Diameter of 3-Colorable Graphs and Some Remarks on the Midrange Crossing Constant

Inne Singgih

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Diameter of 3-colorable Graphs and
Some Remarks on the Midrange Crossing Constant

by

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DEDICATION

To people who actually read this dissertation:

Hello!! *waves* \(\sim o \sim/)\n
To people who fought and made things possible amid COVID-19 pandemic:

Thank you!! *deep bow in gratitude* m(_ _)m
ACKNOWLEDGMENTS

I would like to thank my advisors, Éva Czabarka and László Székely, for adopting me into your research family, for your overwhelming patience, unique perspectives, and amazing ideas toward finishing this dissertation. Éva, thank you for all the encouragement and supports during my doctoral study, both emotionally and psychologically. Thank you for all the reminders and constant pushes to do what needs to be done. László, thank you for the grant support for my last summer and for your thorough dedication for the case analyses in our work. Thank you both, for not giving up on me when I was about to give up on myself.

Thank you to Jerry Griggs for introducing me to the Department of Mathematics of University of South Carolina, and for believing in me during qualifying and comprehensive exam periods. Thank you to fellow graduate students, who are impossible to be mentioned one by one, for being awesome friends and allies through this long journey. I hope to know all of you for many years to come. A special thanks Zhiyu Wang, for helping me so many times on so many subjects during our study, particular to this dissertation for your collaboration in the results on Chapter 3 and for the program in Section A.1 I would also like to express my gratitude to the members of my committee, Linyuan Lu, Joshua Cooper, and Nathan Carnes, for their suggestions and helps in finalizing this dissertation.
Abstract

The first part of this dissertation discussing the problem of bounding the diameter of a graph in terms of its order and minimum degree. The initial problem was solved independently by several authors between 1965 – 1989. They proved that for fixed \( \delta \geq 2 \) and large \( n \), \( \text{diam}(G) \leq \frac{3n}{\delta+1} + O(1) \). In 1989, Erdős, Pach, Pollack, and Tuza conjectured that the upper bound on the diameter can be improved if \( G \) does not contain a large complete subgraph \( K_k \).

Let \( r, \delta \geq 2 \) be fixed integers and let \( G \) be a connected graph with \( n \) vertices and minimum degree \( \delta \). In general, Erdős et al. conjectured tight upper bounds for \( K_{2r} \)-free and \( K_{2r+1} \)-free graphs that are better than the known \( \frac{3n}{\delta+1} + O(1) \). Particular to this dissertation, their conjecture stated that \( K_5 \)-free graphs with \( 5 \mid \delta \) have diameter \( \leq \frac{5n}{25} + O(1) \), while \( K_4 \)-free graphs with \( 8 \mid \delta \) have diameter \( \leq \frac{16n}{75} + O(1) \).

The first progress towards this conjecture was published by Czabarka, Dankelmann, and Székely in 2008. They worked under the stronger assumption for when \( r = 2 \), that the graphs are 4-colorable rather than \( K_5 \)-free. They showed that for every connected 4-colorable graph \( G \) of order \( n \) and \( \delta \geq 1 \), \( \text{diam}(G) \leq \frac{5n}{25} - 1 \).

We provide a counterexample to this 30 years old unsolved conjecture for \( K_4 \)-free graphs by showing classes of 3-colorable graphs with diameter \( \frac{7n}{38+3} + O(1) \). From here we conjectured that 3-colorable graphs has diameter at most \( \frac{7n}{38} + O(1) \). We use the Duality of Linear Programming to prove 3-colorable graphs have diameter at most \( \frac{5n}{25} + O(1) \). We then utilize inclusion-exclusion into a different linear programming approach to prove a smaller upper bound that for every connected 3-colorable graph \( G \) of order \( n \) and \( \delta \geq 1 \), \( \text{diam}(G) \leq \frac{18n}{765} + O(1) \).
The second part of this dissertation gives some remarks on the midrange crossing constant. The celebrated Crossing Lemma states that for any graph on $n$ vertices and $m \geq 4n$ edges we have $\text{cr}(G) \frac{n^2}{m^3}$ is at least $\frac{1}{64}$. A decade before the Crossing Lemma, Erdős and Guy made the bold conjecture that, if we denote by $\kappa(n,m)$ the minimum crossing number of $n$-vertex graph with at least $m = m(n)$ edges, then there is a positive constant $\gamma$, dubbed as the midrange crossing constant, such that $\gamma = \lim_{n \to \infty} \kappa(n,m) \frac{n^2}{m^3}$ as long as $m$ is both superlinear in $n$. Pach, Spencer and Tóth showed that the Erdős-Guy conjecture is true with the additional (and needed) assumption that $m$ is subquadratic. Pach, Radoičič, Tardos and Tóth gave a construction yielding $\gamma \leq \frac{8}{9\pi^2} \approx 0.0900633$ for the rectilinear midrange crossing constant. Details of neither of these calculations, which are said to be long and unpleasant, are available to the public. We provide a simple alternative construction that yields the same upper bound.
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Chapter 1

Background

1.1 Terminology

In this section we gives definitions and notations used in Chapter 2 and Chapter 3.

Definition 1.1. We write \( f(x) = O(g(x)) \) if there is a positive real number \( M \) and a real number \( x_0 \) such that \( |f(x)| \leq M|g(x)| \) for all \( x \geq x_0 \).

Notation 1.2. Given an integer \( k > 0 \), let \( [k] = \{1, 2, \ldots, k\} \).

Notation 1.3. Given a set \( A \), let \( \binom{A}{k} \) denote all \( k \)-element subsets of \( A \).

Definition 1.4. Let \( G = (V, E) \) denote a finite graph with vertex set \( V \) and edge set \( E \subseteq \binom{V}{2} \). In this dissertation we only consider simple graphs, that is, graphs without loops and multiple edges.

Notation 1.5. We denote the order of \( G \) by \( |G| \). Unless stated otherwise, \( |G| = n \).

Definition 1.6. The complete graph \( K_n \) is a graph \( G = (V, \binom{V}{2}) \) with \( |G| = n \).

Definition 1.7. The complement of graph \( G = (V, E) \) is the graph \( \bar{G} = (V, \binom{V}{2} \setminus E) \). Note that the complement of \( K_n \) is a set of \( n \) independent (isolated) vertices.

Definition 1.8. Given graphs \( G = (V, E) \) and \( G' = (V', E') \). \( G' \) is a subgraph of \( G \) (in notation \( G' \subseteq G \)) if \( V' \subseteq V \) and \( E' \subseteq E \).

