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# Methods for Cancellation of Apparent Cerenkov Radiation Arising From SME Models and Separability of Schrödinger's Equation Using Exotic Potentials in Parabolic Coordinates

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# Abstract

In an attempt to merge the two prominent areas of physics: The Standard Model and General Relativity, there have been many theories for the underlying physics that may lead to Lorentz- and CPT-symmetry violations. At the present moment, technology allows numerous types of Planck-sensitive tests of these symmetries in a range of physical systems.

We address a curiosity in isotropic CPT- and Lorentz-violating electrodynamics where there is a kinematic allowance for Cerenkov radiation of a charged particle in a vacuum moving with uniform motion. This however, should not be the case as it is known that constant motion in a vacuum should not cause the particle to lose any energy. Taking Fourier transforms of the modified magnetic field confirms the cancellation of the apparent radiation. The Fourier transform can be used to show that modes for short and long wavelengths cancel.

In the second area of research we focus on solutions of the Schrödinger equation which may be found by separation of variables in more than one coordinate system. This class of potentials includes a number of important examples, including the isotropic harmonic oscillator and the Coulomb potential. There are multiple separable Hamiltonians that exhibit a number of interesting features, including "accidental" degeneracies in their bound state spectra and often classical bound state orbits that always close. We examine another potential, for which the Schrödinger equation is separable in both cylindrical and parabolic coordinates: a z-independent $V \propto 1/\rho^2 = 1/\left(x^2 + y^2\right)$ in three dimensions. All the persistent, bound classical orbits in this potential close, because all other orbits with negative energies fall to the

center at $\rho = 0$. When separated in parabolic coordinates, the Schrödinger equation splits into three individual equations, two of which are equivalent to the radial equation in a Coulomb potential—one equation with an attractive potential, the other with an equally strong repulsive potential.

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# Chapter 1

# Introduction

Mathematics is the language in which our universe is written. Tuned by the fundamental constants of nature, the laws of physics are written down in accordance with the equations of motion that are exhibited by constituents of the reality that surrounds us.

The standard model of particle physics is the most accurate and successful physical theory ever developed. It has been able to describe the fundamental quantum building blocks that make up our universe with amazing precision and accuracy. Its discovery came from looking at the fundamental symmetries that are found in the equations of motion. In physics, a symmetry of a system is a physical or mathematical feature of the system (observed or intrinsic) that is preserved or remains unchanged under some transformation. These transformations may be continuous or discrete. In the work of Emmy Noether, continuous symmetries were shown to lead to the existence of conserved quantities such as energy and momentum. Yet there also exist some symmetries that, while they were once thought to be unbreakable, have since been discovered may be broken.

The foundations of quantum field theory rest on its symmetries. In the search for a unifying theory between general relativity and the standard model, some theories have arisen that may break either one or both of Lorentz and CPT symmetries. If a theory lacks either one or both of these symmetries, it may lead to the discovery of new physics present. This understanding is important in the search for a unifying theory of quantum gravity. If violations of these symmetries were to be discovered

experimentally, it would certainly require new investigations of fundamental physics. Investigating these violations might tell us a great deal about new physics at its smallest scale.

The first area of research will focus on a extension of the standard model, where in isotropic CPT- and Lorentz-violating electrodynamics there exists a kinematic allowance for Cerenkov radiation by a charged particle in a vacuum moving with uniform motion. This issue is addressed by taking Fourier transforms of the modified magnetic field to confirm the cancellation of the apparent radiation. These Fourier transforms of the modified $\vec{B}$ field along with other important terms can be used to show that the energies of the modes for short and long wavelengths ultimately cancel.

The second area of research focuses on solutions of the Schrödinger equation using separation of variables in more than one coordinate system. This involves potentials that include important examples such as the isotropic harmonic oscillator and the Coulomb potential. Separable Hamiltonians exhibit interesting features, including "accidental" degeneracies in their bound state spectra and often classical bound state orbits that always close. We address a particular potential, for which the Schrödinger equation is separable in both cylindrical and parabolic coordinates: a $z$-independent $V \propto 1/\rho^2 = 1/(x^2+y^2)$ in three dimensions. All of the bound classical orbits in this potential close, because all other orbits with negative energies fall to the center at $\rho = 0$. When separated in parabolic coordinates, the Schrödinger equation splits into three individual equations, two of which are equivalent to the radial equation in a Coulomb potential——one equation with an attractive potential, the other with an equally strong repulsive potential. Investigating these potentials and solutions aids in providing an understanding of the structure of mechanics in parabolic coordinates and the behavior of a number of separable quantum systems.

# Chapter 2

# Lorentz Transformation and Invariance

In physics, Lorentz transformations are a family of linear transformations that may be used to switch from one specified coordinate frame in spacetime to another moving with a relative velocity. Such transformations are needed when an inertial observer witnesses identical experiments that are occurring in different reference frames. In this case the matter and field configurations between the two can vary.

Lorentz transformations can be used to show that the electric and magnetic fields are different aspects of a single electromagnetic field generated by electric charges. They are used in the context of special relativity, relating measurements made between reference frames that may differ by orientation and/or velocity. These transformations come in the form of velocity-changing boosts, spatial rotations, or combinations of both. In special relativity, this transformation in coordinates leaves the laws of physics unchanged in such a way that both observers shall agree on the same physical laws.

## 2.1 Transformation Properties For Four-Vectors

It is important to explore some important fundamental transformation properties. The spacetime position four-vector $x^\mu$ contains four components $(ct, \vec{x})$, where the first term in parentheses is the $x^0$ time component while the second term contains the three spatial components. Together, this object may be written as a 4-dimensional

column vector

$$x^\mu = \begin{pmatrix} x^0 \\ x^1 \\ x^2 \\ x^3 \end{pmatrix} \tag{2.1}$$

while the its transpose is clearly a 4-dimensional row vector

$$(x^\mu)^T = \begin{pmatrix} x^0 & x^1 & x^2 & x^3 \end{pmatrix}. \tag{2.2}$$

We can write a general linear transformation as

$$x'^\alpha = \sum_{\beta=0}^{3} \frac{\partial x'^\alpha}{\partial x^\beta} x^\beta \equiv A^\alpha{}_\beta \, x^\beta. \tag{2.3}$$

We shall mostly use the Einstein summation notation shown on the right-hand side of (2.3), which implies a summation over the repeated Greek indices.

In special relativity, the proper time separation between two events is fundamentally invariant. The square of this quantity is a scalar and should be invariant in all reference frames such that

$$ds^2 \equiv c^2dt^2 - dx^2 - dy^2 - dz^2 \equiv c^2dt^2 - d\vec{x}^{\,2} = c^2dt'^2 - d\vec{x}'^2 \equiv ds'^2. \tag{2.4}$$

This equation looks as if $ds^2$ is the Cartesian inner product of a four-vector containing imaginary spatial components $(ct, i\vec{x})$ with itself. This might suggest that a transformation relating a primed with an unprimed coordinate system may be described by a complex or imaginary rotation.

The rotation matrix for a standard rotation around the $\hat{z}$-direction by an angle $\phi$ looks like

$$A = \begin{pmatrix} \cos\phi & -\sin\phi & 0 \\ \sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{2.5}$$

so that

$$x'_i = A_{ij} x_j. \tag{2.6}$$

(A Latin letter such as $j$ for an index indicates that it represents a component of a spatial three-vector or one of the three spatial components or a spacetime four-vector.) Summing over $j$, the rotation transformation equations read

$$x_1' = \cos\phi\, x_1 - \sin\phi\, x_2 \tag{2.7}$$

$$x_2' = \sin\phi\, x_1 + \cos\phi\, x_2 \tag{2.8}$$

$$x_3' = x_3. \tag{2.9}$$

For an imaginary $\phi \to i\phi$, $\cos\phi \to \cosh\phi$ and $\sin\phi \to -i\sinh\phi$. Redefining the vector as $\widetilde{x}' = (x_1, ix_2, ix_3)$ such that

$$\widetilde{x}_i' = a_{ij}\widetilde{x}_j. \tag{2.10}$$

With these modifications, the primed coordinates now read

$$x_1' = \cosh\phi\, x_1 - \sinh\phi\, x_2 \tag{2.11}$$

$$x_2' = -\sinh\phi\, x_1 + \cosh\phi\, x_2 \tag{2.12}$$

$$x_3' = x_3, \tag{2.13}$$

so that

$$x_1'^{\,2} - x_2'^{\,2} - x_3'^{\,2} = x_1^{\,2} - x_2^{\,2} - x_3^{\,2}. \tag{2.14}$$

With the use of the hyperbolic identity $\cosh^2\phi - \sinh^2\phi = 1$, substituting the explicit primed equations into (2.14), the equivalence can be verified; and thus, we have generated the proper type of invariant to correspond to the proper time in special relativity.

In order to generalize for a four-vector, consider a vector $\tilde{x}^\mu \equiv (x^0, i\vec{x})$, thus having the property

$$(\tilde{x}^\mu)^2 = (x^0)^2 - \vec{x}\cdot\vec{x}, \tag{2.15}$$

where the inner product should remain invariant under a transformation of $\vec{x}$ and $t$. A rotation through an imaginary angle $i\phi$ has a matrix representation:

$$\mathbf{A} = \begin{pmatrix} \cosh\phi & i\sinh\phi & 0 & 0 \\ -i\sinh\phi & \cosh\phi & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.16}$$

Applying this imaginary rotation to $\tilde{x}^\mu$ to produce $\tilde{x}^{\mu\prime}$, the equations have a familiar structure.

$$x^{0\prime} = \cosh\phi\, x^0 - \sinh\phi\, x^1 \tag{2.17}$$

$$x^{1\prime} = -\sinh\phi\, x^0 + \cos\phi\, x^1 \tag{2.18}$$

$$x^{2\prime} = x^2 \tag{2.19}$$

$$x^{3\prime} = x^3. \tag{2.20}$$

These have the form of a real linear transformation acting on the the original (real-valued) four-vector $x^\mu$.

Once again using the hyperbolic identity formulas, it can be shown in the same way that

$$(x^\mu)^2 \equiv (x^0)^2 - \vec{x}\cdot\vec{x} = (x^{0\prime})^2 - \vec{x}\,'\cdot\vec{x}\,', \tag{2.21}$$

verifying that the transformation preserves the Lorentz scalar quantity.

Rewriting $\sinh\phi$ as $\cosh\phi\tanh\phi$, the modified primed equations are

$$x^{0\prime} = \cosh\phi\left(x^0 - \tanh\phi\, x^1\right) \tag{2.22}$$

$$x^{1\prime} = \cosh\phi\left(x^1 - \tanh\phi\, x^0\right) \tag{2.23}$$

If we now consider an object at rest in the $O'$ frame at the point $(ct, \vec{x})$, in this case, $x^{1\prime} = 0$ for all $t'$. As seen from the $O$ frame, the object is at a position $x^1 - vt$, where $v$ is the relative velocity between the two frames. For these equations to be

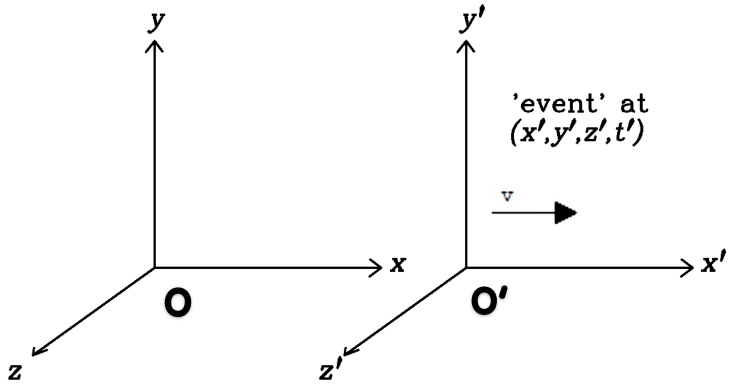


Figure 2.1 Reference frames for different velocities

consistent, we find

$$\tanh\phi = \frac{v}{c} \equiv \beta, \tag{2.24}$$

where $\beta$ is the speed $v$ in units of the speed of light $c$. From the equations we can also see that

$$\cosh\phi = \frac{1}{\sqrt{1-\beta^2}} \equiv \gamma. \tag{2.25}$$

It is known that in special relativity that time times shown on relative clocks for observers are velocity dependent. $\gamma$ is a parameter that is comes out of the derivation of "proper time" that is a byproduct of this principle.

Ultimately, when the coordinate basis of the two frames remain parallel, (e.g. $\hat{z} = \hat{z'}$), the relationship between the frames when $K'$ moves in one direction with speed $v$ is described by

$$x^{0\prime} = \gamma\left(x^0 - \beta x^1\right) \tag{2.26}$$

$$x^{1\prime} = \gamma\left(x^1 - \beta x^0\right) \tag{2.27}$$

$$x^{2\prime} = x^2 \tag{2.28}$$

$$x^{3\prime} = x^3. \tag{2.29}$$

More generally, taking into account rotations, boosts, and transformations that invert the spatial or temporal coordinates, the Lorentz transformation matrix $\mathbf{A}$ has the property that $\det \mathbf{A} = \pm 1$, which can be shown by a simple exercise in linear algebra.

## 2.2 Transformation of the Metric and Other Tensors

In order to write the invariance of the proper time interval $ds^2 = ds'^2$ as an inner product $dx_\alpha dx^\alpha$, the covariant four-vector (with lowered indices) must have components

$$dx_0 = dx^0 \tag{2.30}$$

and

$$dx_j = -dx^j. \tag{2.31}$$

The metric tensor $g_{\alpha\beta}$ is a rank-two tensor that may be expressed in covariant, contravariant, or mixed form. The covariant metric tensor can be defined by the condition

$$ds^2 \equiv g_{\alpha\beta} dx^\alpha dx^\beta \tag{2.32}$$

It should be noted that the double contraction with $dx$ is an invariant and the tensor is symmetric, which proves that it is a rank-two covariant tensor. In matrix form, $g_{\alpha\beta}$ reads

$$g_{\alpha\beta} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}. \tag{2.33}$$

For the flat four-dimensional spacetime that is described by special relativity, the elements of the contravariant $g^{\alpha\beta}$ will be the same. The metric tensor converts contravariant tensor indices into covariant ones or conversely, by raising or lowering

indices. For example,

$$x^\alpha = g^{\alpha\beta}x_\beta = g^{\alpha\beta}g_{\beta\gamma}x^\gamma = \delta^\alpha_\gamma x^\gamma = x^\alpha \tag{2.34}$$

From this we may easily show that the mixed-rank tensor looks like

$$g_\alpha{}^\beta = g_{\alpha\gamma}g^{\gamma\beta} = \delta^\beta_\alpha. \tag{2.35}$$

The symbol $\delta^\beta_\alpha$ is the Kronecker delta and has the property

$$\delta^\beta_\alpha = \begin{cases} 1 & \alpha = \beta \\ 0 & \alpha \neq \beta \end{cases}. \tag{2.36}$$

Use of the metric tensors allows numerous ways to express the inner product:

$$a^\alpha b_\alpha = g_{\alpha\gamma}a^\alpha b^\gamma = g^{\alpha\gamma}a_\gamma b_\alpha. \tag{2.37}$$

These properties shall prove useful when considering transformations of the electromagnetic fields.

Now let us address how other tensors transform. A four-vector is a rank-one Lorentz tensor. Transformations for higher rank tensors follow in a similar straightforward manner through the introduction of additional copies of the transformation matrix $\mathbf{A}$, with each index of the tensor contracted with a separate transformation matrix. A rank-two contravariant tensor is comprised of sixteen objects $T^{\alpha\beta}$ and transforms as

$$T'^{\alpha\beta} = A^\alpha{}_\gamma A^\beta{}_\delta \, T^{\gamma\delta}. \tag{2.38}$$

Similarly, the covariant counterpart transforms as

$$T'_{\alpha\beta} = A_\alpha{}^\gamma A_\beta{}^\delta \, T_{\gamma\delta}. \tag{2.39}$$

For a mixed rank-two tensor, the transformation looks similar:

$$T'^\alpha{}_\beta = A^\alpha{}_\gamma A_\beta{}^\delta \, T^\gamma{}_\delta. \tag{2.40}$$

Looking at how the inner product of two four-vectors transforms, we have

$$a'^{\alpha} b'_{\alpha} = A_{\alpha}{}^{\gamma} A^{\alpha}{}_{\delta}\, b_{\delta} a^{\gamma}. \tag{2.41}$$

Looking at the transformation operators, we find

$$A_{\alpha}{}^{\gamma} A^{\alpha}{}_{\delta} = \frac{\partial x^{\gamma}}{\partial x'^{\alpha}} \frac{\partial x'^{\alpha}}{\partial x^{\delta}} = \frac{\partial x^{\gamma}}{\partial x^{\delta}} = \delta^{\gamma}_{\delta}. \tag{2.42}$$

Thus, as a consequence of (2.42), the scalar inner product is invariant as expected,

$$a'^{\alpha} b'_{\alpha} = \delta^{\gamma}_{\delta} a^{\delta} b_{\gamma} = a^{\gamma} b_{\gamma}. \tag{2.43}$$

## 2.3 Transformation Properties For Differential Operators

Now lets turn our attention to the covariant four-gradient operator (which is actually a derivative with respect to a contravariant spacetime component),

$$\partial_{\alpha} = \frac{\partial}{\partial x^{\alpha}}. \tag{2.44}$$

When transforming from the unprimed to primed coordinate system, the transformation can be written as

$$\frac{\partial}{\partial x'^{\alpha}} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}} \frac{\partial}{\partial x^{\beta}} \equiv A_{\alpha}{}^{\beta} \frac{\partial}{\partial x^{\beta}}, \tag{2.45}$$

which has components that transform the same as the components of a covariant vector forming a covariant four-vector with elements

$$A_{\alpha}{}^{\beta} = \frac{\partial x^{\beta}}{\partial x'^{\alpha}}, \tag{2.46}$$

where the object $A_{\alpha}{}^{\beta}$ is the usual transformation matrix. The contravariant version of the four-operator transforms similarly,

$$\begin{aligned} \frac{\partial}{\partial x'_{\alpha}} &= \frac{\partial x_{\beta}}{\partial x'_{\alpha}} \frac{\partial}{\partial x_{\beta}} \\ &= g_{\beta\gamma} \frac{\partial x^{\gamma}}{\partial x'^{\delta}} \frac{\partial x^{\delta}}{\partial x'_{\alpha}} \frac{\partial}{\partial x_{\beta}} \\ &= g_{\beta\gamma} g^{\delta\alpha} A_{\delta}{}^{\gamma} \frac{\partial}{\partial x_{\beta}} \\ &= A^{\alpha}{}_{\beta} \frac{\partial}{\partial x_{\beta}}. \end{aligned} \tag{2.47}$$

This operator is a contravariant four-vector as it is clearly transforms as such. We shall often use the the shorter notation for these four-divergence operators

$$\partial_\alpha \equiv \frac{\partial}{\partial x^\alpha} \tag{2.48}$$

$$\partial^\alpha \equiv \frac{\partial}{\partial x_\alpha}, \tag{2.49}$$

which are covariant and contravariant, respectively. The inner product of either four-operator with a four-vector is an invariant (scalar),

$$\partial^\alpha A_\alpha = \partial_\alpha A^\alpha = \frac{\partial A^0}{\partial x^0} + \vec{\nabla} \cdot \vec{A} \tag{2.50}$$

as is the four-dimensional Laplacian

$$\partial^\alpha \partial_\alpha \equiv \Box = \frac{\partial^2}{\partial (x^0)^2} - \vec{\nabla} \cdot \vec{\nabla} \tag{2.51}$$

which is also referred to as the d'Alembertian or the wave operator.

## 2.4 Covariance Of Electrodynamics

One of the key underlying principles that is true (so far as we know) is that the fundamental properties of the physical world should transform in a well defined way, such that the physical laws remain unchanged in all inertial reference frames. When these transformations are Lorentz transformations, an equation is said to be (Lorentz) "covariant." In order to show this for Maxwell's equations, we must explore how each of the physical objects in electrodynamics will transform. These objects of interest are the electric and magnetic fields, the charge density, and the current ($\vec{E}$, $\vec{B}$, $\rho$, and $\vec{J}$).

First, let us turn our attention to the transformation of $\rho$ and $\vec{J}$. It takes nothing more than a simple thought experiment to postulate that the total charge of a system should not change when measured from different reference frames. In a co-moving frame, one should count the same number of charges as one would in a frame where

the observer is at rest. This happens to be a fundamental principle in physics (conservation of charge) and will aid in determining how the charge density and current transform.

