n, then $f(x)$ is irreducible? Is the base 10 special, or do similar results hold when 10 is replaced by a different base $b \geq 2$?

Some answers to these questions can be found in the literature. The result of Cohn has been extended to all bases $b \geq 2$ by J. Brillhart, M. Filaseta, and A. Odlyzko in [2]. In [9], M. Filaseta extended this to base b representations of kp where k is a positive integer less than b and p is a prime, and M. R. Murty [30] has obtained an analog in function fields over finite fields. Furthermore, [2] allows the coefficients d_j in Cohn's theorem to satisfy $0 \leq d_j \leq 167$ rather than $0 \leq d_j \leq 9$; and later M. Filaseta [10] showed that the d_j need only satisfy $0 \leq d_j \leq 10^{30} d_n$, and further that simply $d_j \geq 0$ suffices if $n \leq 31$.

Recent work by M. Filaseta and S. Gross [12] extended this last line of investigation even further. They showed that if $f(x)$ is a polynomial with non-negative coefficients bounded above by

$$49598666989151226098104244512918$$

and $f(10)$ is prime, then $f(x)$ is irreducible over $\mathbb{Z}$. They also showed that if the coefficients are instead bounded above by

$$8592444743529135815769545955936773,$$

then $f(x)$ is either irreducible over $\mathbb{Z}[x]$ or divisible by $x^2 - 20x + 101$. Furthermore, they showed that these values are sharp, in that they exhibited polynomials having non-negative integer coefficients with $f(10)$ prime and maximum coefficient one more than each of these numbers where in each case the polynomial factors in $\mathbb{Z}[x]$ and in the latter case is divisible by $x^2 - 19x + 91$.

In [4], M. Cole, S. Dunn, and M. Filaseta extended these results and exhibited bounds $M_1(b)$ such that if the coefficients of $f(x)$ are bounded above by $M_1(b)$ and $f(b)$ is prime for an integer $b \in [2, 20]$, then $f(x)$ is irreducible in $\mathbb{Z}[x]$. They also found bounds $M_2(b)$ such that if the coefficients of $f(x)$ are bounded above by $M_2(b)$ and $f(b)$ is prime for $3 \leq b \leq 5$, then $f(x)$ is either irreducible or divisible by $\Phi_3(x - b)$, where $\Phi_n(x)$ is the n^{th} cyclotomic polynomial. Similarly, if $6 \leq b \leq 20$, and the coefficients of $f(x)$ are bounded above by $M_2(b)$, then $f(x)$ is either irreducible or divisible by $\Phi_4(x - b)$. Furthermore, they established that the upper bounds $M_1(b)$ are sharp for $3 \leq b \leq 20$, and that the upper bounds $M_2(b)$ are sharp for $4 \leq b \leq 20$.

Our main goal for this portion of the dissertation is to extend the results in [4] to all integers $b \geq 2$. We rely heavily on the basic techniques established in both [12] and [4], though some care is needed to handle the generality to all $b \geq 2$. We establish the following due to M. Filaseta, J. Foster, J. Southwick, and the author.

Theorem 1.1. *Let $b \in \mathbb{Z}$ with $b > 2$. Let $f(x)$ be a polynomial with non-negative integer coefficients and $f(b)$ prime. Let $\Phi_n(x)$ be the n^{th} cyclotomic polynomial and define $D = D_b = \left\lfloor \dfrac{\pi}{\arctan(1/b)} \right\rfloor$. Then the following hold.*

- *For $b > 5$, define $M_1(b) = max\left(M_1^{(1)}(b), M_1^{(2)}(b)\right)$ where*

$$M_1^{(1)}(b) = \left(\sum_{0 \leq k \leq \frac{D}{2}} \binom{D}{2k+1} b^{D-2k-1}(-1)^k \right)(b^2 - 2b + 2)$$

  and

$$M_1^{(2)}(b) = \left(\sum_{0 \leq k \leq \frac{D-1}{2}} \binom{D-1}{2k+1} b^{D-2k-2}(-1)^k \right)(b^2 - 2b + 2).$$

  If each coefficient of $f(x)$ is less than or equal to $M_1(b)$, then $f(x)$ is irreducible.

- *There exists a similarly explicit bound, $M_2(b)$, such that if the coefficients of $f(x)$ are less than or equal to $M_2(b)$ and $f(x)$ is reducible, then $f(x)$ is divisible by $\Phi_3(x-b)$ if $3 \leq b \leq 5$, and divisible by $\Phi_4(x-b)$ if $b > 5$.*

- *For $b > 2$, there exists a similarly explicit bound, $M_3(b)$, such that if the coefficients of $f(x)$ are less than or equal to $M_3(b)$ and $f(x)$ is reducible, then $f(x)$ is divisible by at least one of $\Phi_3(x-b)$ and $\Phi_4(x-b)$.*

- *For $b > 69$, there exists an explicit bound, $M_4(b)$, such that if the coefficients of $f(x)$ are less than or equal to $M_4(b)$ and $f(x)$ is reducible, then $f(x)$ is divisible by at least one of $\Phi_3(x-b)$, $\Phi_4(x-b)$, and $\Phi_6(x-b)$.*

Expressions for $M_1(b)$, $M_2(b)$, $M_3(b)$, and $M_4(b)$ are given in Chapter 2. Whereas M. Cole, S. Dunn, and M. Filaseta were able to show for a fixed $b \in [4, 20] \cap \mathbb{Z}$

that the given upper bounds are sharp, we were not able to do so for general $b \geq 2$. However, the bounds in Theorem 1.1 agree with the prior sharp bounds obtained for $4 \leq b \leq 20$, and we conjecture the bounds $M_1(b)$, $M_2(b)$, and $M_3(b)$ in Theorem 1.1 are sharp for all $b \geq 4$. Furthermore, the values $M_1(b)$, $M_2(b)$, and $M_3(b)$ are sharp in another way: For $n \in \{3, 4, 6\}$, if $\Phi_n(x - b)$ is a factor of $f(x)$ and $f(x)$ has non-negative coefficients (and where we no longer require that $f(b)$ is prime), then the largest coefficient must be at least as large as $M_1(b)$, $M_2(b)$, or $M_3(b)$, respectively, and there is a polynomial that achieves this bound.

The proof of Theorem 1.1 is discussed in Chapter 2, noting that some of the details that are omitted are made explicit in Joseph C Foster's dissertation. The proof relies heavily on the techniques established in [12] and [4]. Throughout Chapter 2, irreducibility will refer to irreducibility in $\mathbb{Z}[x]$.

On the degree side of the questions, S. Gross and M. Filaseta [12] showed that for $f(x)$ a polynomial with non-negative coefficients with $f(10)$ prime, if the degree of $f(x)$ is less than or equal to 31, then $f(x)$ is irreducible. They also showed that if the degree of $f(x)$ is instead less than or equal to 34, then $f(x)$ is reducible only in the case that $\Phi_4(x - 10) = x^2 - 20x + 101$ divides $f(x)$. They further showed that if 34 is instead replaced by 36, then $f(x)$ is reducible only if it is divisible by one of $\Phi_3(x - 10)$ and $\Phi_4(x - 10)$.

Continuing to explore the degree of $f(x)$, M. Cole, S. Dunn, and M. Filaseta [4] showed that for $f(x)$ a polynomial with non-negative coefficients such that $f(b)$ is prime for $2 \leq b \leq 20$, there are sharp bounds $D(b), D_1(b)$, and $D_2(b)$ on the degree of $f(x)$ such that if $f(x)$ has degree less than or equal to $D(b)$, then $f(x)$ is irreducible; if $f(x)$ has degree less than or equal to $D_1(b)$, then $f(x)$ is only reducible if it is divisible by $\Phi_4(x - b)$; while if $f(x)$ has degree less than or equal to $D_2(b)$, then $f(x)$ must be divisible by $\Phi_3(x - b)$ or $\Phi_4(x - b)$.

We state here an extension of the results in [4] due to M. Filaseta, J. Foster, J. Southwick, and the author for $b \geq 5$ that integrates a similar divisibility condition with $\Phi_6(x-b)$ that is joint work with .

Theorem 1.2. *Fix an integer $b \geq 5$, and for $n \in \{3, 4, 6\}$ set*

$$D_n = D_n(b) = \left\lfloor \frac{\pi}{\arg(b+\zeta_n)} \right\rfloor \qquad \text{and} \qquad D = D(b) = \left\lfloor \frac{\pi}{\arctan\left(\frac{1732}{1000(2b+1)}\right)} \right\rfloor.$$

Let $f(x) \in Z[x]$ with non-negative coefficients and with $f(b)$ prime. If the degree of $f(x)$ is $\leq D_4$, then $f(x)$ is irreducible. Additionally, if the degree of $f(x)$ is $\leq D_3$ and $f(x)$ is reducible, then $f(x)$ is divisible by $\Phi_4(x-b)$ and not divisible by $\Phi_3(x-b)$ or $\Phi_6(x-b)$. Furthermore, in the case that $b \geq 27$, if the degree of $f(x)$ is $\leq D_6$ and $f(x)$ is reducible, then $f(x)$ is divisible by either $\Phi_4(x-b)$ or $\Phi_3(x-b)$ and not by $\Phi_6(x-b)$. Lastly, in the case that $b \geq 27$, if the degree of $f(x)$ is $\leq D$ and $f(x)$ is reducible, then $f(x)$ is divisible by $\Phi_3(x-b)$, $\Phi_4(x-b)$, or $\Phi_6(x-b)$.

The proof of Theorem 1.2 is discussed further in Jeremiah Southwick's dissertation [36].

1.2 Results pertaining to the truncated binomial polynomials

In 2018, M. Filaseta and R. Moy [16] studied the Galois group of a truncated binomial expansion. They start by defining for t and r non-negative integers, the polynomial

$$p_{r,t}(x) = \sum_{j=0}^{r} \binom{t+j}{j} x^j.$$

This polynomial arises from a normalization of the t^{th} derivative of $1 + x + \cdots + x^{t+r}$ and has been conjectured to be irreducible for all values of r and t. In [1], the polynomial is shown to be irreducible for t sufficiently large when r is fixed. The polynomial is considered in a different form in [15] where M. Filaseta, A. Kumchev, and D. Pasechnik study a truncated binomial expansion of $(x+1)^n$, that is

$$q_{r,n}(x) = \sum_{j=0}^{r} \binom{n}{j} x^j \qquad \text{for } r < n.$$

In [1] and [15], identities involving $p_{r,t}(x)$ and $q_{r,n}(x)$ are established. If we define

$$\widetilde{p}_{r,t}(x) = x^r p_{r,t}(1/x) = \sum_{j=0}^{r} \binom{t+j}{j} x^{r-j},$$

then according to [1] we have

$$\widetilde{p}_{r,t}(x+1) = \sum_{j=0}^{r} \binom{t+r+1}{j} x^{r-j}.$$

They show that these polynomials are related to a truncated binomial expansion via $\widetilde{p}_{r,t}(x+1) = x^r q_{r,t+r+1}(1/x)$.

The irreducibility over $\mathbb{Q}$ of one of $p_{r,t}(x)$, $\widetilde{p}_{6,t}(x)$, $\widetilde{p}_{6,t}(x+1)$, and $q_{r,t+r+1}(x-1)$ implies the irreducibility of the other three. Also, since the roots for each all generate the same number field, we have that the Galois groups over $\mathbb{Q}$ associated with these polynomials are all the same.

In [16], M. Filaseta and R. Moy show that the Galois group over $\mathbb{Q}$ of most of these polynomials are the symmetric group. In particular, they show that for the t values that make $p_{r,t}(x)$ irreducible, $p_{r,t}(x)$ has Galois group S_r for each r a positive integer not equal to 6. For $r = 6$, they show that there are at most $O(\log T)$ values of $t \leq T$ for which the Galois group of $p_{6,t}(x)$ is not S_6 and is instead $PGL_2(5)$, a transitive subgroup of S_6 isomorphic to S_5. Notice that the set of values for t such that $p_{6,t}(x)$ has Galois group $PGL_2(5)$ is not necessarily finite. They are able to give an explicit example of a polynomial having Galois group $PGL_2(5)$, namely $q_{6,10}(x)$.

In Chapter 3, we show, based on ideas of Jeremy Rouse (private communications), that the following theorem holds.

Theorem 1.3. *There are at most finitely many values for t such that $p_{6,t}(x)$ has Galois group $PGL_2(5)$ over $\mathbb{Q}$.*

Notice that this is an improvement over the result found in [16] for $r = 6$. For any progress in determining explicitly the finitely many values of t indicated in Theorem 1.3, we would need to know when $p_{6,t}(x)$ is irreducible. In Section 3.2, following ideas of [1] and [15], we show that the polynomial $p_{6,t}(x)$ is irreducible for all values of t using Newton polygons to create Thue equations and elliptic curves whose integer solutions are values of t where $\widetilde{p}_{6,t}(x)$ or $\widetilde{p}_{6,t}(x+1)$ could potentially factor. As there are a finite number of equations to solve, each with a finite number of integer solutions, we reduce showing the irreducibility of $p_{6,t}(x)$ for all t, to showing the irreducibility of $p_{6,t}(x)$ for a finite list of t values.

Section 3.3 uses the irreducibility of $p_{6,t}(x)$ to prove Theorem 1.3. This is achieved by creating the resolvent polynomial, a polynomial that holds information about the Galois group over $\mathbb{Q}$ of a given polynomial. The resolvent polynomial is created using elementary symmetric polynomials and an extension of the elementary symmetric polynomials. Appendix C exhibits the resolvent polynomial. Observing that the resolvent polynomial has genus 7, we use Siegel's Theorem to see that their are finitely many integer roots of the resolvent polynomial. Each integer root corresponds to a t value for which the Galois group of $p_{6,t}(x)$ is $PGL_2(5)$, thus proving Theorem 1.3.

1.3 Results pertaining to widely digitally stable composite numbers

In 1979, P. Erdős [8], answering a problem posed by M. S. Klamkin [24], showed that there are infinitely many prime numbers N, with the property that you can change any digit in the base 10 representation of N, and the resulting number will be composite. The first such prime, called a *digitally delicate* prime in [23], is 294001. Later, T. Tao [38] showed a positive proportion of the primes are digitally delicate; and J. Hopper and P. Pollack [23] resolved a question of Tao's allowing for an arbitrary but fixed number of digit changes to the beginning and end of the prime.

As Konyagin [25] (also see [14]) has pointed out, the methods of the others above imply that a positive proportion of composite numbers N, coprime to 10, satisfy the property that if any digit in the base 10 representation of N is changed, then the resulting number remains composite. For example, the number $N = 212159$ satisfies this property. That is, every number in the set

$$\{d12159, 2d2159, 21d159, 212d59, 2121d9, 21215d : d \in \{0, 1, 2, \ldots, 9\}\}$$

is composite. Note that if we remove the requirement that N be coprime to 10, then the results are less intriguing. For example, for m a positive integer, the number

$$N = (11 \cdot 13 \cdot 17 \cdot 19 \cdot 10)m + 15$$

is composite and remains composite when replacing any digit with $d \in \{0, 1, \ldots, 9\}$. This can be easily seen since changing any digit besides the right-most digit will result in a number divisible by 5. A similar conclusion can be obtained by looking at the number

$$N = (11 \cdot 13 \cdot 17 \cdot 19 \cdot 10)m + 12.$$

When starting with a composite number N, M. Filaseta, M. Kozek, C. Nicol, and J. Selfridge [14] noted that you can get an existence result for inserting a digit instead of replacing a digit. In other words, they showed that there are infinitely many composite numbers N, coprime to 10, with the property that if you insert any digit into the base 10 representation of N, then the resulting number remains composite. The first composite number with this property is $N = 25011$; that is, each number in the set

$$\{d25011, 2d5011, 25d011, 250d11, 2501d1, 25011d : d \in \{0, 1, 2, \ldots, 9\}\}$$

is composite. We call such numbers *digitally stable* composite numbers. A similar result is not currently known to hold for prime numbers N.

In 2020, M. Filaseta and J. Southwick [17] showed that a positive proportion of the primes N, are such that you can replace any digit in the base 10 representation of N, *including one of the infinitely many leading zeros*, and the resulting number will be composite. They called such numbers *widely digitally delicate* primes. Notice that the first digitally delicate prime, 294001, is not widely digitally delicate since 10294001 is prime. In fact, even though a positive proportion of the primes are widely digitally delicate, no specific examples of widely digitally delicate primes are known.

In this dissertation, we also consider the leading zeros, but in the case of composite numbers N.

Definition 1.4. A *widely digitally stable* composite number is a composite number, coprime to 10, that remains composite when any digit is inserted in the decimal expansion of N, including between two of the infinitely many leading zeros of N and to the right of the units digit of N.

In Chapter 4, we focus on widely digitally stable composite numbers in base 10. We establish the following theorem [26] due to M. Filastea, J. Southwick, and the author.

Theorem 1.5. *There are infinitely many widely digitally stable composite numbers.*

Providing an explicit example of a widely digitally stable composite number does not appear to be easy. For example, the smallest digitally stable composite number 25011 is not a widely digitally stable composite number as inserting a one to the right of the third leading zero results in the prime number 10025011. The proof of Theorem 1.5 is nevertheless constructive, so in theory one can construct such a number from our proof. Furthermore, we emphasize that unlike the prior results which give that a positive proportion of the primes or numbers satisfy a certain property, we do not know if a positive proportion of the composite numbers are widely digitally stable.

In Section 4.1, we start with a number N defined to have a string of leading sevens. We then break the argument up into three cases: inserting a digit into the leading zeros, inserting a digit into the leading sevens, and inserting a digit into the right most digits of N. For the first two cases, we exhibit specific covering systems in a manner similar to [14] to show that for sufficiently large composite numbers N satisfying certain congruences, inserting a digit into the leading sevens or zeros results in a number divisible by at least one prime in a finite set $\mathcal{P}$. Section 4.2 looks at the coverings used for inserting a digit in the leading zeros, while Section 4.3 and Appendix A looks at the coverings used for inserting a digit in the leading sevens. In Section 4.4, we show that inserting a digit in the right-most digits of N results in a composite number by observing that their are a fixed number of insertions that correspond to the right-most digits. Since there are a fixed number of insertions, we guarantee that each insertion results in a composite number. Theorem 1.5 then follows.

Because our argument depends on exhibiting particular covering systems which allow for inserting a digit in base 10, our proof does not generalize directly to other bases. However, the rest of our argument only depends on the existence of such a covering system, so to prove the statement analogous to Theorem 1.5 for a given base b it suffices to exhibit enough coverings to take care of each digit insertion. We have found such covering systems for bases $b \in \{2, 3, \ldots, 11\}$, which are listed in Appendix B. As b gets larger, the coverings become harder to establish. We also looked at the base $b = 31$ since finding a covering is simplified in the case that $b - 1$ has distinct small prime factors. That is, we use the primes 2, 3, and 5 in such a way that we only need 16 coverings instead of the initial 62 coverings. Constructing the composite number N to have a string of leading sevens, we are able to utilize the prime 7 to reduce this count to only 12 coverings. We were able to show that their are infinitely many widely digitally stable composite numbers in base 31, and the details for that argument can be found in Appendix B as well. We emphasize again though that we do not know whether a result similar to Theorem 1.5 holds for an arbitrary base.

The corresponding case of $b = 2$ in Theorem 1.5 has received prior attention in the literature in a different form. Examples of widely digitally stable composite numbers can be obtained by looking through a list of Sierpiński numbers. A Sierpiński number is a number n such that $n \cdot 2^k + 1$ is composite for all non-negative integers k.

The notion of a Sierpiński number relates to the notion of a widely digitally stable composite number in the following way: Let $n \in \mathbb{Z}^+$. One can show, as was shown for Riesel numbers in Lemma 4 of [11], that if there is a finite set of primes $\mathcal{P}$ such that for all sufficiently large k, some prime $p \in \mathcal{P}$ divides $n \cdot 2^k + 1$, then for all k in $\mathbb{Z}$, the number $n + 2^k$ is divisible by some prime $p \in \mathcal{P}$.

When looking at the base 2 representation of each Sierpiński number, we know that inserting a one (or zero) into the leading zeros results in a composite number. To find an example of a widely digitally stable composite number in base 2, we found composite Sierpiński numbers, N, that remain composite when inserting a zero or one between any two digits of N and to the right of N. Since we know that inserting a digit in the leading zeros results in a composite number, each Sierpiński number only has a finite number of insertions to check.

We note that the number

$$6135559 = (10111011001111100000111)_2$$

is an example of a number N in base 2 which remains composite when any binary digit 0 or 1 is inserted in the number including between two of the infinitely many leading binary zeros of N and to the right of the units digit of N. We searched the numbers listed through [35] for digitally stable composite numbers and found 6135559. It is widely digitally stable since 6135559 is digitally stable and has been proven to be a Sierpiński number with covering set

$$\mathcal{P} = \{3, 5, 7, 13, 17, 241\},$$

so that the numbers $6135559 + 2^k$ are also composite for all non-negative integers k since any base 2 digit insertion in 6135559 results in a number larger than 241. This provides an explicit example of a widely digitally stable number in base 2. Other examples in base 2 include the base 2 representations of the base 10 numbers 7134623, 8629967, 9454129, 16010419, 16907749, 34158143, and 34629797.

There are a number of interesting open questions related to Theorem 1.5 that can be asked, which we address in Section 4.5. One such open question merges the idea of inserting a digit with that of replacing a digit of N.

1.4 Other Results

We now proceed to other results due to the author whose details do not appear elsewhere in this dissertation. In [13], M. Filaseta and the author use methods similar to those used for Theorem 1.5 to prove the following theorem.

Theorem 1.6. *For any positive integer k, there exist k consecutive primes all of which are widely digitally delicate.*

The proof of Theorem 1.6 considers increasing and decreasing a digit in the $k+1^{\text{st}}$ digit by x of a number N, denoted $N \pm x \cdot 10^k$. For each digit change, we construct a covering of the integers in a similar manner to that of Chapter 4, where the congruences in the coverings each correspond to a congruence condition on N that guarantees that the corresponding digit change results in a composite number. Many of the coverings used in the proof of Theorem 1.6 are the same as the coverings found in Chapter 4 and Appendix B.

The resulting numbers N that satisfy the congruence conditions can be written as $N = Bx + A$ for fixed positive integers A and B with $\gcd(A, B) = 1$. That is, every number in the arithmetic progression $N \equiv A \pmod{B}$ is such that changing any digit, including any one of the infinitely many leading zeros, results in a composite number. Since $\gcd(A, B) = 1$, we know that there are infinitely many prime numbers in this arithmetic progression. D. Shiu [33] showed that in any arithmetic progression containing infinitely many primes, that is, $Bx + A$ with $\gcd(A, B) = 1$ and $B > 0$, there are arbitrarily long sequences of consecutive primes. Thus, we establish through covering systems that such an arithmetic progression exists where every prime in the arithmetic progression is widely digitally delicate, and then D. Shiu's result immediately applies to finish the proof of Theorem 1.6.

We can also use the work done in [13] to prove the following theorem related to widely digitally stable composite numbers.

Theorem 1.7. *A positive proportion of the composite numbers coprime to 10 remain composite when we replace any digit in the decimal expansion, including any of the infinitely many leading zeros.*

In [13], we showed that there are relatively prime positive integers A and B with $\gcd(10, B) = 1$ such that if N is a positive integer for which $N \equiv A \pmod{B}$, then every digit change in N, including any one of the infinitely many leading zeros of N, results in a composite number. Taking D to be any positive integer and K to be the number of digits of D, then almost all numbers $N \equiv A \pmod{B}$ and $N \equiv D \pmod{10^K}$ are composite. Thus, we are able to deduce that there is a positive density of positive composite integers N ending with the digits in D and satisfying the property that they remain composite after changing any single digit of N, including any one of its infinitely many leading zeros. That is, if we let $S_{w,D}(x)$ be the number of composite numbers up to x ending with the digits in D that remain composite when any single digit, including any of its infinitely many leading zeros, is changed, and let $S_D(x)$ be the number of composite numbers up to x ending with the digits in D, then

$$\liminf_{x\to\infty} \frac{S_{w,D}(x)}{S_D(x)} \geq \frac{1}{B} > 0.$$

Theorem 1.7 follows by taking D to be coprime to 10.

Another result due to the author utilizes Newton polygons. In [19], J. Foster, J. Southwick, and the author use Newton polygons to establish that a certain family of polynomials is irreducible. The work stems from [20], in which Heim, Luca, and Neuhauser study the functions

$$\exp\left(x \sum_{n\geq 1} g(n) \frac{q^n}{n}\right) = \sum_{n\geq 0} P_n^g(x) q^n,$$

where $q = e^{2\pi i \tau}$ with τ in the upper complex half-plane and $g : \mathbb{N} \to \mathbb{N}$ is an arithmetic function normalised such that $g(1) = 1$. Under these conditions, the polynomials $P_n^g(x)$ can be shown to satisfy the recursive relation

$$P_n^g(x) = \frac{x}{n} \sum_{k=1}^{n} g(k) P_{n-k}^g(x)$$

for all $n \geq 1$ with $P_0^g(x) = 1$. In the case that $g(n) = \sigma(n) = \sum_{d|n} d$, the roots of the polynomials $P_n^\sigma(x)$ dictate the vanishing properties of the n-th Fourier coefficients of powers of the Dedekind eta function (see, for example, [22], [27]).

Using the fact that $n \leq \sigma(n) \leq n^2$, Heim, Luca and Neuhauser consider the family of polynomials $P_n^g(x)$ where $g(n) = n$ and $g(n) = n^2$. When $g(n) = n$, the recursive formula for $P_n^g(x)$ gives the closed form

$$P_n^g(x) = x \sum_{k=0}^{n-1} \binom{n-1}{k} \frac{x^k}{(k+1)!}.$$

Using a result of Schur [31], they show that for $g(n) = n$ the polynomial $P_n^g(x)$ is x times an irreducible polynomial (over $\mathbb{Q}$).

Schur's result does not apply for $g(n) = n^2$, in which case Heim, Luca, and Neuhauser show that $P_n^g(x) = x\widetilde{P}_n(x)$ where

$$\widetilde{P}_n(x) = \sum_{j=0}^{n-1} \frac{1}{(j+1)!} \binom{n+j}{2j+1} x^j.$$

They establish that $\widetilde{P}_n(x)$ is Eisenstein when $n-1$ is prime, and hence, in this case, $\widetilde{P}_n(x)$ is irreducible. By constructing the Newton polygons with respect to primes that divide $n-1$, J. Foster, J. Southwick, and the author establish the following theorem.

Theorem 1.8. *The polynomials $\widetilde{P}_n(x)$ are irreducible over $\mathbb{Q}$ for all integers $n \geq 2$.*

Using methods similar to those established by Heim and Neuhauser in [21], J. Foster and J. Southwick have generalized Theorem 1.8 to hold when $g(n) = n^k$ for $k \in \mathbb{Z}^+$.

Chapter 2

Irreducibility Criteria for Polynomials with Prime Values

2.1 Root Bounding Function

The following lemma can be found in [10, Lemma 1] and is meant to motivate the steps we take.

Lemma 2.1. *Fix an integer $b \geq 2$. Let $f(x)$ be a polynomial with non-negative integer coefficients such that $f(b)$ is prime. If $f(x)$ is reducible, then $f(x)$ has a non-real root in the disc $\mathfrak{D}_b = \{z \in \mathbb{C} : |b - z| \leq 1\}$.*

Proof. Assume that $f(x)$ is reducible. Then $f(x) = g(x)h(x)$ where $g(x)$ and $h(x)$ are in $\mathbb{Z}[x]$, have positive leading coefficients, and are not identically ± 1. Since $f(b)$ is prime, we may take, without loss of generality, $g(b) = \pm 1$ and $h(b) = \pm p$. Let c be the leading coefficient of $g(x)$, and denote its roots including multiplicity $\beta_1, \ldots, \beta_m$. Thus, the degree of $g(x)$ is m and we have

$$1 = |g(b)| = |c| \prod_{j=1}^{m} |b - \beta_j| \geq \prod_{j=1}^{m} |b - \beta_j|.$$

Thus, we conclude that at least one of the roots of $g(x)$, and hence of $f(x)$, is in $\mathfrak{D}_b$. The lemma then follows when we recall that $f(x)$ has no positive real roots since it has non-negative coefficients. $\square$

A motivating idea for the remainder of this chapter is to replace the disc $\mathfrak{D}_b$ in Lemma 2.1 with a different region such that if $\alpha = re^{i\theta}$ is in this region, then $|\theta|$ is bounded above by a small number.

For a given integer $b \geq 6$, our main goal is to establish the upper bounds $M_n(b)$ in Theorem 1.1. We will utilize three main methods. First, we introduce certain rational functions that will give us information about the location of possible roots of $f(x)$ assuming $f(x)$ is reducible. While better rational functions can be chosen, as in [4], we will make choices to simplify the results in later sections. Second, we will obtain four upper bounds for the coefficients of $f(x)$ such that if a bound is satisfied, then $f(x)$ cannot have a root at a certain location. Third, we will show that the minimum of these four bounds is $M_1(b)$; hence, if the coefficients of $f(x)$ are bounded above by $M_1(b)$, $f(x)$ cannot have roots at the locations required for $f(x)$ to be reducible. In the remainder of this section, we focus on the first of these ideas.

Recall that $\Phi_n(x)$ denotes the n^{th} cyclotomic polynomial, and let $\zeta_n = e^{2\pi i/n}$. As usual, for $z \in \mathbb{C}$, the notation $\overline{z}$ will refer to the complex conjugate of z. Thus, $\overline{\zeta_n} = e^{-2\pi i/n}$. Fix an integer $b \geq 2$, and let $f(x)$ be a non-constant polynomial with non-negative integer coefficients such that $f(b)$ is prime. Suppose $f(x) = g(x)h(x)$, where $g(x)$ and $h(x)$ are in $\mathbb{Z}[x]$, have positive leading coefficients, and are not identically ± 1. Since $f(b)$ is prime, we may take, without loss of generality, $g(b) = \pm 1$ and $h(b) = \pm f(b)$. Using the ideas of [12], we want to show that either $g(x)$ has a root in common with one of

$$\Phi_3(x-b) = x^2 - (2b-1)x + b^2 - b + 1,$$
$$\Phi_4(x-b) = x^2 - 2bx + b^2 + 1,$$
$$\Phi_6(x-b) = x^2 - (2b+1)x + b^2 + b + 1,$$

or $g(x)$ has a root in a certain region $\mathcal{R}_b$ to be defined shortly.

We define

$$\mathcal{F}_b(z) = \frac{\mathcal{N}_b(z)}{\mathcal{D}_b(z)}, \tag{2.1}$$

where

$$\mathcal{N}_b(z) = |b-1-z|^{2e_2}(|b+\zeta_3 - z||b+\overline{\zeta_3} - z|)^{2e_3} \cdot (|b+i-z||b-i-z|)^{2e_4}(|b+\zeta_6 - z||b+\overline{\zeta_6} - z|)^{2e_6},$$

$$\mathcal{D}_b(z) = |b-z|^{4(e_3+e_4+e_6)+2(e_2+d+1)},$$

and e_2, e_3, e_4, e_6, and d are all non-negative integers that could depend on b. Although we want some flexibility on the choices for e_2, e_3, e_4, e_6, and d for a given b, for clarity, we indicate in Table 2.1 the choices for these variables we use to establish Theorem 1.1. Note that the values for $b \leq 20$ are the same as the values chosen in [4]. The values

Table 2.1 Numbers used in $\mathcal{F}_b(z)$ for b

b	2	3	4	5	$6 \leq b \leq 20$	$b \geq 21$
$e_2(b)$	20	0	0	0	0	0
$e_3(b)$	4	15	9	6	4	1
$e_4(b)$	0	2	2	2	2	1
$e_6(b)$	0	0	3	3	3	1
$d(b)$	0	3	3	3	3	1

we chose for $b \geq 21$ achieve our purposes for all $b \geq 7$; however, the bounds acquired in Section 2.4 are not good enough to include $b \leq 20$. Thus, we will refer to [4] to make a statement about all $b \geq 2$. Our choices for $b \geq 21$ were based on trial and error to give us our desired results.

Setting $z = x + iy$, direct computations show that the expressions in $\mathcal{N}_b$ and $\mathcal{D}_b$ simplify to

$$|b-1-z|^2 = y^2 + (x-b)^2 + 2(x-b) + 1,$$

$$(|b+\zeta_3 - z||b+\overline{\zeta_3} - z|)^2 = y^4 + (2(x-b)^2 + 2(x-b) - 1)y^2 + \big((x-b)^2 + (x-b) + 1\big)^2,$$

$$(|b+i-z||b-i-z|)^2 = y^4 + (2(x-b)^2 - 2)y^2 + ((x-b)^2 + 1)^2,$$

$$(|b+\zeta_6-z||b+\overline{\zeta_6}-z|)^2 = y^4+(2(x-b)^2-2(x-b)-1)y^2$$
$$+((x-b)^2-(x-b)+1)^2,$$

and

$$|b-z|^2 = y^2+(x-b)^2.$$

Notice that each one of these expressions is in $\mathbb{Z}[b,x,y^2]$, namely in $\mathbb{Z}[x-b,y^2]$. Thus, $\mathcal{N}_b(z)$ and $\mathcal{D}_b(z)$ are in $\mathbb{Z}[b,x,y^2]$, making $\mathcal{F}_b(z)$ a rational function in b, x, and y^2. Moreover, we observe that for each integer $b \geq 3$, the polynomial

$$\mathcal{P}_b(x,y) = \mathcal{D}_b(x+iy) - \mathcal{N}_b(x+iy) \tag{2.2}$$

can be written as

$$\mathcal{P}_b(x,y) = \sum_{j=0}^{r} a_j(b,x)y^{2j} \tag{2.3}$$

where $r = 2(e_3+e_4+e_6)+e_2+d+1$ and each $a_j(b,x)$ is an integer polynomial in b and x. We write the factor $g(x)$ of $f(x)$ in the form

$$g(x) = c\prod_{j=1}^{m}(x-\beta_j),$$

where c is the leading coefficient of $g(x)$ and $\beta_1,\ldots,\beta_m$ are the roots of $g(x)$ and, hence, roots of $f(x)$. One can check that

$$\frac{|g(b-1)|^{2e_2}|g(b+\zeta_3)g(b+\overline{\zeta_3})|^{2e_3}|g(b+i)g(b-i)|^{2e_4}|g(b+\zeta_6)g(b+\overline{\zeta_6})|^{2e_6}}{|g(b)|^{4(e_3+e_4+e_6)+2(e_2+d+1)}}$$

and

$$\frac{1}{c^{2(d+1)}}\prod_{j=1}^{m}\mathcal{F}_b(\beta_j)$$

are equal. We denote this common value by $V = V_b(g)$.

Since each of $g(b+\zeta_3)g(b+\overline{\zeta_3})$, $g(b+i)g(b-i)$, and $g(b+\zeta_6)g(b+\overline{\zeta_6})$ are symmetric polynomialsw in the roots of an irreducible monic quadratic in $\mathbb{Z}[x]$, we conclude that each of these expressions are themselves integers. Also, $g(b-1)$ is an integer, and by assumption, $g(b) = \pm 1$. Thus, by looking at the first expression for V, either $V = 0$ or $V \in \mathbb{Z}^+$.

We can say more about when $V = 0$. Since $f(x)$ has non-negative integer coefficients, it cannot have a positive real root, and neither can its factor $g(x)$. Therefore, $g(b-1) \neq 0$. Either expression for V now implies that $V = 0$ if and only if $g(b+\zeta_3)g(b+\overline{\zeta_3})$, $g(b+i)g(b-i)$, or $g(b+\zeta_6)g(b+\overline{\zeta_6})$ is zero, which happens precisely when $g(x)$ is divisible by at least one of $\Phi_3(x-b)$, $\Phi_4(x-b)$, or $\Phi_6(x-b)$. If one of these is not a factor of $g(x)$, we have $V \in \mathbb{Z}^+$. Observe that $\mathcal{F}_b(z)$ is a non-negative real number for all $z \in \mathbb{C}$ with $z \neq b$. By looking at the product in the second expression for V, we see that if $V \neq 0$, then $\mathcal{F}_b(\beta_j) \geq 1$ for at least one value of $j \in \{1, \ldots, r\}$. Said differently, if $V \neq 0$, then there is a root β_j of $g(x)$, and thus of $f(x)$, such that $\mathcal{F}_b(\beta_j) \geq 1$.

More generally the above ideas imply, given only that $g(x) \in \mathbb{Z}[x]$, $g(b-1) \neq 0$, $g(x) \not\equiv \pm 1$, and $g(b) = \pm 1$, either $g(x)$ is divisible by at least one of $\Phi_3(x-b)$, $\Phi_4(x-b)$, and $\Phi_6(x-b)$, or $g(x)$ has a root β in the region

$$\mathcal{R}_b = \{z \in \mathbb{C} : \mathcal{F}_b(z) \geq 1\}. \tag{2.4}$$

Figure 2.1 is an illustration of our choice for $\mathcal{R}_b$ when $b \geq 21$.

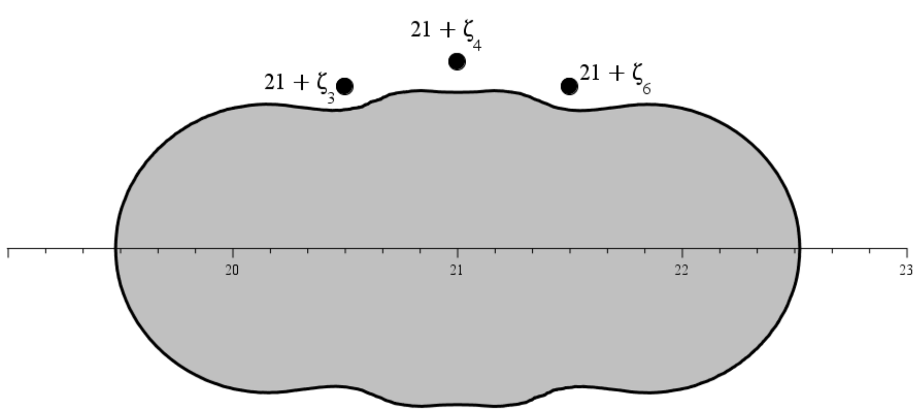

Figure 2.1: The region $\mathcal{R}_{21}$ along with possible roots of $g(x)$

While analyzing the region $\mathcal{R}_b$, we will sometimes refer to points (x, y) in $\mathcal{R}_b$. This is to be interpreted as the point $z = x + iy$ in the complex plane in $\mathcal{R}_b$. To further help analyze $\mathcal{R}_b$, we consider $\mathcal{P}_b(x, y)$ defined in equations (2.2) and (2.3). The definition of $\mathcal{D}_b(z)$ implies that $\mathcal{D}_b(z) > 0$ for all complex $z \neq b$. Thus

$$\mathcal{F}_b(x + iy) \geq 1 \quad \text{and} \quad \mathcal{P}_b(x, y) \leq 0$$

are equivalent for $z \neq b$, and $\mathcal{F}_b(x+iy) = 1$ and $\mathcal{P}_b(x, y) = 0$ are equivalent for $z \neq b$. Note that $\mathcal{P}_b(b, 0) = \mathcal{D}_b(b) - \mathcal{N}_b(b) = 0 - 1 = -1$. Therefore, the $z \in \mathbb{C}$ such that $\mathcal{F}_b(z) = 1$ correspond exactly to the points (x, y) where $\mathcal{P}_b(x, y) = 0$.

The following lemma corresponds to [12, Lemma 2], and the variation [4, Lemma 3.1].

Lemma 2.2. *Fix an integer $b \geq 2$, and let (e_2, e_3, e_4, e_6, d) be either as in Table 2.1 or with $(e_2, e_3, e_4, e_6, d) = (0, 1, 1, 1, 1)$ for $b \geq 5$. Then there exist real numbers $a_0 = a_0(b)$, $a_1 = a_1(b)$, and a non-negative real-valued function $\rho_b(x)$ defined on the interval $I_b = [b - a_0, b + a_1]$ such that:*

(i) $\mathcal{P}_b(x, y) \neq 0$ *for all* $x \notin I_b$ *and* $y \in \mathbb{R}$.

(ii) $\mathcal{P}_b(x, \rho_b(x)) = 0$ *for all* $x \in I_b$.

(iii) $\rho_b(b - a_0) = 0$ *and* $\rho_b(b + a_1) = 0$.

(iv) *The function $\rho_b(x)$ is continuously differentiable on the interior of I_b and is continuous on I_b.*

(v) *If x and y are real numbers for which $\mathcal{P}_b(x, y) \leq 0$, then $x \in I_b$ and $|y| \leq \rho_b(x)$.*

In view of the above lemma, complex numbers of the form $x + i\rho_b(x)$ are boundary points of $\mathcal{R}_b$, which are on or above the real axis. Since $\mathcal{P}_b(x, y)$ is a polynomial in y^2 with coefficients in $\mathbb{Z}[b, x]$, the region $\mathcal{R}_b$ is symmetric about the real axis. Thus, the points $x - i\rho_b(x)$ are boundary points of $\mathcal{R}_b$ which lie on or below the real axis. The points $b - a_0$ and $b + a_1$ are boundary points on the real axis.

A proof of Lemma 2.2 can be found in [4, Lemma 3.1]. In their proof, they reduce the dependence on b to strictly a dependence on $e_2(b)$, $e_3(b)$, $e_4(b)$, $e_6(b)$, and $d(b)$. They then use Sturm sequences on the resulting polynomials in x to prove the lemma. One can verify computationally that the desired properties hold for our choice of $(e_2, e_3, e_4, e_6, d) = (0, 1, 1, 1, 1)$ when $b \geq 21$, with $a_0 = a_1 = 1.522009\ldots$. Thus, the lemma follows. For further details, the reader should see [4].

In the next section, we will use Lemma 2.2 to prove irreducibility criteria based on the coefficients of $f(x)$ when $f(x)$ has a root in $\mathcal{R}_b$ for $b \geq 21$.

2.2 A First Bound on the Coefficients

Throughout this section, $\mathcal{R}_b$ is as defined in (2.4), with $\mathcal{F}_b(z)$ given by (2.1) and $\mathcal{P}_b(x, y)$ given by (2.2). We use the values for e_2, e_3, e_4, e_6, and d given in Table 2.1.

We summarize what we have done so far. Let b be an integer ≥ 2, and let $f(x)$ be in $\mathbb{Z}[x]$ with non-negative coefficients and $f(b)$ prime. We write $f(x) = g(x)h(x)$ with $g(x) \not\equiv \pm 1$, $h(x) \not\equiv \pm 1$, and both $g(x)$ and $h(x)$ having positive leading coefficients. Using the fact that $f(b)$ is prime, we reduced our considerations to $g(b) = \pm 1$. We then showed that either $g(x)$, and hence $f(x)$, is divisible by at least one of $\Phi_3(x - b)$, $\Phi_4(x - b)$, or $\Phi_6(x - b)$, or $g(x)$ has a root $\beta \in \mathcal{R}_b$.

A goal of ours is to establish bounds on the coefficients of $f(x)$ based on the location of a root of $g(x)$. We start by considering the case that $g(x)$ has the root $\beta \in \mathcal{R}_b$. Since M. Cole, S. Dunn, and M. Filaseta already found bounds for $2 \leq b \leq 20$, we restrict our attention to $b \geq 21$. We rely heavily on the following lemma.

Lemma 2.3. *Let* $f(x) = \sum_{j=0}^{m} a_j x^j \in \mathbb{Z}[x]$*, where* $a_j \geq 0$ *for* $j \in \{0, 1, \ldots, m\}$ *and* $a_m > 0$*. Suppose* $\alpha = re^{i\theta}$ *is a root of* $f(x)$ *with* $0 < \theta < \pi/2$ *and* $r > 1$*. Let*

$$B = \max_{\pi/(2\theta) < k < \pi/\theta} \left\{ \frac{r^k(r-1)}{1 + \cot(\pi - k\theta)} \right\},$$

where k *denotes an integer. Then there is some* $j \in \{0, 1, \ldots, m-1\}$ *such that* $a_j > Ba_m$*.*

The proof of Lemma 2.3 is similar to that of [10, Theorem 5], and is established in the above form in [12]. We use Lemma 2.3 to prove the following corollary.

Corollary 2.4. *Let* $b \geq 21$ *be an integer. Let* $f(x) = \sum_{j=0}^{m} a_j x^j \in \mathbb{Z}[x]$ *be such that* $a_j \geq 0$ *for each* j *and* $a_m > 0$*. Let* $\phi = \arctan(0.8444/(b - 0.2))$*. If* $f(x)$ *has a root* $\beta \in \mathcal{R}_b$*, then* $f(x)$ *has a coefficient* $a_j > \mathcal{B}_b$ *where*

$$\mathcal{B}_b = \frac{(b - 1.5221)^{\kappa}(b - 2.5221)}{1 + \cot(\pi/b^2)} \qquad \textit{with} \qquad \kappa = \kappa(b) = \left\lfloor \frac{(b^2 - 1)\pi}{b^2 \phi} \right\rfloor. \tag{2.5}$$

Before proceeding to the proof of Corollary 2.4, we note that the value for $\mathcal{B}_b$ is not the best bound that can be achieved. The values $b - 1.5221$ and 0.8444 appear respectively as bounds for the least real part and the largest imaginary part of elements in $\mathcal{R}_b$. The region chosen for values $b \leq 20$ proved difficult to generalize for all b, which is why we have different values for d and the e_i's for $b \geq 21$.

Proof of Corollary 2.4. Suppose that $f(x)$ is as described with a root $\beta \in \mathcal{R}_b$. Taking B as in Lemma 2.3, we have $B \geq B_k$, where

$$B_k = \frac{r^k(r-1)}{1 + \cot(\pi - k\theta)}, \tag{2.6}$$

for each $k \in \mathbb{Z} \cap (\pi/(2\theta), \pi/\theta)$.

Fix an integer $b \geq 21$. Let $\beta = re^{i\theta}$ be a root of $f(x)$ in the region $\mathcal{R}_b$. Since $f(x)$ has no positive real roots, we have $\theta \neq 0$. Since the conjugate of β is also a root of $f(x)$ and $\mathcal{R}_b$ is symmetric about the real axis, we may take $\theta > 0$. Denote the imaginary part of β by $\mathrm{Im}(\beta)$, and the real part of β by $\mathrm{Re}(\beta)$. Using Sturm sequences for $\mathcal{P}_b(x+b, 8444/10000)$ and $\mathcal{P}_b(b-15221/10000, y)$, which do not depend on b, we see that $0 < \mathrm{Im}(\beta) < 0.8444$ and $b - 1.5221 < \mathrm{Re}(\beta)$. Then $r = \sqrt{\mathrm{Im}(\beta)^2 + \mathrm{Re}(\beta)^2}$ satisfies $r > b - 1.5221$. Note that in Maple, one must use fractions instead of decimals when finding Sturm sequences.

We claim that the line L, defined by $y = 0.8444x/(b-0.2)$, lies completely above the region $\mathcal{R}_b$ for $b \geq 21$. We use a Sturm sequence to see that

$$\mathcal{P}_{21}(x, 8444x/(10000(21-0.2))) = \mathcal{P}_{21}(x, 2111x/52000)$$

has no real roots. Using Lemma 2.2 (ii), we deduce that the line L does not intersect the region $\mathcal{R}_{21}$. Using a Sturm sequence with $\mathcal{P}_{21}(x, 0.8444)$, we deduce similarly that the line $y = 0.8444$ does not intersect the region $\mathcal{R}_{21}$. Since $\mathcal{R}_b$ is simply a horizontal translation of $\mathcal{R}_{21}$, we see that the line $y = 0.8444$ lies strictly above $\mathcal{R}_b$ for $b \geq 21$. The lines $y = 0.8444$ and L intersect at the point $(b-0.2, 0.8444)$. The angle between these two lines is $\arctan(0.8444/(b-0.2))$. It follows that, since the line L lies strictly above the region $\mathcal{R}_{21}$, we have also that the line L lies strictly above the region $\mathcal{R}_b$. Thus, we have an upper bound for the angle θ, given by $0 < \theta < \phi < \pi/2$, where

$$\phi := \arctan\left(\frac{0.8444}{b-0.2}\right).$$

Let

$$\kappa = \kappa(b) = \left\lfloor \frac{(b^2-1)\pi}{b^2\theta} \right\rfloor.$$

From the first inequality in [36, Equation 3.8], we see that

$$\theta < \arg(b+\zeta_4) = \arctan\left(\frac{1}{b}\right) \leq \arctan\left(\frac{1}{5}\right) < \frac{\pi}{6} \quad \text{for } b \geq 9. \tag{2.7}$$

We now argue that κ is in the desired range noting that

$$\kappa\theta < \frac{(b^2-1)\pi}{b^2\theta}\theta = \frac{(b^2-1)\pi}{b^2} < \pi,$$

and

$$\kappa\theta > \left(\frac{(b^2-1)\pi}{b^2\theta} - 1\right)\theta = \frac{(b^2-1)\pi}{b^2} - \theta \geq \frac{24\pi}{25} - \theta > \frac{\pi}{2}.$$

Putting these together we see that

$$\frac{\pi}{2\theta} < \kappa < \frac{\pi}{\theta},$$

so $\kappa \in \mathbb{Z} \cap (\pi/(2\theta), \pi/\theta)$. We now bound the denominator of B_k, for $k = \kappa$, in (2.6). We have

$$\frac{\pi}{2} > \pi - \kappa\theta > \pi - \frac{b^2-1}{b^2}\pi = \frac{\pi}{b^2} > 0,$$

which gives us $\cot(\pi - \kappa\theta) < \cot(\pi/b^2)$.

Thus we have established that

$$B > \frac{r^\kappa(r-1)}{1 + \cot(\pi - \kappa\theta)} > \mathcal{B}_b,$$

for κ defined above and $\mathcal{B}_b$ as in (2.5). In conclusion, Lemma 2.3 implies that if $f(x)$ has a root in $\mathcal{R}_b$, then $f(x)$ has a coefficient $a_j > Ba_m > \mathcal{B}_b a_m \geq \mathcal{B}_b$, completing the proof. □

2.3 Bounds Based on Recurrence Relations

In this section we will establish results that will help us find bounds $\mathcal{B}_b^{(n)}$ for $n \in \{3, 4, 6\}$ such that if $f(x)$ is divisible by $\Phi_n(x-b)$, then $f(x)$ must have a coefficient $\geq \mathcal{B}_b^{(n)}$. We take $b \geq 5$.

Much of this section is based on the work done in [4] and [12]. We give enough background from these to describe our work for general b.

Fix positive integers A and B and integers b_j such that

$$f(x) = h(x)g(x) = (b_0x^s + b_1x^{s-1} + \cdots + b_{s-1}x + b_s)(x^2 - Ax + B)$$

is a polynomial of degree $m = s+2$ with non-negative integer coefficients. We restrict ourselves to the case where

$$x^2 - Ax + B = \Phi_n(x-b), \quad \text{with } n \in \{3, 4, 6\}$$

so that

$$(A, B) \in \{(2b-1, b^2-b+1), (2b, b^2+1), (2b+1, b^2+b+1)\}.$$

Define $b_j = 0$ for all $j < 0$ and all $j > s$. Since the coefficients of $f(x)$ are all non-negative, we deduce that $b_0 \geq 1$ and $b_j \geq Ab_{j-1} - Bb_{j-2}$ for all $j \in \mathbb{Z}$. Define

$$\beta_j = \begin{cases} 0 & \text{if } j < 0, \\ 1 & \text{if } j = 0, \\ A\beta_{j-1} - B\beta_{j-2} & \text{if } j \geq 1, \end{cases} \tag{2.8}$$

so that the β_j satisfy a recurrence relation for $j \geq -1$. In particular, $\beta_1 = A$ and $\beta_2 = A^2 - B$. Also, with our restriction on our choice of $x^2 - Ax + B$ above, we have

$$\beta_1 \in \{2b-1, 2b, 2b+1\},$$

so the sequence $\beta_0, \beta_1, \ldots$ is initially increasing. We obtain a closed form for the solution to this recurrence relation. The recurrence relation has characteristic polynomial $x^2 - Ax + B$, which has roots $b + \zeta_n$ and $b + \overline{\zeta_n}$ for some $n \in \{3, 4, 6\}$. So β_j has the closed form

$$\beta_j = c_1(b + \zeta_n)^j + c_2(b + \overline{\zeta_n})^j,$$

for some constants c_1 and c_2 depending on A and B. Taking $j = 0$, we obtain $c_2 = 1 - c_1$; and taking $j = -1$, we see that $c_1 = (b + \zeta_n) / \left(\zeta_n - \overline{\zeta_n}\right)$. Substituting these values for c_1 and c_2 and reducing, we deduce

$$\begin{aligned} \beta_j &= \frac{1}{\zeta_n - \overline{\zeta_n}} \left[(b+\zeta_n)^{j+1} - (b+\overline{\zeta_n})^{j+1}\right] \\ &= \frac{|b+\zeta_n|^{j+1} e^{i(j+1)\arg(b+\zeta_n)} - |b+\overline{\zeta_n}|^{j+1} e^{-i(j+1)\arg(b+\zeta_n)}}{\zeta_n - \overline{\zeta_n}} \\ &= \frac{|b+\zeta_n|^{j+1}}{\sin(2\pi/n)} \sin((j+1)\arg(b+\zeta_n)). \end{aligned} \tag{2.9}$$

We note that B is the constant term of the minimal polynomial for $b + \zeta_n$, so

$$B = |b + \zeta_n|^2. \tag{2.10}$$

For ease of notation, we set

$$\theta = \arg(b + \zeta_n) \in (0, \pi/2) \quad \text{and} \quad D = \lfloor \pi/\theta \rfloor \tag{2.11}$$

where θ and D depend on both b and n. From (2.9), we obtain

$$\beta_j = \frac{\sqrt{B}^{j+1}}{\sin(2\pi/n)} \sin((j+1)\theta). \tag{2.12}$$

By taking $j = 0$ in (2.12), with (2.8), we see that

$$\sin(\theta) = \frac{\sin(2\pi/n)}{\sqrt{B}}. \tag{2.13}$$

Define J to be the smallest positive integer such that

$$\beta_{J+1} < \beta_J \qquad \text{and} \qquad \beta_{J-1} < \beta_J.$$

Analyzing the derivative of β_j with respect to j one can show that J is well-defined and

$$J = D - 2 \qquad \text{or} \qquad J = D - 1, \tag{2.14}$$

with explicit details found in Joseph C Foster's dissertation.

We are interested in A and B with $f(x)$ divisible by $\Phi_n(x - b) = x^2 - Ax + B$, where $n \in \{3, 4, 6\}$. We view A and B as fixed. We want $f(x)$ to have non-negative integer coefficients but with the largest coefficient as small as possible. Theorem 3.8 of [36] implies such $f(x)$ exist (also, see Lemma 3 in [10]). Let

$$M = M(A, B) \text{ be the maximum coefficient for such an } f(x).$$

Let $\ell \in \mathbb{Z}^+$. We consider he matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & \dots & 1 & 1 & 1 \\ -A & B & 0 & \dots & 0 & 0 & 0 \\ 1 & -A & B & \dots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & B & 0 & 0 \\ 0 & 0 & 0 & \dots & -A & B & 0 \\ 0 & 0 & 0 & \dots & 1 & -A & B \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \vdots \\ \mu_{\ell-3} \\ \mu_{\ell-2} \\ \mu_{\ell-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{2.15}$$

in the unknowns $\mu_0, \mu_1, \dots, \mu_{\ell-1}$. Let $\ell = J + 1$. We make some observations about the solutions to (2.15), as well as produce a closed form for such a solution.

Lemma 2.5. *Let b be an integer ≥ 5. Let A and B be such that $x^2 - Ax + B$ is $\Phi_n(x-b)$ where $n \in \{3, 4, 6\}$. The above matrix equation has a solution where $\mu_j > 0$ for all $0 \leq j \leq J$.*

Proof. Let $\mathcal{M}$ be the matrix in (2.15) with $\ell = J + 1$. The equation in (2.15) arising from the second row of $\mathcal{M}$ is equivalent to the condition $\mu_1 = A\mu_0/B$. The next $\ell - 2 = J - 1$ rows of $\mathcal{M}$ correspond to a recurrence relation for $\mu_0, \mu_1, \dots, \mu_J$ beginning with μ_0 and μ_1 and satisfying

$$\mu_j = \frac{A\mu_{j-1}}{B} - \frac{\mu_{j-2}}{B} \qquad \text{for } 2 \leq j \leq J.$$

We will have a solution in μ_j to (2.15) provided then that we can find $\mu_0 > 0$ for which $\mu_1 = A\mu_0/B$ and the above recurrence gives $\sum_{0 \leq j \leq J} \mu_j = 1$. Observe that with μ_0 defined arbitrarily, this solution gives each μ_j as a multiple of μ_0. With this in mind, we define $\mu_j^* = \mu_j/\mu_0$.

As before, we find the characteristic polynomial for this recurrence relation to find the general term. This recurrence has characteristic polynomial $x^2 - Ax/B + 1/B$, which is the reciprocal polynomial of $x^2 - Ax + B$ divided by B; hence, the roots of the characteristic polynomial are $1/(b + \zeta_n)$ and $1/(b + \overline{\zeta_n})$. So

$$\mu_j^* = c_1 \left(\frac{1}{b + \zeta_n} \right)^j + c_2 \left(\frac{1}{b + \overline{\zeta_n}} \right)^j \tag{2.16}$$

for $j \geq 0$, where c_1 and c_2 are constants to be determined. Taking $j = 0$ in (2.16), we see that since $\mu_0^* = 1$ we obtain $c_2 = 1 - c_1$. Next, we take $j = 1$ and use both $\overline{\zeta_n} - \zeta_n = -2i \sin(2\pi/n)$ and (2.10). Since $\mu_1^* = A/B$, we obtain

$$c_1 = \frac{\frac{A}{B} - \frac{1}{b+\overline{\zeta_n}}}{\frac{1}{b+\zeta_n} - \frac{1}{b+\overline{\zeta_n}}} = \frac{A - (b + \zeta_n)}{-2i \sin(2\pi/n)}$$

and

$$c_2 = 1 - c_1 = \frac{\overline{\zeta_n} - \zeta_n}{-2i \sin(2\pi/n)} - \frac{A - (b + \zeta_n)}{-2i \sin(2\pi/n)} = \frac{A - (b + \overline{\zeta_n})}{2i \sin(2\pi/n)}.$$

Thus, for $0 \leq j \leq J$, we have

$$\begin{aligned}
\mu_j^* &= \frac{A - (b + \zeta_n)}{-2i \sin(2\pi/n)} \left(\frac{1}{b + \zeta_n} \right)^j + \frac{A - (b + \overline{\zeta_n})}{2i \sin(2\pi/n)} \left(\frac{1}{b + \overline{\zeta_n}} \right)^j \\
&= \frac{1}{2B \sin(2\pi/n)} \left[\frac{B - A(b + \overline{\zeta_n})}{i(b + \zeta_n)^{j-1}} + \frac{A(b + \zeta_n) - B}{i(b + \overline{\zeta_n})^{j-1}} \right].
\end{aligned}$$

Recall (2.11). As in (2.9), for any $t \in \mathbb{Z}$, we deduce

$$(b + \overline{\zeta_n})^t - (b + \zeta_n)^t = -2i|b + \zeta_n|^t \sin(t\theta).$$

Taking $t = j$ and $t = j - 1$ with $0 \leq j \leq J$, we obtain

$$\begin{aligned}
\mu_j^* &= \frac{1}{2B \sin(2\pi/n)} \left[\frac{2Ai|b + \zeta_n|^j \sin(j\theta) - 2Bi|b + \zeta_n|^{j-1} \sin((j-1)\theta)}{i|b + \zeta_n|^{2(j-1)}} \right] \\
&= \frac{1}{B \sin(2\pi/n)} \left[\frac{A|b + \zeta_n|^j \sin(j\theta) - B|b + \zeta_n|^{j-1} \sin((j-1)\theta)}{B^{j-1}} \right].
\end{aligned}$$

Using (2.8) and (2.9), we get

$$\mu_j^* = \frac{A\beta_{j-1} - B\beta_{j-2}}{B^j} = \frac{\beta_j}{B^j}$$

for $1 \leq j \leq J$. As $\mu_0^* = 1 = \beta_0/B^0$, we see that $\mu_j^* = \beta_j/B^j$ for all $j \in [0, J] \cap \mathbb{Z}$. The definition of J implies μ_j^* is positive for $0 \leq j \leq J$.

Recall that we want $\sum_{0 \le j \le J} \mu_j = 1$, which is equivalent to $\sum_{0 \le j \le J} \mu_j^* = 1/\mu_0$. Set

$$K = \sum_{j=0}^{J} \mu_j^* = \sum_{j=0}^{J} \frac{\beta_j}{B^j}.$$

As $K > 0$, we can take $\mu_0 = 1/K$ to deduce that the lemma holds. □

From the proof of Lemma 2.5, we have a closed form for $\mu_j > 0$ satisfying (2.15). Using geometric series, one can write a closed form for K as

$$K = \frac{B^{J+1} - \beta_{J+1} + \beta_J}{B^{J-1}|b + \zeta_n - B|^2},$$

with explicit details found in Joseph C Foster's dissertation.

To finish this section, we establish three bounds, $\mathcal{B}_b^{(3)}$, $\mathcal{B}_b^{(4)}$, and $\mathcal{B}_b^{(6)}$ such that if $f(x)$ has coefficients less than or equal to $\mathcal{B}_b^{(n)}$ with $f(b)$ prime, then $f(x)$ cannot be divisible by $\Phi_n(x - b)$ for $n \in \{3, 4, 6\}$. Recall that $M(A, B)$ is the smallest value that the largest coefficient of $f(x)$ can be given that $f(x)$ is a polynomial with non-negative integer coefficients divisible by $\Phi_n(x - b)$ for some $n \in \{3, 4, 6\}$. Note the difference between $\mathcal{B}_b^{(n)}$ and $M(A, B)$; $M(A, B)$ does not require $f(b)$ to be prime while $\mathcal{B}_b^{(n)}$ does. To establish such bounds, we show both of the following:

(i) The value of $M(A, B)$ is $(1 - A + B) \cdot \beta_J$ for each $n \in \{3, 4, 6\}$.

(ii) If the maximal coefficient of $f(x)$ equals $M(A, B)$, then $f(b)$ is composite.

Assuming (i) and (ii), we explain now that we can take $\mathcal{B}_b^{(n)} = M(A, B)$. If $f(x)$ has each coefficient less than $M(A, B)$, then $f(x)$ cannot be divisible by $\Phi_n(x - b)$ by the minimality of $M(A, B)$. Note that this conclusion does not require $f(b)$ to be prime. If we further require $f(b)$ to be prime, then by (ii), we would also have that the largest coefficient of $f(x)$ cannot equal $M(A, B)$. Hence, we can take $\mathcal{B}_b^{(n)} = M(A, B)$.

Note the dependence on n; A and B are the integers such that $x^2 - Ax + B$ is $\Phi_n(x - b)$; hence, A and B depend on n. Also, β_J depends on D and θ, both defined in (2.11), which depend on n. The details of the proofs of (i) and (ii) are made explicit in Joseph C Foster's dissertation. Thus, for $n \in \{3,4,6\}$, we have bounds $\mathcal{B}_b^{(n)}$ such that if $f(x)$ is a polynomial with non-negative integer coefficients with $f(b)$ a prime, with each coefficient less than or equal to

$$\mathcal{B}_b^{(n)} = (1 - A + B)\beta_J, \tag{2.17}$$

then $f(x)$ cannot be divisible by $\Phi_n(x-b)$.

2.4 Inequalities on the bounds

We summarize what we have done so far. Let b be an integer greater than or equal to 2, and let $f(x)$ be in $\mathbb{Z}[x]$ with non-negative coefficients and $f(b)$ prime. We write $f(x) = g(x)h(x)$ with $g(x) \not\equiv \pm 1$, $h(x) \not\equiv \pm 1$, and both $g(x)$ and $h(x)$ having positive leading coefficients. Using the fact that $f(b)$ is prime, we reduced our considerations to $g(b) = \pm 1$. We then showed that either $g(x)$, and hence $f(x)$, is divisible by at least one of $\Phi_3(x-b)$, $\Phi_4(x-b)$, and $\Phi_6(x-b)$, or $g(x)$ has a root $\beta \in \mathcal{R}_b$. In the previous sections we established that if the coefficients of $f(x)$ are less than or equal to $\mathcal{B}_b$ defined in (2.5) from Section 2.2, then $f(x)$ does not have a root in the region $\mathcal{R}_b$, and if the coefficients of $f(x)$ are less than or equal to $\mathcal{B}_b^{(n)}$ defined in (2.17) from Section 2.3, then $f(x)$ does not have a root in common with $\Phi_n(x-b)$ for $n \in \{3,4,6\}$.