Definition 1.9. If a graph \( G \) does not contain graph \( H \) as its subgraph, we say that graph \( G \) is \( H \)-free.
Definition 1.10. Two vertices \( x, y \in V \) are adjacent if \( xy \in E \) in \( G = (V, E) \). If \( x \) and \( y \) are adjacent, then \( x \) is a neighbour of \( y \) and vice versa.

Notation 1.11. If \( x, y \) are nonadjacent vertices in the graph \( G = (V, E) \), then \( G + xy \) denotes the graph \( G \) with the edge \( xy \) added to it.

Definition 1.12. A set \( A = \{A_1, A_2, \ldots, A_k\} \) of disjoint subsets of a set \( A \) is a partition of \( A \) if the union \( \bigcup A \) of all the sets \( A_i \in A \) is \( A \) and \( A_i \neq \emptyset \) for every \( i \).

Definition 1.13. Let \( r \geq 2 \) be an integer. A graph \( G = (V, E) \) is called \( r \)-partite if \( V \) admits a partition into \( r \) classes such that every edge has its ends in different classes: vertices in the same partition class must not be adjacent. Instead of ‘2-partite’ one usually says bipartite.

Definition 1.14. We call \( G = (V, E) \) edge-maximal with respect to some properties \( P \) if \( G \) itself has \( P \) but no graph \( G + xy \) does, for any nonadjacent \( x, y \in V \).

Definition 1.15. The open neighbourhood of a vertex \( v \) in a graph \( G = (V, E) \) is \( N(v) = \{x \in V : xv \in E\} \). The closed neighbourhood of \( v \) is \( N[v] = N(v) \cup \{v\} \).

Definition 1.16. The degree \( \deg(v) \) of a vertex \( v \) is the number of neighbours of \( v \), that is, \( \deg(v) = |N(v)| \).

Definition 1.17. The minimum degree of a graph \( G \) is \( \delta(G) = \min\{\deg(v) : v \in V(G)\} \). If no ambiguity arises, we write \( \delta = \delta(G) \).

Definition 1.18. A path is a non-empty graph \( P = (V, E) \) where \( V = \{x_0, x_1, \ldots, x_k\} \) and \( E = \{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\} \) where the \( x_i \) are distinct. Vertices \( x_0 \) and \( x_k \) are called the endpoints. \( |E| \) is called the length of \( P \). Path with two vertices \( a, b \) as its endpoints are called the \( a - b \) path.

Definition 1.19. The distance \( d(x, y) = d_G(x, y) \) of two vertices \( x, y \) in \( G \) is the length of the shortest \( x - y \) path in \( G \). If no such path exists, we set \( d(x, y) = \infty \).
Definition 1.20. The diameter of a graph $G$, denoted by $\text{diam}(G)$, is the greatest distance between any two vertices in $G$.

Definition 1.21. A non-empty graph $G$ is called connected if there is an $x$-$y$ path between any $x, y \in V(G)$.

Definition 1.22. The eccentricity of a vertex $v$ in a connected graph $G$ is the maximum distance between $v$ and any other vertex of $G$.

Definition 1.23. A vertex coloring of a graph $G = (V, E)$ is a map $c : V \rightarrow S$ such that $c(v) \neq c(w)$ whenever $v$ and $w$ are adjacent. The elements of the set $S$ are called the available colors.

Definition 1.24. The smallest integer $k$ such that $G$ has a vertex coloring $c : V \rightarrow [k]$ is the chromatic number of $G$, denoted by $\chi(G)$. A graph $G$ with $\chi(G) = k$ is called $k$-chromatic. If $\chi(G) \leq k$, then $G$ is $k$-colorable.

Definition 1.25. A $k$-coloring is a vertex partition into $k$ independent sets, called color classes.

Definition 1.26. A drawing of an graph $G = (V, E)$ is a mapping from $V$ to disjoint points in the plane, and from $E$ to curves connecting their two endpoints. No vertex should be mapped onto an edge that it is not an endpoint of, and whenever two edges have curves that intersect (other than at a shared endpoint) their intersections should form a finite set of proper crossings, where the two curves are transverse.

Definition 1.27. The crossing number of a graph $G$, denoted $\text{cr}(G)$ is the minimum, over all such drawings, of the number of crossings in a drawing.

Definition 1.28. A straight-line drawing of a graph $G$ is a drawing of $G$ where the edges are mapped to straight-line segments. We will assume that in all such drawings, no three vertices are collinear, and no point lies in the relative interior of three distinct edges.
**Definition 1.29.** The rectilinear crossing number of a graph $G$, denoted $\text{cr}(G)$ is the minimum number of pairs of crossing edges over all straight-line drawings of $G$.

**Definition 1.30.** A great circle of a sphere is the intersection of the sphere and a plane that passes through the center point of the sphere.

![Figure 1.1: The great circle of two points $P$ and $Q$ on a sphere.](image)

1.2 **Simplex Method: Maximization**

This section covers a standard case of Simplex Method used in the Linear Programming approach on Section 2.7.

**Definition 1.31.** Linear programming (LP, or optimization) is a method to achieve the optimal (maximum or minimum) value of a linear function of a point-set satisfying a set of linear inequalities. Linear programming problems can be expressed in canonical form as

\[
\begin{align*}
\text{Maximize} & \quad c^T x \\
\text{subject to} & \quad Ax \leq b \\
\text{and} & \quad x \geq 0
\end{align*}
\]

where $x$ represents the column vector of variables (unknowns), $c$ and $b$ are column vectors of known coefficients, $A$ is a known matrix of coefficients, and $(\cdot)^T$ is the matrix transpose.

One method to solve a linear programming problem that is adaptable to computers is the simplex method, developed by George Dantzig in 1946. We use simplex method
to solve linear programming problem in Section 2.7. In this section we adapt a
description of simplex method given in [8].

A linear programming problem is in \textit{standard form} if it seeks to maximize the
objective function \( z = c^T x \) subject to the constraints \( Ax \leq b \) where \( x \geq 0 \) and \( b \geq 0 \).
That is:

Maximize \( z = c_1 x_1 + c_2 x_2 + \ldots + c_n x_n \)

Subject to the constraints

\[
\begin{align*}
    a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n & \leq b_1 \\
    a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n & \leq b_2 \\
    & \vdots \\
    a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n & \leq b_m
\end{align*}
\]

where \( x_i \geq 0 \) and \( b_j \geq 0 \). After adding slack variables, the corresponding system of
constraint equation is

\[
\begin{align*}
    a_{11} x_1 + a_{12} x_2 + \ldots + a_{1n} x_n + s_1 & = b_1 \\
    a_{21} x_1 + a_{22} x_2 + \ldots + a_{2n} x_n + s_2 & = b_2 \\
    & \vdots \\
    a_{m1} x_1 + a_{m2} x_2 + \ldots + a_{mn} x_n + s_m & = b_m
\end{align*}
\]

A \textit{basic solution} of a standard form LP problem is as solution

\[
(x_1, x_2, \ldots, x_n, s_1, s_2, \ldots, s_m)
\]

of the constraint equations in which at most \( m \) variables are nonzero. These nonzero
variables are called \textit{basic variables}. A basic solution for which all variables are non-
negative is called a \textit{basic feasible solution}.