Consider a system of charges ($q$) with density $\rho$ in a rest frame $O$ and $\rho'$ in the moving frame $O'$ (figure 2.1). The charge element is defined by

$$dq = \rho \, d^3x \, \frac{dt}{dt} \tag{2.52}$$

with a similar expression for the primed coordinates in the $O'$ frame

$$dq' = \rho' \, d^3x' \, \frac{dt'}{dt'}. \tag{2.53}$$

We know that in the transformation, the charge element will be unchanged ($dq = dq'$). If the time element as well as the spatial element are transformed to the primed moving system, then

$$c \, d^3x' \, dt' \equiv d^4x' = \left| \frac{\partial \left(x'^0, x'^1, x'^2, x'^3\right)}{\partial \left(x^0, x^1, x^2, x^3\right)} \right| d^{\,4}x \equiv |\det \mathbf{A}| \, d^{\,4}x, \tag{2.54}$$

and because the determinant of $\mathbf{A}$ is $\pm 1$, the overall spacetime volume element is invariant as

$$d^4x = d^4x'. \tag{2.55}$$

Since the charge element is unchanged ($dq = dq'$), and given the assumption that $c$ is invariant, it ultimately leads us to

$$\frac{\rho}{dt} = \frac{\rho'}{dt'} \tag{2.56}$$

This equation can only be valid if and only if the charge density transforms in the same way as the time element. This means it must be the $0^{\text{th}}$ component of a four-vector. Since the current $\vec{J}$ is the product of the velocity with the charge density, it may be written

$$\vec{J} = \rho \vec{v} = \rho \frac{d\vec{x}}{dt}. \tag{2.57}$$

We know that $\rho/dt$ is invariant, therefore $\vec{J}$ must transform the same as the contravariant spatial components of the four-vector. The current may be written in its contravariant form

$$J^{\alpha} = \left(c\rho, \vec{J}\right) \tag{2.58}$$

or its covariant form, which simply differs by the sign of the spatial components

$$J_{\alpha} = \left(c\rho, -\vec{J}\right). \tag{2.59}$$

The charge conservation is expressed using the four-divergence

$$\partial_{\alpha} J^{\alpha} = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0, \tag{2.60}$$

which is "covariant" in the sense that both sides of the equation are scalars (and so actually Lorentz invariant).

Starting with the equations for the potentials in the Lorenz gauge, we should expect that all relevant equations be Lorentz covariant, and that this holds when investigating the outcomes for the $\vec{E}$ and $\vec{B}$ fields. The sourced wave equations in the Lorenz gauge (and in Heavyside-Lorentz units) are

$$\Box \vec{A}\left(\vec{x}, t\right) = \frac{1}{c} \vec{J}\left(\vec{x}, t\right) \tag{2.61}$$

$$\Box \phi\left(\vec{x}, t\right) = \rho\left(\vec{x}, t\right), \tag{2.62}$$

which may be written in four-vector form as

$$\Box A^{\alpha}\left(\vec{x}, t\right) = \frac{1}{c} J^{\alpha}\left(\vec{x}, t\right) \tag{2.63}$$

where the potential is a four-vector defined as follows:

$$A^{\alpha} \equiv \left(\phi, \vec{A}\right). \tag{2.64}$$

In fact, for the equations of motions to hold in all inertial frames, this potential must be a contravariant four-vector. The Lorenz condition, specifying that a potential is in the Lorenz gauge, is

$$\vec{\nabla} \cdot \vec{A}\left(\vec{x}, t\right) + \frac{1}{c} \frac{\partial \phi}{\partial t} = \partial_{\alpha} A^{\alpha} = 0, \tag{2.65}$$

which is yet another four-divergence of a four-vector.

We now turn our attention to $\vec{E}$ and $\vec{B}$. The transformations of electric and magnetic field under Lorentz boosts were partially understood even before Einstein developed his special theory of relativity. To show that the electric field can transform into magnetic fields and vice versa, as an example, one need only consider a charge at rest which gives off an electric field. By boosting into a frame in which the charge is moving, there will be an observed electric current and thus a magnetic field. This implies that the electric field cannot be a Lorentz invariant quantity and that the electric and magnetic fields must be combined into one object to handle a Lorentz transformation properly. This mathematical object is composed of six components: an antisymmetric second-rank tensor called the electromagnetic field strength tensor $F^{\mu\nu}$.

The standard approach for calculating the electric and magnetic fields from a potential is given by

$$\vec{E}\,(\vec{x},t) = -\vec{\nabla}\phi\,(\vec{x},t) - \frac{1}{c}\frac{\partial}{\partial t}\vec{A}\,(\vec{x},t) \tag{2.66}$$

$$\vec{B}\,(\vec{x},t) = \vec{\nabla}\times\vec{A}\,(\vec{x},t) \tag{2.67}$$

For simplification purposes, we will adopt the relativistic convention of setting $c=1$ and drop the $(ct,\vec{x}\,)$ notation for the most part from this point on.

From (2.66) the $x$-component of the electric field reads as

$$\begin{aligned} E_x &= -\frac{\partial A_x}{\partial t} - \frac{\partial \phi}{\partial x} \\ &= -\frac{\partial A^1}{\partial x^0} + \frac{\partial A^0}{\partial x^1} \\ &= -\partial^0 A^1 + \partial^1 A^0. \end{aligned} \tag{2.68}$$

The $x$-component of the induction magnetic field (caused by the moving $\vec{E}$-field) may be read off in a similar fashion. From (2.67), the equation reads

$$B_x = -\partial^2 A^3 + \partial^3 A^2. \tag{2.69}$$

As previously stated, the potential and the differential operator are both four-vectors, which means that the electric and induced magnetic field elements are that of a rank-two tensor—the field strength tensor, which in its contravariant form may be written as

$$F^{\alpha\beta} = \partial^{\alpha} A^{\beta} - \partial^{\beta} A^{\alpha}. \tag{2.70}$$

Explicitly, this tensor and its components look like

$$F^{\alpha\beta} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix} \tag{2.71}$$

The field-strength tensor is antisymmetric and contains only six independent values for the $\vec{E}$ and $\vec{B}$ components. The covariant version of this tensor can be written by simply switching the entries in the first column and first row.

The "dual" of the field strength tensor is defined by

$$\widetilde{F}^{\alpha\beta} = \frac{1}{2} \epsilon^{\alpha\beta\mu\nu} F_{\mu\nu}, \tag{2.72}$$

which is a fully antisymmetric rank-four pseudotensor. In the $O$ rest frame, the components are specified by the conditions of the four-dimensional Levi-Civita symbol:

$$\epsilon^{\alpha\beta\mu\nu} \equiv \begin{cases} 1 & \text{if } (\alpha\beta\mu\nu) \text{ 1s an even permutation of (0123)} \\ -1 & \text{if } (\alpha\beta\mu\nu) \text{ is an odd permutation of (0123)} \\ 0 & \text{otherwise} \end{cases} \tag{2.73}$$

Explicitly, the dual field strength tensor may also be written in its matrix form as

$$\widetilde{F}^{\alpha\beta} = \begin{pmatrix} 0 & -B_x & -B_y & -B_z \\ B_x & 0 & E_z & -E_y \\ B_y & -E_z & 0 & E_x \\ B_z & E_y & -E_x & 0 \end{pmatrix}. \tag{2.74}$$

## 2.5 Invariance of Maxwell's Equations

Having covered all of the transformation properties in the previous section, we may now test the covariance of Maxwell's equations. The inhomogeneous equations are

$$\vec{\nabla} \cdot \vec{E} = \rho \tag{2.75}$$

$$\vec{\nabla} \times \vec{B} - \frac{\partial \vec{E}}{\partial t} = \vec{J} \tag{2.76}$$

Using the field strength tensor, (2.75) may be written

$$\partial_1 F^{10} + \partial_2 F^{20} + \partial_3 F^{30} = J^0. \tag{2.77}$$

Given that $F^{00} = 0$ we may freely add $\partial_0 F^{00}$ to the left-hand side. so that the whole expression reads

$$\partial_\alpha F^{\alpha 0} = J^0 \tag{2.78}$$

The left-hand side is a divergence of the $0^{\text{th}}$ column of rank-two tensor, while the right hand side is simply the time component of a four-vector. The Lorentz indices of these objects match. The other inhomogeneous equations for $\beta = 1, 2, 3$ are manifestly covariant and together, all four equations may be written as

$$\partial_\alpha F^{\alpha\beta} = J^\beta. \tag{2.79}$$

The homogeneous equations are

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{2.80}$$

$$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0. \tag{2.81}$$

For these equations, the first one can be written using the dual field strength tensor in a similar fashion as before, where

$$\partial_1 \widetilde{F}^{10} + \partial_2 \widetilde{F}^{20} + \partial_3 \widetilde{F}^{30} = 0. \tag{2.82}$$

Again, because $\widetilde{F}^{00} = 0$, we have

$$\partial_\alpha \widetilde{F}^{\alpha 0} = 0; \tag{2.83}$$

and similarly, for $\beta = 1, 2, 3$, we are left with

$$\partial_\alpha \widetilde{F}^{\alpha\beta} = 0, \tag{2.84}$$

which is clearly covariant, as the right-hand side is equal to zero. The covariance of both the inhomogeneous and homogeneous equations verify that Maxwell's equations remain invariant under these transformations. These equations are components of a rank-one pseudotensor, which is formed by summing certain indices of a rank-three tensor $\partial^\alpha F^{\beta\gamma}$. Consider the equation for the divergence of the magnetic field (2.80). In tensor notation, this equation reads

$$\partial_1 F^{23} + \partial_2 F^{31} + \partial_3 F^{12} = 0. \tag{2.85}$$

The other three homogeneous equations may be written in a similar fashion. All four equations may be written completely as

$$\partial^\alpha F^{\beta\gamma} + \partial^\gamma F^{\alpha\beta} + \partial^\beta F^{\gamma\alpha} = 0, \tag{2.86}$$

where $\alpha, \beta, \gamma$ represent any 3 combinations of $0, 1, 2, 3$—which leads to a total of four equations. Due to the properties of the Levi-Civita symbol ($\epsilon$), all other permutations will ultimately give a result of 0.

## 2.6 Transformation of the Electromagnetic Field

Knowing how a general rank-two tensor transforms we may now investigate the specific transformation properties for $\vec{E}$ and $\vec{B}$. It must transform as

$$(F')^{\alpha\beta} = A^\alpha{}_\gamma A^\beta{}_\delta F^{\gamma\delta}. \tag{2.87}$$

In matrix notation

$$\mathbf{F}' = \mathbf{A}\mathbf{F}\mathbf{A}^T, \tag{2.88}$$

where $\mathbf{A}^T$ is the transpose of $\mathbf{A}$.

If $O'$ moves at velocity $\vec{v} = \beta\hat{x}$ relative to $K$, then the transformation matrix looks like

$$\mathbf{A} = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \mathbf{A}^T \tag{2.89}$$

From (2.71), we find

$$\mathbf{FA} = \begin{pmatrix} \beta\gamma E_x & \gamma E_x & -E_y & -E_z \\ \gamma E_x & -\beta\gamma E_x & -B_z & B_y \\ \gamma E_y - \beta\gamma B_z & -\beta\gamma E_y + \gamma B_z & 0 & -B_x \\ \gamma E_z + \beta\gamma B_y & -\beta\gamma E_z - \gamma B_y & B_x & 0 \end{pmatrix}. \tag{2.90}$$

It follows that

$$\mathbf{F}' = \begin{pmatrix} 0 & -E_x & -\gamma E_y + \beta\gamma B_z & -\gamma E_z - \beta\gamma B_y \\ E_x & 0 & \beta\gamma E_y - \gamma B_z & \beta\gamma E_z + \gamma B_y \\ \gamma E_y - \beta\gamma B_z & -\beta\gamma E_y + \gamma B_z & 0 & -B_x \\ \gamma E_z + \beta\gamma B_y & -\beta\gamma E_z - \gamma B_y & B_x & 0 \end{pmatrix}. \tag{2.91}$$

This is an antisymmetric tensor, from which we may now write down the appropriate primed components $\vec{B}'$ and $\vec{E}'$.

$$B'_x = B_x \tag{2.92}$$

$$B'_y = \gamma\left(B_y + \beta E_z\right) \tag{2.93}$$

$$B'_z = \gamma\left(B_z - \beta E_y\right) \tag{2.94}$$

$$E'_x = E_x \tag{2.95}$$

$$E'_y = \gamma\left(E_y - \beta B_z\right) \tag{2.96}$$

$$E'_z = \gamma\left(E_z + \beta B_y\right) \tag{2.97}$$

From this we can write down the explicit relationships between the projections of the fields parallel and perpendicular to the direction of motion $\hat{x}$:

$$\vec{E}'_{\parallel} = \vec{E}_{\parallel} \tag{2.98}$$

$$\vec{E}'_{\perp} = \gamma\left[\vec{E}_{\perp} + \left(\vec{\beta} \times \vec{B}\right)\right] \tag{2.99}$$

$$\vec{B}'_{\parallel} = \vec{B}_{\parallel} \tag{2.100}$$

$$\vec{B}'_{\perp} = \gamma\left[\vec{B}_{\perp} + \left(\vec{\beta} \times \vec{E}\right)\right]. \tag{2.101}$$

## 2.7 Fields of a Point Charge With Constant Velocity

Consider a simple example of a point charge moving with a constant velocity $\vec{v} = \beta\hat{x}$ relative to the frame $O$. $O'$ is moving at the same velocity ($\vec{v}$), such that the particle is at rest in the primed frame and located at its origin (figure 2.1). The fields in $O'$ are

$$\vec{B}'\left(\vec{x}', t'\right) = 0 \tag{2.102}$$

$$\vec{E}'\left(\vec{x}', t'\right) = \frac{q}{4\pi r'^3}\vec{x}\,'. \tag{2.103}$$

In the $z' = 0$ plane, the electric field is

$$\vec{E}' = \frac{q}{4\pi\left(x'^2 + y'^2\right)^{3/2}}\left(x'\hat{x}' + y'\hat{y}'\right) \tag{2.104}$$

Transforming the electromagnetic field, the nonvanishing components in the $O$ frame are

$$E_x = \frac{qx'}{4\pi\left(x'^2 + y'^2\right)^{3/2}} \tag{2.105}$$

$$E_y = \frac{\gamma q y'}{4\pi\left(x'^2 + y'^2\right)^{3/2}} \tag{2.106}$$

and

$$B_z = \frac{\gamma\beta q y'}{4\pi\left(x'^2 + y'^2\right)^{3/2}}. \tag{2.107}$$

Consider an observer at the point $(0, b, 0)$ in the unprimed $O$ frame. Using the Lorentz transformations, the position translates into $O'$ as

$$(x', y', z') = (-\gamma v t, b, 0). \tag{2.108}$$

Substituting these in for $\vec{E}$ and $\vec{B}$, the equations become

$$E_x = \frac{qvt}{4\pi \left[b^2 + (\gamma v t)^2\right]^{3/2}} \tag{2.109}$$

$$E_y = \frac{\gamma q b}{4\pi \left[b^2 + (\gamma v t)^2\right]^{3/2}} \tag{2.110}$$

$$B_z = \frac{\gamma \beta q b}{4\pi \left[b^2 + (\gamma v t)^2\right]^{3/2}}. \tag{2.111}$$

Evaluating these equations we can see that $E_x$ falls to zero at $t = 0$. The other two fields have a maximum at $t = 0$ and then fall to zero at large times. The maximum field strengths are $\gamma q/4\pi b^2$ for $E_y$ and $\gamma \beta q/4\pi b^2$ for $B_z$. For a highly relativistic particle ($\beta \to 1$), $B_z$ approaches $E_y$ in strength.

# Chapter 3

# CPT Symmetry And Its Violations

Another symmetry that plays an important role in theoretical physics is CPT symmetry. A CPT transformation refers to a simultaneous transformation of charge conjugation (C), a parity transformation (P) (also referred to as helicity, chirality or handedness), and a time reversal (T). CPT is the only observed discrete spacetime symmetry that appears to be an exact symmetry of nature at a fundamental level. In the case of flat Minkowski spacetime in which gravity may be neglected, the established discrete symmetry is given by CPT invariance. Charge conjugation can be understood as exchanging a particle for an antiparticle. A parity transformation is a reflection of the three spatial directions ($\vec{x} \to -\vec{x}$), while T refers to the reversal of the time coordinates ($t \to -t$).

## 3.1 Discrete Symmetry Background

In the early 1950s, it was thought that all fundamental particles obeyed the C, P and T symmetries. In 1956, a paper by Lee and Yang [1] observed that parity conservation was yet to be tested in experiments involving the weak force. In the same year, Chinese-American experimental physicist Chien-Shiung Wu performed a test on radioactive $^{60}$Co atoms at extremely low temperature. Applying a strong magnetic field causes the atoms to align their spins in a uniform direction. When the $^{60}$Co decays it emits a $\beta$-particle, and from the experiment it was observed that these particles were emitted preferentially in a direction opposite that of the orientation of the spin axis. This simple observation would indicate that parity is violated, as

the experiment shows that if one were to perform a mirror-image version of this setup, the spin axis (angular momenta such as the spin being axial vectors that do not change sign under P) would be unchanged, but the $\beta$-particle emission direction would become inverted. If parity were conserved, the expectation of the dot product between the spin orientation (an axial vector) and the decay particle direction (a polar vector) should vanish, which it does not. The results imply that the universe appears to care about right- and left-handedness. As a result of this work, in 1957 Lee and Yang were awarded the Nobel Prize for Physics for showing that the weak force indeed violates parity.

In the following years physicists determined that maybe this parity violation could be rectified if it were not necessarily fundamental as suspected, but were part of a larger symmetry. This would lead to the formulation of the combined charge conjugation-parity CP symmetry conservation. Revisiting the $^{60}$Co experiment, the spin axis as well as the emitted particle both depend on the magnetic field present. By reversing the spin axis direction as well as changing the charge of the emitted particle we would find that it resolves the issue, as a positron would be emitted in the same preferred direction. This temporary fix would allow the physics community to relax until 1964, when a paper by Christenson, Fitch, and Cronin [2] showed that some particles can also violate combined CP symmetry. Once again it was theorized that this combination by itself might not be fundamental either, and as a result the third transformation was added: time reversal.

If CP may be violated, then T symmetry violation must also be allowed in order to preserve the overall CPT symmetry. T symmetry breaking has been observed in mesons (quark-antiquark bound state particles, such as $K^0$ and $\bar{K}^0$) which, through weak interactions, can oscillate back and forth between particle and antiparticle flavors. It was observed that this swapping is a time-dependent process that takes different amounts of time going forward ($K^0 \to \bar{K}^0$) and backward ($\bar{K}^0 \to K^0$). This

would mean that the process does not play the same forward in time as it would backward in time, and so it is not time symmetric.

To this date, the combined trio of C, P and T together is believed to be a fundamental symmetry, and there has yet to have an experiment to prove otherwise.

## 3.2 The CPT Theorem

The CPT theorem states that under mild technical assumptions, any unitary, Lorentz-invariant, point-particle quantum field theory within a flat Minkowski space is CPT invariant. A version of it was first established in the 1950s in the context of Lagrangian field theory. The operator $\Theta$ is used to describe a discrete reflection containing C, P, and T such that $\Theta = \text{CPT}$ and is used as a means to establish the invariance of quantum field theories under these unified reflections. As it turns out, the order in which the individual C, P, and T reflections are performed is irrelevant.

The mathematical setup for the transformation of $\Theta$ on the physical state $|\psi\rangle$ in Hilbert space returns a CPT-conjugate state $\left|\bar{\psi}\right\rangle$:

$$|\Theta\psi\rangle = \Theta\,|\psi\rangle \equiv \left|\bar{\psi}\right\rangle \tag{3.1}$$

Note that we consider both particle and antiparticle states at the same time, as $\Theta$ involves charge conjugation. The $\Theta$ operator also possesses the intuitive property

$$\Theta\Theta = 1 \tag{3.2}$$

and thus

$$\Theta = \Theta^{-1} = \Theta^{\dagger}. \tag{3.3}$$

which should seem quite reasonable as performing any pair of the contained transformations: two parity transformations, two charge conjugations, and two time reversals would render the physics unchanged. For the last step we note that quantum symmetry operations are known to be unitary or antiunitary. An antiunitary operator

is one that includes a time reversal component (thus obviously including $\Theta$). These operators apply an additional complex conjugation to any state vector that they act on; this is necessary in order ensure that any $e^{-i\omega t}$ time evolution factors are changed to $e^{i\omega t}$, in accord with taking $t \to -t$.

If we consider a quantum operator $\boldsymbol{A}$, a general form of the transformation law for quantum operators may be taken as

$$\boldsymbol{A}_{CPT} = \Theta \boldsymbol{A} \Theta^{\dagger} = \Theta^{\dagger} \boldsymbol{A} \Theta \tag{3.4}$$

Using these operators and their properties, various proofs have shown the validity of the theorem. These approaches include a proof based on Lagrangian formalism in quantum field theory performed by Bell [3], Luders [4] [5], Pauli [6] and implicitly by Schwinger [7]. In 1957, Jost would produce a more technical proof based on axiomatic field theory involving the complexified version of Lorentz transformations. His work would display a closer connection between Lorentz and CPT Symmetry [8].

## 3.3 CPT Violation

The proofs of the CPT theorem highlight the status of CPT as a fundamental symmetry. Testing this symmetry principle will either allow us to solidify our theories further or possibly uncover new underlying unknown physics, just as violations of P and CP led to improved understandings of the weak interactions. The question then arises as to whether CPT may be evaded physically in a manner that is acceptable. To address this query, there are three approaches to consider.