Letting $M(b)$ be the minimum of these four bounds, then if the coefficients of $f(x)$ are less than or equal to $M(b)$, $f(x)$ is irreducible. In this section we finish the proof of Theorem 1.1 by showing for $b \geq 6$ the inequalities

$$\mathcal{B}_b^{(4)} < \mathcal{B}_b^{(3)} < \mathcal{B}_b^{(6)}$$

hold, and then by finding, for each $n \in \{3,4,6\}$, a lower bound for b satisfying $\mathcal{B}_b^{(n)} < \mathcal{B}_b$. Having established these inequalities, we will have that for b large enough, each of the following hold; if the coefficients of $f(x)$ are less than or equal to $\mathcal{B}_b^{(4)}$, then $f(x)$ is irreducible; if the coefficients of $f(x)$ are less than or equal to $\mathcal{B}_b^{(3)}$ and $f(x)$ is reducible, then $f(x)$ is divisible by $\Phi_4(x-b)$; if the coefficients of $f(x)$ are less than or equal to $\mathcal{B}_b^{(6)}$ and $f(x)$ is reducible, then $f(x)$ is divisible by $\Phi_4(x-b)$ or $\Phi_3(x-b)$; and if the coefficients of $f(x)$ are less than or equal to $\mathcal{B}_b$ and $f(x)$ is reducible, then $f(x)$ is divisible by $\Phi_4(x-b)$, $\Phi_3(x-b)$, or $\Phi_6(x-b)$.

We begin by showing that $\mathcal{B}_b^{(4)} < \mathcal{B}_b^{(3)}$. Recall that $\mathcal{B}_b^{(n)} = M(A,B) = (1-A+B)\beta_J$. Also recall that A, B, D, J, β_j, and θ all depend on n and b. For clarity, we introduce subscripts on these values to indicate which n we are considering. Thus, we want to establish the values of b such that

$$\frac{\mathcal{B}_b^{(4)}}{\mathcal{B}_b^{(3)}} = \frac{(1-A_4+B_4)\beta_{J_4}}{(1-A_3+B_3)\beta_{J_3}} < 1.$$

By the definition of J in (2.14), we have $\beta_{J_3} > \beta_{J_3-1}$. Since $\Phi_n(x) = x^2 - A_n x + B_n$ and recalling the definition of β_j in (2.12) for $n=4$ and $n=3$, we obtain

$$\frac{\mathcal{B}_b^{(4)}}{\mathcal{B}_b^{(3)}} < \frac{(1-A_4+B_4)\beta_{J_4}}{(1-A_3+B_3)\beta_{J_3-1}} = \frac{\sqrt{3}(b^2-2b+2)\sqrt{b^2+1}^{J_4+1}\sin((J_4+1)\theta_4)}{2(b^2-3b+3)\sqrt{b^2-b+1}^{J_3}\sin(J_3\theta_3)}.$$

Using (2.13) and the inequality $\pi/2 < J_3\theta_3 \leq (D_3-1)\theta_3 < \pi - \theta_3 < \pi$ arising from $D = \lfloor \pi/\theta \rfloor$, we have

$$\sin(J_3\theta_3) > \sin(\theta_3) = \frac{\sqrt{3}}{2\sqrt{b^2-b+1}} > \frac{1}{2\sqrt{b^2-b+1}}. \tag{2.18}$$

Since $\pi > (J_4+1)\theta_4 \geq (D_4-1)\theta_4 > \pi - 2\theta_4 > \pi/2$, we obtain from (2.13) that

$$\sin((J_4+1)\theta_4) < \sin(2\theta_4) = 2\sin\theta_4\cos\theta_4 = \frac{2b}{b^2+1}. \tag{2.19}$$

Using these approximations, we acquire

$$\begin{aligned}\frac{\mathcal{B}_b^{(4)}}{\mathcal{B}_b^{(3)}} &< \frac{(b^2-2b+2)\sqrt{b^2+1}^{J_4+1}\dfrac{2b}{b^2+1}}{(b^2-3b+3)\sqrt{b^2-b+1}^{J_3}\dfrac{1}{2\sqrt{b^2-b+1}}} \\ &= \frac{4b(b^2-2b+2)}{(b^2+1)(b^2-3b+3)} \cdot \frac{\sqrt{b^2+1}^{J_4+1}}{\sqrt{b^2-b+1}^{J_3-1}}.\end{aligned}$$

Direct calculations show that for $b \geq 6$, the first fraction is less than 1. In the way of some details, viewing the first fraction as a function of b, we find the critical points of the first fraction and see that its derivative is negative past the largest critical point, which is < 2. The fraction is then decreasing for $b \geq 6$, and we see that evaluating it at $b = 6$ results in a number less than 1; thus, we conclude that the first fraction is less than 1 for all values of $b \geq 6$. We consider the second fraction. If we can show that the second fraction is less than 1, then we will have shown $\mathcal{B}_b^{(4)} < \mathcal{B}_b^{(3)}$. Consider the Shafer-Fink inequalities (see [18] and [32])

$$\frac{3x}{1+2\sqrt{1+x^2}} < \arctan(x) < \frac{\pi x}{1+2\sqrt{1+x^2}}, \quad \text{for } x > 0. \tag{2.20}$$

From (2.20), we deduce

$$J_4 + 1 < \frac{\pi}{\theta_4} = \frac{\pi}{\arctan(1/b)} < \frac{\pi}{3}(b + 2\sqrt{b^2+1}) < \pi(b+1)$$

and

$$J_3 > \frac{\pi}{\theta_3} - 3 = \frac{\pi}{\arctan\left(\sqrt{3}/(2b-1)\right)} - 3 > \frac{1}{\sqrt{3}}(2b - 1 + 4\sqrt{b^2-b+1}) - 3.$$

Utilizing these approximations on the second fraction, we obtain

$$\begin{aligned}\frac{\sqrt{b^2+1}^{J_4+1}}{\sqrt{b^2-b+1}^{J_3-1}} &\leq \frac{\sqrt{b^2+1}^{\pi(b+1)}}{\sqrt{b^2-b+1}^{(2b-1+4\sqrt{b^2-b+1})/\sqrt{3}-4}} \\ &\leq \frac{\sqrt{b^2+1}^{\pi(b+1)}}{\sqrt{b^2-b+1}^{(2b-1+4(b-1))/\sqrt{3}-8/\sqrt{3}}} \\ &= \frac{\sqrt{b^2+1}^{\pi(b+1)}}{\sqrt{b^2-b+1}^{(6b-13)/\sqrt{3}}} \leq \frac{\sqrt{b^2+1}^{\pi(b+1)}}{\sqrt{b^2-b+1}^{2\sqrt{3}(b-3)}}.\end{aligned}$$

For $b \geq 47$, one checks that

$$\begin{aligned}50\pi\left(1+\frac{1}{b}\right) &\leq 50\pi\left(1+\frac{1}{47}\right) < 161 \quad \text{and} \\ 100\sqrt{3}\left(1-\frac{3}{b}\right) &\geq 100\sqrt{3}\left(1-\frac{3}{47}\right) > 162.\end{aligned}$$

Thus, we deduce that for $b \geq 47$, the inequality

$$\frac{\sqrt{b^2+1}^{\pi(b+1)}}{\sqrt{b^2-b+1}^{2\sqrt{3}(b-3)}} = \left(\frac{\sqrt{b^2+1}^{\pi(b+1)}}{\sqrt{b^2-b+1}^{2\sqrt{3}(b-3)}}\right)^{\frac{50}{b}\cdot\frac{b}{50}} = \left(\frac{\sqrt{b^2+1}^{50\pi(1+1/b)}}{\sqrt{b^2-b+1}^{100\sqrt{3}(1-3/b)}}\right)^{\frac{b}{50}}$$

$$\leq \left(\frac{\sqrt{b^2+1}^{161}}{\sqrt{b^2-b+1}^{162}}\right)^{\frac{b}{50}} = \left(\frac{1}{\sqrt{b^2-b+1}} \cdot \left(\sqrt{\frac{b^2+1}{b^2-b+1}}\right)^{161}\right)^{\frac{b}{50}}$$

$$\leq \left(\frac{1}{8}\right)^{\frac{b}{50}} \leq \left(\frac{24}{25}\right)^{b}$$

holds where we have used that b^2-b+1 is increasing and $(b^2+1)/(b^2-b+1)$ is decreasing for $b \geq 47$. Since $24/25$ is less than 1, we have $\mathcal{B}_b^{(3)} > \mathcal{B}_b^{(4)}$ for $b \geq 47$. Direct calculations show that $\mathcal{B}_b^{(3)} > \mathcal{B}_b^{(4)}$ for $b \geq 6$.

We now use similar methods to show $\mathcal{B}_b^{(3)} < \mathcal{B}_b^{(6)}$. From the definitions of J in (2.14) and β_j in (2.12) for $n = 3$ and $n = 6$, we obtain

$$\frac{\mathcal{B}_b^{(3)}}{\mathcal{B}_b^{(6)}} < \frac{(1-A_3+B_3)\beta_{J_3}}{(1-A_6+B_6)\beta_{J_6-1}} = \frac{(b^2-3b+3)\sqrt{b^2-b+1}^{J_3+1}\sin((J_3+1)\theta_3)}{(b^2-b+1)\sqrt{b^2+b+1}^{J_6}\sin(J_6\theta_6)}$$

$$< \frac{(b^2-3b+3)\sqrt{b^2-b+1}^{J_3-J_6+1}\sin((J_3+1)\theta_3)}{(b^2-b+1)\sin(J_6\theta_6)}.$$

Using similar approximations to (2.18) and (2.19) with $n = 6$ and $n = 3$ respectively, we establish the inequalities

$$\sin(J_6\theta_6) > \frac{1}{2\sqrt{b^2+b+1}} \qquad \text{and} \qquad \sin((J_3+1)\theta_3) < \frac{\sqrt{3}(2b-1)}{2(b^2-b+1)}.$$

From these approximations, we see that

$$\frac{\mathcal{B}_b^{(3)}}{\mathcal{B}_b^{(6)}} < \frac{(b^2-3b+3)\sqrt{b^2-b+1}^{J_3-J_6+1}\dfrac{\sqrt{3}(2b-1)}{2(b^2-b+1)}}{(b^2-b+1)\dfrac{1}{2\sqrt{b^2+b+1}}}$$

$$< \frac{2(b^2-3b+3)(2b-1)\sqrt{b^2+b+1}\sqrt{b^2-b+1}^{J_3-J_6+1}}{(b^2-b+1)^2}$$

$$< \frac{2(b^2-3b+3)(2b-1)\sqrt{b^2+b+1}^{J_3-J_6+2}}{(b^2-b+1)^2}.$$

If we can show that $J_3 - J_6 + 2 < 1/2$, then

$$\frac{\mathcal{B}_b^{(3)}}{\mathcal{B}_b^{(6)}} < \frac{2(b^2-3b+3)(2b-1)\sqrt{b^2+b+1}^{1/2}}{(b^2-b+1)^2} < \frac{2(b^2-3b+3)(2b-1)\sqrt{b+1}}{(b^2-b+1)^2}.$$

One checks that this is a decreasing function of b for $b \geq 5$ that is less than 1 for $b \geq 14$. Thus, by showing that $J_3 - J_6 + 2 < 1/2$, we will have $\mathcal{B}_b^{(3)} < \mathcal{B}_b^{(6)}$ for $b \geq 14$. Using the definition of J in (2.14) and the definition $D = \lfloor \pi/\theta \rfloor$ to deduce

$$J_3 - J_6 + 2 \leq D_3 - D_6 + 3 \leq \frac{\pi}{\theta_3} - \frac{\pi}{\theta_6} + 4.$$

Consider the function $\Omega(b) = \pi/\arctan\left(\sqrt{3}/(2b-1)\right)$. Then we have $\Omega(b) = \pi/\theta_3$ and $\Omega(b+1) = \pi/\theta_6$. Differentiating with respect to b and using $\arctan(x) \leq x$ for $x \geq 0$, we obtain

$$\frac{d\Omega(b)}{db} = \frac{\pi\sqrt{3}}{2(b^2-b+1)\left(\arctan\left(\frac{\sqrt{3}}{2b-1}\right)\right)^2} \geq \frac{\pi\sqrt{3}}{2(b^2-b+1)\left(\frac{\sqrt{3}}{2b-1}\right)^2} = \frac{\pi\sqrt{3}(2b-1)^2}{6(b^2-b+1)}.$$

One checks that this last expression is an increasing function of b that approaches $2\pi/\sqrt{3} > 7/2$ as b gets large; furthermore, direct calculations show that it is greater than $7/2$ for $b \geq 6$. Thus, for $b \geq 6$ we have $\Omega(b+1) - \Omega(b) > 7/2$, which is $\frac{\pi}{\theta_3} - \frac{\pi}{\theta_6} < -\frac{7}{2}$ proving that $J_3 - J_6 + 2 < 1/2$ for $b \geq 6$. Hence, we have shown that $\mathcal{B}_b^{(6)} > \mathcal{B}_b^{(3)}$ for $b \geq 14$. Direct calculations show that $\mathcal{B}_b^{(6)} > \mathcal{B}_b^{(3)}$ also holds for $b \geq 2$.

Having established $\mathcal{B}_b^{(4)} < \mathcal{B}_b^{(3)} < \mathcal{B}_b^{(6)}$, we now find the values of b such that $\mathcal{B}_b > \mathcal{B}_b^{(4)}$, $\mathcal{B}_b > \mathcal{B}_b^{(3)}$, and $\mathcal{B}_b > \mathcal{B}_b^{(6)}$. As each inequality involves $\mathcal{B}_b$, we first make approximations on $\mathcal{B}_b$. Recall that the equation for $\mathcal{B}_b$ in (2.5) is

$$\mathcal{B}_b = \frac{(b-1.5221)^\kappa(b-2.5221)}{1+\cot\left(\pi/b^2\right)} \quad \text{with} \quad \kappa = \kappa(b) = \left\lfloor \frac{(b^2-1)\pi}{b^2\phi} \right\rfloor$$

for $\phi = \arctan(0.8444/(b-0.2))$. Using $\tan x \geq x$ for $x \in [0, \pi/2)$, we obtain

$$1 + \cot(\pi/b^2) \leq 1 + b^2/\pi < b^2 + 1$$

and

$$\kappa + 1 > \frac{(b^2-1)\pi}{b^2\phi} > \frac{b^2-1}{b^2} \cdot \frac{\pi}{\frac{0.8444}{b-0.2}} = \frac{b^2-1}{b^2} \cdot \frac{\pi(b-0.2)}{0.8444}.$$

Using these approximations, we see that

$$\mathcal{B}_b > \frac{(b-2)^{k_1-1}(b-3)}{b^2+1} \quad \text{for} \quad k_1 = \frac{b^2-1}{b^2} \cdot \frac{\pi(b-0.2)}{0.8444}. \tag{2.21}$$

One checks that k_1/b is increasing for $b \geq 1$.

We now show $\mathcal{B}_b > \mathcal{B}_b^{(4)}$. Using the approximations $\sin((J_4+1)\theta_4) < 2b/(b^2+1)$ and $J_4 + 1 < \pi(b+1)$ previously established with the above approximation for $\mathcal{B}_b$, we see that

$$\begin{aligned}
\frac{\mathcal{B}_b^{(4)}}{\mathcal{B}_b} &< \frac{2b(b^2-2b+2)\sqrt{b^2+1}^{\,\pi(b+1)}}{b^2+1} \cdot \frac{b^2+1}{(b-3)(b-2)^{k_1-1}} \\
&= \frac{2b(b^2-2b+2)}{b-3} \cdot \frac{\sqrt{b^2+1}^{\,\pi(b+1)}}{(b-2)^{k_1-1}} \\
&< \frac{2b(b^2-2b+2)}{(b-3)^4} \cdot \frac{\sqrt{b^2+1}^{\,\pi b+4}}{(b-2)^{k_1-4}},
\end{aligned}$$

where the last inequality comes from moving three powers of $b-2$ over to $b-3$ noting that $b-2 > b-3$.

One checks that $2b(b^2-2b+2)/(b-3)^4$ is less than 1 for $b \geq 9$. Note that for $b \geq 24$, we have both

$$\begin{aligned}
10\left(\pi + \frac{4}{b}\right) &\leq 10\left(\pi + \frac{4}{24}\right) < 34 \quad \text{and} \\
10\left(\frac{k_1}{b} - \frac{4}{b}\right) &\geq 10\left(\frac{k_1(24)}{24} - \frac{4}{24}\right) > 35
\end{aligned}$$

where we use the fact that k_1/b is an increasing function of b. Thus, for $b \geq 24$ we deduce that the inequality

$$\begin{aligned}
\frac{\mathcal{B}_b^{(4)}}{\mathcal{B}_b} &< \left(\frac{\sqrt{b^2+1}^{\,\pi b+4}}{(b-2)^{k_1-4}}\right)^{\frac{10}{b}\cdot\frac{b}{10}} < \left(\frac{1}{b-2} \cdot \left(\frac{b^2+1}{b^2-4b+4}\right)^{17}\right)^{b/10} \\
&< \left(\frac{91}{100}\right)^{b/10} < \left(\frac{991}{1000}\right)^{b} < 1
\end{aligned}$$

holds where we have used that $b-2$ is increasing and $(b^2+1)/(b^2-4b+4)$ is decreasing for $b \geq 24$. Since 9991/1000 is less than 1, we have $\mathcal{B}_b > \mathcal{B}_b^{(4)}$ for $b \geq 24$. Direct calculations show that $\mathcal{B}_b > \mathcal{B}_b^{(4)}$ for $b \geq 8$. Note that we only need to show this inequaltiy holds for $b > 20$ since our choice for the $e_i(b)$'s and $d(b)$, and thus our approximation $\mathcal{B}_b$, change when $b \leq 20$. For $b \leq 20$, our choice for the $e_i(b)$'s and $d(b)$ are the same as those found in [4]. Using these values, we see that $\mathcal{B}_b > \mathcal{B}_b^{(4)}$ for $b \geq 4$.

The method used for determining the values of b such that $\mathcal{B}_b > \mathcal{B}_b^{(3)}$ is similar. We start with the analogous argument to (2.19), except with $n = 3$ instead of $n = 4$. That is, we have the inequality

$$\sin((J_3 + 1)\theta_3) < \frac{\sqrt{3}(2b - 1)}{2(b^2 - b + 1)}.$$

Using the Shafer-Fink inequalities in (2.20), we have the upper bound

$$J_3 + 1 < \frac{\pi}{\theta_3} < \frac{\pi}{3\sqrt{3}}(2b - 1 + 4\sqrt{b^2 - b + 1}) < \frac{\pi}{3\sqrt{3}}(6b + 3) = \frac{\pi}{\sqrt{3}}(2b + 1).$$

Thus, using the approximation on $\mathcal{B}_b$ found in (2.21), we obtain

$$\begin{aligned}
\frac{\mathcal{B}_b^{(3)}}{\mathcal{B}_b} &< \frac{(2b-1)(b^2-3b+3)\sqrt{b^2-b+1}^{\pi(2b+1)/\sqrt{3}}}{b^2-b+1} \cdot \frac{b^2+1}{(b-3)(b-2)^{k_1-1}} \\
&= \frac{(2b-1)(b^2-3b+3)}{(b-3)(b^2-b+1)} \cdot \frac{\sqrt{b^2-b+1}^{\pi(2b+1)/\sqrt{3}}}{(b-2)^{k_1-1}} \\
&< \frac{(2b-1)(b^2-3b+3)}{(b-3)^2(b^2-b+1)} \cdot \frac{\sqrt{b^2-b+1}^{\pi(2b+1)/\sqrt{3}}}{(b-2)^{k_1-2}}.
\end{aligned}$$

One checks that the first fraction in the last expression above is less than 1 for $b \geq 6$. Note that for $b \geq 82$, we have both

$$\begin{aligned}
&\frac{40\pi}{\sqrt{3}}\left(2 + \frac{1}{b}\right) \leq \frac{40\pi}{\sqrt{3}}\left(2 + \frac{1}{82}\right) < 146 \quad \text{and} \\
&40\left(\frac{k_1}{b} - \frac{2}{b}\right) \geq 40\left(\frac{k_1(82)}{82} - \frac{2}{82}\right) > 147.
\end{aligned}$$

Thus, for $b \geq 82$ we deduce the inequality

$$\frac{\mathcal{B}_b^{(3)}}{\mathcal{B}_b} < \left(\frac{\sqrt{b^2-b+1}^{\pi/\sqrt{3}(2b+1)}}{(b-2)^{k_1-2}}\right)^{\frac{40}{b}\cdot\frac{b}{40}} < \left(\frac{1}{b-2}\left(\frac{b^2-b+1}{b^2-4b+4}\right)^{73}\right)^{b/40}$$

$$< \left(\frac{1}{5}\right)^{b/40} < \left(\frac{97}{100}\right)^{b}$$

holds where we have used that $b-2$ is increasing and $(b^2+b+1)/(b^2-4b+4)$ is decreasing for $b \geq 82$. Since $97/100$ is less than 1, we have $\mathcal{B}_b > \mathcal{B}_b^{(3)}$ for $b \geq 82$. Direct calculations show that $\mathcal{B}_b > \mathcal{B}_b^{(3)}$ for $b \geq 3$ where we recall that $\mathcal{B}_b$ are the values determined in [4] for $2 \leq b \leq 20$.

We again use a similar method for determining the values of b such that $\mathcal{B}_b > \mathcal{B}_b^{(6)}$. We start with the analogous argument to (2.19), except with $n = 6$ instead of $n = 4$. That is, we have the inequalities

$$\sin((J_6+1)\theta_6) < \frac{\sqrt{3}(2b+1)}{2(b^2+b+1)},$$

and

$$J_6 + 1 < \frac{\pi}{\theta_6} < \frac{\pi}{3\sqrt{3}}(2b+1+4\sqrt{b^2+b+1}) < \frac{\pi}{3\sqrt{3}}(6b+5) < \frac{\pi}{\sqrt{3}}(2b+2).$$

Thus, using the approximation on $\mathcal{B}_b$ found in (2.21), we obtain

$$\frac{\mathcal{B}_b^{(6)}}{\mathcal{B}_b} < \frac{(2b+1)(b^2-b+1)\sqrt{b^2+b+1}^{\pi(2b+2)/\sqrt{3}}}{b^2+b+1} \cdot \frac{b^2+1}{(b-3)(b-2)^{k_1-1}}$$

$$= \frac{(2b+1)(b^2-b+1)}{(b-3)(b^2+b+1)} \cdot \frac{\sqrt{b^2+b+1}^{\pi(2b+2)/\sqrt{3}}}{(b-2)^{k_1-1}}$$

$$< \frac{(2b+1)(b^2-b+1)}{(b-3)^2(b^2+b+1)} \cdot \frac{\sqrt{b^2+b+1}^{\pi(2b+2)/\sqrt{3}}}{(b-2)^{k_1-2}}.$$

One checks that the first fraction in the last expression above is less than 1 for $b \geq 5$. Note that for $b \geq 112$, we have both

$$\frac{50\pi}{\sqrt{3}}\left(2+\frac{2}{b}\right) \leq \frac{50\pi}{\sqrt{3}}\left(2+\frac{2}{112}\right) < 183 \quad \text{and}$$

$$50\left(\frac{k_1}{b}-\frac{2}{b}\right) \geq 50\left(\frac{k_1(112)}{112}-\frac{2}{112}\right) > 184.$$

Thus, for $b \geq 112$ we deduce the inequality

$$\frac{\mathcal{B}_b^{(6)}}{\mathcal{B}_b} < \left(\frac{\sqrt{b^2+b+1}^{\pi(2b+2)/\sqrt{3}}}{(b-2)^{k_1-2}}\right)^{\frac{50}{b}\cdot\frac{b}{50}} < \left(\frac{1}{\sqrt{b^2+b+1}}\left(\frac{b^2+b+1}{b^2-4b+4}\right)^{92}\right)^{b/50}$$

$$< \left(\frac{14}{25}\right)^{b/50} < \left(\frac{99}{100}\right)^{b}$$

holds where we have used that $\sqrt{b^2+b+1}$ is increasing and $(b^2+b+1)/(b^2-4b+4)$ is decreasing for $b \geq 112$. Since $99/100$ is less than 1, we have $\mathcal{B}_b > \mathcal{B}_b^{(6)}$ for $b \geq 112$. Direct calculations show that $\mathcal{B}_b > \mathcal{B}_b^{(3)}$ for $b \geq 70$. Here we note that in [4], M. Cole, S. Dunn, and M. Filaseta showed that $\mathcal{B}_b < \mathcal{B}_b^{(6)}$ for $2 \leq b \leq 20$. Thus, we know that while the bound $b \geq 70$ might not be sharp, it cannot be better than $b \geq 21$.

Having established these inequalities, we note that $M_1(b) = \mathcal{B}_b^{(3)}$ for $3 \leq b \leq 5$ and $M_1(b) = \mathcal{B}_b^{(4)}$ for $b > 5$ in Theorem 1.1. This is the case because $\mathcal{B}_b^{(4)} < \mathcal{B}_b^{(3)}$ for $b > 5$. Continuing along Theorem 1.1, $M_2(b) = \mathcal{B}_b^{(4)}$ for $3 \leq b \leq 5$ and $M_2(b) = \mathcal{B}_b^{(3)}$ for $b > 5$, $M_3(b) = \mathcal{B}_b^{(6)}$, and $M_4(b) = \mathcal{B}_b$.

We now put the finishing touches on the proof of Theorem 1.1. From the definition of β_j in (2.9), we have

$$\begin{aligned}\beta_j &= \frac{1}{2i\,\mathrm{Im}(\zeta_n)}\left[(b+\zeta_n)^{j+1}-(b+\overline{\zeta}_n)^{j+1}\right]\\ &= \frac{1}{2i\,\mathrm{Im}(\zeta_n)}\left[(b+\mathrm{Re}(\zeta_n)+i\,\mathrm{Im}(\zeta_n))^{j+1}-(b+\mathrm{Re}(\zeta_n)-i\,\mathrm{Im}(\zeta_n))^{j+1}\right].\end{aligned}$$

Writing out the binomial expansion of each term and combining like terms yields

$$\begin{aligned}\beta_j = \frac{1}{2i\,\mathrm{Im}(\zeta_n)}\Bigg[&\sum_{k=0}^{j+1}\binom{j+1}{k}(i\,\mathrm{Im}(\zeta_n))^k(b+\mathrm{Re}(\zeta_n))^{j+1-k}\\ &-\sum_{k=0}^{j+1}\binom{j+1}{k}(-i\,\mathrm{Im}(\zeta_n))^k(b+\mathrm{Re}(\zeta_n))^{j+1-k}\Bigg]\\ = \frac{1}{2i\,\mathrm{Im}(\zeta_n)}\Bigg[&\sum_{k=0}^{j+1}\binom{j+1}{k}(b+\mathrm{Re}(\zeta_n))^{j+1-k}\left((i\,\mathrm{Im}(\zeta_n))^k-(-i\,\mathrm{Im}(\zeta_n))^k\right)\Bigg]\end{aligned}$$

$$= \sum_{0 \le k \le \frac{j+1}{2}} \binom{j+1}{2k+1} (b + \mathrm{Re}(\zeta_n))^{j-2k} (-\mathrm{Im}(\zeta_n)^2)^k.$$

Recall that J_n was shown to be either $D_n - 1$ or $D_n - 2$. Evaluating β_j at $j = D_n - 1$ and $j = D_n - 2$, we get

$$\beta_{D_n-1} = \sum_{0 \le k \le \frac{D_n}{2}} \binom{D_n}{2k+1} (b + \mathrm{Re}(\zeta_n))^{D_n-2k-1} (-\mathrm{Im}(\zeta_n)^2)^k \quad \text{and}$$
$$\beta_{D_n-2} = \sum_{0 \le k \le \frac{D_n-1}{2}} \binom{D_n-1}{2k+1} (b + \mathrm{Re}(\zeta_n))^{D_n-2k-2} (-\mathrm{Im}(\zeta_n)^2)^k.$$

Taking J_n to be such that $\beta_{J_n} > \beta_{J_n-1}$ and $\beta_{J_n} > \beta_{J_n+1}$, we define $\beta_{J_n} = \max(\beta_{D_n-1}, \beta_{D_n-2})$. Thus, $\mathcal{B}_b^{(n)}$ is the maximum of

$$\mathcal{B}_{b,1}^{(n)} = \left(\sum_{0 \le k \le \frac{D_n}{2}} \binom{D_n}{2k+1} (b + \mathrm{Re}(\zeta_n))^{D_n-2k-1} (-\mathrm{Im}(\zeta_n)^2)^k \right) (1 - A_n + B_n)$$

and

$$\mathcal{B}_{b,2}^{(n)} = \left(\sum_{0 \le k \le \frac{D_n-1}{2}} \binom{D_n-1}{2k+1} (b + \mathrm{Re}(\zeta_n))^{D_n-2k-2} (-\mathrm{Im}(\zeta_n)^2)^k \right) (1 - A_n + B_n)$$

for each $n \in \{3, 4, 6\}$, completing the proof of Theorem 1.1.

Chapter 3

Truncated Binomial Polynomials

3.1 Preliminaries

We first introduce the notion of Newton polygons. Let $f(x) = \sum_{j=0}^{r} a_j x^j$ be a polynomial in $\mathbb{Z}[x]$ with $a_0 a_r \neq 0$ and fix a prime p. For an integer $m \neq 0$, denote $\nu_p(m)$ to be the p-adic valuation of m, that is, the exponent in the largest power of p dividing m. Let S be the set of lattice points $(j, \nu_p(a_{r-j}))$ for $0 \leq j \leq r$ with $a_{r-j} \neq 0$. The Newton polygon of $f(x)$ with respect to the prime p is the polygonal path along the lower convex hull of these points from $(0, \nu_p(a_r))$ to $(r, \nu_p(a_0))$. The endpoints of every edge belong to the set S, and the slopes of the edges strictly increase as we move from left to right along the Newton polygon.

Newton polygons hold a wealth of information regarding the irreduciblity of a polynomial. The main result we use regarding Newton polygons is due to Dumas ([7] and [29]) and relates the Newton polygon of two polynomials to the Newton polygon of their product.

Theorem 3.1. *Let $g(x)$ and $h(x)$ be in $\mathbb{Z}[x]$ with $g(0)h(0) \neq 0$, and let p be a prime. Let k be a non-negative integer such that p^k divides the leading coefficient of $g(x)h(x)$ but p^{k+1} does not. Then the edges of the Newton polygon for $g(x)h(x)$ with respect to p can be formed by constructing a polygonal path beginning at $(0, k)$ and using translates of the edges in the Newton polygons for $g(x)$ and $h(x)$ with respect to the prime p, using exactly one translate for each edge of the Newton polygons for $g(x)$*

and $h(x)$. Necessarily, the translated edges are translated in such a way as to form a polygonal path with the slopes of the edges increasing from left to right.

Theorem 3.1 will be utilized in the following way. Let $f(x)$ be a polynomial of degree 6 such that there are primes p and q where the Newton polygon with respect to p has edges of horizontal length divisible by 2, and the Newton polygon with respect to q has edges of horizontal length divisible by 3. From Theorem 3.1, we see that if $f(x) = g(x)h(x)$ where $g(x)$ and $h(x)$ are in $\mathbb{Z}[x]$ with $\deg g(x) > 0$, then the total horizontal length of the Newton polygon of $g(x)$ with respect p must be 2, 4, or 6; and the total horizontal length of the Newton polygon of $g(x)$ with respect to q must be 3 or 6. The only case that these two Newton polygons agree are when the total lengths are 6, which means that $g(x)$ has degree 6. Thus, $f(x)$ must be irreducible over $\mathbb{Q}$.

To prove Theorem 1.3, we utilize work by R. Stauduhar in [37] on computational Galois theory. In particular we have the following definition related to finding the Galois group of a polynomial.

Definition 3.2. Let $F(x_1, \ldots, x_n)$ be a polynomial with integer coefficients in the indeterminants $x_1, \ldots, x_n$. Let G be a group of permutations on $1, \ldots, n$. If F is left unchanged by precisely the permutations of G, we say that F *belongs to* G.

The following Theorem combines Theorems 4 and 5 from [37].

Theorem 3.3. *Let $p(x)$ be a monic irreducible polynomial of degree n with integer coefficients. Let $r_1, r_2, \ldots, r_n$ be a fixed ordering of the roots of $p(x)$. Suppose H is a transitive subgroup of S_n, and suppose that, with respect to the given ordering of the roots, the Galois group Γ of $p(x)$ is a subgroup of H. Let G be a subgroup of H and $F(x_1, \ldots, x_n)$ a function belonging to G in H. Let $\pi_1, \ldots, \pi_k$ be representative for the* *right cosets of G in H. Then the resolvent polynomial*

$$Q_{(H,G)}(y) = \prod_{i=1}^{k}(y - \pi_i(F(r_1, \ldots, r_n)))$$

has integer coefficients with $F(r_1, \ldots, r_n)$ a root. Moreover, if we assume $F(r_1, \ldots, r_n)$ is not a repeated root of $Q_{(H,G)}(y)$, then $\Gamma \subset G$ if and only if $F(r_1, \ldots, r_n)$ is a rational integer.

Recall from Section 1.2 with $r = 6$ that for t a non-negative integer the polynomials

$$p_{6,t}(x) = \sum_{j=0}^{6} \binom{t+j}{j} x^j, \qquad \widetilde{p}_{6,t}(x) = \sum_{j=0}^{6} \binom{t+j}{j} x^{6-j},$$

$$\text{and} \quad \widetilde{p}_{6,t}(x+1) = \sum_{j=0}^{6} \binom{t+7}{j} x^{6-j}$$

each have the same Galois group over $\mathbb{Q}$ since the roots for each generate the same number field. To prove Theorem 1.3, we focus on the polynomials $\widetilde{p}_{6,t}(x)$ and $\widetilde{p}_{6,t}(x+1)$.

Our goal is to utilize the first part of Theorem 3.3 to construct the resolvent polynomial to then use the latter half of Theorem 3.3 to say something about the Galois group of $p_{6,t}(x)$ over $\mathbb{Q}$. In order to do so, we must have the conditions of Theorem 3.3 hold; in particular, we will want $p_{6,t}(x)$ to be irreducible for all values of t. Notice that the irreducibility over $\mathbb{Q}$ of one of $p_{6,t}(x)$, $\widetilde{p}_{6,t}(x)$, and $\widetilde{p}_{6,t}(x+1)$ implies the irreducibility of the other two. For example, if $\widetilde{p}_{6,t}(x+1)$ is irreducible, then so is $p_{6,t}(x)$. Indeed, if $p_{6,t}(x) = g_t(x)h_t(x)$, then $\widetilde{p}_{6,t}(x+1) = \widetilde{g}_t(x+1)\widetilde{h}_t(x+1)$. Similarly, if $\widetilde{p}_{6,t}(x)$ is irreducible, then so is $p_{6,t}(x)$.

3.2 Irreducibility of $p_{6,t}(x)$

We start by considering the polynomials with integer coefficients defined by

$$q(x,t) = 6!\widetilde{p}_{6,t}(x) = 6! \sum_{j=0}^{6} \binom{t+j}{j} x^{6-j}$$

and

$$q(x+1,t) = 6!\widetilde{p}_{6,t}(x+1) = 6!\sum_{j=0}^{6}\binom{t+7}{j}x^{6-j}.$$

To make our argument easier to follow, we write out the terms of each polynomial below.

$$\begin{aligned} q(x,t) = 6!\,\widetilde{p}_{6,t}(x) = 6!\,x^6 &+ 6!\,(t+1)x^5 + 3\cdot 5!\,(t+2)(t+1)x^4 \\ &+ 5!\,(t+3)(t+2)(t+1)x^3 \\ &+ 6\cdot 5(t+4)(t+3)(t+2)(t+1)x^2 \\ &+ 6(t+5)(t+4)(t+3)(t+2)(t+1)x \\ &+ (t+6)(t+5)(t+4)(t+3)(t+2)(t+1), \end{aligned}$$

and

$$\begin{aligned} q(x+1,t) = 6!\,\widetilde{p}_{6,t}(x+1) = 6!\,x^6 &+ 6!\,(t+7)x^5 + 3\cdot 5!\,(t+6)(t+7)x^4 \\ &+ 5!\,(t+5)(t+6)(t+7)x^3 \\ &+ 6\cdot 5(t+4)(t+5)(t+6)(t+7)x^2 \\ &+ 6(t+3)(t+4)(t+5)(t+6)(t+7)x \\ &+ (t+2)(t+3)(t+4)(t+5)(t+6)(t+7). \end{aligned}$$

Let $t+7 = p_1^{n_1}p_2^{n_2}\dots p_{j_t}^{n_{j_t}}P_t$ where the p_k, for $1 \le k \le j_t$, are the distinct primes greater than 5 dividing $t+7$ and $P_t = 2^{e_2}3^{e_3}5^{e_5}$ with $e_i \ge 0$. The choice of letting $p_k > 5$ is to ensure that the 6! and the other constant factors shown in the terms above do not affect the Newton polygon of $q(x+1,t)$ with respect to the primes p_k for each k. If there is a k such that $p_k \mid (t+7)$ while $p_k^2 \nmid (t+7)$, then $q(x+1,t)$ is Eisenstein with respect to p and hence irreducible. So we do not need to consider the case that $t+7$ has a prime divisor > 5 with multiplicity 1; thus, we suppose that for each k, we have $n_k > 1$. The Newton polygons of $q(x+1,t)$ with respect to the primes p_k will each have vertices $(0,0)$ and $(6,n_k)$. If there is a k such that

$\gcd(n_k, 6) = 1$, then the Newton polygon of $q(x+1,t)$ with respect to p_k will contain only two lattice points, $(0,0)$ and $(6, n_k)$. In this case, $q(x+1,t)$ will be irreducible for such values of t.

Thus, if $t+7$ has prime factors p_k larger than 5 with multiplicity n_k at least 2, we know that $q(x+1,t)$ will be irreducible unless $\gcd(n_k, 6) > 1$ for each such k. We have three remaining cases when $t+7$ has a prime factor larger than 5: n_k is divisible by 2 for each k; n_k is divisible by 3 for each k; or there is at least one prime p_k satisfying $\gcd(n_k, 6) = 2$ and at least one prime p_ℓ satisfying $\gcd(n_\ell, 6) = 3$. While the first two cases are not mutually exclusive, the fact that they could overlap is irrelevant for our argument.

Consider the third case. In this case, write $n_k = 2a$ with $3 \nmid a$, and $n_\ell = 3b$ with $2 \nmid b$. The Newton polygon of $q(x+1,t)$ with respect to p_k is a line segment with endpoints $(0,0)$ and $(6, 2a)$, containing the lattice points $(0,0)$, $(3,a)$, and $(6,2a)$. Similarly, the Newton polygon of $q(x+1,t)$ with respect to p_ℓ is a line segment with endpoints $(0,0)$ and $(6,3b)$, containing the lattice points $(0,0)$, $(2,b)$, $(4,2b)$, and $(6,3b)$. Using Theorem 3.1, we have that such a $q(x+1,t)$ is irreducible over $\mathbb{Q}$. Thus, when $t+7$ has at least one prime factor $p_k^{n_k}$ with $p_k > 5$, we have shown that $q(x+1,t)$ is irreducible except possibly when n_k is divisible by 2 for each such k or when n_k is divisible by 3 for each such k.

For these two cases, we look at the Newton polygons of $q(x,t)$ with respect to various primes dividing $t+1$. Notice that the same arguments hold for $q(x,t)$ and $t+1$. That is, let $t+1 = q_1^{m_1} q_2^{m_2} \dots q_{r_t}^{m_{r_t}} Q_t$ where the q_k are distinct primes greater than 5 with $m_k > 1$ for each $k \in \{1, 2, \dots, r_t\}$ and $Q_t = 2^{f_2} 3^{f_3} 5^{f_5}$ for some $f_i \geq 0$. Similar to the argument for $q(x+1,t)$ above, we obtain that if $r_t > 0$, then $q(x,t)$ is irreducible except possibly when m_k is divisible by 2 for each k or when m_k is divisible by 3 for each k. Since $q(x,t)$ is irreducible for a given t exactly when $q(x+1,t)$ is irreducible for the same values of t, we are interested in the values of t where these cases occur simultaneously for $q(x,t)$ and $q(x+1,t)$. That is, denoting the horizontal lengths between lattice points along a Newton polygon as a list, we have the four cases where $q(x,t)$ has a Newton polygon with horizontal lengths either $[2,2,2]$ or $[3,3]$, and $q(x+1,t)$ has a Newton polygon with horizontal lengths either $[2,2,2]$ or $[3,3]$.

Assume that we are in one of these four cases. We now shift our focus from the coefficients of x^5 to the coefficients of x^4. Using many of the same ideas as before, if $p > 3$ is a prime such that $p^n \mid (t+6)$ but $p^{n+1} \nmid (t+6)$, then the Newton polygon of $q(x+1,t)$ with respect to the prime p has vertices $(0,\epsilon)$, $(1,\epsilon)$, and $(6,n+\epsilon)$ where $\epsilon = 0$ if $p > 5$, and $\epsilon = 1$ if $p = 5$. Here, we are able to include the prime 5 because the constant term will not add an additional power of 5 since $t+1$ is not a factor. If n is not divisible by 5, then their will be no other lattice points on the Newton polygon, except in the case that 5 strictly divides $t+6$. When this happens, the Newton polygon is a horizontal line segment from $(0,1)$ to $(6,1)$ and we can not conclude that $q(x+1,t)$ is irreducible. For $p^n \neq 5$, the horizontal distance between lattice points on the Newton polygon of $q(x+1,t)$ with respect to p will be $[1,5]$. Combining this with the fact that the Newton polygon with respect to a different prime will have horizontal distances between lattice points $[2,2,2]$ or $[3,3]$, we conclude that the polynomial $q(x+1,t)$ will be irreducible by Theorem 3.1. If, however, there is a prime $p > 3$ such that $p^{5n} \mid (t+6)$ but $p^{5n+1} \nmid (t+6)$, the lattice points on the Newton Polygon with respect to p will have horizontal lengths $[1,1,1,1,1,1]$. When this happens we can not conclude that $q(x+1,t)$ is irreducible.

Again, the same arguments hold when looking at $q(x,t)$ and $t+2$. The only instances where $q(x,t)$ can factor occur when there is a prime $q > 3$ such that $q^{5m} \mid (t+2)$ but $q^{5m+1} \nmid (t+2)$, or when 5 strictly divides $t+2$. Since $q(x,t)$ is irreducible for a given t exactly when $q(x+1,t)$ is irreducible for the same t, we need only consider the values of t where the exponents of each prime divisor greater than

3 for both $t+2$ and $t+6$ are each divisible by 5. Notice that both $t+6$ and $t+2$ are not divisible by 5. Thus, we reduce the four cases above to the case that $t+6 = ax^5$ and $t+2 = by^5$ with $a = 2^{e_2}3^{e_3}5^{\delta_1}$ and $b = 2^{f_2}3^{f_3}5^{\delta_1}$ where $e_j \in \{0,1,2,3,4\}$, $f_j \in \{0,1,2,3,4\}$, $\delta_i \in \{0,1\}$, and $\delta_1\delta_2 \neq 1$. Since these two conditions must be satisfied simultaneously for $q(x,t)$, and thus $q(x+1,t)$, to potentially factor, we subtract the two and find the integer solutions to the Thue equations

$$ax^5 - by^5 = 4.$$

Using built-in programs to find integer solutions to Thue equations in Magma V2.22, we obtain integer solutions to these 1875 Thue equations. Notice that since $t \geq 0$, we must have $ax^5 \geq 6$ and $by^5 \geq 2$. Of the 1875 Thue equations, only 126 have integer solutions, with only 12 having solutions where $t \geq 0$. Table 3.1 exhibits these 12 Thue equations, along with their solution that corresponds to a non-negative t. A computation shows for each of these 12 values of t, the sextic polynomial $q(x,t)$ is irreducible.

Table 3.1 Solutions to $ax^5 - by^5 = 4$ corresponding to positive t

Thue Equation	Solution	t
$8x^5 - 4y^5 = 4$	$(1,1)$	2
$36x^5 - y^5 = 4$	$(1,2)$	30
$6x^5 - 2y^5 = 4$	$(1,1)$	0
$16x^5 - 12y^5 = 4$	$(1,1)$	10
$12x^5 - 8y^5 = 4$	$(1,1)$	6
$10x^5 - 6y^5 = 4$	$(1,1)$	4

Thue Equation	Solution	t
$40x^5 - 36y^5 = 4$	$(1,1)$	34
$20x^5 - 16y^5 = 4$	$(1,1)$	14
$9x^5 - 5y^5 = 4$	$(1,1)$	3
$324x^5 - 10y^5 = 4$	$(1,2)$	318
$2x^5 - 60y^5 = 4$	$(2,1)$	58
$24x^5 - 20y^5 = 4$	$(1,1)$	18

Notice that we are looking at values of t such that $t+6$ and $t+2$ each have a prime factor greater than 3. The Thue equations above also include the case that $t+6$ or $t+2$ do not have any prime factors greater than 3. When this happens, $t+6 = 2^{a_2}3^{a_3}$ or $t+2 = 2^{b_2}3^{b_3}$. Taking the difference still results in $ax^5 - by^5 = 4$ with x or y products of powers of 2 and 3. Hence, this completes the case that $t+7$ and $t+1$ each have at least one prime factor greater than 5.

We now have two possibilities left; exactly one of $t+7$ or $t+1$ has at least one prime factor larger than 5, and neither $t+7$ nor $t+1$ have a prime factor larger than 5. In the case that $t+7$ and $t+1$ each do not have a prime divisor larger than 5, write $t+7 = 2^{e_2}3^{e_3}5^{e_5}$ and $t+1 = 2^{f_2}3^{f_3}5^{f_5}$. Subtracting the two, we get the Thue equations

$$ax^3 - by^3 = 6,$$

where $ax^3 = t+7 = 2^{e_2}3^{e_3}5^{e_5}$ and $by^3 = t+1 = 2^{f_2}3^{f_3}5^{f_5}$, with $e_j \in \{0,1,2\}$ and $f_j \in \{0,1,2\}$ for each j. Using Magma V2.22, we obtain integer solutions to these 729 Thue equations. Notice that since $t \geq 0$, we must have $ax^3 \geq 7$ and $by^3 \geq 1$. Of the 729 Thue equations, only 86 have integer solutions, with only 18 having solutions with $t \geq 0$. Table 3.2 exhibits these 18 Thue equations, along with their solution that corresponds to a positive t. A computation shows for each of these 18 values of t, the sextic polynomial $q(x,t)$ is irreducible.

Table 3.2 Solutions to $ax^3 - by^3 = 6$ corresponding to positive t

Thue Equation	Solution	t
$x^3 - 2y^3 = 6$	$(2,1)$	1
$2x^3 - 6y^3 = 6$	$(3,2)$	47
$2x^3 - 10y^3 = 6$	$(2,1)$	9
$3x^3 - 18y^3 = 6$	$(2,1)$	17
$3x^3 - 75y^3 = 6$	$(3,1)$	74
$4x^3 - 2y^3 = 6$	$(4,5)$	249
$9x^3 - 3y^3 = 6$	$(1,1)$	2
$10x^3 - 4y^3 = 6$	$(1,1)$	3
$12x^3 - 6y^3 = 6$	$(1,1)$	5

Thue Equation	Solution	t
$12x^3 - 90y^3 = 6$	$(2,1)$	89
$15x^3 - 6y^3 = 6$	$(14,19)$	41153
$15x^3 - 9y^3 = 6$	$(1,1)$	8
$18x^3 - 12y^3 = 6$	$(1,1)$	11
$18x^3 - 60y^3 = 6$	$(3,2)$	479
$30x^3 - 3y^3 = 6$	$(1,2)$	23
$36x^3 - 30y^3 = 6$	$(1,1)$	29
$60x^3 - 2y^3 = 6$	$(1,3)$	53
$150x^3 - 18y^3 = 6$	$(1,2)$	143

In the case that exactly one of $t+7$ or $t+1$ has a prime factor larger than 5, we use much of what was discussed above. That is, if $t+7$ is divisible by a prime larger than 5, we have shown that $q(x+1,t)$ is irreducible except possibly when each of the primes larger than 5 dividing $t+7$ have exponents divisible by 2 or when each of the primes larger than 5 dividing $t+7$ have exponents divisible by 3. That is, we can write $t+7$ as either $t+7 = a_1x^2$ or $t+7 = a_2x^3$ for $a_1 = 2^{e_2}3^{e_3}5^{e_5}$ and $a_2 = 2^{f_2}3^{f_3}5^{f_5}$

with $e_j \in \{0,1\}$ and $f_j \in \{0,1,2\}$. In either case, since now $t+1$ has no prime divisors > 5, we can write $t+1 = by^3$ with $b = 2^{g_2}3^{g_3}5^{g_5}$ where $g_j \in \{0,1,2\}$ and y is divisible by no primes larger than 5. Taking the difference, the only possibilities for $q(x,t)$ to factor occur at integer solutions to

$$a_1x^2 - by^3 = 6 \quad \text{or} \quad a_2x^3 - by^3 = 6.$$

The same must be true for the case that $t+1$ is divisible by a prime larger than 5 while $t+7$ is not. That is, for $b_1 = 2^{e_2}3^{e_3}5^{e_5}$ and $b_2 = 2^{f_2}3^{f_3}5^{f_5}$ with $e_j \in \{0,1\}$ and $f_j \in \{0,1,2\}$, and when $a = 2^{g_2}3^{g_3}5^{g_5}$, we look for integer solutions to

$$ax^3 - b_1y^2 = 6 \quad \text{and} \quad ax^3 - b_2y^3 = 6.$$

We have already exhibited solutions to $a_2x^3 - by^3 = 6$ and $ax^3 - b_2y^3 = 6$. Using built-in programs to find integer points on elliptic curves in Magma V2.22, out of the 216 elliptic curves of the form $a_1x^2 - by^3 = 6$, only 50 have integer solutions, with just 23 satisfying $t \geq 0$. Of these 23 equations, only 7 have solutions that correspond to t values such that $t+7$ is divisible by at least one prime > 5 and $t+1$ is divisible by only primes ≤ 5. And out of the 216 elliptic curves of the form $ax^3 - b_1y^2 = 6$, only 29 have integer solutions, with just 21 satisfying $t \geq 0$. Of these 21 equations, only 2 have solutions that correspond to t values such that $t+7$ has no large prime divisor and $t+1$ has at least one large prime divisor. Table 3.3 displays these equations along with their solutions that correspond to appropriate t. For convenience, we list only the solutions (x,y) such that $x \geq 0$ and $y \geq 0$.

In the way of some details, observe that the equation $ax^2 - by^3 = 6$ is equivalent to $X^2 - Y^3 = 6a^3b^2$ for $X = a^2bx$ and $Y = aby$ for $ab \neq 0$. Using Magma V2.22, we find integer solutions to $X^2 - Y^3 = 6a^3b^2$, then make the substitution $x = X/(ba^2)$ and $y = Y/(ab)$. Notice that not all integer solutions of $X^2 - Y^3 = 6a^3b^2$ are equivalent to integer solutions to $ax^2 - by^3 = 6$; however, if (x,y) is an integer solution of $ax^2 - by^3 = 6$, then (a^2bx, aby) is an integer solution to $X^2 - Y^3 = 6a^3b^2$. Thus we do

Table 3.3 Solutions to $a_1x^2 - by^3 = 6$ and $ax^3 - b_1y^2 = 6$

Equation	Solutions (x, y)	t
$2x^3 - 3y^2 = 6$	$(9, 22)$	1451
$6x^2 - y^3 = 6$	$(17, 12)$	1727
$6x^2 - 10y^3 = 6$	$(19, 6)$	2159
$6x^2 - 36y^3 = 6$	$(7, 2)$	287
$6x^2 - 90y^3 = 6$	$(31, 4)$, $(11, 2)$, $(161, 12)$	5759, 719, 155519
$6x^2 - 150y^3 = 6$	$(26, 3)$	4049
$6x^2 - 225y^3 = 6$	$(49, 4)$	14399
$15x^2 - y^3 = 6$	$(7, 9)$	728
$300x^3 - 6y^2 = 6$	$(1, 7)$	293

not exclude any integer solutions of $ax^2 - by^3 = 6$ by making these substitutions. For $ax^3 - by^2 = 6$, we consider the equation $Y^2 - X^3 = -6a^2b^3$ for $X = abx$ and $Y = ab^2y$. For each of these values for t, we show that $q(x, t)$ or $q(x + 1, t)$ is irreducible by a direct computation.

3.3 The Galois group of $p_{6,t}(x)$

The rest of the proof of Theorem 1.3 is dedicated to showing that there are only finitely many values of $t \in \mathbb{Z}_{\geq 0}$ such that $p_{6,t}(x)$ has Galois group $PGL_2(5)$. Since the ploynomials $p_{6,t}(x)$, $\widetilde{p}_{6,t}(x)$, and $\widetilde{p}_{6,t}(x + 1)$ each have the same Galois group, we choose to work with $\widetilde{p}_{6,t}(x)$. We make use of the following lemma from [6] and [5].

Lemma 3.4. *Let $f(x)$ be an irreducible polynomial of degree $r \geq 2$. If the Galois group of $f(x)$ over $\mathbb{Q}$ contains a 2-cycle and a q-cycle for some prime $q > r/2$, then the Galois group is S_r. Alternatively, if the Galois group of $f(x)$ over $\mathbb{Q}$ contains a 3-cycle and a q-cycle for some prime $q > r/2$, then the Galois group is either the alternating group A_r or the symmetric group S_r.*

Using Lemma 3.4, we obtain the following Lemma which we will use to construct the resolvent polynomial.

Lemma 3.5. *Let H be the subgroup $PGL_2(5)$ in S_6. Let G be the subgroup generated by the 2-cycle $(1,2)$ and the 3-cycle $(1,2,3)$ in S_6. Then the elements of G can be used as representatives for the 6 distinct cosets of H in S_6.*

Proof. From [16], we know that the Galois group of $\widetilde{p}_{6,t}(x)$ is either S_6 or $PGL_2(5)$. Assume that it is $PGL_2(5)$. From Section 3.2, the polynomial $\widetilde{p}_{6,t}(x)$ is irreducible with degree 6 for every $t \in \mathbb{Z}_{\geq 0}$.

Since any n-cycle and a 2-cycle generate S_n, we see that $G = S_3$; thus, the size of G is 6. It is not difficult to check that

$$G = \{(1), (1,2), (2,3), (1,3), (1,2,3), (1,3,2)\}.$$

To show that the six cosets H, $(1,2)H$, $(2,3)H$, $(1,3)H$, $(1,2,3)H$, $(1,3,2)H$ are distinct, we show that $H \cap G = \{(1)\}$.

By Cauchy's Theoreom, since 5 divides $120 = |H|$, H has an element of order 5. Since $H \subset S_6$ and 5 is prime, this element must be a 5-cycle. Since H contains a 5-cycle, if any of $(1,2)$, $(2,3)$ or $(1,3)$ are in H, then $H = S_6$ by Lemma 3.4, which contradicts $H = PGL_2(5)$. Similarly, if $(1,2,3)$ or $(1,3,2)$ are in H, then Lemma 3.4 states that H is either A_6 or S_6, neither of which is $PGL_2(5)$. Thus it must be the case that $H \cap G = \{(1)\}$.

To finish the proof, if $a, b \in G$ and $aH = bH$, then $a^{-1}b \in H$. Since $a^{-1}b$ is also in G, we have that $a^{-1}b = (1)$ because $H \cap G = \{(1)\}$. Thus, $a = b$ implying that $aH = bH$ only if $a = b$. Hence the six cosets represented by the elements in G are distinct. □

Using Lemma 3.5, we follow [37] closely to finish Theorem 1.3. We start by finding the polynomial that belongs to $PGL_2(5)$; that is, the polynomial that is fixed by precisely $PGL_2(5)$. Following the proof of [37, Theorem 1], set $F^* = \alpha_2\alpha_3^2\alpha_4^3\alpha_5^4\alpha_6^5$.

Then

$$F = F(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6) = \sum_{\sigma \in PGL_2(5)} \sigma(F^*) = \sum_{\sigma \in PGL_2(5)} \alpha_{\sigma(2)} \alpha_{\sigma(3)}^2 \alpha_{\sigma(4)}^3 \alpha_{\sigma(5)}^4 \alpha_{\sigma(6)}^5$$

belongs to $PGL_2(5)$. Notice that F, by definition, is not a symmetric polynomial in the α_i's.

We now construct the resolvent polynomial $Q(x,t) = Q_{(S_6, PGL_2(5))}(x,t)$ using Theorem 3.3. Let π_i be representatives for the right cosets of $PGL_2(5)$ in S_6, and $r_1, ..., r_6$ be the roots of $\widetilde{p}_{6,t}(x)$. By Lemma 3.5, we know that the right cosets of $PGL_2(5)$ can be represented by elements in the subgroup generated by $(1,2)$ and $(1,2,3)$. Then $Q(x)$ is defined by

$$Q(x,t) = \prod_{i=1}^{6} (x - F(r_{\pi_i(1)}, r_{\pi_i(2)}, r_{\pi_i(3)}, r_{\pi_i(4)}, r_{\pi_i(5)}, r_{\pi_i(6)})).$$

Notice that $Q(x,t)$ is a symmetric polynomial in the roots $r_1, ..., r_6$ of $\widetilde{p}_{6,t}(x)$. Using the fundamental theorem of symmetric polynomials, we can write $Q(x,t)$ in terms of the elementary symmetric polynomials in the variables $r_1, \ldots, r_6$, and use the correlation between the elementary symmetric polynomials and the coefficients of $\widetilde{p}_{6,t}(x)$ to get a closed form for the resolvent $Q(x,t)$. This calculation proved too cumbersome for Maple 2019. Instead, we define a family of symmetric polynomials that also relate to the coefficients of $\widetilde{p}_{6,t}(x)$. Let $\sigma_{i,j}$ be the sum of all possible i-tuples of x_k for $k \leq n$ raised to the j-th power where $i \leq m$ and j is a positive integer. For example, for $n = 4$ we have the four variables x_1, x_2, x_3, and x_4, and we have the symmetric polynomials

$$\sigma_{1,1} = x_1 + x_2 + x_3 + x_4,$$

$$\sigma_{1,2} = x_1^2 + x_2^2 + x_3^2 + x_4^2,$$

$$\sigma_{2,1} = x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4,$$

$$\sigma_{2,2} = x_1^2x_2^2 + x_1^2x_3^2 + x_1^2x_4^2 + x_2^2x_3^2 + x_2^2x_4^2 + x_3^2x_4^2,$$

$$\sigma_{4,3} = x_1^3x_2^3x_3^3x_4^3, \quad \text{etc.}$$

Since we have the six variables $r_1, \ldots, r_6$, we will let i range from 1 to 6. Notice that if $j = 1$, the polynomials $\sigma_{i,1}$ for $i \in \{1, 2, \ldots, 6\}$ are the elementary symmetric polynomials, and are thus the coefficients of $\widetilde{p}_{6,t}(x)$ up to sign. Also notice that $\sigma_{6,j} = \sigma_{6,1}^j$, so $\sigma_{6,j}$ is $\widetilde{p}_{6,t}(0)^j$. To get $\sigma_{1,j}$, we use Newton's power sum formulas. That is, we relate each $\sigma_{1,j}$ to the elementary symmetric polynomials and $\sigma_{1,k}$ for $k < j$ by

$$\begin{aligned}
\sigma_{1,2} &= \sigma_{1,1}^2 - 2\sigma_{2,1}, \\
\sigma_{1,3} &= \sigma_{1,1}\sigma_{1,2} - \sigma_{2,1}\sigma_{1,1} + 3\sigma_{3,1}, \\
\sigma_{1,4} &= \sigma_{1,1}\sigma_{1,3} - \sigma_{2,1}\sigma_{1,2} + \sigma_{3,1}\sigma_{1,1} - 4\sigma_{4,1}, \\
\sigma_{1,5} &= \sigma_{1,1}\sigma_{1,4} - \sigma_{2,1}\sigma_{1,3} + \sigma_{3,1}\sigma_{1,2} - \sigma_{4,1}\sigma_{1,1} + 5\sigma_{5,1}, \\
\sigma_{1,j} &= \sum_{k=1}^{6} (-1)^{k+1}\sigma_{k,1}\sigma_{1,j-k} \quad \text{for } j \geq 6.
\end{aligned}$$

Working sequentially, we are able to use these formulas to write $\sigma_{1,j}$ in terms of the coefficients of $\widetilde{p}_{6,t}(x)$ for any positive integer j.

For $\sigma_{2,j}$, we write $\sigma_{1,2}$ in terms of the elementary symmetric polynomials

$$\sigma_{1,2} = \sigma_{1,1}^2 - 2\sigma_{2,1} \quad \text{or} \quad \sigma_{2,1} = \frac{\sigma_{1,1}^2 - \sigma_{1,2}}{2}.$$

Replacing r_i with r_i^n gives us the formula

$$\sigma_{2,n} = \frac{\sigma_{1,n}^2 - \sigma_{1,2n}}{2}$$

that we use to write $\sigma_{2,n}$ in terms of the coefficients of $\widetilde{p}_{6,t}(x)$ for any positive integer j. Using the same approach, we can express $\sigma_{3,n}$, $\sigma_{4,n}$, and $\sigma_{5,n}$ in terms of the elementary symmetric polynomials and the $\sigma_{i,j}$ that we have already calculated using

$$\begin{aligned}
\sigma_{3,n} &= \frac{\sigma_{1,3n} - \sigma_{1,1}^3 + 3\sigma_{1,n}\sigma_{2,n}}{3}, \\
\sigma_{4,n} &= \frac{\sigma_{1,n}^4 - 4\sigma_{1,n}^2\sigma_{2,n} + 4\sigma_{1,n}\sigma_{3,n} + 2\sigma_{2,n}^2 - \sigma_{1,4n}}{4}, \quad \text{and} \\
\sigma_{5,n} &= \frac{\sigma_{1,5n} - \sigma_{1,n}^5 + 5\sigma_{1,n}^3\sigma_{2,n} - 5\sigma_{1,n}^2\sigma_{3,n} - 5\sigma_{1,n}\sigma_{2,n}^2 + 5\sigma_{1,n}\sigma_{4,n} + 5\sigma_{2,n}\sigma_{3,n}}{5}.
\end{aligned}$$

Thus, we are able to get all the $\sigma_{i,j}$ in terms of the coefficients of $\widetilde{p}_{6,t}(x)$. Instead of writing $Q(x,t)$ in terms of the elementary symmetric polynomials, we write it in terms of these $\sigma_{i,j}$. When writing $Q(x,t)$ in terms of these $\sigma_{i,j}$, we go out to $\sigma_{6,17}$, which requires calculating up to $\sigma_{1,90}$. To mitigate the work for Maple, we write the $\sigma_{i,j}$ in terms of the coefficients of $\widetilde{p}_{6,t-3}(x)$ instead of $\widetilde{p}_{6,t}(x)$ as this makes the coefficients have fewer terms. For example, the coefficient of x becomes $t(t^2-1)(t^2-4)/120$ instead of $(t+5)(t+4)\dots(t+1)$. Hence, we are able to write the resolvent $Q(x,t-3)$ in terms of x and t, which can be found in Appendix C.

To finish the proof of Theorem 1.3, we utilize the second half of Theorem 3.3. Notice that $F(r_1,...,r_6)$ is a root of $Q(x,t-3)$ since the identity element is a representative of the right cosets of $PGL_2(5)$ in S_6. Using Maple 2019, one can check that $\gcd\left(\frac{\partial Q(x,t-3)}{\partial x}, Q(x,t-3)\right)=1$, so $Q(x,t-3)$ has no repeated roots. Theorem 3.3 states that the Galois group of $\widetilde{p}_{6,t-3}(x)$ is a subset of $PGL_2(5)$ if and only if $F(r_1,...,r_6)$ is a rational integer. Since we know from [16] that $\widetilde{p}_{6,t-3}(x)$ has Galois group either S_6 or $PGL_2(5)$, we actually have that $F(r_1,...,r_6)$ is a rational integer if and only if the Galois group of $\widetilde{p}_{6,t-3}(x)$ is $PGL_2(5)$. Using Siegel's Theorem [34], the curve $Q(x,t-3)=0$ of genus 7 has only a finite number of integer solutions. Since $F(r_1,...,r_6)$ is a solution to $Q(x,t-3)=0$, there are only finitely many values for $t\in\mathbb{Z}$ such that $F(r_1,...,r_6)$ is an integer, so there are finitely many integers t such that $\widetilde{p}_{6,t-3}(x)$ has Galois group $PGL_2(5)$. This concludes the proof of Theorem 1.3.