\textit{Simplex tableau} consists of the augmented matrix corresponding to the constant
equations together with the coefficients of the objective function, written in the form

\[
-c_1 x_1 - c_2 x_2 - \ldots - c_n x_n + (0)s_1 + (0)s_2 + \ldots + (0)s_m + z = 0
\]
An example of an LP problem before (left) and after (right) slack variables are added:

Maximize $z = 4x_1 + 6x_2$  
Subject to:

- $-x_1 + x_2 \leq 11$
- $x_1 + x_2 \leq 27$
- $2x_1 + 5x_2 \leq 90$

Maximize $z = 4x_1 + 6x_2$  
Constraints:

- $-x_1 + x_2 + s_1 = 11$
- $x_1 + x_2 + s_2 = 27$
- $2x_1 + 5x_2 + s_3 = 90$

Table 1.1 is the corresponding simplex tableau.

Table 1.1: Initial simplex tableau for the example LP problem.

<table>
<thead>
<tr>
<th>Tableau 1</th>
<th>$c_i$</th>
<th>$b$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>11</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>27</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>90</td>
<td>2</td>
<td>5</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$z$</td>
<td>0</td>
<td>-4</td>
<td>-6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

For this initial tableau, the basic variables (included in the rows) are $s_1, s_2, s_3$ while the nonbasic variables (which have a value of zero) are $x_1$ and $x_2$. Column $c_i$ is the coefficient of the row variables in the objective function. Column $b$ gives the current solution $z = 0$ with $(x_1, x_2, s_1, s_2, s_3) = (0, 0, 11, 27, 90)$. If any entries in the $z$ row are negative, then the solution is not yet optimal, and we can improve the current solution by *pivoting*.

To improve the current solution, we bring a new basic variable into the solution, we call this the *entering variable*. Consequently, one of the current basic variables must leave, we call this the *departing variable*. We will describe how to choose the entering and departing variables by continuing above example.

Since the objective function is $z = 4x_1 + 6x_2$, it appears that a unit change in $x_2$ produces a change of 6 in $z$, whereas a unit change in $x_1$ produces a change of 4 in $z$. Hence we choose $x_2$ as the entering variable.
To decide on the departing variable, we see the ratio between the entries in the $b$ column with the entries in the entering $x_2$ column: $\frac{11}{1} = 11, \frac{27}{1} = 27, \frac{90}{5} = 18$. The smallest non-negative ratio is 11, which corresponds to the $s_1$ row. Thus $s_1$ is the departing variable.

The intersection entry between the entering variable’s column and the departing variable’s row is the pivot. Next we use Gauss-Jordan elimination corresponding to the pivot in order to obtain the improved solution. Table 1.2 shows the improved simplex tableau.

Table 1.2: Simplex tableau after 1 pivoting.

<table>
<thead>
<tr>
<th>Tableau 2</th>
<th>$c_i$</th>
<th>$b$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>6</td>
<td>11</td>
<td>-1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>16</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>1</td>
<td>0</td>
</tr>
<tr>
<td>$s_3$</td>
<td>0</td>
<td>35</td>
<td>7</td>
<td>0</td>
<td>-5</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$z$</td>
<td>66</td>
<td>-10</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

From Table 1.2 we can see that the current improved solution $z = 66$ with $(x_1, x_2, s_1, s_2, s_3) = (0, 11, 0, 16, 35)$. Note that in the $z$ row there is still a negative entry “−10”. So we repeat the pivoting process to get the next improved solution. Choose $x_1$ as the entering variable, and then compare the ratio $\frac{11}{1} = -11, \frac{16}{2} = 8, \frac{35}{7} = 5$ to see that $s_3$ is the departing variable. The intersection entry “7” is now the pivot, and Gauss-Jordan elimination produces Table 1.3

Table 1.3: Simplex tableau after 2 pivoting.

<table>
<thead>
<tr>
<th>Tableau 3</th>
<th>$c_i$</th>
<th>$b$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_2$</td>
<td>6</td>
<td>16</td>
<td>0</td>
<td>1</td>
<td>$\frac{2}{7}$</td>
<td>0</td>
<td>$\frac{1}{7}$</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>$\frac{3}{7}$</td>
<td>1</td>
<td>$\frac{2}{7}$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>4</td>
<td>5</td>
<td>1</td>
<td>0</td>
<td>$-\frac{5}{7}$</td>
<td>0</td>
<td>$\frac{1}{7}$</td>
</tr>
<tr>
<td>$z$</td>
<td>116</td>
<td>0</td>
<td>0</td>
<td>$-\frac{8}{7}$</td>
<td>0</td>
<td>$\frac{10}{7}$</td>
<td></td>
</tr>
</tbody>
</table>
From Table 1.3 we can see that the current improved solution \( z = 116 \) with 
\((x_1, x_2, s_1, s_2, s_3) = (5, 16, 0, 6, 0)\). Note that in the \( z \) row there is still a negative entry \( \frac{-8}{7} \). So we repeat the pivoting process once again and get Table 1.4.

Table 1.4: Simplex tableau after 3 pivoting.

<table>
<thead>
<tr>
<th>Tableau 4</th>
<th>( c_i )</th>
<th>( b )</th>
<th>( x_1 )</th>
<th>( x_2 )</th>
<th>( s_1 )</th>
<th>( s_2 )</th>
<th>( s_3 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( x_2 )</td>
<td>6</td>
<td>12</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>(-\frac{2}{3})</td>
<td>( \frac{1}{3} )</td>
</tr>
<tr>
<td>( s_1 )</td>
<td>0</td>
<td>14</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>( \frac{7}{3} )</td>
<td>(-\frac{2}{3} )</td>
</tr>
<tr>
<td>( x_1 )</td>
<td>4</td>
<td>15</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>( \frac{5}{3} )</td>
<td>(-\frac{1}{3} )</td>
</tr>
<tr>
<td>( z )</td>
<td>132</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>( \frac{8}{3} )</td>
<td>( \frac{2}{3} )</td>
<td></td>
</tr>
</tbody>
</table>

There is no negative entry in the \( z \) row of Table 1.4, we have therefore determined the optimal solution to be \( z = 132 \) with 
\((x_1, x_2, s_1, s_2, s_3) = (15, 12, 14, 0, 0)\).