The first approach is based on the fact that CPT holds for point particles in quantum field theory. CPT symmetry might be avoidable if the fundamental physics requires a broader framework that contains quantum field theory only as a limiting case. In string theory, there exists a case in which the field theory for open superstrings may allow for spontaneous CPT violation [9]. String theory contains quantum

field theory in certain limits, so it may be possible to see these CPT-violating effects in the quantum field theory limit. The SME (standard model extension) is an effective field theory utilized to investigate leading, low-energy remnants of a CPT breakdown in underlying models, such as string theory

Generally speaking, the most fundamental models are theories that consider a large number of degrees of freedom. The standard model is a many-body quantum theory, and thus it is assumed that its underlying physics also contains many degrees of freedom. It has been shown that successful predictions do not necessarily require detailed knowledge of all of the dynamics for these degrees of freedom, as can be seen in areas that use statistical mechanics and thermodynamics. In certain cases, the collective excitations will have the behavior of many degrees of freedom be correlated, as seen in the physics of fluid mechanics, phonons, etc. Thus a physical description for the mechanics of these systems does not necessarily require details about the motion of particles at the deepest level. This has shown to be successful in some field-theoretic methods, such as the Navier-Stokes equations for viscous fluids and the Beltrami-Michell equations for elasticity. So it is reasonable to think that because collective excitations in the large-distance limit are well descriped in field theory, it might be the case that the dominant low-energy effects of general CPT violation may be from yet unknown underlying physics similar to the other cases, meaning a well-defined effective quantum field theory describing CPT violation.

Whitin the framework of quantum field theory, there currently still remain two distinct options for CPT violation. They arise from the axiomatic proof of CPT theorem pertaining to Lorentz symmetry and from a certain degree of analytic smoothness in the laws of physics, so that they remain valid for complex-valued Lorentz boosts. If CPT is violated, one would either have to give up Lorentz symmetry or give up on some of the smoothness properties in quantum field theory. From the work of Greenberg [10], CPT violations in theory with a smooth and well-defined time evolution

structure imply the breakdown of both microcausality and Lorentz symmetry.

Relaxing the condition of Lorentz symmetry has become a prominent area of research in physics. The idea has been studied by Colladay and Kostelecký to capture leading effects of the spontaneous breaking in string theory [32]. Spontaneous symmetry breaking is well understood and is correspondingly benign. The consistency of quantum field theory does not seem to require Lorentz symmetry, as many non-relativistic effective field theories have been successful. This has led to the SME as a general field theory framework which describes perturbative Lorentz violation. Although about half of the Lorentz-breaking contributions to the SME Lagrangian also violate CPT invariance, Lorentz violation without CPT breaking is possible. The SME aims to provide a framework for analyzing experimental and observational studies of Lorentz and CPT symmetry [11].

The second field theory option requires giving up certain smoothness properties of physics. The theories developed this way make similar assumptions to the prior ones, in that the properties are based on energy positivity, microcausality, the validity of closed-system quantum theory, and other natural assumptions such as finite spin. Another related approach suggests that a nontrivial spacetime topology evades one of the prerequisite for CPT Theorem [12]. Models of bosons have been investigated in which particle and antiparticle states are contained within the same multiplet [13]. This implies that the conventional quantization methods of imposing the standard commutation relations on the creation and annihilation operators leads to non-point interactions for half-integer isospin. This nonlocality violates microcausality and is the source for CPT violation in these theories; however, these multiplets have never been observed in nature.

Hawking has also claimed that CPT-violating effects may arrive in quantum mechanics due to gravity [14]. His claims were that the gravitational field in addition to the uncertainty principle may cause limitations on the predictability of future events.

Typically, given initial conditions along with a known set of physical laws, one may make an accurate prediction of future observables. In the presence of a black hole, interaction regions may contain event horizons beyond which the observer has limited physical knowledge, rendering part of the the systems' information inaccessible, which may cause the systems' quantum evolution to involve thermodynamic features such as nonunitarity. Other works of Hawking make arguments for a different type of loss of quantum-mechanical coherence in other approaches to quantum gravity. He argues that observable low-energy particles coupled to unobserved high-energy string states has an affect on the physics [15]. This requires that the observable sector be regarded as an open quantum system that leads to a decoherence phenomena that has departures from CPT symmetry.

Lastly, one of the technical requirements from CPT theorem is that the fields have spin that is both definite and finite. It can be shown that in the case of the spin-zero scalar field, this requirement is not specifically needed. However, in these models, an investigation of Lorentz-symmetric infinite-spin fields by Abers, Grodsky, and Norton has shown CPT violation occurs, as well as apparent negative-energy states [16]. Oksak and Todorav have since developed a model that contains the CPT violation while maintaining energy positivity [17]. These efforts emphasize the requirements for finite-spin for CPT symmetry, however they have also yet to be seen experimentally.

# Chapter 4

# Standard Model Extensions (SME)

Spontaneous symmetry breaking (SSB) occurs whenever a physical system in a symmetric state ends up in an asymmetric state. It describes a system in which the equations of motion or Lagrangian obey symmetries that are not preserved in the lowest-energy state. When this SSB occurs, the lowest energy state of the system (usually the vacuum), requires the value of a field to be non-vanishing. Often, these non-zero vacuum expectation values (VEVs) will not exhibit all of the dynamical symmetries of a system and can lead to the main observable imprints from the symmetry breaking. For Lorentz symmetry breaking, these VEVs are non-zero tensor or vector fields. In the specific case of flat spacetime, the VEVs are usually taken as constant, showing a preferred direction which ultimately which leads to a loss of Lorentz (and CPT) invariance.

In 1997–1998 [18], Donald Colladay and Alan Kostelecký were responsible for the foundation of the standard model extension (SME) in flat spacetime, the workings of which would give rise to experimental searches for types of signals that break the symmetries. Today it is understood that new physics beyond the standard model may involve interesting forms of symmetry breaking. The SME is a general framework for studying these violations and contains operators capable of breaking these symmetries. The SME incorporates the standard model and general relativity and includes particular operators that both preserve and break CPT symmetry. The general SME has an infinite number of parameters, containing nonrenormalizable operators that may be of arbitrarily high number of dimensions. However, it can be far more

practical to focus attention on a finite subset of these operators. Physicists call this the minimal SME, and it describes the forms of Lorentz violation of greatest likely importance at lower energies.

For Lorentz and CPT violation in string field theory, the symmetry breaking is spontaneous, and as a result, certain features such as microcausality and positivity of the energy are expected to hold in the low-energy effective theory. Energy-momentum conservation is also preserved so long as the tensor expectation values are spacetime-position independent. In addition, quantization methods remain uneffected allowing the relativistic Dirac and nonrelativistic Schrödinger equations emerge at the limits. With spontaneous breaking, the fundamental theory and the effective low-energy theory should remain invariant under observer Lorentz transformations, including rotations or boosts of the observer's inertial frame [18]. Non-zero tensor expectation values in the vacuum will only affect invariance properties under particle Lorentz transformations, i.e., rotations or boosts of a localized particle or field that leave the background expectation values unchanged. This approach has been used to obtain a general extension of the minimal $SU(3) \times SU(2) \times U(1)$ standard model that violates both Lorentz invariance and CPT [18]. In addition to energy-momentum conservation, observed Lorentz invariance, hermiticity, quantization, microcausality and energy positivity, the minimal SME also preserves gauge invariance and power-counting renormalizability.

The SME provides a quantitative microscopic theory of Lorentz and CPT violation. Physicists can consider potentially observable signals and attempt to establish bounds from various experiments, in the photon sector and elsewhere. Many tests of Lorentz invariance and CPT exist, and although many experiments are not sensitive enought to detect suppressed effects motivating these investigations, certain high-precision ones may display observable signals within this framework.

## 4.1 Lorentz-Violating Experimental Searches

The results of the SME model have been used to examine possible bounds on CPT and Lorentz violations from measurements of neutral-meson oscillations [19, 20, 21, 22], from tests of quantum electrodynamics in Penning traps [23, 24], and from baryogenesis [25]. One study tests the possibility of Lorentz and CPT effects on hydrogen and antihydrogen spectroscopy [26], while many others address limits attainable in clock-comparison experiments [27, 28]. There also exists a number of unaddressed significant theoretical issues arising at scales between the electroweak mass and the Planck mass. The "dimension problem" aims to address whether spontaneous Lorentz breaking in the fundamental theory near the Planck scale (which may contain more than four dimensinos) indeed extends to the four physical spacetime dimensions; and, if so, what might be the mechanism for its suppression; and if not, why exactly four spacetime dimensions are spared. Other areas address the effects of mode fluctuations around the tensor expectation values and possible constraints and effects arising from a nonminimal standard model or (super)unification below the Planck scale. Stemming from the works of Dirac and Heisenberg, physicists have considered an unphysical spontaneous Lorentz breaking in an effort to interpret the photon as a Nambu-Goldstone boson [29]. Conversely, Nielsen and his colleagues suggest that the observed Lorentz symmetry in nature might be a low-energy manifestation of a fundamental theory without Lorentz invariance [30].

## 4.2 Lorentz-Violating Terms

The most genereal form of a Lorentz-violating term in the SME contains two parts, one that acts as a coupling coefficient and another that is constructed from the basis fields of the standard model. The form is constrained by the limitations placed by requirements for the structure of each part. The tensor-valued coupling coefficient

carries spacetime indices and reflects the properties of the relevant nonzero expectation values from the fundamental theory under observer Lorentz transformations. The coupling coefficient may be complex; however, it is constrained by the requirement that the Lagrangian be Hermitian. If a coupling coefficient contains an even number of spacetime indices, the pure trace component is irrelevant as it maintains Lorentz invariance. Therefore, a coupling coefficient of this type may taken traceless.

The field part for fermions may involve covariant derivatives and gamma matrices. Gauge invariance requires that the field part is a singlet under $SU(3) \times SU(2) \times U(1)$, while power-counting renormalizability requires it to have mass dimension $n \leq 4$. The requirement that the SME originates from spontaneous Lorentz breaking in a covariant fundamental theory implies the entire Lorentz-violating term be a singlet under observer Lorentz transformations. Thus the field part requires indices that will match those of the coupling coefficient.

The field part of each term having mass dimension $n$ must have a coupling coefficient with mass dimension $4 - n$. The relevant scale for these effects may be roughly that of the Planck mass. In the low-energy theory a field term of mass dimension $n + 1$ would have coupling coefficient suppressed by an additional power of relative to the coefficient of a field term of mass dimension $n$.

For the purposes of the research explored in this paper, we shall forgo the complete SME and focus the electron and photon sectors and their specific field redefinition effects. The minimal photon sector contains a combination of CPT-even and CPT-odd Lorentz-violating terms. The free photon Lagrangian is

$$\mathcal{L}_{photon} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}(k_F)^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \frac{1}{2}(k_{AF})^{\mu}\epsilon_{\mu\nu\rho\sigma}F^{\nu\rho}A^{\sigma}, \tag{4.1}$$

where $k_F$ and $k_{AF}$ represent the CPT-even and -odd terms respectively. The parameter $k_F$ is a real, dimensionless coupling that may be taken to have the symmetries of the Riemann tensor and to be double traceless, and $k_{AF}$ is real and has mass dimension 1.

The four-component electron field $\psi$ with electron mass $m$, has a lagrangian that contains Lorentz-violating interactions arising from spontaneous symmetry breaking.

$$\mathcal{L}_{electron}^{QED} = i\overline{\psi}\left(\Gamma_\nu D^\nu - M\right)\psi, \tag{4.2}$$

where $D_\mu = \partial_\mu + iqA_\mu$ is the standard covariant derivative,

$$\Gamma_\nu = \gamma_\nu + c_{\mu\nu}\gamma^\mu + d_{\mu\nu}\gamma_5\gamma^\mu + e_\nu + if_\nu\gamma_5 + \frac{1}{2}g_{\lambda\mu\nu}\sigma^{\lambda\mu} \tag{4.3}$$

and

$$M = m + im_5\gamma_5 + a_\mu\gamma^\mu + b_\mu\gamma_5\gamma^\mu + \frac{1}{2}H_{\mu\nu}\sigma^{\mu\nu}. \tag{4.4}$$

Any Lorentz-preserving terms that arise from spontaneous symmetry breaking may be absorbed into the bare mass terms $m$, $m_5$, and the overall normalization of the Lagrangian. The fields are normalized so that the $\gamma_\nu$ term in (4.3) is one. The coupling coefficients $a, b, c, d, e, f, g, m_5$, and $H$ are all real, constant parameters that are related to the vacuum expectation value of the contributing tensor fields from the underlying theory. It can be noted that some of these parameters are only apparently Lorentz-violating, as through the use of the appropriate field redefinition, the Lagrangian containing them can be shown to be equivalent to the standard Lagrangian [31].

# Chapter 5

# Electromagnetic SME and Chern-Simons Term

Both Lorentz and gauge invariance are important aspects of Maxwell electrodynamics. Lorentz-invariant dynamics describes the properties of electromagnetic radiation, while gauge invariance, which implies that the photon is massless, may be validated by placing strict limits on the photon mass. Lorentz-invariant violations of gauge invariance are typically parameterized by a mass parameter $\mu$ for the photon field $A_\nu$. Including a mass term modifies the standard non-free electromagnetic Lagrange density

$$\mathcal{L} = -\frac{1}{4}F^{\nu\lambda}F_{\nu\lambda} - J_\nu A^\nu, \tag{5.1}$$

so that when a massive photon is considered, this picks up an additional term

$$\mathcal{L} = -\frac{1}{4}F^{\nu\lambda}F_{\nu\lambda} - J_\nu A^\nu + \frac{\mu^2}{2}A^\nu A_\nu. \tag{5.2}$$

In the presence of a conserved current $J_\nu$, the field equations are

$$\Box A_\nu + \mu^2 A_\nu = J_\nu \tag{5.3}$$

$$\partial_\nu A^\nu = 0, \tag{5.4}$$

where the d'Alembertian is $\Box = \partial_t^2 - \vec{\nabla}^2$ and the light speed $c$ is normalized to 1. It can be seen that gauge invariance of $\mathcal{L}$ under

$$A_\nu \to A_\nu + \partial_\nu \chi \tag{5.5}$$

is clearly lost, because of the explicit dependance of the free Lagrange density on the potential $A^\mu$, rather than on the field strength $F^{\mu\nu}$.

A different particular modification of Maxwell's theory respects gauge invariance but violates Lorentz invariance. The modification involves adding a Chern-Simons-like term to the Lagrange density given by

$$\mathcal{L}_{CS} = -\left(k_{AF}\right)_{\alpha} A_{\beta} \widetilde{F}^{\alpha\beta}, \tag{5.6}$$

where the dual electromagnetic tensor $\widetilde{F}^{\alpha\beta}$ is

$$\widetilde{F}^{\alpha\beta} = \frac{1}{2}\epsilon^{\alpha\beta\mu\nu} F_{\mu\nu} \tag{5.7}$$

This modification couples the axial vector $\left(k_{AF}\right)_{\alpha}$ background to the electromagnetic field. (The subscript letters "$AF$" simply indicates that the term involves both the four-vector potential $A$ and the field strength $F$.)

The behavior of $\Delta\mathcal{L}_{CS}$ under a gauge transformation is somewhat peculiar: $\mathcal{L}_{CS} \rightarrow \mathcal{L}_{CS} + \Delta\mathcal{L}_{CS}$, where

$$\Delta\mathcal{L}_{CS} = -\partial_{\beta}\left(k_{AF}\right)_{a} \chi \tilde{F}^{\alpha\beta} + \frac{1}{2}\chi \widetilde{F}^{\beta\alpha}\left[\partial_{\alpha}\left(k_{AF}\right)_{\beta} - \partial_{\beta}\left(k_{AF}\right)_{\alpha}\right] \tag{5.8}$$

where gauge invariance of $\mathcal{L}$ would demand that this term vanish for arbitrary $\chi(x)$. This does not occur; however, because the first term in $\Delta\mathcal{L}_{CS}$ is a four-divergence, the integrated action $S = \int dx\, \mathcal{L}$ will be gauge invariant if the second term vanishes. This will occur if $\left(k_{AF}\right)_{\alpha}$ happens to be a constant of nature. If spacetime is flat, when $\partial_{\alpha}\left(k_{AF}\right)_{\beta} = 0$ in one frame, it will vanish in all frames and thus $\Delta\mathcal{L}_{CS}$ is a total derivative, and we shall have sufficient gauge invariance. However, if it is constant, this may be problematic as it would pick out a preferred direction in spacetime when coupled to an observable field. If the spatial component is non-vanishing, it would break rotational invariance while a non-vanishing time component is not invariant under Lorentz boosts.

The universe does have a preferred reference frame, such as that of a typical galaxy or the cosmic microwave background. In this frame, the universe will appear isotropic and homogeneous. It will have a preferred time direction and notion of simultaneity.

It is not necessarily implied that $(k_{AF})_\alpha$ points along this preferred time direction; however it is not unreasonable to consider specifically this case. This causes $(k_{AF})_\alpha$ to be timelike, such that

$$(k_{AF})_\alpha (k_{AF})^\alpha \equiv k_0^2 > 0. \tag{5.9}$$

The rest frame for the axial vector is frequently considered, where

$$\mathcal{L}_{CS} = -k_0 \vec{B} \cdot \vec{A}. \tag{5.10}$$

In a more general frame, the full vector is retained: $(k_{AF})_\alpha = \left(k^0, \vec{k}\right)$. Effects are not isotropic when the spatial components of the vector are not zero; thus absolute motion of the observer is detectable. This can show the breaking of Lorentz invariance of the Chern-Simons term in 4-dimensions. The following derivation places limits on the $(k_{AF})_\alpha$ as a parameterization of violating the Lorentz invariance. We should also note that there is parity and CPT violation additionally. This is due to the structure of the scalar product between the axial magnetic vector $\vec{B}$ and polar vector potential $\vec{A}$.

The Chern-Simons term replaces the source current four-vector $J^\nu \rightarrow J^\nu + (2k_{AF})_\mu \widetilde{F}^{\mu\nu}$. The resulting field equations become

$$\partial_\mu F^{\mu\nu} = J^\nu + (2k_{AF})_\mu \widetilde{F}^{\mu\nu}, \tag{5.11}$$

which leaves the two homogeneous Maxwell's equations unchanged. The other two modified Maxwell's equations are

$$\vec{\nabla} \cdot \vec{E} = \rho - 2\vec{k} \cdot \vec{B} \tag{5.12}$$

$$-\partial_t \vec{E} + \vec{\nabla} \times \vec{B} = \vec{J} - 2k_0 \vec{B} + 2\vec{k} \times \vec{E}, \tag{5.13}$$

while the homogeneous equations that remain unchanged are

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{5.14}$$

$$\partial_t \vec{B} + \vec{\nabla} \times \vec{E} = 0 \tag{5.15}$$

With the additon of this Chern-Simons term, it can be clearly seen that these equations will still reduce to the usual Maxwell's equations when $(k_{AF})_\alpha$ is set to zero. Moreover, the field equations involve only the guage-invariant quantities $\vec{E}$, and $\vec{B}$, rather than the potentials; therefore gauge invariance is preserved.

The energy momentum tensor can be written as

$$\Theta^{\mu\nu} = -F^{\mu\alpha}F^{\nu}{}_{\alpha} + \frac{1}{4}g^{\mu\nu}F^{\alpha\beta}F_{\alpha\beta} - \frac{1}{2}\left(k_{AF}\right)^{\nu}\epsilon^{\mu\alpha\beta\gamma}F_{\beta\gamma}A_{\alpha}. \tag{5.16}$$

The energy density $\Theta^{00}$ and momentum density $\Theta^{0j}$ are not gauge invariant

$$\mathcal{E} = \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 + k^0\vec{A}\cdot\vec{B} \tag{5.17}$$

$$\mathcal{P}^i = \left(\vec{E}\times\vec{B}\right)^i + k^i\vec{A}\cdot\vec{B}. \tag{5.18}$$

Under a gauge transformation they are each (like $\mathcal{L}$) changed by a total derivative, as

$$\vec{B}\cdot\vec{A} \to \vec{B}\cdot\left(\vec{A} - \vec{\nabla}\chi\right) = \vec{B}\cdot\vec{A} - \vec{\nabla}\cdot\left(\vec{B}\chi\right); \tag{5.19}$$

consequently, the electromagnetic energy and momentum, which are obtained by performing integrals over all space will be be gauge invariant. Spatial integrals can be shown to be explicitly gauge invariant by making use of the formula

$$\int d^3r\,\left[\vec{B}(\vec{r})\cdot\vec{A}(\vec{r})\right] = \int d^3r\,\vec{B}(\vec{r})\cdot\int d^3r'\,\frac{1}{4\pi\left|\vec{r}-\vec{r}'\right|}\vec{\nabla}\times\vec{B}(\vec{r}'), \tag{5.20}$$

which is valid assuming the field falls off rapidly enough for large distances, so that there are no surface terms produced by an integration by parts. Performing the integration over (5.17) and (5.18) we find

$$U = \int d^3r\,\mathcal{E} \tag{5.21}$$

$$= \frac{1}{2}\int d^3r\,\left[\vec{E}^2(\vec{r}) + \vec{B}^2(\vec{r})\right] + k^0\int d^3r\,d^3r'\,B^n(\vec{r})K^{nm}\left(\vec{r}-\vec{r}'\right)B^m(\vec{r}') \tag{5.22}$$

$$\vec{P} = \int d^3\,\mathcal{P} \tag{5.23}$$

$$= \frac{1}{2}\int d^3r\,\left[\vec{E}(\vec{r})\times\vec{B}(\vec{r})\right] \tag{5.24}$$
$$+\vec{k}\int d^3r\,d^3r'\,B^n(\vec{r})K^{nm}\left(\vec{r}-\vec{r}\,'\right)B^m(\vec{r}\,'),$$

where the kernel $K^{nm}$ is given by

$$K^{nm}(\vec{r}) = \epsilon^{nmi}\partial_i\frac{1}{4\pi r} \tag{5.25}$$

It should be noted that the energy is not positive definite. Eq. (5.17) implies

$$U = \frac{1}{2}\int d^3r\,\left[\vec{E}^2+\left(\vec{B}+k^0\vec{A}\right)^2\right]-(k_0)^2\int d^3r\,\vec{A}^{\,2} \tag{5.26}$$

By taking $|\vec{A}|$ sufficient large, and varying only on sufficiently large spatial scales (so that $k_0\vec{A}$ can overwhelm $\vec{B}=\vec{\nabla}\times\vec{A}$), the final term in (5.26) may drive the total energy $U$ to arbitrarily negative values. Energies unbounded below are generally indicative of an instability, because the field $\vec{A}$ may grow unboundedly in time, while the energy only becomes more negative as this happens.