In theory, one can find the integer values of t such that $Q(x,t-3)=0$; however, this is quite difficult in practice. In the context of our problem, we are more restrictive than finding integer values of t; we want integer values of t such that $t\geq 3$ so that $t-3$ is nonnegative.

Chapter 4

Widely Digitally Stable Composite Numbers

4.1 Preliminaries

We begin by considering

$$N = 7 \cdot \frac{10^n - 1}{9} + M$$

where n and M are large natural numbers to be determined and n is large enough that the left-most digit of N is 7. Notice that N can be written as

$$N = 00\ldots00\underbrace{77\ldots77}_{n \text{ total 7's}}+M,$$

where $77\ldots77$ is a string of n digits that are all 7's.

To prove Theorem 1.5, we will determine a positive integer M and a set of primes $\mathcal{P}$ such that for infinitely many choices of the positive integer n, when we insert any $x \in \{0, 1, \ldots, 9\}$ between any two digits of N, including any of the infinitely many leading zeros, or to the right of the right-most digit of N, the resulting number will be divisible by at least one prime in $\mathcal{P}$. We will do so by forming covering systems of the integers where the moduli of the congruences in the coverings correspond to unique primes.

The role of 7 above is motivated by the idea of taking $7 \in \mathcal{P}$ and $M \equiv 0 \pmod 7$ so that any insertion of the digit 0 or the digit 7 into the leading 0's or leading 7's of N will produce a number divisible by 7. Thus, we do not need to worry about these particular insertions. We will also require M to be congruent to one of 0, 2, 4 or 6 modulo 10 to ensure $\gcd(N, 10) = 1$.

For a nonnegative integer k, let $N^{(k)}(x)$ denote inserting a digit $x \in \{0, ..., 9\}$ to the right of the $k + 1^{st}$ digit of N, that is to the right of d_k. For example,

$$N^{(2)}(x) = 00 \ldots 00 d_{n-1} d_{n-2} \ldots d_2 x d_1 d_0.$$

We set K to be the number of digits of M; thus, initially, K is unknown to us. Observe that $d_k = 7$ for $K \leq k \leq n - 1$. We will break the argument up into three cases: $k \geq n$ corresponding to the leading zeros, $K \leq k \leq n - 1$ corresponding to the leading sevens, and $0 \leq k < K$ corresponding to M. We can formulate $N^{(k)}(x)$ above nicely for the first two cases. Specifically, with $x \in \{0, 1, \ldots, 9\}$, for $k \geq n$, the value of $N^{(k)}(x)$ takes the form

$$N_0^{(k)}(x) = 7 \cdot \frac{10^n - 1}{9} + M + x \cdot 10^k\,;$$

and for $K \leq k < n$, the value of $N^{(k)}(x)$ takes the form

$$N_7^{(k)}(x) = 7 \cdot 10^n + 7 \cdot \frac{10^n - 1}{9} + M + (x - 7) \cdot 10^k.$$

For $k \geq n$, we see that $N_0^{(k)}(x)$ is the result of inserting the digit x to the right of a leading zero; for $K \leq k < n$, we see that $N_7^{(k)}(x)$ is the result of inserting the digit x to the right of a leading seven.

Recall that by taking $M \equiv 0 \pmod 7$, we have that both $N^{(k)}(0)$ and $N^{(k)}(7)$ are divisible by the prime $7 \in \mathcal{P}$ for all $k \geq K$. We now also take $3 \in \mathcal{P}$ and $M \equiv 1 \pmod 3$. Then if in addition $n \equiv 0 \pmod 3$, one checks that

$$N^{(k)}(2) \equiv N^{(k)}(5) \equiv N^{(k)}(8) \equiv 0 \pmod 3$$

for every nonnegative integer k. Thus, inserting an $x \in \{2, 5, 8\}$ into the number N results in a number divisible by 3. At this point, we have that inserting an $x \in \{0, 2, 5, 7, 8\}$ into the number N produces a number divisible by a prime in $\{3, 7\} \subseteq \mathcal{P}$.

To handle $x \in \{1, 3, 4, 6, 9\}$, we will do a bit more work, and this is where we will consider $N_0^{(k)}(x)$ and $N_7^{(k)}(x)$ separately, as well as inserting x in the right-most K digits of N. We begin with the following definition.

Definition 4.1. A finite system of congruences $x \equiv a_i \pmod{m_i}$, $1 \leq i \leq t$, is called a *covering of the integers* (or simply a *covering*) if each integer satisfies at least one congruence in the system.

For example, one can check that the system

$$\begin{aligned} x &\equiv 1 \pmod 2 & x &\equiv 0 \pmod 3 \\ x &\equiv 2 \pmod 6 & x &\equiv 4 \pmod 9 \\ x &\equiv 10 \pmod{18} & x &\equiv 16 \pmod{18} \end{aligned}$$

is a covering of the integers. This covering, as well as the coverings used in this dissertation, will not require that the moduli be distinct.

The following lemma is similar to [14, Lemma 1] and clarifies our use of covering systems. This lemma will be used for inserting a digit in the leading zeros. By way of notation, we define $c(p)$ to be the multiplicative order of 10 modulo a prime with $\gcd(10, p) = 1$.

Lemma 4.2. *Let N and M be natural numbers such that*

$$N = 7 \cdot \frac{10^n - 1}{9} + M,$$

where N has the decimal expansion

$$N = d_{n-1}d_{n-2}\ldots d_1d_0, \quad d_i \in \{0, 1, \ldots, 9\} \quad n \geq 1, \quad d_{n-1} = 7.$$

Let K be a nonnegative integer such that $d_k = 7$ for $K \leq k \leq n-1$, and set $d_k = 0$ for $k \geq n$. For a fixed $x \in \{0, 1, \ldots, 9\}$, suppose we have distinct primes $p_1, \ldots, p_t$, each > 5, for which *(i) there exists a covering of the integers*

$$k \equiv b_i \pmod{c(p_i)}, \quad 1 \le i \le t,$$

(ii) $n \equiv 0 \pmod{lcm(c(p_1), \ldots, c(p_t))}$*,*

(iii) M *is a solution to the system of congruences*

$$M \equiv -x \cdot 10^{b_i} \pmod{p_i}, \quad 1 \le i \le t.$$

Then, for all nonnegative integers k*, we have*

$$N_0^{(k)}(x) = 7 \cdot \frac{10^n - 1}{9} + M + x \cdot 10^k$$

is divisible by at least one of the primes p_i *where* $1 \le i \le t$*.*

Proof. Suppose the conditions in the lemma hold and let $k \ge n$. By (i), there is an $i \in \{1, \ldots, t\}$ such that $k \equiv b_i \pmod{c(p_i)}$. Since $c(p_i)$ is the order of 10 modulo the prime p_i, and $n \equiv 0 \pmod{\mathrm{lcm}(c(p_1), \ldots, c(p_t))}$, we have $10^n - 1 \equiv 0 \pmod{p_i}$ and thus $(10^n - 1)/9 \equiv 0 \pmod{p_i}$ since $p_i > 5$. Thus

$$N_0^{(k)}(x) = 7 \cdot \frac{10^n - 1}{9} + M + x \cdot 10^k \equiv M + x \cdot 10^{b_i} \pmod{p_i}.$$

From (iii), we deduce that $N_0^{(k)}(x) \equiv 0 \pmod{p_i}$. □

Although the conclusion holds for all nonnegative integers k, we are interested only in the case $k \ge n$ of the lemma.

Recall that we previously took $M \equiv 1 \pmod 3$, $3 \in \mathcal{P}$, and $n \equiv 0 \pmod 3$ to handle inserting an $x \in \{2, 5, 8\}$ into the number N. This corresponds to taking $t = 1$ and $p_1 = 3$ in Lemma 4.2. As $c(3) = 1$, we take $b_1 = 0$. Observe then that (i) of Lemma 4.2 hold. This application of Lemma 4.2 can be viewed as a degenerate case, where the covering of the integers in (i) is the single congruence $k \equiv 0 \pmod 1$ and taking $n \equiv 0 \pmod 3$ is in lieu of having the prime $p_1 = 3 < 5$.

For each $x \in \{1, 3, 4, 6, 9\}$, we want to create a covering system to use with Lemma 4.2. The covering system will be of the form given in (i) corresponding to inserting x to the right of the digit d_k. This lemma is specifically to address the insertion of x into the leading 0's; we will want a similar but different lemma to address inserting a digit into the leading 7's and yet a different argument for inserting a digit to the right of d_k for $k < K$. To obtain the primes p of a given order $c = c(p)$, one merely needs to look at the primes p which divide $\Phi_c(10)$, where $\Phi_c(x)$ is the c^{th} cyclotomic polynomial. Every prime divisor p of $\Phi_c(10)$ either will be such that the order of 10 modulo p is c or will be equal to the largest prime divisor of c. In [3, p. III C 1] it is discussed that the largest prime divisor of c can only divide $\Phi_c(10)$ to the first power. There are also no other primes besides those primes dividing $\Phi_c(10)$ for which the order of 10 modulo p is c. It follows that for the covering system given in (i), we can use the modulus c multiple times, where the multiplicity is equal to the number of distinct prime divisors of $\Phi_c(10)$ other than the largest prime divisor of c. What complicates the covering systems as we proceed is that the primes considered in general cannot be used for different $x \in \{1, 3, 4, 6, 9\}$. More precisely, a congruence involving a prime p_i in (i) can only be used for two different values of x if the congruence condition on M in (iii) is the same for those values of x. In practice, this rarely happens, but we will definitely take advantage of some instances where this is the case.

The prime divisors of $\Phi_c(10)$ were obtained through factorizations or partial factorizations largely using Magma V2.22 and checked using Maple 2019. To clarify, we can use the modulus c, $L(c)$ times where $L(c)$ is the number of distinct prime divisors of $\Phi_c(10)$ found, minus 1 if the largest prime factor of c divides $\Phi_c(10)$, plus τ, where $\tau = 0$ if the prime factorization we obtained was a complete factorization, $\tau = 1$ if the prime factorization we obtained was incomplete and the remaining factor is relatively prime to the product of the known prime divisors of $\Phi_c(10)$ and c and the remaining factor is a probable prime, and $\tau = 2$ if the prime factorization we obtained was incomplete and the remaining factor is relatively prime to the product of the known prime divisors of $\Phi_c(10)$ and c and the remaining factor is neither a probable prime nor a prime power.

4.2 Inserting a digit into the leading zeros

Set $\mathcal{P}_2 = \mathcal{P}_5 = \mathcal{P}_8 = \{3\}$ and $\mathcal{P}_0 = \mathcal{P}_7 = \{7\}$. For $x \in \{1, 3, 4, 6, 9\}$, let $\mathcal{P}_x$ be a set of primes, to be determined, as in Lemma 4.2.

For this section, we make use of covering systems developed in [14] and [17], with some minor modifications. Thus, the tables given in this section are not identical to the tables in [14] and [17]. Some minor edits to the congruences were made so that some of the same moduli can be used for more congruences when we look at inserting a digit into the leading sevens in the next section.

Table 4.1 Covering used in Lemma 4.2 (i) for $x = 9$

row	congruence	prime p_i
1	$k \equiv 0 \pmod 2$	11
2	$k \equiv 3 \pmod 4$	101

row	congruence	prime p_i
3	$k \equiv 1 \pmod 8$	73
4	$k \equiv 5 \pmod 8$	137

When $x = 9$, the covering system used is given in Table 4.1. To clarify, the covering system consists of the congruences in the middle columns. One checks that this is a covering of the integers. One can also see that

$$2 = c(11) = \text{ord}_{11}(10), \qquad 4 = c(101) = \text{ord}_{101}(10),$$
$$8 = c(73) = \text{ord}_{73}(10), \qquad 8 = c(137) = \text{ord}_{137}(10).$$

What Table 4.1 is indicating is that

$$N_0^{(k)}(9) = 7 \cdot \frac{10^n - 1}{9} + M + 9 \cdot 10^k \equiv 0 \pmod p,$$

for some $p \in \{11, 101, 73, 137\}$, depending on which congruence k satisfies from the covering. With regard to Lemma 4.2, we take

$$t = 4, \quad p_i \in \mathcal{P}_9 = \{11, 73, 101, 137\}, \quad n \equiv 0 \pmod 8,$$

and M satisfying all the congruences

$$M \equiv -9 \cdot 10^0 \equiv 2 \pmod{11}, \qquad M \equiv -9 \cdot 10^3 \equiv 90 \pmod{101}$$
$$M \equiv -9 \cdot 10^1 \equiv 56 \pmod{73}, \qquad M \equiv -9 \cdot 10^5 \equiv 90 \pmod{137}.$$

Since we have different prime moduli above, the Chinese Remainder Theorem ensures us that there is a single congruence for M modulo $11 \cdot 101 \cdot 73 \cdot 137$ that is equivalent to the above congruences on M. Thus, Lemma 4.2 says that under these conditions, the number $N_0^{(k)}(9)$ is composite for all $k \geq n$.

Combining the information so far, we see that if $n \equiv 0 \pmod{24}$ and M satisfies all of the congruences on M modulo $11, 101, 73$ and 137 above as well as $M \equiv 0 \pmod 7$ and $M \equiv 1 \pmod 3$, then $N_0^{(k)}(x)$ will be divisible by at least one prime in $\{3, 7, 11, 73, 101, 137\}$ for each $x \in \{0, 2, 5, 7, 8, 9\}$ and $k \geq n$.

For $x = 3$, we use the primes found in [14, Table 1]. While the covering is similar to that of [14, Table 1], we have again made some modifications to the congruences so that some of the same moduli can be used for more congruences when we look at inserting a digit into the leading sevens in the next section. Table 4.2 exhibits the covering used for $x = 3$ where we have made use of the notation

$$p_7 = 440334654777631, \qquad p_{14} = 3199044596370769.$$

The set $\mathcal{P}_3$ is the set of 14 primes in the columns "prime p_i". One can check that the congruences in the columns "congruence" form a covering of the integers. The least common multiple of the moduli found in Table 4.2 is 216, so verifying that the congruneces in Table 4.2 form a covering amounts to verifying that each of 0, 1, . . . , 215 satisfies at least one of the congruences. One further checks that each prime p

listed in the "prime p_i" columns of Table 4.2 has $c(p)$ equal to the corresponding modulus in that row. We can then apply Lemma 4.2 as was done previously to see that the number $N_0^{(k)}(3)$ is composite for all $k \geq n$.

Table 4.2 Covering used in Lemma 4.2 (i) for $x = 3$

row	congruence	prime p_i
1	$k \equiv 0 \pmod{3}$	37
2	$k \equiv 1 \pmod{6}$	13
3	$k \equiv 2 \pmod{9}$	333667
4	$k \equiv 14 \pmod{18}$	19
5	$k \equiv 5 \pmod{18}$	52579
6	$k \equiv 17 \pmod{27}$	757
7	$k \equiv 8 \pmod{27}$	p_7

row	congruence	prime p_i
8	$k \equiv 53 \pmod{54}$	70541929
9	$k \equiv 26 \pmod{54}$	14175966169
10	$k \equiv 4 \pmod{12}$	9901
11	$k \equiv 10 \pmod{24}$	99990001
12	$k \equiv 22 \pmod{72}$	3169
13	$k \equiv 46 \pmod{72}$	98641
14	$k \equiv 70 \pmod{72}$	p_{14}

We use the same technique for $x \in \{1, 4, 6\}$. For $x \in \{1, 6\}$, we use similar coverings to those found in [14], and for $x = 4$ we use a similar covering to that found in Tables 6 and 7 from [17], which we reproduce here.

In general, one can verify that the congruences given in a table for a given x is a covering by setting ℓ to be the least common multiple of the moduli in the set of congruences and verifying that each integer in $[0, \ell-1]$ satisfies one of the congruences. Then it is not difficult to see that every integer will satisfy one of the congruences.

For $x = 6$, we use the covering found in Table 4.3, and for $x = 1$ we use the covering found in Table 4.4. The following notation for primes is used in these tables:

$$p_{10} = 3199044596370769, \quad p_{11} = 102598800232111471,$$

$$p_{13} = 265212793249617641, \quad p_{14} = 30703738801, \quad p_{15} = 625437743071,$$

$$p_{16} = 57802050308786191965409441, \quad p_{17} = 4185502830133110721$$

$$p_{21} = 4458192223320340849, \quad p_{27} = 127522001020150503761, \quad p_{31} = 60368344121,$$

$$p_{32} = 848654483879497562821, \quad p_{34} = 73765755896403138401,$$

$$p_{35} = 11189053009, \quad p_{36} = 603812429055411913, \quad p_{37} = 148029423400750506553.$$

Table 4.3 Covering used in Lemma 4.2 (i) for $x = 6$

row	congruence	prime p_i
1	$k \equiv 0 \pmod{5}$	41
2	$k \equiv 1 \pmod{5}$	271
3	$k \equiv 2 \pmod{10}$	9091
4	$k \equiv 3 \pmod{20}$	3541
5	$k \equiv 13 \pmod{20}$	27961
6	$k \equiv 7 \pmod{30}$	211
7	$k \equiv 17 \pmod{30}$	241
8	$k \equiv 27 \pmod{30}$	2161
9	$k \equiv 8 \pmod{40}$	1676321
10	$k \equiv 28 \pmod{40}$	p_{10}
11	$k \equiv 18 \pmod{60}$	61
12	$k \equiv 58 \pmod{60}$	4188901
13	$k \equiv 38 \pmod{60}$	39526741
14	$k \equiv 9 \pmod{15}$	31
15	$k \equiv 4 \pmod{15}$	2906161
16	$k \equiv 29 \pmod{45}$	238681
17	$k \equiv 14 \pmod{45}$	p_{17}
18	$k \equiv 89 \pmod{90}$	29611
19	$k \equiv 44 \pmod{90}$	3762091

Table 4.4 Covering used in Lemma 4.2 (i) for $x = 1$

row	congruence	prime p_i
1	$k \equiv 0 \pmod{7}$	239
2	$k \equiv 1 \pmod{7}$	4649
3	$k \equiv 9 \pmod{21}$	43
4	$k \equiv 2 \pmod{21}$	1933
5	$k \equiv 16 \pmod{21}$	10838689
6	$k \equiv 3 \pmod{14}$	909091
7	$k \equiv 10 \pmod{28}$	29
8	$k \equiv 24 \pmod{28}$	281
9	$k \equiv 4 \pmod{35}$	71
10	$k \equiv 11 \pmod{35}$	123551
11	$k \equiv 18 \pmod{35}$	p_{11}
12	$k \equiv 25 \pmod{70}$	4147571
13	$k \equiv 60 \pmod{70}$	p_{13}
14	$k \equiv 32 \pmod{105}$	p_{14}
15	$k \equiv 67 \pmod{105}$	p_{15}
16	$k \equiv 102 \pmod{105}$	p_{16}
17	$k \equiv 5 \pmod{42}$	127
18	$k \equiv 26 \pmod{42}$	2689
19	$k \equiv 12 \pmod{42}$	459691
20	$k \equiv 33 \pmod{84}$	226549
21	$k \equiv 75 \pmod{84}$	p_{21}
22	$k \equiv 19 \pmod{63}$	10837
23	$k \equiv 40 \pmod{63}$	23311
24	$k \equiv 61 \pmod{63}$	45613
25	$k \equiv 6 \pmod{28}$	121499449
26	$k \equiv 13 \pmod{56}$	7841
27	$k \equiv 41 \pmod{56}$	p_{27}
28	$k \equiv 20 \pmod{140}$	421
29	$k \equiv 48 \pmod{140}$	3471301
30	$k \equiv 76 \pmod{140}$	13489841
31	$k \equiv 104 \pmod{140}$	p_{31}
32	$k \equiv 132 \pmod{140}$	p_{32}
33	$k \equiv 27 \pmod{112}$	113
34	$k \equiv 83 \pmod{112}$	p_{34}
35	$k \equiv 55 \pmod{168}$	p_{35}
36	$k \equiv 111 \pmod{168}$	p_{36}
37	$k \equiv 167 \pmod{168}$	p_{37}

For inserting a 4 in the leading zeros, we turn to Tables 6 and 7 in [17]. Rows 1 and 2 of [17, Table 6] utilize the primes 73 and 7, which we have already used for $x = 9$ in Table 4.1 and when inserting a 0 or 7 in the number N, respectively. Hence, we must replace these congruences with congruences that correspond to primes that we have yet to use.

For $x = 4$ we use the covering found in Table 4.6. For this table, and future tables, we utilize partial factorizations of $\Phi_n(10)$. Table 4.5 displays such partial factorizations, as well as some full factorizations of $\Phi_n(10)$ as a way to exhibit large primes. In Table 4.5, we write $\Phi_n(10) = p_1 p_2 \cdots p_r C_n$, where C_n is a composite factor of $\Phi_n(10)$ having at least 2 distinct prime divisors different from $p_1, p_2, \ldots, p_r$ and n. We write $\Phi_n(10) = p_1 p_2 \cdots p_r P_n$, where P_n is a prime factor of $\Phi_n(10)$ different from $p_1, p_2, \ldots, p_r$. Computationally, P_n was determined to be a prime power and then verified to be a prime. For these tables, and future tables, we denote P_x to be a probable prime too large to include comfortably where $\text{ord}_{P_x}(10) = x$. We also use $c_{x,1}$ to denote one prime factor from the composite number C_x where $\text{ord}_{c_{x,1}}(10) = x$ that we were unable to factor, and $c_{x,2}$ to denote the other prime factor from the same composite number. We did not compute the values of $c_{n,1}$ and $c_{n,2}$, but we know they exist.

Table 4.5 Partial/Full factorizations of $\Phi_n(10)$ for large n

n	Factorization of $\Phi_n(10)$
121	$15973 \cdot 38237 \cdot 274187 \cdot P_{121}$
242	$11 \cdot 4357 \cdot 25169 \cdot 1485397 \cdot 1025029814313591715988 93 \cdot P_{242}$
275	$7151 \cdot 15401 \cdot 59951 \cdot C_{275}$
363	$622001227 \cdot 1830142890743707 \cdot C_{363}$
396	$79082656489 \cdot 1538607523068637497164701 \cdot P_{396}$
484	$56629 \cdot 170369 \cdot 29606281 \cdot 1491164086760128255001869 \cdot C_{484}$
528	$7567515354198286020285 8401 \cdot 1178630028484491047954681 5969 \cdot P_{528}$
605	$9666954991 \cdot C_{605}$
726	$727 \cdot 1453 \cdot 3481311540961 \cdot C_{726}$
1210	$10891 \cdot 131891 \cdot C_{1210}$
1452	P_{1452}
1584	$249357075126193 \cdot C_{1584}$
2904	C_{2904}
4356	$949609 \cdot 384538969 \cdot C_{4356}$

To conserve space, we denote

$$p_{1c} = 2838308265222322798939727777, \quad p_{1e} = 117863002848449104795468159969,$$

$$p_{18} = 16205834846012967584927082656402106953,$$

$$p_{42} = 138267770127916457629034873443951,$$

$$p_{43} = 1703548913892494075097664562023844278044121,$$

$$p_{44} = 1395900370916327245555441901,$$

$$p_{45} = 36380545029953205956377406702261,$$

$$p_{48} = 1112314101311286003379752617807870409611285281,$$

$$p_{52} = 1366146685760023293714964475559157409101810043,$$

$$p_{62} = 3628537243429904693247662354742688697863118860538883,$$

$$p_{64} = 14112252487788618228223353931779614493830511168717,$$

$$p_{75} = 7907009307594694001053552000588658391100974093457603716419437,$$

$$p_{77} = 892724462394118139823253, \quad p_{78} = 1866763546567680996103417376059,$$

$$p_{79} = 11297030838243985940172694734174070455495173740851135457З,$$

$$p_{80} = 745075571220969645310060665147889844234389319509119324219479473339.$$

For $x \in \{1, 3, 4, 6, 9\}$, the set of primes $\mathcal{P}_x$ corresponding to a given x used in Lemma 4.2 is the set of primes appearing in the last column of the table given for x. For example, as noted earlier, $\mathcal{P}_9 = \{11, 73, 101, 137\}$. We now have a congruence condition on M of the form given in Lemma 4.2 (iii) for each prime in $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_9$. In the next section, we will develop more congruence conditions on M to handle inserting a digit into one of the leading sevens of N.

Table 4.6 Covering used in Lemma 4.2 (i) for $x = 4$

row	congruence	prime p_i
1a	$k \equiv 16 \pmod{66}$	183411838171
1b	$k \equiv 136 \pmod{264}$	126197002179733470481
1c	$k \equiv 202 \pmod{264}$	p_{1c}
1d	$k \equiv 70 \pmod{528}$	75675153541982860202858401
1e	$k \equiv 268 \pmod{528}$	p_{1e}
1f	$k \equiv 334 \pmod{528}$	P_{528}
1g	$k \equiv 532 \pmod{1584}$	249357075126193
1h	$k \equiv 1060 \pmod{1584}$	$c_{1584,1}$
1i	$k \equiv 4 \pmod{1584}$	$c_{1584,2}$
2a	$k \equiv 3 \pmod{16}$	5882353
2b	$k \equiv 11 \pmod{32}$	353
2c	$k \equiv 27 \pmod{32}$	449
3	$k \equiv 0 \pmod{11}$	21649
4	$k \equiv 1 \pmod{11}$	513239
5	$k \equiv 10 \pmod{16}$	17
6	$k \equiv 2 \pmod{22}$	23
7	$k \equiv 13 \pmod{22}$	4093
8	$k \equiv 3 \pmod{22}$	8779
9	$k \equiv 14 \pmod{44}$	89
10	$k \equiv 36 \pmod{44}$	1052788969
11	$k \equiv 15 \pmod{33}$	67
12	$k \equiv 26 \pmod{33}$	1344628210313298373

row	congruence	prime p_i
13	$k \equiv 37 \pmod{66}$	599144041
14	$k \equiv 38 \pmod{132}$	5419170769
15	$k \equiv 126 \pmod{132}$	789390798020221
16	$k \equiv 60 \pmod{132}$	2361000305507449
17	$k \equiv 5 \pmod{88}$	617
18	$k \equiv 49 \pmod{88}$	p_{18}
19	$k \equiv 104 \pmod{264}$	2377
20	$k \equiv 236 \pmod{264}$	16369
21	$k \equiv 71 \pmod{264}$	432961
22	$k \equiv 159 \pmod{264}$	6796152793
23	$k \equiv 247 \pmod{264}$	24387741577
24	$k \equiv 6 \pmod{55}$	1321
25	$k \equiv 17 \pmod{55}$	62921
26	$k \equiv 28 \pmod{55}$	83251631
27	$k \equiv 39 \pmod{55}$	1300635692678058358830121
28	$k \equiv 50 \pmod{110}$	331
29	$k \equiv 105 \pmod{110}$	5171
30	$k \equiv 7 \pmod{110}$	20163494891
31	$k \equiv 62 \pmod{110}$	318727841165674579776721
32	$k \equiv 18 \pmod{220}$	661
33	$k \equiv 73 \pmod{220}$	18041
34	$k \equiv 128 \pmod{220}$	148721

Table 4.6 cont. Covering used in Lemma 4.2 (i) for $x = 4$

row	congruence	prime p_i
35	$k \equiv 183 \pmod{220}$	1121407321
36	$k \equiv 40 \pmod{275}$	7151
37	$k \equiv 95 \pmod{275}$	15401
38	$k \equiv 150 \pmod{275}$	59951
39	$k \equiv 205 \pmod{275}$	$c_{275,1}$
40	$k \equiv 260 \pmod{275}$	$c_{275,2}$
41	$k \equiv 29 \pmod{165}$	471241
42	$k \equiv 84 \pmod{165}$	p_{42}
43	$k \equiv 139 \pmod{165}$	p_{43}
44	$k \equiv 51 \pmod{220}$	p_{44}
45	$k \equiv 161 \pmod{220}$	p_{45}
46	$k \equiv 106 \pmod{330}$	4124507971
47	$k \equiv 216 \pmod{330}$	19835636682880495867311241
48	$k \equiv 326 \pmod{330}$	p_{48}
49	$k \equiv 8 \pmod{77}$	5237
50	$k \equiv 19 \pmod{77}$	42043
51	$k \equiv 30 \pmod{77}$	29920507
52	$k \equiv 41 \pmod{77}$	p_{52}
53	$k \equiv 52 \pmod{154}$	463
54	$k \equiv 129 \pmod{154}$	24179
55	$k \equiv 63 \pmod{154}$	590437
56	$k \equiv 140 \pmod{154}$	7444361

row	congruence	prime p_i
57	$k \equiv 74 \pmod{154}$	4539402627853030477
58	$k \equiv 151 \pmod{154}$	4924630160315726207887
59	$k \equiv 9 \pmod{99}$	199
60	$k \equiv 20 \pmod{99}$	397
61	$k \equiv 31 \pmod{99}$	34849
62	$k \equiv 42 \pmod{99}$	p_{62}
63	$k \equiv 53 \pmod{198}$	7093127053
64	$k \equiv 152 \pmod{198}$	p_{64}
65	$k \equiv 64 \pmod{396}$	79082656489
66	$k \equiv 163 \pmod{396}$	1538607523068637497164701
67	$k \equiv 262 \pmod{396}$	P_{396}
68	$k \equiv 361 \pmod{792}$	761113
69	$k \equiv 757 \pmod{792}$	440718109921
70	$k \equiv 75 \pmod{297}$	55243
71	$k \equiv 174 \pmod{297}$	198397
72	$k \equiv 273 \pmod{297}$	1981560241
73	$k \equiv 86 \pmod{297}$	31600574312077
74	$k \equiv 185 \pmod{297}$	165426670443186506567467
75	$k \equiv 284 \pmod{297}$	p_{75}
76	$k \equiv 97 \pmod{594}$	7129
77	$k \equiv 196 \pmod{594}$	p_{77}
78	$k \equiv 295 \pmod{594}$	p_{78}

Table 4.6 cont. Covering used in Lemma 4.2 (i) for $x = 4$

row	congruence	prime p_i
79	$k \equiv 394 \pmod{594}$	p_{79}
80	$k \equiv 493 \pmod{594}$	p_{80}
81	$k \equiv 592 \pmod{1188}$	765144469
82	$k \equiv 1186 \pmod{1188}$	3205591041505249
83	$k \equiv 10 \pmod{121}$	15973
84	$k \equiv 21 \pmod{121}$	38237
85	$k \equiv 32 \pmod{121}$	274187
86	$k \equiv 43 \pmod{121}$	P_{121}
87	$k \equiv 54 \pmod{242}$	4357
88	$k \equiv 175 \pmod{242}$	25169
89	$k \equiv 65 \pmod{242}$	1485397
90	$k \equiv 186 \pmod{242}$	P_{242}
91	$k \equiv 76 \pmod{242}$	1025029814313591715988893
92	$k \equiv 197 \pmod{484}$	56629
93	$k \equiv 439 \pmod{484}$	170369
94	$k \equiv 87 \pmod{484}$	29606281
95	$k \equiv 208 \pmod{484}$	1491164086760128255001869
96	$k \equiv 329 \pmod{484}$	$c_{484,1}$
97	$k \equiv 450 \pmod{484}$	$c_{484,2}$
98	$k \equiv 98 \pmod{363}$	622001227
99	$k \equiv 219 \pmod{363}$	1830142890743707
100	$k \equiv 340 \pmod{363}$	$c_{363,1}$
101	$k \equiv 109 \pmod{726}$	727

row	congruence	prime p_i
102	$k \equiv 230 \pmod{726}$	1453
103	$k \equiv 351 \pmod{726}$	3481311540961
104	$k \equiv 472 \pmod{726}$	$c_{726,1}$
105	$k \equiv 593 \pmod{1452}$	P_{1452}
106	$k \equiv 1319 \pmod{4356}$	949609
107	$k \equiv 2771 \pmod{4356}$	384538969
108	$k \equiv 4223 \pmod{4356}$	$c_{4356,1}$
109	$k \equiv 714 \pmod{2904}$	$c_{2904,1}$
110	$k \equiv 1440 \pmod{2904}$	$c_{2904,2}$
111	$k \equiv 2166 \pmod{5808}$	21582529
112	$k \equiv 5070 \pmod{5808}$	114690577
113	$k \equiv 2892 \pmod{5808}$	1070459998625917444769
114	$k \equiv 5796 \pmod{5808}$	48098483178241
115	$k \equiv 120 \pmod{605}$	9666954991
116	$k \equiv 241 \pmod{605}$	$c_{605,1}$
117	$k \equiv 362 \pmod{1210}$	10891
118	$k \equiv 967 \pmod{1210}$	131891
119	$k \equiv 483 \pmod{1210}$	$c_{1210,1}$
120	$k \equiv 1088 \pmod{1210}$	$c_{1210,2}$
121	$k \equiv 604 \pmod{2420}$	1006721
122	$k \equiv 1209 \pmod{2420}$	2323201
123	$k \equiv 1814 \pmod{2420}$	1754328181
124	$k \equiv 2419 \pmod{2420}$	151620139001

4.3 Inserting a digit into the leading sevens

We now work on an analogous argument for inserting a digit x in the string of leading sevens. Recall that the case $x \in \{0, 2, 5, 7, 8\}$ has already been handled; given that $M \equiv 7 \pmod{21}$, inserting any one of the digits in $\{0, 2, 5, 7, 8\}$ anywhere in N will produce a number divisible by 3 or 7. So now we are interested in the case where one of $x \in \{1, 3, 4, 6, 9\}$ is inserted in the leading sevens. We start with a Lemma similar to Lemma 4.2.

Lemma 4.3. *Let N and M be natural numbers such that*

$$N = 7 \cdot \frac{10^n - 1}{9} + M,$$

where N has the decimal expansion

$$N = d_{n-1}d_{n-2}\ldots d_1d_0, \quad d_i \in \{0, 1, \ldots, 9\} \quad n \geq 1, \quad d_{n-1} = 7.$$

Let K be a nonnegative integer such that $d_k = 7$ for $K \leq k \leq n-1$, and set $d_k = 0$ for $k \geq n$. For a fixed $d \in \{0, 1, \ldots, 9\}$, suppose we have distinct primes $p_1, \ldots, p_t$, each > 5, for which

(i) there exists a covering of the integers

$$k \equiv b_i \pmod{c(p_i)}, \quad 1 \leq i \leq t,$$

(ii) $n \equiv 0 \pmod{lcm(c(p_1), \ldots, c(p_t))}$,

(iii) M is a solution to the system of congruences

$$M \equiv -7 - (d-7) \cdot 10^{b_i} \pmod{p_i}, \quad 1 \leq i \leq t.$$

Then, for all nonnegative integers k, we have

$$N_7^{(k)}(d) = 7 \cdot 10^n + 7 \cdot \frac{10^n - 1}{9} + M + (d-7) \cdot 10^k$$

is divisible by at least one of the primes p_i where $1 \leq i \leq t$.

We omit the details of the proof as it is similar to the proof of Lemma 4.2. We are interested in the conclusion of the lemma for $K \leq k \leq n-1$, though as indicated the conclusion follows for all nonnegative integers k. Notice that in Lemma 4.3, we refer to the digit to be inserted as d rather than x. From now forward, this will help clarify when we are talking about tables where the digit d is inserted into the leading sevens (for applying Lemma 4.3) and when we are talking about tables where the digit x is inserted into the leading zeros (for applying Lemma 4.2).

In the previous section, when making our coverings for $x \in \{1, 3, 4, 6, 9\}$, we wanted the primes considered to be distinct so that we could apply the Chinese Remainder Theorem to justify the existence of M satisfying the congruences simultaneously that arise in Lemma 4.2 (iii). Due to the fact that $N_7^{(k)}(d)$ has the additional $7 \cdot 10^n$ term and $d-7$ instead of x, we can reuse some primes for $d \in \{1, 3, 4, 6, 9\}$ as long as we use them in a specific way. For example, consider the prime 101. We used this prime in Table 4.1 for inserting a 9 into the leading zeros via the congruence $k \equiv 3 \pmod 4$. From this, we get that

$$n \equiv 0 \pmod 4 \quad \text{and} \quad M \equiv -9 \cdot 10^3 \equiv 90 \pmod{101}.$$

When this happens, $N_0^{(k)}(9) \equiv 0 \pmod{101}$. Looking at $N_7^{(k)}(d)$, if

$$k \equiv 2 \pmod 4, \quad n \equiv 0 \pmod 4, \quad \text{and} \quad d = 3,$$

then using $M \equiv 90 \pmod{101}$, we have

$$N_7^{(k)}(3) \equiv 7 + 0 + 90 + (3-7) \cdot 10^2 \equiv 97 - 400 \equiv 0 \pmod{101}.$$

Thus, we can reuse the prime 101 assuming we use it via the congruence $k \equiv 2 \pmod 4$ for $d = 3$. Table 4.7 lists the primes that we will use again and how we will use them when making coverings for various values of d.

The coverings used for inserting the digits $d \in \{1, 3, 4, 6, 9\}$ in the leading sevens using Lemma 4.3 can be found in Appendix A. In the way of some details, we note

Table 4.7 Reusable primes for inserting a digit in the leading sevens

prime p_i	x	x congruence	$M \pmod{p_i}$	d	d congruence
11	9	$k \equiv 0 \pmod{2}$	$M \equiv 2 \pmod{11}$	9	$k \equiv 0 \pmod{2}$
13	3	$k \equiv 1 \pmod{6}$	$M \equiv 9 \pmod{13}$	3	$k \equiv 5 \pmod{6}$
13	3	$k \equiv 1 \pmod{6}$	$M \equiv 9 \pmod{13}$	4	$k \equiv 0 \pmod{6}$
19	3	$k \equiv 14 \pmod{18}$	$M \equiv 9 \pmod{19}$	4	$k \equiv 9 \pmod{18}$
29	1	$k \equiv 10 \pmod{28}$	$M \equiv 23 \pmod{29}$	1	$k \equiv 18 \pmod{28}$
31	6	$k \equiv 9 \pmod{15}$	$M \equiv 28 \pmod{31}$	6	$k \equiv 12 \pmod{15}$
37	3	$k \equiv 0 \pmod{3}$	$M \equiv 34 \pmod{37}$	3	$k \equiv 0 \pmod{3}$
37	3	$k \equiv 0 \pmod{3}$	$M \equiv 34 \pmod{37}$	4	$k \equiv 2 \pmod{3}$
41	6	$k \equiv 0 \pmod{5}$	$M \equiv 35 \pmod{41}$	6	$k \equiv 0 \pmod{5}$
43	1	$k \equiv 9 \pmod{21}$	$M \equiv 2 \pmod{43}$	1	$k \equiv 10 \pmod{21}$
61	6	$k \equiv 18 \pmod{60}$	$M \equiv 59 \pmod{61}$	3	$k \equiv 40 \pmod{60}$
71	1	$k \equiv 4 \pmod{35}$	$M \equiv 11 \pmod{71}$	1	$k \equiv 9 \pmod{35}$
101	9	$k \equiv 3 \pmod{4}$	$M \equiv 90 \pmod{101}$	3	$k \equiv 2 \pmod{4}$
137	9	$k \equiv 5 \pmod{8}$	$M \equiv 90 \pmod{137}$	3	$k \equiv 5 \pmod{8}$
211	6	$k \equiv 7 \pmod{30}$	$M \equiv 171 \pmod{211}$	3	$k \equiv 26 \pmod{30}$
239	1	$k \equiv 0 \pmod{7}$	$M \equiv 238 \pmod{239}$	1	$k \equiv 0 \pmod{7}$

that in Table A.2 for $d = 1$, the least common multiple of the moduli is

$$78218300160 = 2^8 \cdot 3^3 \cdot 5 \cdot 7^2 \cdot 11 \cdot 13 \cdot 17 \cdot 19.$$

Checking directly that every integer in the interval $[0, 78218300160)$ satisfies one of the 358 congruences in Table A.2 is unreasonable. A more indirect approach is as follows. We check the covering by addressing k separately depending on its residue class modulo some factor of the least common multiple of the moduli. The choice for such a factor is arbitrary, but in general we used the product of the largest prime used and some of the smaller primes used. For $d = 1$ we consider the residue classes modulo $3990 = 19 \cdot 7 \cdot 5 \cdot 3 \cdot 2$. Fix r from 0 to 3989. We don't want to look at all of the congruences in Table A.2 as most of them will be inconsistent with being $r \pmod{3990}$. For each r, define the set $\mathcal{C}_r$ to be the set of congruences that are consistent with $r \pmod{3990}$. That is, let $\mathcal{C}$ be the complete set of 358 congruences $k \equiv b \pmod{c}$ in Table A.2. Then

$$\mathcal{C}' = \{k \equiv b \pmod{c} : k \equiv b \pmod{c} \text{ is in } \mathcal{C},\ 3990 \nmid c\},$$

and, for $r \in \{0, 1, ..., 3989\}$,

$$\mathcal{C}'_r = \{k \equiv b \pmod{c} : k \equiv b \pmod{c} \text{ is in } \mathcal{C},\ 3990|c \text{ and } b \equiv r \pmod{3990}\}.$$

Hence $\mathcal{C}_r = \mathcal{C}'_r \cup \mathcal{C}'$. For each $r \in \{0, 1, \ldots, 3989\}$, we computed the least common multiple ℓ_r of the moduli in $\mathcal{C}_r$. The goal now is to determine whether, for each r, every integer in the interval $[0, \ell_r)$ that is r modulo 3990 satisfies one of the congruences in $\mathcal{C}_r$. The largest value of ℓ_r is $\ell_{25} = 286513920$. We only need to look at the integers in $[0, \ell_{25})$ that are 25 modulo 3990, so we are left with 71808 integers to verify satisfy one of the congruences in $\mathcal{C}_{25}$ and then to do a similar but smaller computation with the other values of $r \in \{0, 1, \ldots, 3989\}$. After the analogous computation for each $r \in \{0, 1, \ldots, 3989\}$, the verification that the congruences in Table A.2 form a covering is justified.

For the case $d = 3$ in Table A.4, there are 74 congruences and the least common multiple of the moduli for these congruences is 299520. A direct analysis as done with inserting a digit x into the leading zeros can be done in this case. This is also true for Table A.6 that corresponds to $d = 4$, where there are 126 congruences and the least common multiple of the moduli is 8648640.

When $d = 6$, we have the covering found in Table A.8 containing 291 congruences with least common multiple of the moduli for these congruences being 1272348000. An analysis similar to that above for $d = 1$ can be done where the largest prime dividing the least common multiple of the moduli is 17 instead of 19.

For $d = 9$ and Table A.10, there are 329 congruences and the least common multiple of the moduli for these congruences is 127242949680. To verify that the 329 congruences are indeed a covering, we found it easier to look at how we constructed the covering. When constructing this covering we considered the smaller covering defined by the congruences

$$x \equiv 0 \pmod{2}, \quad x \equiv 1 \pmod{6}, \quad x \equiv 5 \pmod{6},$$
$$x \equiv 3 \pmod{12}, \quad \text{and} \quad x \equiv 9 \pmod{12}.$$

We then found a collection of congruences that is equivalent to each of the above congruences in the smaller covering. Observe that $k \equiv 0 \pmod{2}$ is the first congruence listed in Table A.10, so every integer satisfying the first of the five congruences above (the even integers) satisfy a congruence in Table A.10. Now, we want to consider each of the four remaining congruences above and show that the integers satisfying each of these also satisfy a congruence in Table A.10. One can accomplish this as follows. First, we check that every integer satisfying $x \equiv 1 \pmod{6}$ satisfies one of the congruences in rows 1-57 of Table A.10. These congruences have moduli dividing 57960. Observe that 57960 is divisible by 6 and since every integer congruent to 1 modulo 6 in $[0, 57960)$ satisfies one of these 57 congruences, we can deduce every integer congruent to 1 modulo 6 not in $[0, 57960)$ also does. Indeed, if $k \geq 57960$ with $k \equiv 1 \pmod{6}$ and $k \equiv k_0 \pmod{57960}$, then $k_0 \equiv 1 \pmod{6}$ and thus satisfies one of the 57 congruences in Table A.10. Hence, $k_0 \equiv b \pmod{c}$ for some c dividing 57960, then $k \equiv b \pmod{c}$ also. Thus, every integer satisfying $x \equiv 1 \pmod{6}$ also satisfies a congruence in Table A.10. Similarly, one can check that every integer satisfying $x \equiv 5 \pmod{6}$ satisfies one of the 87 congruences in rows 58-144 of Table A.10 with moduli dividing 438480; every integer satisfying $x \equiv 3 \pmod{12}$ satisfies one of the 81 congruences in rows 145-225 of Table A.10 with moduli dividing 468720; and every integer satisfying $x \equiv 9 \pmod{12}$ satisfies one of the 103 congruences in rows 226-328 of Table A.10 with moduli dividing 2051280. Thus, the 329 congruences in Table A.10 form a covering.

4.4 The right-most digits

Define B to be the least common multiple of the moduli appearing in the congruences $k \equiv b \pmod{c}$ in the tables obtained from using Lemma 4.2 (i) in Section 4.2 and from using Lemma 4.3 (i) in Section 4.3. Note that from Table 4.2, we see that 6 divides B; in particular, the order of 10 modulo 3 and 7 also divides B. For Lemma 4.2 (ii) and Lemma 4.3 (ii), we take $n \equiv 0 \pmod{B}$.

Let $\mathcal{P}'_d$ be the set of primes appearing in the tables associated with $d \in \{1, 3, 4, 6, 9\}$ in the previous section. Recalling the sets $\mathcal{P}_j$ in the previous section, we set

$$\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \cdots \cup \mathcal{P}_9 \cup \mathcal{P}'_1 \cup \mathcal{P}'_3 \cup \mathcal{P}'_4 \cup \mathcal{P}'_6 \cup \mathcal{P}'_9,$$

and let P denote the product of the primes in $\mathcal{P}$.

From Lemma 4.2 (iii) and Lemma 4.3 (iii), we have congruence conditions on M where the moduli are the primes $p \in \mathcal{P}$. We deduce from these lemmas that if any digit is inserted in any of the leading zeros or leading sevens, then N is divisible by a prime in $\mathcal{P}$. We impose the further condition on M that $M \equiv 0 \pmod{10}$. From the definition of N in Lemma 4.2 and Lemma 4.3, we see then that $N \equiv 7 \pmod{10}$ so that N is coprime to 10.

We fix M as above, and modify n, using an argument similar to that used in [14], to obtain an N for which inserting a digit in the remaining right-most digits of N also results in a composite number. The basic strategy we describe next is to consider primes, possibly not in $\mathcal{P}$, that divide the number we obtain after inserting a digit in the remaining right-most digits of N. Each insertion will correspond to a prime, though these primes need not be distinct. As M is fixed, there are a fixed number of such insertions and hence a fixed number of these primes to consider.

Recall that $N^{(k)}(x)$ is the number obtained by inserting a digit $x \in \{0, 1, \ldots, 9\}$ to the right of the $(k+1)^{\text{st}}$ digit of N. We are now interested in the case that $k \in \{0, \ldots, K-1\}$, where M is fixed as above and K is the number of digits of M.

Fix $n_0 \equiv 0 \pmod{B}$ satisfying $10^{n_0-2} > M$. We let n vary with $n \equiv 0 \pmod{B}$ and $n \geq n_0$. For any such n, the number N has a string of leading sevens. Since M is fixed, the natural number K is also fixed and does not vary as n varies. Therefore, there are finitely many ways to insert a digit in M; that is, there are finitely many choices of $k \in \{0, 1, \ldots, K-1\}$ and $x \in \{0, 1, \ldots, 9\}$ independent of n.

Recall M and n_0 are fixed. Momentarily, fix also $k \in \{0, 1, \ldots, K-1\}$ and $x \in \{0, 1, \ldots, 9\}$. Set

$$\overline{N} = 7 \cdot \frac{10^{n_0} - 1}{9} + M.$$

Observe that $\overline{N}$ is coprime to 10 since, as noted above, we take $M \equiv 0 \pmod{10}$. We denote by $\overline{N}^{(k)}(x)$ the number obtained by inserting $x \in \{0, 1, \ldots, 9\}$ to the right of the $(k+1)^{\text{st}}$ digit of $\overline{N}$. Let $q = q(k, x)$ denote the least prime $q = q(k, x)$ dividing $\overline{N}^{(k)}(x)$. If $q \notin \{2, 5\}$ and $n \equiv n_0 \pmod{c(q)}$, then

$$N^{(k)}(x) - \overline{N}^{(k)}(x) = 7 \cdot 10^{n_0} \cdot \frac{10^{n-n_0} - 1}{9} \equiv 0 \pmod{q}.$$

Thus, we also have q divides $N^{(k)}(x)$ whenever $n \equiv n_0 \pmod{c(q)}$. If $q \in \{2, 5\}$, then the units digit of $\overline{N}^{(k)}(x)$ will be even or 5, which implies that, for $n \geq n_0$, the units digit of $N^{(k)}(x)$ is even or 5. Thus, in this case, $N^{(k)}(x)$ is composite.

Letting $k \in \{0, 1, \ldots, K-1\}$ and $x \in \{0, 1, \ldots, 9\}$ vary, we let B' be the least common multiple of B and the numbers $c(q)$ where q varies over the primes $q(k, x)$. Taking $n \equiv n_0 \pmod{B'}$, we see that after inserting any digit anywhere in N, including the leading zeros and to the right of the units digit, the resulting number will be divisible by one of the primes $q(k, x)$ above or one of the primes $p \in \mathcal{P}$. Theorem 1.5 follows from the fact that N goes to infinity as n increases, while the primes $q(k, x)$ and $\mathcal{P}$ remain finite.

4.5 Related topics and open problems

Questions related to Theorem 1.5 are abundant. One such question looks at combining the notion of replacing a digit as mentioned in the introduction with widely digitally stable composite numbers. Consider the following theorem.

Theorem 4.4. *There exist infinitely many widely digitally stable composite numbers, N, coprime to 10, that remain composite when we replace any digit in the decimal expansion of N, including any of the infinitely many leading zeros.*

We expect Theorem 4.4 to hold for base 10. In the way of some details, we provide two outlines for such a proof. Both outlines follow the same process as was used for Theorem 1.5, but will be looking at different starting composite numbers, N.

Approach 1 considers N, $N_0^{(k)}(x)$, and $N_7^{(k)}(d)$ as was used in the proof of Theorem 1.5. That is,

$$N = 7 \cdot \frac{10^n - 1}{9} + M,$$

with $N_0^{(k)}(x)$ defined as

$$N_0^{(k)}(x) = 7 \cdot \frac{10^n - 1}{9} + M + x \cdot 10^k,$$

and

$$N_7^{(k)}(d) = 7 \cdot 10^n + 7 \cdot \frac{10^n - 1}{9} + M + (d - 7) \cdot 10^k.$$

In Sections 4.1 and 4.2 we require $k \geq n$ to ensure that we are inserting a digit in the leading zeros; however, the conclusions of Lemmas 4.2 and 4.3 hold for all k. If we drop this requirement we see that $N_0^{(k)}(x)$ accomplishes three things. If $x \in \{0, 1, \ldots, 9\}$ and $k \geq n$ corresponding to the leading zeros, then $N_0^{(k)}(x)$ represents inserting a digit to the right of the $k + 1^{st}$ digit of N; if $x \in \{-7, -6, \ldots, 2\}$ and $K \leq k \leq n$ corresponding to the leading sevens, then $N_0^{(k)}(x)$ represents replacing the $k + 1^{st}$ digit of N with $x \in \{0, 1, \ldots, 9\}$; and if $x \in \{0, 1, \ldots, 9\}$ and $k > n$, then $N_0^{(k)}(x)$

represents replacing the $k+1^{st}$ digit of N with x. To prove Theorem 4.4, we need coverings for $x \in \{-7, -6, \ldots, 9\}$ and $d \in \{0, 1, \ldots, 9\}$.

Taking $M \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 3$, we have $N_0^{(k)}(x) \equiv 0 \pmod 3$ for $x \equiv 2 \pmod 3$ and $N_7^{(k)}(d) \equiv 0 \pmod 3$ for $d \equiv 2 \pmod 3$. That is, we do not need coverings for $x \in \{-7, -4, -1, 2, 5, 8\}$ or $d \in \{2, 5, 8\}$. Utilizing the prime 7 in the same way as in Section 4.2, we also do not need coverings for $x, d \in \{0, 7\}$. Using the ten coverings in the same way as in the previous sections, we need only provide coverings for $x \in \{-6, -5, -3, -2\}$.

For approach 2, consider N to contain a string of leading ones. That is,

$$N = \frac{10^n - 1}{9} + M$$

with $N_0^{(k)}(x)$ defined by

$$N_0^{(k)}(x) = \frac{10^n - 1}{9} + M + x \cdot 10^k,$$

and $N_1^{(k)}(d)$ defined by

$$N_1^{(k)}(d) = 10^n + \frac{10^n - 1}{9} + M + (d - 1) \cdot 10^k.$$

We use the same idea as in method 1. That is, if $x \in \{0, 1, \ldots, 9\}$ and $k \geq n$, then $N_0^{(k)}(x)$ represents inserting a digit to the right of the $k+1^{st}$ digit of N; if $x \in \{-1, 0, \ldots, 8\}$ and $K \leq k \leq n$, corresponding to the leading ones, then $N_0^{(k)}(x)$ represents replacing the $k+1^{st}$ digit of N with $x \in \{0, 1, \ldots, 9\}$; and if $x \in \{0, 1, \ldots, 9\}$ and $k > n$, then $N_0^{(k)}(x)$ represents replacing the $k+1^{st}$ digit of N with x. Thus, we want coverings for $x \in \{-1, 0, \ldots, 9\}$ and $d \in \{0, 1, \ldots, 9\}$.

Taking $M \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 3$, we have $N_0^{(k)}(x) \equiv 0 \pmod 3$ for $x \equiv 2 \pmod 3$ and $N_1^{(k)}(d) \equiv 0 \pmod 3$ for $d \equiv 2 \pmod 3$. That is, we now need coverings for $x \in \{0, 1, 3, 4, 6, 7, 9\}$ and $d \in \{0, 1, 3, 4, 6, 7, 9\}$. Notice that $N_0^{(k)}(0) = N$ for all k. Thus, we do not need a full covering for $x = 0$. We will instead use a prime not used in any covering to guarantee that N is composite.

We outlined two approaches as they both have their merits. Approach 1 requires four more coverings, for a total of 14, but we do not have to modify the coverings used for Theorem 1.5. The downside, however, is that we will likely be able to reuse very few of the smaller primes when constructing the four new coverings. The second method requires one less covering for a total of 13 coverings. Since the N used in approach 2 has a string of leading ones instead of sevens, we will not be able to reuse the same primes in the same way as in Section 4.3. Hence, we will have to recreate the coverings used for $d \in \{0, 1, 3, 4, 6, 7, 9\}$. Fortunately, the second method allows us to include the prime 7, which will likely be usable in multiple coverings. Both approaches will require a modified approach when looking at insertions and replacements in the right most digits.

In [17], a positive proportion of prime numbers, n, were shown to have the property that if you replace an arbitrary digit of n, including any of the infinitely many leading zeros, then the resulting number is composite. This leads to several questions. Originally appearing in [14], one could ask if there are infinitely many primes p, or a positive proportion of primes p, such that if an arbitrary digit is inserted between any two digits of p, or to the right of the units digit of p, then the resulting number is composite. Are a positive proportion of composite numbers widely digitally stable? We do not know the answers to these questions and expect that methods different than those used in this dissertation are required to establish such results.

Bibliography

[1] Alexander Borisov, Michael Filaseta, Tsit Yuen Lam, and Ognian Trifonov. “Classes of polynomials having only one non-cyclotomic irreducible factor”. In: *Acta Arithmetica* 90.2 (1999), pp. 121–153.

[2] John Brillhart, Michael Filaseta, and Andrew Odlyzko. “On an irreducibility theorem of A. Cohn”. In: *Canadian J. Math.* 33.5 (1981), pp. 1055–1059.

[3] John Brillhart, Derrick Lehmer, John Selfridge, Bryant Tuckerman, and Samuel Wagstaff Jr. *Factorizations of* $b^n \pm 1$. Third. Vol. 22. Contemporary Mathematics. $b = 2, 3, 5, 6, 7, 10, 11, 12$ up to high powers. American Mathematical Society, Providence, RI, 2002, pp. xcvi+236.

[4] Morgan Cole, Scott Dunn, and Michael Filaseta. “Further irreducibility criteria for polynomials with non-negative coefficients”. In: *Acta Arith.* 175.2 (2016), pp. 137–181.

[5] Keith Conrad, ed. *Recognizing Galois groups Sn and An*. URL: `https://citeseerx.ist.psu.edu/viewdoc/download?doi=10.1.1.691.9929&rep=rep1&type=pdf` (visited on 03/03/2021).

[6] John D. Dixon and Brian Mortimer. *Permutation Groups*. Vol. 163. Graduate Texts in Mathematics. Springer-Verlag, New York, 1996.

[7] Gustave Dumas. “Sur quelques cas d’irréductibilité des polynômes à coefficients rationnels”. In: *Journal de Mathématiques Pures et Appliquées* 2 (1906), pp. 191–258.

[8] Paul Erdős. “Solution to problem 1029: Erdős and the computer”. In: *Math. Mag* 52 (1979), pp. 180–181.

[9] Michael Filaseta. “A further generalization of an irreducibility theorem of A. Cohn”. In: *Canad. J. Math.* 34.6 (1982), pp. 1390–1395.

[10] Michael Filaseta. “Irreducibility criteria for polynomials with nonnegative coefficients”. In: *Canad. J. Math.* 40.2 (1988), pp. 339–351.

[11] Michael Filaseta, Carrie Finch, and Mark Kozek. "On powers associated with Sierpiński numbers, Riesel numbers and Polignac's conjecture". In: *Journal of Number Theory* 128 (July 2008), pp. 1916–1940.

[12] Michael Filaseta and Samuel Gross. "49598666989151226098104244512918". In: *J. Number Theory* 137 (2014), pp. 16–49.

[13] Michael Filaseta and Jacob Juillerat. "Consecutive primes which are widely digitally delicate". In: *arXiv preprint arXiv:2101.08898* (2021).

[14] Michael Filaseta, Mark Kozek, Charles Nicol, and John Selfridge. "Composites that remain composite after changing a digit". In: *J. Comb. Number Theory* 2.1 (2010), pp. 25–36.

[15] Michael Filaseta, Angel Kumchev, and Dmitrii V. Pasechnik. "On the irreducibility of a truncated binomial expansion". In: *The Rocky Mountain Journal of Mathematics* 37.2 (2007), pp. 455–464.

[16] Michael Filaseta and Richard Moy. "On the Galois group over $\mathbb{Q}$ of a truncated binomial expansion". In: *Colloquium Mathematicum*. Vol. 154. Instytut Matematyczny Polskiej Akademii Nauk. 2018, pp. 295–308.

[17] Michael Filaseta and Jeremiah Southwick. "Primes that become composite after changing an arbitrary digit". In: *Mathematics of Computation* 90.328 (2021), pp. 979–993.

[18] Arlington M. Fink. "Two inequalities". In: *Univ. Beograd. Publ. Elektrotehn. Fak. Ser. Mat.* 6 (1995), pp. 48–49.

[19] Joseph Foster, Jacob Juillerat, and Jeremiah Southwick. "The irreducibility of polynomials arising from the study of Fourier coefficients of powers of the Dedekind-eta function". In: *Journal of Combinatorics and Number Theory* 10.3 (2018), pp. 161–167.

[20] Bernhard Heim, Florian Luca, and Markus Neuhauser. "Recurrence relations for polynomials obtained by arithmetic functions". In: *Int. J. Number Theory* 15.6 (2019), pp. 1291–1303.

[21] Bernhard Heim, Markus Neuhauser, and PC Athaibah. "Log-Concavity of Recursively Defined Polynomials." In: *J. Integer Seq.* 22.1 (2019), pp. 19–1.

[22] Bernhard Heim, Markus Neuhauser, and Florian Rupp. "Fourier coefficients of powers of the Dedekind eta function". In: *The Ramanujan Journal* 48.1 (2019), pp. 1–11.

[23] Jackson Hopper and Paul Pollack. "Digitally delicate primes". In: *J. Number Theory* 168 (2016), pp. 247–256.

[24] Murray S. Klamkin. "Problem 1029". In: *Mathematics Magazine* 51 (1978), p. 69.

[25] Sergei Konyagin. "Numbers that become composite after changing one or two digits". In: *Pioneer Jour. of Algebra, Number Theory and Appl* 6 (2013), pp. 1–7.

[26] Melvyn Nathanson. *Combinatorial and Additive Number Theory IV*. Springer, 2021.

[27] Ken Ono. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*. Vol. 102. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 2004, pp. viii+216. ISBN: 0-8218-3368-5.

[28] Georg Pólya and Gabor Szegő. *Aufgaben und Lehrsätze aus der Analysis. Band II: Funktionentheorie, Nullstellen, Polynome Determinanten, Zahlentheorie.* Vierte Auflage, Heidelberger Taschenbücher, Band 74. Springer-Verlag, Berlin-New York, 1971, xii+407 pp. (loose errata).

[29] Victor V. Prasolov. *Polynomials*. Vol. 11. Algorithms and Computation in Mathematics. Springer-Verlag, New York, 2004.

[30] Maruti Ram Murty. "Prime numbers and irreducible polynomials". In: *Amer. Math. Monthly* 109.5 (2002), pp. 452–458.

[31] Issai Schur. "Affektlose Gleichungen in der Theorie der Laguerreschen und Hermiteschen Polynome". In: *J Reine Angew. Math.* 165 (1931), pp. 52–58.

[32] Robert E. Shafer. "Problem E 1867". In: *Amer. Math. Monthly* 73 (1966), p. 309.

[33] Daniel Shiu. "Strings of congruent primes". In: *Journal of the London Mathematical Society* 61.2 (2000), pp. 359–373.

[34] Carl Ludwig Siegel. "Uber einige anwendungen diophantischer approximationen". In: *Abh. Pruess. Akad. Wiss., Phsy.-Math. Kl.* 1 (1929), pp. 14–72.

[35] Neil J. A. Sloane, ed. *The On-Line Encyclopedia of Integer Sequences.* URL: `https://oeis.org/A076336` (visited on 03/06/2020).

[36] Jeremiah Southwick. “Two inquiries related to the digits of prime numbers”. In: *Doctoral Dissertation* (2020).

[37] Richard P. Stauduhar. “The determination of Galois groups”. In: *Mathematics of computation* 27.124 (1973), pp. 981–996.

[38] Terence Tao. “A remark on primality testing and decimal expansions”. In: *J. Aust. Math. Soc.* 91.3 (2011), pp. 405–413.

Appendix A

Coverings for inserting a digit in the leading sevens in base 10

We produce the coverings used in Lemma 4.3 for inserting a digit $d \in \{1, 3, 4, 6, 9\}$ into the leading sevens. As with Table 4.6, we will make use of partial factorizations of $\Phi_n(10)$ for inserting a digit into the leading sevens. Recall that if we are unable to fully factor $\Phi_n(10)$, then we write $\Phi_n(10) = p_1 p_2 \cdots p_r C_n$, where C_n is a composite factor of $\Phi_n(10)$ having at least 2 distinct prime divisors different from $p_1, p_2, \ldots, p_r$ and n. We also write $\Phi_n(10) = p_1 p_2 \cdots p_r P_n$, where P_n is a prime factor of $\Phi_n(10)$ different from $p_1, p_2, \ldots, p_r$. Computationally, P_n was determined to be a prime power and then verified to be a prime. For these tables, and future tables, we denote P_x to be a probable prime too large to include comfortably where $\mathrm{ord}_{P_x}(10) = x$. We also use $c_{x,1}$ to denote one prime factor from the composite number C_x where $\mathrm{ord}_{c_{x,1}}(10) = x$ that we were unable to factor, and $c_{x,2}$ to denote the other prime factor from the same composite number. We did not compute the values of $c_{n,1}$ and $c_{n,2}$, but we know they exist.