Remark: When choosing the departing variable, if all entries in the entering variable’s row are 0 or negative, then there is no maximum solution to the LP problem. If there is a ties when comparing the non-negative ratios, then choose either entry.

In Section 2.7 our method resulting to LP problem with much more variables and constraints. To do the process described in this section we utilize the open source online tool “PHPSimplex” ([http://www.phpsimplex.com/en/](http://www.phpsimplex.com/en/)).
Chapter 2

Diameter of 3-colorable Graphs

2.1 History and Conjecture

Let \( G = (V,E) \) be a simple, finite, connected graph on \( n \) vertices, with minimum degree \( \delta \geq 2 \) and diameter \( \text{diam}(G) \). The natural problem of bounding the diameter of graph in terms of its order and minimum degree was solved by several authors \([3, 6, 7, 9]\), who independently proved the following result.

**Theorem 2.1.** \([3, 6, 7, 9]\) For a fixed \( \delta \geq 2 \) and large \( n \),

\[
\text{diam}(G) \leq \frac{3n}{\delta + 1} + O(1). \tag{2.1}
\]

In 1989, Erdős, Pach, Pollack, and Tuza \([6]\) conjectured that the upper bound \(2.1\) can be improved if the graph \( G \) does not contain a large complete subgraph \( K_k \).

**Conjecture 2.2.** Let \( r, \delta \geq 2 \) be fixed integers and let \( G \) be a connected graph with \( n \) vertices and minimum degree \( \delta \).

(1) If \( G \) is \( K_{2r} \)-free and \( \delta \) is a multiple of \( (r-1)(3r+2) \) then, for large \( n \),

\[
\text{diam}(G) \leq \frac{2(r-1)(3r+2)}{(2r^2-1)} \frac{n}{\delta} + O(1)
\]

(2) If \( G \) is \( K_{2r+1} \)-free and \( \delta \) is a multiple of \( (3r-1) \) then, for large \( n \),

\[
\text{diam}(G) \leq \frac{3r-1}{r} \frac{n}{\delta} + O(1)
\]

They also constructed graphs showing that, if the upper bounds hold, then they are sharp, apart from an additive constant. For convenience, we introduce in Table 2.1 a ‘ball’ notations to simplify the explanation to their construction.
Table 2.1: Notations used to simplify Erdős et al. construction.

<table>
<thead>
<tr>
<th>Notation</th>
<th>Multi-partite graph</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x$</td>
<td>$K_{n}^{*}$</td>
</tr>
<tr>
<td>$2$</td>
<td>$K_3^{*}$</td>
</tr>
</tbody>
</table>

Their construction is given below. Dashed edges between two balls denote the edges of a complete bipartite graph between the corresponding two vertex sets.

(1) Construction or $K_{2r}$-free and $(r - 1)(3r + 2) | \delta$:

![Figure 2.1: Construction for Conjecture 2.2(1).](image)

Note that each ball in this construction can be expressed in our notation as well: We have clumps of independent vertices where any pair of clumps is connected. This leads to the following alternative (clump-graph) representation of the Erdős et al. construction: We have clumps (representing independent vertex sets) arranged in a row of columns; the size of each clump is written inside the clump, and any clump is connected to all other clump in the same column, in the column immediately before and immediately after. We will use this alternative representation in a slightly modified form for our later purposes.

Ordering the columns from left to right, there are $r$ clumps on each odd numbered column, and $r - 1$ clumps on each even numbered column. It follows
immediately that such graph is $K_{2r}$-free.

WLOG let there be $(k + 1)$ columns where $k$ is even. Since the conjecture assumes $n$ is large, then $k$ is sufficiently large. To verify the minimum degree, first observe that any vertex inside any clump that located in the first, second, second to last, or last columns have degree greater than $\delta$.

Next observe the columns of $y$-clumps that are not the second or second to last: for a vertex $w$ inside a $y$-clump, we have $\deg(w) = 2rx + (r - 2)y$. Then observe the columns of $x$-clumps: for a vertex $v$ inside an $x$-clump, we have $\deg(v) = (r - 1)x + 2(r - 1)y = (r - 1)(x + 2y)$. Maximize $\delta$ by setting $\deg(v) = \delta = \deg(w)$:

- setting $\deg(v) = \deg(w)$:

\[
(r - 1)(x + 2y) = 2rx + (r - 2)y \\
rx + 2ry - x - 2y = 2rx + ry - 2y \\
ry = rx + x \\
ry = r(x + 1) \\
y = x\left(\frac{r + 1}{r}\right)
\]
setting $\delta = \deg(v)$:

\[
\delta = (r - 1)(x + 2y) \\
= (r - 1)
\left[
\left(x + 2 \cdot \frac{r + 1}{r}ight) \cdot x
\right] \\
= (r - 1)
\left[
\left(\frac{3r + 2}{r}\right)x
\right] \\
x = \frac{r\delta}{(r - 1)(3r + 2)} , \text{ hence } y = \frac{(r + 1)\delta}{(r - 1)(3r + 2)}
\]

To get the conjectured upper bound, we count the number of vertices across all $k + 1$ columns:

\[
n = 2r\delta + \frac{k}{2} \cdot y(r - 1) + \frac{k - 2}{2} \cdot xr \\
= 2r(r - 1)
\left[
\left(\frac{3r + 2}{r}\right)x + \frac{k}{2}
\left(\frac{r + 1}{r}ight) \cdot x(r - 1) + \frac{k - 2}{2} \cdot xr
\right] \\
= 2(r - 1)(3r + 2)
\left[
\frac{k}{2}
\left(\frac{r + 1}{r}\right)(r - 1)x + \frac{k - 2}{2} \cdot xr
\right] \\
= k
\left[
\frac{(2r^2 - 1)}{2r}
\right] \\
\geq k\frac{(2r^2 - 1)}{2r} \\
\geq k\frac{r\delta}{(r - 1)(3r + 2)} \frac{2r^2 - 1}{2r} \\
\frac{n}{\delta} \geq \frac{2r^2 - 1}{2(r - 1)(3r + 2)} \cdot k \\
\text{diam}(G) = k \leq \frac{2(r - 1)(3r + 2)}{(2r^2 - 1)\delta} n
\]

$O(1)$ inaccuracy might occur from the first and last columns.