We may find the wave solutions to the source-free modified Maxwell's equations by using a phase-exponential ansatz [39], which leads to

$$\omega^2\vec{E}-Q^2\vec{E}+\left(\vec{Q}\cdot\vec{E}\right)\vec{Q} = i\left(-2k_0\vec{Q}\times\vec{E}+\omega 2\vec{k}\times\vec{E}\right), \tag{5.27}$$

where $\omega$ is the frequency and $\vec{Q}$ is the wave vector. These form the four-vector $Q^\alpha=\left(\omega,\vec{Q}\right)$; $Q=\left|\vec{Q}\right|$. The dispersion relation is then given by

$$\left(Q^\alpha Q_\alpha\right)^2+4\left(Q^\alpha Q_\alpha\right)\left[\left(k_{AF}\right)_\beta\left(k_{AF}\right)^\beta\right] = 4\left[Q^\alpha\left(k_{AF}\right)_\alpha\right]^2 \tag{5.28}$$

which may be rewritten in the implicit form

$$\omega^2-Q^2 = \mp 2\left(k_0Q-\omega k\cos\theta\right)\left[1-\frac{4k^2\sin^2\theta}{\omega^2-Q^2}\right]^{-1/2}, \tag{5.29}$$

where $k=\left|\vec{k}\right|$. The angle $\theta$ is between $\vec{k}$ and $\vec{Q}$. Right-and left-handed circularly polarization for the waves are represented by the $-$ and $+$, respectively. The introduction of $\left(k_{AF}\right)_\alpha$ causes the photons to split into two modes with different dispersion

relations. The waves travels with a group velocity that differs from the speed of light at second order in $(k_{AF})_\alpha$,

$$\frac{\partial\,\omega}{\partial Q} = 1 \pm \mathcal{O}\left[{(k_{AF})_\alpha}^2\right] \tag{5.30}$$

The fact that the two polarization modes propagate at different velocities is evidence that there is violation of parity and Lorentz invariance.

The wave four-vector can become spacelike in which case we need exponentially unstable modes to solve the field equations. It can be seen that in the rest frame, the dispersion relation for circularly polarized photons of momentum $\vec{Q}$ becomes

$$\omega_\pm^2 = Q(Q \mp 2k_0) \tag{5.31}$$

so $\omega$ becomes imaginary for $Q < k_0$. However, this will not violate energy conservation because even though the two terms from (5.26) may become arbitrarily large, their difference remains a time-independent, finite quantity.

## 5.1 Vacuum Radiation

In theoretical physics, Lorentz violating field theories can have some very interesting properties, in that they posess new features that do not occur in the ordinary Lorentz invariant ones. When Lorentz symmetry is exact, some processes are kinematically forbidden. However when this symmetry is broken, even weakly, some of these processes can become allowable. One of these processes is vacuum Cerenkov radiation, $e^- \to e^- + \gamma$, which is analogous to ordinary Cerenkov radiation in matter.

Cerenkov radiation is expected when the phase speed for a given mode of the electromagnetic field is found to be lesser than the speed at which the particle is moving at. A familiar example being instances where a relativistic particle passes through a medium or material with a significant index of refraction. As a result, the phase speed of light becomes slower than that of the particle that emitted it and causes a blue shift effect which can be visibly observed in experiments such as nuclear reactors

housed in water. Vacuum radiation contains certain aspects that distinguish it from regular textbook radiation, including the existance of dispersion, birefringence, and directional dependence. Besides Cerenkov radiation, the first aspect, dispersion, can be found in real materials as well as Lorentz-violating vacua. The other two aspects require directional- and polarization- dependent light speeds and may be observed experimentally in some asymmetric crystals [34]. These two are much more complicated and important when it comes to Lorentz-violating theories. A change in the dispersion relation of the electromagnetic waves without a change in the polariztion structure may lead to possible Cerenkov radiation, while conversely, a change in the polarization states without a change in the dispersion relation will not. Many of the calculations performed will be on a mode to mode basis. As a result, there will be certain aspects of the Cerenkov radiation that we will not investigate. This includes the back reaction on a moving charge. This quantity ultimately depends on the total emissions in all modes of the electromagnetic field. The resulting effect is a limitation in the ability for us to study the high energy spectrum where recoil becomes an important factor. So we are limited to investigating velocities far beyond the Cerenkov threshold. Directly at the threshold, an emitted photon could push back the velocity of the particle just below the threshold, which then disturbs the entire process. We shall neglect considerations of finite-duration effects and treat Cerenkov radiation as though it were a steady state process. As a result, investigating some phenomena of interest, such as diffraction, will be neglected.

## 5.2 Solution By Green's Function

Tachyonic modes do not need to be excited by well-behaved sources. For $\vec{k} = 0$ the modified Maxwell equations imply

$$\Box\vec{E}_T + 2k_0\vec{\nabla} \times \vec{E}_T = -\partial_i \vec{J}_T \tag{5.32}$$

$$\Box\vec{B} + 2k_0\vec{\nabla} \times \vec{B} = \vec{\nabla} \times \vec{J}_T, \tag{5.33}$$

where $\vec{E}_T$ is the transverse part of the electric field and $\vec{J}_T$ is the transverse current. In the Coulomb gauge

$$\Box\vec{A} + 2k_0\vec{\nabla} \times \vec{A} = \vec{J}_T \tag{5.34}$$

A transverse Green's function that is free of any unstable exponential growth may be constructed to deal with (5.32), (5.33) and is given by

$$G^{ij}(t, \vec{r}) = \left[\left(\delta^{ij} - \frac{\partial_i\partial_j}{\vec{\nabla}^2}\right)\Box + 2k_0\epsilon^{ijk}\partial_k\right] g(t, \vec{r}) \tag{5.35}$$

where $g$ satisfies

$$\Box^2 g + 4(k_0)^2\vec{\nabla}^2 g = \delta^4(x). \tag{5.36}$$

In Fourier space, where

$$\widetilde{g}\left(\omega, \vec{Q}\right) = \int dt\, d^3r\, e^{-i\left(\omega t - \vec{Q}\cdot\vec{r}\right)} g(t, \vec{r}), \tag{5.37}$$

the equation becomes

$$\left[\left(-\omega^2 + Q^2\right)^2 - 4(k_0)^2Q^2\right]\tilde{g} = 1. \tag{5.38}$$

A tachyonic pole arises in $g$ for wavelengths $Q < 2|k_0|$. However, there is a solution that does not grow in time and reduces to the Lienard-Wiechert formula when $k_0 = 0$

$$\begin{aligned} g(t, \vec{r}) = \frac{\sin k_0 r}{\pi k_0 r}\Bigg\{&\theta(t)\int_0^\infty dz\, \frac{\cos r\left[(k_0)^2 + z^2\right]^{1/2}}{\left[(k_0)^2 + z^2\right]^{1/2}} \sin tz \\ &-\frac{1}{2}\int_0^{k_0} dz\, \frac{\cos r\left[(k_0)^2 + z^2\right]^{1/2}}{\left[(k_0)^2 + z^2\right]^{1/2}} e^{-|t|/z}\Bigg\} \end{aligned} \tag{5.39}$$

We can see that the stability of the solution comes at the cost of this expression being noncausal. When $k_0 \neq 0$, the second term acts even for $t < 0$, before the $\delta$-function disturbance occurs. If a solution were to be causal for $t < 0$, it would possess exponential growth in time. This would be a runaway solution.

# Chapter 6

# Radiation Cancelations in Chern-Simons Theory

The electromagnetic Chern-Simons term in the minimal SME Lagrange density is not gauge invariant, depending explicitly on the potentials and not just the field strengths. However the integrated action is in fact gauge invariant. The Chern-Simons term changes the dispersion relations for electromagnetic waves, slowing down one of the polarizations. As a result, vacuum Cerenkov radiation becomes possible. This possible radiation can be studied using an iterative algorithm for finding the relationships between the between electric and magnetic fields of a moving charge. The result of the calculations it that for a timelike Chern-Simons coefficient, there is zero radiative power lost for a charged particle in uniform motion.

We consider the minimal SME Lagrange density for electrodynamics,

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} - \frac{1}{4}k_F^{\mu\nu\rho\sigma}F_{\mu\nu}F_{\rho\sigma} + \frac{1}{2}k_{AF}^{\mu}\epsilon_{\mu\nu\rho\sigma}F^{\nu\rho}A^{\sigma} - J^{\mu}A_{\mu}. \tag{6.1}$$

The $k_F$ term is CPT-invariant and gives rise to a completely different set of phenomena, which we shall not consider further. Since $k_{AF}$, the four-dimensional Lorentz-violating Chern-Simons term, will henceforth be taken to be purely timelike, we may write it as $k_{AF}^{\mu} = (k, \vec{0}\,)$. This form for $k_{AF}$ makes $\mathcal{L}_{CS}$ proportional to $\vec{A}\cdot\vec{B}$

The dispersion relation for circularly polarized photons (5.31) of momentum $\vec{Q}$ becomes, in the current notation,

$$\omega_{\pm}^2 = Q(Q \mp 2k). \tag{6.2}$$

One set of the helicity modes can have imaginary frequencies. In general, this may be problematic, as it can lead to the existence of runaway solutions. However, with the

use of the properly chosen acausal Green's function (5.35) [39], only real frequencies exist, not exponentially growing imaginary frequencies. However, an unattractive result of the acausal form is that a charged particle will start to emit radiation before it even starts to move. The purely electromagnetic part of the energy-momentum tensor containing an arbitrary Chern-Simons term is given by (5.16) [39]. Due to Lorentz violation, this tensor is not symmetric. With a purely timelike $k_{AF}$, $\Theta^{\mu\nu}$ the energy density ($\Theta^{00}$), momentum density ($\Theta^{0j}$), and energy flux ($\Theta^{j0}$) can be calculated

$$\mathcal{E} = \frac{1}{2}\vec{E}^2 + \frac{1}{2}\vec{B}^2 - k\vec{A}\cdot\vec{B} \tag{6.3}$$

$$\vec{\mathcal{P}} = \vec{E}\times\vec{B} \tag{6.4}$$

$$\vec{S} = \vec{E}\times\vec{B} - kA_0\vec{B} + k\vec{A}\times\vec{E}. \tag{6.5}$$

The momentum density is the only one of these explicit equations that (for purely timelike $k_{AF}$) is gauge invariant; however, the total energy obtained from integrating the energy density is in fact gauge invariant. The $-k\vec{A}\cdot\vec{B}$ term is not bounded from below. This is a peculiar property that causes instability in the theory. By increasing the amplitude of $\vec{A}$ this term may be made arbitrarily negative; this is the potential instability.

The purely timelike $k_{AF}$ means that the only modified eq will be the Ampere-Maxwell Law

$$\vec{\nabla}\times\vec{B} - \frac{\partial\vec{E}}{\partial t} = 2k\vec{B} + \vec{J} \tag{6.6}$$

where we can see from the right-hand side that the magnetic field becomes a source for itself. The equation is typically solved as a power series in the Lorentz violating parameter $k$, since Lorentz violation as a physical phenomenon is known to be small.

The Chern-Simons theory allows for modes of the radiated field with arbitrary small wave phase velocities $\omega_\pm/Q$, so that Cerenkov radiation is expected. To analyze this problem we shall, for simplicity, consider a point-like charge with constant

velocity $\vec{v}$. It should be noted that if a charge with constant velocity were actually radiating energy, this effect would cause it to lose energy and momentum, thus slowing it down causing an acceleration that would produce additional radiation, further modifying the the equations. However, acceleration-dependent radiation is not truly Cerenkov and shall be omitted. This will be seen to be a consistent approximation.

## 6.1 Deriving the Modified Electric and Magnetic Fields

The fields $\vec{E}$ and $\vec{B}$ can be described by power series in the Lorentz-violation parameter $k$ and the speed $v$. The particular advantage of this is that with a uniformly moving charge, the time derivatives in Maxwell's equations may be replaced with strictly spatial ones.

We shall consider a charged particle $q$ moving with trajectory $\vec{r}(t) = vt\hat{z}$ at the time $t = 0$. Neglecting any recoil (which does not produce true Cerenkov radiation) and considering a steady velocity, all fields must be moving along the $\hat{z}$-direction, also having speed $v$. This scenario is what simplifies the time derivatives in Maxwell's equations.

The electric and magnetic fields may be written as power series

$$\vec{E} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\vec{E}^{(m,n)} \tag{6.7}$$

$$\vec{B} = \sum_{m=0}^{\infty}\sum_{n=1}^{\infty}\vec{B}^{(m,n)}, \tag{6.8}$$

where the indices $m$ and $n$ indicate the powers of $k$ and $v$, respectively (that is—$k^m v^n$). The conventional terms present when $k = 0$ are the $\vec{E}^{(0,n)}$ with $n$ even and $\vec{B}^{(0,n)}$ with $n$ odd.

By demanding steady motion we avoid needing to use an acausal Green's function. The electric and magnetic fields must have the form $\vec{W}(\vec{r}, t) = \vec{W}(\vec{r} - v\hat{z}t, 0)$ so we replace $\frac{\partial}{\partial t}$ with $-v\frac{\partial}{\partial z}$. The modified fields equations, considered order by order,

become

$$\vec{\nabla} \times \vec{E}^{(m,n)} = v\frac{\partial \vec{B}^{(m,n-1)}}{\partial z} \tag{6.9}$$

$$\vec{\nabla} \cdot \vec{E}^{(m,n)} = 0 \tag{6.10}$$

$$\vec{\nabla} \times \vec{B}^{(m,n)} = -v\frac{\partial \vec{E}^{(m,n-1)}}{\partial z} + 2k\vec{B}^{(m-1,n)} \tag{6.11}$$

$$\vec{\nabla} \cdot \vec{B}^{(m,n)} = 0. \tag{6.12}$$

We may start with the known initial fields $\vec{E}^{(0,n)}$ and $\vec{B}^{(0,n)}$ and solve for the fields successively, order by order in $k$ and $v$.

However the total power radiated must have mass dimensions of 2 (provided it is finite). Therefore, no terms beyond $m = 2$ may contribute to a finite expression for the energy energy loss, and this offers a useful simplification.

The components for $\vec{E}^{(m,n)}$ and $\vec{B}^{(m,n)}$ will take one of two geometric forms; azimuthal or toroidal. An azimuthal field is divergenceless and points in the $\hat{\phi}$-direction. All of the $\vec{B}^{(0,n)}$ terms take this form. The divergenceless toroidal field points in the $\hat{r}$- and $\hat{\theta}$-directions. Neither of the two forms have any components with a $\phi$ dependance. The type of structure that $\vec{W}$ has will remain the same for its $z$-derivative $\partial\vec{W}/\partial z$. In physics, it is known that a current loop produces a toroidal magnetic field, while a toroid produces an azimuthal magnetic field. This is evident in the equations, in that the source from the right-hand sides of (6.9) and (6.11) produce a field with the opposite structure on the left-hand side and vice versa.

Its important to understand whether the components of the field are even or odd functions of $\cos\theta$ (or equivalently $z$). It can be referred to as $z$-parity [40]. It will become important, because if $\vec{S} \cdot \hat{\vec{r}}$ as $r \to \infty$ is an odd function of $\cos\theta$, there will be no net outflow of energy.

If a toroidal $\vec{B}^{(m,n)}$ has odd $z$-parity in the $r$-component and even $z$-parity in the $\theta$-component, it produces new fields—generated through (6.9) and (6.11)—that will be an azimuthal $\vec{B}^{(m+1,n)}$ with even $z$-parity and an azimuthal $\vec{E}^{(m,n+1)}$ with odd

Table 6.1 How the $z$-parity values specified for four different types of field terms: $+$ and $-$ denote even and odd parity, respectively; $\emptyset$ denotes that the corresponding term is zero for fields of the indicated type.

| Field Term | $z$-Parity $\hat{r}$ | $\hat{\theta}$ | $\hat{\phi}$ |
|---|---|---|---|
| Toroidal $\vec{E}$ | $+$ | $-$ | $\emptyset$ |
| Azimuthal $\vec{E}$ | $\emptyset$ | $\emptyset$ | $-$ |
| Toroidal $\vec{B}$ | $-$ | $+$ | $\emptyset$ |
| Azimuthal $\vec{B}$ | $\emptyset$ | $\emptyset$ | $+$ |

$z$-parity. For a toroidal $\vec{E}^{(m,n)}$ with the opposite conditions (odd for $r$-component and even for $\theta$-component) the new term will be an azimuthal $\vec{B}^{(m,n+1)}$ with even $z$-parity. This determines how toroidal fields act as sources for azimuthal fields.

How azimuthal fields act as sources for toroidal fields is trickier. Because the toroidal fields have two components, they cannot be determined using Amperian loop methods. Instead, a study of coupled differential equations is required. The $z$-derivative operator

$$\frac{\partial}{\partial z} = \cos\theta \frac{\partial}{\partial r} + \frac{\sin^2\theta}{r} \frac{\partial}{\partial \cos\theta} \tag{6.13}$$

reverses the $z$-parity of the field it acts on. So that the fields will have the correct dimensionality, $k$ and $r$ will always be combined into a factor of $\frac{k^m}{r^{2-m}}$. Thus, $\vec{E}^{(m,n)}$ may be written as

$$\vec{E}^{(m,n)} = \frac{1}{r^{2-m}} \left[ X(\theta)\hat{r} + Y(\theta)\hat{\theta} \right] \tag{6.14}$$

The curl and divergence can be reduced to

$$\vec{\nabla} \times \vec{E}^{(m,n)} = \frac{1}{r^{3-m}} [-X'(\theta) + (m-1)Y(\theta)]\hat{\phi} = v \frac{\partial B_\phi^{(m,n-1)}}{\partial z} \hat{\phi} \tag{6.15}$$

$$\vec{\nabla} \cdot \vec{E}^{(m,n)} = \frac{1}{r^{3-m}} [mX(\theta) + \cot\theta\, Y(\theta) + Y'(\theta)] = 0. \tag{6.16}$$

The $X'$ function is the derivative with respect to the argument $\theta$. It can be noted that if $X$ and $Y$ are functions of well-defined $z$-parity, (6.16) implies that the two component functions $X$ and $Y$ have opposite $z$-parities.

$Y$ may be eliminated from (6.15) for $m \neq 1$, using $\frac{\partial}{\partial\theta} = -\sin\theta \frac{\partial}{\partial \cos\theta}$; the resulting differential equation becomes

$$X'' + \cot\theta X' + m(m-1)X = vr^{3-m}\left[\cot\theta \frac{\partial B_\phi^{(m,n-1)}}{\partial z} - \sin\theta \frac{\partial}{\partial\cos\theta}\frac{\partial B_\phi^{(m,n-1)}}{\partial z}\right], \tag{6.17}$$

If the $B_\phi^{(m,n-1)}$ on the right-hand-side of (6.17) has a well-defined $z$-parity, then the entire equation (including all the $X$-dependent terms on the left-hand side) preserves that same $z$-parity.

Since the $\vec{B}^{(0,n)}$ (the terms that all other Lorentz-violating terms must ultimately be derived from) are toroidal, we see that the generated fields are toroidal when $m+n$ is even and azimuthal when $m+n$ is odd. Each of the generated fields uniquely has one of the two geometry types; none have a mixture. As a result, all the solutions for $\vec{E}^{(m,n)}$ and $\vec{B}^{(m,n)}$ will necessarily have the $z$-parities given in Table 6.1.

## 6.2 Calculation of $\vec{B}^{(1,1)}$

As an example of this iterative method, we may see how the $\vec{B}^{(1,1)}$ field term is calculated.