For $d = 1$ we use the covering in Table A.2, where the letter rows arise from reusing a prime shown in Table 4.7. Table A.1 provides the partial factorizations of $\Phi_n(10)$ for large n used in these coverings, as well as some full factorizations as a way to exhibit large primes. Some of the composite numbers and large primes that appear in Table A.2 are not included in Table A.1. These composite numbers can be determined by dividing $\Phi_n(10)$ by the primes associated with n that can be found in adjacent rows in the table. For example, rows 35 and 36 of Table A.2 utilize the composite number associated with $n = 245$. This means that C_{245} is $\Phi_{245}(10)$ divided by the primes 336737801, 23609565631, and 8274923251622825388807721, corresponding to rows 32, 33, and 34, respectively. The same is used for the large primes P_n that do not appear in Table A.1. To save space we denote some of the larger primes that still fit comfortably below.

$$p_4 = 18977242267323558587448573265 9,$$

$$p_{21} = 197673014459819096356802301467933 3,$$

$$p_{23} = 5076141624365532994918781726395939035533,$$

$$p_{28} = 4769337181464959147997704753876850429427,$$

$$p_{69} = 3694204838266898054461991356 1,$$

$$p_{96} = 2824480286120668942569844248692643858 01,$$

$$p_{134} = 6717658458758041138199521, \quad p_{284} = 965194617121640791456070347951751,$$

$$p_{286} = 75320180627132846254797791940 7,$$

$$p_{310} = 1844928855004465413924191940 1, \quad p_{320} = 30701085207038248431740137 3.$$

Table A.1 Partial/Full factorizations of $\Phi_n(10)$ for $d = 1$

n	Factorization of $\Phi_n(10)$
294	$7 \cdot P_{294}$
448	C_{448}
456	$27817 \cdot 376102873$ $\cdot\ 36120612721 \cdot C_{456}$
588	C_{588}
616	C_{616}

n	Factorization of $\Phi_n(10)$
665	C_{665}
1064	$260827749862793 \cdot C_{1064}$
1197	C_{1197}
1596	$539449 \cdot 3301756921 \cdot C_{1596}$
2394	C_{2394}
3192	C_{3192}

Table A.2 Covering used in Lemma 4.3 (i) for $d = 1$

row	congruence	prime p_i
a	$k \equiv 0 \pmod{7}$	239
b	$k \equiv 10 \pmod{21}$	43
c	$k \equiv 18 \pmod{28}$	29
d	$k \equiv 9 \pmod{35}$	71
1	$k \equiv 59 \pmod{63}$	45121231
2	$k \equiv 45 \pmod{63}$	1921436048294281
3	$k \equiv 38 \pmod{126}$	5274739
4	$k \equiv 101 \pmod{126}$	p_4
5	$k \equiv 213 \pmod{252}$	1009
6	$k \equiv 150 \pmod{252}$	43266855241
7	$k \equiv 87 \pmod{252}$	P_{252}
8	$k \equiv 24 \pmod{504}$	41593295521
9	$k \equiv 276 \pmod{504}$	7064204436768358634702473
10	$k \equiv 17 \pmod{189}$	878075126908698927928483
11	$k \equiv 80 \pmod{189}$	33030636037992147205820927521
12	$k \equiv 143 \pmod{189}$	P_{189}
13	$k \equiv 255 \pmod{315}$	631
14	$k \equiv 192 \pmod{315}$	142809770881
15	$k \equiv 129 \pmod{315}$	P_{315}
16	$k \equiv 381 \pmod{630}$	131041
17	$k \equiv 66 \pmod{630}$	525326613841
18	$k \equiv 318 \pmod{630}$	2521
19	$k \equiv 3 \pmod{630}$	$c_{630,1}$
20	$k \equiv 48 \pmod{49}$	505885997
21	$k \equiv 41 \pmod{49}$	p_{21}
22	$k \equiv 34 \pmod{98}$	197
23	$k \equiv 83 \pmod{98}$	p_{23}
24	$k \equiv 27 \pmod{147}$	63799
25	$k \equiv 76 \pmod{147}$	4715467
26	$k \equiv 125 \pmod{147}$	267652966241599
27	$k \equiv 118 \pmod{147}$	2603941883787374089
28	$k \equiv 69 \pmod{147}$	p_{28}
29	$k \equiv 167 \pmod{294}$	P_{294}
30	$k \equiv 314 \pmod{588}$	$c_{588,1}$
31	$k \equiv 20 \pmod{588}$	$c_{588,2}$
32	$k \equiv 209 \pmod{245}$	336737801
33	$k \equiv 160 \pmod{245}$	23609565631
34	$k \equiv 111 \pmod{245}$	8274923251622825388807721
35	$k \equiv 62 \pmod{245}$	$c_{245,1}$
36	$k \equiv 13 \pmod{245}$	$c_{245,2}$
37	$k \equiv 153 \pmod{196}$	P_{196}
38	$k \equiv 104 \pmod{392}$	52781463673
39	$k \equiv 300 \pmod{392}$	$c_{392,1}$
40	$k \equiv 251 \pmod{784}$	54881
41	$k \equiv 447 \pmod{784}$	403539336813078648113
42	$k \equiv 643 \pmod{784}$	$c_{784,1}$
43	$k \equiv 55 \pmod{784}$	$c_{784,2}$
44	$k \equiv 986 \pmod{1176}$	737353
45	$k \equiv 790 \pmod{1176}$	1482937
46	$k \equiv 594 \pmod{1176}$	70251889

Table A.2 cont. Covering used in Lemma 4.3 (i) for $d = 1$

row	congruence	prime p_i
47	$k \equiv 398 \pmod{1176}$	14498348929
48	$k \equiv 202 \pmod{1176}$	3343677109249
49	$k \equiv 6 \pmod{1176}$	604681084564369
50	$k \equiv 89 \pmod{91}$	547
51	$k \equiv 82 \pmod{91}$	14197
52	$k \equiv 75 \pmod{91}$	17837
53	$k \equiv 68 \pmod{91}$	4262077
54	$k \equiv 61 \pmod{91}$	43442141653
55	$k \equiv 54 \pmod{91}$	316877365766624209
56	$k \equiv 47 \pmod{91}$	110742186470530054291318013
57	$k \equiv 40 \pmod{182}$	21705503
58	$k \equiv 131 \pmod{182}$	P_{182}
59	$k \equiv 33 \pmod{273}$	32786209
60	$k \equiv 124 \pmod{273}$	139708570703521
61	$k \equiv 215 \pmod{273}$	1093
62	$k \equiv 117 \pmod{273}$	$c_{273,1}$
63	$k \equiv 208 \pmod{273}$	$c_{273,2}$
64	$k \equiv 572 \pmod{819}$	31123
65	$k \equiv 299 \pmod{819}$	P_{819}
66	$k \equiv 845 \pmod{1638}$	24571
67	$k \equiv 26 \pmod{1638}$	280099
68	$k \equiv 19 \pmod{455}$	175631
69	$k \equiv 110 \pmod{455}$	p_{69}
70	$k \equiv 201 \pmod{455}$	911
71	$k \equiv 292 \pmod{455}$	$c_{455,1}$
72	$k \equiv 383 \pmod{455}$	$c_{455,2}$
73	$k \equiv 285 \pmod{364}$	654721485601
74	$k \equiv 194 \pmod{364}$	$c_{364,1}$
75	$k \equiv 103 \pmod{364}$	$c_{364,2}$
76	$k \equiv 12 \pmod{728}$	$c_{728,1}$
77	$k \equiv 376 \pmod{728}$	$c_{728,2}$
78	$k \equiv 642 \pmod{728}$	574393
79	$k \equiv 551 \pmod{728}$	10727809
80	$k \equiv 460 \pmod{728}$	286569193
81	$k \equiv 733 \pmod{1092}$	37275014869
82	$k \equiv 369 \pmod{1092}$	428085457801
83	$k \equiv 5 \pmod{1092}$	4902137591818741
84	$k \equiv 278 \pmod{1456}$	10193
85	$k \equiv 1006 \pmod{1456}$	436801
86	$k \equiv 187 \pmod{1456}$	64348368449
87	$k \equiv 915 \pmod{1456}$	$c_{1456,1}$
88	$k \equiv 1552 \pmod{2184}$	3624275167969
89	$k \equiv 824 \pmod{2184}$	64365563809
90	$k \equiv 96 \pmod{2184}$	$c_{2184,1}$

Table A.2 cont. Covering used in Lemma 4.3 (i) for $d = 1$

row	congruence	prime p_i
91	$k \equiv 32 \pmod{112}$	1199683691448463702260833377
92	$k \equiv 81 \pmod{420}$	1384194841
93	$k \equiv 165 \pmod{420}$	42681134161
94	$k \equiv 249 \pmod{420}$	424451728681
95	$k \equiv 333 \pmod{420}$	1397909410136282227711346421
96	$k \equiv 417 \pmod{420}$	p_{96}
97	$k \equiv 179 \pmod{210}$	29970369241
98	$k \equiv 95 \pmod{210}$	1661378260814161
99	$k \equiv 11 \pmod{210}$	18276168846821336356291
100	$k \equiv 137 \pmod{840}$	134401
101	$k \equiv 557 \pmod{840}$	575794801
102	$k \equiv 53 \pmod{840}$	67501680066009758096678991121
103	$k \equiv 473 \pmod{840}$	$c_{840,1}$
104	$k \equiv 109 \pmod{672}$	1454209
105	$k \equiv 277 \pmod{672}$	9396577
106	$k \equiv 445 \pmod{672}$	77700458351603754241
107	$k \equiv 613 \pmod{672}$	465149401935712615628055361
108	$k \equiv 529 \pmod{672}$	$c_{672,1}$
109	$k \equiv 25 \pmod{1680}$	3361
110	$k \equiv 361 \pmod{1680}$	20161
111	$k \equiv 697 \pmod{1680}$	75500478859681
112	$k \equiv 1033 \pmod{1680}$	$c_{1680,1}$
113	$k \equiv 1369 \pmod{1680}$	$c_{1680,2}$
114	$k \equiv 193 \pmod{2016}$	2017
115	$k \equiv 865 \pmod{2016}$	518113
116	$k \equiv 1537 \pmod{2016}$	56763828581443201
117	$k \equiv 95 \pmod{224}$	673
118	$k \equiv 207 \pmod{224}$	43735845217
119	$k \equiv 88 \pmod{224}$	$c_{224,1}$
120	$k \equiv 200 \pmod{224}$	$c_{224,2}$
121	$k \equiv 39 \pmod{336}$	337
122	$k \equiv 151 \pmod{336}$	$c_{336,1}$
123	$k \equiv 263 \pmod{336}$	$c_{336,2}$
124	$k \equiv 67 \pmod{448}$	$c_{448,1}$
125	$k \equiv 179 \pmod{448}$	$c_{448,2}$
126	$k \equiv 291 \pmod{1344}$	205633
127	$k \equiv 739 \pmod{1344}$	4546753
128	$k \equiv 1187 \pmod{1344}$	1342657
129	$k \equiv 403 \pmod{1344}$	10418689
130	$k \equiv 851 \pmod{1344}$	$c_{1344,1}$
131	$k \equiv 1299 \pmod{1344}$	$c_{1344,2}$
132	$k \equiv 60 \pmod{560}$	1378721
133	$k \equiv 172 \pmod{560}$	5758943337281
134	$k \equiv 284 \pmod{560}$	p_{134}

Table A.2 cont. Covering used in Lemma 4.3 (i) for $d = 1$

row	congruence	prime p_i
135	$k \equiv 396 \pmod{560}$	$c_{560,1}$
136	$k \equiv 508 \pmod{560}$	$c_{560,2}$
137	$k \equiv 459 \pmod{896}$	105528193
138	$k \equiv 683 \pmod{896}$	$c_{896,1}$
139	$k \equiv 123 \pmod{1120}$	48102033281
140	$k \equiv 347 \pmod{1120}$	555204874561
141	$k \equiv 571 \pmod{1120}$	29232317046703681
142	$k \equiv 795 \pmod{1120}$	$c_{1120,1}$
143	$k \equiv 1019 \pmod{1120}$	$c_{1120,2}$
144	$k \equiv 235 \pmod{1792}$	7710977
145	$k \equiv 1131 \pmod{1792}$	12939800833
146	$k \equiv 907 \pmod{1792}$	45501964033
147	$k \equiv 11 \pmod{1792}$	$c_{1792,1}$
148	$k \equiv 1124 \pmod{1232}$	30357713
149	$k \equiv 1012 \pmod{1232}$	3697
150	$k \equiv 900 \pmod{1232}$	$c_{1232,1}$
151	$k \equiv 788 \pmod{1232}$	$c_{1232,2}$
152	$k \equiv 60 \pmod{308}$	39511854229
153	$k \equiv 256 \pmod{308}$	P_{308}
154	$k \equiv 452 \pmod{616}$	$c_{616,1}$
155	$k \equiv 340 \pmod{616}$	$c_{616,2}$
156	$k \equiv 228 \pmod{231}$	2311
157	$k \equiv 74 \pmod{231}$	22187551
158	$k \equiv 151 \pmod{231}$	535212994471849
159	$k \equiv 1348 \pmod{2464}$	29569
160	$k \equiv 116 \pmod{2464}$	232722337
161	$k \equiv 1236 \pmod{2464}$	11229398470301331937
162	$k \equiv 4 \pmod{2464}$	210732524523217313
163	$k \equiv 113 \pmod{119}$	923441
164	$k \equiv 106 \pmod{119}$	3924966376871
165	$k \equiv 99 \pmod{119}$	$c_{119,1}$
166	$k \equiv 92 \pmod{119}$	$c_{119,2}$
167	$k \equiv 85 \pmod{238}$	1868879293
168	$k \equiv 204 \pmod{238}$	5673320472670315859129
169	$k \equiv 197 \pmod{238}$	P_{238}
170	$k \equiv 316 \pmod{476}$	$c_{476,1}$
171	$k \equiv 78 \pmod{476}$	$c_{476,2}$
172	$k \equiv 428 \pmod{476}$	2381
173	$k \equiv 309 \pmod{476}$	275855329893529
174	$k \equiv 190 \pmod{476}$	20087794479102305428621
175	$k \equiv 547 \pmod{952}$	$c_{952,1}$
176	$k \equiv 71 \pmod{952}$	$c_{952,2}$
177	$k \equiv 302 \pmod{952}$	371281
178	$k \equiv 540 \pmod{952}$	6514537

Table A.2 cont. Covering used in Lemma 4.3 (i) for $d = 1$

row	congruence	prime p_i
179	$k \equiv 778 \pmod{952}$	953
180	$k \equiv 1016 \pmod{1904}$	7024789118033
181	$k \equiv 64 \pmod{1904}$	46502552676497
182	$k \equiv 1135 \pmod{1190}$	42841
183	$k \equiv 897 \pmod{1190}$	14281
184	$k \equiv 659 \pmod{1190}$	674731
185	$k \equiv 421 \pmod{1190}$	$c_{1190,1}$
186	$k \equiv 183 \pmod{1190}$	$c_{1190,2}$
187	$k \equiv 295 \pmod{357}$	4999
188	$k \equiv 176 \pmod{357}$	143556841
189	$k \equiv 57 \pmod{357}$	3329808692827
190	$k \equiv 288 \pmod{357}$	43451713720907517733
191	$k \equiv 169 \pmod{357}$	$c_{357,1}$
192	$k \equiv 50 \pmod{357}$	$c_{357,2}$
193	$k \equiv 638 \pmod{714}$	17851
194	$k \equiv 281 \pmod{714}$	P_{714}
195	$k \equiv 876 \pmod{1071}$	43834194307
196	$k \equiv 519 \pmod{1071}$	7094621664153616951
197	$k \equiv 162 \pmod{1071}$	6427
198	$k \equiv 1114 \pmod{1428}$	898827469
199	$k \equiv 757 \pmod{1428}$	1429
200	$k \equiv 400 \pmod{1428}$	$c_{1428,1}$
201	$k \equiv 43 \pmod{1428}$	$c_{1428,2}$
202	$k \equiv 512 \pmod{595}$	10711
203	$k \equiv 393 \pmod{595}$	9521
204	$k \equiv 274 \pmod{595}$	$c_{595,1}$
205	$k \equiv 155 \pmod{595}$	$c_{595,2}$
206	$k \equiv 1226 \pmod{1785}$	149866505006632681
207	$k \equiv 631 \pmod{1785}$	$c_{1785,1}$
208	$k \equiv 36 \pmod{1785}$	$c_{1785,2}$
209	$k \equiv 743 \pmod{833}$	17241400681
210	$k \equiv 624 \pmod{833}$	1667
211	$k \equiv 505 \pmod{833}$	$c_{833,1}$
212	$k \equiv 386 \pmod{833}$	$c_{833,2}$
213	$k \equiv 1100 \pmod{1666}$	609757
214	$k \equiv 267 \pmod{1666}$	548804832033845773
215	$k \equiv 981 \pmod{1666}$	$c_{1666,1}$
216	$k \equiv 148 \pmod{1666}$	$c_{1666,2}$
217	$k \equiv 1695 \pmod{2499}$	2962284613
218	$k \equiv 862 \pmod{2499}$	$c_{2499,1}$
219	$k \equiv 29 \pmod{2499}$	$c_{2499,2}$
220	$k \equiv 5 \pmod{17}$	5363222357
221	$k \equiv 15 \pmod{17}$	2071723
222	$k \equiv 8 \pmod{34}$	103

Table A.2 cont. Covering used in Lemma 4.3 (i) for $d = 1$

row	congruence	prime p_i
223	$k \equiv 25 \pmod{34}$	21993833369
224	$k \equiv 18 \pmod{34}$	4013
225	$k \equiv 35 \pmod{68}$	2324557465671829
226	$k \equiv 1 \pmod{68}$	1491383821
227	$k \equiv 128 \pmod{133}$	1597
228	$k \equiv 121 \pmod{133}$	2021015460335957
229	$k \equiv 114 \pmod{133}$	P_{133}
230	$k \equiv 107 \pmod{266}$	247025236977306025681323889
231	$k \equiv 240 \pmod{266}$	$c_{266,1}$
232	$k \equiv 233 \pmod{266}$	$c_{266,2}$
233	$k \equiv 100 \pmod{798}$	425991366045253
234	$k \equiv 366 \pmod{798}$	$c_{798,1}$
235	$k \equiv 632 \pmod{798}$	$c_{798,2}$
236	$k \equiv 93 \pmod{399}$	6316969
237	$k \equiv 226 \pmod{399}$	34282879
238	$k \equiv 359 \pmod{399}$	1473464802559
239	$k \equiv 352 \pmod{399}$	$c_{399,1}$
240	$k \equiv 219 \pmod{399}$	$c_{399,2}$
241	$k \equiv 485 \pmod{1197}$	$c_{1197,1}$
242	$k \equiv 884 \pmod{1197}$	$c_{1197,2}$
243	$k \equiv 86 \pmod{2394}$	$c_{2394,1}$
244	$k \equiv 1283 \pmod{2394}$	$c_{2394,2}$
245	$k \equiv 79 \pmod{532}$	14691812549
246	$k \equiv 212 \pmod{532}$	2129
247	$k \equiv 345 \pmod{532}$	172809571660534248409
248	$k \equiv 478 \pmod{532}$	$c_{532,1}$
249	$k \equiv 471 \pmod{532}$	$c_{532,2}$
250	$k \equiv 338 \pmod{1064}$	$c_{1064,1}$
251	$k \equiv 870 \pmod{1064}$	$c_{1064,2}$
252	$k \equiv 205 \pmod{1596}$	3301756921
253	$k \equiv 737 \pmod{1596}$	$c_{1596,1}$
254	$k \equiv 1269 \pmod{1596}$	$c_{1596,2}$
255	$k \equiv 1668 \pmod{2128}$	57457
256	$k \equiv 1136 \pmod{2128}$	1676106728215569889
257	$k \equiv 604 \pmod{2128}$	$c_{2128,1}$
258	$k \equiv 72 \pmod{2128}$	$c_{2128,2}$
259	$k \equiv 863 \pmod{931}$	16759
260	$k \equiv 730 \pmod{931}$	83791
261	$k \equiv 597 \pmod{931}$	13071241
262	$k \equiv 464 \pmod{931}$	7645033117
263	$k \equiv 331 \pmod{931}$	65385402019201951
264	$k \equiv 198 \pmod{931}$	$c_{931,1}$
265	$k \equiv 65 \pmod{931}$	$c_{931,2}$
266	$k \equiv 541 \pmod{665}$	$c_{665,1}$

Table A.2 cont. Covering used in Lemma 4.3 (i) for $d = 1$

row	congruence	prime p_i
267	$k \equiv 408 \pmod{665}$	$c_{665,2}$
268	$k \equiv 275 \pmod{1330}$	1258323641
269	$k \equiv 940 \pmod{1330}$	$c_{1330,1}$
270	$k \equiv 1472 \pmod{1995}$	496774951
271	$k \equiv 807 \pmod{1995}$	127681
272	$k \equiv 142 \pmod{1995}$	$c_{1995,1}$
273	$k \equiv 1 \pmod{19}$	1111111111111111111
274	$k \equiv 32 \pmod{38}$	909090909090909091
275	$k \equiv 51 \pmod{76}$	1369778187490592461
276	$k \equiv 13 \pmod{76}$	722817036322379041
277	$k \equiv 44 \pmod{57}$	21319
278	$k \equiv 6 \pmod{57}$	10749631
279	$k \equiv 25 \pmod{57}$	3931123022305129377976519
280	$k \equiv 18 \pmod{95}$	191
281	$k \equiv 37 \pmod{95}$	59281
282	$k \equiv 56 \pmod{95}$	63841
283	$k \equiv 75 \pmod{95}$	1289981231950849543985493631
284	$k \equiv 94 \pmod{95}$	p_{284}
285	$k \equiv 87 \pmod{114}$	1458973
286	$k \equiv 49 \pmod{114}$	p_{286}
287	$k \equiv 11 \pmod{228}$	33950736420661075851541
288	$k \equiv 125 \pmod{228}$	739653893349540289
289	$k \equiv 30 \pmod{152}$	P_{152}
290	$k \equiv 68 \pmod{152}$	5240808656722481737
291	$k \equiv 106 \pmod{152}$	1403417
292	$k \equiv 144 \pmod{152}$	457
293	$k \equiv 156 \pmod{190}$	1812604116731
294	$k \equiv 118 \pmod{190}$	121450506296081
295	$k \equiv 80 \pmod{190}$	P_{190}
296	$k \equiv 4 \pmod{380}$	417332341
297	$k \equiv 190 \pmod{380}$	74861
298	$k \equiv 42 \pmod{380}$	761
299	$k \equiv 232 \pmod{380}$	1901
300	$k \equiv 213 \pmod{228}$	229
301	$k \equiv 175 \pmod{228}$	2281
302	$k \equiv 137 \pmod{228}$	4789
303	$k \equiv 99 \pmod{228}$	304077901
304	$k \equiv 61 \pmod{228}$	52875286008709
305	$k \equiv 23 \pmod{456}$	$c_{456,1}$
306	$k \equiv 251 \pmod{456}$	$c_{456,2}$
307	$k \equiv 54 \pmod{171}$	229162071140015324614111
308	$k \equiv 111 \pmod{171}$	$c_{171,1}$
309	$k \equiv 168 \pmod{171}$	$c_{171,2}$
310	$k \equiv 263 \pmod{285}$	p_{310}

Table A.2 cont. Covering used in Lemma 4.3 (i) for $d = 1$

row	congruence	prime p_i
311	$k \equiv 206 \pmod{285}$	$c_{285,1}$
312	$k \equiv 149 \pmod{285}$	$c_{285,2}$
313	$k \equiv 35 \pmod{570}$	571
314	$k \equiv 320 \pmod{570}$	11382273828031938028891
315	$k \equiv 92 \pmod{570}$	43135513763991692454656851
316	$k \equiv 377 \pmod{570}$	P_{570}
317	$k \equiv 301 \pmod{342}$	2053
318	$k \equiv 244 \pmod{342}$	4410785971
319	$k \equiv 187 \pmod{342}$	2911579215499
320	$k \equiv 130 \pmod{342}$	p_{320}
321	$k \equiv 73 \pmod{342}$	P_{342}
322	$k \equiv 16 \pmod{684}$	126541
323	$k \equiv 358 \pmod{684}$	2050658332436114459080741
324	$k \equiv 2529 \pmod{2660}$	85121
325	$k \equiv 1997 \pmod{2660}$	73700621
326	$k \equiv 1465 \pmod{2660}$	$c_{2660,1}$
327	$k \equiv 933 \pmod{2660}$	$c_{2660,2}$
328	$k \equiv 401 \pmod{5320}$	202797315945355601
329	$k \equiv 3061 \pmod{5320}$	$c_{5320,1}$
330	$k \equiv 1332 \pmod{1596}$	539449
331	$k \equiv 800 \pmod{3192}$	$c_{3192,1}$
332	$k \equiv 2396 \pmod{3192}$	$c_{3192,2}$
333	$k \equiv 268 \pmod{4788}$	71821
334	$k \equiv 1864 \pmod{4788}$	682320298511881
335	$k \equiv 3460 \pmod{4788}$	$c_{4788,1}$
336	$k \equiv 667 \pmod{1064}$	260827749862793
337	$k \equiv 135 \pmod{456}$	27817
338	$k \equiv 287 \pmod{456}$	376102873
339	$k \equiv 439 \pmod{456}$	36120612721
340	$k \equiv 534 \pmod{608}$	3031489
341	$k \equiv 458 \pmod{608}$	$c_{608,1}$
342	$k \equiv 382 \pmod{608}$	$c_{608,2}$
343	$k \equiv 306 \pmod{1216}$	665153
344	$k \equiv 914 \pmod{1216}$	1601473
345	$k \equiv 230 \pmod{1216}$	65384321
346	$k \equiv 838 \pmod{1216}$	911712031611457
347	$k \equiv 154 \pmod{1216}$	1217
348	$k \equiv 762 \pmod{1216}$	$c_{1216,1}$
349	$k \equiv 686 \pmod{1216}$	$c_{1216,2}$
350	$k \equiv 1294 \pmod{2432}$	7297
351	$k \equiv 78 \pmod{2432}$	$c_{2432,1}$
352	$k \equiv 1218 \pmod{1824}$	1252017313
353	$k \equiv 610 \pmod{1824}$	524026007884633812193
354	$k \equiv 2 \pmod{1824}$	$c_{1824,1}$

Notable factorizations of $\Phi_n(10)$ for large n used in the covering for $d = 3$ can be found in Table A.3. Table A.4 is the covering used for $d = 3$. Here we note some primes that did not fit into the table while maintaining proper formatting guidelines.

$$p_6 = 10000999999998999989999000000010001,$$

$$p_{13} = 153431681888891378183369,$$

$$p_{14} = 5152175252652132674478699068158 73,$$

$$p_{40} = 5376349118996722135857554610727903470 9697,$$

$$p_{64} = 7323941687838105624847742701467683590147273.$$

Table A.3 Partial/Full factorizations of $\Phi_n(10)$ for $d = 3$

n	Factorization of $\Phi_n(10)$
256	$257 \cdot 15361 \cdot 453377 \cdot P_{256}$
288	$13249 \cdot 1067329 \cdot P_{288}$
384	$3457 \cdot 12289 \cdot 418725889 \cdot C_{384}$
480	$177601 \cdot C_{480}$
624	$148068337 \cdot 10662171313 \cdot 325807801400017 \cdot C_{624}$
936	$937 \cdot 368255828264024377 \cdot 5030608786526082241$ $\cdot 22088303808535743236641 \cdot P_{936}$
1560	$66365521 \cdot 4487398067478973921 \cdot C_{1560}$

Table A.4 Covering used in Lemma 4.3 (1) for $d = 3$

row	congruence	prime p_i
a	$k \equiv 2 \pmod{4}$	101
b	$k \equiv 5 \pmod{8}$	137
c	$k \equiv 0 \pmod{3}$	37
d	$k \equiv 5 \pmod{6}$	13
e	$k \equiv 26 \pmod{30}$	211
f	$k \equiv 40 \pmod{60}$	61
1	$k \equiv 1 \pmod{96}$	97
2	$k \equiv 25 \pmod{96}$	206209
3	$k \equiv 49 \pmod{96}$	66554101249
4	$k \equiv 73 \pmod{96}$	75118313082913
5	$k \equiv 7 \pmod{48}$	9999999900000001
6	$k \equiv 31 \pmod{120}$	p_6
7	$k \equiv 79 \pmod{240}$	2812599852484377900051014401
8	$k \equiv 127 \pmod{240}$	31388506438433752927908678241
9	$k \equiv 175 \pmod{240}$	1132716961
10	$k \equiv 223 \pmod{480}$	177601
11	$k \equiv 463 \pmod{480}$	$c_{480,1}$
12	$k \equiv 116 \pmod{128}$	1265011073

row	congruence	prime p_i
13	$k \equiv 100 \pmod{128}$	p_{13}
14	$k \equiv 84 \pmod{128}$	p_{14}
15	$k \equiv 196 \pmod{256}$	257
16	$k \equiv 68 \pmod{256}$	15361
17	$k \equiv 180 \pmod{256}$	453377
18	$k \equiv 52 \pmod{256}$	P_{256}
19	$k \equiv 420 \pmod{512}$	10753
20	$k \equiv 292 \pmod{512}$	8253953
21	$k \equiv 164 \pmod{512}$	9524994049
22	$k \equiv 36 \pmod{512}$	73171503617
23	$k \equiv 276 \pmod{384}$	3457
24	$k \equiv 148 \pmod{384}$	12289
25	$k \equiv 20 \pmod{384}$	418725889
26	$k \equiv 260 \pmod{384}$	$c_{384,1}$
27	$k \equiv 132 \pmod{384}$	$c_{384,2}$
28	$k \equiv 388 \pmod{768}$	434689
29	$k \equiv 4 \pmod{768}$	859393
30	$k \equiv 16 \pmod{32}$	641

Table A.4 cont. Covering used in Lemma 4.3 (i) for $d = 3$

row	congruence	prime p_i
31	$k \equiv 28 \pmod{32}$	1409
32	$k \equiv 12 \pmod{32}$	69857
33	$k \equiv 8 \pmod{64}$	834427406578561
34	$k \equiv 40 \pmod{64}$	976193
35	$k \equiv 24 \pmod{64}$	6187457
36	$k \equiv 56 \pmod{64}$	19841
37	$k \equiv 0 \pmod{192}$	193
38	$k \equiv 96 \pmod{192}$	769
39	$k \equiv 32 \pmod{192}$	1253224535459902849
40	$k \equiv 128 \pmod{192}$	p_{40}
41	$k \equiv 64 \pmod{288}$	13249
42	$k \equiv 160 \pmod{288}$	1067329
43	$k \equiv 256 \pmod{288}$	P_{288}
44	$k \equiv 3115 \pmod{3120}$	38609381516161
45	$k \equiv 1555 \pmod{3120}$	686402214033121
46	$k \equiv 307 \pmod{1560}$	66365521
47	$k \equiv 619 \pmod{1560}$	4487398067478973921
48	$k \equiv 931 \pmod{1560}$	$c_{1560,1}$
49	$k \equiv 1243 \pmod{1560}$	$c_{1560,2}$

row	congruence	prime p_i
50	$k \equiv 907 \pmod{1872}$	1873
51	$k \equiv 1843 \pmod{1872}$	7489
52	$k \equiv 595 \pmod{936}$	937
53	$k \equiv 283 \pmod{936}$	368255828264024377
54	$k \equiv 883 \pmod{936}$	5030608786526082241
55	$k \equiv 571 \pmod{936}$	22088303808535743236641
56	$k \equiv 259 \pmod{936}$	P_{936}
57	$k \equiv 235 \pmod{624}$	148068337
58	$k \equiv 523 \pmod{624}$	325807801400017
59	$k \equiv 211 \pmod{624}$	10662171313
60	$k \equiv 547 \pmod{624}$	$c_{624,1}$
61	$k \equiv 19 \pmod{156}$	3121
62	$k \equiv 43 \pmod{156}$	6060517860310398033985611921721
63	$k \equiv 67 \pmod{156}$	53397071018461
64	$k \equiv 91 \pmod{312}$	p_{64}
65	$k \equiv 163 \pmod{312}$	291593563046646669491593
66	$k \equiv 115 \pmod{312}$	1358074433371719716641
67	$k \equiv 187 \pmod{312}$	1101673
68	$k \equiv 139 \pmod{312}$	313

We produce the coverings used in Lemma 4.3 for inserting $d = 4$ into the leading sevens. Table A.5 provides notable factorizations of $\Phi_n(10)$ for large n used in the congruences found in Table A.6. The following primes are used.

$$p_{17} = 754309323029578981494450523147961,$$

$$p_{18} = 4832227158939716614468345433242909,$$

$$p_{84} = 36096800156828895568286578224818258719817914995401933354161,$$

$$p_{89} = 632527440202150745090622412245443923049201,$$

$$p_{96} = 5538396997364024056286510640780600481,$$

$$p_{98} = 8396862596258693901610602298557167100076327481,$$

$$p_{119} = 8610583349234340055547908764091017276717091.$$

Table A.5 Partial/Full factorizations of $\Phi_n(10)$ for $d = 4$

n	Factorization of $\Phi_n(10)$
143	$2823679 \cdot 180523201 \cdot C_{143}$
286	$51767 \cdot 22144088539 \cdot 264752347289 \cdot 104730101107272149081 \cdot P_{286}$
360	$265183201 \cdot C_{360}$
390	$15601 \cdot 925081 \cdot P_{390}$
429	$5018491662756 \cdot C_{429}$
468	$11761957262582764997761 \cdot C_{468}$
546	$102103 \cdot P_{546}$
572	$17576772101461 \cdot 754309323029578981494450523147961$ $\cdot\, 4832227158939716614468345433242909 \cdot C_{572}$
715	C_{715}
780	$19501 \cdot 9160020509281917601 \cdot 201721695849323521 \cdot C_{780}$
858	$6007 \cdot 210991557708361 \cdot 80216851991399964961367677 \cdot C_{858}$
1170	$1171 \cdot 75417940111411 \cdot C_{1170}$
1248	$57653857 \cdot 4694971009 \cdot 4155351649525441 \cdot C_{1248}$
1430	$708433441 \cdot 146582316035503921 \cdot C_{1430}$
2730	$2731 \cdot 5528251 \cdot 6097697971 \cdot C_{2730}$

Table A.6 Covering used in Lemma 4.3 (i) for $d = 4$

row	congruence	prime p_i
a	$k \equiv 2 \pmod{3}$	37
b	$k \equiv 0 \pmod{6}$	13
c	$k \equiv 9 \pmod{18}$	19
1	$k \equiv 11 \pmod{13}$	53
2	$k \equiv 8 \pmod{13}$	79
3	$k \equiv 5 \pmod{13}$	265371653
4	$k \equiv 28 \pmod{39}$	9009009009009909909909991
5	$k \equiv 12 \pmod{26}$	859
6	$k \equiv 25 \pmod{26}$	1058313049
7	$k \equiv 139 \pmod{143}$	2823679
8	$k \equiv 126 \pmod{143}$	180523201
9	$k \equiv 113 \pmod{143}$	$c_{143,1}$
10	$k \equiv 100 \pmod{143}$	$c_{143,2}$
11	$k \equiv 87 \pmod{286}$	51767
12	$k \equiv 230 \pmod{286}$	22144088539
13	$k \equiv 74 \pmod{286}$	264752347289
14	$k \equiv 217 \pmod{286}$	104730101107272149081
15	$k \equiv 204 \pmod{286}$	p_{15}
16	$k \equiv 347 \pmod{572}$	17576772101461
17	$k \equiv 477 \pmod{572}$	p_{17}
18	$k \equiv 334 \pmod{572}$	p_{18}

row	congruence	prime p_i
19	$k \equiv 191 \pmod{572}$	$c_{572,1}$
20	$k \equiv 48 \pmod{572}$	$c_{572,2}$
21	$k \equiv 633 \pmod{1144}$	119424449
22	$k \equiv 61 \pmod{1144}$	503186110849316156410721
23	$k \equiv 321 \pmod{429}$	5018491662756
24	$k \equiv 178 \pmod{429}$	$c_{429,1}$
25	$k \equiv 35 \pmod{429}$	$c_{429,2}$
26	$k \equiv 737 \pmod{858}$	210991557708361
27	$k \equiv 594 \pmod{858}$	80216851991399964961367677
28	$k \equiv 451 \pmod{858}$	6007
29	$k \equiv 308 \pmod{858}$	$c_{858,1}$
30	$k \equiv 165 \pmod{858}$	$c_{858,2}$
31	$k \equiv 880 \pmod{1716}$	1063921
32	$k \equiv 22 \pmod{1716}$	152980598629
33	$k \equiv 581 \pmod{715}$	$c_{715,1}$
34	$k \equiv 438 \pmod{715}$	$c_{715,2}$
35	$k \equiv 1010 \pmod{1430}$	708433441
36	$k \equiv 295 \pmod{1430}$	146582316035503921
37	$k \equiv 867 \pmod{1430}$	$c_{1430,1}$
38	$k \equiv 152 \pmod{1430}$	$c_{1430,2}$
39	$k \equiv 1439 \pmod{2145}$	51481

Table A.6 cont. Covering used in Lemma 4.3 (i) for $d = 4$

row	congruence	prime p_i	row	congruence	prime p_i
40	$k \equiv 724 \pmod{2145}$	6224791	61	$k \equiv 1027 \pmod{1248}$	4694971009
41	$k \equiv 9 \pmod{2145}$	2832665551	62	$k \equiv 871 \pmod{1248}$	4155351649525441
42	$k \equiv 331 \pmod{390}$	15601	63	$k \equiv 715 \pmod{1248}$	$c_{1248,1}$
43	$k \equiv 253 \pmod{390}$	925081	64	$k \equiv 559 \pmod{1248}$	$c_{1248,2}$
44	$k \equiv 175 \pmod{390}$	P_{390}	65	$k \equiv 1651 \pmod{2496}$	192193
45	$k \equiv 97 \pmod{1170}$	75417940111411	66	$k \equiv 403 \pmod{2496}$	965953
46	$k \equiv 487 \pmod{1170}$	1171	67	$k \equiv 1495 \pmod{2496}$	22389121
47	$k \equiv 877 \pmod{1170}$	$c_{1170,1}$	68	$k \equiv 247 \pmod{2496}$	199369365313
48	$k \equiv 175 \pmod{546}$	102103	69	$k \equiv 1339 \pmod{2496}$	324200647681
49	$k \equiv 331 \pmod{546}$	P_{546}	70	$k \equiv 91 \pmod{2496}$	4969602289694017
50	$k \equiv 1579 \pmod{2730}$	5528251	71	$k \equiv 169 \pmod{780}$	19501
51	$k \equiv 1189 \pmod{2730}$	6097697971	72	$k \equiv 325 \pmod{780}$	9160020509281917601
52	$k \equiv 799 \pmod{2730}$	2731	73	$k \equiv 481 \pmod{780}$	201721695849323521
53	$k \equiv 409 \pmod{2730}$	$c_{2730,1}$	74	$k \equiv 637 \pmod{780}$	$c_{780,1}$
54	$k \equiv 474 \pmod{910}$	2475034612051	75	$k \equiv 13 \pmod{2340}$	25741
55	$k \equiv 19 \pmod{910}$	1081846114760321	76	$k \equiv 793 \pmod{2340}$	257764562641
56	$k \equiv 58 \pmod{78}$	157	77	$k \equiv 1573 \pmod{2340}$	6253963297921
57	$k \equiv 16 \pmod{78}$	388847808493	78	$k \equiv 10 \pmod{117}$	240396841140769
58	$k \equiv 55 \pmod{78}$	216451	79	$k \equiv 49 \pmod{117}$	537947698126879
59	$k \equiv 52 \pmod{78}$	6397	80	$k \equiv 88 \pmod{117}$	3352825314499987
60	$k \equiv 1183 \pmod{1248}$	57653857	81	$k \equiv 85 \pmod{117}$	2304017384484085131816292573

Table A.6 cont. Covering used in Lemma 4.3 (i) for $d = 4$

row	congruence	prime p_i
82	$k \equiv 46 \pmod{234}$	461917
83	$k \equiv 163 \pmod{234}$	60034573
84	$k \equiv 124 \pmod{234}$	p_{84}
85	$k \equiv 7 \pmod{468}$	117619572625827649977761
86	$k \equiv 241 \pmod{468}$	$c_{468,1}$
87	$k \equiv 30 \pmod{52}$	521
88	$k \equiv 43 \pmod{52}$	1900381976777332243781
89	$k \equiv 17 \pmod{104}$	p_{89}
90	$k \equiv 69 \pmod{104}$	1580801
91	$k \equiv 4 \pmod{208}$	1249
92	$k \equiv 56 \pmod{208}$	49297
93	$k \equiv 108 \pmod{208}$	300977
94	$k \equiv 160 \pmod{208}$	648961
95	$k \equiv 40 \pmod{65}$	162503518711
96	$k \equiv 53 \pmod{65}$	p_{96}
97	$k \equiv 27 \pmod{130}$	131
98	$k \equiv 92 \pmod{130}$	p_{98}
99	$k \equiv 14 \pmod{260}$	2311921
100	$k \equiv 79 \pmod{260}$	1031498834064949381
101	$k \equiv 144 \pmod{260}$	12763852652999774041
102	$k \equiv 209 \pmod{260}$	121119730504567977254081

row	congruence	prime p_i
103	$k \equiv 1 \pmod{195}$	1951
104	$k \equiv 66 \pmod{195}$	35081393881
105	$k \equiv 131 \pmod{195}$	360924572424391
106	$k \equiv 21 \pmod{36}$	999999000001
107	$k \equiv 3 \pmod{108}$	109
108	$k \equiv 39 \pmod{108}$	5977957715633453386654838281
109	$k \equiv 75 \pmod{108}$	153469
110	$k \equiv 87 \pmod{90}$	8985695684401
111	$k \equiv 69 \pmod{180}$	181
112	$k \equiv 159 \pmod{180}$	4999437541453012143121
113	$k \equiv 141 \pmod{180}$	1105097795002994798105101
114	$k \equiv 51 \pmod{360}$	265183201
115	$k \equiv 231 \pmod{360}$	$c_{360,1}$
116	$k \equiv 33 \pmod{270}$	6481
117	$k \equiv 123 \pmod{270}$	577603663291
118	$k \equiv 213 \pmod{270}$	31023833790241
119	$k \equiv 195 \pmod{270}$	p_{119}
120	$k \equiv 105 \pmod{540}$	329941
121	$k \equiv 375 \pmod{540}$	68189581
122	$k \equiv 15 \pmod{540}$	49229101
123	$k \equiv 285 \pmod{540}$	13029637224192121671301

We use the covering found in Table A.8 for inserting the digit $d = 6$ into the leading sevens. Table A.7 provides notable factorizations of $\Phi_n(10)$ for large n used in these coverings. As with the previous tables, we omit the factorizations corresponding to the composite numbers and large primes in Table A.7 that can be determined by dividing $\Phi_n(10)$ by the nearby primes included in Table A.8. To save space we denote some of the larger primes below.

$$p_6 = 15763985553739191709164170940063151,$$

$$p_{59} = 48341841859722067723851735391523196183 1,$$

$$p_{104} = 1000009999999998999989999900000000100001,$$

$$p_{110} = 38654658795718156456729958859629701,$$

$$p_{192} = 846160494149365798478410729, \quad p_{193} = 1125442817557827326736713670 61,$$

$$p_{196} = 341944731161595469798186890 31,$$

$$p_{262} = 37932032823724801, \quad p_{269} = 10527797306161,$$

$$p_{276} = 6315203673292075607, \quad p_{277} = 1257973994926769177 21,$$

$$p_{280} = 111262583033346559, \quad p_{281} = 155623169021.$$

Table A.7 Partial/Full factorizations of $\Phi_n(10)$ for $d = 6$

n	Factorization of $\Phi_n(10)$
250	$21001 \cdot 162251 \cdot 10893295001 \cdot P_{250}$
272	$17 \cdot 13355595217 \cdot P_{272}$
765	C_{765}
816	$54673 \cdot 5637065089 \cdot C_{816}$
1530	$1531 \cdot 7947357582331 \cdot 40984651817371 \cdot 12804651623971 \cdot C_{1530}$
1575	C_{1575}
1870	C_{1870}

Table A.8 Covering used in Lemma 4.3 (i) for $d = 6$

row	congruence	prime p_i
a	$k \equiv 0 \pmod{5}$	41
b	$k \equiv 12 \pmod{15}$	31
1	$k \equiv 21 \pmod{25}$	21401
2	$k \equiv 16 \pmod{25}$	25601
3	$k \equiv 11 \pmod{25}$	182521213001
4	$k \equiv 6 \pmod{75}$	151
5	$k \equiv 31 \pmod{75}$	4201
6	$k \equiv 56 \pmod{75}$	p_6
7	$k \equiv 76 \pmod{100}$	60101
8	$k \equiv 51 \pmod{100}$	7019801
9	$k \equiv 26 \pmod{100}$	14103673319201
10	$k \equiv 1 \pmod{100}$	1680588011350901
11	$k \equiv 23 \pmod{50}$	251
12	$k \equiv 48 \pmod{50}$	5051
13	$k \equiv 13 \pmod{50}$	78875943472201
14	$k \equiv 188 \pmod{200}$	401
15	$k \equiv 138 \pmod{200}$	1201
16	$k \equiv 88 \pmod{200}$	1601
17	$k \equiv 38 \pmod{200}$	P_{200}
18	$k \equiv 118 \pmod{125}$	751
19	$k \equiv 93 \pmod{125}$	1797655751
20	$k \equiv 68 \pmod{125}$	176144543406001
22	$k \equiv 43 \pmod{125}$	P_{125}
22	$k \equiv 143 \pmod{250}$	21001
23	$k \equiv 18 \pmod{250}$	162251
24	$k \equiv 208 \pmod{225}$	2002877551
25	$k \equiv 133 \pmod{225}$	2636899200194401
26	$k \equiv 58 \pmod{225}$	97671987485517534751
27	$k \equiv 183 \pmod{225}$	$c_{225,1}$
28	$k \equiv 108 \pmod{225}$	$c_{225,2}$
29	$k \equiv 483 \pmod{675}$	40588518151
30	$k \equiv 258 \pmod{675}$	$c_{675,1}$
31	$k \equiv 33 \pmod{675}$	$c_{675,2}$
32	$k \equiv 308 \pmod{375}$	21751
33	$k \equiv 233 \pmod{375}$	47001751
34	$k \equiv 158 \pmod{375}$	4471031976001
35	$k \equiv 83 \pmod{375}$	$c_{375,1}$
36	$k \equiv 8 \pmod{375}$	$c_{375,2}$
37	$k \equiv 153 \pmod{175}$	1852584391849069588675 1
38	$k \equiv 128 \pmod{175}$	991474271662986957800680951
39	$k \equiv 103 \pmod{175}$	P_{175}
40	$k \equiv 228 \pmod{350}$	3612546001
41	$k \equiv 253 \pmod{350}$	299547376801
42	$k \equiv 78 \pmod{350}$	P_{350}
43	$k \equiv 403 \pmod{700}$	422100001
44	$k \equiv 53 \pmod{700}$	701
45	$k \equiv 378 \pmod{525}$	79801
46	$k \equiv 203 \pmod{525}$	35120401
47	$k \equiv 28 \pmod{525}$	435288001
48	$k \equiv 353 \pmod{525}$	$c_{525,1}$

Table A.8 cont. Covering used in Lemma 4.3 (i) for $d = 6$

row	congruence	prime p_i
49	$k \equiv 178 \pmod{525}$	$c_{525,2}$
50	$k \equiv 1053 \pmod{1575}$	$c_{1575,1}$
51	$k \equiv 528 \pmod{1575}$	$c_{1575,2}$
52	$k \equiv 3153 \pmod{4725}$	286826401
53	$k \equiv 1578 \pmod{4725}$	918984151
54	$k \equiv 3 \pmod{4725}$	7650670784401
55	$k \equiv 127 \pmod{135}$	1577071
56	$k \equiv 112 \pmod{135}$	16357951
57	$k \equiv 97 \pmod{135}$	310362841
58	$k \equiv 82 \pmod{135}$	258360989311
59	$k \equiv 67 \pmod{135}$	p_{59}
60	$k \equiv 52 \pmod{405}$	21871
61	$k \equiv 187 \pmod{405}$	61561
62	$k \equiv 322 \pmod{405}$	5222071
63	$k \equiv 37 \pmod{405}$	40435201
64	$k \equiv 172 \pmod{405}$	$c_{405,1}$
65	$k \equiv 307 \pmod{405}$	$c_{405,2}$
66	$k \equiv 697 \pmod{810}$	811
67	$k \equiv 562 \pmod{810}$	213872067091
68	$k \equiv 427 \pmod{810}$	$c_{810,1}$
69	$k \equiv 292 \pmod{810}$	$c_{810,2}$
70	$k \equiv 967 \pmod{1620}$	1621
71	$k \equiv 157 \pmod{1620}$	19441
72	$k \equiv 832 \pmod{1620}$	27541

row	congruence	prime p_i
73	$k \equiv 22 \pmod{1620}$	119881
74	$k \equiv 1492 \pmod{1620}$	68041
75	$k \equiv 1087 \pmod{1620}$	29639179139212862101
76	$k \equiv 682 \pmod{1620}$	8101
77	$k \equiv 277 \pmod{1620}$	$c_{1620,1}$
78	$k \equiv 952 \pmod{1215}$	369361
79	$k \equiv 547 \pmod{1215}$	$c_{1215,1}$
80	$k \equiv 142 \pmod{1215}$	$c_{1215,2}$
81	$k \equiv 1627 \pmod{2025}$	180860878351
82	$k \equiv 1222 \pmod{2025}$	$c_{2025,1}$
83	$k \equiv 817 \pmod{2025}$	$c_{2025,2}$
84	$k \equiv 2437 \pmod{4050}$	4051
85	$k \equiv 412 \pmod{4050}$	1931851
86	$k \equiv 2032 \pmod{4050}$	71764372708675651
87	$k \equiv 7 \pmod{4050}$	$c_{4050,1}$
88	$k \equiv 237 \pmod{250}$	10893295001
89	$k \equiv 162 \pmod{250}$	P_{250}
90	$k \equiv 87 \pmod{500}$	4001
91	$k \equiv 337 \pmod{500}$	76001
92	$k \equiv 12 \pmod{500}$	1610501
93	$k \equiv 262 \pmod{500}$	$c_{500,1}$
94	$k \equiv 437 \pmod{500}$	$c_{500,2}$
95	$k \equiv 362 \pmod{750}$	25005172023963824973001
96	$k \equiv 287 \pmod{750}$	$c_{750,1}$

Table A.8 cont. Covering used in Lemma 4.3 (i) for $d = 6$

row	congruence	prime p_i
97	$k \equiv 212 \pmod{750}$	$c_{750,2}$
98	$k \equiv 687 \pmod{1000}$	24001
99	$k \equiv 187 \pmod{1000}$	1378001
100	$k \equiv 887 \pmod{1500}$	3001
101	$k \equiv 137 \pmod{1500}$	473217092140501
102	$k \equiv 812 \pmod{1500}$	89285503565971501
103	$k \equiv 62 \pmod{1500}$	$c_{1500,1}$
104	$k \equiv 122 \pmod{150}$	p_{104}
105	$k \equiv 257 \pmod{300}$	601
106	$k \equiv 107 \pmod{300}$	261301
107	$k \equiv 242 \pmod{300}$	3903901
108	$k \equiv 92 \pmod{300}$	168290119201
109	$k \equiv 227 \pmod{300}$	25074091038628125301
110	$k \equiv 77 \pmod{300}$	p_{110}
111	$k \equiv 797 \pmod{900}$	1801
112	$k \equiv 647 \pmod{900}$	139501
113	$k \equiv 497 \pmod{900}$	33301
114	$k \equiv 347 \pmod{900}$	560701
115	$k \equiv 197 \pmod{900}$	5030101
116	$k \equiv 47 \pmod{900}$	$c_{900,1}$
117	$k \equiv 332 \pmod{450}$	270001
118	$k \equiv 182 \pmod{450}$	P_{450}
119	$k \equiv 932 \pmod{1350}$	371251
120	$k \equiv 482 \pmod{1350}$	$c_{1350,1}$

row	congruence	prime p_i
121	$k \equiv 32 \pmod{1350}$	$c_{1350,2}$
122	$k \equiv 917 \pmod{1050}$	1051
123	$k \equiv 767 \pmod{1050}$	22771401850597648936801
124	$k \equiv 617 \pmod{1050}$	3146327779391852182051
125	$k \equiv 17 \pmod{1050}$	17911809459808314248342142601
126	$k \equiv 167 \pmod{1050}$	$c_{1050,1}$
127	$k \equiv 1517 \pmod{2100}$	1194901
128	$k \equiv 467 \pmod{2100}$	62134801
129	$k \equiv 1367 \pmod{2100}$	219345525001
130	$k \equiv 317 \pmod{2100}$	5455568729101
131	$k \equiv 452 \pmod{600}$	32401
132	$k \equiv 302 \pmod{600}$	$c_{600,1}$
133	$k \equiv 152 \pmod{600}$	$c_{600,2}$
134	$k \equiv 602 \pmod{1200}$	32233075296001
135	$k \equiv 2 \pmod{1200}$	$c_{1200,1}$
136	$k \equiv 84 \pmod{85}$	262533041
137	$k \equiv 79 \pmod{85}$	8119594779271
138	$k \equiv 74 \pmod{85}$	P_{85}
139	$k \equiv 69 \pmod{170}$	87211
140	$k \equiv 154 \pmod{170}$	787223761
141	$k \equiv 149 \pmod{170}$	P_{170}
142	$k \equiv 574 \pmod{680}$	1361
143	$k \equiv 404 \pmod{680}$	787131281
144	$k \equiv 234 \pmod{680}$	$c_{680,1}$

Table A.8 cont. Covering used in Lemma 4.3 (i) for $d = 6$

row	congruence	prime p_i
145	$k \equiv 64 \pmod{680}$	$c_{680,2}$
146	$k \equiv 25 \pmod{68}$	28559389
147	$k \equiv 127 \pmod{136}$	152533657
148	$k \equiv 59 \pmod{136}$	P_{136}
149	$k \equiv 178 \pmod{204}$	409
150	$k \equiv 144 \pmod{204}$	3061
151	$k \equiv 110 \pmod{204}$	5969449
152	$k \equiv 76 \pmod{204}$	134703241
153	$k \equiv 42 \pmod{204}$	225974065503889
154	$k \equiv 8 \pmod{204}$	44398000479007997569751764249
155	$k \equiv 37 \pmod{51}$	613
156	$k \equiv 20 \pmod{51}$	210631
157	$k \equiv 3 \pmod{51}$	52986961
158	$k \equiv 49 \pmod{51}$	13168164561429877
159	$k \equiv 83 \pmod{102}$	291078844423
160	$k \equiv 32 \pmod{102}$	377526955309799110357
161	$k \equiv 270 \pmod{408}$	2857
162	$k \equiv 219 \pmod{408}$	13266937
163	$k \equiv 168 \pmod{408}$	1119527384827710553
164	$k \equiv 117 \pmod{408}$	53043011765949537815976769
165	$k \equiv 66 \pmod{408}$	P_{408}
166	$k \equiv 423 \pmod{816}$	54673
167	$k \equiv 15 \pmod{816}$	5637065089
168	$k \equiv 372 \pmod{1224}$	282803977
169	$k \equiv 780 \pmod{1224}$	3777811937749537921
170	$k \equiv 1188 \pmod{1224}$	$c_{1224,1}$
171	$k \equiv 1137 \pmod{1224}$	$c_{1224,2}$
172	$k \equiv 3177 \pmod{3672}$	3673
173	$k \equiv 1953 \pmod{3672}$	1813969
174	$k \equiv 729 \pmod{3672}$	161569
175	$k \equiv 2769 \pmod{3672}$	87140233
176	$k \equiv 1545 \pmod{3672}$	322941468457
177	$k \equiv 321 \pmod{3672}$	629674149824204775337
178	$k \equiv 146 \pmod{153}$	307
179	$k \equiv 129 \pmod{153}$	18973
180	$k \equiv 112 \pmod{153}$	11910133
181	$k \equiv 95 \pmod{153}$	25332185271529
182	$k \equiv 78 \pmod{153}$	41331541464123787
183	$k \equiv 61 \pmod{153}$	P_{153}
184	$k \equiv 350 \pmod{459}$	12853
185	$k \equiv 197 \pmod{459}$	919
186	$k \equiv 44 \pmod{459}$	2735641
187	$k \equiv 486 \pmod{612}$	54469
188	$k \equiv 333 \pmod{612}$	158963941
189	$k \equiv 180 \pmod{612}$	2709009355501
190	$k \equiv 27 \pmod{612}$	3626707988341
191	$k \equiv 469 \pmod{612}$	157538980319816607121
192	$k \equiv 316 \pmod{612}$	p_{192}

Table A.8 cont. Covering used in Lemma 4.3 (i) for $d = 6$

row	congruence	prime p_i
193	$k \equiv 163 \pmod{612}$	p_{193}
194	$k \equiv 10 \pmod{612}$	P_{612}
195	$k \equiv 209 \pmod{255}$	77967508765681
196	$k \equiv 124 \pmod{255}$	p_{196}
197	$k \equiv 39 \pmod{255}$	$c_{255,1}$
198	$k \equiv 204 \pmod{255}$	$c_{255,2}$
199	$k \equiv 374 \pmod{765}$	$c_{765,1}$
200	$k \equiv 629 \pmod{765}$	$c_{765,2}$
201	$k \equiv 884 \pmod{1530}$	$c_{1530,1}$
202	$k \equiv 119 \pmod{1530}$	$c_{1530,2}$
203	$k \equiv 289 \pmod{306}$	2142001
204	$k \equiv 238 \pmod{306}$	5364487
205	$k \equiv 187 \pmod{306}$	832339891
206	$k \equiv 136 \pmod{306}$	276402747619
207	$k \equiv 85 \pmod{306}$	2405782797823
208	$k \equiv 34 \pmod{306}$	P_{306}
209	$k \equiv 284 \pmod{340}$	26861
210	$k \equiv 199 \pmod{340}$	7568346838961
211	$k \equiv 114 \pmod{340}$	237612993541791006121
212	$k \equiv 29 \pmod{340}$	$c_{340,1}$
213	$k \equiv 279 \pmod{340}$	$c_{340,2}$
214	$k \equiv 1554 \pmod{1700}$	5101
215	$k \equiv 1214 \pmod{1700}$	3221501
216	$k \equiv 874 \pmod{1700}$	7239775671637271201

row	congruence	prime p_i
217	$k \equiv 534 \pmod{1700}$	$c_{1700,1}$
218	$k \equiv 194 \pmod{1700}$	$c_{1700,2}$
219	$k \equiv 1129 \pmod{1360}$	72547841
220	$k \equiv 789 \pmod{1360}$	174323368765921
221	$k \equiv 449 \pmod{1360}$	32594452803395687056 1
222	$k \equiv 109 \pmod{1360}$	3031447252741466336597441
223	$k \equiv 228 \pmod{272}$	13355595217
224	$k \equiv 160 \pmod{272}$	P_{272}
225	$k \equiv 364 \pmod{544}$	58627969
226	$k \equiv 92 \pmod{544}$	$c_{544,1}$
227	$k \equiv 568 \pmod{816}$	$c_{816,1}$
228	$k \equiv 296 \pmod{816}$	$c_{816,2}$
229	$k \equiv 840 \pmod{1632}$	3156663361
230	$k \equiv 24 \pmod{1632}$	$c_{1632,1}$
231	$k \equiv 359 \pmod{425}$	2551
232	$k \equiv 274 \pmod{425}$	1076120401
233	$k \equiv 189 \pmod{425}$	P_{425}
234	$k \equiv 529 \pmod{850}$	2254201
235	$k \equiv 104 \pmod{850}$	$c_{850,1}$
236	$k \equiv 869 \pmod{1275}$	122401
237	$k \equiv 444 \pmod{1275}$	4029001
238	$k \equiv 19 \pmod{1275}$	10947151
239	$k \equiv 439 \pmod{510}$	102001
240	$k \equiv 354 \pmod{510}$	5516286288241

Table A.8 cont. Covering used in Lemma 4.3 (i) for $d = 6$

row	congruence	prime p_i
241	$k \equiv 269 \pmod{510}$	4270914986978327797975291
242	$k \equiv 184 \pmod{510}$	$c_{510,1}$
243	$k \equiv 99 \pmod{510}$	$c_{510,2}$
244	$k \equiv 524 \pmod{1020}$	1021
245	$k \equiv 14 \pmod{1020}$	855781
246	$k \equiv 944 \pmod{1020}$	2586721
247	$k \equiv 774 \pmod{1020}$	5071197096181
248	$k \equiv 604 \pmod{1020}$	8161
249	$k \equiv 434 \pmod{1020}$	$c_{1020,1}$
250	$k \equiv 1284 \pmod{2040}$	1664644599118801
251	$k \equiv 264 \pmod{2040}$	10122798208400401
252	$k \equiv 1114 \pmod{2040}$	$c_{2040,1}$
253	$k \equiv 94 \pmod{2040}$	$c_{2040,2}$
254	$k \equiv 1369 \pmod{1530}$	1531
255	$k \equiv 1199 \pmod{1530}$	7947357582331
256	$k \equiv 1029 \pmod{1530}$	40984651817371
257	$k \equiv 859 \pmod{1530}$	12804651623971
258	$k \equiv 2219 \pmod{3060}$	9181
259	$k \equiv 689 \pmod{3060}$	302941
260	$k \equiv 2049 \pmod{3060}$	49281384122461
261	$k \equiv 519 \pmod{3060}$	239693526486549721
262	$k \equiv 1879 \pmod{3060}$	p_{262}
263	$k \equiv 349 \pmod{3060}$	6121
264	$k \equiv 1709 \pmod{3060}$	$c_{3060,1}$

row	congruence	prime p_i
265	$k \equiv 3239 \pmod{6120}$	36721
266	$k \equiv 179 \pmod{6120}$	24481
267	$k \equiv 4599 \pmod{6120}$	12241
268	$k \equiv 3069 \pmod{6120}$	1211761
269	$k \equiv 1539 \pmod{6120}$	p_{269}
270	$k \equiv 9 \pmod{6120}$	$c_{6120,1}$
271	$k \equiv 106 \pmod{187}$	143899867
272	$k \equiv 21 \pmod{187}$	$c_{187,1}$
273	$k \equiv 123 \pmod{187}$	$c_{187,2}$
274	$k \equiv 225 \pmod{374}$	192611
275	$k \equiv 38 \pmod{374}$	1284680342573
276	$k \equiv 327 \pmod{374}$	p_{276}
277	$k \equiv 140 \pmod{374}$	p_{277}
278	$k \equiv 242 \pmod{561}$	1123
279	$k \equiv 55 \pmod{561}$	51613
280	$k \equiv 429 \pmod{561}$	p_{280}
281	$k \equiv 718 \pmod{748}$	p_{281}
282	$k \equiv 531 \pmod{748}$	7481
283	$k \equiv 344 \pmod{748}$	$c_{748,1}$
284	$k \equiv 157 \pmod{748}$	$c_{748,2}$
285	$k \equiv 259 \pmod{935}$	1871
286	$k \equiv 174 \pmod{935}$	82986894209449271801
287	$k \equiv 89 \pmod{935}$	189055214380376875384151
288	$k \equiv 939 \pmod{1870}$	$c_{1870,1}$
289	$k \equiv 4 \pmod{1870}$	$c_{1870,2}$

For inserting the last digit, $d = 9$, into the leading sevens, we use the covering found in Table A.10 . Table A.9 provides the notable factorizations of $\Phi_n(10)$ for large n used in these coverings. As with the previous tables, we omit the factorizations corresponding to the composite numbers and primes in Table A.9 that can be determined by dividing $\Phi_n(10)$ by the nearby primes included in Table A.10. To save space we list some of the larger primes below.

$$p_{27} = 233910282064172736373583 80573, \quad p_{64} = 1540832049306625577812 01849,$$

$$p_{234} = 900778143960550179382572 37117, \quad p_{240} = 405481405140627747580718 40361,$$

$$p_{243} = 216060644986915052462000 58094681,$$

$$p_{244} = 489116891108913037061741 93415115219.$$

Table A.9 Partial/Full factorizations of $\Phi_n(10)$ for $d = 9$

n	Factorization of $\Phi_n(10)$
93	P_{93}
161	$6763 \cdot 472341157 \cdot 112731707711317503 91 \cdot C_{161}$
203	$18620680471 \cdot 686168051734005022 43 \cdot 223072327648229020879 \cdot P_{203}$
259	$2591 \cdot 64638631 \cdot 3663802449983927390483 \cdot C_{259}$
290	$1450 \cdot 30104611 \cdot 58765601 \cdot 2433146345771 \cdot 17996132431060961 \cdot P_{290}$
465	$31 \cdot 50718862231 \cdot 151063549389365288761 \cdot C_{465}$
518	C_{518}
558	$1117 \cdot C_{558}$
620	C_{620}
783	C_{783}
1116	$4967159761 \cdot 28975286089 \cdot 206251228717780861 \cdot P_{1116}$
2232	P_{2232}
4464	C_{4464}