Recall that the goal is to improve the known upper bound (2.1). Comparing the conjectured bound and the known bound for $r, \delta, n > 1$ we have

\[
\frac{2(r - 1)(3r + 2)}{(2r^2 - 1)\delta} n \leq \frac{3}{\delta + 1} n \Rightarrow \delta \geq \frac{2(r - 1)(3r + 2)}{2r + 1}.
\]

Hence the the assumption $(r - 1)(3r + 2) | \delta$ is needed.
(2) Construction for $K_{2r+1}$-free and $(3r-1) \mid \delta$:

$$
\begin{array}{cccccc}
\delta & x & x & x & \cdots & \delta \\
K^*_r & K^*_r & K^*_r & K^*_r & \cdots & K^*_r
\end{array}
$$

Figure 2.3: Construction for Conjecture 2.2(2).

Similar to previous case, the alternative representation of the graph is given in Figure 2.4. To verify the minimum degree, first observe that any vertex inside any clump that located in the first, second, second to last, or last columns have degree greater than $\delta$. Next observe the columns of $x$-clumps that are not the second or second to last: for a vertex $v$ inside a $x$-clump, we have $\deg(v) = (r-1)x + 2rx = (3r-1)x$. Maximize $\delta$ by setting $\delta = \deg(v) = (3r-1)x \Rightarrow x = \frac{\delta}{3r-1}$.

Count the number of vertices across $(k+1)$ columns:

$$
n = 2r\delta + (k-1)xr = 3r\delta + (k-1)\frac{r\delta}{3r-1}
$$

$$
\frac{n}{\delta} = 3r + \frac{(k-1)r}{3r-1}
$$

$$
= 2r + \frac{kr}{3r-1} - \frac{r}{3r-1}
$$

$$
= \frac{kr}{3r-1} + \frac{3r-1}{3r-1} > 0 \text{ since } r > 1
$$

$$
\geq \frac{r}{3r-1}k
$$

$$
diam(G) = k \leq \frac{3r-1}{3r-1}n
$$

$O(1)$ inaccuracy might occur from the first and last columns.

Recall that the goal is to improve the known upper bound (2.1). Comparing the conjectured bound and the known bound for $r, \delta, n > 1$ we have

$$
\frac{3r-1}{r\delta}n \leq \frac{3}{\delta + 1}n \Rightarrow \delta \geq 3r - 1.
$$

Hence the assumption $(3r-1) \mid \delta$ is needed.
Figure 2.4: Construction for Conjecture 2.2 as clumps without edges.

Note that in these construction, as shown in Figure 2.2 and Figure 2.4, having $\delta$ vertices in each clump in both leftmost and rightmost columns is too generous. This way the vertices in two leftmost and two rightmost columns has minimum degree well beyond $\delta$. However, in terms of the conjectured bound for the diameter, they work.

As mentioned in the abstract, Czabarka et al. replaced the requirement that $G$ has no $K_k$ subgraph by the assumption that $G$ is $(k-1)$-colorable. Note that any upper bound for the $K_k$-free graphs is also an upper bound for the $(k-1)$-colorable graphs. Under this stronger assumption, they stated Conjecture 2.3, the colorability version of Conjecture 2.2.

**Conjecture 2.3** (Czabarka, Dankelmann, Székely, 2008). *Let $r, \delta \geq 2$ be fixed integers and let $G$ be a connected graph with $n$ vertices and minimum degree $\delta$.

1. If $G$ is $(2r-1)$-colorable then

$$diam(G) \leq \frac{2(r-1)(3r+2)}{(2r^2-1)} \frac{n}{\delta} + O(1)$$

2. If $G$ is $2r$-colorable then

$$diam(G) \leq \frac{3r-1}{r} \frac{n}{\delta} + O(1)$$
In this dissertation we are particularly looking at the case \( r = 2 \). Let \( \delta \geq 2 \) be fixed integers and \( G \) be a connected graph with \( n \) vertices and minimum degree \( \delta \). Conjecture 2.2 stated that:

- If \( G \) is \( K_4 \)-free and \( 8 \mid \delta \), then for large \( n \) we have \( \text{diam}(G) \leq \frac{16n}{7\delta} + O(1) \).
- If \( G \) is \( K_5 \)-free and \( 5 \mid \delta \), then for large \( n \) we have \( \text{diam}(G) \leq \frac{5n}{2\delta} + O(1) \).

Hence we worked on the 3-colorable and 4-colorable version of Conjecture 2.3:

**Conjecture 2.4.** Let \( \delta \geq 2 \) be fixed integers and \( G \) be a connected graph with \( n \) vertices and minimum degree \( \delta \). Conjecture 2.2 stated that:

- If \( G \) is 3-colorable then for large \( n \) we have \( \text{diam}(G) \leq \frac{16n}{7\delta} + O(1) \).
- If \( G \) is 4-colorable then for large \( n \) we have \( \text{diam}(G) \leq \frac{5n}{2\delta} + O(1) \).

In Section 2.2 we describe the vertex-coloring approach of Czabarka, Dankelmann, and Székely [4]. Using this approach, they showed that the conjecture holds for all \( \delta \geq 1 \) under a stronger assumption that \( G \) is 4-colorable instead of \( K_5 \)-free. This result is discussed in more details in Section 2.5.

### 2.2 Clump Decomposition

Given a \( k \)-colorable connected graph \( G \) with \( n \) vertices and minimum degree at least \( \delta \). Take a vertex \( x \) with eccentricity \( \text{diam}(G) \), and a fixed \( k \)-coloring of \( G \). \( L_i \) denotes the set of vertices of distance \( i \) from \( x \), and a clump in \( L_i \) are the set of precisely those vertices in \( L_i \) that have the same color. Let \( c(i) \in \{1,2,3\} \) denotes the number of colors used in \( L_i \) by our fixed coloring.

**Theorem 2.5.** Let \( G \) a graph on \( n \) vertices, diameter \( D \) and minimum degree at least \( \delta \), given a fixed \( k \)-coloration with corresponding vertex sets \( L_0 = \{x\}, L_1, \ldots, L_D \). There is graph \( G^* \) on \( n \) vertices, diameter \( D \) and minimum degree at least \( \delta \) with a \( k \)-coloration with with corresponding vertex sets \( L_0 = \{x\}, L_1, \ldots, L_D \) such that \( G \) is a subgraph of \( G^* \) and:
(1) The set of color classes in $L_i$ in $G$ and $G^*$ are the same (though not necessarily the colors used).

(2) For every $i : 1 \leq i \leq D$ and every vertex in $x \in L_i$ there is at least one vertex $y \in L_{i-1}$ such that $xy \in E(G) \subseteq E(G^*)$.