An arbitrary field in spherical coordinates will have the structure $A_r\hat{z} + A_\theta\hat{\theta} + A_\phi\hat{\phi}$, the operators needed for Maxwell's equations will be the divergence and curl

$$\vec{\nabla}\cdot\vec{A} = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2 A_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(A_\theta\sin\theta) + \frac{1}{\sin\theta}\frac{\partial}{\partial\phi}(A_\phi) \tag{6.18}$$

$$\begin{aligned}\vec{\nabla}\times\vec{A} = {} & \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(A_\phi\sin\theta) - \frac{\partial}{\partial\phi}A_\theta\right]\hat{r} \\ & + \frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}A_r - \frac{\partial}{\partial r}(rA_\phi)\right]\hat{\theta} + \frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial}{\partial\theta}A_r\right]\hat{\phi}.\end{aligned} \tag{6.19}$$

$\vec{B}^{(1,1)}$ will have the form $\frac{k}{r}\left[X(\theta)\hat{r} + Y(\theta)\hat{\theta}\right]$ as in (6.14). For the primary step, the

curl is taken

$$\begin{aligned}
\vec{\nabla}\times\vec{B}^{(1,1)} &= \frac{k}{r\sin\theta}\left\{\left[-\frac{\partial}{\partial\phi}B_\theta^{(1,1)}\right]\hat{r}+\frac{1}{r}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\phi}B_r^{(1,1))}\right]\hat{\theta}\right\} && (6.20)\\
&\quad +\frac{k}{r}\left\{\frac{\partial}{\partial r}\left[rB_\theta^{(1,1)}\right]-\frac{\partial}{\partial\theta}B_r^{(1,1)}\right\}\hat{\theta}\\
&= \frac{k}{r}\left\{\left[\frac{\partial}{\partial r}Y(\theta)\right]-\frac{X'(\theta)}{r}\right\}\hat{\theta} && (6.21)\\
&= -\frac{k}{r^2}X'(\theta)\,\hat{\theta}. && (6.22)
\end{aligned}$$

Plugging this into (6.11), we can use this to solve for the radial function

$$-\frac{k}{r^2}X'(\theta) = -v\frac{\partial}{\partial z}\vec{E}^{(1,0)}+2k\vec{B}^{(0,1)}, \tag{6.23}$$

noting that due to its $z$-parity, $\frac{\partial\vec{E}^{(1,0)}}{\partial z}=0$ and the $\vec{B}^{(0,1)}$ is just the ordinary Biot-Savart term (6.34), the equation becomes simplified

$$X'(\theta) = -\frac{qv}{2\pi}\sin\theta, \tag{6.24}$$

which when integrated, gives us the general solution

$$X(\theta) = \frac{qv}{2\pi}\cos\theta + C, \tag{6.25}$$

with some as yet unknown constant $C$.

The next step is to take the divergence of the partially solved $\vec{B}^{(1,1)}$ and set it equal to 0.

$$\begin{aligned}
\vec{\nabla}\cdot\vec{B}^{(1,1)} &= \frac{k}{r^2}\left\{\frac{\partial}{\partial r}\left[r^2\left(\frac{qv}{2\pi r}\cos\theta+\frac{C}{r}\right)\right]+\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}Y(\theta)\sin\theta\right]\right\} && (6.26)\\
&= \frac{k}{r^2}\left\{\left(\frac{qv}{2\pi}\cos\theta+C\right)+\frac{1}{\sin\theta}\left[\frac{\partial}{\partial\theta}Y(\theta)\sin\theta\right]\right\}. && (6.27)
\end{aligned}$$

Isolating terms to one side allows us to write an ordinary differential equation for the $\theta$-component function $Y(\theta)$,

$$\frac{\partial}{\partial\theta}\left[Y(\theta)\sin\theta\right] = -\sin\theta\left(\frac{qv}{2\pi}\cos\theta - C\right). \tag{6.28}$$

Then integrate with respect to $\theta$ with a standard substitution for $\sin\theta\cos\theta$

$$Y(\theta)\sin\theta = \int d\theta \left( C\sin\theta - \frac{qv}{2\pi}\sin\theta\cos\theta \right) \tag{6.29}$$

$$= -C\cos\theta - \frac{qv}{2\pi}\left(\frac{1}{2}\sin^2\theta + D\right), \tag{6.30}$$

with another unknown constant $D$. So the $Y(\theta)$ is then

$$Y(\theta) = -C\frac{\cos\theta}{\sin\theta} - \frac{qv}{4\pi}\left(\sin\theta + \frac{D}{\sin\theta}\right). \tag{6.31}$$

Evaluating the $\theta = 0$ limit of this equation, we see that $C$ and $D$ must be dropped; otherwise the $C$ and $D$ terms would contain a sin function in the denominator, which would lead to the terms blowing up when $\sin\theta$ goes to zero. Therefore, our solution for the $\theta$-component includes only the middle term, and thus the final solution is

$$Y(\theta) = -\frac{qv}{4\pi}\sin\theta. \tag{6.32}$$

Putting the pieces together

$$\vec{B}^{(1,1)} = \frac{kqv}{4\pi r}\left(2\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}\right) \tag{6.33}$$

There is also another method by which $\vec{B}^{(1,1)}$ may be calculated, which is less systematic, but which may give additional insight, since it is closer to some of the elementary methods used to calculate electrostatic and magnetostatic fields in the conventional Maxwell theory.

For a charge $q$ located at the origin moving with nonrelativistic velocity $\vec{v} = v\hat{z}$ the conventional magnetic field is

$$\vec{B}^{(0,1)}(\vec{r}) = \frac{qv}{4\pi}\frac{\sin\theta}{r^2}\hat{\phi}. \tag{6.34}$$

However, in the presence of $k_{AF}$ this is not the complete field even at lowest order in $v$. There is further magnetic field generated from the modified Ampere's Law, with an additional effective current $\vec{J}_{\text{eff}} = 2k\vec{B}$. When $\vec{B}$ is given by (6.34), the field of $\vec{J}_{\text{eff}}$ can be found exactly. This is similar to finding the vector potential from a uniform

moving charge, but in the equivalent of the Coulomb gauge ($\vec{\nabla} \cdot \vec{B} = 0$) instead of the more commonly applied Lorenz gauge.

The calculation proceeds by splitting $\vec{J}_{\text{eff}}$ into a set of infinitesimally-thin spherical shells. Between $R$ and $R + dr$, there is an effective surface current

$$\vec{K}_{\text{eff}} = \vec{J}_{\text{eff}}\, dr \equiv K_0(R) \sin\theta\, dr\, \hat{\phi} \tag{6.35}$$

and the field from the surface current is a standard result,

$$\vec{B}_K(\vec{r}) = \begin{cases} \frac{2}{3} K_0 \hat{z}, & r < R \\ \frac{R^3}{3r^3} K_0 (3\cos\theta \hat{r} - \hat{z}), & r > R \end{cases} \tag{6.36}$$

To calculate the total contribution, from all the shells, we must integrate from $R = 0$ to $\infty$, noting that the equatorial surface density is a function of $R$.

$$\begin{aligned} \vec{B}(\vec{r}) &\approx \vec{B}^{(0,1)}(\vec{r}) + \int_0^r dR \left(\frac{kqv}{2\pi R^2}\right) \left[\frac{R^3}{3r^3}(3\cos\theta\, \hat{r} - \hat{z})\right] && (6.37) \\ &\quad + \int_r^\infty dR \left(\frac{kqv}{2\pi R^2}\right) \left(\frac{2}{3}\hat{z}\right) \\ &= \vec{B}^{(0,1)}(\vec{r}) + \frac{kqv}{4\pi r}(\cos\theta\, \hat{r} + \hat{z}) && (6.38) \\ &= \vec{B}^{(0,1)}(\vec{r}) + \frac{kqv}{4\pi r}\left(2\cos\theta\, \hat{r} - \sin\theta\, \hat{\theta}\right), && (6.39) \end{aligned}$$

where the second term in (6.39) which is the same $\vec{B}^{(1,1)}$, previously calculated using the modified Maxwell's equations from the standard fields. From this point using these methods as well as those presented in the next section, we will have the tools needed to derive all other fields that will be of importance to us.

## 6.3 Pseudo-Amperian Loop Calculation Method

Having showed how to determine $\vec{B}^{(1,1)}$ by two different methods, we can now show how to use the simpler method of using pseudo-Amperian loop calculations to find $\vec{B}^{(2,1)}$ and other fields with azimuthal symmetry. If the sources appearing on the right-hand sides of (6.9) and (6.11) are toroidal, they produce azimuthal on the left

hand side and vice versa. When a source, such as $\vec{B}^{(1,1)}$, is toroidal, we consider a circle $\mathcal{C}$ with radius $\rho$ that is centered at $(0,0,z)$ and parallel to the $xy$-plane. We can calculate $\vec{B}^{(2,1)}$ by an analogy to Ampere's Law,

$$\int_{\mathcal{C}} d\vec{l}\cdot\vec{B}^{(2,1)} = 2\pi\rho B_{\phi}^{(2,1)} = \int_0^{\rho}(2\pi\rho'\,d\rho')\vec{B}^{(1,1)}\cdot\hat{z} \tag{6.40}$$

$$2\pi\rho B_{\phi}^{(2,1)} = \frac{kqv}{2}\int_0^{\rho} d\rho'\frac{\rho'}{\sqrt{\rho'^2+z^2}}\left(1+\frac{z^2}{\rho'^2+z^2}\right) \tag{6.41}$$

$$2\pi\rho B_{\phi}^{(2,1)} = \frac{kqv}{2}\frac{\rho^2}{\sqrt{\rho^2+z^2}} \tag{6.42}$$

$$\vec{B}^{(2,1)} = \frac{kqv}{4\pi}\sin\theta\,\hat{\phi} \tag{6.43}$$

The two methods we have outlined can be used, in purely mechanical fashion, to solve for all the higher fields $\vec{E}^{(m,n)}$ and $\vec{B}^{(m,n)}$, although the complexity of the integrals and ordinary differential equations involved does increase with $m$ and $n$.

## 6.4 Power Cancellation

Looking at (6.39), we can see that, at very large distances, the second term is larger than the first by a dimensionless factor $kr$ and thus decays as $1/r$, which is obviously a less rapid falloff than the usual Biot-Savart term (6.34). This inherently means that the field may potentially produce outgoing energy and momentum fluxes at infinity. To address whether or not this is the case, we look at the energy flux $\vec{S}$ of the modified pointing vector at large $r$. The presence of such outward energy flux would be a signature of energy loss from vacuum Cerenkov radiation.

Only one of the three terms from $\vec{S}$ (6.5) can have any contribution to an outward energy flux at infinity up to $\mathcal{O}(v^2)$. The two non-contributing terms are those involving cross products of $\vec{E}$ with $\vec{B}$ and $\vec{A}$. The magnetic quantities $\vec{A}$ and $\vec{B}$ are each proportional to at least one power of $v$, and the inductive part of $\vec{E}$ [the part sourced by $\partial B/\partial t = -v(\partial B/\partial z)$] is $\mathcal{O}(v^2)$; so the products $(\vec{E}\times\vec{B})\cdot\hat{r}$ and $(\vec{A}\times\vec{E})\cdot\hat{r}$ will

be of $\mathcal{O}(v^3)$ at least. Therefore, the only term in $\vec{S}$ remaining that may contribute to radiative energy loss at $\mathcal{O}(v^2)$ is $-kA_0\vec{B}$.

With the terms of up to $\mathcal{O}(v^2)$, the radial flux at spatial infinity is

$$\vec{S}\cdot\hat{r} = -k\left(\frac{q}{4\pi r}\right)\left[\frac{kqv}{4\pi r}(2\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})\right]\cdot\hat{r} = -\frac{k^2q^2v}{8\pi^2r^2}\cos\theta, \tag{6.44}$$

involving the unmodified scalar potential $A_0$ of a stationary charge. Like the electric field, $A_0$ is itself only modified from this form by terms of $\mathcal{O}(v^2)$ and higher. We recall that the local expressions for the energy densities are not gauge invariant in this theory; however, the integral of this flux over a spherical surface $\Sigma$ approaching spatial infinity is. And so the invariant quantity

$$\int_{r\to\infty} d\Sigma\,\left(\vec{S}\cdot\hat{r}\right) = 0 \tag{6.45}$$

represents a vanishing net outward flow of energy at this order.

This result clearly holds up to $\mathcal{O}(v^2)$, but the behavior at higher orders may be unclear. However, a key point in the calculation was that the integrand in (6.45), being a linear function of $\cos\theta$, had odd $z$-parity. In fact, using the $z$-parities tabulated in Table 6.1, it is actually evident that $\vec{S}\cdot\hat{r}$ will always have odd $z$-parity. Therefore, we see there is no net flux away from the particle and thus no net vacuum Cerenkov radiation, although this global result provides limited insight into the question of why the fields of the moving charge do not carry away any energy.

## 6.5 Fourier Transform of the Flux

A better picture of how the total energy loss always vanishes can be found by taking Fourier transforms of the fields and fluxes involved. The power can vanish in all orders in $k$, because long-wavelength modes of the field carrying negative energies can cancel the energy carried by the real-frequency positive energy modes. For dimensional reasons, since $k$ is the only dimensional quantity in the problem (as we are using

natural units with $\hbar = c = 1$), any finite power must be proportional to exactly two powers of $k$. We shall therefore concentrate on terms in $\vec{S}$ that are $\mathcal{O}(k^2)$. However, we shall see that the Fourier transform methods can also be used to see the relationships between contributions to the fields at different orders in $k$.

As noted, there are many quantities that are not gauge invariant, so we will continue to use the Coulomb gauge $\vec{\nabla} \cdot \vec{A} = 0$, as it simplifies the forms of $\vec{A}$ and $A_0$. In particular, $A_0$ is completely independent of $k$,

$$A_0 = \frac{q}{4\pi r\sqrt{1 - v^2\left[1 - (\hat{v} \cdot \hat{r})^2\right]}}, \tag{6.46}$$

which to zeroth order becomes

$$A_0^{(0,0)} = \frac{q}{4\pi r}. \tag{6.47}$$

Looking at the leading order $k$-dependent magnetic field

$$\vec{B}^{(1,1)} = \frac{kqv}{4\pi r}\left[\hat{v} + (\hat{v} \cdot \hat{r})\hat{r}\right], \tag{6.48}$$

we see that we will need the Fourier transforms of functions—specifically $\vec{B}^{(1,1)}$ itself, as well as $A_0^{(0,0)}\vec{B}^{(1,1)}$—with the general form

$$r^{-n}[\hat{v} + (\hat{v} \cdot \hat{r})\hat{r}]. \tag{6.49}$$

Using the usual coordinate system, with the $z$-axis pointed along the direction of the wave number variable $\vec{Q}$, the quantities $\vec{r}$ and $\vec{v}$ are expressed in spherical coordinates with respect to this axis as $\vec{r} = (r, \theta, \phi)$ and $\vec{v} = (v, \theta', \phi')$. In these coordinates, the general Fourier transform is expressed by

$$\mathcal{I}_n = \int d^3r \frac{e^{-\mu r}e^{i\vec{Q}\cdot\vec{r}}}{(r^2 + \lambda^2)^{n/2}}\left[\hat{v} + (\hat{v} \cdot \hat{r})\,\hat{r}\right], \tag{6.50}$$

where the quantities $\lambda$ and $\mu$ are used to regularize the integrals at $r = 0$ and $r \to \infty$ limits and shall be sent to zero at some point during the calculation.

The three-dimensional iterated integral for $\mathcal{I}_n$ is written as

$$\mathcal{I}_n = \int_0^\infty r^2\,dr \frac{e^{-\mu r}}{(r^2+\lambda^2)^{n/2}} \int_0^\pi \sin\theta\,d\theta\, e^{iQr\cos\theta} \int_0^{2\pi} d\phi\, \{\hat{v} + [\sin\theta\sin\theta'\,(\cos\phi\cos\phi' + \sin\phi\sin\phi') + \cos\theta\cos\theta']\,(\sin\theta\cos\phi\,\hat{x} + \sin\theta\sin\phi\,\hat{y} + \cos\theta\,\hat{z})\}\,. \tag{6.51}$$

We need only consider non-zero contributions in the $\phi$ integration, which is elementary; only terms with even powers of $\cos\phi$ and $\sin\phi$ give nonvanishing results, so

$$\mathcal{I}_n = \int_0^\infty dr \frac{r^2 e^{-\mu r}}{(r^2+\lambda^2)^{n/2}} \int_0^\pi \sin\theta\,d\theta\, e^{iQr\cos\theta}(\pi)\,[2\hat{v} + \sin^2\theta\sin\theta'\,(\cos\phi'\,\hat{x} + \sin\phi'\,\hat{y}) + 2\cos^2\theta\cos\theta'\,\hat{z}\Big]\,. \tag{6.52}$$

Using the standard substitution $u = \cos\theta$,

$$\mathcal{I}_n = \pi \int_0^\infty dr \frac{r^2 e^{-\mu r}}{(r^2+\lambda^2)^{n/2}} \int_{-1}^1 du\, e^{iQru}\,[2\hat{v} + (1-u^2)\sin\theta'\,(\cos\phi'\,\hat{x} + \sin\phi'\,\hat{y}) + 2u^2\cos\theta'\,\hat{z}\Big]\,. \tag{6.53}$$

The elementary integrals with the exponential $e^{iQru}$ read

$$\int_{-1}^1 du\, e^{iQru} = \frac{2\sin Qr}{Qr} \tag{6.54}$$

$$\int_{-1}^1 du\, u^2 e^{iQru} = \frac{2\sin Qr}{Qr} + \frac{4\cos Qr}{(Qr)^2} - \frac{4\sin Qr}{(Qr)^3}, \tag{6.55}$$

Performing these calculations, (6.53) becomes

$$\mathcal{I}_n = 4\pi \int_0^\infty dr \frac{r^2 e^{-\mu r}}{(r^2+\lambda^2)^{n/2}} \left\{(\hat{v} + \cos\theta'\,\hat{z})\left(\frac{\sin Qr}{Qr}\right) + [2\cos\theta'\,\hat{z} - \sin\theta'\,(\cos\phi'\,\hat{x} + \sin\phi'\,\hat{y})]\left[\frac{Qr\cos Qr - \sin Qr}{(Qr)^3}\right]\right\} \tag{6.56}$$

The general structure of $\mathcal{I}_n$ allows for the calculation at all orders $n$. For $n = 1$, for the first-order magnetic field, the regulation at small $r$ is not needed, and $\lambda$ may be set to zero. The first term inside the French brackets of (6.56) corresponds to the same kind of elementary integral as (6.54), but with different limits. The second term is determined by

$$\frac{4\pi}{Q^3}\,(3\cos\theta'\,\hat{z} - \hat{v}) \int_0^\infty dr \frac{e^{-\mu r}(Qr\cos Qr - \sin Qr)}{r^2} = \frac{4\pi}{Q^3}\,(3\cos\theta'\,\hat{z} - \hat{v}) \times \left[-Q + \mu\tan^{-1}\left(\frac{Q}{\mu}\right)\right]. \tag{6.57}$$

Sending $\mu \to 0$,

$$\mathcal{I}_1 = \frac{8\pi}{Q^2}(\hat{v} - \cos\theta'\,\hat{z}). \tag{6.58}$$

$\vec{Q}$ is in the $\hat{z}$-direction, so the Fourier transform for $\vec{B}^{(1,1)}$ may be expressed in coordinate-independent fashion as

$$\widetilde{\vec{B}^{(1,1)}} = \frac{2kqv}{Q^2}\left[\hat{v} - \left(\hat{v}\cdot\hat{Q}\right)\hat{Q}\right]. \tag{6.59}$$

For the $n = 2$ case, the first term in (6.56)—the one with $(\sin Qr)/Qr$—again requires only $\mu$ for it to remain finite, so we send $\lambda \to 0$ and are left with

$$\frac{4\pi}{Q}\left(\hat{v} + \cos\theta'\,\hat{z}\right)\int_0^\infty dr \frac{e^{-\mu r}\sin Qr}{r} = \frac{4\pi}{Q}\left(\hat{v} + \cos\theta'\,\hat{z}\right)\tan^{-1}\left(\frac{Q}{\mu}\right) \to \frac{2\pi^2}{|Q|}\left(\hat{v} + \cos\theta'\,\hat{z}\right), \tag{6.60}$$

where the expression on the far right applies as $\mu \to 0$. The second term will be finite without the $\mu$ regularization, but $\lambda$ is required to keep the integration well defined. In this case, with $\mu = 0$ the term is

$$\begin{aligned}\frac{4\pi}{Q^2}\left(3\cos\theta'\,\hat{z} - \hat{v}\right)\int_0^\infty dr \frac{\cos Qr - \frac{\sin Qr}{Qr}}{r^2+\lambda^2} &= \frac{2\pi^2}{\lambda^2 Q^3}\left(3\cos\theta'\,\hat{z} - \hat{v}\right) \\ &\quad\times\left[|\lambda|Qe^{-|\lambda Q|} + \frac{|Q|}{Q}\left(e^{-|\lambda Q|} - 1\right)\right].\end{aligned} \tag{6.61}$$

Now taking $\lambda \to 0$, this becomes

$$\frac{4\pi}{Q^2}\left(3\cos\theta'\,\hat{z} - \hat{v}\right)\int_0^\infty dr \frac{\cos Qr - \frac{\sin Qr}{Qr}}{r^2+\lambda^2} \to -\frac{\pi^2}{|Q|}\left(3\cos\theta'\,\hat{z} - \hat{v}\right). \tag{6.62}$$

Taking the two terms together,

$$\mathcal{I}_2 = \frac{\pi^2}{\left|\vec{Q}\right|}\left[3\hat{v} - \left(\hat{v}\cdot\hat{Q}\right)\hat{Q}\right], \tag{6.63}$$

and we arrive at

$$-k\widetilde{A_0^{(0,0)}\vec{B}^{(1,1)}} = -\frac{k^2q^2v}{16Q}\left[3\hat{v} - \left(\hat{v}\cdot\hat{Q}\right)\hat{Q}\right]. \tag{6.64}$$