Table A.10 Covering used in Lemma 4.3 (i) for $d = 9$

row	congruence	prime p_i
a	$k \equiv 0 \pmod{2}$	11
1	$k \equiv 18 \pmod{23}$	11111111111111111111111
2	$k \equiv 35 \pmod{46}$	47
3	$k \equiv 29 \pmod{46}$	139
4	$k \equiv 23 \pmod{46}$	2531
5	$k \equiv 17 \pmod{46}$	549797184491917
6	$k \equiv 34 \pmod{69}$	277
7	$k \equiv 28 \pmod{69}$	203864078068831
8	$k \equiv 22 \pmod{69}$	1595352086329224644348978893
9	$k \equiv 39 \pmod{92}$	1289
10	$k \equiv 85 \pmod{92}$	18371524594609
11	$k \equiv 79 \pmod{92}$	4181003300071669867932658901
12	$k \equiv 125 \pmod{184}$	2393
13	$k \equiv 33 \pmod{184}$	P_{184}
14	$k \equiv 96 \pmod{115}$	31511
15	$k \equiv 73 \pmod{115}$	19707665921
16	$k \equiv 50 \pmod{115}$	20414137203567631
17	$k \equiv 27 \pmod{115}$	5799951513941382144830754391
18	$k \equiv 4 \pmod{115}$	122403569491783662720773144041
19	$k \equiv 67 \pmod{138}$	31051
20	$k \equiv 61 \pmod{138}$	143574021480139
21	$k \equiv 55 \pmod{138}$	24649445347649059192745899
22	$k \equiv 3 \pmod{161}$	6763
23	$k \equiv 49 \pmod{161}$	472341157

row	congruence	prime p_i
24	$k \equiv 279 \pmod{322}$	967
25	$k \equiv 233 \pmod{322}$	1569936761
26	$k \equiv 187 \pmod{322}$	1203881882727712699967
27	$k \equiv 141 \pmod{322}$	p_{27}
28	$k \equiv 95 \pmod{322}$	$c_{322,1}$
29	$k \equiv 181 \pmod{207}$	109908191603107
30	$k \equiv 112 \pmod{207}$	$c_{207,1}$
31	$k \equiv 43 \pmod{207}$	$c_{207,2}$
32	$k \equiv 175 \pmod{276}$	829
33	$k \equiv 37 \pmod{276}$	1569889
34	$k \equiv 169 \pmod{276}$	536430531035337769
35	$k \equiv 31 \pmod{276}$	757108543129939106221
36	$k \equiv 157 \pmod{276}$	P_{276}
37	$k \equiv 295 \pmod{552}$	1657
38	$k \equiv 19 \pmod{552}$	1469409649
39	$k \equiv 301 \pmod{345}$	5521
40	$k \equiv 232 \pmod{345}$	74521
41	$k \equiv 163 \pmod{345}$	487831
42	$k \equiv 94 \pmod{345}$	19277911
43	$k \equiv 25 \pmod{345}$	$c_{345,1}$
44	$k \equiv 289 \pmod{414}$	37916801893
45	$k \equiv 151 \pmod{414}$	143409436964525899
46	$k \equiv 13 \pmod{414}$	831759677425747570837717
47	$k \equiv 283 \pmod{414}$	$c_{414,1}$

Table A.10 cont. Covering used in Lemma 4.3 (i) for $d = 9$

row	congruence	prime p_i
48	$k \equiv 145 \pmod{414}$	$c_{414,2}$
49	$k \equiv 421 \pmod{828}$	4969
50	$k \equiv 7 \pmod{828}$	111781
51	$k \equiv 415 \pmod{483}$	14461987
52	$k \equiv 346 \pmod{483}$	176686231
53	$k \equiv 277 \pmod{483}$	4828646712337104427
54	$k \equiv 208 \pmod{483}$	575760557843831535447049
55	$k \equiv 139 \pmod{483}$	$c_{483,1}$
56	$k \equiv 70 \pmod{161}$	11273170771131750391
57	$k \equiv 1 \pmod{161}$	$c_{161,1}$
58	$k \equiv 28 \pmod{29}$	3191
59	$k \equiv 22 \pmod{29}$	16763
60	$k \equiv 16 \pmod{29}$	43037
61	$k \equiv 10 \pmod{29}$	62003
62	$k \equiv 4 \pmod{29}$	77843839397
63	$k \equiv 27 \pmod{58}$	59
64	$k \equiv 21 \pmod{58}$	p_{64}
65	$k \equiv 44 \pmod{87}$	4003
66	$k \equiv 38 \pmod{87}$	72559
67	$k \equiv 32 \pmod{87}$	p_{67}
68	$k \equiv 113 \pmod{116}$	349
69	$k \equiv 55 \pmod{116}$	38861
70	$k \equiv 107 \pmod{116}$	618049
71	$k \equiv 49 \pmod{116}$	p_{71}

row	congruence	prime p_i
72	$k \equiv 101 \pmod{174}$	638453709757
73	$k \equiv 95 \pmod{174}$	135080726389891
74	$k \equiv 89 \pmod{174}$	1274194732898148471766404179653
75	$k \equiv 257 \pmod{348}$	997369
76	$k \equiv 83 \pmod{348}$	15565833747318241
77	$k \equiv 251 \pmod{348}$	157002934023127338801841
78	$k \equiv 77 \pmod{348}$	$c_{348,1}$
79	$k \equiv 593 \pmod{696}$	13921
80	$k \equiv 419 \pmod{696}$	11090099157944399977
81	$k \equiv 245 \pmod{696}$	$c_{696,1}$
82	$k \equiv 71 \pmod{696}$	$c_{696,2}$
83	$k \equiv 413 \pmod{522}$	155557
84	$k \equiv 239 \pmod{522}$	2762580729671077856014459
85	$k \equiv 65 \pmod{522}$	$c_{522,1}$
86	$k \equiv 349 \pmod{406}$	19489
87	$k \equiv 291 \pmod{406}$	243720583
88	$k \equiv 233 \pmod{406}$	2563963356465898129
89	$k \equiv 175 \pmod{406}$	13569612563320403017443683179
90	$k \equiv 117 \pmod{406}$	P_{406}
91	$k \equiv 59 \pmod{203}$	18620680471
92	$k \equiv 1 \pmod{203}$	68616805173400502243
93	$k \equiv 227 \pmod{232}$	233
94	$k \equiv 169 \pmod{232}$	355193
95	$k \equiv 111 \pmod{232}$	21591416633

Table A.10 cont. Covering used in Lemma 4.3 (i) for $d = 9$

row	congruence	prime p_i
96	$k \equiv 53 \pmod{232}$	$c_{232,1}$
97	$k \equiv 221 \pmod{232}$	$c_{232,2}$
98	$k \equiv 395 \pmod{464}$	929
99	$k \equiv 163 \pmod{464}$	23201
100	$k \equiv 337 \pmod{464}$	182353
101	$k \equiv 105 \pmod{464}$	4465073
102	$k \equiv 279 \pmod{464}$	$c_{464,1}$
103	$k \equiv 47 \pmod{464}$	$c_{464,2}$
104	$k \equiv 215 \pmod{261}$	523
105	$k \equiv 128 \pmod{261}$	670249
106	$k \equiv 41 \pmod{261}$	44974999
107	$k \equiv 209 \pmod{261}$	9113995243
108	$k \equiv 122 \pmod{261}$	7299238406959
109	$k \equiv 35 \pmod{261}$	$c_{261,1}$
110	$k \equiv 203 \pmod{261}$	$c_{261,2}$
111	$k \equiv 638 \pmod{783}$	$c_{783,1}$
112	$k \equiv 377 \pmod{783}$	$c_{783,2}$
113	$k \equiv 899 \pmod{1566}$	1567
114	$k \equiv 116 \pmod{1566}$	32887
115	$k \equiv 812 \pmod{1044}$	2089
116	$k \equiv 551 \pmod{1044}$	31753261
117	$k \equiv 290 \pmod{1044}$	1452031741
118	$k \equiv 29 \pmod{1044}$	803137587589
119	$k \equiv 255 \pmod{290}$	1451
120	$k \equiv 197 \pmod{290}$	30104611
121	$k \equiv 139 \pmod{290}$	58765601
122	$k \equiv 81 \pmod{290}$	2433146345771
123	$k \equiv 23 \pmod{290}$	17996132431060961
124	$k \equiv 133 \pmod{145}$	9605671
125	$k \equiv 104 \pmod{145}$	15589280974996818911
126	$k \equiv 75 \pmod{145}$	$c_{145,1}$
127	$k \equiv 46 \pmod{145}$	$c_{145,2}$
128	$k \equiv 162 \pmod{290}$	P_{290}
129	$k \equiv 307 \pmod{580}$	1767955366388381478541
130	$k \equiv 17 \pmod{580}$	$c_{580,1}$
131	$k \equiv 359 \pmod{435}$	80911
132	$k \equiv 272 \pmod{435}$	102019681
133	$k \equiv 185 \pmod{435}$	8538216023012034296071
134	$k \equiv 98 \pmod{435}$	P_{435}
135	$k \equiv 446 \pmod{870}$	33931
136	$k \equiv 11 \pmod{870}$	70350656881
137	$k \equiv 527 \pmod{609}$	41413
138	$k \equiv 440 \pmod{609}$	1840603883162311
139	$k \equiv 353 \pmod{609}$	$c_{609,1}$
140	$k \equiv 266 \pmod{609}$	$c_{609,2}$
141	$k \equiv 788 \pmod{1218}$	2437
142	$k \equiv 179 \pmod{1218}$	4315282651
143	$k \equiv 92 \pmod{203}$	223072327648229020879

Table A.10 cont. Covering used in Lemma 4.3 (i) for $d = 9$

row	congruence	prime p_i
144	$k \equiv 5 \pmod{203}$	P_{203}
145	$k \equiv 22 \pmod{31}$	2791
146	$k \equiv 10 \pmod{31}$	6943319
147	$k \equiv 29 \pmod{31}$	57336415063790604359
148	$k \equiv 17 \pmod{62}$	909090909090909090909090909091
149	$k \equiv 36 \pmod{93}$	P_{93}
150	$k \equiv 55 \pmod{124}$	2049349
151	$k \equiv 43 \pmod{124}$	p_{151}
152	$k \equiv 93 \pmod{186}$	373
153	$k \equiv 81 \pmod{186}$	44641
154	$k \equiv 69 \pmod{186}$	3590254957
155	$k \equiv 57 \pmod{186}$	p_{155}
156	$k \equiv 231 \pmod{372}$	5209
157	$k \equiv 219 \pmod{372}$	$c_{372,1}$
158	$k \equiv 207 \pmod{372}$	$c_{372,2}$
159	$k \equiv 195 \pmod{248}$	1489
160	$k \equiv 71 \pmod{248}$	640543322297
161	$k \equiv 183 \pmod{248}$	27908132670449
162	$k \equiv 59 \pmod{248}$	3847054441822302911 05649
163	$k \equiv 171 \pmod{248}$	P_{248}
164	$k \equiv 295 \pmod{496}$	789592121167489
165	$k \equiv 47 \pmod{496}$	$c_{496,1}$
166	$k \equiv 252 \pmod{279}$	18989357081041
167	$k \equiv 159 \pmod{279}$	$c_{279,1}$

row	congruence	prime p_i
168	$k \equiv 66 \pmod{279}$	$c_{279,2}$
169	$k \equiv 271 \pmod{310}$	11161
170	$k \equiv 209 \pmod{310}$	3925963357681
171	$k \equiv 147 \pmod{310}$	5167617497664851
172	$k \equiv 85 \pmod{310}$	P_{310}
173	$k \equiv 333 \pmod{620}$	$c_{620,1}$
174	$k \equiv 23 \pmod{620}$	$c_{620,2}$
175	$k \equiv 507 \pmod{744}$	700849
176	$k \equiv 135 \pmod{744}$	11110153
177	$k \equiv 495 \pmod{744}$	1220725699657
178	$k \equiv 123 \pmod{744}$	5419392721
179	$k \equiv 483 \pmod{744}$	42367299139993
180	$k \equiv 111 \pmod{744}$	$c_{744,1}$
181	$k \equiv 471 \pmod{744}$	$c_{744,2}$
182	$k \equiv 843 \pmod{1488}$	7573921
183	$k \equiv 99 \pmod{1488}$	$c_{1488,1}$
184	$k \equiv 831 \pmod{868}$	8681
185	$k \equiv 707 \pmod{868}$	106300489
186	$k \equiv 583 \pmod{868}$	120995729
187	$k \equiv 459 \pmod{868}$	363981764441
188	$k \equiv 335 \pmod{868}$	$c_{868,1}$
189	$k \equiv 211 \pmod{868}$	$c_{868,2}$
190	$k \equiv 87 \pmod{434}$	240437
191	$k \equiv 385 \pmod{434}$	1142669053

Table A.10 cont. Covering used in Lemma 4.3 (i) for $d = 9$

row	congruence	prime p_i
192	$k \equiv 323 \pmod{434}$	313360308665807383
193	$k \equiv 261 \pmod{434}$	5792107074602303416893
194	$k \equiv 199 \pmod{434}$	P_{434}
195	$k \equiv 137 \pmod{217}$	3662093
196	$k \equiv 75 \pmod{217}$	$c_{217,1}$
197	$k \equiv 13 \pmod{217}$	$c_{217,2}$
198	$k \equiv 435 \pmod{465}$	50718862231
199	$k \equiv 342 \pmod{465}$	1510635493893652888761
200	$k \equiv 249 \pmod{465}$	$c_{465,1}$
201	$k \equiv 156 \pmod{465}$	$c_{465,2}$
202	$k \equiv 63 \pmod{155}$	311
203	$k \equiv 423 \pmod{558}$	$c_{558,1}$
204	$k \equiv 237 \pmod{558}$	$c_{558,2}$
205	$k \equiv 1167 \pmod{1674}$	5023
206	$k \equiv 609 \pmod{1674}$	261315626851
207	$k \equiv 51 \pmod{1674}$	$c_{1674,1}$
208	$k \equiv 783 \pmod{1116}$	4967159761
209	$k \equiv 411 \pmod{1116}$	28975286089
210	$k \equiv 39 \pmod{1116}$	206251228717780861
211	$k \equiv 1515 \pmod{1860}$	1861
212	$k \equiv 1143 \pmod{1860}$	459421
213	$k \equiv 771 \pmod{1860}$	105966383429834881
214	$k \equiv 399 \pmod{1860}$	26482637738395021
215	$k \equiv 27 \pmod{1860}$	3069934683332551531 81

row	congruence	prime p_i
216	$k \equiv 1503 \pmod{1860}$	$c_{1860,1}$
217	$k \equiv 1131 \pmod{1860}$	$c_{1860,2}$
218	$k \equiv 759 \pmod{930}$	22672589441232691
219	$k \equiv 387 \pmod{930}$	$c_{930,1}$
220	$k \equiv 15 \pmod{930}$	$c_{930,2}$
221	$k \equiv 747 \pmod{1116}$	P_{1116}
222	$k \equiv 375 \pmod{558}$	1117
223	$k \equiv 1119 \pmod{2232}$	P_{2232}
224	$k \equiv 2235 \pmod{4464}$	$c_{4464,1}$
225	$k \equiv 3 \pmod{4464}$	$c_{4464,2}$
226	$k \equiv 34 \pmod{37}$	2028119
227	$k \equiv 22 \pmod{37}$	247629013
228	$k \equiv 10 \pmod{37}$	2212394296770203368013
229	$k \equiv 35 \pmod{74}$	7253
230	$k \equiv 23 \pmod{74}$	422650073734453
231	$k \equiv 11 \pmod{74}$	296557347313446299
232	$k \equiv 36 \pmod{111}$	30557051518647307
233	$k \equiv 24 \pmod{111}$	8845981170865629119271997
234	$k \equiv 12 \pmod{111}$	p_{234}
235	$k \equiv 37 \pmod{148}$	149
236	$k \equiv 25 \pmod{148}$	3109
237	$k \equiv 13 \pmod{148}$	111149
238	$k \equiv 1 \pmod{148}$	708840373781
239	$k \equiv 137 \pmod{148}$	6690316866614278428 29

Table A.10 cont. Covering used in Lemma 4.3 (i) for $d = 9$

row	congruence	prime p_i
240	$k \equiv 125 \pmod{148}$	p_{240}
241	$k \equiv 39 \pmod{222}$	223
242	$k \equiv 27 \pmod{222}$	4663
243	$k \equiv 15 \pmod{222}$	p_{243}
244	$k \equiv 3 \pmod{222}$	p_{244}
245	$k \equiv 213 \pmod{444}$	23653713304547869
246	$k \equiv 201 \pmod{444}$	$c_{444,1}$
247	$k \equiv 189 \pmod{444}$	$c_{444,2}$
248	$k \equiv 621 \pmod{888}$	54615849443593
249	$k \equiv 177 \pmod{888}$	31495401976675753
250	$k \equiv 609 \pmod{888}$	$c_{888,1}$
251	$k \equiv 165 \pmod{888}$	$c_{888,2}$
252	$k \equiv 1041 \pmod{1332}$	38629
253	$k \equiv 597 \pmod{1332}$	215018101
254	$k \equiv 153 \pmod{1332}$	14003835481
255	$k \equiv 1029 \pmod{1332}$	$c_{1332,1}$
256	$k \equiv 585 \pmod{1332}$	$c_{1332,2}$
257	$k \equiv 1473 \pmod{2664}$	7993
258	$k \equiv 141 \pmod{2664}$	6920827178401
259	$k \equiv 1461 \pmod{1776}$	1777
260	$k \equiv 1017 \pmod{1776}$	7228321
261	$k \equiv 573 \pmod{1776}$	14645119755678294049
262	$k \equiv 129 \pmod{1776}$	c_{1776}
263	$k \equiv 154 \pmod{185}$	6511230041186560022095681
264	$k \equiv 117 \pmod{185}$	1936570114827069923550119591
265	$k \equiv 80 \pmod{185}$	$c_{185,1}$
266	$k \equiv 43 \pmod{185}$	$c_{185,2}$
267	$k \equiv 191 \pmod{370}$	49173925574833481
268	$k \equiv 6 \pmod{370}$	P_{370}
269	$k \equiv 253 \pmod{296}$	61544617
270	$k \equiv 105 \pmod{296}$	16444765848115921
271	$k \equiv 315 \pmod{333}$	96455449
272	$k \equiv 204 \pmod{333}$	$c_{333,1}$
273	$k \equiv 93 \pmod{333}$	$c_{333,2}$
274	$k \equiv 525 \pmod{555}$	14431
275	$k \equiv 414 \pmod{555}$	169831
276	$k \equiv 303 \pmod{555}$	994561
277	$k \equiv 192 \pmod{555}$	5153778841
278	$k \equiv 81 \pmod{555}$	20786617240933593976831
279	$k \equiv 513 \pmod{666}$	902659997773
280	$k \equiv 291 \pmod{666}$	7116181512528105949933
281	$k \equiv 69 \pmod{666}$	62562101977662753776437
282	$k \equiv 649 \pmod{740}$	1481
283	$k \equiv 501 \pmod{740}$	15541
284	$k \equiv 353 \pmod{740}$	68821
285	$k \equiv 205 \pmod{740}$	6336499381
286	$k \equiv 57 \pmod{740}$	4098464044501
287	$k \equiv 637 \pmod{740}$	74615611921
288	$k \equiv 489 \pmod{740}$	107771345624719321

Table A.10 cont. Covering used in Lemma 4.3 (i) for $d = 9$

row	congruence	prime p_i
289	$k \equiv 341 \pmod{740}$	16206170837408945509907221
290	$k \equiv 193 \pmod{740}$	$c_{740,1}$
291	$k \equiv 45 \pmod{740}$	$c_{740,2}$
292	$k \equiv 699 \pmod{777}$	79003664587
293	$k \equiv 588 \pmod{777}$	531660478441
294	$k \equiv 477 \pmod{777}$	315050417550355 21
295	$k \equiv 366 \pmod{777}$	$c_{777,1}$
296	$k \equiv 255 \pmod{777}$	$c_{777,2}$
297	$k \equiv 144 \pmod{259}$	$c_{259,1}$
298	$k \equiv 33 \pmod{259}$	$c_{259,2}$
299	$k \equiv 909 \pmod{1036}$	9392332326728043169
300	$k \equiv 761 \pmod{1036}$	$c_{1036,1}$
301	$k \equiv 613 \pmod{1036}$	$c_{1036,2}$
302	$k \equiv 465 \pmod{518}$	$c_{518,1}$
303	$k \equiv 58 \pmod{259}$	2591
304	$k \equiv 169 \pmod{259}$	64638631
305	$k \equiv 21 \pmod{259}$	3663802449983927390483
306	$k \equiv 379 \pmod{407}$	46399
307	$k \equiv 342 \pmod{407}$	390721
308	$k \equiv 305 \pmod{407}$	2377517312347

row	congruence	prime p_i
309	$k \equiv 268 \pmod{407}$	1492218407878727 6001107
310	$k \equiv 231 \pmod{407}$	$c_{407,1}$
311	$k \equiv 194 \pmod{407}$	$c_{407,2}$
312	$k \equiv 564 \pmod{814}$	409038414731
313	$k \equiv 157 \pmod{814}$	$c_{814,1}$
314	$k \equiv 527 \pmod{814}$	$c_{814,2}$
315	$k \equiv 934 \pmod{1628}$	24421
316	$k \equiv 120 \pmod{1628}$	115589
317	$k \equiv 1304 \pmod{1628}$	7716573481
318	$k \equiv 897 \pmod{1628}$	26860773000292529
319	$k \equiv 490 \pmod{1628}$	$c_{1628,1}$
320	$k \equiv 1711 \pmod{3256}$	3257
321	$k \equiv 83 \pmod{3256}$	1655532160846814535 13
322	$k \equiv 860 \pmod{1221}$	3306121237
323	$k \equiv 453 \pmod{1221}$	37232500009
324	$k \equiv 46 \pmod{1221}$	171055055020477
325	$k \equiv 823 \pmod{1221}$	$c_{1221,1}$
326	$k \equiv 416 \pmod{1221}$	$c_{1221,2}$
327	$k \equiv 1230 \pmod{2442}$	18216009861391358 2339
328	$k \equiv 9 \pmod{2442}$	$c_{2442,1}$

Appendix B

Coverings for bases other than 10

For bases other than 10, we wish to establish a statement analogous to Theorem 1.5. While Theorem 1.5 requires the widely digitally stable composite number, N, to be coprime to 10, we will require N to be coprime to the base, b. We generalize much of the notation used in Chapter 4. For any base b, consider the number

$$N = a \cdot \frac{b^n - 1}{b - 1} + M,$$

with $a \in \{1, 2, \ldots, b-1\}$ and where n and M are large natural numbers to be determined with n large enough that the left-most digit of N is a. Notice that the base b representation of N can be written as

$$N = 00\ldots00\underbrace{aa\ldots aa}_{n \text{ total } a\text{'s}}+M,$$

where $aa\ldots aa$ is a string of n digits that are all a's.

For a nonnegative integer k, let $N^{(k)}(x)$ denote inserting a digit $x \in \{0, ..., b-1\}$ to the right of the $k+1^{st}$ digit of N. We set K to be the number of digits of M; thus, initially, K is unknown to us. Similar to the method used in Chapter 4, we break the argument up into three cases: $k \geq n$ corresponding to the leading zeros, $K \leq k \leq n-1$ corresponding to the leading a's, and $0 \leq k < K$ corresponding to M. We can formulate $N^{(k)}(x)$ above nicely for the first two cases. Specifically, with $x \in \{0, 1, \ldots, b-1\}$, for $k \geq n$, the value of $N^{(k)}(x)$, representing inserting x into the leading zeros, takes the form

$$N_0^{(k)}(x) = a \cdot \frac{b^n - 1}{b - 1} + M + x \cdot b^k\,;$$

and for $K \leq k < n$, the value of $N^{(k)}(d)$, representing inserting d into the leading a's, takes the form

$$N_a^{(k)}(d) = a \cdot b^n + a \cdot \frac{b^n - 1}{b - 1} + M + (d - a) \cdot b^k,$$

where we use d to indicate that we are looking at the leading a's and x to indicate that we are looking at the leading zeros. For base 10 we let $b = 10$ and $a = 7$; however, we will pick different values for a to better utilize small primes for each base.

To prove an analogous statement to Theorem 1.5 for base b, we will follow the same method outlined in Chapter 4. We will exhibit coverings of the integers corresponding to the following generalizations of Lemma 4.2 and Lemma 4.3. Each covering corresponds to inserting a digit x in the leading zeros or inserting a digit d in the leading a's. By way of notation, we define $c(p)$ to be the multiplicative order of b modulo a prime with $\gcd(p, b) = 1$.

Lemma B.1. *Fix an integer $b > 1$. Let N and M be natural numbers, and $a \in \{1, 2, \ldots, b-1\}$, such that*

$$N = a \cdot \frac{b^n - 1}{b - 1} + M,$$

where N has base b expansion

$$N = d_{n-1}d_{n-2}\ldots d_1d_0, \quad d_i \in \{0, 1, \ldots, b\} \quad n \geq 1, \quad d_{n-1} = a.$$

Let K be a nonnegative integer such that $d_k = a$ for $K \leq k \leq n - 1$, and set $d_k = 0$ for $k \geq n$. For a fixed $x \in \{0, 1, \ldots, b-1\}$, suppose we have distinct primes $p_1, \ldots, p_t$, each coprime to b, for which

(i) there exists a covering of the integers $k \equiv b_i \pmod{c(p_i)}$, for $1 \leq i \leq t$,

(ii) $n \equiv 0 \pmod{lcm(c(p_1), \ldots, c(p_t))}$,

(iii) M is a solution to the system of congruences

$$M \equiv -x \cdot b^{b_i} \pmod{p_i}, \quad 1 \leq i \leq t.$$

Then, for all nonnegative integers k, we have

$$N_0^{(k)}(x) = a \cdot \frac{b^n - 1}{b - 1} + M + x \cdot b^k$$

is divisible by at least one of the primes p_i where $1 \le i \le t$.

Lemma B.2. *Fix an integer $b > 1$. Let N and M be natural numbers, and $a \in \{1, 2, \ldots, b-1\}$, such that*

$$N = a \cdot \frac{b^n - 1}{b - 1} + M,$$

where N has base b expansion

$$N = d_{n-1} d_{n-2} \ldots d_1 d_0, \quad d_i \in \{0, 1, \ldots, b\} \quad n \ge 1, \quad d_{n-1} = a.$$

Let K be a nonnegative integer such that $d_k = a$ for $K \le k \le n-1$, and set $d_k = 0$ for $k \ge n$. For a fixed $d \in \{0, 1, \ldots, b-1\}$, suppose we have distinct primes $p_1, \ldots, p_t$, each coprime to b, for which

(i) *there exists a covering of the integers $k \equiv b_i \pmod{c(p_i)}$, for $1 \le i \le t$,*

(ii) $n \equiv 0 \pmod{lcm(c(p_1), \ldots, c(p_t))}$,

(iii) *M is a solution to the system of congruences*

$$M \equiv -a - (d - a) \cdot b^{b_i} \pmod{p_i}, \quad 1 \le i \le t.$$

Then, for all nonnegative integers k, we have

$$N_a^{(k)}(d) = a \cdot b^n + a \cdot \frac{b^n - 1}{b - 1} + M + (d - a) \cdot b^k$$

is divisible by at least one of the primes p_i where $1 \le i \le t$.

The proofs of Lemma B.1 and Lemma B.2 are identical to that of Lemma 4.2. For the right-most digits, M, the same argument used in Section 4.4 of Chapter 4 is used with the role of a and b replacing that of 7 and 10.

The rest of this appendix will be used to exhibit the coverings used in (i) of Lemmas B.1 and B.2 for each base. Many of the coverings used in this appendix can be found in Jeremiah Southwick's dissertation, [36], but with slight modifications. We will use the same notation for displaying the coverings as was used in Chapter 4 and Appendix A. We have at times truncated the presentation of the congruences chosen so as to meet formatting guidelines. For example, in Table B.12 we have removed the columns with heading 'row' to allow for more horizontal space and have also shortened the presentation of each residue class chosen. Notice that to obtain the primes p of a given order $c = c(p)$ for a given base, one looks at the primes p which divide $\Phi_c(b)$, where $\Phi_c(x)$ is the c^{th} cyclotomic polynomial.

B.1 Covering systems for $b = 2$

For base 2, we let $b = 2$ and $a = 1$. That is, we consider

$$N = 2^n - 1 + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = 2^n - 1 + M + x \cdot 2^k \quad \text{and} \quad N_1^{(k)}(d) = 2^{n+1} - 1 + M + (d-1) \cdot 2^k.$$

We want coverings for $x \in \{0, 1\}$ and $d \in \{0, 1\}$; however, inserting a zero into the leading zeros does not change N. That is, we do not need a covering for $x = 0$. To guarantee that N is composite, we use a single congruence instead of a full covering.

Table B.1 is the covering used for $x = 1$, and Table B.2 is the covering used for $d = 0$. Recall that we can use a prime p more than once as long as the congruence condition on M from (iii) of Lemmas B.1 and B.2 are equivalent. For $d = 0$, we do not reuse any primes from Table B.1. We reuse the prime 3 in the congruence $k \equiv 0 \pmod 2$ for the covering found in Table B.3 corresponding to $d = 1$. One can check that the congruences involving the prime 3 correspond to the congruence condition $M \equiv 2 \pmod 3$.

Table B.1 Covering for $x = 1$ in Base 2

row	congruence	prime p_i
1	$k \equiv 0 \pmod{2}$	3
2	$k \equiv 1 \pmod{4}$	5
3	$k \equiv 3 \pmod{8}$	17
4	$k \equiv 7 \pmod{16}$	257
5	$k \equiv 15 \pmod{32}$	65537
6	$k \equiv 31 \pmod{64}$	641
7	$k \equiv 63 \pmod{64}$	6700417

Table B.2 Covering for $d = 0$ in Base 2

row	congruence	prime p_i
1	$k \equiv 0 \pmod{3}$	7
2	$k \equiv 7 \pmod{9}$	73
3	$k \equiv 13 \pmod{18}$	19
4	$k \equiv 4 \pmod{36}$	37
5	$k \equiv 22 \pmod{36}$	109
6	$k \equiv 19 \pmod{27}$	262657
7	$k \equiv 37 \pmod{54}$	87211
8	$k \equiv 10 \pmod{108}$	246241
9	$k \equiv 64 \pmod{108}$	279073
10	$k \equiv 1 \pmod{81}$	2593
11	$k \equiv 28 \pmod{81}$	71119

row	congruence	prime p_i
12	$k \equiv 55 \pmod{81}$	97685839
13	$k \equiv 11 \pmod{12}$	13
14	$k \equiv 17 \pmod{24}$	241
15	$k \equiv 5 \pmod{48}$	97
16	$k \equiv 29 \pmod{48}$	673
17	$k \equiv 2 \pmod{5}$	31
18	$k \equiv 8 \pmod{10}$	11
19	$k \equiv 14 \pmod{15}$	151
20	$k \equiv 20 \pmod{30}$	331
21	$k \equiv 26 \pmod{60}$	61
22	$k \equiv 56 \pmod{60}$	1321

Table B.3 Covering for $d = 1$ in Base 2

congruence	prime p_i
$k \equiv 0 \pmod{2}$	3
$k \equiv 19 \pmod{20}$	41
$k \equiv 29 \pmod{40}$	61681
$k \equiv 49 \pmod{80}$	4278255361
$k \equiv 89 \pmod{160}$	414721
$k \equiv 9 \pmod{160}$	44479210368001
$k \equiv 67 \pmod{70}$	281
$k \equiv 57 \pmod{70}$	86171
$k \equiv 117 \pmod{140}$	7416361
$k \equiv 47 \pmod{140}$	47392381
$k \equiv 177 \pmod{210}$	211
$k \equiv 107 \pmod{210}$	664441
$k \equiv 37 \pmod{210}$	1564921
$k \equiv 6 \pmod{7}$	127

congruence	prime p_i
$k \equiv 10 \pmod{14}$	43
$k \equiv 17 \pmod{28}$	29
$k \equiv 3 \pmod{28}$	113
$k \equiv 14 \pmod{21}$	337
$k \equiv 28 \pmod{42}$	5419
$k \equiv 49 \pmod{84}$	1429
$k \equiv 7 \pmod{84}$	14449
$k \equiv 42 \pmod{63}$	92737
$k \equiv 21 \pmod{63}$	649657
$k \equiv 63 \pmod{126}$	77158673929
$k \equiv 126 \pmod{252}$	40388473189
$k \equiv 0 \pmod{252}$	118750098349
$k \equiv 43 \pmod{50}$	251
$k \equiv 33 \pmod{50}$	4051

Table B.3 cont. Covering for $d = 1$ in Base 2

congruence	prime p_i
$k \equiv 23 \pmod{25}$	601
$k \equiv 13 \pmod{25}$	1801
$k \equiv 53 \pmod{75}$	100801
$k \equiv 28 \pmod{75}$	10567201
$k \equiv 153 \pmod{225}$	115201
$k \equiv 78 \pmod{225}$	617401
$k \equiv 3 \pmod{225}$	1348206751
$k \equiv 91 \pmod{100}$	101
$k \equiv 41 \pmod{100}$	8101
$k \equiv 81 \pmod{100}$	268501
$k \equiv 231 \pmod{300}$	1201
$k \equiv 131 \pmod{300}$	63901
$k \equiv 31 \pmod{300}$	13334701
$k \equiv 121 \pmod{150}$	1133836730401
$k \equiv 371 \pmod{450}$	4714696801
$k \equiv 221 \pmod{450}$	281941472953710177758647201
$k \equiv 521 \pmod{900}$	695701
$k \equiv 71 \pmod{900}$	307116398490301
$k \equiv 171 \pmod{300}$	1182468601
$k \equiv 621 \pmod{900}$	413150254353901
$k \equiv 321 \pmod{900}$	6269989892198401
$k \equiv 21 \pmod{900}$	3192261504216112476901
$k \equiv 161 \pmod{200}$	401
$k \equiv 111 \pmod{200}$	340801
$k \equiv 61 \pmod{200}$	2787601
$k \equiv 11 \pmod{200}$	3173389601

congruence	prime p_i
$k \equiv 301 \pmod{350}$	1051
$k \equiv 251 \pmod{350}$	110251
$k \equiv 201 \pmod{350}$	347833278451
$k \equiv 151 \pmod{350}$	34010032331525251
$k \equiv 31 \pmod{35}$	71
$k \equiv 16 \pmod{35}$	122921
$k \equiv 71 \pmod{105}$	29191
$k \equiv 36 \pmod{105}$	106681
$k \equiv 1 \pmod{105}$	152041
$k \equiv 6 \pmod{11}$	23
$k \equiv 7 \pmod{11}$	89
$k \equiv 19 \pmod{22}$	683
$k \equiv 31 \pmod{44}$	397
$k \equiv 9 \pmod{44}$	2113
$k \equiv 10 \pmod{55}$	881
$k \equiv 0 \pmod{55}$	3191
$k \equiv 45 \pmod{55}$	201961
$k \equiv 57 \pmod{66}$	67
$k \equiv 35 \pmod{66}$	20857
$k \equiv 79 \pmod{132}$	312709
$k \equiv 13 \pmod{132}$	4327489
$k \equiv 25 \pmod{110}$	2971
$k \equiv 15 \pmod{110}$	48912491
$k \equiv 115 \pmod{220}$	415878438361
$k \equiv 5 \pmod{220}$	3630105520141

For the case $x = 1$ in Table B.1, the least common multiple of the moduli for the congruences is 64. A direct analysis as done in Section 4.2 for base 10 can be done in this case to verify the covering. That is, one can check that every integer in $[0, 63)$ satisfies at least one congruence in Table B.1. This is also true for $d = 0$, where the least common multiple of the moduli is 6480; and $d = 1$, where the least common multiple of the moduli is 554400.

To guarantee that N is composite, we use the prime 47 which satisfies $\text{ord}_{47}(2) = 23$ and does not appear in any of the coverings used for the other insertions. Let n satisfy the congruences in (ii) of Lemmas B.1 and B.2, as well as $n \equiv 0 \pmod{23}$, and let M satisfy the congruences in (iii) of Lemmas B.1 and B.2, as well as $M \equiv 0 \pmod{47}$. Then $N \equiv 0 \pmod{47}$, and hence is composite.

B.2 Covering systems for $b = 3$

For base 3, we let $b = 3$ and $a = 2$. That is, we consider

$$N = 3^n - 1 + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = 3^n - 1 + M + x \cdot 3^k \quad \text{and} \quad N_2^{(k)}(d) = 3^{n+1} - 1 + M + (d-2) \cdot 3^k.$$

We want coverings for $x \in \{0, 1, 2\}$ and $d \in \{0, 1, 2\}$. We start by utilizing smaller primes. Notice that if $M \equiv 0 \pmod{2}$, then both $N_0^{(k)}(x) \equiv 0 \pmod{2}$ and $N_2^{(k)}(d) \equiv 0 \pmod{2}$ for $x \equiv 0 \pmod{2}$ and $d \equiv 0 \pmod{2}$. Thus, we only need to find coverings for $x = 1$ and $d = 1$. Notice that inserting a 0 into the leading zeros does not change the number, so N is a composite number.

Table B.4 is the covering used for $x = 1$, and Table B.5 is the covering used for $d = 1$. Table B.5 reuses the primes 13 and 19 from Table B.4 in the congruences $k \equiv 0 \pmod{3}$ and $k \equiv 5 \pmod{18}$. One checks that these are equivalent to the congruences $M \equiv 12 \pmod{13}$ and $M \equiv 13 \pmod{19}$.

Table B.4 Covering for $x = 1$ in Base 3

congruence	prime p_i	congruence	prime p_i
$k \equiv 0 \pmod{3}$	13	$k \equiv 17 \pmod{72}$	282429005041
$k \equiv 1 \pmod{6}$	7	$k \equiv 53 \pmod{144}$	1418632417
$k \equiv 2 \pmod{9}$	757	$k \equiv 125 \pmod{144}$	56227703611393
$k \equiv 8 \pmod{18}$	19	$k \equiv 23 \pmod{216}$	2161
$k \equiv 14 \pmod{18}$	37	$k \equiv 95 \pmod{216}$	15121
$k \equiv 0 \pmod{4}$	5	$k \equiv 167 \pmod{216}$	10512289
$k \equiv 10 \pmod{12}$	73	$k \equiv 71 \pmod{108}$	150094634909578633
$k \equiv 5 \pmod{36}$	530713	$k \equiv 143 \pmod{216}$	16569793
$k \equiv 3 \pmod{8}$	41	$k \equiv 215 \pmod{216}$	3958044610033

Table B.5 Covering for $d = 1$ in Base 3

congruence	prime p_i	congruence	prime p_i
$k \equiv 0 \pmod{3}$	13	$k \equiv 50 \pmod{54}$	19441
$k \equiv 5 \pmod{18}$	19	$k \equiv 32 \pmod{54}$	19927
$k \equiv 3 \pmod{5}$	11	$k \equiv 122 \pmod{162}$	163
$k \equiv 5 \pmod{10}$	61	$k \equiv 68 \pmod{162}$	1297
$k \equiv 10 \pmod{20}$	1181	$k \equiv 14 \pmod{162}$	208657
$k \equiv 20 \pmod{40}$	42521761	$k \equiv 0 \pmod{7}$	1093
$k \equiv 40 \pmod{80}$	14401	$k \equiv 12 \pmod{14}$	547
$k \equiv 0 \pmod{80}$	128653413121	$k \equiv 19 \pmod{28}$	29
$k \equiv 7 \pmod{15}$	4561	$k \equiv 5 \pmod{28}$	16493
$k \equiv 19 \pmod{30}$	31	$k \equiv 17 \pmod{21}$	368089
$k \equiv 4 \pmod{30}$	271	$k \equiv 29 \pmod{42}$	43
$k \equiv 31 \pmod{45}$	181	$k \equiv 8 \pmod{42}$	2269
$k \equiv 16 \pmod{45}$	1621	$k \equiv 20 \pmod{63}$	144542918285300809
$k \equiv 1 \pmod{45}$	927001	$k \equiv 74 \pmod{126}$	127
$k \equiv 26 \pmod{27}$	109	$k \equiv 11 \pmod{126}$	883
$k \equiv 17 \pmod{27}$	433	$k \equiv 65 \pmod{126}$	2521
$k \equiv 8 \pmod{27}$	8209	$k \equiv 2 \pmod{126}$	550554229

For the case $x = 1$ in Table B.4, the least common multiple of the moduli for the congruences is 432. The least common multiple of the moduli for the 34 congruences found in Table B.5 is 45360. A direct analysis can be done to verify these coverings.

B.3 Covering systems for $b = 4$

For base 4, we let $b = 4$ and $a = 1$. That is, we consider

$$N = \frac{4^n - 1}{3} + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = \frac{4^n - 1}{3} + M + x \cdot 4^k \quad \text{and} \quad N_1^{(k)}(d) = 4^n + \frac{4^n - 1}{3} + M + (d-1) \cdot 4^k.$$

We want coverings for $x, d \in \{0, 1, 2, 3\}$. Setting $M \equiv 0 \pmod 3$ and $n \equiv 0 \pmod 3$, we see that both $N_0^{(k)}(x)$ and $N_1^{(k)}(d)$ are divisible by 3 for $x, d \equiv 0 \pmod 3$. Thus, we only need to find coverings for $x, d \in \{1, 2\}$.

One checks that the congruences involving the primes 5 and 7 correspond to the congruence conditions $M \equiv 4 \pmod 5$ and $M \equiv 5 \pmod 7$, respectively. The largest of the least common multiples of the moduli for the coverings is 7560. A direct analysis on each covering can be used for verification. For $d = 2$, we use the covering found in Table B.9. To preserve space, denote $p_{25} = 469775495062434961$.

Table B.6 Covering for $x = 1$ in Base 4

row	congruence	prime p_i
1	$k \equiv 0 \pmod 2$	5
2	$k \equiv 1 \pmod 4$	17
3	$k \equiv 3 \pmod 8$	257

row	congruence	prime p_i
4	$k \equiv 7 \pmod{16}$	65537
5	$k \equiv 15 \pmod{32}$	641
6	$k \equiv 31 \pmod{32}$	6700417

Table B.7 Covering for $x = 2$ in Base 4

row	congruence	prime p_i
1	$k \equiv 0 \pmod 3$	7
2	$k \equiv 1 \pmod 6$	13
3	$k \equiv 4 \pmod{12}$	241
4	$k \equiv 10 \pmod{24}$	97

row	congruence	prime p_i
5	$k \equiv 22 \pmod{24}$	673
6	$k \equiv 2 \pmod 9$	19
7	$k \equiv 5 \pmod 9$	73
8	$k \equiv 8 \pmod{18}$	37
9	$k \equiv 17 \pmod{18}$	109

Table B.8 Covering for $d = 1$ in Base 4

row	congruence	prime p_i
a	$k \equiv 0 \pmod{2}$	5
1	$k \equiv 4 \pmod{5}$	11
2	$k \equiv 2 \pmod{5}$	31
3	$k \equiv 5 \pmod{10}$	41
4	$k \equiv 13 \pmod{15}$	151
5	$k \equiv 8 \pmod{15}$	331

row	congruence	prime p_i
6	$k \equiv 18 \pmod{30}$	61
7	$k \equiv 3 \pmod{30}$	1321
8	$k \equiv 11 \pmod{20}$	61681
9	$k \equiv 21 \pmod{40}$	4278255361
10	$k \equiv 41 \pmod{80}$	414721
11	$k \equiv 1 \pmod{80}$	44479210368001

Table B.9 Covering for $d = 2$ in Base 4

row	congruence	prime p_i
a	$k \equiv 0 \pmod{3}$	7
1	$k \equiv 40 \pmod{42}$	1429
2	$k \equiv 34 \pmod{42}$	14449
3	$k \equiv 70 \pmod{84}$	3361
4	$k \equiv 28 \pmod{84}$	88959882481
5	$k \equiv 106 \pmod{126}$	40388473189
6	$k \equiv 64 \pmod{126}$	118750098349
7	$k \equiv 148 \pmod{252}$	1009
8	$k \equiv 22 \pmod{252}$	21169
9	$k \equiv 2 \pmod{14}$	29

B.4 Covering systems for $b = 5$

For base 5, let $b = 5$ and $a = 3$. That is, we consider

$$N = 3 \cdot \frac{5^n - 1}{4} + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = 3 \cdot \frac{5^n - 1}{4} + M + x \cdot 5^k \quad \text{and} \quad N_3^{(k)}(d) = 3 \cdot 5^n + 3 \cdot \frac{5^n - 1}{4} + M + (d - 3) \cdot 5^k.$$

We want coverings for $x, d \in \{0, 1, 2, 3, 4\}$. Setting $M \equiv 0 \pmod{3}$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod{3}$ and $N_3^{(k)}(d) \equiv 0 \pmod{3}$ for $x, d \equiv 0 \pmod{3}$. Thus, we only need to find coverings for $x, d \in \{1, 2, 4\}$. Moreover, setting $M \equiv 0 \pmod{2}$ and $n \equiv 0 \pmod{2}$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod{2}$ and $N_3^{(k)}(d) \equiv 0 \pmod{2}$ for $x, d \equiv 0 \pmod{2}$. So we need only find coverings for $x = 1$ and $d = 1$.

Table B.9 cont. Covering for $d = 2$ in Base 4

row	congruence	prime p_i
10	$k \equiv 10 \pmod{14}$	113
11	$k \equiv 18 \pmod{28}$	15790321
12	$k \equiv 32 \pmod{56}$	5153
13	$k \equiv 4 \pmod{56}$	54410972897
14	$k \equiv 26 \pmod{27}$	87211
15	$k \equiv 17 \pmod{27}$	262657
16	$k \equiv 35 \pmod{54}$	246241
17	$k \equiv 8 \pmod{54}$	279073
18	$k \equiv 41 \pmod{45}$	631
19	$k \equiv 32 \pmod{45}$	23311
20	$k \equiv 23 \pmod{45}$	18837001
21	$k \equiv 59 \pmod{90}$	181
22	$k \equiv 14 \pmod{90}$	54001
23	$k \equiv 50 \pmod{90}$	29247661
24	$k \equiv 95 \pmod{180}$	168692292721
25	$k \equiv 5 \pmod{180}$	469775495062434961
26	$k \equiv 56 \pmod{63}$	92737
27	$k \equiv 47 \pmod{63}$	649657
28	$k \equiv 38 \pmod{63}$	77158673929
29	$k \equiv 1 \pmod{7}$	43
30	$k \equiv 6 \pmod{7}$	127
31	$k \equiv 11 \pmod{21}$	337
32	$k \equiv 2 \pmod{21}$	5419

row	congruence	prime p_i
33	$k \equiv 31 \pmod{36}$	433
34	$k \equiv 25 \pmod{36}$	38737
35	$k \equiv 55 \pmod{72}$	577
36	$k \equiv 19 \pmod{72}$	487824887233
37	$k \equiv 85 \pmod{108}$	33975937
38	$k \equiv 49 \pmod{108}$	138991501037953
39	$k \equiv 121 \pmod{216}$	209924353
40	$k \equiv 13 \pmod{216}$	4261383649
41	$k \equiv 115 \pmod{144}$	1153
42	$k \equiv 79 \pmod{144}$	6337
43	$k \equiv 43 \pmod{144}$	38941695937
44	$k \equiv 7 \pmod{144}$	278452876033
45	$k \equiv 289 \pmod{324}$	1297
46	$k \equiv 253 \pmod{324}$	3889
47	$k \equiv 217 \pmod{324}$	30433969
48	$k \equiv 181 \pmod{324}$	1164777409
49	$k \equiv 145 \pmod{324}$	3718266498433
50	$k \equiv 109 \pmod{324}$	134921168163073
51	$k \equiv 73 \pmod{324}$	1174029487714513
52	$k \equiv 361 \pmod{648}$	10369
53	$k \equiv 37 \pmod{648}$	259201
54	$k \equiv 325 \pmod{648}$	1824045366145909775041
55	$k \equiv 1 \pmod{648}$	524833094849624730914401

One checks that the congruences involving the prime 7 correspond to the congruence condition $M \equiv 1 \pmod 7$. The least common multiple of the moduli in the covering found in Table B.10 is 1800 and the least common multiple of the module found in Table B.11 is 288. A direct analysis on each covering can be used for verification.

Table B.10 Covering for $x = 1$ in Base 5

congruence	prime p_i
$k \equiv 0 \pmod 5$	11
$k \equiv 1 \pmod 5$	71
$k \equiv 3 \pmod 6$	7
$k \equiv 2 \pmod{15}$	181
$k \equiv 7 \pmod{15}$	1741
$k \equiv 4 \pmod{10}$	521
$k \equiv 8 \pmod{20}$	41
$k \equiv 18 \pmod{20}$	9161
$k \equiv 12 \pmod{30}$	61
$k \equiv 13 \pmod{30}$	7621
$k \equiv 19 \pmod{60}$	2281
$k \equiv 49 \pmod{60}$	69566521

congruence	prime p_i
$k \equiv 23 \pmod{150}$	118801
$k \equiv 53 \pmod{150}$	20775901
$k \equiv 83 \pmod{150}$	24665701
$k \equiv 113 \pmod{150}$	149439601
$k \equiv 143 \pmod{300}$	14401
$k \equiv 293 \pmod{300}$	299541552154912341601
$k \equiv 29 \pmod{90}$	60081451169922001
$k \equiv 59 \pmod{180}$	20478961
$k \equiv 149 \pmod{180}$	6794091374761
$k \equiv 89 \pmod{180}$	25535754811081
$k \equiv 179 \pmod{360}$	8641
$k \equiv 359 \pmod{360}$	440641

Table B.11 Covering for $d = 1$ in Base 5

row	congruence	prime p_i
a	$k \equiv 4 \pmod 6$	7
1	$k \equiv 2 \pmod 3$	31
2	$k \equiv 7 \pmod{12}$	601
3	$k \equiv 13 \pmod{24}$	390001
4	$k \equiv 25 \pmod{48}$	152587– 500001
5	$k \equiv 49 \pmod{96}$	97

row	congruence	prime p_i
6	$k \equiv 1 \pmod{96}$	2400315913– 94168814433
7	$k \equiv 6 \pmod 9$	19
8	$k \equiv 3 \pmod 9$	829
9	$k \equiv 9 \pmod{18}$	5167
10	$k \equiv 18 \pmod{36}$	37
11	$k \equiv 0 \pmod{36}$	6597973

B.5 Covering systems for $b = 6$

For base 6, let $b = 6$ and $a = 1$. That is, we consider

$$N = \frac{6^n - 1}{5} + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = \frac{6^n - 1}{5} + M + x \cdot 6^k \quad \text{and} \quad N_1^{(k)}(d) = 6^n + \frac{6^n - 1}{5} + M + (d-1) \cdot 6^k.$$

We want coverings for $x, d \in \{0, 1, 2, 3, 4, 5\}$. Setting $M \equiv 0 \pmod 5$ and $n \equiv 0 \pmod 5$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod 5$ and $N_1^{(k)}(d) \equiv 0 \pmod 5$ for $x, d \equiv 0 \pmod 5$. Thus, we only need to find coverings for $x, d \in \{1, 2, 3, 4\}$.

The coverings used for $x \in \{1, 2, 3, 4\}$ can each be verified using a direct analysis. The largest of the least common multiples of the moduli for the four coverings is that used for $x = 4$, which has 75 congruences with the least common multiple of the moduli being 176400.

As this base has more complicated coverings, we recall some notation used in Appendix A. Recall that if we are unable to fully factor $\Phi_n(b)$, then we write $\Phi_n(b) = p_1 p_2 \cdots p_r C_n$, where C_n is a composite factor of $\Phi_n(b)$ having at least 2 distinct prime divisors different from $p_1, p_2, \ldots, p_r$ and n. If we are able to fully factor $\Phi_n(b)$, then we write $\Phi_n(b) = p_1 p_2 \cdots p_r P_n$, where P_n is a large prime factor of $\Phi_n(b)$ different from $p_1, p_2, \ldots, p_r$. Computationally, P_n is determined to be a prime power and then verified to be a prime. We denote P_n to be a probable prime too large to include comfortably where $\text{ord}_{P_n}(b) = n$. We also use $c_{n,1}$ to denote one prime factor from the composite number C_n where $\text{ord}_{c_{n,1}}(b) = n$ that we were unable to factor, and $c_{n,2}$ to denote the other prime factor from the same composite number. We did not compute the values of $c_{n,1}$ and $c_{n,2}$, but we know they exist.

When necessary, we provide a table of notable factorizations of $\Phi_n(b)$ used in the covering that follows. It can be assumed that the large primes P_n and composite numbers C_n that do not appear in a table or list of large primes can be determined by dividing $\Phi_n(b)$ by the primes associated with n that can be found in adjacent rows in the tables.

Table B.12 Covering for $x = 1$ in Base 6

congruence	prime p_i
0 (mod 5)	311
3 (mod 10)	11
8 (mod 10)	101
2 (mod 15)	1171
7 (mod 15)	1201
12 (mod 45)	2161
27 (mod 45)	112771
42 (mod 45)	19353635731
16 (mod 20)	241
11 (mod 20)	6781
1 (mod 40)	41
21 (mod 40)	68754507401
6 (mod 60)	61
26 (mod 60)	74161
46 (mod 60)	181
4 (mod 25)	18198701
9 (mod 25)	40185601
14 (mod 50)	3655688315536801
39 (mod 100)	343801
89 (mod 100)	22243201
19 (mod 75)	601
44 (mod 75)	82051
69 (mod 75)	271041511600591342728451
24 (mod 125)	9536585501
49 (mod 125)	117811792772681609501
74 (mod 125)	105875321588567599765751
99 (mod 125)	1098445767808750903973251
124 (mod 250)	251
249 (mod 250)	751

Table B.13 Covering for $x = 2$ in Base 6

row	congruence	prime p_i
1	$k \equiv 1 \pmod{2}$	7
2	$k \equiv 0 \pmod{4}$	37
3	$k \equiv 6 \pmod{8}$	1297
4	$k \equiv 10 \pmod{16}$	17
5	$k \equiv 2 \pmod{16}$	98801

Table B.14 Covering for $x = 3$ in Base 6

row	congruence	prime p_i
1	$k \equiv 4 \pmod{6}$	31
2	$k \equiv 2 \pmod{9}$	19
3	$k \equiv 5 \pmod{9}$	2467
4	$k \equiv 1 \pmod{12}$	13
5	$k \equiv 7 \pmod{12}$	97
6	$k \equiv 8 \pmod{18}$	46441
7	$k \equiv 17 \pmod{36}$	73
8	$k \equiv 35 \pmod{36}$	541
9	$k \equiv 0 \pmod{3}$	43

To conserve space we denote some of the larger primes used for the covering found in Table B.15 associated with the digit insertion $x = 4$.

$$p_{52} = 19100900655540830489319601,$$

$$p_{58} = 6151465354203683883830062906173218740292621286631,$$

$$p_{60} = 7215934956451622094949379164309254404855094763766767407121,$$

$$p_{66} = 10189440239012883075423169,$$

$$p_{68} = 802354924671793395266725600744864562917483072464133108417.$$

Table B.15 Covering for $x = 4$ in Base 6

row	congruence	prime p_i
1	$k \equiv 6 \pmod{7}$	55987
2	$k \equiv 9 \pmod{14}$	29
3	$k \equiv 2 \pmod{14}$	197
4	$k \equiv 18 \pmod{21}$	1822428931
5	$k \equiv 53 \pmod{63}$	379
6	$k \equiv 32 \pmod{63}$	8387947
7	$k \equiv 11 \pmod{63}$	616332907
8	$k \equiv 67 \pmod{84}$	804146449
9	$k \equiv 25 \pmod{84}$	6055984789
10	$k \equiv 88 \pmod{126}$	127
11	$k \equiv 46 \pmod{126}$	154260982009
12	$k \equiv 4 \pmod{126}$	528921402377887
13	$k \equiv 24 \pmod{28}$	421
14	$k \equiv 17 \pmod{28}$	5030761
15	$k \equiv 38 \pmod{56}$	281
16	$k \equiv 10 \pmod{56}$	337
17	$k \equiv 31 \pmod{56}$	617
18	$k \equiv 3 \pmod{56}$	81035189089
19	$k \equiv 33 \pmod{35}$	71
20	$k \equiv 26 \pmod{35}$	37863211
21	$k \equiv 19 \pmod{35}$	1469029031
22	$k \equiv 47 \pmod{70}$	631
23	$k \equiv 12 \pmod{70}$	701
24	$k \equiv 40 \pmod{70}$	2311
25	$k \equiv 5 \pmod{70}$	9241
26	$k \equiv 64 \pmod{70}$	585131

Table B.15 cont. Covering for $x = 4$ in Base 6

row	congruence	prime p_i
27	$k \equiv 99 \pmod{140}$	13509594555661
28	$k \equiv 29 \pmod{140}$	1708114263204222806519161
29	$k \equiv 92 \pmod{105}$	211
30	$k \equiv 57 \pmod{105}$	35281
31	$k \equiv 22 \pmod{105}$	58171
32	$k \equiv 85 \pmod{105}$	6192110479195032209415801l
33	$k \equiv 155 \pmod{210}$	71191
34	$k \equiv 50 \pmod{210}$	2716136029771530996493785866668l
35	$k \equiv 225 \pmod{315}$	423123121
36	$k \equiv 120 \pmod{315}$	1076836800079531
37	$k \equiv 15 \pmod{315}$	7272568060810890512170608l
38	$k \equiv 148 \pmod{175}$	3822701
39	$k \equiv 113 \pmod{175}$	114265201
40	$k \equiv 78 \pmod{175}$	141218351
41	$k \equiv 43 \pmod{175}$	7292423951
42	$k \equiv 8 \pmod{175}$	34840572551
43	$k \equiv 141 \pmod{175}$	35107498301
44	$k \equiv 106 \pmod{175}$	2370825139201
45	$k \equiv 71 \pmod{175}$	18308895187508844830498555551
46	$k \equiv 211 \pmod{350}$	3851
47	$k \equiv 36 \pmod{350}$	12601
48	$k \equiv 176 \pmod{350}$	1305192701
49	$k \equiv 1 \pmod{350}$	66631795301
50	$k \equiv 35 \pmod{42}$	2527867231
51	$k \equiv 147 \pmod{168}$	1176362433121

row	congruence	prime p_i
52	$k \equiv 63 \pmod{168}$	p_{52}
53	$k \equiv 189 \pmod{252}$	1009
54	$k \equiv 105 \pmod{252}$	17389
55	$k \equiv 21 \pmod{252}$	10339309
56	$k \equiv 259 \pmod{294}$	32670457
57	$k \equiv 217 \pmod{294}$	1152319183
58	$k \equiv 175 \pmod{294}$	p_{58}
59	$k \equiv 133 \pmod{147}$	32093041
60	$k \equiv 91 \pmod{147}$	p_{60}
61	$k \equiv 0 \pmod{49}$	6527977
62	$k \equiv 7 \pmod{49}$	122694573317
63	$k \equiv 98 \pmod{112}$	113
64	$k \equiv 70 \pmod{112}$	4817
65	$k \equiv 42 \pmod{112}$	4048129
66	$k \equiv 14 \pmod{112}$	p_{66}
67	$k \equiv 168 \pmod{196}$	288628033
68	$k \equiv 140 \pmod{196}$	p_{68}
69	$k \equiv 308 \pmod{392}$	352409
70	$k \equiv 112 \pmod{392}$	171103297
71	$k \equiv 280 \pmod{392}$	110495360641
72	$k \equiv 84 \pmod{392}$	P_{392}
73	$k \equiv 56 \pmod{98}$	762332681442053
74	$k \equiv 28 \pmod{98}$	9017961693638401l
75	$k \equiv 0 \pmod{49}$	600827908214213

For inserting a digit $d \in \{1,2,3,4\}$, we use the coverings that follow. For $d = 1$, we use the covering found in Table B.16. This covering has 101 congruences with the least common multiple of the moduli being 720720 and can be verified by a direct analysis. Row a contains the prime 29 which is used in the covering for $x = 4$. One checks that both congruences correspond to the congruence condition $M \equiv 28 \pmod{29}$. The following are larger primes that do not fit in the tables comfortably.

$$p_{24} = 570190871341512540616666217177286691672798621 51,$$

$$p_{39} = 22252087149176585501952276278615134188 79,$$

$$p_{71} = 1095110988527465328644161,$$

$$p_{72} = 1948613642347230963768492935356381,$$

$$p_{90} = 618089408508629562797739046557629942711588422796977.$$

Table B.16 Covering for $d = 1$ in Base 6

row	congruence	prime p_i
a	$k \equiv 4 \pmod{14}$	29
1	$k \equiv 9 \pmod{11}$	23
2	$k \equiv 7 \pmod{11}$	3154757
3	$k \equiv 16 \pmod{22}$	51828151
4	$k \equiv 25 \pmod{33}$	67
5	$k \equiv 14 \pmod{33}$	45686117391553
6	$k \equiv 69 \pmod{99}$	16633
7	$k \equiv 36 \pmod{99}$	18380539
8	$k \equiv 3 \pmod{99}$	18414001

Table B.16 cont. Covering for $d = 1$ in Base 6

row	congruence	prime p_i
9	$k \equiv 34 \pmod{44}$	58477
10	$k \equiv 12 \pmod{44}$	70489
11	$k \equiv 32 \pmod{44}$	863017
12	$k \equiv 54 \pmod{88}$	89
13	$k \equiv 10 \pmod{88}$	150080764792922988676714149209
14	$k \equiv 41 \pmod{55}$	3675127061
15	$k \equiv 19 \pmod{55}$	3031462959351050977391
16	$k \equiv 52 \pmod{110}$	1031141
17	$k \equiv 30 \pmod{110}$	16336066781
18	$k \equiv 8 \pmod{110}$	84155540944421
19	$k \equiv 50 \pmod{66}$	463
20	$k \equiv 28 \pmod{66}$	72073
21	$k \equiv 6 \pmod{66}$	127236649
22	$k \equiv 59 \pmod{77}$	484847574510970082567
23	$k \equiv 37 \pmod{77}$	84002092056248016278479853
24	$k \equiv 92 \pmod{154}$	p_{24}
25	$k \equiv 224 \pmod{308}$	40590776689
26	$k \equiv 70 \pmod{308}$	P_{308}
27	$k \equiv 202 \pmod{231}$	174090854323
28	$k \equiv 125 \pmod{231}$	1526323866435523
29	$k \equiv 48 \pmod{231}$	$c_{231,1}$
30	$k \equiv 488 \pmod{616}$	65725969
31	$k \equiv 334 \pmod{616}$	59649652128793520533433

row	congruence	prime p_i
32	$k \equiv 180 \pmod{616}$	P_{616}
33	$k \equiv 642 \pmod{1232}$	3015937
34	$k \equiv 26 \pmod{1232}$	70475637120255553017217
35	$k \equiv 79 \pmod{99}$	226407819331
36	$k \equiv 57 \pmod{99}$	38167293140100433
37	$k \equiv 134 \pmod{198}$	199
38	$k \equiv 112 \pmod{198}$	110881
39	$k \equiv 90 \pmod{198}$	p_{39}
40	$k \equiv 266 \pmod{396}$	397
41	$k \equiv 68 \pmod{396}$	7129
42	$k \equiv 244 \pmod{396}$	23761
43	$k \equiv 46 \pmod{396}$	13705107769
44	$k \equiv 222 \pmod{396}$	697344975757
45	$k \equiv 24 \pmod{396}$	1184960471459158549
46	$k \equiv 200 \pmod{396}$	254253859796489462713
47	$k \equiv 2 \pmod{396}$	12335988891985043955517
48	$k \equiv 110 \pmod{132}$	3037
49	$k \equiv 88 \pmod{132}$	96493
50	$k \equiv 66 \pmod{132}$	622513
51	$k \equiv 44 \pmod{132}$	4629769
52	$k \equiv 22 \pmod{132}$	16266405013
53	$k \equiv 132 \pmod{264}$	602977
54	$k \equiv 0 \pmod{264}$	87917281