(3) If $X, Y$ denotes the set of vertices in two clumps that are colored differently and appear in the same or consecutive $L_i$’s of $G^*$, then all edges between $X$ and $Y$ appear in $G^*$.

(4) For $i : 1 \leq i \leq D$ if $c(i-1) + c(i) \leq k$, then the colors used in $L_i$ and $L_{i-1}$ are disjoint in $G^*$; in particular if $c(i-1) = 1$ then $c(i) \leq k - 1$.

Proof. We will prove all statements by either changing our fixed coloring such that within an $L_i$ we recolor one or more clumps with a new color, or add edges between two clumps that belong to the same or consecutive $L_i$’s. This will ensure that $G^*$ is a subgraph of $G$, $G$ has $n$ vertices, minimum degree at least $\delta$, diameter $D$ and same distance classes $L_i$, and (1) is satisfied. (2) is immediate from the definition of the $L_i$. (3) follows from the fact that we can add edges between differently colored clumps that are in the same or consecutive $L_i$’s without decreasing the diameter or increasing the chromatic number. Finally, if for some $i : 1 \leq i \leq D$ we have that $c(i-1) + c(i) \leq k$, then let $S$ be the set of shared colors in $L_{i-1}$ and $L_i$, and let $T$ be the set of colors not used in $L_{i-1} \cup L_i$. Since $c(i-1) + c(i) \leq k$, and $|T| \geq k - (c(i-1) + c(i) - |S|)$, there is a $T_0 \subseteq T$ with $|T_0| = |S|$. Using $S = \{s_1, \ldots, s_c\}$ and $T_0 = \{t_1, \ldots, t_c\}$, we recolor vertices in the vertex set $\bigcup_{j \geq i} L_j$, as follows: we exchange colors $s_p$ and $t_p$ for all $1 \leq p \leq c$. This is obviously still a valid $k$-coloring of the graph, but (4) is satisfied for $L_{i-1}$ and $L_i$ (we may need to add new edges for (3)).

The clump graph $H$ of a connected graph $G$ has vertices representing the clumps of $G^*$. Two vertices of $H$ are connected by an edge if there were edges between the corresponding clumps in $G^*$. $H$ is naturally 3-colored and layered based on $G^*$. 

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By the layering-by-distance construction of $H$, each clump in $L_i$ must be connected to some clumps in $L_{i-1}$. This gives Proposition 2.6 and Proposition 2.7.

**Proposition 2.6.** Let $G$ be a connected graph of order $n$, $\text{diam}(G) = D$, and minimum degree $\delta$. Let $H$ be a the clump graph of $G$. A consecutive layer of a single clump and layer of 2 clumps in $H$ does not share colors.

**Proposition 2.7.** Let $G$ be a connected graph of order $n$, $\text{diam}(G) = D$, and minimum degree $\delta$. Let $H$ be a the clump graph of $G$. A layer of a single clump in $H$ is not immediately followed by a layer of 3 clumps.

Figure 2.5 (left) and (right) show an example for the scenarios described in Proposition 2.6 and Proposition 2.7, respectively. In both figures there is a colored $A$ clump that is not connected to any clump in the preceding layer, contradicting the layering-by-distance construction. For convenience, WLOG, we call Figure 2.5 (left) as an $A-AB$ pattern and Figure 2.5 (right) as an $A-ABC$ pattern.

![Diagram of Layering-by-Distance Constructions](image-url)

Figure 2.5: Example: impossible layering-by-distance constructions.

**Lemma 2.8.** For positive integers $b_2, c_2, b_3, c_3$, we have:

$$\left\lceil \frac{b_2 + c_2}{2} \right\rceil + \left\lceil \frac{b_3 + c_3}{2} \right\rceil \geq \min\{b_2 + b_3, c_2 + c_3\}.$$ 

**Proof.** Consider 4 cases:

1. $b_2 \equiv c_2 \pmod{2}$ and $b_3 \equiv c_3 \pmod{2}$

   $$b_2 \equiv c_2 \pmod{2} \Rightarrow \left\lceil \frac{b_2 + c_2}{2} \right\rceil = \frac{b_2 + c_2}{2}$$

   $$b_3 \equiv c_3 \pmod{2} \Rightarrow \left\lceil \frac{b_3 + c_3}{2} \right\rceil = \frac{b_3 + c_3}{2}$$

2. (other cases can be handled similarly)
Hence \[ \left\lfloor \frac{b_2 + c_2}{2} \right\rfloor + \left\lfloor \frac{b_3 + c_3}{2} \right\rfloor = \frac{1}{2} \left[ (b_2 + c_2) + (b_3 + c_3) \right] = \frac{1}{2} \left[ (b_2 + b_3) + (c_2 + c_3) \right]. \] This case is done as the minimum of two numbers is at most their average.

(2) \( b_2 \not\equiv c_2 \pmod{2} \) and \( b_3 \not\equiv c_3 \pmod{2} \)

\[
\begin{align*}
\frac{b_2 + c_2}{2} &\not\equiv \frac{b_2 + c_2}{2} + 1 \\
\frac{b_3 + c_3}{2} &\not\equiv \frac{b_3 + c_3}{2} - 1
\end{align*}
\]

Hence \[ \left\lfloor \frac{b_2 + c_2}{2} \right\rfloor + \left\lfloor \frac{b_3 + c_3}{2} \right\rfloor = \frac{1}{2} \left[ (b_2 + c_2) + (b_3 + c_3) \right] = \frac{1}{2} \left[ (b_2 + b_3) + (c_2 + c_3) \right], \] and the desired inequality obtained by similar argument with Case (1).