We can achieve this same result through an alternative method by convolving the Fourier transforms of $A_0^{(0,0)}$ and $\vec{B}^{(1,1)}$. The convolution leading to (6.64) requires the integral

$$\widetilde{A_0^{(0,0)}\vec{B}^{(1,1)}} = \widetilde{A_0^{(0,0)}}\left(\vec{Q}\right) * \widetilde{\vec{B}^{(1,1)}}\left(\vec{Q}\right) = \frac{1}{2\pi}\int d^3\ell\,\widetilde{A_0^{(0,0)}}\left(\vec{Q} - \vec{\ell}\right)\widetilde{\vec{B}^{(1,1)}}\left(\vec{\ell}\right). \tag{6.65}$$

We may split the full convolution into two separate terms, since $\widetilde{\vec{B}^{(1,1)}}$ contains terms proportional to $Q^{-2}$ and $Q^{-2}\left(\hat{v}\cdot\hat{Q}\right)\hat{Q}$. Using the well-known Fourier transform of the nonrelativistic $A_0^{(0,0)}$, which is also proportional to $Q^{-2}$, the first term is determined by

$$\frac{1}{Q^2} * \frac{1}{Q^2} = \frac{1}{2\pi}\int_0^\infty \ell^2\, d\ell \int_0^\pi \sin\vartheta\, d\vartheta \int_0^{2\pi} d\varphi \frac{1}{Q^2+\ell^2-2Q\ell\cos\vartheta}\frac{1}{\ell^2}. \tag{6.66}$$

The spherical coordinates of the integration variable $\vec{\ell}$ are $(\ell,\vartheta,\varphi)$. The $\varphi$ integration is simple, and with the substitution $\upsilon=\cos\vartheta$, the remaining two integrals are

$$\frac{1}{Q^2} * \frac{1}{Q^2} = \int_0^\infty d\ell \int_{-1}^1 d\upsilon \frac{1}{Q^2+\ell^2-2Q\ell\upsilon}. \tag{6.67}$$

The integration over $\upsilon$ ranges over a region that is symmetric about zero, and therefore the integrand of the outermost $\ell$ integration is an even function of $\ell$. Changing the sign of $\ell$ will change the value of the integrand. However, simultaneously changing the sign of $\upsilon$ returns the integrand to its original value, and all values of $\upsilon$ between $-1$ and 1 are included in the integration. Which means that the $\ell$ integration may be extended to run from $-\infty$ to $\infty$ (and then halved). We do this and then reverse the order of the integration and we get

$$\begin{aligned}
\frac{1}{Q^2} * \frac{1}{Q^2} &= \frac{1}{2}\int_{-1}^1 d\upsilon \int_{-\infty}^\infty d\ell \frac{1}{(\ell-Q\upsilon)^2+(Q^2-Q^2\upsilon^2)} && (6.68)\\
&= \frac{1}{2}\int_{-1}^1 d\upsilon \left[\frac{1}{Q\sqrt{1-\upsilon^2}}\tan^{-1}\left(\frac{l-Q\upsilon}{Q\sqrt{1-\upsilon^2}}\right)\right]\Bigg|_{-\infty}^{\infty} && (6.69)\\
&= \frac{\pi}{2Q}\int_{-1}^1 d\upsilon \frac{1}{\sqrt{1-\upsilon^2}} && (6.70)\\
&= \frac{\pi^2}{2Q}. && (6.71)
\end{aligned}$$

This accounts for the term in the final convolution that is proportional to $\hat{v}$

For the convolution with $Q^{-2}\left(\hat{v}\cdot\hat{Q}\right)\hat{Q}$, we simply combine the results obtained for the other calculated elements. The $\varphi$ integration is more difficult than in (6.66), but it has already been done in (6.53),

$$\int_0^\pi d\varphi\, \left(\hat{v}\cdot\hat{\ell}\right)\hat{\ell} = \pi\left[(1-\upsilon^2)\left(\sin\theta'\cos\phi'\,\hat{x}+\sin\theta'\sin\phi'\,\hat{y}\right)+2\upsilon^2\cos\theta'\,\hat{z}\right]. \tag{6.72}$$

The remaining integrations proceed as in (6.68). Thus we have, again reversing the order of the iterated integrals,

$$\begin{aligned}
\frac{1}{Q^2} * \frac{\left(\hat{v}\cdot\hat{Q}\right)\hat{Q}}{Q^2} &= \frac{1}{4}\int_{-1}^{1} dv\, \Big[(1-v^2)\left(\sin\theta'\cos\phi'\,\hat{x} + \sin\theta'\sin\phi'\,\hat{y} - 2\cos\theta'\,\hat{z}\right) \\
&\quad + 2\cos\theta'\,\hat{z}\Big]\int_{-\infty}^{\infty} d\ell \frac{1}{(\ell - Qv)^2 + (Q^2 - Q^2v^2)} \qquad (6.73) \\
&= \frac{\pi}{4Q}\int_{-1}^{1} dv \left\{\sqrt{1-v^2}\left[\hat{v} - 3\left(\hat{v}\cdot\hat{Q}\right)\hat{Q}\right] + \frac{2\left(\hat{v}\cdot\hat{Q}\right)\hat{Q}}{\sqrt{1-v^2}}\right\} \qquad (6.74) \\
&= \frac{\pi^2}{8Q}\left[\hat{v} + \left(\hat{v}\cdot\hat{Q}\right)\hat{Q}\right]. \qquad (6.75)
\end{aligned}$$

Inserting the proper multiplicative factors and taking a difference of (6.71) and (6.75), we then recover the result (6.64).

## 6.6 Cancelations Between Low- and High-$Q$ Modes

What the Fourier transform method will make possible is a comparison of the moving energy densities carried by long-wavelength and short-wavelength modes (low and high $Q$ values, respectively). At this point, the Fourier transforms calculated thus far are not sufficient to display the cancelation behavior as they all depend only on $k$ as simple powers. This kind of $k$ dependence cannot have any cancelations between effects at small and large $Q$; whether $|k|$ is greater than or less than $Q/2$ will not affect the sign of a term with this form. To find cancelation between different $Q$ ranges, we must look at interference between terms at different orders in $k$. To do this, we must extend our calculation of the Fourier transform beyond first order in $k$. In fact, the calculation can be extended to every $\widetilde{\vec{B}^{(m,1)}}$. Although the terms with even $m$ are azimuthal (and so will not contribute to $\vec{S}\cdot\hat{r}$), they may also be included in this determination.

Using the same conventions describing the vectors $\vec{r}$ and $\vec{v}$ in spherical coordinates that we used previously, the total integral for the first $\widetilde{\vec{B}^{(m,1)}}$ term with an even $m > 0$

is

$$\widetilde{\vec{B}^{(2,1)}} = \frac{k^2 qv}{2\pi} \int_0^\infty r^2\, dr\, e^{-\mu r} \int_0^\pi \sin\theta\, d\theta\, e^{iQr\cos\theta} \int_0^{2\pi} d\phi\ (\hat{v} \times \hat{r})\,. \tag{6.76}$$

Included is a regularization factor $e^{-\mu r}$, to eliminate divergences at large $r$. However, it can be noted that the regularization at small $r$ is not needed. There is no power law divergence in (6.76) in the vicinity of $r = 0$, but a singularity still exists, because of the presence of the $\sin\theta$ factor in $\vec{B}^{(2,1)}$—$\theta$ being undefined at $r = 0$.

Any term from (6.76) that is linear in $\sin\phi$ or $\cos\phi$ will give zero after integration. This means that in the cross product, the contributions proportional to the $x$- and $y$-components of $\hat{r}$ must vanish. The remaining $\phi$ integral is

$$\int_0^{2\pi} d\phi\ (\hat{v} \times \cos\theta\, \hat{z}) = 2\pi \cos\theta\, (\sin\theta' \sin\phi'\, \hat{x} - \sin\theta' \cos\phi'\, \hat{y})\,. \tag{6.77}$$

Substitution $u = \cos\theta$, the full Fourier transform is (recalling $\hat{z} = \hat{Q}$)

$$\widetilde{\vec{B}^{(2,1)}} = k^2 qv \int_0^\infty dr\, r^2 e^{-\mu r} \int_{-1}^1 du\, u e^{iQru} \left(\hat{v} \times \hat{Q}\right). \tag{6.78}$$

The $u$ integration is simple, as in (6.54) and (6.55), the result leaves us with

$$\begin{aligned}
\widetilde{\vec{B}^{(2,1)}} &= \frac{2ik^2 qv}{Q^2} \left(\hat{v} \times \hat{Q}\right) \int_0^\infty dr\ (\sin Qr - Qr\cos Qr)\, e^{-\mu r} && (6.79)\\
&= \frac{4ik^2 qvQ}{(Q^2+\mu^2)^2} \left(\hat{v} \times \hat{Q}\right) && (6.80)\\
&\to \frac{4ik^2 qv}{Q^3} \left(\hat{v} \times \hat{Q}\right), && (6.81)
\end{aligned}$$

where the last limit in (6.81) obviously applies as we send $\mu$ to zero.

The magnetic field terms that are of the lowest (linear) order in the speed $v$ are determined by the system of differential equations

$$\begin{aligned}
\vec{\nabla} \times \vec{B}^{(m,1)} &= 2k\vec{B}^{(m-1,1)} && (6.82)\\
\vec{\nabla} \cdot \vec{B}^{(m,1)} &= 0. && (6.83)
\end{aligned}$$

No contributions come from $\partial\vec{E}/\partial t$, because a $k$-dependent $\vec{E}$ term can itself only be generated by the time dependence of a $k$-dependent $\vec{B}$ term, which makes the $\vec{E}$ term

involved necessarily of higher order in $v$. Taking another curl of the curl equation (6.82) and applying the solenoidal field condition (6.83), we get

$$-\vec{\nabla}^2 \vec{B}^{\,(m+2,1)} = (2k)^2 \vec{B}^{(m,1)}. \tag{6.84}$$

In Fourier space, this becomes simply $Q^2 \widetilde{\vec{B}^{(m+2,1)}} = (2k)^2 \widetilde{\vec{B}^{(m,1)}}$, or, resumming all the terms with odd powers of $k$,

$$\widetilde{\vec{B}^{(odd,1)}} = \frac{Q^2}{Q^2-(2k)^2} \widetilde{\vec{B}^{(1,1)}} = \frac{2kqv}{Q^2-(2k)^2} \left[\hat{v} - \left(\hat{v}\cdot\hat{Q}\right)\hat{Q}\right]. \tag{6.85}$$

From here we see that the difference of the signs for the Fourier modes with $Q$ above and below $|2k|$ is clearly apparent. For each individual term $\widetilde{\vec{B}^{(m,1)}}$ with odd $m > 0$, the contribution to the $-kA_0\vec{B}$ energy outflow vanishes for symmetry reasons. This is the case for coordinate space as well as Fourier space—the latter as shown below in (6.87). When combining the terms of different orders in $k$, there exists a singularity and sign change at $Q = |2k|$, confirming the earlier inferences about low- and high-$Q$ cancelations. (A principal value integration through the pole at $Q = |2k|$ will always yield a finite result, and so the infinity encountered in the transform will not be a problem.)

It should be noted that the expression (6.85) is necessarily an even function of $Q$, because without knowing the sign of $k$, it is impossible to determine whether the pole in $Q$ occurs at $2k$ or $-2k$. The denominator involving $Q^2$ automatically captures both possible pole locations in a single expression.

We can actually now express the Fourier transform of the magnetic field at $\mathcal{O}(v)$, to all orders in the Chern-Simons coefficient $k$,

$$\widetilde{\vec{B}^{(all,1)}} = \frac{qv}{Q^2-(2k)^2} \left[2k\,\hat{v} - 2k\left(\hat{v}\cdot\hat{Q}\right)\hat{Q} + iQ\left(\hat{v}\times\hat{Q}\right)\right]. \tag{6.86}$$

Transforming back to position space, the pole at $Q = |2k|$ will lead to sign-changing spatial oscillations in field strength at large $r$, with characteristic length $\sim |2k|^{-1}$.

The keys to (6.86) having the required form were that the Fourier transforms (6.59) and (6.81) do not have any singularities at $Q = |2k|$.

It does not appear that a mode (with the wave vector $\vec{Q}$) proportional to $\hat{v}$ carries away any energy. The first term from (6.64) corresponds with the initial term in (6.48) which is proportional to $\vec{v}/r$. It represents an apparent flow of energy that propagates in the same as the direction of travel of the particle which corresponds to a similar energy flow in coordinate space. No net energy is deposited. This suggests the curious notion that there is an overall energy flow in the system, from $+\infty$ to $-\infty$ along the $\hat{v}$-direction, with equal amounts moving in from positive spatial infinity and out to negative spatial infinity, so that there is no net change in the energy surrounding the charge. However, any local statement like this about the location of the energy is actually going to be gauge dependent.

We may also explicitly show that there will be no net energy outflow at spatial infinity for the second term in (6.64) as well. To calculate the total radiating power from the moving charge as the integral of $\vec{S} \cdot \hat{r}$, we may use the equivalent three-dimensional integral of $\vec{\nabla} \cdot \vec{S}$ over all space. In Fourier space, this means the outflow of energy is proportional to an integral over all $\vec{Q}$ of the dot product of the wave vector $\vec{Q}$ with the Fourier transform of $\vec{S}$. Remembering that only the $-kA_0\vec{B}$ term in $\vec{S}$ is capable of describing energy outflow in this chosen gauge, the power radiated at this order must be proportional to

$$P \propto \int d^3Q \left[\frac{3\hat{v} - \left(\hat{v} \cdot \hat{Q}\right)\hat{Q}}{Q}\right] \cdot \vec{Q} = \int d^3Q \left(2\hat{v} \cdot \hat{Q}\right) = 0. \tag{6.87}$$

So both of the terms describes a distribution of energy among the Fourier modes that does not actually represent radiation from the moving charge out to infinity.

In general, when $-k\widetilde{A_0\vec{B}^{(m,l)}}$ takes the form $X(Q,\theta')\hat{v} + Y(Q,\theta')\hat{Q}$, where $\theta'$ is still the angle between $\vec{Q}$ and $\vec{v}$, the net energy outflow vanishes if $X$ is an even function of $\cos\theta' = \hat{v} \cdot \hat{Q}$ and $Y$ is an odd function of $\cos\theta'$. The results vanish when integrating

over $\vec{Q}$, as the dot product of $\vec{Q}$ with either term is an odd function of $\cos\theta'$. The same symmetry argument applies for higher order $\widetilde{\vec{B}^{(m,l)}}$ terms. Similar to real space, where the cancelation arguments were based on the parity of the field components with respect to $\hat{v}\cdot\hat{r}$, in Fourier space cancelation is dependent on whether terms are even or odd functions of $\hat{v}\cdot\hat{Q}$.

Thus the Fourier transforms are basically translations of results that were previously known in position space into Fourier space. However, the calculations lead to a much better understanding of Fourier decomposition of the energy flow, and thus it is possible to derive some further results that may not be so easily expressed in coordinate space.

Since the total power emitted by the charge is zero, the $Q < |2k|$ modes must be carrying negative energy. This amount must exactly cancel the energy carried by the shorter-wavelength $Q > |2k|$ modes. The propagating modes in the kinematically allowed region $|2k| < Q < |2k|/(1-v^2)$ really are excited. However, the negative-energy modes with $Q < |2k|$ cancel out the energy flow carried in the higher modes. Normally the involvement of the $Q < |2k|$ modes would would lead us to suspect instabilities, as their imaginary frequencies would give the field an exponentially increasing time dependence. However, by studying field configurations with constant uniform motion, we have forced the modes to behave as propagating modes. As a result, instead of growing exponentially, these unstable modes are associated with propagating solutions carrying negative energies. The very same troublesome modes which necessitate the use of an acausal Green's function to ensure stability actually ensure that the timelike Chern-Simons theory is actually free of vacuum Cerenkov radiation

# Chapter 7

# Separability and Degeneracy in Quantum Mechanics

There are certain special Hamiltonians for which the Schrödinger equation is separable in more than one coordinate system. The spectra of these Hamiltonians exhibit what are known as "accidental" degeneracies, and a number of the Hamiltonians are extremely important. There are a few well known examples of systems that are separable in more than one set of coordinates, including the free particle (separable in any coordinates for which the Robertson condition is satisfied [41, 42]), the three-dimensional isotropic harmonic oscillator (separable in spherical, ellipsoidal, cylindrical, and rectangular coordinates), the Coulomb potential (separable in spherical, prolate ellipsoidal [43], and parabolic coordinates), and the constant magnetic field (separable in rectangular or cylindrical coordinates, depending on the gauge, but in either case with a free choice of the location of the origin). All these examples are also well known for the degeneracies present in their spectra. A slightly less well known system is a particular anisotropic harmonic oscillator, with potential

$$V(\vec{r}) = \frac{1}{2}M\omega_0^2(x^2 + y^2 + 4z^2). \tag{7.1}$$

That this energy spectrum possesses accidental degeneracy is obvious; however, it is not as well known that, just like the Coulomb problem, this potential problem is separable in parabolic coordinates.

Separability in parabolic coordinates is guaranteed if the Hamiltonian involves

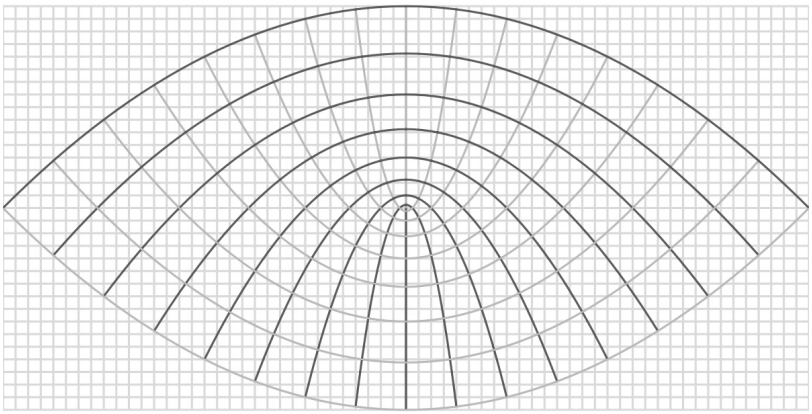

Figure 7.1 Parabolic coordinate system showing the curves of constant $\eta$ and $\xi$. The horizontal and vertical axes are the $x$ and $y$ coordinates, respectively. Coordinates are projected along the $z$-axis, and so this diagram holds for any value of the $z$ coordinate.

just a standard nonrelativistic kinetic term and a potential $V$ of the form

$$V(\vec{r}) = \frac{1}{r}\left[f(\eta) + g(\xi)\right], \tag{7.2}$$

for some functions $f$ and $g$. The quantities $\eta$ and $\xi$ are two of the parameters of the parabolic coordinate system $(\eta, \xi, \phi)$, where

$$\eta = r + z \tag{7.3}$$

$$\xi = r - z. \tag{7.4}$$

(Surfaces of constant $\eta$ or $\xi$ are orthogonal paraboloids of revolution, each with its focus at the origin.) For the anisotropic harmonic oscillator (7.1), the two functions are $f(s) = g(s) = \frac{1}{2}M\omega_0^2 s^3$. The use of parabolic coordinates in the Coulomb problem is also especially convenient for dealing with the Stark effect. While the Hamiltonian remains separable in spherical coordinates when an external magnetic field is applied, it remains separable in the parabolic coordinates with a external electric field present.

Classically, all the potentials we have mentioned with accidental degeneracy are well known for another feature. All their bound orbits close. In this paper, we shall look at another potential that, in a sense, shares this classical feature. Bertrand's

theorem is normally taken to hold that the only two central potentials for which all the bound orbits close are the attractive Coulomb potential and the isotropic harmonic oscillator potential. However, this is not quite accurate; there are other examples of potentials with the stated property, but they are typically discounted because they do not have a full spectrum of bound states. For example, all the bound orbits in a constant potential, which exerts no force, close—precisely because there are no bound states. A system with a charged particle in a constant magnetic field also evades the strong restriction imposed by Bertrand's Theorem [44], since the force in this instance is not derived from a central potential. The orbits in the magnetic field are right circular helices. The velocity parallel to the magnetic field is a constant of the motion; only when the velocity in that direction vanishes are the orbits truly bound—in which case they are closed circles. The potential considered in this paper is in a similar category to the two examples just mentioned. We shall see that the attractive $V \propto 1/\rho^2$ potential possesses few bound orbits (for an appropriate interpretation of the meaning of "bound"), but those that it does possess are circular, with exactly zero energy (as expected from the virial theorem).

Separability of the Schrödinger equation in a given set of coordinates means that the energy of a given state may be expressed as a function of three quantum numbers, one corresponding to each coordinate. If separations in multiple coordinate systems are possible, there must be multiple formulas for the energy, based on different sets on quantum numbers. The eigenstates with fixed quantum numbers are generally different in different coordinates; an eigenfunction in one coordinate system must be a linear combination of eigenfunctions in the other coordinates. The only way this is possible for an energy eigenstate is if there are multiple eigenstates with exactly the same energy. This shows why separability in multiple coordinate systems requires the presence of accidental degeneracy. Moreover, it is worth noting that the same argument can be applied even to the degeneracy of system with a generic central

potential $V(r)$, which is separable in spherical coordinates only. A system with angular momentum $\ell$ possesses a $(2\ell + 1)$-fold degeneracy, which is actually related to the fact that the spherical coordinates may be chosen with their polar axis pointing in any direction. The Schrödinger equation is thus separable in an infinite number of different spherical coordinate systems.