Table B.16 cont. Covering for $d = 1$ in Base 6

row	congruence	prime p_i
55	$k \equiv 12 \pmod{13}$	3433
56	$k \equiv 10 \pmod{13}$	760891
57	$k \equiv 21 \pmod{26}$	53
58	$k \equiv 19 \pmod{26}$	937
59	$k \equiv 17 \pmod{26}$	37571
60	$k \equiv 28 \pmod{39}$	3143401
61	$k \equiv 2 \pmod{39}$	1262014275211
62	$k \equiv 15 \pmod{78}$	79
63	$k \equiv 65 \pmod{78}$	9049
64	$k \equiv 39 \pmod{78}$	868999
65	$k \equiv 13 \pmod{78}$	8857759
66	$k \equiv 37 \pmod{52}$	313
67	$k \equiv 11 \pmod{52}$	2341
68	$k \equiv 35 \pmod{52}$	6291946695217
69	$k \equiv 217 \pmod{260}$	425101
70	$k \equiv 165 \pmod{260}$	571140885901
71	$k \equiv 113 \pmod{260}$	p_{71}
72	$k \equiv 61 \pmod{260}$	p_{72}
73	$k \equiv 269 \pmod{520}$	380641
74	$k \equiv 9 \pmod{520}$	98735521
75	$k \equiv 46 \pmod{65}$	11831
76	$k \equiv 20 \pmod{65}$	1420901
77	$k \equiv 59 \pmod{65}$	49398961
78	$k \equiv 33 \pmod{65}$	22533649654910414281
79	$k \equiv 7 \pmod{130}$	131
80	$k \equiv 83 \pmod{104}$	192193
81	$k \equiv 57 \pmod{104}$	14090441
82	$k \equiv 31 \pmod{104}$	8284434950526240125727017
83	$k \equiv 109 \pmod{208}$	36265841
84	$k \equiv 5 \pmod{208}$	141365953
85	$k \equiv 133 \pmod{156}$	157
86	$k \equiv 107 \pmod{156}$	8893
87	$k \equiv 81 \pmod{156}$	197743936282933
88	$k \equiv 55 \pmod{156}$	6429178169720749
89	$k \equiv 185 \pmod{312}$	8162133795225884317 07233
90	$k \equiv 29 \pmod{312}$	p_{90}
91	$k \equiv 315 \pmod{468}$	21529
92	$k \equiv 159 \pmod{468}$	1044927469
93	$k \equiv 3 \pmod{468}$	5806612933322769708940 9
94	$k \equiv 66 \pmod{91}$	482159105638327986 97
95	$k \equiv 40 \pmod{91}$	183873846016889600127566887259284192 3
96	$k \equiv 105 \pmod{182}$	2003
97	$k \equiv 79 \pmod{182}$	2549
98	$k \equiv 53 \pmod{182}$	36947
99	$k \equiv 27 \pmod{182}$	52769627
100	$k \equiv 1 \pmod{182}$	12468702878009806771287543538567443817

For the insertion $d = 2$, we use the covering found in Table B.18. This covering has 224 congruences with the least common multiple of the moduli being 4727479680. One checks that the primes 37 and 13 correspond to the congruence conditions $M \equiv 35 \pmod{37}$ and $M \equiv 8 \pmod{13}$, which agree with the previous uses of these primes. To verify that this is indeed a covering, we consider the congruence classes $x \equiv 0 \pmod{2}$ and $x \equiv 1 \pmod{2}$. One can verify that every integer satisfying $x \equiv 0 \pmod{2}$ satisfies one of the congruences in rows a-121 of Table B.18. These congruences have moduli dividing 30898560. Similarly, one can check that every integer satisfying $x \equiv 1 \pmod{2}$ satisfies one of the 101 congruences in rows 122-222 of Table B.18 with moduli dividing 5654880. Thus, the 224 congruences in Table B.18 form a covering. Table B.17 lists notable factorizations of $\Phi_n(6)$ for large n used in the covering for $d = 2$. We list a few of the larger primes here.

$$p_6 = 23471108401585638160281863182465 61,$$

$$p_{21} = 2683619343579360102399832385 2801,$$

$$p_{115} = 151866410185957186292900 23681,$$

$$p_{116} = 357800878806910504804240000283796111411697784953797 3223,$$

$$p_{148} = 942601233869792838725476754 18641,$$

$$p_{155} = 514205763748113814036094 45473,$$

$$p_{172} = 678389915020055546314233943801.$$

Table B.17 Partial/Full factorizations of $\Phi_n(6)$ for $d = 2$

n	Factorization of $\Phi_n(10)$
272	$17 \cdot 2376897969771200667937073 \cdot P_{272}$
352	C_{352}
1064	C_{1064}

Table B.18 Covering for $d = 2$ in Base 6

row	congruence	prime p_i
a	$k \equiv 0 \pmod{4}$	37
b	$k \equiv 10 \pmod{12}$	13
1	$k \equiv 78 \pmod{80}$	17761
2	$k \equiv 62 \pmod{80}$	3696985841
3	$k \equiv 46 \pmod{80}$	121206120881
4	$k \equiv 110 \pmod{160}$	9601
5	$k \equiv 30 \pmod{160}$	2810800069601
6	$k \equiv 94 \pmod{160}$	p_6
7	$k \equiv 174 \pmod{320}$	82241
8	$k \equiv 14 \pmod{320}$	15954097282309262360999041
9	$k \equiv 13 \pmod{19}$	191
10	$k \equiv 5 \pmod{19}$	638073026189
11	$k \equiv 16 \pmod{38}$	1787
12	$k \equiv 8 \pmod{38}$	48713705333
13	$k \equiv 30 \pmod{76}$	1030762781149
14	$k \equiv 38 \pmod{76}$	9736145643041809
15	$k \equiv 98 \pmod{152}$	4561
16	$k \equiv 90 \pmod{152}$	355518408146401
17	$k \equiv 82 \pmod{152}$	5028187486478069273
18	$k \equiv 74 \pmod{152}$	13038313680704041577
19	$k \equiv 218 \pmod{304}$	417827633
20	$k \equiv 66 \pmod{304}$	505981859041
21	$k \equiv 210 \pmod{304}$	p_{21}
22	$k \equiv 58 \pmod{304}$	P_{304}
23	$k \equiv 506 \pmod{608}$	440123297
24	$k \equiv 354 \pmod{608}$	$c_{608,1}$
25	$k \equiv 202 \pmod{608}$	$c_{608,2}$
26	$k \equiv 658 \pmod{1216}$	1217
27	$k \equiv 50 \pmod{1216}$	$c_{1216,1}$
28	$k \equiv 650 \pmod{760}$	761
29	$k \equiv 498 \pmod{760}$	689321
30	$k \equiv 346 \pmod{760}$	3939048167443707601
31	$k \equiv 194 \pmod{760}$	$c_{760,1}$
32	$k \equiv 42 \pmod{760}$	$c_{760,2}$
33	$k \equiv 15 \pmod{133}$	11971
34	$k \equiv 129 \pmod{133}$	188861
35	$k \equiv 110 \pmod{133}$	1101773
36	$k \equiv 91 \pmod{133}$	164331230229374083
37	$k \equiv 72 \pmod{133}$	P_{133}
38	$k \equiv 186 \pmod{266}$	1050169
39	$k \equiv 34 \pmod{266}$	2783850843587543905 1
40	$k \equiv 140 \pmod{266}$	6110613202111672866319
41	$k \equiv 254 \pmod{266}$	P_{266}
42	$k \equiv 102 \pmod{532}$	1597

Table B.18 cont. Covering for $d = 2$ in Base 6

row	congruence	prime p_i
43	$k \equiv 482 \pmod{532}$	28729
44	$k \equiv 330 \pmod{532}$	181514505515281
45	$k \equiv 178 \pmod{532}$	19900271591604097
46	$k \equiv 26 \pmod{532}$	74795600265930240481
47	$k \equiv 398 \pmod{532}$	$c_{532,1}$
48	$k \equiv 246 \pmod{532}$	$c_{532,2}$
49	$k \equiv 626 \pmod{1064}$	$c_{1064,1}$
50	$k \equiv 474 \pmod{1064}$	$c_{1064,2}$
51	$k \equiv 322 \pmod{399}$	109928491
52	$k \equiv 189 \pmod{399}$	$c_{399,1}$
53	$k \equiv 56 \pmod{399}$	$c_{399,2}$
54	$k \equiv 702 \pmod{798}$	9951571519
55	$k \equiv 436 \pmod{798}$	1268637449760199
56	$k \equiv 170 \pmod{798}$	$c_{798,1}$
57	$k \equiv 1082 \pmod{1596}$	10453801
58	$k \equiv 550 \pmod{1596}$	26846778030654037
59	$k \equiv 18 \pmod{1596}$	$c_{1596,1}$
60	$k \equiv 770 \pmod{912}$	7297
61	$k \equiv 618 \pmod{912}$	$c_{912,1}$
62	$k \equiv 466 \pmod{912}$	$c_{912,2}$
63	$k \equiv 1226 \pmod{1824}$	14593
64	$k \equiv 314 \pmod{1824}$	378866839993921
65	$k \equiv 1074 \pmod{1824}$	4947871864850689
66	$k \equiv 162 \pmod{1824}$	$c_{1824,1}$
67	$k \equiv 59 \pmod{209}$	18873119
68	$k \equiv 116 \pmod{209}$	84125065520563
69	$k \equiv 173 \pmod{209}$	414230477
70	$k \equiv 230 \pmod{418}$	419
71	$k \equiv 78 \pmod{418}$	87713957
72	$k \equiv 762 \pmod{836}$	178069
73	$k \equiv 610 \pmod{836}$	93714790877401
74	$k \equiv 458 \pmod{1672}$	41801
75	$k \equiv 306 \pmod{1672}$	45597113
76	$k \equiv 154 \pmod{1672}$	8627680513
77	$k \equiv 2 \pmod{1672}$	53686006051666649
78	$k \equiv 166 \pmod{176}$	9307950433
79	$k \equiv 150 \pmod{176}$	P_{176}
80	$k \equiv 310 \pmod{352}$	$c_{352,1}$
81	$k \equiv 134 \pmod{352}$	$c_{352,2}$
82	$k \equiv 470 \pmod{528}$	3169
83	$k \equiv 294 \pmod{528}$	2508001
84	$k \equiv 118 \pmod{528}$	3537601
85	$k \equiv 454 \pmod{528}$	8245249
86	$k \equiv 278 \pmod{528}$	$c_{528,1}$

Table B.18 cont. Covering for $d = 2$ in Base 6

row	congruence	prime p_i
87	$k \equiv 102 \pmod{528}$	$c_{528,2}$
88	$k \equiv 614 \pmod{704}$	978494460213024200513
89	$k \equiv 438 \pmod{704}$	$c_{704,1}$
90	$k \equiv 262 \pmod{704}$	$c_{704,2}$
91	$k \equiv 790 \pmod{1408}$	2889217
92	$k \equiv 86 \pmod{1408}$	1409
93	$k \equiv 246 \pmod{264}$	1581003091009
94	$k \equiv 158 \pmod{264}$	332526664667473
95	$k \equiv 70 \pmod{264}$	6416538652864923578689
96	$k \equiv 758 \pmod{880}$	881
97	$k \equiv 582 \pmod{880}$	21121
98	$k \equiv 406 \pmod{880}$	975773245921
99	$k \equiv 230 \pmod{880}$	18444967576384646881
100	$k \equiv 54 \pmod{880}$	$c_{880,1}$
101	$k \equiv 742 \pmod{880}$	$c_{880,2}$
102	$k \equiv 126 \pmod{440}$	8247361
103	$k \equiv 390 \pmod{440}$	778055521
104	$k \equiv 214 \pmod{440}$	254055515561
105	$k \equiv 38 \pmod{440}$	20695248084558521
106	$k \equiv 286 \pmod{440}$	518060072335148988041
107	$k \equiv 110 \pmod{440}$	6560970004839878965676927281
108	$k \equiv 374 \pmod{440}$	278651896702502332482621071375321

row	congruence	prime p_i
109	$k \equiv 198 \pmod{220}$	423852369601
110	$k \equiv 22 \pmod{220}$	P_{220}
111	$k \equiv 72 \pmod{121}$	4163261521
112	$k \equiv 17 \pmod{121}$	1304380464883
113	$k \equiv 83 \pmod{121}$	P_{121}
114	$k \equiv 28 \pmod{242}$	727
115	$k \equiv 94 \pmod{242}$	p_{115}
116	$k \equiv 160 \pmod{242}$	p_{116}
117	$k \equiv 226 \pmod{484}$	1453
118	$k \equiv 534 \pmod{968}$	68729
119	$k \equiv 358 \pmod{968}$	2095721
120	$k \equiv 182 \pmod{968}$	1771133748439529
121	$k \equiv 6 \pmod{968}$	4802007937
122	$k \equiv 16 \pmod{17}$	239
123	$k \equiv 14 \pmod{17}$	409
124	$k \equiv 12 \pmod{17}$	1123
125	$k \equiv 10 \pmod{17}$	30839
126	$k \equiv 25 \pmod{34}$	190537
127	$k \equiv 23 \pmod{34}$	12690943
128	$k \equiv 89 \pmod{102}$	103
129	$k \equiv 55 \pmod{102}$	919
130	$k \equiv 21 \pmod{102}$	980146969

Table B.18 cont. Covering for $d = 2$ in Base 6

row	congruence	prime p_i
131	$k \equiv 87 \pmod{102}$	99617785207
132	$k \equiv 2 \pmod{51}$	307
133	$k \equiv 19 \pmod{51}$	927037099
134	$k \equiv 34 \pmod{51}$	23412002806867
135	$k \equiv 119 \pmod{153}$	9757142011
136	$k \equiv 68 \pmod{153}$	12644443230579801886843521019
137	$k \equiv 17 \pmod{153}$	P_{153}
138	$k \equiv 34 \pmod{85}$	26807981
139	$k \equiv 153 \pmod{255}$	1199699502336395787186091
140	$k \equiv 102 \pmod{255}$	P_{255}
141	$k \equiv 306 \pmod{510}$	47431
142	$k \equiv 51 \pmod{510}$	241317924973591
143	$k \equiv 255 \pmod{510}$	$c_{510,1}$
144	$k \equiv 0 \pmod{510}$	$c_{510,2}$
145	$k \equiv 49 \pmod{68}$	934117
146	$k \equiv 15 \pmod{68}$	8289713345361373993
147	$k \equiv 64 \pmod{85}$	20891158391
148	$k \equiv 30 \pmod{85}$	p_{148}
149	$k \equiv 81 \pmod{170}$	1383638161
150	$k \equiv 47 \pmod{170}$	72085651321561
151	$k \equiv 13 \pmod{170}$	740797672014674927371043791
152	$k \equiv 113 \pmod{136}$	137
153	$k \equiv 79 \pmod{136}$	5849
154	$k \equiv 45 \pmod{136}$	1536052010629489
155	$k \equiv 11 \pmod{136}$	p_{155}
156	$k \equiv 179 \pmod{204}$	1785001
157	$k \equiv 145 \pmod{204}$	12346985995648844989
158	$k \equiv 111 \pmod{204}$	2953728137900959095135649
159	$k \equiv 281 \pmod{408}$	8715697
160	$k \equiv 77 \pmod{408}$	35540881
161	$k \equiv 247 \pmod{408}$	P_{408}
162	$k \equiv 451 \pmod{816}$	6529
163	$k \equiv 43 \pmod{816}$	70177
164	$k \equiv 621 \pmod{816}$	2823361
165	$k \equiv 417 \pmod{816}$	380776609
166	$k \equiv 213 \pmod{816}$	158141084859073
167	$k \equiv 9 \pmod{816}$	P_{816}
168	$k \equiv 92 \pmod{119}$	16661
169	$k \equiv 58 \pmod{119}$	29032062767
170	$k \equiv 24 \pmod{119}$	103198889691409
171	$k \equiv 109 \pmod{119}$	12405291558509977
172	$k \equiv 75 \pmod{119}$	p_{172}
173	$k \equiv 41 \pmod{238}$	22885730141
174	$k \equiv 7 \pmod{238}$	2681104713967
175	$k \equiv 243 \pmod{272}$	2376897969771200667937073
176	$k \equiv 175 \pmod{272}$	P_{272}

Table B.18 cont. Covering for $d = 2$ in Base 6

row	congruence	prime p_i
177	$k \equiv 379 \pmod{544}$	257857
178	$k \equiv 107 \pmod{544}$	80189953
179	$k \equiv 311 \pmod{544}$	350828321
180	$k \equiv 39 \pmod{544}$	P_{544}
181	$k \equiv 277 \pmod{340}$	1021
182	$k \equiv 209 \pmod{340}$	372810476994982432801
183	$k \equiv 141 \pmod{340}$	22448072997003469050001
184	$k \equiv 73 \pmod{340}$	41819674674441587529015061
185	$k \equiv 5 \pmod{340}$	115394656437025419824224817341
186	$k \equiv 275 \pmod{306}$	147799
187	$k \equiv 241 \pmod{306}$	29274753335383
188	$k \equiv 207 \pmod{306}$	1888411753890127
189	$k \equiv 173 \pmod{306}$	P_{306}
190	$k \equiv 445 \pmod{612}$	613
191	$k \equiv 139 \pmod{612}$	6121
192	$k \equiv 411 \pmod{612}$	12241
193	$k \equiv 105 \pmod{612}$	57372656104261
194	$k \equiv 377 \pmod{612}$	$c_{612,1}$
195	$k \equiv 71 \pmod{612}$	$c_{612,2}$
196	$k \equiv 955 \pmod{1224}$	847009
197	$k \equiv 649 \pmod{1224}$	247245516556366227393
198	$k \equiv 343 \pmod{1224}$	$c_{1224,1}$
199	$k \equiv 37 \pmod{1224}$	$c_{1224,2}$

row	congruence	prime p_i
200	$k \equiv 615 \pmod{918}$	4591
201	$k \equiv 309 \pmod{918}$	3673
202	$k \equiv 3 \pmod{918}$	$c_{918,1}$
203	$k \equiv 154 \pmod{187}$	382229
204	$k \equiv 120 \pmod{187}$	4208612395681
205	$k \equiv 86 \pmod{187}$	$c_{187,1}$
206	$k \equiv 52 \pmod{187}$	$c_{187,2}$
207	$k \equiv 205 \pmod{374}$	480615433
208	$k \equiv 171 \pmod{374}$	P_{374}
209	$k \equiv 511 \pmod{748}$	6733
210	$k \equiv 137 \pmod{748}$	6773155698277
211	$k \equiv 477 \pmod{748}$	2501850879498651781
212	$k \equiv 103 \pmod{748}$	$c_{748,1}$
213	$k \equiv 817 \pmod{1122}$	465631
214	$k \equiv 443 \pmod{1122}$	2148631
215	$k \equiv 69 \pmod{1122}$	52249945639
216	$k \equiv 783 \pmod{1122}$	1397005264183
217	$k \equiv 409 \pmod{1122}$	$c_{1122,1}$
218	$k \equiv 35 \pmod{1122}$	$c_{1122,2}$
219	$k \equiv 1123 \pmod{1496}$	1319465750666449
220	$k \equiv 749 \pmod{1496}$	202506717689
221	$k \equiv 375 \pmod{1496}$	$c_{1496,1}$
222	$k \equiv 1 \pmod{1496}$	$c_{1496,2}$

For $d = 3$, we use the covering found in Table B.20. This covering has 92 congruences with the least common multiple of the moduli being 9028800. One checks that the primes 43, 31, 13, and 61 correspond to the congruence conditions $M \equiv 40 \pmod{43}$, $M \equiv 18 \pmod{31}$, $M \equiv 8 \pmod{13}$, and $M \equiv 9 \pmod{61}$, which agree with the previous uses of these primes. For $d = 4$, we use the covering found in Table B.21 . This covering has 87 congruences with the least common multiple of the moduli being 98167680. One checks that the primes 7, 17, and 13 correspond to the congruence conditions $M \equiv 2 \pmod{7}$, $M \equiv 4 \pmod{17}$, and $M \equiv 8 \pmod{13}$, which agree with the previous uses of these primes. A direct analysis on each can be used to verify that they are indeed coverings. Table B.19 exhibits notable factorizations of $\Phi_n(6)$ for large n used in both the coverings for $d = 3$ and $d = 4$. To make the tables fit nicely, we list some of the larger primes used for both the coverings.

$$p_8 = 22452257707354557235348829785471057921,$$

$$p_{32} = 4458139843615061052376640769667 21,$$

$$p_{51} = 4697579356329885130202590555110526850 9881,$$

$$p_{58} = 12363858534320578896678437 39281,$$

$$p_{67} = 4075260804919014932181635118615 4151.$$

Table B.19 Partial/Full factorizations of $\Phi_n(6)$ for $d = 3$ and $d = 4$

n	Factorization of $\Phi_n(10)$
171	$19 \cdot C_{171}$
360	P_{360}
380	$300961 \cdot 21905019726901 \cdot 347101070179037778781 \cdot P_{380}$
552	C_{552}

Table B.20 Covering for $d = 3$ in Base 6

row	congruence	prime p_i
a	$k \equiv 0 \pmod{3}$	43
b	$k \equiv 1 \pmod{6}$	31
c	$k \equiv 5 \pmod{12}$	13
d	$k \equiv 16 \pmod{60}$	61
1	$k \equiv 23 \pmod{24}$	1678321
2	$k \equiv 35 \pmod{48}$	5953
3	$k \equiv 11 \pmod{48}$	473896897
4	$k \equiv 32 \pmod{36}$	55117
5	$k \equiv 56 \pmod{72}$	577
6	$k \equiv 20 \pmod{72}$	3313
7	$k \equiv 44 \pmod{72}$	2478750186961
8	$k \equiv 80 \pmod{144}$	p_8
9	$k \equiv 152 \pmod{288}$	115777
10	$k \equiv 8 \pmod{288}$	31057921
11	$k \equiv 98 \pmod{108}$	109
12	$k \equiv 62 \pmod{108}$	591841
13	$k \equiv 26 \pmod{108}$	171467713
14	$k \equiv 86 \pmod{108}$	932461936453
15	$k \equiv 158 \pmod{216}$	115963921
16	$k \equiv 50 \pmod{216}$	433
17	$k \equiv 122 \pmod{216}$	8781208996949976153601
18	$k \equiv 14 \pmod{216}$	241282001155985351966017
19	$k \equiv 146 \pmod{180}$	9001
20	$k \equiv 110 \pmod{180}$	211501
21	$k \equiv 74 \pmod{180}$	2106930961
22	$k \equiv 38 \pmod{180}$	559778011272683406 1
23	$k \equiv 362 \pmod{540}$	4861
24	$k \equiv 182 \pmod{540}$	39326041
25	$k \equiv 2 \pmod{540}$	51353541541
26	$k \equiv 106 \pmod{120}$	13441
27	$k \equiv 46 \pmod{120}$	5925751096274000042641
28	$k \equiv 28 \pmod{30}$	1950271
29	$k \equiv 82 \pmod{240}$	55201
30	$k \equiv 202 \pmod{240}$	122401
31	$k \equiv 172 \pmod{240}$	21027841
32	$k \equiv 142 \pmod{240}$	p_{32}
33	$k \equiv 352 \pmod{360}$	P_{360}
34	$k \equiv 592 \pmod{720}$	13599361
35	$k \equiv 232 \pmod{720}$	6776234506081
36	$k \equiv 472 \pmod{720}$	$c_{720,1}$
37	$k \equiv 112 \pmod{720}$	$c_{720,2}$
38	$k \equiv 772 \pmod{960}$	31313147521
39	$k \equiv 532 \pmod{960}$	244465222594290613929804801
40	$k \equiv 292 \pmod{960}$	$c_{960,1}$
41	$k \equiv 52 \pmod{960}$	$c_{960,2}$
42	$k \equiv 982 \pmod{1200}$	62401

Table B.20 cont. Covering for $d = 3$ in Base 6

row	congruence	prime p_i
43	$k \equiv 742 \pmod{1200}$	295201
44	$k \equiv 502 \pmod{1200}$	78180664801
45	$k \equiv 262 \pmod{1200}$	$c_{1200,1}$
46	$k \equiv 22 \pmod{1200}$	$c_{1200,2}$
47	$k \equiv 310 \pmod{330}$	661982172984001
48	$k \equiv 280 \pmod{330}$	p_{48}
49	$k \equiv 85 \pmod{165}$	331
50	$k \equiv 55 \pmod{165}$	13724862774476458441
51	$k \equiv 25 \pmod{165}$	p_{51}
52	$k \equiv 490 \pmod{660}$	661
53	$k \equiv 160 \pmod{660}$	1321
54	$k \equiv 460 \pmod{660}$	144541
55	$k \equiv 130 \pmod{660}$	110221
56	$k \equiv 430 \pmod{660}$	188074921
57	$k \equiv 100 \pmod{660}$	14966414761
58	$k \equiv 400 \pmod{660}$	$c_{660,1}$
59	$k \equiv 70 \pmod{660}$	$c_{660,2}$
60	$k \equiv 370 \pmod{495}$	2971
61	$k \equiv 205 \pmod{495}$	59941224796217491
62	$k \equiv 40 \pmod{495}$	P_{495}
63	$k \equiv 1000 \pmod{1320}$	4812721
64	$k \equiv 670 \pmod{1320}$	2573249895018001
65	$k \equiv 340 \pmod{1320}$	$c_{1320,1}$

row	congruence	prime p_i
66	$k \equiv 10 \pmod{1320}$	$c_{1320,2}$
67	$k \equiv 69 \pmod{95}$	571
68	$k \equiv 39 \pmod{95}$	1901
69	$k \equiv 9 \pmod{95}$	825838991
70	$k \equiv 74 \pmod{95}$	1298704628041
71	$k \equiv 44 \pmod{95}$	25425408247171
72	$k \equiv 14 \pmod{95}$	2995523312517361
73	$k \equiv 174 \pmod{190}$	6271
74	$k \equiv 144 \pmod{190}$	2666738161
75	$k \equiv 114 \pmod{190}$	37260330001
76	$k \equiv 84 \pmod{190}$	61671024221
77	$k \equiv 54 \pmod{190}$	3229504809106404383981
78	$k \equiv 214 \pmod{285}$	47562842881
79	$k \equiv 184 \pmod{285}$	270596289241
80	$k \equiv 154 \pmod{285}$	209547179344816511252551051
81	$k \equiv 124 \pmod{285}$	P_{285}
82	$k \equiv 94 \pmod{570}$	40844746349999464436020322521
83	$k \equiv 64 \pmod{570}$	P_{570}
84	$k \equiv 34 \pmod{380}$	300961
85	$k \equiv 224 \pmod{380}$	21905019726901
86	$k \equiv 1144 \pmod{1710}$	61561
87	$k \equiv 574 \pmod{1710}$	3062755351
88	$k \equiv 4 \pmod{1710}$	2410655182800230692840951

Table B.21 Covering for $d = 4$ in Base 6

row	congruence	prime p_i
a	$k \equiv 1 \pmod{2}$	7
b	$k \equiv 4 \pmod{16}$	17
c	$k \equiv 2 \pmod{12}$	13
1	$k \equiv 28 \pmod{32}$	353
2	$k \equiv 12 \pmod{32}$	1697
3	$k \equiv 24 \pmod{32}$	4709377
4	$k \equiv 72 \pmod{96}$	193
5	$k \equiv 40 \pmod{96}$	8641
6	$k \equiv 8 \pmod{96}$	688490113
7	$k \equiv 48 \pmod{64}$	2753
8	$k \equiv 32 \pmod{64}$	145601
9	$k \equiv 16 \pmod{64}$	19854979505843329
10	$k \equiv 64 \pmod{128}$	4926056449
11	$k \equiv 0 \pmod{128}$	447183309836853377
12	$k \equiv 55 \pmod{57}$	47881
13	$k \equiv 43 \pmod{57}$	820459
14	$k \equiv 31 \pmod{57}$	219815829325921729
15	$k \equiv 76 \pmod{114}$	457
16	$k \equiv 64 \pmod{114}$	137713
17	$k \equiv 52 \pmod{114}$	190324492938225748951
18	$k \equiv 154 \pmod{171}$	$c_{171,1}$

row	congruence	prime p_i
19	$k \equiv 97 \pmod{171}$	$c_{171,2}$
20	$k \equiv 382 \pmod{513}$	432482774993234980723
21	$k \equiv 211 \pmod{513}$	697343318106716202287329
22	$k \equiv 40 \pmod{513}$	$c_{513,1}$
23	$k \equiv 142 \pmod{228}$	229
24	$k \equiv 130 \pmod{228}$	25309
25	$k \equiv 118 \pmod{228}$	43321
26	$k \equiv 106 \pmod{228}$	2197693
27	$k \equiv 94 \pmod{228}$	300537265358917
28	$k \equiv 82 \pmod{228}$	659334264781883756771821
29	$k \equiv 298 \pmod{456}$	8502577
30	$k \equiv 70 \pmod{456}$	$c_{456,1}$
31	$k \equiv 286 \pmod{456}$	$c_{456,2}$
32	$k \equiv 970 \pmod{1368}$	12896362376641
33	$k \equiv 514 \pmod{1368}$	48893768514888932017
34	$k \equiv 58 \pmod{1368}$	$c_{1368,1}$
35	$k \equiv 502 \pmod{684}$	60204997
36	$k \equiv 274 \pmod{684}$	4765941001
37	$k \equiv 46 \pmod{684}$	37187713
38	$k \equiv 490 \pmod{684}$	473164737913969822009
39	$k \equiv 262 \pmod{684}$	235011794248567117

Table B.21 cont. Covering for $d = 4$ in Base 6

row	congruence	prime p_i
40	$k \equiv 34 \pmod{684}$	$c_{684,1}$
41	$k \equiv 934 \pmod{1140}$	2281
42	$k \equiv 706 \pmod{1140}$	2358661
43	$k \equiv 478 \pmod{1140}$	6197208621830041
44	$k \equiv 250 \pmod{1140}$	3817262181721
45	$k \equiv 22 \pmod{1140}$	18805516694127001
46	$k \equiv 922 \pmod{1140}$	888066870806284147801
47	$k \equiv 694 \pmod{1140}$	$c_{1140,1}$
48	$k \equiv 466 \pmod{1140}$	$c_{1140,2}$
49	$k \equiv 238 \pmod{380}$	347101070179037778781
50	$k \equiv 10 \pmod{380}$	P_{380}
51	$k \equiv 17 \pmod{23}$	47
52	$k \equiv 5 \pmod{23}$	139
53	$k \equiv 16 \pmod{23}$	3221
54	$k \equiv 4 \pmod{23}$	7505944891
55	$k \equiv 38 \pmod{46}$	113958101
56	$k \equiv 26 \pmod{46}$	990000731
57	$k \equiv 60 \pmod{69}$	11731
58	$k \equiv 48 \pmod{69}$	p_{58}
59	$k \equiv 82 \pmod{92}$	6073
60	$k \equiv 70 \pmod{92}$	2259889
61	$k \equiv 58 \pmod{92}$	9564781

row	congruence	prime p_i
62	$k \equiv 46 \pmod{92}$	128407494947883673
63	$k \equiv 103 \pmod{115}$	461
64	$k \equiv 80 \pmod{115}$	1151
65	$k \equiv 57 \pmod{115}$	44851
66	$k \equiv 34 \pmod{115}$	257917526307345124973861
67	$k \equiv 11 \pmod{115}$	p_{67}
68	$k \equiv 114 \pmod{138}$	24648570768391
69	$k \equiv 102 \pmod{138}$	816214079084081564521
70	$k \equiv 182 \pmod{184}$	14537
71	$k \equiv 90 \pmod{184}$	433176829049
72	$k \equiv 170 \pmod{230}$	23126990778420152651
73	$k \equiv 55 \pmod{230}$	P_{230}
74	$k \equiv 216 \pmod{345}$	691
75	$k \equiv 147 \pmod{345}$	241541401
76	$k \equiv 78 \pmod{345}$	139954341051841
77	$k \equiv 9 \pmod{345}$	P_{345}
78	$k \equiv 66 \pmod{276}$	277
79	$k \equiv 54 \pmod{276}$	12344559539431993
80	$k \equiv 42 \pmod{276}$	14030036355387001
81	$k \equiv 30 \pmod{276}$	23027140435639321
82	$k \equiv 18 \pmod{276}$	279219519230141641
83	$k \equiv 282 \pmod{552}$	$c_{552,1}$
84	$k \equiv 6 \pmod{552}$	$c_{552,2}$

B.6 Covering systems for $b = 7$

For base 7, let $b = 7$ and $a = 5$. That is, we consider

$$N = 5 \cdot \frac{7^n - 1}{6} + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = 5 \cdot \frac{7^n - 1}{6} + M + x \cdot 7^k \quad \text{and} \quad N_5^{(k)}(d) = 5 \cdot 7^n + 5 \cdot \frac{7^n - 1}{6} + M + (d-5) \cdot 7^k.$$

We want coverings for $x, d \in \{0, 1, 2, 3, 4, 5, 6\}$. Setting $M \equiv 0 \pmod 5$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod 5$ and $N_5^{(k)}(d) \equiv 0 \pmod 5$ for $x, d \equiv 0 \pmod 5$. Additionally, letting $n \equiv 0 \pmod 2$ and $M \equiv 0 \pmod 2$, then both $N_0^{(k)}(x)$ and $N_5^{(k)}(d)$ are even for $x, d \equiv 0 \pmod 2$. Lastly, setting $n \equiv 0 \pmod 3$ and $M \equiv 0 \pmod 3$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod 3$ and $N_5^{(k)}(d) \equiv 0 \pmod 3$ for $x, d \equiv 0 \pmod 3$. Thus, we only need to find coverings for $x = 1$ and $d = 1$.

The covering used for $x = 1$ is displayed in Table B.22 with the least common multiple of the moduli being 96. Table B.23 exhibits the covering for $d = 1$ where $p_{120} = 85560261859655897641$. The least common multiple of the moduli in Table B.23 is 4320. Both coverings can be verified with a direct analysis.

Table B.22 Covering for $x = 1$ in Base 7

row	congruence	prime p_i
1	$k \equiv 0 \pmod 3$	19
2	$k \equiv 4 \pmod 8$	1201
3	$k \equiv 8 \pmod{16}$	17
4	$k \equiv 0 \pmod{16}$	169553
5	$k \equiv 1 \pmod 6$	43
6	$k \equiv 2 \pmod{12}$	13
7	$k \equiv 10 \pmod{12}$	181

row	congruence	prime p_i
8	$k \equiv 5 \pmod{24}$	73
9	$k \equiv 17 \pmod{24}$	193
10	$k \equiv 11 \pmod{24}$	409
11	$k \equiv 23 \pmod{48}$	33232924804801
12	$k \equiv 47 \pmod{96}$	97
13	$k \equiv 95 \pmod{96}$	104837857

Table B.23 Covering for $d = 1$ in Base 7

congruence	prime p_i	congruence	prime p_i
$k \equiv 0 \pmod{3}$	19	$k \equiv 7 \pmod{10}$	191
$k \equiv 11 \pmod{12}$	13	$k \equiv 0 \pmod{5}$	2801
$k \equiv 56 \pmod{60}$	61	$k \equiv 26 \pmod{36}$	13841169553
$k \equiv 44 \pmod{60}$	555915824341	$k \equiv 50 \pmod{72}$	42409
$k \equiv 2 \pmod{30}$	6568801	$k \equiv 14 \pmod{72}$	137089
$k \equiv 5 \pmod{15}$	31	$k \equiv 38 \pmod{72}$	32952799801
$k \equiv 8 \pmod{15}$	159871	$k \equiv 74 \pmod{144}$	129169
$k \equiv 113 \pmod{120}$	12913561	$k \equiv 2 \pmod{144}$	25726609
$k \equiv 53 \pmod{120}$	p_{120}	$k \equiv 7 \pmod{9}$	37
$k \equiv 161 \pmod{180}$	9901	$k \equiv 4 \pmod{9}$	1063
$k \equiv 101 \pmod{180}$	1795168434777002101	$k \equiv 19 \pmod{27}$	109
$k \equiv 41 \pmod{180}$	2065025164427492401	$k \equiv 10 \pmod{27}$	811
$k \equiv 9 \pmod{10}$	11	$k \equiv 1 \pmod{27}$	2377

B.7 Covering systems for $b = 8$

For base 8, let $b = 8$ and $a = 3$. That is, we consider

$$N = 3 \cdot \frac{8^n - 1}{7} + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = 3 \cdot \frac{8^n - 1}{7} + M + x \cdot 8^k \quad \text{and} \quad N_3^{(k)}(d) = 3 \cdot 8^n + 3 \cdot \frac{8^n - 1}{7} + M + (d-3) \cdot 8^k.$$

We want coverings for $x, d \in \{0, 1, 2, 3, 4, 5, 6, 7\}$. Setting $M \equiv 0 \pmod{3}$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod{3}$ and $N_3^{(k)}(d) \equiv 0 \pmod{3}$ for $x, d \equiv 0 \pmod{3}$. Additionally, by letting $n \equiv 0 \pmod{7}$ and $M \equiv 0 \pmod{7}$, both $N_0^{(k)}(x) \equiv 0 \pmod{7}$ and $N_3^{(k)}(d) \equiv 0 \pmod{7}$ for $x, d \equiv 0 \pmod{7}$. Lastly, setting $n \equiv 0 \pmod{4}$ and $M \equiv 0 \pmod{5}$, we see that $N_0^{(k)}(x)$ is divisible by 5 for $x \equiv 0 \pmod{5}$. Thus, we only need to find coverings for $x \in \{1, 2, 4\}$ and $d \in \{1, 2, 4, 5\}$.

For the insertions $x \in \{1, 2, 4\}$, we use the coverings found in Table B.24, B.25, and B.26, respectively. The largest of the least common multiples of the moduli for the coverings is 1440. A direct analysis on each covering can be used for verification.

We list a few of the larger primes below.

$$p_{120} = 469775495062434961,$$

$$p_{240} = 750016890283777055704738227247474485366338380663681,$$

$$p_{360,1} = 1117180440577441,$$

$$p_{360,2} = 47322686948898415351505582462576221839235677646571281.$$

Table B.24 Covering for $x = 1$ in Base 8

congruence	prime p_i
$k \equiv 0 \pmod{4}$	13
$k \equiv 6 \pmod{8}$	17
$k \equiv 2 \pmod{8}$	241
$k \equiv 13 \pmod{16}$	97
$k \equiv 9 \pmod{16}$	257
$k \equiv 5 \pmod{16}$	673
$k \equiv 17 \pmod{32}$	193
$k \equiv 1 \pmod{32}$	65537
$k \equiv 23 \pmod{24}$	433
$k \equiv 19 \pmod{24}$	38737
$k \equiv 39 \pmod{48}$	577
$k \equiv 15 \pmod{48}$	487824887233
$k \equiv 59 \pmod{72}$	33975937
$k \equiv 35 \pmod{72}$	138991501037953
$k \equiv 83 \pmod{144}$	209924353

congruence	prime p_i
$k \equiv 11 \pmod{144}$	4261383649
$k \equiv 79 \pmod{96}$	1153
$k \equiv 55 \pmod{96}$	6337
$k \equiv 31 \pmod{96}$	38941695937
$k \equiv 7 \pmod{96}$	278452876033
$k \equiv 99 \pmod{120}$	168692292721
$k \equiv 75 \pmod{120}$	p_{120}
$k \equiv 171 \pmod{240}$	8369281
$k \equiv 51 \pmod{240}$	p_{240}
$k \equiv 267 \pmod{360}$	2161
$k \equiv 147 \pmod{360}$	21601
$k \equiv 27 \pmod{360}$	201519653761
$k \equiv 243 \pmod{360}$	$p_{360,1}$
$k \equiv 123 \pmod{360}$	$p_{360,2}$
$k \equiv 3 \pmod{60}$	29247661

Table B.25 Covering for $x = 2$ in Base 8

row	congruence	prime p_i
1	$k \equiv 4 \pmod{5}$	31
2	$k \equiv 3 \pmod{5}$	151
3	$k \equiv 7 \pmod{10}$	11
4	$k \equiv 2 \pmod{10}$	331
5	$k \equiv 11 \pmod{15}$	631
6	$k \equiv 6 \pmod{15}$	23311
7	$k \equiv 16 \pmod{30}$	18837001

row	congruence	prime p_i
8	$k \equiv 31 \pmod{60}$	181
9	$k \equiv 1 \pmod{60}$	54001
10	$k \equiv 15 \pmod{20}$	41
11	$k \equiv 10 \pmod{20}$	61
12	$k \equiv 5 \pmod{20}$	1321
13	$k \equiv 20 \pmod{40}$	61681
14	$k \equiv 0 \pmod{40}$	4562284561

For $d = 1$, we use the covering found in Table B.27. The least common multiple of the moduli for this covering is 224, so a direct analysis is used for verification. One

Table B.26 Covering for $x = 4$ in Base 8

row	congruence	prime p_i
1	$k \equiv 0 \pmod{3}$	73
2	$k \equiv 5 \pmod{6}$	19
3	$k \equiv 2 \pmod{12}$	37
4	$k \equiv 8 \pmod{12}$	109
5	$k \equiv 7 \pmod{9}$	262657

row	congruence	prime p_i
6	$k \equiv 13 \pmod{18}$	87211
7	$k \equiv 22 \pmod{36}$	246241
8	$k \equiv 4 \pmod{36}$	279073
9	$k \equiv 19 \pmod{27}$	2593
10	$k \equiv 10 \pmod{27}$	71119
11	$k \equiv 1 \pmod{27}$	97685839

checks that the primes 5, 13, and 17 correspond to the congruence conditions $M \equiv 0 \pmod{5}$, $M \equiv 12 \pmod{13}$, and $M \equiv 13 \pmod{17}$, which agree with the previous uses of these primes. To conserve space, we denote $p_{224} = 358429848460993$.

Table B.27 Covering for $d = 1$ in Base 8

congruence	prime p_i
$k \equiv 2 \pmod{4}$	5
$k \equiv 0 \pmod{4}$	13
$k \equiv 1 \pmod{8}$	17
$k \equiv 53 \pmod{56}$	3361
$k \equiv 45 \pmod{56}$	15790321
$k \equiv 37 \pmod{56}$	88959882481
$k \equiv 85 \pmod{112}$	2017
$k \equiv 29 \pmod{112}$	5153
$k \equiv 77 \pmod{112}$	25629623713
$k \equiv 21 \pmod{112}$	54410972897
$k \equiv 69 \pmod{112}$	1538595959564161
$k \equiv 125 \pmod{224}$	449

congruence	prime p_i
$k \equiv 13 \pmod{224}$	2689
$k \equiv 173 \pmod{224}$	47886721
$k \equiv 117 \pmod{224}$	183076097
$k \equiv 61 \pmod{224}$	p_{224}
$k \equiv 5 \pmod{224}$	P_{224}
$k \equiv 27 \pmod{28}$	29
$k \equiv 23 \pmod{28}$	113
$k \equiv 19 \pmod{28}$	1429
$k \equiv 15 \pmod{28}$	14449
$k \equiv 11 \pmod{14}$	43
$k \equiv 7 \pmod{14}$	5419
$k \equiv 3 \pmod{7}$	127

We use the covering found in Table B.28 for $d = 2$. The least common multiple of the moduli for this covering is 45760, so a direct analysis is used for verification. One checks that the primes 5, 17, and 4562284561 correspond to the congruence conditions $M \equiv 0 \pmod{5}$, $M \equiv 13 \pmod{17}$, and $M \equiv -2 \pmod{4562284561}$, which agree with the previous uses of these primes.

Table B.28 Covering for $d = 2$ in Base 8

row	congruence	prime p_i
a	$k \equiv 1 \pmod{4}$	5
b	$k \equiv 4 \pmod{8}$	17
c	$k \equiv 0 \pmod{40}$	4562284561
1	$k \equiv 32 \pmod{80}$	394783681
2	$k \equiv 72 \pmod{80}$	4278255361
3	$k \equiv 64 \pmod{80}$	46908728641
4	$k \equiv 264 \pmod{320}$	4855681
5	$k \equiv 184 \pmod{320}$	26881
6	$k \equiv 104 \pmod{320}$	3602561
7	$k \equiv 24 \pmod{320}$	137603804161
8	$k \equiv 136 \pmod{160}$	23041
9	$k \equiv 96 \pmod{160}$	414721
10	$k \equiv 56 \pmod{160}$	44479210368001
11	$k \equiv 16 \pmod{160}$	1476878430700906164431823695804160 1
12	$k \equiv 188 \pmod{220}$	661
13	$k \equiv 148 \pmod{220}$	3301
14	$k \equiv 108 \pmod{220}$	8581
15	$k \equiv 68 \pmod{220}$	391249826881
16	$k \equiv 28 \pmod{220}$	415878438361
17	$k \equiv 208 \pmod{220}$	3630105520141
18	$k \equiv 168 \pmod{220}$	12127627350301
19	$k \equiv 128 \pmod{220}$	13379250952981
20	$k \equiv 88 \pmod{110}$	2971
21	$k \equiv 48 \pmod{110}$	48912491
22	$k \equiv 8 \pmod{110}$	415365721
23	$k \equiv 42 \pmod{44}$	397
24	$k \equiv 38 \pmod{44}$	2113
25	$k \equiv 34 \pmod{44}$	312709
26	$k \equiv 30 \pmod{44}$	4327489
27	$k \equiv 70 \pmod{88}$	353
28	$k \equiv 26 \pmod{88}$	7393
29	$k \equiv 0 \pmod{22}$	67
30	$k \equiv 18 \pmod{22}$	683
31	$k \equiv 14 \pmod{22}$	20857
32	$k \equiv 10 \pmod{11}$	23
33	$k \equiv 6 \pmod{11}$	89
34	$k \equiv 2 \pmod{11}$	599479
35	$k \equiv 51 \pmod{52}$	53
36	$k \equiv 47 \pmod{52}$	157
37	$k \equiv 43 \pmod{52}$	313
38	$k \equiv 39 \pmod{52}$	1249
39	$k \equiv 35 \pmod{52}$	1613

Table B.28 cont. Covering for $d = 2$ in Base 8

row	congruence	prime p_i
40	$k \equiv 31 \pmod{52}$	3121
41	$k \equiv 27 \pmod{52}$	21841
42	$k \equiv 75 \pmod{104}$	858001
43	$k \equiv 23 \pmod{104}$	308761441
44	$k \equiv 19 \pmod{26}$	2731

row	congruence	prime p_i
45	$k \equiv 15 \pmod{26}$	22366891
46	$k \equiv 11 \pmod{13}$	79
47	$k \equiv 7 \pmod{13}$	8191
48	$k \equiv 3 \pmod{13}$	121369

For $d = 4$, we use the covering found in Table B.29. The least common multiple of the moduli for this covering is 105840, so a direct analysis is used for verification. One checks that the primes 73, 19, and 109 correspond to the congruence conditions $M \equiv 69 \pmod{73}$, $M \equiv 9 \pmod{19}$, and $M \equiv 38 \pmod{109}$, which agree with the previous uses of these primes. The covering used for $d = 5$ can be found in Table B.30. The least common multiple of the moduli for this covering is 4200, so a direct analysis is also used for this covering. One checks that the congruence conditions on M arising from the primes 13, 5, and 11, $M \equiv 12 \pmod{13}$, $M \equiv 0 \pmod{5}$, and $M \equiv 7 \pmod{11}$, agree with the previous uses of these primes.

For ease of notation, denote $p_{20} = 29728307155963706810228435378401$ and $p_{22} = 11247702599676505481447137991664348691$.

Table B.29 Covering for $d = 4$ in Base 8

row	congruence	prime p_i
a	$k \equiv 0 \pmod{3}$	73
b	$k \equiv 2 \pmod{6}$	19
c	$k \equiv 5 \pmod{12}$	109
1	$k \equiv 19 \pmod{21}$	92737
2	$k \equiv 16 \pmod{21}$	649657
3	$k \equiv 55 \pmod{63}$	1560007
4	$k \equiv 34 \pmod{63}$	207617485544258392970753527

Table B.29 cont. Covering for $d = 4$ in Base 8

row	congruence	prime p_i
5	$k \equiv 76 \pmod{126}$	379
6	$k \equiv 13 \pmod{126}$	119827
7	$k \equiv 31 \pmod{42}$	77158673929
8	$k \equiv 94 \pmod{126}$	127391413339
9	$k \equiv 52 \pmod{126}$	56202143607667
10	$k \equiv 262 \pmod{378}$	562873504411
11	$k \equiv 136 \pmod{378}$	4744655685883
12	$k \equiv 10 \pmod{378}$	P_{378}
13	$k \equiv 28 \pmod{35}$	71
14	$k \equiv 21 \pmod{35}$	29191
15	$k \equiv 14 \pmod{35}$	106681
16	$k \equiv 7 \pmod{35}$	122921
17	$k \equiv 0 \pmod{35}$	152041
18	$k \equiv 88 \pmod{105}$	870031
19	$k \equiv 67 \pmod{105}$	983431
20	$k \equiv 46 \pmod{105}$	p_{20}
21	$k \equiv 130 \pmod{210}$	1765891
22	$k \equiv 25 \pmod{210}$	p_{22}
23	$k \equiv 319 \pmod{420}$	2521
24	$k \equiv 214 \pmod{420}$	1711081
25	$k \equiv 109 \pmod{420}$	430839361

row	congruence	prime p_i
26	$k \equiv 4 \pmod{420}$	17369459529909057773233442461
27	$k \equiv 127 \pmod{147}$	126127
28	$k \equiv 106 \pmod{147}$	309583
29	$k \equiv 85 \pmod{147}$	5828257
30	$k \equiv 64 \pmod{147}$	448753375334630583898 5313
31	$k \equiv 43 \pmod{147}$	7086423574853972147970086088434689
32	$k \equiv 22 \pmod{49}$	4432676798593
33	$k \equiv 1 \pmod{49}$	2741672362528725535068727
34	$k \equiv 83 \pmod{84}$	40388473189
35	$k \equiv 71 \pmod{84}$	118750098349
36	$k \equiv 143 \pmod{168}$	1009
37	$k \equiv 59 \pmod{168}$	21169
38	$k \equiv 131 \pmod{168}$	2627857
39	$k \equiv 47 \pmod{168}$	269389009
40	$k \equiv 119 \pmod{168}$	1475204679190128571777
41	$k \equiv 203 \pmod{336}$	34273
42	$k \equiv 35 \pmod{336}$	P_{336}
43	$k \equiv 191 \pmod{252}$	757
44	$k \equiv 107 \pmod{252}$	4563764310536263394735333320957
45	$k \equiv 23 \pmod{252}$	304832756195865229284807891468769
46	$k \equiv 4 \pmod{7}$	337

Table B.30 Covering for $d = 5$ in Base 8

row	congruence	prime p_i
a	$k \equiv 2 \pmod{4}$	13
b	$k \equiv 0 \pmod{4}$	5
c	$k \equiv 3 \pmod{10}$	11
1	$k \equiv 41 \pmod{50}$	251
2	$k \equiv 31 \pmod{50}$	4051
3	$k \equiv 21 \pmod{50}$	1133836730401
4	$k \equiv 61 \pmod{100}$	101
5	$k \equiv 11 \pmod{100}$	1201
6	$k \equiv 51 \pmod{100}$	8101
7	$k \equiv 1 \pmod{100}$	63901
8	$k \equiv 20 \pmod{25}$	601
9	$k \equiv 15 \pmod{25}$	1801
10	$k \equiv 10 \pmod{25}$	100801
11	$k \equiv 5 \pmod{25}$	10567201
12	$k \equiv 50 \pmod{75}$	115201
13	$k \equiv 25 \pmod{75}$	617401
14	$k \equiv 0 \pmod{75}$	1348206751
15	$k \equiv 17 \pmod{20}$	61
16	$k \equiv 87 \pmod{100}$	268501
17	$k \equiv 67 \pmod{100}$	13334701
18	$k \equiv 47 \pmod{100}$	1182468601
19	$k \equiv 127 \pmod{200}$	401
20	$k \equiv 27 \pmod{200}$	340801
21	$k \equiv 107 \pmod{200}$	2787601
22	$k \equiv 7 \pmod{200}$	3173389601
23	$k \equiv 69 \pmod{70}$	211
24	$k \equiv 59 \pmod{70}$	281
25	$k \equiv 49 \pmod{70}$	86171
26	$k \equiv 39 \pmod{70}$	664441
27	$k \equiv 29 \pmod{70}$	1564921
28	$k \equiv 89 \pmod{140}$	421
29	$k \equiv 19 \pmod{140}$	7416361
30	$k \equiv 79 \pmod{140}$	47392381
31	$k \equiv 9 \pmod{140}$	146919792181

B.8 Covering systems for $b = 9$

For base 9, let $b = 9$ and $a = 7$. That is, we consider

$$N = 7 \cdot \frac{9^n - 1}{8} + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = 7 \cdot \frac{9^n - 1}{8} + M + x \cdot 9^k \quad \text{and} \quad N_7^{(k)}(d) = 7 \cdot 9^n + 3 \cdot \frac{9^n - 1}{8} + M + (d - 7) \cdot 9^k.$$

We want coverings for $x, d \in \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$. Setting $M \equiv 1 \pmod 2$ and $n \equiv 0 \pmod 2$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod 2$ and $N_7^{(k)}(d) \equiv 0 \pmod 2$ for $x, d \equiv 0 \pmod 2$. Additionally, by letting $M \equiv 0 \pmod 7$, then both $N_0^{(k)}(x)$ and $N_7^{(k)}(d)$ are divisible by 7 for $x, d \equiv 0 \pmod 7$. Lastly, setting $n \equiv 0 \pmod 4$ and $M \equiv 0 \pmod 5$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod 5$ and $N_7^{(k)}(d) \equiv 0 \pmod 5$ for $x \equiv 0 \pmod 5$. Thus, we only need to find coverings for $x, d \in \{1, 3, 5\}$.

For the insertions $x \in \{1, 3, 5\}$, we use the coverings found in Tables B.31, B.32, and B.33, respectively. The largest of the least common multiples of the moduli for the coverings is 1200. A direct analysis on each covering can be used for verification.

Table B.31 Covering for $x = 1$ in Base 9

row	congruence	prime p_i
1	$k \equiv 0 \pmod 2$	5
2	$k \equiv 1 \pmod 4$	41

row	congruence	prime p_i
3	$k \equiv 7 \pmod 8$	17
4	$k \equiv 3 \pmod 8$	193

Table B.32 Covering for $x = 3$ in Base 9

row	congruence	prime p_i
1	$k \equiv 0 \pmod 3$	7
2	$k \equiv 1 \pmod 3$	13
3	$k \equiv 2 \pmod 6$	73

row	congruence	prime p_i
4	$k \equiv 5 \pmod{12}$	6481
5	$k \equiv 11 \pmod{24}$	97
6	$k \equiv 23 \pmod{24}$	577

We use the coverings found in Tables B.34, B.35, and B.36 for $d \in \{1, 3, 5\}$, respectively. The largest of the least common multiples of the moduli for the coverings is 3960. A direct analysis on each covering can be used for verification.

Table B.33 Covering for $x = 5$ in Base 9

row	congruence	prime p_i
1	$k \equiv 0 \pmod{5}$	11
2	$k \equiv 1 \pmod{5}$	61
3	$k \equiv 3 \pmod{10}$	1181
4	$k \equiv 18 \pmod{20}$	42521761
5	$k \equiv 8 \pmod{40}$	14401
6	$k \equiv 68 \pmod{80}$	8194721
7	$k \equiv 28 \pmod{80}$	700984481
8	$k \equiv 2 \pmod{15}$	31
9	$k \equiv 7 \pmod{15}$	271
10	$k \equiv 12 \pmod{15}$	4561
11	$k \equiv 4 \pmod{25}$	151
12	$k \equiv 9 \pmod{25}$	8951
13	$k \equiv 14 \pmod{25}$	391151
14	$k \equiv 19 \pmod{25}$	22996651
15	$k \equiv 24 \pmod{50}$	101
16	$k \equiv 49 \pmod{50}$	394201

One checks that the primes 5, 7, and 13 correspond to the congruence conditions $M \equiv 4 \pmod{5}$, $M \equiv 4 \pmod{7}$, and $M \equiv 12 \pmod{13}$, which agree with the previous congruences involving these primes found in Tables B.31 and B.32. Denote $p_{66} = 13490012358249728401$ for Table B.34.

Table B.34 Covering for $d = 1$ in Base 9

congruence	prime p_i
$k \equiv 0 \pmod{2}$	5
$k \equiv 21 \pmod{22}$	5501
$k \equiv 19 \pmod{22}$	570461
$k \equiv 6 \pmod{11}$	23
$k \equiv 4 \pmod{11}$	67
$k \equiv 2 \pmod{11}$	661
$k \equiv 0 \pmod{11}$	3851
$k \equiv 31 \pmod{33}$	25411
$k \equiv 20 \pmod{33}$	176419
$k \equiv 9 \pmod{33}$	2413941289
$k \equiv 29 \pmod{44}$	89
$k \equiv 7 \pmod{44}$	2382953
$k \equiv 27 \pmod{44}$	56625998353
$k \equiv 49 \pmod{88}$	6922081
$k \equiv 5 \pmod{88}$	15656839738849
$k \equiv 36 \pmod{55}$	1321
$k \equiv 14 \pmod{55}$	659671
$k \equiv 47 \pmod{55}$	24472341743191
$k \equiv 25 \pmod{55}$	560088668384411
$k \equiv 3 \pmod{110}$	177101
$k \equiv 56 \pmod{66}$	p_{66}
$k \equiv 89 \pmod{132}$	660001
$k \equiv 23 \pmod{132}$	11096576833
$k \equiv 78 \pmod{99}$	199
$k \equiv 45 \pmod{99}$	397
$k \equiv 12 \pmod{99}$	4357
$k \equiv 67 \pmod{99}$	3186217
$k \equiv 34 \pmod{99}$	337448233
$k \equiv 1 \pmod{99}$	378450588583

Table B.35 Covering for $d = 3$ in Base 9

congruence	prime p_i
$k \equiv 1 \pmod{2}$	5
$k \equiv 12 \pmod{14}$	29
$k \equiv 10 \pmod{14}$	16493
$k \equiv 1 \pmod{7}$	547
$k \equiv 6 \pmod{7}$	1093
$k \equiv 18 \pmod{28}$	430697
$k \equiv 4 \pmod{28}$	647753
$k \equiv 30 \pmod{35}$	71
$k \equiv 23 \pmod{35}$	2664097031
$k \equiv 16 \pmod{35}$	374857981681
$k \equiv 79 \pmod{105}$	211

congruence	prime p_i
$k \equiv 44 \pmod{105}$	421
$k \equiv 9 \pmod{105}$	1051
$k \equiv 72 \pmod{105}$	6301
$k \equiv 37 \pmod{105}$	24151
$k \equiv 2 \pmod{105}$	1616161
$k \equiv 56 \pmod{70}$	28596961
$k \equiv 42 \pmod{70}$	32839661
$k \equiv 28 \pmod{70}$	94373861
$k \equiv 84 \pmod{140}$	281
$k \equiv 14 \pmod{140}$	18481
$k \equiv 70 \pmod{140}$	369879560116990841
$k \equiv 0 \pmod{140}$	P_{140}

Table B.36 Covering for $d = 5$ in Base 9

congruence	prime p_i
$k \equiv 1 \pmod{3}$	7
$k \equiv 2 \pmod{3}$	13
$k \equiv 18 \pmod{21}$	43
$k \equiv 15 \pmod{21}$	2269
$k \equiv 12 \pmod{21}$	368089
$k \equiv 30 \pmod{42}$	2857
$k \equiv 9 \pmod{42}$	109688713
$k \equiv 48 \pmod{63}$	127
$k \equiv 27 \pmod{63}$	883

congruence	prime p_i
$k \equiv 6 \pmod{63}$	2521
$k \equiv 45 \pmod{63}$	550554229
$k \equiv 24 \pmod{63}$	144542918285300809
$k \equiv 66 \pmod{126}$	1180369
$k \equiv 3 \pmod{126}$	475110761833
$k \equiv 63 \pmod{84}$	337
$k \equiv 42 \pmod{84}$	673
$k \equiv 21 \pmod{84}$	1009
$k \equiv 0 \pmod{84}$	167329

B.9 Covering systems for $b = 11$

For base 11, let $b = 11$ and $a = 3$. That is, we consider

$$N = 3 \cdot \frac{11^n - 1}{10} + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = 3 \cdot \frac{11^n - 1}{10} + M + x \cdot 11^k \quad \text{and} \quad N_3^{(k)}(d) = 3 \cdot 11^n + 3 \cdot \frac{11^n - 1}{10} + M + (d-3) \cdot 11^k.$$

We want coverings for $x, d \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$. Setting $M \equiv 0 \pmod{2}$ and $n \equiv 0 \pmod{2}$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod{2}$ and $N_3^{(k)}(d) \equiv 0 \pmod{2}$ for

$x, d \equiv 0 \pmod{2}$. Additionally, by letting $M \equiv 0 \pmod{3}$, then both $N_0^{(k)}(x)$ and $N_3^{(k)}(d)$ are divisible by 3 for $x, d \equiv 0 \pmod{3}$. Lastly, setting $n \equiv 0 \pmod{5}$ and $M \equiv 0 \pmod{5}$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod{5}$ and $N_3^{(k)}(d) \equiv 0 \pmod{5}$ for $x, d \equiv 0 \pmod{5}$. Thus, we only need to find coverings for $x, d \in \{1, 7\}$.

Tables B.37 through B.40 exhibit these coverings. Each covering can be verified directly where the largest of the least common multiples of the moduli for the coverings is 18480. One checks that the primes 7, 19, and 61 correspond to the congruence conditions $M \equiv 5 \pmod{7}$, $M \equiv 18 \pmod{19}$, and $M \equiv 54 \pmod{61}$. Denote

$$p_{45} = 9842332430037465033595921, \quad p_{112} = 3090443962383595123379137,$$

$$p_{44} = 2649263870814793,$$

$$p_{88} = 1298256794387169996154165633, \quad \text{and} \quad p_{33} = 637265428480297.$$

Table B.37 Covering for $x = 1$ in Base 11

congruence	prime p_i	congruence	prime p_i
$k \equiv 2 \pmod{3}$	7	$k \equiv 46 \pmod{63}$	8317
$k \equiv 0 \pmod{3}$	19	$k \equiv 37 \pmod{63}$	867259
$k \equiv 7 \pmod{9}$	1772893	$k \equiv 28 \pmod{63}$	106431697
$k \equiv 13 \pmod{18}$	590077	$k \equiv 19 \pmod{63}$	316825425410373433
$k \equiv 22 \pmod{36}$	3138426605161	$k \equiv 73 \pmod{126}$	3304981
$k \equiv 40 \pmod{72}$	73	$k \equiv 10 \pmod{126}$	468843103
$k \equiv 4 \pmod{72}$	40177	$k \equiv 64 \pmod{126}$	71596275661
$k \equiv 55 \pmod{63}$	127	$k \equiv 1 \pmod{126}$	278853374647

Table B.38 Covering for $x = 7$ in Base 11

row	congruence	prime p_i
1	$k \equiv 0 \pmod{4}$	61
2	$k \equiv 6 \pmod{8}$	7321
3	$k \equiv 10 \pmod{16}$	17
4	$k \equiv 2 \pmod{16}$	6304673
5	$k \equiv 3 \pmod{6}$	37
6	$k \equiv 7 \pmod{12}$	13
7	$k \equiv 1 \pmod{12}$	1117
8	$k \equiv 17 \pmod{24}$	10657
9	$k \equiv 11 \pmod{24}$	20113
10	$k \equiv 29 \pmod{48}$	97
11	$k \equiv 5 \pmod{48}$	241
12	$k \equiv 23 \pmod{48}$	1777
13	$k \equiv 47 \pmod{48}$	1106131489

Table B.39 Covering for $d = 1$ in Base 11

congruence	prime p_i
$k \equiv 1 \pmod{3}$	7
$k \equiv 0 \pmod{3}$	19
$k \equiv 14 \pmod{15}$	195019441
$k \equiv 1 \pmod{5}$	3221
$k \equiv 23 \pmod{30}$	31
$k \equiv 8 \pmod{30}$	7537711
$k \equiv 5 \pmod{10}$	13421
$k \equiv 10 \pmod{20}$	212601841

congruence	prime p_i
$k \equiv 20 \pmod{40}$	41
$k \equiv 0 \pmod{40}$	1120648576818041
$k \equiv 32 \pmod{45}$	p_{45}
$k \equiv 62 \pmod{90}$	181
$k \equiv 17 \pmod{90}$	631
$k \equiv 47 \pmod{90}$	86306335830799838011
$k \equiv 92 \pmod{180}$	9001
$k \equiv 2 \pmod{180}$	16921

Table B.40 Covering for $d = 7$ in Base 11

congruence	prime p_i
$k \equiv 0 \pmod{4}$	61
$k \equiv 6 \pmod{7}$	43
$k \equiv 4 \pmod{7}$	45319
$k \equiv 9 \pmod{14}$	1623931
$k \equiv 21 \pmod{28}$	29
$k \equiv 7 \pmod{28}$	1933
$k \equiv 19 \pmod{28}$	55527473
$k \equiv 89 \pmod{112}$	337
$k \equiv 61 \pmod{112}$	394129
$k \equiv 33 \pmod{112}$	236352238647181441
$k \equiv 5 \pmod{112}$	p_{112}
$k \equiv 45 \pmod{56}$	113
$k \equiv 31 \pmod{56}$	449
$k \equiv 17 \pmod{56}$	2521
$k \equiv 3 \pmod{56}$	77001139434480073
$k \equiv 57 \pmod{70}$	71
$k \equiv 43 \pmod{70}$	7561

congruence	prime p_i
$k \equiv 29 \pmod{70}$	17011
$k \equiv 15 \pmod{70}$	1649341
$k \equiv 1 \pmod{70}$	10047871
$k \equiv 9 \pmod{11}$	15797
$k \equiv 5 \pmod{11}$	1806113
$k \equiv 12 \pmod{22}$	23
$k \equiv 8 \pmod{22}$	89
$k \equiv 4 \pmod{22}$	199
$k \equiv 0 \pmod{22}$	58367
$k \equiv 18 \pmod{44}$	251857
$k \equiv 14 \pmod{44}$	p_{44}
$k \equiv 54 \pmod{88}$	353
$k \equiv 10 \pmod{88}$	72689
$k \equiv 50 \pmod{88}$	13585441
$k \equiv 6 \pmod{88}$	p_{88}
$k \equiv 24 \pmod{33}$	661
$k \equiv 13 \pmod{33}$	1453
$k \equiv 2 \pmod{33}$	p_{33}

B.10 Covering systems for $b = 31$

For base 31, let $b = 31$ and $a = 7$. That is, we consider

$$N = 7 \cdot \frac{31^n - 1}{30} + M,$$

where the digit insertions are

$$N_0^{(k)}(x) = 7\cdot\frac{31^n - 1}{30} + M + x\cdot 31^k \quad \text{and} \quad N_7^{(k)}(d) = 7\cdot 31^n + 7\cdot\frac{31^n - 1}{30} + M + (d-7)\cdot 31^k.$$

We want coverings for $x, d \in \{0, 1, 2, \ldots, 29, 30\}$. Setting $M \equiv 0 \pmod 2$ and $n \equiv 0 \pmod 2$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod 2$ and $N_7^{(k)}(d) \equiv 0 \pmod 2$ for $x, d \equiv 0 \pmod 2$. Additionally, by letting $M \equiv 1 \pmod 3$ and $n \equiv 0 \pmod 3$, then both $N_0^{(k)}(x)$ and $N_7^{(k)}(d)$ are divisible by 3 for $x, d \equiv 2 \pmod 3$. Moreover, by setting $n \equiv 0 \pmod 5$ and $M \equiv 0 \pmod 5$, we have that both $N_0^{(k)}(x) \equiv 0 \pmod 5$ and $N_7^{(k)}(d) \equiv 0 \pmod 5$ for $x, d \equiv 0 \pmod 5$. Lastly, setting $n \equiv 0 \pmod 7$ and $M \equiv 0 \pmod 7$, we see that both $N_0^{(k)}(x) \equiv 0 \pmod 7$ and $N_7^{(k)}(d) \equiv 0 \pmod 7$ for $x, d \equiv 0 \pmod 7$. Thus, we only need to find coverings for $x, d \in \{1, 3, 9, 13, 19, 27\}$.