(3) \( b_2 \equiv c_2 \pmod{2} \) and \( b_3 \not\equiv c_3 \pmod{2} \)

\[
\begin{align*}
\frac{b_2 + c_2}{2} &\equiv \frac{b_2 + c_2}{2} \equiv 0 \\
\frac{b_3 + c_3}{2} &\not\equiv \frac{b_3 + c_3}{2} - 1
\end{align*}
\]

Hence \[ \left\lfloor \frac{b_2 + c_2}{2} \right\rfloor + \left\lfloor \frac{b_3 + c_3}{2} \right\rfloor = \frac{1}{2} \left[ (b_2 + c_2) + (b_3 + c_3) \right] - \frac{1}{2}
\]

Here \( \frac{1}{2} \left[ (b_2 + b_3) + (c_2 + c_3) \right] = a + \frac{1}{2} \geq \min\{b_2 + b_3, c_2 + c_3\} \) for some \( a \in \mathbb{Z}^+ \). Since \( \min\{b_2 + b_3, c_2 + c_3\} \in \mathbb{Z}^+ \), we must have \( a \geq \min\{b_2 + b_3, c_2 + c_3\} \). Thus

\[
\left\lfloor \frac{b_2 + c_2}{2} \right\rfloor + \left\lfloor \frac{b_3 + c_3}{2} \right\rfloor = a + \frac{1}{2} - \frac{1}{2} = a \geq \min\{b_2 + b_3, c_2 + c_3\}
\]

(4) \( b_2 \not\equiv c_2 \pmod{2} \) and \( b_3 \equiv c_3 \pmod{2} \)

\[
\begin{align*}
\frac{b_2 + c_2}{2} &\not\equiv \frac{b_2 + c_2}{2} + 1 \\
\frac{b_3 + c_3}{2} &\equiv \frac{b_3 + c_3}{2}
\end{align*}
\]

Hence immediately we have

\[
\left\lfloor \frac{b_2 + c_2}{2} \right\rfloor + \left\lfloor \frac{b_3 + c_3}{2} \right\rfloor = \frac{1}{2} \left[ (b_2 + b_3) + (c_2 + c_3) \right] + \frac{1}{2}
\]

\[
\geq \min\{b_2 + b_3, c_2 + c_3\} + \frac{1}{2} \geq \min\{b_2 + b_3, c_2 + c_3\}
\]

\[\square\]
Theorem 2.9. Let \( G \) be a connected graph of order \( n \), \( \text{diam}(G) = D \), and minimum degree \( \delta \). By surgery (reconstruction) and recoloring of the clump graph \( H \) of \( G \), we can remove any \( ABC - A, AB - AB, \) and \( AB - A \) patterns. The result is a graph \( G' \) of order \( n \) with minimum degree \( \delta \) and diameter \( D \) that does not have these patterns.

Proof. Let \( \#(X) \) denote the number of color pattern \( X \) in a clump graph. Consider those clump graphs and 3-colorations that minimize

\[
\#(ABC - A) + \#(AB - AB) + \#(AB - A).
\]

We claim that this minimum is zero. This proves the existence of clump graph that simultaneously satisfies all our requirements.

First, we must have \( \#(AB - AB) = 0 \). Otherwise, finding any pair of \( AB - AB \) colored consecutive layers \( L_i \) and \( L_{i+1} \), we can switch the colors \( B \) and \( C \) starting in layer \( i + 1 \) through layer \( D \), decreasing \( \#(AB - AB) \) by one, and not changing the other two terms in the minimization, hence contradict the minimum value.

Next, we also must have \( \#(AB - A) = 0 \). Otherwise, finding any pair of \( AB - A \) colored consecutive layers \( L_i \) and \( L_{i+1} \), we can switch the colors \( A \) and \( C \) starting in layer \( i + 1 \) through layer \( D \), decreasing \( \#(AB - A) \) by one, and not changing the other two terms in the minimization, hence also contradict the minimum value.

Assume now that \( \#(ABC - A) > 0 \). WLOG Let \( L_i \) of 3 clumps colored \( A, B, C \) is immediately followed by \( L_{i+1} \) of a single clump colored \( A \). From Proposition 2.7, we know that \( L_{i-1} \) can not be a single clump. Consider cases:

(1) \( L_{i-1} \) has \( ABC \) or \( AB \) or \( AC \) pattern.

From Proposition 2.6, \( L_{i-2} \) is not a single clump colored \( A \). In this case, we move the clump colored \( A \) from \( L_i \) to be merged in the clump colored \( A \) in \( L_{i-1} \). Doing this might add some edges between \( L_{i-2} \) and \( L_{i-1} \), but preserves \( D, n, \delta \).

An example of this procedure is described in Figure 2.6.
Figure 2.6: Example: eliminating $ABC - A$ pattern in Case 1.

(2) $L_{i-1}$ has $BC$ pattern and $L_{i-2}$ is not a single clump colored $A$.

By Proposition 2.6, $L_{i-2}$ can not be a single clump either colored $B$ or $C$, so $L_{i-2}$ is either a layer of 2 or 3 clumps. In this case, move the clump colored $A$ from $L_i$ and make a clump colored $A$ in $L_{i-1}$. Similar with Case 1, this step might add some edges between $L_{i-2}$ and $L_{i-1}$, but preserve $D, n, \delta$. This step is described in Figure 2.7.

Figure 2.7: Example: eliminating $ABC - A$ pattern in Case 2.

(3) $L_{i-1}$ has $BC$ pattern and $L_{i-2}$ is a single clump colored $A$.

In this case, from $L_{i-2}$ to $L_{i+1}$ we have the pattern $A - BC - ABC - A$. Note that moving the clump colored $A$ from $L_i$ and make a clump colored $A$ in $L_{i-1}$ as in Case 2 contradict Proposition 2.7, so we need to consider a different elimination surgery. Assume that the cardinalities of the clump from $L_{i-2}$ to $L_{i+1}$ corresponds to the respective color classes are $a_1, b_2, c_2, a_3, b_3, c_3, a_4$. We consider cases:

(3.1) $a_3 \geq b_3$ or $a_3 \geq c_3$

If $a_3 \geq b_3$: Recolor the clump colored $A$ in $L_{i+1}$ using color $B$, then inter-
change the color $A$ and $B$ in all clumps in $L_j$ for $j \geq i + 1$. Since $a_3 \geq b_3$ we might add some edges but preserve $n$ and $D$. This cause an $ABC - B$ pattern to appear on $L_i$ and $L_{i+1}$, but we can apply the elimination surgery for $ABC - B$ as described in Case (1). Similar argument works if $a_3 \geq c_3$.

(3.2) $a_3 < b_3$ and $a_3 < c_3$, and either $a_3 \geq b_2$ or $a_3 \geq c_2$

WLOG assume $a_3 \geq b_2$. Interchange color $A$ and $B$ in $L_{i-2}$ and $L_{i-1}$, and all $L_j$ for $j < i - 2$. Within our pattern, change the cardinalities as shown in Figure 2.8.

![Figure 2.8: Re-coloration on case $a_3 \geq b_2$ before (left) and after (right).](image)

In Figure 2.8 the label of each clump is their cardinality. Note that the new coloration satisfies the required numerical conditions: we do not change $n$ and $D$, and we can verify the $\delta$ condition by comparing the degree list of vertices in each pattern as shown in Table 2.2 and Table 2.3.