However, the existence of accidental degeneracy does not absolutely require that a system be separable in multiple coordinate systems. Any anisotropic three-dimensional harmonic oscillator for which the frequencies of the motions along the three coordinate directions are rational multiples will have classical orbits that eventually close and degeneracies in its quantum mechanical spectrum. So there are systems for which the degeneracy is seemingly too "sporadic" to be indicative of any deeper underlying symmetry principle at work.

The existence of alternative bases of quantum numbers is also related, of course, to the existence of additional observables that commute with the Hamiltonian. For the spherical harmonic oscillator and the Coulomb potential, these extra conserved quantities are well known. The harmonic oscillator has separate, commuting Hamiltonians governing the motion along the three orthogonal axes. For the Coulomb problem, there is the Runge-Lenz vector, which points out the direction of the major axis of a bound state elliptical orbit. For the free particle, with its extensive degeneracy, the additional conserved quantity is the momentum itself. Using the algebras generated with the inclusion of any of these conserved quantities, it is possible to determine the bound state spectra of these problems using operator algebra alone.

In the Coulomb problem, part of the accidental degeneracy associated with the additional operators that commute with the Hamiltonian persists even in the relativistic Dirac theory, although the separability in parabolic coordinates actually does not carry over. Another potential, albeit a potential in only one dimension, that is also amenable to similar operator methods is the $V = V_0 \operatorname{sech}^2 ax$ potential; the eigen-

states for potentials of different depths $V_0$ are related by operators [45], and as with the other potentials solvable by operator methods, the operators involved may be interpreted as elements of a $(0+1)$-dimensional supersymmetry algebra [46, 47, 48, 49]. (An excellent introductory treatment of quantum-mechanical supersymmetry, especially as applied to the Coulomb problem, may be found in the lecture notes [50].) The harmonic oscillator, Coulomb, and $\operatorname{sech}^2 ax$ potentials just mentioned are all among the shape-independent potentials, and the main object of study in this paper will be yet another such one. There are a wide variety of mathematical tools that may be useful for solving and addressing questions about these shape-invariant potentials [51, 52].

The harmonic oscillator and Coulomb systems are also known for the fact that certain classical and semiclassical approximations yield exact results when applied to these systems. The Bohr-Sommerfeld quantization rule derived from the Wentzel-Kramers-Brillouin (WKB) approximation gives the exact energies for a harmonic oscillator system in one dimension—and hence also for an isotropic or anisotropic harmonic oscillator in any number of dimensions. Harmonic oscillators also have coherent states, with the zero-point uncertainties in position and momentum added onto a classically orbiting wave function centroid. For the Coulomb potential, there is the fact that the full nonperturbative scattering cross section is the same as the cross sections derived classically or from the first-order Born approximation. Moreover, for the $V = V_0 \operatorname{sech}^2 ax$ potential also, a certain approximation is exact; the potential is reflectionless, so the classical reflection coefficient is always precisely correct. The exact successes of these various approximations are highly appealing features of these special potential types, although it is not clear whether we should expect anything similar when dealing with more esoteric multiply separable potentials.

This particular approach will examine the $1/\rho^2 = 1/(x^2 + y^2)$ potential in three spatial dimensions. The attractive version of the potential has been observed physi-

cally, in the interaction of a long charged wire with a polarizable atom [53]. Chapter 8 introduces the features that make this potential special classically. Finding analytical solutions of the equations of motion is easy, and there are relatively few bound, stable orbits, but those that do exist close. The simplicity of the classical problem leads us to suspect that the corresponding quantum mechanical theory may also exhibit special properties.

In chapter 9, it will be shown that the Hamiltonian with a $1/\rho^2$ potential belongs to the elite family that are separable in multiple coordinate systems—cylindrical and parabolic, in this case. The scattering state wave functions of this potential in two space dimensions have already been studied [54], but it is only with the inclusion of the third dimension that the dual separations become possible. So new symmetry phenomena are expected to be found in the three-dimensional axisymmetric potential, although it is not really clear in advance what special behavior could be expected. If the potential possessed bound states, we would obviously expect them to have additional degeneracies. Unfortunately, however, the attractive $1/\rho^2$ potential does not support a stable spectrum of bound negative-energy states; the attractive singularity at $\rho = 0$ is too strong. For the scattering states of a repulsive potential, we might hope, based on the behavior of other multiply separable Hamiltonian systems, to uncover some interesting behavior. The special features of the $1/\rho^2$ potential that we have identified are discussed and summarized in chapter 10.

# Chapter 8

# Classical Features of the $1/\rho^2$ Potential

Central potential problems in three (or more) dimensions may be reduced to two-dimensional problems using the conservation of angular momentum. The differential equation for the orbital curve $r(\phi) = [u(\phi)]^{-1}$ then has the simple, well-known form

$$\frac{d^2u}{d\phi^2} = -u - \frac{M}{L^2u^2}F\left(\frac{1}{u}\right), \tag{8.1}$$

where $F(r) = -dV/dr$ is the radial force. The $u$ term on the right-hand side of (8.1) corresponds to the centripetal term in the effective potential governing the radial motion. For potentials $V \propto r^n$, the orbital shape can be expressed in terms of elementary trigonometric functions for $n = 2,\ 0,\ -1$, or $-2$ [44]. These are, respectively, the harmonic oscillator, the free particle, the Coulomb potential, and the inverse square potential of interest here. For certain other integral and rational values of the exponent $n$, the solutions may be expressed in terms of elliptic, hypergeometric, or other progressively more general functions.

Because it is straightforward to reduce the central force problem from three to two dimensions, there is very little practical difference at the classical level between computations with a three-dimensional central potential $V(r)$ and the two-dimensional analogue $V(\rho)$. (The same cannot be said in quantum mechanics though; for example, a two-dimensional attractive potential always possesses at least one bound state, but in three dimensions the bound state need not be present.) The classical radial equation of motion in two-dimensional space, with a potential $V(\rho) = \kappa/\rho^2$, using

the effective potential (depending on the angular momentum $L = L_z$), is

$$M\frac{d^2\rho}{dt^2} = -\frac{d}{d\rho}\left(\frac{\kappa}{\rho^2} + \frac{L^2}{2M\rho^2}\right) \tag{8.2}$$

$$= \frac{2\kappa + L^2/M}{\rho^3}. \tag{8.3}$$

This is clearly just as solvable with $\kappa \neq 0$ as it is for the free particle ($\kappa = 0$) case. The equation for the orbital curve is correspondingly

$$\frac{d^2u}{d\phi^2} = -\left(1 + \frac{2\kappa M}{L^2}\right)u. \tag{8.4}$$

The full solutions in three dimensions are simply the two-dimensional $(\rho, \phi)$ motion superimposed upon uniform motion in the $z$-direction, $z(t) = z(0) + \dot{z}(0)t$. The classical approach we are using corresponds to the solution of the Schrödinger problem in cylindrical coordinates.

The nature of the solutions for $\rho(\phi) = [u(\phi)]^{-1}$ depends on the sign of $1+2\kappa M/L^2$, and thus on the sign of $\kappa$. If $\kappa > 0$, the coefficient in parentheses on the right-hand side of (8.4) is automatically positive. The only possible trajectories in this repulsive potential are scattering orbits

$$\rho = A\sec\left(\sqrt{1 + 2\kappa M/L^2}\,\phi + \delta\right). \tag{8.5}$$

In a coordinate system with the phase angle $\delta = 0$, the radial coordinate diverges at $\phi = \pm\pi/2\sqrt{1 + 2\kappa M/L^2}$, corresponding to a classical scattering angle

$$\varphi_{\text{scat}} = \pi\left|1 - \frac{1}{\sqrt{1 + 2\kappa M/L^2}}\right|, \tag{8.6}$$

which depends on the angular momentum, but not separately on the energy—a consequence of the scale invariance of the problem. This expression can also be cast in terms of the impact parameter $b$ in the $xy$-plane, via $L = \sqrt{2M\mathcal{E}'}b$, where $\mathcal{E}' > 0$ is the energy of the in-plane motion (so that the total energy is $\mathcal{E} = \mathcal{E}' + \frac{1}{2}M\dot{z}^2$). For attractive potentials with $\kappa < 0$, there are similar scattering orbits when the energy

is positive (which means $L^2 > 2|\kappa|M$). The scattering angle is again given by (8.6); the absolute value present in that formula, which was superfluous for the repulsive potential, is needed in the attractive case to give a nonnegative $\varphi_{\text{scat}}$.

The angle $\varphi_{\text{scat}}$ represents the scattering angle in the $xy$-plane. When the uniform motion in the third dimension is included, it is also possible to describe the total scattering angle $\vartheta_{\text{scat}}$. Since the potential in three dimensions is not spherically symmetric, the scattering behavior does not depend solely on an impact parameter (or equivalently, for fixed energy, on an angular momentum). Instead, we shall describe the incoming trajectory of a particle by a direction $\hat{\Theta}$, together with the angular momentum component $L = L_z$. Choosing an appropriate orientation for the $x$ and $y$ coordinates, $\hat{\Theta}$ is

$$\hat{\Theta} = \frac{1}{\sqrt{2\mathcal{E}'/M + \dot{z}^2}} \left( \sqrt{2\mathcal{E}'/M}\, \hat{x} + \dot{z}\, \hat{z} \right). \tag{8.7}$$

The $z$-axis around which the potential $V(\rho)$ is symmetric and the incoming trajectory (along which the particle would travel if it were not deflected) are generally skew lines. Their distance of closest approach to one-another is given by the in-plane impact parameter $b = L/\sqrt{2M\mathcal{E}'}$. After the scattering, the in-plane component of the velocity has been rotated through an angle $\pm\varphi_{\text{scat}}$, making the outgoing direction vector

$$\hat{\Theta}' = \frac{1}{\sqrt{2\mathcal{E}'/M + \dot{z}^2}} \left( \sqrt{2\mathcal{E}'/M} \cos\varphi_{\text{scat}}\, \hat{x} \pm \sqrt{2\mathcal{E}'/M} \sin\varphi_{\text{scat}}\, \hat{y} + \dot{z}\, \hat{z} \right). \tag{8.8}$$

Therefore, the three-dimensional scattering angle $\vartheta_{\text{scat}}$ is

$$\vartheta_{\text{scat}} = \cos^{-1}\left( \hat{\Theta} \cdot \hat{\Theta}' \right) = \cos^{-1}\left( \cos\varphi_{\text{scat}} + \frac{2\dot{z}^2}{2\mathcal{E}'/M + \dot{z}^2} \sin^2\frac{\varphi_{\text{scat}}}{2} \right), \tag{8.9}$$

where $\varphi_{\text{scat}}$ is still a function of $L$ or $b$, according to (8.6). Naturally, when the motion is planar ($\dot{z} = 0$), this gives $\vartheta_{\text{scat}} = \varphi_{\text{scat}}$. Conversely, when the velocity in the $z$-direction (which does not change) predominates, $\vartheta_{\text{scat}} \to 0$.

Apart from the sign difference inside the absolute value in (8.6), there is another important difference between the attractive and repulsive regimes. When $\kappa > 0$, the

scattering angle is limited to the range $0 \leq \varphi_{\text{scat}} < \pi$; the trajectory never crosses itself. In contrast, when $\kappa < 0$, the angle $\varphi_{\text{scat}}$ may be arbitrarily large. When the potential is attractive, the particle may orbit around the center any number of times before it escapes again to infinity. The resulting two-dimensional trajectories cross over themselves repeatedly. This is quite different than the behavior seen in classical Rutherford scattering, in which the trajectories for attractive and repulsive potentials are represented by the two disjoint branches of the same hyperbola. However, this behavior, with the number of times the orbital curve intersects itself increasing as the total energy approaches zero, is by no means unique to the attractive $1/\rho^2$ potential, but is in fact fairly generic.

The $\kappa < 0$ scattering orbits, with more and more revolutions around the origin as the energy decreases, are approaching the limit of perfectly circular orbits, which occur when the energy vanishes at $L^2 = -2\kappa M$. Any attractive potential will have classical circular orbits. However, for the potential we are interested in, it turns out that these circular orbits are, in a certain meaningful sense, the only bound orbits. If the total energy is negative, then the quantity in parentheses in (8.4) is negative, and the orbital solution $u(\phi)$ becomes a linear combination of equiangular spirals, so that

$$\rho(\phi) = A \operatorname{sech}\left(\sqrt{2|\kappa|M/L^2 - 1}\,\phi + \delta\right), \tag{8.10}$$

or

$$\rho(\phi) = A \operatorname{csch}\left(\sqrt{2|\kappa|M/L^2 - 1}\,\phi + \delta\right), \tag{8.11}$$

depending on whether the both endpoints lie at $\rho = 0$ or one at $\rho = 0$ and the other at $\rho = \infty$. Note that all the bound orbits in two dimensions do therefore (in a certain sense) close, because the circular orbits are the only persistent bound orbits.

Alternatively, taking the negative-energy solutions of (8.4) as linear combinations

$$u(\phi) = B \cosh\left(\sqrt{2|\kappa|M/L^2 - 1}\,\phi\right) + C \sinh\left(\sqrt{2|\kappa|M/L^2 - 1}\,\phi\right), \tag{8.12}$$

$\rho(\phi)$ has the form (8.10), with both endpoints at the center, if $|B| > |C|$; it has the form (8.11) if $|B| < |C|$. The intermediate cases, with $B = \pm C$, yields pure inward or outward equiangular spirals. The spirals are also limiting forms of the other expressions, with $\delta \to \infty$ while $Ae^{-\delta}$ is held finite.

Most generally, for states with $\kappa + L^2/2M < 0$, energy $\mathcal{E}' < 0$, and initial radial velocity inward [$\dot{\rho}(0) \leq 0$ at time $t = 0$], the equation of motion (8.2) has the implicit solution

$$t = \frac{\rho}{2|\mathcal{E}'|}\sqrt{\frac{|2\kappa M + L^2|}{\rho^2} - 2M|\mathcal{E}'|} + \frac{M\rho(0)\dot{\rho}(0)}{2|\mathcal{E}'|}. \tag{8.13}$$

In this regime, the time $t_f$ required for the particle to reach $\rho = 0$ is

$$t_f = \frac{\sqrt{|2\kappa M + L^2|}}{2|\mathcal{E}'|} - \frac{M\rho(0)|\dot{\rho}(0)|}{2|\mathcal{E}'|} \tag{8.14}$$

This is behavior is known as "falling to the center." Note that making $\dot{\rho}(0)$ more negative while keeping $\rho(0)$ fixed decreases $t_f$, which is clearly correct; the effects of varying $\rho(0)$ independently are less intuitively obvious.

# Chapter 9

# Two Separations of the Schrödinger Equation

We now turn our attention to the quantum theory. The classical theory with an attractive potential was dominated by the falling to the center. The quantum mechanical wave functions in the presence of the $\kappa < 0$ potential exhibit their own manifestation of this phenomenon. If $\psi(\vec{\rho})$ is an eigenfunction of the two-dimensional time-independent Schrödinger equation (with energy eigenvalue $\mathcal{E}'_0$), then for any real number $\alpha$, $\psi(\alpha\vec{\rho})$ is also an eigenfunction, with energy $\alpha^2\mathcal{E}'_0$. Thus, if there is a normalizable eigenstate with energy $\mathcal{E}'_0 < 0$, then there must be eigenstates with arbitrarily negative energies. By compressing the wave function closer to the attractive singularity at $\rho = 0$, the energy may be made as negative as we wish, meaning that there cannot be a stable Hilbert space of quantum states; the energy is not bounded below.

In contrast, when $\kappa > 0$, the energy eigenvalues are always positive. We may still dilate the wave function to decrease its energy, but the energies remain bounded from below by zero; and it is, of course, no surprise that all positive energies are allowed in this scattering system.

It is possible, via one of several renormalization procedures, to let the strength of the attractive $1/r^2$ potential go to zero, in such a way that there is a reasonable physical spectrum (with exactly one bound state) [55, 56, 57]. However, the resulting Hamiltonian is not self-adjoint, in spite of it having a naively Hermitian appearance [58]; moreover, the scale invariance of the solutions is broken by an anomaly. Most analyses of the regularized Hamiltonian have focused on the three-dimensional

$1/r^2$ potential, although the general character of the solutions appears to be independent of the dimensionality [55, 59, 60, 61]. Our results in this paper might be extended to this renormalized regime; in fact, it would be very interesting to see how the renormalization would interact with the separation of the Schrödinger equation in parabolic coordinates. However, this regime lies beyond the scope of the present work.

## 9.1 Cylindrical Coordinates

In order to have a quantum theory with well-defined wave functions, without additional regularization of the potential in the vicinity of $\rho = 0$, the potential we shall consider in the remainder of our analysis is

$$V(\vec{r}) = \frac{\hbar^2 K}{2M}\frac{1}{\rho^2}, \tag{9.1}$$

with repulsive $K > 0$. (The potential strength $\kappa$ has been rescaled to avoid unnecessary factors of $\hbar$ and $M$.) With this potential, the Schrödinger equation in cylindrical coordinates is

$$\left[-\frac{1}{\rho}\frac{\partial}{\partial\rho}\left(\rho\frac{\partial}{\partial\rho}\right) - \frac{1}{\rho^2}\frac{\partial^2}{\partial\phi^2} - \frac{\partial^2}{\partial z^2} + \frac{K}{\rho^2}\right]\psi = E\psi, \tag{9.2}$$

where $E = 2M\mathcal{E}/\hbar^2$, with $\mathcal{E}$ being the total energy. With a separable ansatz,

$$\psi = P(\rho)e^{ikz}e^{im\phi}, \tag{9.3}$$

this reduces to a single-variable Schrödinger equation for $P(\rho)$,

$$\left[-\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}\right) + \frac{m^2+K}{\rho^2}\right]P = \left(E - k^2\right)P. \tag{9.4}$$

This is just the usual Bessel's equation that arises for a free particle in two dimensions, except with the indices of the Bessel function solutions changed to $\sqrt{m^2+K}$. (Note that $m^2 + K$ just corresponds to the classical quantity $L_z^2 + 2M\kappa$, measured in units of $\hbar$.) The general solution is thus

$$P(\rho) = AJ_{\sqrt{m^2+K}}\left(\sqrt{E-k^2}\,\rho\right) + BJ_{-\sqrt{m^2+K}}\left(\sqrt{E-k^2}\,\rho\right). \tag{9.5}$$

If the Bessel function index $\sqrt{m^2+K}$ happens to be an integer, the usual replacement of the linearly dependent $J_{-\sqrt{m^2+K}}$ by the Neumann function $N_{\sqrt{m^2+K}}$ is required. However, only the Bessel function with positive index is regular at $\rho = 0$ (and thus permitted).

The scattering theory of these solutions is straightforward. The scattering by a three-dimensional $1/r^2$ potential is worked out in [62]. One surprising result is that the classical limit only exists for strong potentials, as the classical scattering cross section is linear (never quadratic) in the strength of the potential. The two-dimensional $1/\rho^2$ version is completely analogous, merely using the formalism for partial wave scattering in two dimensions [63]. The partial wave expansion for an incoming plane wave is

$$e^{iqx} = J_0(q\rho) + 2\sum_{m=1}^{\infty} i^m \cos(m\phi) J_m(q\rho), \tag{9.6}$$

in terms of the free wave radial functions. The Bessel functions have the limiting behavior $J_\nu(s) = \sqrt{\frac{2}{\pi s}}\cos\left(s - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$, and the scattering state wave function may be written

$$\begin{aligned} \psi &= e^{iq\rho} + \psi_{\text{scat}} \\ &\to e^{iq\rho} + \sqrt{\frac{2}{\pi q\rho}}\left[\cos\left(q\rho - \frac{\pi}{4} + \delta_0\right) + 2\sum_{m=1}^{\infty} i^m \cos(m\phi)\cos\left(q\rho - \frac{m\pi}{2} - \frac{\pi}{4} + \delta_m\right)\right], \end{aligned} \tag{9.7}$$

where $q = \sqrt{E-k^2}$. The non-free wave functions with $J_{\sqrt{m^2+K}}(q\rho)$ are simply phase shifted by

$$\delta_m = -\frac{\pi}{2}\left(\sqrt{m^2+K} - |m|\right). \tag{9.8}$$

The fact that $\delta_m$ is independent of the energy for each partial wave is another consequence of the scale invariance. Moreover, as noted, the classical limit corresponds to $K \gg |m|$.

## 9.2 Parabolic Coordinates

Unlike the scattering solution in cylindrical coordinates, the solution of the $1/\rho^2$ potential in parabolic coordinates is not a standard problem. A $1/\rho^2$ potential has, however, previously been considered algebraically, as a perturbation added to the Coulomb Hamiltonian (which, as noted above, is also parabolic separable) [64].