Tables B.41, B.42, and B.43 correspond to the insertions $x \in \{1, 3, 9\}$, respectively. Each of these coverings can be verified using a direct analysis. For Table B.42, let $p_6 = 7499207440683838894753$.

Table B.41 Covering for $x = 1$ in Base 31

row	congruence	prime p_i	row	congruence	prime p_i
1	$k \equiv 0 \pmod 4$	13	6	$k \equiv 14 \pmod{16}$	25085030513
2	$k \equiv 3 \pmod 4$	37	7	$k \equiv 26 \pmod{32}$	1889
3	$k \equiv 5 \pmod 8$	409	8	$k \equiv 10 \pmod{32}$	1347329
4	$k \equiv 1 \pmod 8$	1129	9	$k \equiv 18 \pmod{32}$	6139297
5	$k \equiv 6 \pmod{16}$	17	10	$k \equiv 2 \pmod{32}$	23277313

Table B.42 Covering for $x = 3$ in Base 31

row	congruence	prime p_i	row	congruence	prime p_i
1	$k \equiv 0 \pmod 3$	331	6	$k \equiv 34 \pmod{48}$	p_6
2	$k \equiv 1 \pmod 6$	19	7	$k \equiv 5 \pmod 9$	3637
3	$k \equiv 4 \pmod{12}$	922561	8	$k \equiv 8 \pmod 9$	81343
4	$k \equiv 22 \pmod{24}$	852890113921	9	$k \equiv 2 \pmod{18}$	577
5	$k \equiv 10 \pmod{48}$	97	10	$k \equiv 11 \pmod{18}$	1538083

Table B.43 Covering for $x = 9$ in Base 31

congruence	prime p_i
$k \equiv 1 \pmod{5}$	11
$k \equiv 0 \pmod{5}$	17351
$k \equiv 2 \pmod{10}$	41
$k \equiv 7 \pmod{10}$	21821
$k \equiv 4 \pmod{15}$	2521
$k \equiv 9 \pmod{15}$	327412201
$k \equiv 29 \pmod{30}$	880374069121

congruence	prime p_i
$k \equiv 44 \pmod{60}$	61
$k \equiv 14 \pmod{60}$	25621
$k \equiv 13 \pmod{25}$	101
$k \equiv 18 \pmod{25}$	4951
$k \equiv 23 \pmod{25}$	13277801
$k \equiv 8 \pmod{25}$	20235942281002951
$k \equiv 28 \pmod{50}$	1901
$k \equiv 3 \pmod{50}$	4726301

Table B.45 forms the covering used for $x = 13$. The least common multiple of the moduli is 997920 and a direct analysis can be used for verification. Table B.46 forms the covering used for $x = 19$. The least common multiple of the moduli is 158760 and a direct analysis can also be used for verification.

As this covering, as well as the ones that follow, are more complicated, we recall some notation used in Appendix A and some of the smaller bases. Recall that if we are unable to fully factor $\Phi_n(31)$, then we write $\Phi_n(31) = p_1 p_2 \cdots p_r C_n$, where C_n is a composite factor of $\Phi_n(31)$ having at least 2 distinct prime divisors different from $p_1, p_2, \ldots, p_r$ and n. If we are able to fully factor $\Phi_n(31)$, then we write $\Phi_n(31) = p_1 p_2 \cdots p_r P_n$, where P_n is a large prime factor of $\Phi_n(31)$ different from $p_1, p_2, \ldots, p_r$. Computationally, P_n is determined to be a prime power and then verified to be a prime. We denote P_n to be a probable prime too large to include comfortably where $\operatorname{ord}_{P_n}(31) = n$. We also use $c_{n,1}$ to denote one prime factor from the composite number C_n where $\operatorname{ord}_{c_{n,1}}(31) = n$ that we were unable to factor, and $c_{n,2}$ to denote the other prime factor from the same composite number. We did not compute the values of $c_{n,1}$ and $c_{n,2}$, but we know they exist.

When necessary, we provide a table of notable factorizations of $\Phi_n(31)$ used in the covering that follows. It can be assumed that the large primes P_n and composite numbers C_n that are omitted from the tables that contain notable factorizations of

$\Phi_n(31)$ can be determined by dividing $\Phi_n(31)$ by the primes associated with n that can be found in adjacent rows in the tables. Table B.44 contains notable factorizations of $\Phi_n(31)$ used in the coverings for $x = 13$ and $x = 19$, and we list a few primes below that do not fit in the tables comfortably.

$$p_2 = 512616735577, \quad p_{10} = 550469850411853,$$

$$p_{19} = 4303134368687145997467938682848881,$$

$$p_{20} = 62322419393153627851729037464684263699383389269055382039663,$$

$$p_{22} = 147882001432537751112306358052999119715341090542830827,$$

$$p_{36} = 104277841808893792264266721,$$

$$p_{48} = 728921581954037396189325850537700569.$$

Table B.44 Partial/Full factorizations of $\Phi_n(31)$ for $x = 13$ and $x = 19$

n	Factorization of $\Phi_n(31)$
81	$3 \cdot 2593 \cdot 13933 \cdot 477739 \cdot 6757669 \cdot 7822981 \cdot P_{81}$
294	$7 \cdot 159937 \cdot 9561579721 \cdot 64636178950385134849 \cdot P_{294}$
924	C_{924}
1134	C_{1134}

Table B.45 Covering for $x = 13$ in Base 31

row	congruence	prime p_i	row	congruence	prime p_i
1	35 (mod 36)	1536553	12	26 (mod 84)	4038949965541
2	17 (mod 36)	p_2	13	62 (mod 84)	939903457569889
3	47 (mod 54)	109	14	104 (mod 168)	337886977
4	29 (mod 54)	163	15	20 (mod 168)	P_{168}
5	11 (mod 54)	541	16	182 (mod 210)	13454000701
6	41 (mod 54)	6427	17	140 (mod 210)	18396393590821
7	23 (mod 54)	18880993	18	98 (mod 210)	350121327433921
8	5 (mod 54)	599329963	19	56 (mod 210)	p_{19}
9	38 (mod 42)	211	20	644 (mod 840)	52081
10	32 (mod 42)	p_{10}	21	434 (mod 840)	30241
11	68 (mod 84)	163598989	22	224 (mod 840)	13842240721

Table B.45 cont. Covering for $x = 13$ in Base 31

row	congruence	prime p_i
23	$k \equiv 14 \pmod{840}$	$c_{840,1}$
24	$k \equiv 218 \pmod{252}$	37323324288470557
25	$k \equiv 92 \pmod{252}$	$c_{252,1}$
26	$k \equiv 176 \pmod{252}$	$c_{252,2}$
27	$k \equiv 302 \pmod{504}$	241921
28	$k \equiv 50 \pmod{504}$	7894153
29	$k \equiv 260 \pmod{378}$	379
30	$k \equiv 134 \pmod{378}$	$c_{378,1}$
31	$k \equiv 8 \pmod{378}$	$c_{378,2}$
32	$k \equiv 296 \pmod{336}$	337
33	$k \equiv 254 \pmod{336}$	2017
34	$k \equiv 212 \pmod{336}$	555073
35	$k \equiv 170 \pmod{336}$	4861797895048710 7729
36	$k \equiv 128 \pmod{336}$	p_{36}
37	$k \equiv 86 \pmod{336}$	P_{336}
38	$k \equiv 380 \pmod{672}$	673
39	$k \equiv 44 \pmod{672}$	20673051169
40	$k \equiv 338 \pmod{672}$	4408464948749687469840 1
41	$k \equiv 2 \pmod{672}$	$c_{672,1}$
42	$k \equiv 25 \pmod{27}$	1836205027201
43	$k \equiv 16 \pmod{27}$	126901881805771
44	$k \equiv 61 \pmod{81}$	2593
45	$k \equiv 34 \pmod{81}$	13933
46	$k \equiv 7 \pmod{81}$	477739
47	$k \equiv 76 \pmod{81}$	6757669

row	congruence	prime p_i
48	$k \equiv 49 \pmod{81}$	7822981
49	$k \equiv 22 \pmod{81}$	P_{81}
50	$k \equiv 94 \pmod{108}$	277477787226853
51	$k \equiv 67 \pmod{108}$	91556360840213317
52	$k \equiv 40 \pmod{108}$	1923555333838297316 10441
53	$k \equiv 121 \pmod{216}$	12733754041
54	$k \equiv 13 \pmod{216}$	$c_{216,1}$
55	$k \equiv 112 \pmod{135}$	1344742561
56	$k \equiv 85 \pmod{135}$	302533008751
57	$k \equiv 58 \pmod{135}$	50358897181
58	$k \equiv 31 \pmod{135}$	$c_{135,1}$
59	$k \equiv 4 \pmod{135}$	$c_{135,2}$
60	$k \equiv 37 \pmod{45}$	271
61	$k \equiv 28 \pmod{45}$	63901
62	$k \equiv 19 \pmod{45}$	106291
63	$k \equiv 10 \pmod{45}$	33704868363348084546 7801
64	$k \equiv 46 \pmod{90}$	2065411
65	$k \equiv 1 \pmod{90}$	300392264044249601502733598251
66	$k \equiv 30 \pmod{33}$	650141690025315305584300036801
67	$k \equiv 93 \pmod{99}$	199
68	$k \equiv 60 \pmod{99}$	991
69	$k \equiv 27 \pmod{99}$	204733
70	$k \equiv 90 \pmod{99}$	36093579787
71	$k \equiv 57 \pmod{99}$	294573316951
72	$k \equiv 24 \pmod{99}$	215792743120601131

Table B.45 cont. Covering for $x = 13$ in Base 31

row	congruence	prime p_i
73	$k \equiv 87 \pmod{99}$	3763784187326467459
74	$k \equiv 54 \pmod{99}$	869535983092745596321
75	$k \equiv 120 \pmod{198}$	8713
76	$k \equiv 21 \pmod{198}$	430057
77	$k \equiv 183 \pmod{198}$	291207313
78	$k \equiv 84 \pmod{198}$	24763965905251
79	$k \equiv 150 \pmod{198}$	P_{198}
80	$k \equiv 249 \pmod{396}$	18217
81	$k \equiv 51 \pmod{396}$	56629
82	$k \equiv 315 \pmod{396}$	312841
83	$k \equiv 216 \pmod{396}$	114678696529
84	$k \equiv 117 \pmod{396}$	9293745221943733
85	$k \equiv 18 \pmod{396}$	P_{396}
86	$k \equiv 48 \pmod{66}$	67
87	$k \equiv 15 \pmod{66}$	297991
88	$k \equiv 45 \pmod{66}$	34731987261785578083133
89	$k \equiv 78 \pmod{132}$	599382278617
90	$k \equiv 12 \pmod{132}$	169301958609793153
91	$k \equiv 207 \pmod{231}$	463
92	$k \equiv 174 \pmod{231}$	13780537
93	$k \equiv 141 \pmod{231}$	816786763717
94	$k \equiv 108 \pmod{231}$	P_{231}
95	$k \equiv 306 \pmod{462}$	427813176370045109029
96	$k \equiv 75 \pmod{462}$	$c_{462,1}$
97	$k \equiv 273 \pmod{462}$	$c_{462,2}$
98	$k \equiv 504 \pmod{924}$	$c_{924,1}$
99	$k \equiv 42 \pmod{924}$	$c_{924,2}$
100	$k \equiv 9 \pmod{77}$	2927
101	$k \equiv 138 \pmod{165}$	1321
102	$k \equiv 105 \pmod{165}$	P_{165}
103	$k \equiv 237 \pmod{330}$	2995081
104	$k \equiv 72 \pmod{330}$	3173617894921
105	$k \equiv 204 \pmod{330}$	13027986803063207491231
106	$k \equiv 39 \pmod{330}$	P_{330}
107	$k \equiv 336 \pmod{495}$	418771
108	$k \equiv 171 \pmod{495}$	377475051691
109	$k \equiv 6 \pmod{495}$	$c_{495,1}$
110	$k \equiv 102 \pmod{132}$	4451983606421686827580205284201
111	$k \equiv 201 \pmod{264}$	183516169
112	$k \equiv 69 \pmod{264}$	6542062289677294884725761
113	$k \equiv 168 \pmod{264}$	P_{264}
114	$k \equiv 300 \pmod{528}$	3169
115	$k \equiv 36 \pmod{528}$	10524479059277959777
116	$k \equiv 531 \pmod{660}$	1149061
117	$k \equiv 399 \pmod{660}$	52061461
118	$k \equiv 267 \pmod{660}$	$c_{660,1}$
119	$k \equiv 135 \pmod{660}$	$c_{660,2}$
120	$k \equiv 663 \pmod{1320}$	19801
121	$k \equiv 3 \pmod{1320}$	215758788030763381958401
122	$k \equiv 0 \pmod{11}$	23

Table B.46 Covering for $x = 19$ in Base 31

row	congruence	prime p_i
1	$k \equiv 6 \pmod{7}$	917087137
2	$k \equiv 12 \pmod{14}$	11971
3	$k \equiv 5 \pmod{14}$	71821
4	$k \equiv 18 \pmod{21}$	43
5	$k \equiv 11 \pmod{21}$	6301
6	$k \equiv 4 \pmod{21}$	2813432694367
7	$k \equiv 24 \pmod{28}$	29
8	$k \equiv 17 \pmod{28}$	7253
9	$k \equiv 10 \pmod{28}$	13469
10	$k \equiv 3 \pmod{28}$	277739477
11	$k \equiv 30 \pmod{35}$	319061
12	$k \equiv 23 \pmod{35}$	203633641
13	$k \equiv 16 \pmod{35}$	9240957640390889951861
14	$k \equiv 44 \pmod{70}$	71
15	$k \equiv 9 \pmod{70}$	149269961
16	$k \equiv 72 \pmod{105}$	421
17	$k \equiv 37 \pmod{105}$	$c_{105,1}$
18	$k \equiv 2 \pmod{105}$	$c_{105,2}$
19	$k \equiv 43 \pmod{49}$	6959
20	$k \equiv 36 \pmod{49}$	p_{20}
21	$k \equiv 78 \pmod{98}$	2932755253
22	$k \equiv 29 \pmod{98}$	p_{22}
23	$k \equiv 120 \pmod{147}$	883
24	$k \equiv 71 \pmod{147}$	6585932048863895071
25	$k \equiv 22 \pmod{147}$	P_{147}
26	$k \equiv 162 \pmod{196}$	197
27	$k \equiv 113 \pmod{196}$	P_{196}
28	$k \equiv 260 \pmod{392}$	29401
29	$k \equiv 64 \pmod{392}$	946681
30	$k \equiv 211 \pmod{392}$	50762136041
31	$k \equiv 15 \pmod{392}$	37737969050722454420873
32	$k \equiv 204 \pmod{245}$	491
33	$k \equiv 155 \pmod{245}$	1325489196571
34	$k \equiv 106 \pmod{245}$	29220445871
35	$k \equiv 57 \pmod{245}$	$c_{245,1}$
36	$k \equiv 8 \pmod{245}$	$c_{245,2}$
37	$k \equiv 246 \pmod{294}$	159937
38	$k \equiv 197 \pmod{294}$	9561579721
39	$k \equiv 148 \pmod{294}$	64636178950385134849
40	$k \equiv 99 \pmod{294}$	P_{294}
41	$k \equiv 344 \pmod{588}$	270647865962041
42	$k \equiv 50 \pmod{588}$	492508466593101040661595493
43	$k \equiv 295 \pmod{588}$	$c_{588,1}$
44	$k \equiv 1 \pmod{588}$	$c_{588,2}$
45	$k \equiv 56 \pmod{63}$	127
46	$k \equiv 49 \pmod{63}$	70309
47	$k \equiv 42 \pmod{63}$	75077698123
48	$k \equiv 35 \pmod{63}$	p_{48}
49	$k \equiv 154 \pmod{189}$	8317
50	$k \equiv 91 \pmod{189}$	657178775852071573297

Table B.46 cont. Covering for $x = 19$ in Base 31

row	congruence	prime p_i
51	$k \equiv 28 \pmod{189}$	175189978713624355909
52	$k \equiv 147 \pmod{189}$	$c_{189,1}$
53	$k \equiv 84 \pmod{189}$	$c_{189,2}$
54	$k \equiv 399 \pmod{567}$	1962171989486844920285764789
55	$k \equiv 210 \pmod{567}$	$c_{567,1}$
56	$k \equiv 21 \pmod{1134}$	$c_{1134,1}$
57	$k \equiv 588 \pmod{1134}$	$c_{1134,2}$
58	$k \equiv 266 \pmod{315}$	631
59	$k \equiv 203 \pmod{315}$	22185451
60	$k \equiv 140 \pmod{315}$	7180239284191
61	$k \equiv 77 \pmod{315}$	11352145269151
62	$k \equiv 14 \pmod{315}$	P_{315}
63	$k \equiv 70 \pmod{126}$	2143
64	$k \equiv 7 \pmod{126}$	45376431752737
65	$k \equiv 63 \pmod{126}$	1652484831253806817
66	$k \equiv 0 \pmod{126}$	3041204060704443103

The covering used for $x = 27$ is found in Table B.48. The least common multiple of the moduli is 2032800 and verification can be done directly. Table B.47 contains notable factorizations of $\Phi_n(31)$ used in the covering for $x = 27$. We denote

$$p_4 = 727422334085254365392641, \quad p_6 = 3889436310686727916228493162492361601,$$

$$p_{12} = 281887891699576309494931758688962111082311886906752520001,$$

$$p_{37} = 24106981477091678423113880081946849059226586740161,$$

$$p_{48} = 236661696642275153056980146191674776616380367693641,$$

$$p_{53} = 263768160996144192120004532942855021486760529559458116494001319.$$

Table B.47 Partial/Full factorizations of $\Phi_n(31)$ for $x = 27$

n	Factorization of $\Phi_n(31)$
121	C_{121}
176	P_{176}

n	Factorization of $\Phi_n(31)$
440	C_{440}

Table B.48 Covering for $x = 27$ in Base 31

row	congruence	prime p_i
1	$k \equiv 0 \pmod{4}$	13
2	$k \equiv 14 \pmod{20}$	181
3	$k \equiv 18 \pmod{20}$	4707206941
4	$k \equiv 30 \pmod{40}$	p_4
5	$k \equiv 50 \pmod{80}$	136046551681
6	$k \equiv 10 \pmod{80}$	p_6
7	$k \equiv 46 \pmod{60}$	1529401
8	$k \equiv 26 \pmod{60}$	304643210761
9	$k \equiv 66 \pmod{120}$	35401
10	$k \equiv 6 \pmod{120}$	1546081
11	$k \equiv 82 \pmod{100}$	1601
12	$k \equiv 62 \pmod{100}$	p_{12}
13	$k \equiv 142 \pmod{200}$	601
14	$k \equiv 42 \pmod{200}$	7137001
15	$k \equiv 122 \pmod{200}$	P_{200}
16	$k \equiv 222 \pmod{400}$	401
17	$k \equiv 22 \pmod{400}$	16001
18	$k \equiv 302 \pmod{400}$	593742117601
19	$k \equiv 202 \pmod{400}$	P_{400}
20	$k \equiv 502 \pmod{800}$	881954401

row	congruence	prime p_i
21	$k \equiv 102 \pmod{800}$	35406260488184458384 01
22	$k \equiv 402 \pmod{800}$	$c_{800,1}$
23	$k \equiv 2 \pmod{800}$	$c_{800,2}$
24	$k \equiv 21 \pmod{22}$	757241
25	$k \equiv 19 \pmod{22}$	1048563011
26	$k \equiv 39 \pmod{44}$	2729
27	$k \equiv 17 \pmod{44}$	245911396799577828131028569
28	$k \equiv 81 \pmod{88}$	89
29	$k \equiv 59 \pmod{88}$	414407390867564627396249
30	$k \equiv 37 \pmod{88}$	12236290645201501169749559350025041
31	$k \equiv 103 \pmod{176}$	P_{176}
32	$k \equiv 191 \pmod{352}$	353
33	$k \equiv 15 \pmod{352}$	136344440321
34	$k \equiv 101 \pmod{110}$	661
35	$k \equiv 79 \pmod{110}$	2531
36	$k \equiv 57 \pmod{110}$	11551
37	$k \equiv 35 \pmod{110}$	p_{37}
38	$k \equiv 123 \pmod{220}$	171161
39	$k \equiv 13 \pmod{220}$	78101
40	$k \equiv 209 \pmod{220}$	208121

Table B.48 cont. Covering for $x = 27$ in Base 31

row	congruence	prime p_i
41	$k \equiv 99 \pmod{220}$	806753201
42	$k \equiv 187 \pmod{220}$	3558024432228170856 61
43	$k \equiv 77 \pmod{220}$	$c_{220,1}$
44	$k \equiv 165 \pmod{220}$	$c_{220,2}$
45	$k \equiv 275 \pmod{440}$	$c_{440,1}$
46	$k \equiv 55 \pmod{440}$	$c_{440,2}$
47	$k \equiv 33 \pmod{55}$	167767051
48	$k \equiv 11 \pmod{55}$	p_{48}
49	$k \equiv 141 \pmod{154}$	818819
50	$k \equiv 119 \pmod{154}$	P_{154}
51	$k \equiv 20 \pmod{77}$	23503054499
52	$k \equiv 75 \pmod{77}$	16169321243923
53	$k \equiv 53 \pmod{77}$	p_{53}
54	$k \equiv 493 \pmod{616}$	3697
55	$k \equiv 339 \pmod{616}$	101641
56	$k \equiv 185 \pmod{616}$	1442057
57	$k \equiv 31 \pmod{616}$	28459299793
58	$k \equiv 471 \pmod{616}$	16461619704377
59	$k \equiv 317 \pmod{616}$	$c_{616,1}$
60	$k \equiv 163 \pmod{616}$	$c_{616,2}$

row	congruence	prime p_i
61	$k \equiv 625 \pmod{1232}$	15089537
62	$k \equiv 9 \pmod{1232}$	689761073
63	$k \equiv 106 \pmod{121}$	$c_{121,1}$
64	$k \equiv 84 \pmod{121}$	$c_{121,2}$
65	$k \equiv 183 \pmod{242}$	1435061
66	$k \equiv 161 \pmod{242}$	105156503
67	$k \equiv 139 \pmod{242}$	639322672850027
68	$k \equiv 117 \pmod{242}$	852408217
69	$k \equiv 95 \pmod{242}$	$c_{242,1}$
70	$k \equiv 73 \pmod{242}$	$c_{242,2}$
71	$k \equiv 293 \pmod{363}$	2546083
72	$k \equiv 172 \pmod{363}$	1217678565551454550039
73	$k \equiv 51 \pmod{363}$	9643438985844684785 9653
74	$k \equiv 271 \pmod{484}$	2186713
75	$k \equiv 29 \pmod{484}$	8547867857
76	$k \equiv 249 \pmod{484}$	6045052521839057
77	$k \equiv 7 \pmod{484}$	6081187466917124436 82529
78	$k \equiv 5 \pmod{11}$	397
79	$k \equiv 3 \pmod{11}$	617
80	$k \equiv 1 \pmod{11}$	150332843

We use the covering found in Table B.50 for $d = 1$. The least common multiple of the moduli for the covering is 4233600, and a direct analysis can be used for verification. The first six rows in Table B.50 correspond to primes that appear in the coverings for $x \in \{1, 3, 9\}$. These congruences correspond to

$$M \equiv 12 \pmod{13}, \qquad M \equiv 46 \pmod{61},$$

$$M \equiv 31 \pmod{37}, \qquad M \equiv 2 \pmod{17},$$

$$M \equiv 2 \pmod{19}, \qquad M \equiv 25 \pmod{97},$$

which agree with the congruence conditions on M arising from the congruences in Tables B.41, B.42, and B.43. Table B.49 contains notable factorizations of $\Phi_n(31)$ for large n used in the covering for $d = 1$. For convenience, denote

$$p_7 = 2602599520578352740951559767101441091988\,9,$$

$$p_{18} = 9667783133425605119410155998541192601,$$

$$p_{25} = 1637872091108040680148042294721,$$

$$p_{35} = 2121734092665157406108976641321341571103440372921057,$$

$$p_{71} = 185963958223287385373849564550 41.$$

Table B.49 Partial/Full factorizations of $\Phi_n(31)$ for $d = 1$

n	Factorization of $\Phi_n(31)$
192	$3457 \cdot 25537 \cdot 4067137 \cdot 8874745729 \cdot C_{192}$
360	C_{360}
864	C_{864}

Table B.50 Covering for $d = 1$ in Base 31

row	congruence	prime p_i
a	$k \equiv 0 \pmod{4}$	13
b	$k \equiv 1 \pmod{4}$	37
c	$k \equiv 2 \pmod{6}$	19
d	$k \equiv 34 \pmod{60}$	61
e	$k \equiv 11 \pmod{16}$	17
f	$k \equiv 39 \pmod{48}$	97
1	$k \equiv 66 \pmod{72}$	73

Table B.50 cont. Covering for $d = 1$ in Base 31

row	congruence	prime p_i
2	$k \equiv 54 \pmod{72}$	4683817
3	$k \equiv 42 \pmod{72}$	18145037636761304494089799921
4	$k \equiv 102 \pmod{144}$	1704673
5	$k \equiv 30 \pmod{144}$	476428033
6	$k \equiv 90 \pmod{144}$	18210235769136721
7	$k \equiv 18 \pmod{144}$	p_7
8	$k \equiv 366 \pmod{432}$	433
9	$k \equiv 222 \pmod{432}$	12097
10	$k \equiv 78 \pmod{432}$	262798661953
11	$k \equiv 294 \pmod{432}$	$c_{432,1}$
12	$k \equiv 150 \pmod{432}$	$c_{432,2}$
13	$k \equiv 438 \pmod{864}$	$c_{864,1}$
14	$k \equiv 6 \pmod{864}$	$c_{864,2}$
15	$k \equiv 178 \pmod{180}$	8728381
16	$k \equiv 118 \pmod{180}$	14398921
17	$k \equiv 58 \pmod{180}$	P_{180}
18	$k \equiv 106 \pmod{120}$	p_{18}
19	$k \equiv 286 \pmod{360}$	1259350392569520313706401
20	$k \equiv 202 \pmod{240}$	241
21	$k \equiv 142 \pmod{240}$	32360641
22	$k \equiv 82 \pmod{240}$	362634922081
23	$k \equiv 22 \pmod{240}$	69916284426778163281
24	$k \equiv 190 \pmod{240}$	864542404017920606719201
25	$k \equiv 130 \pmod{240}$	p_{25}
26	$k \equiv 310 \pmod{480}$	20641

row	congruence	prime p_i
27	$k \equiv 70 \pmod{480}$	330991018976905188481
28	$k \equiv 250 \pmod{480}$	11282208020367154665601
29	$k \equiv 10 \pmod{480}$	P_{480}
30	$k \equiv 166 \pmod{360}$	$c_{360,1}$
31	$k \equiv 46 \pmod{360}$	$c_{360,2}$
32	$k \equiv 99 \pmod{112}$	2129
33	$k \equiv 83 \pmod{112}$	27329
34	$k \equiv 67 \pmod{112}$	3117962927633
35	$k \equiv 51 \pmod{112}$	p_{35}
36	$k \equiv 147 \pmod{224}$	523694468332725375841
37	$k \equiv 35 \pmod{224}$	P_{224}
38	$k \equiv 691 \pmod{784}$	18555784913
39	$k \equiv 579 \pmod{784}$	362194685329
40	$k \equiv 467 \pmod{784}$	861423109554113
41	$k \equiv 355 \pmod{784}$	$c_{784,1}$
42	$k \equiv 243 \pmod{784}$	$c_{784,2}$
43	$k \equiv 915 \pmod{1568}$	2312801
44	$k \equiv 131 \pmod{1568}$	$c_{1568,1}$
45	$k \equiv 803 \pmod{1568}$	$c_{1568,2}$
46	$k \equiv 1587 \pmod{3136}$	3137
47	$k \equiv 19 \pmod{3136}$	$c_{3136,1}$
48	$k \equiv 899 \pmod{1008}$	1009
49	$k \equiv 563 \pmod{1008}$	$c_{1008,1}$
50	$k \equiv 227 \pmod{1008}$	$c_{1008,2}$
51	$k \equiv 1795 \pmod{2016}$	1135009

Table B.50 cont. Covering for $d = 1$ in Base 31

row	congruence	prime p_i
52	$k \equiv 787 \pmod{2016}$	86689
53	$k \equiv 1459 \pmod{2016}$	21615553
54	$k \equiv 451 \pmod{2016}$	$c_{2016,1}$
55	$k \equiv 2131 \pmod{3024}$	794882457090289
56	$k \equiv 1123 \pmod{3024}$	$c_{3024,1}$
57	$k \equiv 115 \pmod{3024}$	$c_{3024,2}$
58	$k \equiv 1347 \pmod{1680}$	104161
59	$k \equiv 1011 \pmod{1680}$	356346481
60	$k \equiv 675 \pmod{1680}$	1389156854401
61	$k \equiv 2019 \pmod{3360}$	3361
62	$k \equiv 339 \pmod{3360}$	13441
63	$k \equiv 1683 \pmod{3360}$	$c_{3360,1}$
64	$k \equiv 3 \pmod{3360}$	$c_{3360,2}$
65	$k \equiv 127 \pmod{128}$	257
66	$k \equiv 111 \pmod{128}$	641
67	$k \equiv 95 \pmod{128}$	2689
68	$k \equiv 79 \pmod{128}$	9601
69	$k \equiv 63 \pmod{128}$	13768516609
70	$k \equiv 47 \pmod{128}$	17777097601059636481
71	$k \equiv 31 \pmod{128}$	72317464957811235585793
72	$k \equiv 15 \pmod{128}$	p_{72}
73	$k \equiv 167 \pmod{192}$	3457
74	$k \equiv 119 \pmod{192}$	25537
75	$k \equiv 71 \pmod{192}$	$c_{192,1}$
76	$k \equiv 23 \pmod{192}$	$c_{192,2}$
77	$k \equiv 679 \pmod{720}$	55441
78	$k \equiv 439 \pmod{720}$	121731121
79	$k \equiv 199 \pmod{720}$	1034802677322639156706561
80	$k \equiv 631 \pmod{720}$	$c_{720,1}$
81	$k \equiv 391 \pmod{720}$	$c_{720,2}$
82	$k \equiv 871 \pmod{1440}$	120879361
83	$k \equiv 151 \pmod{1440}$	706194721
84	$k \equiv 1303 \pmod{1440}$	218070064800374401
85	$k \equiv 583 \pmod{1440}$	$c_{1440,1}$
86	$k \equiv 1063 \pmod{1440}$	$c_{1440,2}$
87	$k \equiv 1783 \pmod{2880}$	152989623361
88	$k \equiv 343 \pmod{2880}$	$c_{2880,1}$
89	$k \equiv 1543 \pmod{2160}$	2161
90	$k \equiv 823 \pmod{2160}$	425521
91	$k \equiv 103 \pmod{2160}$	5398792227695761201
92	$k \equiv 775 \pmod{960}$	2510401
93	$k \equiv 535 \pmod{960}$	13996242437782818818465281
94	$k \equiv 295 \pmod{960}$	$c_{960,1}$
95	$k \equiv 55 \pmod{960}$	$c_{960,2}$
96	$k \equiv 967 \pmod{1200}$	1201
97	$k \equiv 727 \pmod{1200}$	573602076001
98	$k \equiv 487 \pmod{1200}$	$c_{1200,1}$
99	$k \equiv 247 \pmod{1200}$	$c_{1200,2}$
100	$k \equiv 1207 \pmod{2400}$	8937715783648801
101	$k \equiv 7 \pmod{2400}$	$c_{2400,1}$

We use the covering found in Table B.52 for $d = 3$. The least common multiple of the moduli for the covering is 244823040. To verify that the 157 congruences are indeed a covering, we use a similar method to the method used for verifying the covering found in Table A.10 when inserting the digit $d = 9$ into the leading sevens in base 10. That is, we consider the congruence classes modulo three. Observe that $k \equiv 0 \pmod 3$ is the first congruence listed in Table B.52, so every integer that is divisible by three satisfies a congruence in Table B.52. One can check that every integer satisfying $x \equiv 1 \pmod 3$ satisfies one of the congruences in rows b-23 of Table B.52. These congruences have moduli dividing 9216. Observe that 9216 is divisible by 3 and since every integer congruent to 1 modulo 3 in $[0, 9216)$ satisfies one of these 25 congruences, we can deduce every integer congruent to 1 modulo 3 not in $[0, 9216)$ also does. Similarly, one can check that every integer satisfying $x \equiv 2 \pmod 3$ satisfies one of the 131 congruences in rows 24-154 of Table B.52 with moduli dividing 3825360. Thus, the 157 congruences in Table B.52 form a covering.

One checks that the primes 331, 19, and 13 correspond to the congruence conditions $M \equiv -3 \pmod{331}$, $M \equiv 2 \pmod{19}$, and $M \equiv 12 \pmod{13}$, which agree with the previous congruences involving these primes. Table B.51 contains notable factorizations of $\Phi_n(31)$ for large n used in the covering for $d = 3$. Denote

$$p_{34} = 6051086478234010017092361698900113426918339499351 22187,$$

$$p_{148} = 9398244307541469261 3761, \quad p_{154} = 611241183169.$$

Table B.51 Partial/Full factorizations of $\Phi_n(31)$ for $d = 3$

n	Factorization of $\Phi_n(31)$
253	$23 \cdot 36364703 \cdot 82982989 \cdot 1975800971 \cdot C_{253}$
690	C_{690}
1536	C_{1536}

Table B.52 Covering for $d = 3$ in Base 31

row	congruence	prime p_i
a	$k \equiv 0 \pmod{3}$	331
b	$k \equiv 4 \pmod{6}$	19
c	$k \equiv 3 \pmod{4}$	13
1	$k \equiv 85 \pmod{96}$	193
2	$k \equiv 73 \pmod{96}$	7393
3	$k \equiv 61 \pmod{96}$	27333034608226177
4	$k \equiv 49 \pmod{96}$	13567765732542221348764897
5	$k \equiv 133 \pmod{192}$	4067137
6	$k \equiv 37 \pmod{192}$	8874745729
7	$k \equiv 313 \pmod{384}$	769
8	$k \equiv 217 \pmod{384}$	1366657
9	$k \equiv 121 \pmod{384}$	321034406401
10	$k \equiv 25 \pmod{384}$	$c_{384,1}$
11	$k \equiv 289 \pmod{384}$	$c_{384,2}$
12	$k \equiv 205 \pmod{288}$	1829797906352833
13	$k \equiv 109 \pmod{288}$	$c_{288,1}$
14	$k \equiv 13 \pmod{288}$	$c_{288,2}$
15	$k \equiv 577 \pmod{768}$	64513
16	$k \equiv 193 \pmod{768}$	91276251112682001409
17	$k \equiv 481 \pmod{768}$	$c_{768,1}$
18	$k \equiv 97 \pmod{768}$	$c_{768,2}$
19	$k \equiv 1153 \pmod{1536}$	$c_{1536,1}$
20	$k \equiv 769 \pmod{1536}$	$c_{1536,2}$
21	$k \equiv 1921 \pmod{3072}$	18433
22	$k \equiv 385 \pmod{3072}$	750412801
23	$k \equiv 1537 \pmod{3072}$	$c_{3072,1}$
24	$k \equiv 1 \pmod{3072}$	$c_{3072,2}$
25	$k \equiv 22 \pmod{23}$	1509997
26	$k \equiv 19 \pmod{23}$	61562537
27	$k \equiv 16 \pmod{23}$	7176374761323733117
28	$k \equiv 59 \pmod{69}$	139
29	$k \equiv 56 \pmod{69}$	1164859
30	$k \equiv 53 \pmod{69}$	2553526979752336381
31	$k \equiv 50 \pmod{69}$	P_{69}
32	$k \equiv 116 \pmod{138}$	6073
33	$k \equiv 47 \pmod{138}$	117071611
34	$k \equiv 113 \pmod{138}$	p_{34}
35	$k \equiv 458 \pmod{552}$	118681
36	$k \equiv 320 \pmod{552}$	225769
37	$k \equiv 182 \pmod{552}$	$c_{552,1}$
38	$k \equiv 44 \pmod{552}$	$c_{552,2}$
39	$k \equiv 179 \pmod{207}$	15733
40	$k \equiv 110 \pmod{207}$	16798026051215317
41	$k \equiv 41 \pmod{207}$	$c_{207,1}$
42	$k \equiv 176 \pmod{207}$	$c_{207,2}$
43	$k \equiv 521 \pmod{621}$	78688153
44	$k \equiv 314 \pmod{621}$	86221120498695683587
45	$k \equiv 107 \pmod{621}$	$c_{621,1}$
46	$k \equiv 452 \pmod{621}$	$c_{621,2}$
47	$k \equiv 866 \pmod{1242}$	3727

Table B.52 cont. Covering for $d = 3$ in Base 31

row	congruence	prime p_i
48	$k \equiv 245 \pmod{1242}$	223927728227677
49	$k \equiv 659 \pmod{1242}$	$c_{1242,1}$
50	$k \equiv 38 \pmod{1242}$	$c_{1242,2}$
51	$k \equiv 311 \pmod{345}$	6211
52	$k \equiv 242 \pmod{345}$	9661
53	$k \equiv 173 \pmod{345}$	342241
54	$k \equiv 104 \pmod{345}$	1447263271
55	$k \equiv 35 \pmod{345}$	4520673725606114111171
56	$k \equiv 308 \pmod{345}$	P_{345}
57	$k \equiv 9 \pmod{115}$	176542021
58	$k \equiv 55 \pmod{115}$	524818601
59	$k \equiv 101 \pmod{115}$	P_{115}
60	$k \equiv 722 \pmod{1035}$	440911
61	$k \equiv 377 \pmod{1035}$	25932961
62	$k \equiv 32 \pmod{1035}$	$c_{1035,1}$
63	$k \equiv 75 \pmod{92}$	829
64	$k \equiv 52 \pmod{92}$	1955865713101
65	$k \equiv 29 \pmod{92}$	23856848059764277
66	$k \equiv 6 \pmod{92}$	P_{92}
67	$k \equiv 233 \pmod{276}$	277
68	$k \equiv 164 \pmod{276}$	7177
69	$k \equiv 95 \pmod{276}$	21060989233
70	$k \equiv 26 \pmod{276}$	P_{276}
71	$k \equiv 138 \pmod{161}$	3221
72	$k \equiv 115 \pmod{161}$	118492459

row	congruence	prime p_i
73	$k \equiv 92 \pmod{161}$	$c_{161,1}$
74	$k \equiv 69 \pmod{161}$	$c_{161,2}$
75	$k \equiv 207 \pmod{322}$	5153
76	$k \equiv 46 \pmod{322}$	25439
77	$k \equiv 184 \pmod{322}$	2958859
78	$k \equiv 23 \pmod{322}$	9473287662796931849
79	$k \equiv 161 \pmod{322}$	12553064438530403425347011
80	$k \equiv 0 \pmod{322}$	$c_{322,1}$
81	$k \equiv 434 \pmod{483}$	967
82	$k \equiv 365 \pmod{483}$	4971037
83	$k \equiv 296 \pmod{483}$	8738437
84	$k \equiv 227 \pmod{483}$	4310628169
85	$k \equiv 158 \pmod{483}$	35650574938306948573438213
86	$k \equiv 89 \pmod{483}$	$c_{483,1}$
87	$k \equiv 503 \pmod{966}$	32805361
88	$k \equiv 20 \pmod{966}$	$c_{966,1}$
89	$k \equiv 638 \pmod{690}$	$c_{690,1}$
90	$k \equiv 569 \pmod{690}$	$c_{690,2}$
91	$k \equiv 40 \pmod{230}$	691
92	$k \equiv 201 \pmod{230}$	96601
93	$k \equiv 132 \pmod{230}$	114311
94	$k \equiv 63 \pmod{230}$	195271
95	$k \equiv 224 \pmod{230}$	109284961
96	$k \equiv 155 \pmod{230}$	24049043219391687361
97	$k \equiv 86 \pmod{230}$	P_{230}

Table B.52 cont. Covering for $d = 3$ in Base 31

row	congruence	prime p_i
98	$k \equiv 247 \pmod{460}$	461
99	$k \equiv 17 \pmod{460}$	1381
100	$k \equiv 37 \pmod{46}$	47
101	$k \equiv 14 \pmod{46}$	45414448613
102	$k \equiv 34 \pmod{46}$	293006379555093281221
103	$k \equiv 333 \pmod{368}$	59617
104	$k \equiv 287 \pmod{368}$	31301824241
105	$k \equiv 241 \pmod{368}$	1487662250017
106	$k \equiv 195 \pmod{368}$	2408066513
107	$k \equiv 149 \pmod{368}$	1836839958020962733249
108	$k \equiv 103 \pmod{368}$	1515842933140177198097
109	$k \equiv 57 \pmod{368}$	$c_{368,1}$
110	$k \equiv 11 \pmod{368}$	$c_{368,2}$
111	$k \equiv 192 \pmod{253}$	36364703
112	$k \equiv 123 \pmod{253}$	82982989
113	$k \equiv 54 \pmod{253}$	1975800971
114	$k \equiv 238 \pmod{253}$	$c_{253,1}$
115	$k \equiv 169 \pmod{253}$	$c_{253,2}$
116	$k \equiv 353 \pmod{759}$	28843
117	$k \equiv 284 \pmod{759}$	121673117409559
118	$k \equiv 215 \pmod{759}$	125868759920773
119	$k \equiv 146 \pmod{759}$	$c_{759,1}$
120	$k \equiv 77 \pmod{759}$	$c_{759,2}$
121	$k \equiv 1526 \pmod{2277}$	45541
122	$k \equiv 767 \pmod{2277}$	204931

row	congruence	prime p_i
123	$k \equiv 8 \pmod{2277}$	19253961343
124	$k \equiv 764 \pmod{828}$	1657
125	$k \equiv 626 \pmod{828}$	835453
126	$k \equiv 488 \pmod{828}$	163178101
127	$k \equiv 350 \pmod{828}$	101649383628292477
128	$k \equiv 212 \pmod{828}$	$c_{828,1}$
129	$k \equiv 74 \pmod{828}$	$c_{828,2}$
130	$k \equiv 281 \pmod{414}$	1234978860270061
131	$k \equiv 143 \pmod{414}$	$c_{414,1}$
132	$k \equiv 5 \pmod{414}$	$c_{414,2}$
133	$k \equiv 1037 \pmod{1104}$	10882129
134	$k \equiv 761 \pmod{1104}$	30643732417
135	$k \equiv 485 \pmod{1104}$	$c_{1104,1}$
136	$k \equiv 209 \pmod{1104}$	$c_{1104,2}$
137	$k \equiv 1175 \pmod{1380}$	14403061
138	$k \equiv 899 \pmod{1380}$	27224010721
139	$k \equiv 623 \pmod{1380}$	3063980621941
140	$k \equiv 347 \pmod{1380}$	20426703986910901
141	$k \equiv 71 \pmod{1380}$	$c_{1380,1}$
142	$k \equiv 1382 \pmod{1518}$	3069397
143	$k \equiv 1244 \pmod{1518}$	42826631868636661
144	$k \equiv 1106 \pmod{1518}$	1633831932326535943
145	$k \equiv 968 \pmod{1518}$	$c_{1518,1}$
146	$k \equiv 830 \pmod{1518}$	$c_{1518,2}$
147	$k \equiv 186 \pmod{506}$	20236060635541385933

Table B.52 cont. Covering for $d = 3$ in Base 31

row	congruence	prime p_i
148	$k \equiv 48 \pmod{506}$	p_{148}
149	$k \equiv 416 \pmod{506}$	$c_{506,1}$
150	$k \equiv 278 \pmod{506}$	$c_{506,2}$

row	congruence	prime p_i
151	$k \equiv 1658 \pmod{3036}$	3037
152	$k \equiv 140 \pmod{3036}$	9109
153	$k \equiv 1520 \pmod{3036}$	2398441
154	$k \equiv 2 \pmod{3036}$	p_{154}

For $d = 9$, we use the covering found in Table B.55. The least common multiple of the moduli for the covering is 188496000. To verify that the 186 congruences are indeed a covering, we consider the congruence classes modulo five. Observe that $k \equiv 0 \pmod{5}$ and $k \equiv 2 \pmod{5}$ are the first two congruences listed in Table B.55, so every integer satisfying $x \equiv 0 \pmod{5}$ or $x \equiv 2 \pmod{5}$ satisfies a congruence in Table B.55. One can check that every integer satisfying $x \equiv 1 \pmod{5}$ satisfies one of the 33 congruences in rows d and 31-62 of Table B.55 with moduli dividing 63000; every integer satisfying $x \equiv 3 \pmod{5}$ satisfies one of the 31 congruences in rows c and 1-30 of Table B.55 with moduli dividing 67200; and every integer satisfying $x \equiv 4 \pmod{5}$ satisfies one of the 120 congruences in rows 63-182 of Table B.55 with moduli dividing 9424800. Thus, the 186 congruences in Table B.55 form a covering.

One checks that the primes 11, 17351, 41, and 101 correspond to the congruence conditions $M \equiv 7 \pmod{11}$, $M \equiv -9 \pmod{17351}$, $M \equiv 2 \pmod{41}$, and $M \equiv 84 \pmod{101}$, which agree with the previous congruences involving these primes. Table B.53 contains notable factorizations of $\Phi_n(31)$ for large n used in the covering for $d = 9$. Denote

$$p_{33} = 29949383616269163100979009696998921258 51,$$

$$p_{34} = 605108647823401001709236169890011342691833949935122187,$$

$$p_{41} = 1001140147099004096942077589780663810321002740634 51,$$

$$p_{67} = 292415865518548212725264181682005402545390163298379652017348311.$$

Table B.53 Partial/Full factorizations of $\Phi_n(31)$ for $d = 9$

n	Factorization of $\Phi_n(31)$
125	$5 \cdot 251 \cdot 129001 \cdot 12181751 \cdot C_{125}$
272	$17 \cdot 3266233389553$ $\cdot 19220209997787857 \cdot C_{272}$
425	C_{425}

n	Factorization of $\Phi_n(31)$
600	C_{600}
700	$33601 \cdot C_{700}$
3060	C_{3060}
3740	C_{3740}

For $d = 13$, we use the covering found in Table B.56. The least common multiple of the moduli for the covering is 10167474240. To verify that the 196 congruences are indeed a covering, we consider the congruence classes modulo four. Observe that $k \equiv 2 \pmod 4$ and $k \equiv 3 \pmod 4$ are the first two congruences listed in Table B.56. One can check that every integer satisfying $x \equiv 0 \pmod 4$ satisfies one of the 33 congruences in rows 80-193 of Table B.56 with moduli dividing 19293120; and every integer satisfying $x \equiv 1 \pmod 4$ satisfies one of the 31 congruences in rows c-79 of Table B.56 with moduli dividing 1062432. Thus, the 196 congruences in Table B.56 form a covering.

One checks that the primes 13, 37, and 19 correspond to the congruence conditions $M \equiv 12 \pmod{13}$, $M \equiv 31 \pmod{37}$, and $M \equiv 2 \pmod{19}$, which agree with the previous congruences involving these primes. Table B.54 contains notable factorizations of $\Phi_n(31)$ for large n used in the covering for $d = 13$. Denote

$$p_{18} = 2611166636971615423519181335734428 49307,$$

$$p_{26} = 568972471024107865287021434301977158534824481,$$

$$p_{173} = 3146497205537342278271222320801034786923100431720 63.$$

Table B.54 Partial/Full factorizations of $\Phi_n(31)$ for $d = 13$

n	Factorization of $\Phi_n(31)$
203	$537805077322922545931217 \cdot C_{203}$
812	$29 \cdot C_{812}$
1160	C_{1160}

n	Factorization of $\Phi_n(31)$
1392	C_{1392}
1632	C_{1632}

Table B.55 Covering for $d = 9$ in Base 31

row	congruence	prime p_i
a	$k \equiv 2 \pmod{5}$	11
b	$k \equiv 0 \pmod{5}$	17351
c	$k \equiv 8 \pmod{10}$	41
d	$k \equiv 11 \pmod{25}$	101
1	$k \equiv 153 \pmod{160}$	380641
2	$k \equiv 133 \pmod{160}$	1176641
3	$k \equiv 113 \pmod{160}$	8084410241
4	$k \equiv 93 \pmod{160}$	$c_{160,1}$
5	$k \equiv 73 \pmod{160}$	$c_{160,2}$
6	$k \equiv 213 \pmod{320}$	259201
7	$k \equiv 53 \pmod{320}$	59826881
8	$k \equiv 193 \pmod{320}$	177285860161
9	$k \equiv 33 \pmod{320}$	P_{320}
10	$k \equiv 493 \pmod{640}$	4481
11	$k \equiv 333 \pmod{640}$	75465910527404161
12	$k \equiv 173 \pmod{640}$	$c_{640,1}$
13	$k \equiv 13 \pmod{640}$	$c_{640,2}$
14	$k \equiv 123 \pmod{140}$	281
15	$k \equiv 103 \pmod{140}$	106261
16	$k \equiv 83 \pmod{140}$	107941
17	$k \equiv 63 \pmod{140}$	77890519715454496560396129461
18	$k \equiv 43 \pmod{140}$	1534835130260968837905730005181
19	$k \equiv 23 \pmod{70}$	60427990638165876546967231
20	$k \equiv 563 \pmod{700}$	33601

row	congruence	prime p_i
21	$k \equiv 1123 \pmod{1400}$	1499068201
22	$k \equiv 423 \pmod{1400}$	546521676404201
23	$k \equiv 983 \pmod{1400}$	$c_{1400,1}$
24	$k \equiv 283 \pmod{1400}$	$c_{1400,2}$
25	$k \equiv 1543 \pmod{2100}$	4201
26	$k \equiv 843 \pmod{2100}$	287701
27	$k \equiv 143 \pmod{2100}$	1569015001
28	$k \equiv 1403 \pmod{2100}$	3321889201
29	$k \equiv 703 \pmod{2100}$	67545126601
30	$k \equiv 3 \pmod{2100}$	91052684101
31	$k \equiv 71 \pmod{75}$	151
32	$k \equiv 46 \pmod{75}$	997936488044528101
33	$k \equiv 21 \pmod{75}$	p_{33}
34	$k \equiv 116 \pmod{125}$	251
35	$k \equiv 91 \pmod{125}$	129001
36	$k \equiv 66 \pmod{125}$	12181751
37	$k \equiv 41 \pmod{125}$	$c_{125,1}$
38	$k \equiv 16 \pmod{125}$	$c_{125,2}$
39	$k \equiv 131 \pmod{150}$	21001
40	$k \equiv 106 \pmod{150}$	214651
41	$k \equiv 81 \pmod{150}$	p_{41}
42	$k \equiv 206 \pmod{300}$	29777432504101
43	$k \equiv 56 \pmod{300}$	$c_{300,1}$
44	$k \equiv 181 \pmod{300}$	$c_{300,2}$

Table B.55 cont. Covering for $d = 9$ in Base 31

row	congruence	prime p_i
45	$k \equiv 331 \pmod{600}$	$c_{600,1}$
46	$k \equiv 31 \pmod{600}$	$c_{600,2}$
47	$k \equiv 306 \pmod{450}$	43201
48	$k \equiv 156 \pmod{450}$	$c_{450,1}$
49	$k \equiv 6 \pmod{450}$	$c_{450,2}$
50	$k \equiv 151 \pmod{175}$	161408801
51	$k \equiv 126 \pmod{175}$	$c_{175,1}$
52	$k \equiv 101 \pmod{175}$	$c_{175,2}$
53	$k \equiv 251 \pmod{350}$	701
54	$k \equiv 76 \pmod{350}$	224351
55	$k \equiv 226 \pmod{350}$	757751
56	$k \equiv 51 \pmod{350}$	2418095051
57	$k \equiv 201 \pmod{350}$	P_{350}
58	$k \equiv 376 \pmod{700}$	$c_{700,1}$
59	$k \equiv 26 \pmod{700}$	$c_{700,2}$
60	$k \equiv 351 \pmod{525}$	316521451
61	$k \equiv 176 \pmod{525}$	63337748513662351
62	$k \equiv 1 \pmod{525}$	$c_{525,1}$
63	$k \equiv 16 \pmod{17}$	751670559138758105956097
64	$k \equiv 79 \pmod{85}$	108971
65	$k \equiv 74 \pmod{85}$	391206807721
66	$k \equiv 69 \pmod{85}$	21736504684553261
67	$k \equiv 64 \pmod{85}$	p_{67}

row	congruence	prime p_i
68	$k \equiv 144 \pmod{170}$	232122807601
69	$k \equiv 59 \pmod{170}$	P_{170}
70	$k \equiv 224 \pmod{255}$	934831
71	$k \equiv 139 \pmod{255}$	$c_{255,1}$
72	$k \equiv 54 \pmod{255}$	$c_{255,2}$
73	$k \equiv 304 \pmod{340}$	1361
74	$k \equiv 219 \pmod{340}$	3061
75	$k \equiv 134 \pmod{340}$	30941
76	$k \equiv 49 \pmod{340}$	2964461
77	$k \equiv 299 \pmod{340}$	5153381
78	$k \equiv 214 \pmod{340}$	$c_{340,1}$
79	$k \equiv 129 \pmod{340}$	$c_{340,2}$
80	$k \equiv 384 \pmod{680}$	3576121
81	$k \equiv 44 \pmod{680}$	$c_{680,1}$
82	$k \equiv 549 \pmod{595}$	108837401
83	$k \equiv 464 \pmod{595}$	89278561
84	$k \equiv 379 \pmod{595}$	42498121331
85	$k \equiv 294 \pmod{595}$	39338488032189006044000041
86	$k \equiv 209 \pmod{595}$	$c_{595,1}$
87	$k \equiv 124 \pmod{595}$	$c_{595,2}$
88	$k \equiv 634 \pmod{1190}$	2381
89	$k \equiv 39 \pmod{1190}$	6294558137071
90	$k \equiv 374 \pmod{425}$	$c_{425,1}$

Table B.55 cont. Covering for $d = 9$ in Base 31

row	congruence	prime p_i
91	$k \equiv 289 \pmod{425}$	$c_{425,2}$
92	$k \equiv 629 \pmod{850}$	40801
93	$k \equiv 204 \pmod{850}$	82388290001
94	$k \equiv 544 \pmod{850}$	$c_{850,1}$
95	$k \equiv 969 \pmod{1700}$	630701
96	$k \equiv 119 \pmod{1700}$	865301
97	$k \equiv 884 \pmod{1275}$	845686732801
98	$k \equiv 459 \pmod{1275}$	45815896553375206051
99	$k \equiv 34 \pmod{1275}$	$c_{1275,1}$
100	$k \equiv 709 \pmod{765}$	1531
101	$k \equiv 624 \pmod{765}$	6121
102	$k \equiv 539 \pmod{765}$	1074061
103	$k \equiv 454 \pmod{765}$	8197741
104	$k \equiv 369 \pmod{765}$	61624160936101
105	$k \equiv 284 \pmod{765}$	1761543429003124265251
106	$k \equiv 199 \pmod{765}$	$c_{765,1}$
107	$k \equiv 879 \pmod{1530}$	1656910504456939277461
108	$k \equiv 114 \pmod{1530}$	$c_{1530,1}$
109	$k \equiv 794 \pmod{1530}$	$c_{1530,2}$
110	$k \equiv 1559 \pmod{3060}$	$c_{3060,1}$
111	$k \equiv 29 \pmod{3060}$	$c_{3060,2}$
112	$k \equiv 126 \pmod{136}$	137
113	$k \equiv 109 \pmod{136}$	953
114	$k \equiv 92 \pmod{136}$	11833
115	$k \equiv 75 \pmod{136}$	63574561
116	$k \equiv 58 \pmod{136}$	P_{136}
117	$k \equiv 177 \pmod{272}$	3266233389553
118	$k \equiv 41 \pmod{272}$	19220209997787857
119	$k \equiv 160 \pmod{272}$	$c_{272,1}$
120	$k \equiv 24 \pmod{272}$	$c_{272,2}$
121	$k \equiv 415 \pmod{544}$	49824961
122	$k \equiv 279 \pmod{544}$	3688708633601
123	$k \equiv 143 \pmod{544}$	72338290404869473
124	$k \equiv 7 \pmod{544}$	$c_{544,1}$
125	$k \equiv 444 \pmod{510}$	3745479278144701
126	$k \equiv 359 \pmod{510}$	12939956633160901
127	$k \equiv 274 \pmod{510}$	P_{510}
128	$k \equiv 699 \pmod{1020}$	1021
129	$k \equiv 189 \pmod{1020}$	5101
130	$k \equiv 614 \pmod{1020}$	408350881
131	$k \equiv 104 \pmod{1020}$	28830845215221541
132	$k \equiv 529 \pmod{1020}$	$c_{1020,1}$
133	$k \equiv 19 \pmod{1020}$	$c_{1020,2}$
134	$k \equiv 116 \pmod{119}$	239
135	$k \equiv 99 \pmod{119}$	2857
136	$k \equiv 82 \pmod{119}$	78541

Table B.55 cont. Covering for $d = 9$ in Base 31

row	congruence	prime p_i
137	$k \equiv 65 \pmod{119}$	P_{119}
138	$k \equiv 167 \pmod{238}$	11781795397
139	$k \equiv 48 \pmod{238}$	2744226674742050863
140	$k \equiv 150 \pmod{238}$	$c_{238,1}$
141	$k \equiv 31 \pmod{238}$	$c_{238,2}$
142	$k \equiv 371 \pmod{476}$	5237
143	$k \equiv 252 \pmod{476}$	$c_{476,1}$
144	$k \equiv 133 \pmod{476}$	$c_{476,2}$
145	$k \equiv 490 \pmod{952}$	9521
146	$k \equiv 14 \pmod{952}$	$c_{952,1}$
147	$k \equiv 43 \pmod{51}$	1961163283
148	$k \equiv 281 \pmod{306}$	307
149	$k \equiv 230 \pmod{306}$	78198146102753533
150	$k \equiv 179 \pmod{306}$	$c_{306,1}$
151	$k \equiv 128 \pmod{306}$	$c_{306,2}$
152	$k \equiv 383 \pmod{612}$	15913
153	$k \equiv 77 \pmod{612}$	34273
154	$k \equiv 332 \pmod{612}$	$c_{612,1}$
155	$k \equiv 26 \pmod{612}$	$c_{612,2}$
156	$k \equiv 315 \pmod{357}$	637603
157	$k \equiv 264 \pmod{357}$	2112727
158	$k \equiv 213 \pmod{357}$	2346688669627
159	$k \equiv 162 \pmod{357}$	4299055873813
160	$k \equiv 111 \pmod{357}$	2563210372003
161	$k \equiv 60 \pmod{357}$	55648343218960782949
162	$k \equiv 9 \pmod{357}$	$c_{357,1}$
163	$k \equiv 106 \pmod{187}$	1871
164	$k \equiv 21 \pmod{187}$	P_{187}
165	$k \equiv 684 \pmod{935}$	447732031781
166	$k \equiv 599 \pmod{935}$	$c_{935,1}$
167	$k \equiv 514 \pmod{935}$	$c_{935,2}$
168	$k \equiv 1364 \pmod{1870}$	796776797560835338501
169	$k \equiv 429 \pmod{1870}$	$c_{1870,1}$
170	$k \equiv 1279 \pmod{1870}$	$c_{1870,2}$
171	$k \equiv 2214 \pmod{3740}$	$c_{3740,1}$
172	$k \equiv 344 \pmod{3740}$	$c_{3740,2}$
173	$k \equiv 2129 \pmod{2805}$	28051
174	$k \equiv 1194 \pmod{2805}$	1716661
175	$k \equiv 259 \pmod{2805}$	$c_{2805,1}$
176	$k \equiv 361 \pmod{374}$	1123
177	$k \equiv 174 \pmod{374}$	8752362848923
178	$k \equiv 276 \pmod{374}$	$c_{374,1}$
179	$k \equiv 89 \pmod{374}$	$c_{374,2}$
180	$k \equiv 378 \pmod{561}$	29173
181	$k \equiv 191 \pmod{561}$	61694865343
182	$k \equiv 4 \pmod{561}$	$c_{561,1}$

Table B.56 Covering for $d = 13$ in Base 31

row	congruence	prime p_i
a	$k \equiv 2 \pmod{4}$	13
b	$k \equiv 3 \pmod{4}$	37
c	$k \equiv 5 \pmod{6}$	19
1	$k \equiv 201 \pmod{204}$	613
2	$k \equiv 189 \pmod{204}$	1429
3	$k \equiv 177 \pmod{204}$	2157381990573913
4	$k \equiv 165 \pmod{204}$	352607991704483661973
5	$k \equiv 153 \pmod{204}$	P_{204}
6	$k \equiv 345 \pmod{408}$	8161
7	$k \equiv 141 \pmod{408}$	27031633
8	$k \equiv 333 \pmod{408}$	12842210852731378177
9	$k \equiv 129 \pmod{408}$	P_{408}
10	$k \equiv 15 \pmod{102}$	2796214962413636917873
11	$k \equiv 3 \pmod{102}$	195333779873358973907838097
12	$k \equiv 144 \pmod{153}$	17137
13	$k \equiv 93 \pmod{153}$	42834601810502407
14	$k \equiv 42 \pmod{153}$	P_{153}
15	$k \equiv 13 \pmod{68}$	1399577
16	$k \equiv 1 \pmod{68}$	224499664484159761
17	$k \equiv 57 \pmod{68}$	1682325489672499143634073
18	$k \equiv 45 \pmod{51}$	p_{18}
19	$k \equiv 33 \pmod{34}$	103
20	$k \equiv 21 \pmod{34}$	6841661642646463343047
21	$k \equiv 621 \pmod{816}$	1410099665109361
22	$k \equiv 417 \pmod{816}$	$c_{816,1}$
23	$k \equiv 213 \pmod{816}$	$c_{816,2}$
24	$k \equiv 825 \pmod{1632}$	$c_{1632,1}$
25	$k \equiv 9 \pmod{1632}$	$c_{1632,2}$
26	$k \equiv 20 \pmod{31}$	p_{26}
27	$k \equiv 39 \pmod{62}$	373
28	$k \equiv 27 \pmod{62}$	1613
29	$k \equiv 15 \pmod{62}$	62869
30	$k \equiv 3 \pmod{62}$	145577
31	$k \equiv 53 \pmod{62}$	35789156484227
32	$k \equiv 41 \pmod{62}$	2706690202468649
33	$k \equiv 91 \pmod{93}$	15799367012336491417
34	$k \equiv 79 \pmod{93}$	$c_{93,1}$
35	$k \equiv 67 \pmod{93}$	$c_{93,2}$
36	$k \equiv 117 \pmod{124}$	415153
37	$k \equiv 105 \pmod{124}$	P_{124}
38	$k \equiv 31 \pmod{186}$	23251
39	$k \equiv 19 \pmod{186}$	53780227
40	$k \equiv 7 \pmod{186}$	23171696419
41	$k \equiv 181 \pmod{186}$	7513329295414649737
42	$k \equiv 169 \pmod{186}$	7255587057776337278077
43	$k \equiv 157 \pmod{186}$	198129096248543967569450023
44	$k \equiv 145 \pmod{372}$	44641
45	$k \equiv 133 \pmod{372}$	238369488913
46	$k \equiv 121 \pmod{372}$	$c_{372,1}$
47	$k \equiv 109 \pmod{372}$	$c_{372,2}$

Table B.56 cont. Covering for $d = 13$ in Base 31

row	congruence	prime p_i
48	$k \equiv 469 \pmod{744}$	1489
49	$k \equiv 97 \pmod{744}$	$c_{744,1}$
50	$k \equiv 457 \pmod{744}$	$c_{744,2}$
51	$k \equiv 85 \pmod{248}$	1859009
52	$k \equiv 197 \pmod{248}$	12703553
53	$k \equiv 73 \pmod{248}$	3938289601
54	$k \equiv 185 \pmod{248}$	$c_{248,1}$
55	$k \equiv 61 \pmod{248}$	$c_{248,2}$
56	$k \equiv 204 \pmod{217}$	1093681
57	$k \equiv 173 \pmod{217}$	347201
58	$k \equiv 142 \pmod{217}$	68657933
59	$k \equiv 111 \pmod{217}$	5978703277
60	$k \equiv 80 \pmod{217}$	3430295651021
61	$k \equiv 49 \pmod{217}$	$c_{217,1}$
62	$k \equiv 18 \pmod{217}$	$c_{217,2}$
63	$k \equiv 409 \pmod{434}$	14323
64	$k \equiv 347 \pmod{434}$	1547154449
65	$k \equiv 285 \pmod{434}$	70144817
66	$k \equiv 223 \pmod{434}$	37591116739373
67	$k \equiv 161 \pmod{434}$	2522055548449945641667 39
68	$k \equiv 99 \pmod{434}$	$c_{434,1}$
69	$k \equiv 37 \pmod{434}$	$c_{434,2}$
70	$k \equiv 211 \pmod{279}$	1117
71	$k \equiv 118 \pmod{279}$	45757
72	$k \equiv 25 \pmod{279}$	2301193

row	congruence	prime p_i
73	$k \equiv 199 \pmod{279}$	$c_{279,1}$
74	$k \equiv 106 \pmod{279}$	$c_{279,2}$
75	$k \equiv 292 \pmod{558}$	5023
76	$k \equiv 13 \pmod{558}$	2791
77	$k \equiv 373 \pmod{558}$	320852851939
78	$k \equiv 187 \pmod{558}$	1024143471271
79	$k \equiv 1 \pmod{558}$	14301952851224827
80	$k \equiv 25 \pmod{29}$	349
81	$k \equiv 21 \pmod{29}$	10789
82	$k \equiv 17 \pmod{29}$	49823
83	$k \equiv 13 \pmod{29}$	1482570191
84	$k \equiv 9 \pmod{29}$	11242578713
85	$k \equiv 5 \pmod{29}$	189343400041
86	$k \equiv 30 \pmod{58}$	59
87	$k \equiv 26 \pmod{58}$	1838659
88	$k \equiv 22 \pmod{58}$	1671541885847
89	$k \equiv 18 \pmod{58}$	306103768011661849660 3
90	$k \equiv 72 \pmod{116}$	233
91	$k \equiv 68 \pmod{116}$	18329
92	$k \equiv 64 \pmod{116}$	646550190571213
93	$k \equiv 60 \pmod{116}$	P_{116}
94	$k \equiv 172 \pmod{232}$	1176937
95	$k \equiv 56 \pmod{232}$	3502913268292732159 3
96	$k \equiv 168 \pmod{232}$	$c_{232,1}$
97	$k \equiv 52 \pmod{232}$	$c_{232,2}$

Table B.56 cont.   Covering for $d = 13$ in Base 31

row	congruence	prime p_i
98	$k \equiv 280 \pmod{348}$	10438779913
99	$k \equiv 164 \pmod{348}$	1602846874753
100	$k \equiv 48 \pmod{348}$	$c_{348,1}$
101	$k \equiv 276 \pmod{348}$	$c_{348,2}$
102	$k \equiv 160 \pmod{174}$	208898666650411
103	$k \equiv 44 \pmod{174}$	5155437699613779037
104	$k \equiv 69 \pmod{87}$	2437
105	$k \equiv 40 \pmod{87}$	29581
106	$k \equiv 11 \pmod{87}$	6226417
107	$k \equiv 65 \pmod{87}$	24589681
108	$k \equiv 36 \pmod{87}$	19442411479
109	$k \equiv 7 \pmod{87}$	P_{87}
110	$k \equiv 380 \pmod{464}$	5569
111	$k \equiv 264 \pmod{464}$	182353
112	$k \equiv 148 \pmod{464}$	1658801
113	$k \equiv 32 \pmod{464}$	$c_{464,1}$
114	$k \equiv 376 \pmod{464}$	$c_{464,2}$
115	$k \equiv 724 \pmod{928}$	929
116	$k \equiv 260 \pmod{928}$	$c_{928,1}$
117	$k \equiv 608 \pmod{928}$	$c_{928,2}$
118	$k \equiv 1072 \pmod{1856}$	261697
119	$k \equiv 144 \pmod{1856}$	5189377
120	$k \equiv 956 \pmod{1392}$	$c_{1392,1}$
121	$k \equiv 492 \pmod{1392}$	$c_{1392,2}$
122	$k \equiv 1420 \pmod{2784}$	19489
123	$k \equiv 28 \pmod{2784}$	101952662530273
124	$k \equiv 488 \pmod{580}$	1330512111189514381
125	$k \equiv 372 \pmod{580}$	$c_{580,1}$
126	$k \equiv 256 \pmod{580}$	$c_{580,2}$
127	$k \equiv 720 \pmod{1160}$	$c_{1160,1}$
128	$k \equiv 140 \pmod{1160}$	$c_{1160,2}$
129	$k \equiv 24 \pmod{290}$	95543420591
130	$k \equiv 252 \pmod{290}$	$c_{290,1}$
131	$k \equiv 194 \pmod{290}$	$c_{290,2}$
132	$k \equiv 136 \pmod{145}$	140071
133	$k \equiv 78 \pmod{145}$	359311
134	$k \equiv 20 \pmod{145}$	P_{145}
135	$k \equiv 712 \pmod{812}$	$c_{812,1}$
136	$k \equiv 596 \pmod{812}$	$c_{812,2}$
137	$k \equiv 74 \pmod{406}$	404783
138	$k \equiv 364 \pmod{406}$	$c_{406,1}$
139	$k \equiv 248 \pmod{406}$	$c_{406,2}$
140	$k \equiv 132 \pmod{203}$	53780507732292545931217
141	$k \equiv 828 \pmod{1624}$	1781363977
142	$k \equiv 16 \pmod{1624}$	$c_{1624,1}$
143	$k \equiv 940 \pmod{1044}$	78301
144	$k \equiv 824 \pmod{1044}$	26378749
145	$k \equiv 708 \pmod{1044}$	20472841

Table B.56 cont. Covering for $d = 13$ in Base 31

row	congruence	prime p_i
146	$k \equiv 592 \pmod{1044}$	9794139325309
147	$k \equiv 476 \pmod{1044}$	7252412666245773224441
148	$k \equiv 360 \pmod{1044}$	$c_{1044,1}$
149	$k \equiv 244 \pmod{1044}$	$c_{1044,2}$
150	$k \equiv 128 \pmod{522}$	525763621
151	$k \equiv 12 \pmod{522}$	102182193014509
152	$k \equiv 472 \pmod{522}$	78984406417023018433
153	$k \equiv 414 \pmod{522}$	P_{522}
154	$k \equiv 95 \pmod{261}$	523
155	$k \equiv 37 \pmod{261}$	202338644001987940129
156	$k \equiv 240 \pmod{261}$	$c_{261,1}$
157	$k \equiv 182 \pmod{261}$	$c_{261,2}$
158	$k \equiv 646 \pmod{783}$	1567
159	$k \equiv 385 \pmod{783}$	20359
160	$k \equiv 124 \pmod{783}$	25057
161	$k \equiv 588 \pmod{783}$	6251973555615521347
162	$k \equiv 327 \pmod{783}$	$c_{783,1}$
163	$k \equiv 66 \pmod{783}$	$c_{783,2}$
164	$k \equiv 1052 \pmod{1566}$	34638908812126561
165	$k \equiv 530 \pmod{1566}$	1999886726073043
166	$k \equiv 8 \pmod{1566}$	$c_{1566,1}$
167	$k \equiv 410 \pmod{435}$	1741
168	$k \equiv 323 \pmod{435}$	19141
169	$k \equiv 236 \pmod{435}$	$c_{435,1}$

row	congruence	prime p_i
170	$k \equiv 149 \pmod{435}$	$c_{435,2}$
171	$k \equiv 497 \pmod{870}$	270295951
172	$k \equiv 62 \pmod{870}$	510393218641
173	$k \equiv 120 \pmod{174}$	p_{173}
174	$k \equiv 120 \pmod{203}$	$c_{203,1}$
175	$k \equiv 33 \pmod{203}$	$c_{203,2}$
176	$k \equiv 352 \pmod{609}$	32887
177	$k \equiv 265 \pmod{609}$	1387480939340959
178	$k \equiv 178 \pmod{609}$	$c_{609,1}$
179	$k \equiv 91 \pmod{609}$	$c_{609,2}$
180	$k \equiv 613 \pmod{1218}$	173678036123955341833
181	$k \equiv 4 \pmod{1218}$	$c_{1218,1}$
182	$k \equiv 203 \pmod{319}$	19637641
183	$k \equiv 87 \pmod{319}$	$c_{319,1}$
184	$k \equiv 290 \pmod{319}$	$c_{319,2}$
185	$k \equiv 174 \pmod{638}$	4319022352657
186	$k \equiv 58 \pmod{638}$	$c_{638,1}$
187	$k \equiv 580 \pmod{638}$	$c_{638,2}$
188	$k \equiv 464 \pmod{1276}$	1277
189	$k \equiv 348 \pmod{1276}$	38775089
190	$k \equiv 232 \pmod{1276}$	$c_{1276,1}$
191	$k \equiv 116 \pmod{1276}$	$c_{1276,2}$
192	$k \equiv 1276 \pmod{2552}$	20453336796945534241
193	$k \equiv 0 \pmod{2552}$	$c_{2552,1}$

We use the covering found in Table B.58 for $d = 19$. The least common multiple of the moduli for this covering is 720720. A direct analysis can be used for verification. One checks that the prime 23 corresponds to the congruence condition $M \equiv 10 \pmod{23}$, which agrees with the congruence condition on M from the congruence in Table B.45. Table B.57 contains notable factorizations of $\Phi_n(31)$ for large n used in the covering for $d = 19$. To conserve space, denote the following primes appearing in Table B.58 as follows.

$$p_{26} = 11293636368958098920863036926271138717 21,$$

$$p_{32} = 22472120241291866633457681925052319 1369,$$

$$p_{54} = 272620054274111996657517726155748513108418866372275455497 2281,$$

$$p_{80} = 29510535204545262157687088665468191118389698834341366722 5871.$$

Table B.57 Partial/Full factorizations of $\Phi_n(31)$ for $d = 19$

n	Factorization of $\Phi_n(31)$
182	$547 \cdot 1093 \cdot 647011 \cdot 14407667 \cdot 669665808855863 \cdot P_{182}$
273	$506689 \cdot 211173991474447 \cdot 364319063684404 93$ $\cdot\ 81651312342656239 \cdot P_{273}$
312	$313 \cdot 4292809 \cdot 135272593 \cdot 115436220433 \cdot P_{312}$
390	$188431581362701 \cdot 9708046814116951 \cdot C_{390}$
468	$937 \cdot 1015561$ $\cdot\ 880289895387859281 \cdot 604447799126643206661 2797$ $\cdot\ 377280544142980078234 93 \cdot C_{468}$
715	C_{715}
1716	$32312281 \cdot 4296187082124305941$ $\cdot\ 1852852357362091801 \cdot C_{1716}$

Table B.58 Covering for $d = 19$ in Base 31

row	congruence	prime p_i
a	$k \equiv 7 \pmod{11}$	23
1	$k \equiv 12 \pmod{13}$	42407
2	$k \equiv 11 \pmod{13}$	2426789
3	$k \equiv 10 \pmod{13}$	7908811
4	$k \equiv 22 \pmod{26}$	17863
5	$k \equiv 9 \pmod{26}$	42716694944587
6	$k \equiv 47 \pmod{52}$	53
7	$k \equiv 34 \pmod{52}$	116337521
8	$k \equiv 21 \pmod{52}$	76037563733
9	$k \equiv 8 \pmod{52}$	101686136508893
10	$k \equiv 32 \pmod{39}$	79
11	$k \equiv 19 \pmod{39}$	13807
12	$k \equiv 6 \pmod{39}$	39703
13	$k \equiv 31 \pmod{39}$	175500339130677572941801
14	$k \equiv 96 \pmod{117}$	2534689
15	$k \equiv 57 \pmod{117}$	4618898267815261
16	$k \equiv 18 \pmod{117}$	$c_{117,1}$
17	$k \equiv 83 \pmod{117}$	$c_{117,2}$
18	$k \equiv 161 \pmod{234}$	4447
19	$k \equiv 44 \pmod{234}$	7776289
20	$k \equiv 122 \pmod{234}$	21931507729
21	$k \equiv 5 \pmod{234}$	903087677909176579
22	$k \equiv 56 \pmod{65}$	911

row	congruence	prime p_i
23	$k \equiv 43 \pmod{65}$	1951
24	$k \equiv 30 \pmod{65}$	31035996941
25	$k \equiv 17 \pmod{65}$	5979236519649901
26	$k \equiv 4 \pmod{65}$	p_{26}
27	$k \equiv 68 \pmod{78}$	157
28	$k \equiv 42 \pmod{78}$	238213
29	$k \equiv 16 \pmod{78}$	17123370267331917425544180721
30	$k \equiv 81 \pmod{104}$	305688893141113
31	$k \equiv 55 \pmod{104}$	5603212901768856193
32	$k \equiv 29 \pmod{104}$	p_{32}
33	$k \equiv 107 \pmod{208}$	6060328173121
34	$k \equiv 3 \pmod{208}$	P_{208}
35	$k \equiv 80 \pmod{91}$	2549
36	$k \equiv 67 \pmod{91}$	1661479
37	$k \equiv 54 \pmod{91}$	473516688426601
38	$k \equiv 41 \pmod{91}$	$c_{91,1}$
39	$k \equiv 28 \pmod{91}$	$c_{91,2}$
40	$k \equiv 106 \pmod{182}$	547
41	$k \equiv 15 \pmod{182}$	1093
42	$k \equiv 93 \pmod{182}$	647011
43	$k \equiv 2 \pmod{182}$	14407667
44	$k \equiv 261 \pmod{273}$	506689
45	$k \equiv 222 \pmod{273}$	211173991474447

Table B.58 cont.   Covering for $d = 19$ in Base 31

row	congruence	prime p_i
46	$k \equiv 183 \pmod{273}$	36431906368440493
47	$k \equiv 144 \pmod{273}$	81651312342656239
48	$k \equiv 105 \pmod{273}$	P_{273}
49	$k \equiv 157 \pmod{182}$	669665808855863
50	$k \equiv 66 \pmod{182}$	P_{182}
51	$k \equiv 300 \pmod{546}$	30437088338336269167943
52	$k \equiv 27 \pmod{546}$	$c_{546,1}$
53	$k \equiv 131 \pmod{156}$	141336778441
54	$k \equiv 53 \pmod{156}$	p_{54}
55	$k \equiv 248 \pmod{312}$	313
56	$k \equiv 170 \pmod{312}$	4292809
57	$k \equiv 92 \pmod{312}$	135272593
58	$k \equiv 14 \pmod{312}$	115436220433
59	$k \equiv 430 \pmod{468}$	937
60	$k \equiv 352 \pmod{468}$	1015561
61	$k \equiv 274 \pmod{468}$	1880289895387859281
62	$k \equiv 196 \pmod{468}$	37728054414298007823493
63	$k \equiv 118 \pmod{468}$	6044477991266432066612797
64	$k \equiv 40 \pmod{468}$	$c_{468,1}$
65	$k \equiv 547 \pmod{624}$	1249
66	$k \equiv 391 \pmod{624}$	1873
67	$k \equiv 235 \pmod{624}$	275937793
68	$k \equiv 79 \pmod{624}$	$c_{624,1}$
69	$k \equiv 313 \pmod{468}$	$c_{468,2}$
70	$k \equiv 781 \pmod{936}$	7489
71	$k \equiv 625 \pmod{936}$	P_{936}
72	$k \equiv 157 \pmod{312}$	P_{312}
73	$k \equiv 1093 \pmod{1872}$	282419281
74	$k \equiv 157 \pmod{1872}$	12517991911153
75	$k \equiv 937 \pmod{1872}$	$c_{1872,1}$
76	$k \equiv 1 \pmod{1872}$	$c_{1872,2}$
77	$k \equiv 117 \pmod{130}$	131
78	$k \equiv 52 \pmod{130}$	521
79	$k \equiv 104 \pmod{130}$	197271101
80	$k \equiv 39 \pmod{130}$	p_{80}
81	$k \equiv 156 \pmod{195}$	35956831
82	$k \equiv 91 \pmod{195}$	617235858721
83	$k \equiv 26 \pmod{195}$	P_{195}
84	$k \equiv 260 \pmod{390}$	$c_{390,1}$
85	$k \equiv 65 \pmod{390}$	$c_{390,2}$
86	$k \equiv 208 \pmod{260}$	1301
87	$k \equiv 143 \pmod{260}$	22765081
88	$k \equiv 78 \pmod{260}$	P_{260}
89	$k \equiv 273 \pmod{520}$	995048024881
90	$k \equiv 13 \pmod{520}$	P_{520}
91	$k \equiv 325 \pmod{390}$	188431581362701

Table B.58 cont.   Covering for $d = 19$ in Base 31

row	congruence	prime p_i
92	$k \equiv 130 \pmod{390}$	9708046814116951
93	$k \equiv 390 \pmod{585}$	1171
94	$k \equiv 195 \pmod{585}$	633950284120305126391
95	$k \equiv 0 \pmod{585}$	$c_{585,1}$
96	$k \equiv 137 \pmod{143}$	3719
97	$k \equiv 124 \pmod{143}$	P_{143}
98	$k \equiv 254 \pmod{286}$	2861
99	$k \equiv 111 \pmod{286}$	161769686049971436885163
100	$k \equiv 241 \pmod{286}$	24324654055543532347208267
101	$k \equiv 98 \pmod{286}$	$c_{286,1}$
102	$k \equiv 371 \pmod{429}$	859
103	$k \equiv 228 \pmod{429}$	17065706659
104	$k \equiv 85 \pmod{429}$	53096473
105	$k \equiv 358 \pmod{429}$	205040647813
106	$k \equiv 215 \pmod{429}$	$c_{429,1}$
107	$k \equiv 72 \pmod{429}$	$c_{429,2}$
108	$k \equiv 488 \pmod{572}$	14820521
109	$k \equiv 345 \pmod{572}$	152830062529
110	$k \equiv 202 \pmod{572}$	11035333453
111	$k \equiv 59 \pmod{572}$	$c_{572,1}$
112	$k \equiv 475 \pmod{572}$	$c_{572,2}$
113	$k \equiv 904 \pmod{1144}$	243673

row	congruence	prime p_i
114	$k \equiv 332 \pmod{1144}$	4728211489
115	$k \equiv 761 \pmod{1144}$	$c_{1144,1}$
116	$k \equiv 189 \pmod{1144}$	$c_{1144,2}$
117	$k \equiv 1190 \pmod{1716}$	32312281
118	$k \equiv 618 \pmod{1716}$	4296187082124305941
119	$k \equiv 46 \pmod{1716}$	1852852357362091801
120	$k \equiv 605 \pmod{715}$	$c_{715,1}$
121	$k \equiv 462 \pmod{715}$	$c_{715,2}$
122	$k \equiv 1034 \pmod{1430}$	8581
123	$k \equiv 319 \pmod{1430}$	97241
124	$k \equiv 891 \pmod{1430}$	$c_{1430,1}$
125	$k \equiv 176 \pmod{1430}$	$c_{1430,2}$
126	$k \equiv 1463 \pmod{2145}$	1184940901
127	$k \equiv 748 \pmod{2145}$	$c_{2145,1}$
128	$k \equiv 33 \pmod{2145}$	$c_{2145,2}$
129	$k \equiv 735 \pmod{858}$	495067
130	$k \equiv 592 \pmod{858}$	23441183753571013
131	$k \equiv 449 \pmod{858}$	672557508032103409
132	$k \equiv 306 \pmod{858}$	$c_{858,1}$
133	$k \equiv 163 \pmod{858}$	$c_{858,2}$
134	$k \equiv 878 \pmod{1716}$	$c_{1716,1}$
135	$k \equiv 20 \pmod{1716}$	$c_{1716,2}$

We use the covering found in Table B.60 for $d = 27$. The least common multiple of the moduli for this covering is 18404305920. To verify that the 333 congruences are indeed a covering, we consider the way in which we constructed the covering. When constructing this covering we considered the smaller covering defined by the congruences

$$x \equiv 1 \pmod{2}, \quad x \equiv 0 \pmod{4},$$
$$x \equiv 2 \pmod{8}, \quad \text{and} \quad x \equiv 6 \pmod{8}.$$

Observe that $k \equiv 0 \pmod{4}$ is the first congruence listed in Table B.60. One can check that every integer satisfying $x \equiv 1 \pmod{2}$ satisfies one of the 146 congruences in rows 187-332 of Table B.60 with moduli dividing 12640320; every integer satisfying $x \equiv 2 \pmod{8}$ satisfies one of the 100 congruences in rows 87-186 of Table B.60 with moduli dividing 138378240; and every integer satisfying $x \equiv 6 \pmod{8}$ satisfies one of the 86 congruences in rows 1-86 of Table B.60 with moduli dividing 3669120. Thus, the 333 congruences in Table B.60 form a covering.

One checks that the prime 13 corresponds to the congruence condition $M \equiv 12 \pmod{13}$, which agrees with the congruence condition on M from the congruence in Table B.41. Table B.59 contains notable factorizations of $\Phi_n(31)$ for large n used in the covering for $d = 27$. To conserve space, we write

$$p_{88} = 5510772735392838103696416724649435090995481,$$
$$p_{89} = 7181521717145072083078608601358856159842333085909766829569,$$
$$p_{218} = 1434879358379433691210638778172176147818557361098186883210580561,$$
$$p_{246} = 1716439847900062900800798410166938893.$$

Table B.59 Partial/Full factorizations of $\Phi_n(31)$ for $d = 27$

n	Factorization of $\Phi_n(31)$
95	$793926617318201 \cdot P_{95}$
190	C_{190}
256	$2 \cdot p_{89} \cdot P_{256}$
342	C_{342}
352	$353 \cdot 136344440321 \cdot 6766051802154664547351297 \cdot C_{352}$
399	C_{399}
512	$2 \cdot 23583164929 \cdot 99578469377 \cdot P_{512}$
532	C_{532}
672	$673 \cdot 20673051169 \cdot 4408464948749687469840 1 \cdot C_{672}$
840	$52081 \cdot 30241 \cdot 13842240721 \cdot C_{840}$
912	C_{912}
1232	$15089537 \cdot 689761073 \cdot C_{1232}$
1330	C_{1330}
1344	$71233 \cdot 3444673 \cdot C_{1344}$
1520	C_{1520}
1680	$104161 \cdot 356346481 \cdot 1389156854401 \cdot C_{1680}$
1995	C_{1995}
2016	$1135009 \cdot 86689 \cdot 21615553 \cdot C_{2016}$
2184	C_{2184}
2304	C_{2304}
2520	C_{2520}
2736	C_{2736}
3192	C_{3192}

Table B.60 Covering for $d = 27$ in Base 31

row	congruence	prime p_i
a	$k \equiv 0 \pmod{4}$	13
1	$k \equiv 902 \pmod{1176}$	1354007961222841
2	$k \equiv 566 \pmod{1176}$	$c_{1176,1}$
3	$k \equiv 230 \pmod{1176}$	$c_{1176,2}$
4	$k \equiv 1070 \pmod{2352}$	48602776993
5	$k \equiv 902 \pmod{2352}$	$c_{2352,1}$
6	$k \equiv 902 \pmod{2352}$	$c_{2352,2}$
7	$k \equiv 2414 \pmod{4704}$	1354007961222841
8	$k \equiv 62 \pmod{4704}$	1354007961222841
9	$k \equiv 54 \pmod{56}$	113
10	$k \equiv 46 \pmod{56}$	36004683284137
11	$k \equiv 38 \pmod{56}$	152490484148901066281
12	$k \equiv 422 \pmod{448}$	449
13	$k \equiv 366 \pmod{448}$	2107875850753
14	$k \equiv 310 \pmod{448}$	285461959138995558273
15	$k \equiv 254 \pmod{448}$	102910619004244172801
16	$k \equiv 198 \pmod{448}$	P_{448}
17	$k \equiv 590 \pmod{896}$	9857
18	$k \equiv 142 \pmod{896}$	72577
19	$k \equiv 534 \pmod{896}$	$c_{896,1}$
20	$k \equiv 86 \pmod{896}$	$c_{896,2}$
21	$k \equiv 926 \pmod{1344}$	71233
22	$k \equiv 478 \pmod{1344}$	3444673
23	$k \equiv 30 \pmod{1344}$	$c_{1344,1}$
24	$k \equiv 246 \pmod{280}$	2331436501821281
25	$k \equiv 190 \pmod{280}$	$c_{280,1}$
26	$k \equiv 134 \pmod{280}$	$c_{280,2}$
27	$k \equiv 638 \pmod{840}$	$c_{840,1}$
28	$k \equiv 1198 \pmod{1680}$	$c_{1680,1}$
29	$k \equiv 358 \pmod{1680}$	$c_{1680,2}$
30	$k \equiv 1758 \pmod{2520}$	$c_{2520,1}$
31	$k \equiv 918 \pmod{2520}$	$c_{2520,2}$
32	$k \equiv 2598 \pmod{5040}$	35281
33	$k \equiv 78 \pmod{5040}$	34206481
34	$k \equiv 862 \pmod{980}$	10781
35	$k \equiv 722 \pmod{980}$	245981
36	$k \equiv 582 \pmod{980}$	$c_{980,1}$
37	$k \equiv 442 \pmod{980}$	$c_{980,2}$
38	$k \equiv 1282 \pmod{1960}$	78401
39	$k \equiv 302 \pmod{1960}$	3782988161
40	$k \equiv 1142 \pmod{1960}$	1631987626081
41	$k \equiv 162 \pmod{1960}$	$c_{1960,1}$
42	$k \equiv 1982 \pmod{2940}$	135241
43	$k \equiv 1002 \pmod{2940}$	126421
44	$k \equiv 22 \pmod{2940}$	840841
45	$k \equiv 322 \pmod{364}$	17837
46	$k \equiv 266 \pmod{364}$	252253
47	$k \equiv 210 \pmod{364}$	5122828170271382606522777
48	$k \equiv 154 \pmod{364}$	$c_{364,1}$

Table B.60 cont. Covering for $d = 27$ in Base 31

row	congruence	prime p_i
49	$k \equiv 462 \pmod{728}$	6553
50	$k \equiv 406 \pmod{728}$	103196556946426217
51	$k \equiv 350 \pmod{728}$	1602836404032137
52	$k \equiv 294 \pmod{728}$	$c_{728,1}$
53	$k \equiv 238 \pmod{728}$	$c_{728,2}$
54	$k \equiv 910 \pmod{1456}$	48049
55	$k \equiv 182 \pmod{1456}$	277029717329
56	$k \equiv 854 \pmod{1456}$	1137160855847448433
57	$k \equiv 126 \pmod{1456}$	$c_{1456,1}$
58	$k \equiv 798 \pmod{1456}$	$c_{1456,2}$
59	$k \equiv 1526 \pmod{2912}$	953776097
60	$k \equiv 70 \pmod{2912}$	41143649
61	$k \equiv 1470 \pmod{2184}$	$c_{2184,1}$
62	$k \equiv 742 \pmod{2184}$	$c_{2184,2}$
63	$k \equiv 2198 \pmod{4368}$	4333457267176361565697
64	$k \equiv 14 \pmod{4368}$	$c_{4368,1}$
65	$k \equiv 454 \pmod{560}$	42645904104721
66	$k \equiv 342 \pmod{560}$	$c_{560,1}$
67	$k \equiv 230 \pmod{560}$	$c_{560,2}$
68	$k \equiv 678 \pmod{1120}$	54881
69	$k \equiv 118 \pmod{1120}$	$c_{1120,1}$
70	$k \equiv 566 \pmod{1120}$	$c_{1120,2}$
71	$k \equiv 1126 \pmod{2240}$	23109132481
72	$k \equiv 6 \pmod{2240}$	498077325761

row	congruence	prime p_i
73	$k \equiv 510 \pmod{672}$	$c_{672,1}$
74	$k \equiv 846 \pmod{1344}$	$c_{1344,1}$
75	$k \equiv 1518 \pmod{2688}$	26881
76	$k \equiv 174 \pmod{2688}$	29569
77	$k \equiv 2638 \pmod{2688}$	900481
70	$k \equiv 1966 \pmod{2688}$	6128668785793
79	$k \equiv 1294 \pmod{2688}$	1672482152742913
80	$k \equiv 622 \pmod{2688}$	$c_{2688,1}$
81	$k \equiv 1630 \pmod{2016}$	$c_{2016,1}$
82	$k \equiv 2974 \pmod{4032}$	508033
83	$k \equiv 958 \pmod{4032}$	$c_{4032,1}$
84	$k \equiv 2302 \pmod{4032}$	$c_{4032,2}$
85	$k \equiv 4318 \pmod{8064}$	234385062913
86	$k \equiv 286 \pmod{8064}$	$c_{8064,1}$
87	$k \equiv 58 \pmod{64}$	4801
88	$k \equiv 26 \pmod{64}$	p_{88}
89	$k \equiv 242 \pmod{256}$	p_{89}
90	$k \equiv 178 \pmod{256}$	P_{256}
91	$k \equiv 370 \pmod{512}$	23583164929
92	$k \equiv 114 \pmod{512}$	99578469377
93	$k \equiv 306 \pmod{512}$	P_{512}
94	$k \equiv 562 \pmod{1024}$	25601
95	$k \equiv 50 \pmod{1024}$	$c_{1024,1}$
96	$k \equiv 530 \pmod{576}$	20161

Table B.60 cont. Covering for $d = 27$ in Base 31

row	congruence	prime p_i
97	$k \equiv 466 \pmod{576}$	125962561
98	$k \equiv 402 \pmod{576}$	1103426173064775411866689
99	$k \equiv 338 \pmod{576}$	$c_{576,1}$
100	$k \equiv 274 \pmod{576}$	$c_{576,2}$
101	$k \equiv 786 \pmod{1152}$	1153
102	$k \equiv 210 \pmod{1152}$	$c_{1152,1}$
103	$k \equiv 722 \pmod{1152}$	$c_{1152,2}$
104	$k \equiv 1298 \pmod{2304}$	$c_{2304,1}$
105	$k \equiv 146 \pmod{2304}$	$c_{2304,2}$
106	$k \equiv 1234 \pmod{1728}$	8641
107	$k \equiv 658 \pmod{1728}$	36844417
108	$k \equiv 82 \pmod{1728}$	882800641
109	$k \equiv 1170 \pmod{1728}$	14328577
110	$k \equiv 594 \pmod{1728}$	50090204323009
111	$k \equiv 18 \pmod{1728}$	722221930117206721
112	$k \equiv 330 \pmod{352}$	6766051802154664547351297
113	$k \equiv 298 \pmod{352}$	$c_{352,1}$
114	$k \equiv 266 \pmod{352}$	$c_{352,2}$
115	$k \equiv 586 \pmod{704}$	1409
116	$k \equiv 234 \pmod{704}$	127953409
117	$k \equiv 554 \pmod{704}$	P_{704}
118	$k \equiv 906 \pmod{1408}$	34300289
119	$k \equiv 202 \pmod{1408}$	4391075736961
120	$k \equiv 1226 \pmod{1408}$	$c_{1408,1}$
121	$k \equiv 874 \pmod{1408}$	$c_{1408,2}$
122	$k \equiv 1930 \pmod{2816}$	77722149121
123	$k \equiv 522 \pmod{2816}$	7163844781274369
124	$k \equiv 1578 \pmod{2816}$	$c_{2816,1}$
125	$k \equiv 170 \pmod{2816}$	$c_{2816,2}$
126	$k \equiv 842 \pmod{1056}$	2113
127	$k \equiv 490 \pmod{1056}$	47521
128	$k \equiv 138 \pmod{1056}$	18997250623975009
129	$k \equiv 810 \pmod{1056}$	$c_{1056,1}$
130	$k \equiv 458 \pmod{1056}$	$c_{1056,2}$
131	$k \equiv 1162 \pmod{2112}$	6337
132	$k \equiv 106 \pmod{2112}$	$c_{2112,1}$
133	$k \equiv 602 \pmod{880}$	881
134	$k \equiv 250 \pmod{880}$	$c_{880,1}$
135	$k \equiv 778 \pmod{880}$	$c_{880,2}$
136	$k \equiv 2186 \pmod{2640}$	17732881
137	$k \equiv 1306 \pmod{2640}$	$c_{2640,1}$
138	$k \equiv 426 \pmod{2640}$	$c_{2640,2}$
139	$k \equiv 1834 \pmod{3520}$	394241
140	$k \equiv 74 \pmod{3520}$	$c_{3520,1}$
141	$k \equiv 1450 \pmod{1760}$	29921
142	$k \equiv 1098 \pmod{1760}$	14081
143	$k \equiv 746 \pmod{1760}$	113982575019447974441
144	$k \equiv 394 \pmod{1760}$	$c_{1760,1}$

Table B.60 cont. Covering for $d = 27$ in Base 31

row	congruence	prime p_i
145	$k \equiv 42 \pmod{1760}$	$c_{1760,2}$
146	$k \equiv 890 \pmod{1232}$	$c_{1232,1}$
147	$k \equiv 538 \pmod{1232}$	$c_{1232,2}$
148	$k \equiv 1418 \pmod{2464}$	985601
149	$k \equiv 1066 \pmod{2464}$	$c_{2464,1}$
150	$k \equiv 714 \pmod{2464}$	$c_{2464,2}$
151	$k \equiv 2826 \pmod{4928}$	896897
152	$k \equiv 362 \pmod{4928}$	265693121
153	$k \equiv 2474 \pmod{4928}$	$c_{4928,1}$
154	$k \equiv 10 \pmod{4928}$	$c_{4928,2}$
155	$k \equiv 386 \pmod{416}$	1404631073
156	$k \equiv 354 \pmod{416}$	11014021409
157	$k \equiv 322 \pmod{416}$	7480641128031875233
158	$k \equiv 290 \pmod{416}$	$c_{416,1}$
159	$k \equiv 258 \pmod{416}$	$c_{416,2}$
160	$k \equiv 642 \pmod{832}$	13313
161	$k \equiv 226 \pmod{832}$	23297
162	$k \equiv 610 \pmod{832}$	887792562487169
163	$k \equiv 194 \pmod{832}$	800764356289
164	$k \equiv 578 \pmod{832}$	1373547339841
165	$k \equiv 162 \pmod{832}$	$c_{832,1}$
166	$k \equiv 546 \pmod{832}$	$c_{832,2}$
167	$k \equiv 2626 \pmod{3328}$	3329
168	$k \equiv 1794 \pmod{3328}$	39937
169	$k \equiv 962 \pmod{3328}$	$c_{3328,1}$
170	$k \equiv 130 \pmod{3328}$	$c_{3328,2}$
171	$k \equiv 930 \pmod{1248}$	4993
172	$k \equiv 514 \pmod{1248}$	2215201
173	$k \equiv 98 \pmod{1248}$	171684299503393
174	$k \equiv 898 \pmod{1248}$	$c_{1248,1}$
175	$k \equiv 482 \pmod{1248}$	$c_{1248,2}$
176	$k \equiv 1314 \pmod{2496}$	5571073
177	$k \equiv 66 \pmod{2496}$	5880834036481
178	$k \equiv 1282 \pmod{1664}$	632321
179	$k \equiv 866 \pmod{1664}$	127471781962809857
180	$k \equiv 450 \pmod{1664}$	$c_{1664,1}$
181	$k \equiv 34 \pmod{1664}$	$c_{1664,2}$
182	$k \equiv 834 \pmod{1040}$	2081
183	$k \equiv 626 \pmod{1040}$	2057345681
184	$k \equiv 418 \pmod{1040}$	1603690683419761
185	$k \equiv 210 \pmod{1040}$	$c_{1040,1}$
186	$k \equiv 2 \pmod{1040}$	$c_{1040,2}$
187	$k \equiv 18 \pmod{19}$	571
188	$k \equiv 16 \pmod{19}$	14251
189	$k \equiv 14 \pmod{19}$	88770666332610762169
190	$k \equiv 31 \pmod{38}$	191
191	$k \equiv 29 \pmod{38}$	3545592640701962728192781
192	$k \equiv 103 \pmod{114}$	4903

Table B.60 cont. Covering for $d = 27$ in Base 31

row	congruence	prime p_i
193	$k \equiv 65 \pmod{114}$	1553023
194	$k \equiv 27 \pmod{114}$	98595072158281
195	$k \equiv 101 \pmod{114}$	35362755128281368537612757051
196	$k \equiv 6 \pmod{57}$	7639
197	$k \equiv 25 \pmod{57}$	36068660903683
198	$k \equiv 61 \pmod{76}$	1217
199	$k \equiv 23 \pmod{76}$	23126269
200	$k \equiv 59 \pmod{76}$	53489941
201	$k \equiv 21 \pmod{76}$	2491389137
202	$k \equiv 57 \pmod{76}$	1301545806118830206284092O1
203	$k \equiv 95 \pmod{152}$	921121
204	$k \equiv 19 \pmod{152}$	99253763473
205	$k \equiv 131 \pmod{152}$	P_{152}
206	$k \equiv 245 \pmod{304}$	124337
207	$k \equiv 93 \pmod{304}$	2693717249
208	$k \equiv 207 \pmod{304}$	3366543416577233
209	$k \equiv 55 \pmod{304}$	$c_{304,1}$
210	$k \equiv 169 \pmod{304}$	$c_{304,2}$
211	$k \equiv 321 \pmod{608}$	9507314849
212	$k \equiv 17 \pmod{608}$	3178774177
213	$k \equiv 205 \pmod{228}$	229
214	$k \equiv 167 \pmod{228}$	63147793
215	$k \equiv 129 \pmod{228}$	150718033
216	$k \equiv 91 \pmod{228}$	249613717

row	congruence	prime p_i
217	$k \equiv 53 \pmod{228}$	306232251849987913
218	$k \equiv 15 \pmod{228}$	p_{218}
219	$k \equiv 108 \pmod{133}$	775559048587
220	$k \equiv 70 \pmod{133}$	$c_{133,1}$
221	$k \equiv 32 \pmod{133}$	$c_{133,2}$
222	$k \equiv 127 \pmod{266}$	3293879
223	$k \equiv 89 \pmod{266}$	2611420465612378154239
224	$k \equiv 51 \pmod{266}$	$c_{266,1}$
225	$k \equiv 13 \pmod{266}$	$c_{266,2}$
226	$k \equiv 429 \pmod{456}$	457
227	$k \equiv 315 \pmod{456}$	451441
228	$k \equiv 201 \pmod{456}$	134891533840590357 7
229	$k \equiv 87 \pmod{456}$	$c_{456,1}$
230	$k \equiv 391 \pmod{456}$	$c_{456,2}$
231	$k \equiv 733 \pmod{912}$	$c_{912,1}$
232	$k \equiv 277 \pmod{912}$	$c_{912,2}$
233	$k \equiv 1531 \pmod{1824}$	910429537
234	$k \equiv 619 \pmod{1824}$	$c_{1824,1}$
235	$k \equiv 1075 \pmod{1824}$	$c_{1824,2}$
236	$k \equiv 1987 \pmod{3648}$	40129
237	$k \equiv 163 \pmod{3648}$	299137
238	$k \equiv 3241 \pmod{3648}$	94849
239	$k \equiv 2329 \pmod{3648}$	1115197554433
240	$k \equiv 1417 \pmod{3648}$	$c_{3648,1}$

Table B.60 cont. Covering for $d = 27$ in Base 31

row	congruence	prime p_i
241	$k \equiv 505 \pmod{3648}$	$c_{3648,2}$
242	$k \equiv 1873 \pmod{2736}$	$c_{2736,1}$
243	$k \equiv 961 \pmod{2736}$	$c_{2736,2}$
244	$k \equiv 2785 \pmod{5472}$	19461003841
245	$k \equiv 49 \pmod{5472}$	$c_{5472,1}$
246	$k \equiv 11 \pmod{57}$	p_{246}
247	$k \equiv 85 \pmod{95}$	793926617318201
248	$k \equiv 161 \pmod{190}$	$c_{190,1}$
249	$k \equiv 123 \pmod{190}$	$c_{190,2}$
250	$k \equiv 237 \pmod{380}$	1041961
251	$k \equiv 47 \pmod{380}$	27268386181
252	$k \equiv 199 \pmod{380}$	$c_{380,1}$
253	$k \equiv 9 \pmod{380}$	$c_{380,2}$
254	$k \equiv 539 \pmod{570}$	4561
255	$k \equiv 349 \pmod{570}$	612751
256	$k \equiv 159 \pmod{570}$	1177163431
257	$k \equiv 501 \pmod{570}$	63058531
258	$k \equiv 311 \pmod{570}$	$c_{570,1}$
259	$k \equiv 121 \pmod{570}$	$c_{570,2}$
260	$k \equiv 273 \pmod{285}$	2851
261	$k \equiv 178 \pmod{285}$	5271361
262	$k \equiv 83 \pmod{285}$	P_{285}
263	$k \equiv 45 \pmod{95}$	P_{95}

row	congruence	prime p_i
264	$k \equiv 577 \pmod{760}$	761
265	$k \equiv 387 \pmod{760}$	$c_{760,1}$
266	$k \equiv 197 \pmod{760}$	$c_{760,2}$
267	$k \equiv 767 \pmod{1520}$	$c_{1520,1}$
268	$k \equiv 7 \pmod{1520}$	$c_{1520,2}$
269	$k \equiv 138 \pmod{171}$	49267837
270	$k \equiv 100 \pmod{171}$	P_{171}
271	$k \equiv 233 \pmod{342}$	$c_{342,1}$
272	$k \equiv 195 \pmod{342}$	$c_{342,2}$
273	$k \equiv 499 \pmod{684}$	15382971192162601
274	$k \equiv 157 \pmod{684}$	$c_{684,1}$
275	$k \equiv 461 \pmod{684}$	$c_{684,2}$
276	$k \equiv 803 \pmod{1368}$	8209
277	$k \equiv 119 \pmod{1368}$	16417
278	$k \equiv 1107 \pmod{1368}$	28729
279	$k \equiv 765 \pmod{1368}$	4986361
280	$k \equiv 423 \pmod{1368}$	$c_{1368,1}$
281	$k \equiv 81 \pmod{1368}$	$c_{1368,2}$
282	$k \equiv 214 \pmod{513}$	2053
283	$k \equiv 385 \pmod{513}$	2104977965218759
284	$k \equiv 43 \pmod{513}$	$c_{513,1}$
285	$k \equiv 176 \pmod{513}$	$c_{513,2}$
286	$k \equiv 347 \pmod{1026}$	7204633561

Table B.60 cont. Covering for $d = 27$ in Base 31

row	congruence	prime p_i
287	$k \equiv 5 \pmod{1026}$	$c_{1026,1}$
288	$k \equiv 174 \pmod{209}$	224531627
289	$k \equiv 136 \pmod{209}$	5957505709
290	$k \equiv 98 \pmod{209}$	964279537211
291	$k \equiv 60 \pmod{209}$	17110671855861286 5601
292	$k \equiv 22 \pmod{209}$	P_{209}
293	$k \equiv 193 \pmod{418}$	419
294	$k \equiv 155 \pmod{418}$	40657189
295	$k \equiv 117 \pmod{418}$	7292011
296	$k \equiv 79 \pmod{418}$	$c_{418,1}$
297	$k \equiv 41 \pmod{418}$	$c_{418,2}$
298	$k \equiv 421 \pmod{836}$	918891073
299	$k \equiv 3 \pmod{836}$	9000773650776029
300	$k \equiv 495 \pmod{532}$	$c_{532,1}$
301	$k \equiv 229 \pmod{532}$	$c_{532,2}$
302	$k \equiv 989 \pmod{1064}$	8513
303	$k \equiv 723 \pmod{1064}$	141616141088633
304	$k \equiv 457 \pmod{1064}$	$c_{1064,1}$
305	$k \equiv 191 \pmod{1064}$	$c_{1064,2}$
306	$k \equiv 115 \pmod{399}$	$c_{399,1}$
307	$k \equiv 210 \pmod{399}$	$c_{399,2}$
308	$k \equiv 685 \pmod{798}$	1597
309	$k \equiv 419 \pmod{798}$	342517162879291

row	congruence	prime p_i
310	$k \equiv 153 \pmod{798}$	6757367761201
311	$k \equiv 647 \pmod{798}$	$c_{798,1}$
312	$k \equiv 381 \pmod{798}$	$c_{798,2}$
313	$k \equiv 1141 \pmod{1596}$	460447588021
314	$k \equiv 343 \pmod{1596}$	$c_{1596,1}$
315	$k \equiv 875 \pmod{1596}$	$c_{1596,2}$
316	$k \equiv 1673 \pmod{3192}$	$c_{3192,1}$
317	$k \equiv 77 \pmod{3192}$	$c_{3192,2}$
318	$k \equiv 571 \pmod{665}$	6112681
319	$k \equiv 305 \pmod{665}$	8079339031
320	$k \equiv 39 \pmod{665}$	1579235351
321	$k \equiv 400 \pmod{665}$	$c_{665,1}$
322	$k \equiv 134 \pmod{665}$	$c_{665,2}$
323	$k \equiv 1103 \pmod{1330}$	$c_{1330,1}$
324	$k \equiv 837 \pmod{1330}$	$c_{1330,2}$
325	$k \equiv 1863 \pmod{2660}$	372243061
326	$k \equiv 533 \pmod{2660}$	15831936018361
327	$k \equiv 1597 \pmod{2660}$	$c_{2660,1}$
328	$k \equiv 267 \pmod{2660}$	$c_{2660,2}$
329	$k \equiv 1331 \pmod{1995}$	$c_{1995,1}$
330	$k \equiv 666 \pmod{1995}$	$c_{1995,2}$
331	$k \equiv 1996 \pmod{3990}$	63776582941
332	$k \equiv 1 \pmod{3990}$	$c_{3990,1}$

Appendix C

Resolvent polynomial $Q(x, t-3)$

For convenience sake, we write the resolvent

$$Q(x, t-3) = x^6 + a_5x^5 + a_4x^4 + a_3x^3 + a_2x^2 + a_1x + a_0$$

with the coefficients defined below as

$$a_5 = \frac{1}{518400}(t^2(t+2)(t+1)(t^6 + 5t^5 + 33t^4 + 259t^3 + 302t^2 - 1224t + 144)(t-1)^2(t-2)^3),$$

$$a_4 = -\frac{1}{3224862720000}[(t-1)^5(t+2)^2(t-2)^6t^3(t+1)^2(5t^{12} + 199t^{11} + 2323t^{10} + 9931t^9 - 4175t^8 - 192683t^7 - 629071t^6 + 362553t^5 + 6740550t^4 + 11492064t^3 - 1363392t^2 - 3608064t - 1866240)],$$

$$a_3 = -\frac{(t-1)^7t^5(t-2)^8}{10030613004288000000}[(565t^{19} + 17126t^{18} + 237975t^{17} + 1909630t^{16} + 9188870t^{15} + 19818996t^{14} - 68914210t^{13} - 698970740t^{12} - 1688037375t^{11} + 4194678206t^{10} + 32357048155t^9 + 45384279990t^8 - 118515571020t^7 - 386351071704t^6 + 14894176320t^5 + 830049275520t^4 + 140442024960t^3 - 345324570624t^2 - 202628874240t + 16124313600)(t+1)^3(t+2)^3],$$

$$a_2 = -\frac{t^6(t+1)^4(t-2)^{11}(t-1)^9(t+2)^4}{1039973956284579840000000000}(971t^{26} - 22591t^{25} - 1083583t^{24} - 14547383t^{23} - 83466551t^{22} + 47592793t^{21} + 4173298101t^{20}$$

$$+ 28613582757t^{19} + 68460720339t^{18} - 180541237297t^{17}$$
$$- 1446249066469t^{16} - 816561357013t^{15} + 17430769406179t^{14}$$
$$+ 47063847286243t^{13} - 85485729776969t^{12} - 599111235654873t^{11}$$
$$- 616196821798026t^{10} + 1906511196115188t^{9} + 4646891215412856t^{8}$$
$$- 410513547875328t^{7} - 8420432579430912t^{6} - 3385496092686336t^{5}$$
$$+ 3893486111576064t^{4} + 3363925250211840t^{3} + 162533081088000t^{2}$$
$$- 423746961408000t - 174142586880000),$$

$$a_1 = \frac{(t+1)^5 t^9 (t-2)^{14} (t-1)^{11} (t+2)^5}{5391224989379261890560000000000000}(20057t^{31} + 1230698t^{30}$$
$$+ 31775583t^{29} + 484998008t^{28} + 5057677832t^{27} + 38225536428t^{26}$$
$$+ 202415756960t^{25} + 572149272020t^{24} - 1167350005374t^{23}$$
$$- 20726982562288t^{22} - 92753312348986t^{21} - 118039505050572t^{20}$$
$$+ 652668668578976t^{19} + 3052210895296772t^{18} + 1894972913483304t^{17}$$
$$- 16123493520572932t^{16} - 18470256487213219t^{15}$$
$$+ 142822956790453062t^{14} + 362506920521696259t^{13}$$
$$- 545645011247552892t^{12} - 3244885546875680736t^{11}$$
$$- 2345775374118735216t^{10} + 9533883462003027216t^{9}$$
$$+ 19031371767295902720t^{8} - 9098885301823272 96t^{7}$$
$$- 29754674075021561856t^{6} - 17589914238198030336t^{5}$$
$$+ 12105463713031323648t^{4} + 15105787173710069760t^{3}$$
$$+ 1914955473774182400t^{2} - 2794291949076480000t - 1120085118812160000),$$

and

$$a_0 = \frac{(t+2)^6 t^{11} (t+1)^6 (t-1)^{13} (t-2)^{17}}{670754648278610247375912960000000000000}(109147t^{37} + 6378527t^{36}$$
$$+ 158676260t^{35} + 2189527830t^{34} + 17875077487t^{33} + 85348620991t^{32}$$

$$+ 347569166974t^{31} + 4641689399200t^{30} + 65987640867622t^{29}$$
$$+ 552836059458766t^{28} + 2618652684938752t^{27} + 4110917354635588t^{26}$$
$$- 33542867622620378t^{25} - 279655912353961514t^{24} - 983567248456161308t^{23}$$
$$- 1111565086639035656t^{22} + 6592425857181067423t^{21}$$
$$+ 42067117392217407811t^{20} + 118390581547439199852t^{19}$$
$$+ 108325152400511977814t^{18} - 566110420912111523989t^{17}$$
$$- 2837325774310560559541t^{16} - 5490433889537402562258t^{15}$$
$$- 1522595576030079422216t^{14} + 24519482597046680480736t^{13}$$
$$+ 53400338699385797760240t^{12} + 25379810856426907362528t^{11}$$
$$- 85775019351549065358336t^{10} - 152622270374464010778624t^{9}$$
$$- 26729828003045680865280t^{8} + 156339163791201833533440t^{7}$$
$$+ 131522961373278451531776t^{6} - 30854563520158694375424t^{5}$$
$$- 84191031126917775360000t^{4} - 24927107522009487114240t^{3}$$
$$+ 13099860216244076544000t^{2} + 8615413876739014656000t$$
$$+ 1039973956284579840000).$$

Appendix D

Magma and Maple Code

In this appendix we provide the Magma and Maple code that was used throughout this dissertation to find integer solutions to various Thue equations and elliptic curves, construct the resolvent polynomial, find prime divisors of $\Phi_n(b)$, verify a collection of congruences is a covering, and verify that the composite numbers are indeed composite. We have broken the code into sections to provide additional structure.

D.1 Magma code for finding integer solutions to $ax^5 - by^5 = 4$

```
# Range over m=0, 1, 2, 3, 4.
# Create sets for possible values of a and b first.


S:={};
U:={};
m:=0;
for x in [0..4] do
    for y in [0..4] do
    S:={(2^x)*(3^y)} join S; end for; end for;
for x in [0..4] do
    U:={(2^m)*(3^x)} join U; end for;
U;
R<x> := PolynomialRing(Integers());
for t in S do
```

```
    for u in U do
        f := t*x^5 - u;
        T := Thue(f);
        T;
        Solutions(T, 5); end for; end for;
```

D.2 Magma code for finding integer solutions to $ax^3 - by^3 = 6$

```
# Range over m=0, 1, 2.
# Create sets for possible values of a and b first.

S:={};
U:={};
m:=0;
for x in [0..2] do
    for y in [0..2] do
        for z in [0..2] do
    S:={(2^x)*(3^y)*(5^z)} join S; end for; end for; end for;
for x in [0..2] do
    for y in [0..2] do
    U:={(2^m)*(3^x)*(5^y)} join U; end for; end for;
U;
R<x> := PolynomialRing(Integers());
for t in S do
    for u in U do
        f := t*x^3 - u;
        T := Thue(f);
```

```
T;
Solutions(T, 6); end for; end for;
```

D.3 Magma code for finding integer solutions to $ax^2 - by^3 = 6$

```
# Range over m=0, 1, 2.
# Create sets for possible values of a and b first.
# Output: (aby, a^2bx).

S:={};
U:={};
m:=0;
for x in [0..1] do
    for y in [0..1] do
        for z in [0..1] do
    S:={(2^x)*(3^y)*(5^z)} join S; end for; end for; end for;
for x in [0..2] do
    for y in [0..2] do
    U:={(2^m)*(3^x)*(5^y)} join U; end for; end for;
for t in S do
    for u in U do
t; u;
E := EllipticCurve([0,(t^3)*(u^2)*6]);
E;
IntegralPoints(E); end for; end for;
```

D.4 Magma code for finding integer solutions to $ax^3 - by^2 = 6$

```
# Range over m=0, 1, 2.
# Create sets for possible values of a and b first.
# Output: (abx, ab^2y).

S:={};
U:={};
m:=0;
for x in [0..1] do
    for y in [0..1] do
        for z in [0..1] do
    S:={(2^x)*(3^y)*(5^z)} join S; end for; end for; end for;
for x in [0..2] do
    for y in [0..2] do
    U:={(2^m)*(3^x)*(5^y)} join U; end for; end for;
for t in S do
    for u in U do
u; t;
E := EllipticCurve([0,(t^3)*(u^2)*(-6)]);
E;
IntegralPoints(E); end for; end for;
```

D.5 Maple code for constructing the resolvent polynomial

```
# Construct F(x1,x2,x3,x4,x5,x6), labeled F, that belongs to
   PGL_2(2).
```

```
> with(GroupTheory):
> H:=Group(Perm([[1,2,3],[4,5,6]]),Perm([[1,3,4,5,6]]),Perm
    ([[1,2],[3,4],[5,6]])):
> A:=Elements(H):
> F:=expand(sum(alpha[A[j][2]]*alpha[A[j][3]]^2*alpha[A[j
    ][4]]^3*alpha[A[j][5]]^4*alpha[A[j][6]]^5,j=1..120)):

# Construct the resolvent in terms of roots r_1, ..., r_6.

> S:=Group(Perm([[1,2]]),Perm([[1,2,3]])):
> B:=Elements(S):
> Q:=product(x-eval(F,{alpha[1]=alpha[B[j][1]],alpha[2]=alpha
    [B[j][2]],alpha[3]=alpha[B[j][3]],alpha[4]=alpha[B[j][4]],
    alp ha[5]=alpha[B[j][5]],alpha[6]=alpha[B[j][6]]}),j=1..6)
    :
> for j from 0 to 6 do
> w[j]:=coeff(Q,x,j): end do:

# Create the sigma_{i,j} used.

> S:=Elements(SymmetricGroup(6)):
> a[1]:=120: a[2]:=48: a[3]:=36: a[4]:=48: a[5]:=120: a
    [6]:=720:
  for i from 1 to 6 do
    sigma[i,o]:=sum(product(alpha[S[l][k]]^o,k=1..i),l
        =1..720)/a[i]; end do:

# Write coefficients of resolvent in terms of sigma_{i,j}.
# Input: (polynomial to rewrite, number of iterations used)
```

```
# Coefficient of x^5 took 22 iterations, x^4 took 229, x^3
    took 1152, x^2 took 3900, x took 10020, constant term took
     22000.
# Output: (polynomial in terms of roots, polynomial in terms
    of sigma_{i,j})

> ChangeToSigmas:=proc(sympoly)
   local m,j,tempsympoly,sigs,mm,t,sigresult,sigmess:
   tempsympoly:=sympoly: sigs:=NULL: sigresult:=1: sigmess:=1:
   if sympoly=0 then RETURN(0): fi:
   for j from 1 to 6 do
   m:=degree(tempsympoly,alpha[j]):
   tempsympoly:=coeff(tempsympoly,alpha[j],m):
   sigs:=sigs,m:
   od:
   sigs:=[sigs]:
   mm:=nops(sigs):
   for j from 1 to mm-1 do
   t:=sigs[j]-sigs[j+1]:
   if t > 0 then sigresult:=sigresult*eval(sigma[j,o],o=t):
       sigmess:=sigmess*sigma[j,t]:
   fi: od:
   if sigs[mm] > 0 then
   sigresult:=tempsympoly*sigresult*eval(sigma[mm,o],o=sigs[mm
       ]): sigmess:=tempsympoly*sigmess*sigma[mm,sigs[mm]]:
   fi:
   if sigs[mm] = 0 then
```

```
  sigresult:=tempsympoly*sigresult:  sigmess:=sigmess*
     tempsympoly:
  fi:
  RETURN(sigresult,sigmess):
  end:
> CreateSigmas:=proc(sympoly,N)
  local counter,j,check,finalsympoly,tempsympoly, sig,
     finalsigpoly:
  check:=0: finalsympoly:=0:finalsigpoly:=0: tempsympoly:=
     expand(sympoly): counter:=200:
  for j from 1 to N while check=0 do  ## Increase 300 here if
      needed
    sig:=ChangeToSigmas(tempsympoly):
    finalsympoly:=finalsympoly+sig[1]:
    finalsigpoly:=finalsigpoly+sig[2]:
    tempsympoly:=tempsympoly-expand(sig[1]):
  if j>counter then print(counter); counter:=counter+200: fi:
    if tempsympoly=0 then check:=1: fi:
  od:
  if j > N-1 then
    printf("Increase the number of iterations."): RETURN(NULL
       ):
  fi:
  RETURN(finalsympoly,finalsigpoly):
  end:
> pint:=CreateSigmas(w[5],116000):

# Write each sigma_{i,j} in terms of coefficients of poly.
```

```
# "Sigma Replace.txt" contains sigma_{i,j} in terms of
    coefficients.
# Output: coefficient of resolvent in terms of t.

> read "Sigma Replace.txt"
> FinalEval:=proc(poly)
   local a:
   a:=eval(poly,{values}):
   RETURN(a):
   end:
> simplify(FinalEval(pint[2]));
```

D.6 Magma code for finding prime divisors of $\Phi_n(b)$

```
# Either returns the complete factorization,
# Or the factors it found if unable to completely factor
# The nth cyclotomic polynomial at a.

SetVerbose("Factorization", true);
SetVerbose("MPQS", true);
Z := IntegerRing();
n:=1123;                    # Or the order you are looking for.
Q<x> := PolynomialRing(Z);
f:= MinimalPolynomial(RootOfUnity(n));
m:=Evaluate(f, b);          # Replace b with 10 for base 10.
time Factorization(Z!m);
```

D.7 Maple code for checking whether a system of congruences is a covering

```
# Read file containing the covering in question.
# File contains lists of the form table9:=[[a, n, p], ...],
# Where p is a prime factor of the nth cyclotomic polynomial,
# The congruence is k equivalent to a (mod n).

> restart:
> with(NumberTheory);
> read "InsertCoverings.txt":

# Find the least common multiple of the moduli.

> L := 1:
  for y in table9 do
      L := ilcm(L, y[2]):
  end do:
  L;
  ifactor(L);

# Verify the set of congruences is a covering.
# Check that every integer up to L − 1 satisfies a congruence
# Output: every integer in [0, L − 1) that does not satisfy a
    congruence.

> B:=[]:
```

```
> A:=[seq(i,i=000000..L-1)]:
  L:=convert(A,set):
  for x in A do
  P:=0:
  for y in table9
  while x mod y[2]<>y[1] do P:=P+1 od: if P=nops(table9) then
      B:=[op(B),x]
    fi: od:
> B
```

D.8 Maple code for various checks on the coverings

```
# Verify every composite number is not a power of a prime.
# Output: (number of non-prime powers, number of expected
   composites)

> N := 0;
  for y in tabled4c do
      if type(y[3], 'primepower') then ; else N := N + 1; end
          if;
  end do;
  print(N, nops(tabled4c));

# Verify that c(p) = n for each prime used.
# MultOrder2 creates command to check multiplicative order.
# Output: number of incorrect primes/composites.
```

```
> MultOrder2 := proc(s, t)
    local L, y;
    L := 0;
   if 10^s mod t <> 1 then L := L + 1; else
     for y in Divisors(s) do
       if 10^y mod t = 1 then if y = s then L := L; else L
          := L + 1; end if; end if; end do; end if;
   if L = 0 then 0 else 1; end if;
   end proc;
> J := 0;
  for y in tabled4p do
      J := J + MultOrder2(y[2], y[3]);
  end do;
  J;
> J := 0:
  for y in tabled4c do
      J := J + MultOrder2(y[2], y[3]):
  end do:
  J;

# Verifying that no prime is used twice.
# Include all tables with primes in tablebigp used.
# Output: [a, n, p] if the prime p is used more than once.

> tablebigp:=[op(table9), op(table3)]:
> for y in tabled4p do M:=0:
   for x in tablebigp do
```

```
    if x[3]=y[3] then M:=M+1: else fi: od: if M>1 then print(
      y) else fi: od:

# Determine how many times we can use each composite number.
# Include all tables with composites in tablebigc used.
# Include all tables with primes in tablebigp used.

> tablebigc:=[op(table4c), op(tabled3c)]:

# Make sure each composite is used no more than twice.
# Output: composites that are used more than twice.
> for y in tabled4c do
   M:=0: for x in tablebigc do
    if x[3]=y[3] then M:=M+1: else fi: od: if M>2 then print(
      y[3], M) else fi: od:

# Verify composite numbers aren't divisible by primes used.
# Output: [a, n, c] where c is the gcd(composite, primes)
> tablebigp:=[op(table9), op(table3)]:
> for y in tabled4c do
   M:=1: for x in tablebigp do
     if x[2]=y[2] then M:=M*x[3] fi: od:
  print(y[1], y[2], gcd(y[3], M*y[2])) od:
```