In parabolic coordinates, the Laplacian is

$$\vec{\nabla}^2 = \frac{4}{\eta+\xi}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) + \frac{4}{\eta+\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{1}{\eta\xi}\frac{\partial^2}{\partial\phi^2}. \tag{9.9}$$

Whether an eigenfunction is separable in cylindrical coordinates, with the form $\psi = P(\rho)\Phi(\phi)Z(z)$ or in parabolic coordinates $\psi = H(\eta)\Xi(\xi)\Phi(\phi)$, we may take it to be an eigenfunction of $L_z$, $\Phi(\phi) = e^{im\phi}$. Noting that $\eta\xi = \rho^2$, taking this azimuthal dependence reduces the Laplacian plus potential in the parabolic coordinates to

$$\begin{aligned} -\vec{\nabla}^2 + \frac{K}{\rho^2} &= -\frac{4}{\eta+\xi}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) - \frac{4}{\eta+\xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + \frac{m^2+K}{\eta\xi} && (9.10)\\ &= \frac{1}{\eta+\xi}\left[-4\frac{\partial}{\partial\eta}\left(\eta\frac{\partial}{\partial\eta}\right) - 4\frac{\partial}{\partial\xi}\left(\xi\frac{\partial}{\partial\xi}\right) + (m^2+K)\left(\frac{1}{\eta}+\frac{1}{\xi}\right)\right]. && (9.11)\end{aligned}$$

Once again, and not coincidentally, the inclusion of the potential corresponds to the change $m^2 \to m^2 + K$.

With $\Phi(\phi)$ factored out, the remaining Schrödinger equation can be written in the separation form

$$\left[\frac{4}{H}\frac{\partial}{\partial\eta}\left(\eta\frac{\partial H}{\partial\eta}\right) + E\eta - \left(m^2+K\right)\frac{1}{\eta}\right] + \left[\frac{4}{\Xi}\frac{\partial}{\partial\xi}\left(\xi\frac{\partial\Xi}{\partial\xi}\right) + E\xi - \left(m^2+K\right)\frac{1}{\xi}\right] = 0. \tag{9.12}$$

Letting the initial bracketed term in (9.12) be equal to a constant $C$, the ordinary differential equation for $H$ is

$$4\eta^2\frac{d^2H}{d\eta^2} + 4\eta\frac{dH}{d\eta} + \eta^2 EH - \eta CH - \left(m^2+K\right)H = 0, \tag{9.13}$$

and with $C \to -C$ in the equation for $\Xi$.

Since the wave function is complex, it may not be automatically clear whether $C$ should be real or complex. Note that a purely imaginary $C$ gives the real and imaginary parts of the solutions to the ordinary differential equations definite behavior under inversions of the variable, $\eta \to -\eta$ or $\xi \to -\xi$. However, this behavior is not actually physically mandated by the theory, because the physical space is limited to the parameter region where both $\eta$ and $\xi$ are nonnegative. It will, however, necessarily be the case that only a one-parameter family of $C$ values will correspond to physically meaningful states. Any separable energy eigenfunction in three dimensions is determined (up to phase and normalization) by the values of the three real quantum numbers. In this system, we have the physical observables represented by $m$ and $E$, so the choice of $C$ must provide exactly one additional real degree of freedom. Since with a real-valued $C$, the separate differential equations for $H$ and $\Xi$ can be cast as eigenvalue equations for Hermitian operators, a real $C$ is a sufficient condition for having equations that yield bases of wave functions with asymptotic forms that are continuum normalizable. Thus, a real $C$ gives the correct one-parameter family of solutions.

The linearly independent solutions of (9.13) are expressible in terms of the confluent hypergeometric fuctions ${}_1F_1(a;b;s)$,

$$h_\pm(\eta) = \eta^{\pm\frac{1}{2}\sqrt{m^2+K}} e^{-\frac{i}{2}\sqrt{E}\eta}\, {}_1F_1\left(\frac{1}{2} \pm \frac{1}{2}\sqrt{m^2+K} - \frac{iC}{4\sqrt{E}}; 1 \pm \sqrt{m^2+K}; i\sqrt{E}\eta\right). \tag{9.14}$$

In order to have regularity at the origin (where $\eta = \xi = 0$), we must have the solution $H = h_+(\eta)$, and for $\Xi$,

$$\Xi = h'_+(\xi) = \xi^{\frac{1}{2}\sqrt{m^2+K}} e^{-\frac{i}{2}\sqrt{E}\xi}\, {}_1F_1\left(\frac{1}{2} + \frac{1}{2}\sqrt{m^2+K} + \frac{iC}{4\sqrt{E}}; 1 + \sqrt{m^2+K}; i\sqrt{E}\xi\right). \tag{9.15}$$

Unfortunately, the overlap integrals giving the weights needed to write the wave functions $H(\eta)\Xi(\xi)$ as superpositions of the $P(\rho)Z(z)$ in cylindrical coordinates are intractable in the general case.

The asymptotic behavior of ${}_1F_1(a;b;s)$ for $|s|\to\infty$ and $-\frac{3\pi}{2}<\arg s<\frac{\pi}{2}$ is

$$ {}_1F_1(a;b;s)\sim\Gamma(b)\left[\frac{e^s s^{a-b}}{\Gamma(a)}+\frac{e^{i\pi a}s^{-a}}{\Gamma(b-a)}\right]. \tag{9.16}$$

For the solutions $h_+$ and $h'_+$, the relevant values of $a$ and $b-a(=a^*)$ always have real parts $\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}$, which means that the first and second terms in (9.16) are of the same magnitude when $\eta$ or $\xi$ is large. The large $\eta$ behavior of the confluent hypergeometric function appearing in $h_+(\eta)$ is accordingly [using the phase convention that $s=e^{\frac{i\pi}{2}}\sqrt{E}\eta$, corresponding to that in (9.16)],

$$\begin{aligned}
{}_1F_1 \;\sim\;& \Gamma\left(1+\sqrt{m^2+K}\right)\left(i\sqrt{E}\eta\right)^{-\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}\right)}\left[\frac{e^{i\sqrt{E}\eta}\left(e^{\frac{i\pi}{2}}\sqrt{E}\eta\right)^{-\left(\frac{iC}{4\sqrt{E}}\right)}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}-\frac{iC}{4\sqrt{E}}\right)}\right. \qquad (9.17)\\
&\left.+\frac{e^{i\pi\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}-\frac{iC}{4\sqrt{E}}\right)}\left(e^{\frac{i\pi}{2}}\sqrt{E}\eta\right)^{\left(\frac{iC}{4\sqrt{E}}\right)}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}+\frac{iC}{4\sqrt{E}}\right)}\right]\\
=\;& \frac{\Gamma\left(1+\sqrt{m^2+K}\right)e^{\frac{\pi C}{8\sqrt{E}}}\left(i\sqrt{E}\eta\right)^{-\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}\right)}}{\left|\Gamma\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}-\frac{iC}{4\sqrt{E}}\right)\right|}\left[e^{i\left(\sqrt{E}\eta-\arg\Gamma(a)+\frac{C}{4\sqrt{E}}\log\sqrt{E}\eta\right)}\right.\\
&\left.+e^{i\left(\frac{\pi}{2}+\frac{\pi}{2}\sqrt{m^2+K}+\arg\Gamma(a)-\frac{C}{4\sqrt{E}}\log\sqrt{E}\eta\right)}\right] \qquad (9.18)\\
=\;& \frac{2\Gamma\left(1+\sqrt{m^2+K}\right)e^{\frac{\pi C}{8\sqrt{E}}}\left(\sqrt{E}\eta\right)^{-\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}\right)}}{\left|\Gamma\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}-\frac{iC}{4\sqrt{E}}\right)\right|}e^{\frac{i}{2}\sqrt{E}\eta}\cos\left[\frac{1}{2}\sqrt{E}\eta\right. \qquad (9.19)\\
&\left.+\frac{C}{8\sqrt{E}}\log\left(\sqrt{E}\eta\right)-\frac{1}{2}\arg\Gamma\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}-\frac{iC}{4\sqrt{E}}\right)-\frac{\pi}{4}\sqrt{m^2+K}-\frac{\pi}{4}\right].
\end{aligned}$$

In the intermediate formula (9.18), $\arg\Gamma(a)$ has been used to abbreviate the complex argument of $\Gamma\left(\frac{1}{2}+\frac{1}{2}\sqrt{m^2+K}-\frac{iC}{4\sqrt{E}}\right)$.

It follows that the asymptotic behavior of the wave function ($r\to\infty$, but away

from the $z$-axis, where $\eta$ or $\xi$ may vanish) is

$$\begin{aligned} H(\eta)\Xi(\xi) \;&\propto\; \frac{1}{\sqrt{\eta\xi}}\cos\left[\frac{1}{2}\sqrt{E}\eta + \frac{C}{8\sqrt{E}}\log\left(\sqrt{E}\eta\right) - \frac{1}{2}\arg\Gamma(a) - \frac{\pi}{4}\sqrt{m^2+K} - \frac{\pi}{4}\right] \\ &\quad \times\cos\left[\frac{1}{2}\sqrt{E}\xi + \frac{C}{8\sqrt{E}}\log\left(\sqrt{E}\xi\right) - \frac{1}{2}\arg\Gamma(a)^* - \frac{\pi}{4}\sqrt{m^2+K} - \frac{\pi}{4}\right] \\ &= \frac{1}{2\rho}\left\{\cos\left[\sqrt{E}r + \frac{C}{8\sqrt{E}}\log\left(\cot^2\frac{\theta}{2}\right) - \frac{\pi}{2}\sqrt{m^2+K} - \frac{\pi}{2}\right]\right. \\ &\quad \left. + \cos\left[\sqrt{E}z + \frac{C}{8\sqrt{E}}\log\left(E\rho^2\right) - \arg\Gamma(a)\right]\right\}, \end{aligned} \tag{9.20}$$

using $\eta\xi = \rho^2$ and $\frac{\eta}{\xi} = \cot^2\frac{\theta}{2}$.

The limiting form (9.20) away from the $z$-axis is clearly normalizable as a continuum state. On the other hand, in the vicinity of the $z$-axis, either $h_+(\eta)$ or $h'_+(\xi)$ is close to 1, while the other function—and the wave function $\psi$ as a whole—scale as $\sim 1/\sqrt{|z|}$, which is again normalizable behavior. This confirms that our earlier choice of a real separation constant $C$ was the correct one for the physical wave function solutions.

# Chapter 10

# Properties of the $V \propto 1/\rho^2$ Solutions

## 10.1 Special Features

Remarkably, the separated equation (9.13) for $H$ can actually be cast in nearly the same form as the radial Schrödinger equation for a Coulomb potential. Letting $U_1(\eta) = \sqrt{\eta} H(\eta)$, (9.13) becomes

$$-\frac{d^2 U_1}{d\eta^2} + \frac{C}{4\eta} U_1 + \frac{m^2 + K - 1}{4\eta^2} U_1 = \frac{E}{4} U_1. \tag{10.1}$$

The ordinary differential equation of $U_2(\xi) = \sqrt{\xi}\Xi(\xi)$ is identical, except for the switch $C \to -C$, equivalent to interchanging an attractive Coulomb potential with a repulsive one. Moreover, the normalization condition for the wave function,

$$\frac{1}{4}\int_0^\infty d\xi \int_0^\infty d\eta\, (\eta + \xi)\, |H(\eta)\Xi(\xi)|^2 = \frac{1}{4}\int_0^\infty d\xi \int_0^\infty d\eta \left(\frac{1}{\eta} + \frac{1}{\xi}\right) |U_1(\eta) U_2(\xi)|^2 = \frac{1}{2\pi}, \tag{10.2}$$

sets the same kinds of constraints on how quickly the functions $U_1$ and $U_2$ must decay at spatial infinity as in the Coulomb problem. The equivalence also immediately explains the presence of the $\log(\sqrt{E}\eta)$ and $\arg\Gamma(a)$ terms in the argument of the cosine in (9.19), since these same kinds of terms appear in the phases of Coulomb waves.

The transformation of the separated parts of the Schrödinger equation into Coulomb-like forms opens up a number of tools that can be used to further analyze the wave function solutions. However, those tools may play different roles in the analysis of the $1/\rho^2$ potential than in the study of the $1/r$ potential. For example, there are $(0+1)$-dimensional supersymmetry transformations that carry solutions of

the radial Schrödinger equation in the Coulomb problem to other radial solutions with the same energies but different values of the angular momentum (changing $l \leftrightarrow l+1$) [49, 50]. Applied to (10.1), these transformations would still leave the energy unaffected (and also the separation constant $C$ unchanged), but the strength of the potential would be modified through a change to the quantity $m^2+K$, which combines the $z$-component of angular momentum with the strength of the repulsive potential. This is analogous to the situation with the one-dimensional $\mathrm{sech}^2\, ax$ potential, where the supersymmetry transformations connect potentials with the same functional form, but of different depths.

Another interesting feature of the solution in parabolic coordinates stems from the fact that the choice of coordinate system breaks the translation invariance along the $z$-direction. This symmetry is manifestly present in the equations of motion in a cylindrical coordinate system, where $z$ is a cyclic coordinate. Since $z$ never enters the dynamics explicitly, there is nothing special about the location of $z = 0$. The translation symmetry is obscured somewhat in the parabolic coordinates, but it must still exist. If $\psi_1 = H(\eta)\Xi(\xi)e^{im\phi}$ is a solution of the Schrödinger equation, then

$$\psi_2 = H\left[\sqrt{\rho^2+(z-a)^2}+(z-a)\right]\Xi\left[\sqrt{\rho^2+(z-a)^2}-(z-a)\right]e^{im\phi} \tag{10.3}$$

must also be a solution, since it is simply a translate of $\psi_1$ along the $z$-direction. The degeneracy of these states is analogous to the energy degeneracy of the Landau levels for a charged particle moving in the plane perpendicular to a constant magnetic field. The magnitude of the degeneracy is proportional to the area of the plane, since the origin of the coordinates may be located anywhere in the plane.

The presence of $C$ in the eigenfunction equation (10.1) also appears to break the scale invariance of the problem, since the Hermitian operator on the left-hand side contains $C$, which has units of $(\text{length})^{-1}$. However, since $C$ is merely a separation constant, which can take any real value, a rescaling $\vec{\rho} \to \alpha\vec{\rho}$ (and thus $\eta \to \alpha\eta$,

$\xi \to \alpha\xi$) may be accompanied by $C \to \alpha^{-1}C$. Since the same $C$ appears in the equations for $H$ and $\Xi$, this restores the physical scaling invariance.

Finally, the separation of the quantum-mechanical problem in parabolic coordinates can give some insight about the classical behavior in that coordinate system. Because the $\eta$ and $\xi$ portions of the Schrödinger equation are the same as those for two Coulomb problems, one attractive and the other equally repulsive, we can apply the normal methods for solving the Kepler problem to the classical time evolution of a particle's $(\eta, \xi, \phi)$ coordinates. Recalling the that the classical limit applies when $K$ is large, we may neglect the $-1$ in the $m^2 + K - 1$ appearing in (10.1). Then, restoring the factors of $\hbar^2/2M$, (10.1) corresponds to a classical limit of

$$\frac{1}{2}M\dot{\eta}^2 + \frac{\mathcal{C}}{4\eta} + \frac{L^2 + 2M\kappa}{4\eta^2} = \frac{\mathcal{E}}{4}, \tag{10.4}$$

where $C = 2M\mathcal{C}/\hbar^2$. There is no orbital equation that is directly analogous to the one for $r(\phi)$ in the normal Kepler problem, since the relationship between the angular velocity $\dot{\phi}$ and $L = L_z = M\rho^2\dot{\phi} = M\eta\xi\dot{\phi}$ is determined by both $\eta$ and $\xi$ together. However, (10.4) may be solved implicitly for the time $t$ as a function of the coordinate $\eta$,

$$t = \sqrt{\frac{M}{2}} \int_{\eta_{\min}}^{\eta} \frac{d\eta'}{\sqrt{\frac{\mathcal{E}}{4} - \frac{\mathcal{C}}{4\eta'} - \frac{L^2+2M\kappa}{8(\eta')^2}}} \tag{10.5}$$

$$\begin{aligned} = {} & \frac{\sqrt{2M\mathcal{E}\eta^2 - 2M\mathcal{C}\eta - L^2 - 2M\kappa}}{\mathcal{E}} \\ & + \frac{\sqrt{M}\mathcal{C}}{\sqrt{2}\mathcal{E}^{\frac{3}{2}}} \log\left[2\sqrt{\mathcal{E}\left(\mathcal{E}\eta^2 - \mathcal{C}\eta - \frac{L^2}{2M} - \kappa\right)} + 2\mathcal{E}\eta - \mathcal{C}\right] \\ & - \frac{\sqrt{M}\mathcal{C}}{2\sqrt{2}\mathcal{E}^{\frac{3}{2}}} \log\left[\mathcal{C}^2 + 4\mathcal{E}\left(\frac{L^2}{2M} + \kappa\right)\right]. \end{aligned} \tag{10.6}$$

The origin of the time coordinate has been chosen in this case so that $t = 0$ occurs at the turning point

$$\eta(0) = \eta_{\min} = \frac{\mathcal{C} + \sqrt{\mathcal{C}^2 + 4\mathcal{E}\left(\frac{L^2}{2M} + \kappa\right)}}{2\mathcal{E}} \tag{10.7}$$

for the $\eta$. Since condition for $\eta_{\min}$ is $\sqrt{2M\mathcal{E}\eta_{\min}^2 - 2M\mathcal{C}\eta_{\min} - L^2 - 2M\kappa} = 0$, both square roots in (10.6) vanish at $\eta = \eta_{\min}$, and just the last term comes from the lower limit of the integration.

Simultaneously, the $\xi$ coordinate is evolving independently. The time is once again given implicitly, in this instance by

$$\begin{aligned} t &= \frac{\sqrt{2M\mathcal{E}\xi^2 + 2M\mathcal{C}\xi - L^2 - 2M\kappa}}{\mathcal{E}} - \frac{\sqrt{2M\mathcal{E}\xi(0)^2 + 2M\mathcal{C}\xi(0) - L^2 - 2M\kappa}}{\mathcal{E}} \\ &\quad - \frac{\sqrt{M}\mathcal{C}}{\sqrt{2}\mathcal{E}^{\frac{3}{2}}} \log\left\{ \frac{2\sqrt{\mathcal{E}\left(\mathcal{E}\xi^2 + \mathcal{C}\xi - \frac{L^2}{2M} - \kappa\right)} + 2\mathcal{E}\xi + \mathcal{C}}{2\sqrt{\mathcal{E}\left[\mathcal{E}\xi(0)^2 + \mathcal{C}\xi(0) - \frac{L^2}{2M} - \kappa\right]} + 2\mathcal{E}\xi(0) + \mathcal{C}} \right\}. \end{aligned} \tag{10.8}$$

$\xi(0)$ is the value of $\xi$ when $\eta = \eta_{\min}$. If $\eta(t)$ and $\xi(t)$ are determined, then the remaining angular behavior can be found from

$$\phi(t) = \phi(0) + \frac{L}{M}\int_0^t \frac{dt'}{\eta(t')\xi(t')}, \tag{10.9}$$

completing the classical solution.

## 10.2 Discussion

Hamiltonians that are amenable to separation of variables methods in more than one coordinate system have a number of important properties. These include accidental degeneracies in their bound state spectra, and classical behavior that typically involves bound orbits that always close. The nonrelativistic Hamiltonians for a number of important physical systems, such as the hydrogen atom and the charged particle in a constant magnetic field, are multiply separable in this way. These features are also tied to the usefulness of operator methods in solving these Hamiltonians.

The $z$-independent $1/\rho^2$ potential in three dimensions is obviously separable in cylindrical coordinates $(\rho, \phi, z)$, and we have shown that it is also separable in parabolic coordinates $(\eta, \xi, \phi)$. Although parabolic coordinates are not used nearly as frequently as rectangular, spherical, and cylindrical coordinate systems, they were

already known to be useful for addressing certain aspects of the Coulomb problem. The attractive $1/\rho^2$ potential is too strong to support a stable set of bound states, but the repulsive version is well behaved. When separated in parabolic coordinates, the one-dimensional Schrödinger equations for the component functions $H(\eta)$ and $\Xi(\xi)$ have the same forms as the radial Schrödinger equation in the Coulomb problem, although the strength of the Coulomb-like term is set by the separation constant $C$, so that one equation features the effective equivalent of an attractive potential, which the other has a repulsive potential of equal magnitude.

The asymptotic behavior of the of full wave function $\psi(\eta, \xi, \phi)$ in parabolic coordinates is thus determined by the limiting behavior of a product of Coulomb waves, albeit ones that typically have nonintegral values for what would normally be the angular momentum parameter $\ell$. Moreover, although we have focused on the scattering states of a repulsive $1/\rho^2$ potential, it is clear that for sufficiently large values of $L_z = m\hbar$, the scattering wave function in an attractive $1/\rho^2$ potential will have essentially the same structures. They will still be products of Coulomb radial functions in $\eta$ and $\xi$, with unconventional values of $\ell$ and equal and opposite effective potential strengths.

While the separation of variables in cylindrical coordinates keeps the translation symmetry along the $z$-direction and the scale invariance of the problem manifest, both of these invariances are obscured in the parabolic coordinate system, which picks a particular $z = 0$ location about with the $\eta$ and $\xi$ coordinate surfaces are focused. The fact that these important features are hidden in the parabolic coordinates formalism suggests that there may be yet other interesting features of these potentials still to be uncovered. In any case, the planar $1/\rho^2$ potential, as well as being a system of real physical significance [53], appears to be a fruitful theoretical laboratory for understanding the structure of mechanics in parabolic coordinates and the behavior of multiply separable quantum systems.

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