<table>
<thead>
<tr>
<th>Column, Clump</th>
<th>Degree of vertex inside</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{i-2}$, colored $A$</td>
<td>$b_2 + c_2$</td>
</tr>
<tr>
<td>$L_{i-1}$, colored $B$</td>
<td>$a_1 + a_3 + c_2 + c_3$</td>
</tr>
<tr>
<td>$L_{i-1}$, colored $C$</td>
<td>$a_1 + a_3 + b_2 + b_3$</td>
</tr>
<tr>
<td>$L_i$, colored $A$</td>
<td>$b_2 + b_3 + c_2 + c_3$</td>
</tr>
<tr>
<td>$L_i$, colored $B$</td>
<td>$a_3 + a_4 + c_2 + c_3$</td>
</tr>
<tr>
<td>$L_i$, colored $C$</td>
<td>$a_3 + a_4 + b_2 + b_3$</td>
</tr>
<tr>
<td>$L_{i+1}$, colored $A$</td>
<td>$b_3 + c_3$</td>
</tr>
</tbody>
</table>
Table 2.3: Degree list in the pattern from Figure 2.8 (right).

<table>
<thead>
<tr>
<th>Column, Clump</th>
<th>Degree of vertex inside</th>
</tr>
</thead>
<tbody>
<tr>
<td>$L_{i-2}$, colored $B$</td>
<td>$b_2 + c_2$</td>
</tr>
<tr>
<td>$L_{i-1}$, colored $A$</td>
<td>$a_1 + b_2 + b_3 + c_2 + c_3$</td>
</tr>
<tr>
<td>$L_{i-1}$, colored $C$</td>
<td>$a_1 + a_3 + b_2 + b_3$</td>
</tr>
<tr>
<td>$L_i$, colored $A$</td>
<td>$a_3 + a_4 + b_2 + c_2 + c_3$</td>
</tr>
<tr>
<td>$L_i$, colored $B$</td>
<td>$a_3 + a_4 + c_2 + c_3$</td>
</tr>
<tr>
<td>$L_i$, colored $C$</td>
<td>$a_3 + a_4 + b_2 + b_3$</td>
</tr>
<tr>
<td>$L_{i+1}$, colored $A$</td>
<td>$b_2 + b_3 + c_3$</td>
</tr>
</tbody>
</table>

It is easy to verify that the degrees in Table 2.3 is never smaller than some degrees in Table 2.2, so the minimum degree conditions hold. To be exact, we use $d(X_i)$ to denote the degree of a vertex inside clump colored $X$ in column $L_i$ before the surgery and $d'(X_i)$ to denote the degree of a vertex inside clump colored $X$ in column $L_i$ after the surgery, then check the minimum degree condition as follow:

\[
d'(B_{i-2}) = b_2 + c_2 = d(A_{i-2}) \geq \delta \\
d'(A_{i-1}) = a_1 + b_2 + b_3 + c_2 + c_3 > b_2 + b_3 + c_2 + c_3 = d(A_i) \geq \delta \\
d'(C_{i-1}) = a_1 + a_3 + b_2 + b_3 = d(C_{i-1}) \geq \delta \\
d'(A_i) = a_3 + a_4 + b_2 + c_2 + c_3 = d(A_i) \geq \delta \\
d'(B_i) = a_3 + a_4 + c_2 + c_3 = d(B_i) \geq \delta \\
d'(C_i) = a_3 + a_4 + b_2 + b_3 = d(C_i) \geq \delta \\
d'(A_{i+1}) = b_2 + b_3 + c_3 > b_2 + c_3 = d(A_{i+1}) \geq \delta
\]

As the size of column $L_{i+1}$ stays the same, no degrees in column $L_{i+2}$ decreased. If $a_3 - b_2 = 0$, then we no longer have the clump colored $A$ in $L_i$, hence eliminated the $ABC - A$ pattern. If $a_3 - b_2 > 0$, then apply the procedure as described in Case (1) to the columns $L_{i-1}$ and $L_i$. We have proved $a_3 < b_2$. Similar argument proves $a_3 < c_2$. 

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(3.3) $a_3 < b_3$ and $a_3 < c_3$ and $a_3 < b_2$ and $a_3 < c_2$

Consider 3 cases and make the required surgery in each of them.

(3.3.1) $\min\{b_2, c_2\} \geq \max\{b_3, c_3\}$

Change into the following pattern that preserve $n$ and $D$ as shown in Figure 2.9, then check the minimum degree condition.

\begin{align*}
 d'(A_{i-2}) &= b_2 + c_2 = d(A_{i-2}) \geq \delta \\
 d'(B_{i-1}) &= a_1 + b_2 + c_2 + c_3 > a_1 + a_3 + c_2 + c_3 = d(B_{i-1}) \geq \delta \\
 d'(C_{i-1}) &= a_1 + b_2 + b_3 + c_3 > a_1 + a_3 + b_2 + b_3 = d(C_{i-1}) \geq \delta \\
 d'(A_i) &= a_3 + a_4 + b_2 + c_2 \geq a_3 + a_4 + b_2 + b_3 = d(C_i) \geq \delta \\
 d'(B_{i+1}) &= b_3 + c_3 = d(A_{i+1}) \geq \delta
\end{align*}

Inequality (2.3), (2.4), and (2.5) holds because $a_3 < b_2$, $a_3 < c_3$, and $\min\{b_2, c_2\} \geq \max\{b_3, c_3\}$, respectively. We can also replace (2.3) and (2.5) by observing that $d'(B_{i-1})$ and $d'(A_i)$ are each larger than $d(A_{i-2}) \geq \delta$. As the size of column $L_{i+1}$ increased, no degrees in column $L_{i+2}$ decreased.

(3.3.2) $\max\{b_2, c_2\} \leq \min\{b_3, c_3\}$

Change into the following pattern that preserve $n$ and $D$ as shown in Figure 2.10, then check the minimum degree condition. Note that after the surgery this case is a mirror of Case (3.3.1).
In Figure 2.10: Surgery on Case (3.3.2) before (left) and after (right).

\[ d'(B_{i-2}) = b_2 + c_2 = d(A_{i-2}) \geq \delta \]
\[ d'(A_{i-1}) = a_1 + a_3 + b_3 + c_3 \geq a_1 + a_3 + c_2 + c_3 = d(B_{i-1}) \geq \delta \]
\[ d'(B_i) = a_4 + b_2 + c_2 + c_3 > a_3 + a_4 + c_2 + c_3 = d(B_i) \geq \delta \]
\[ d'(C_i) = a_4 + b_2 + b_3 + c_2 > a_3 + a_4 + b_2 + b_3 = d(C_i) \geq \delta \]
\[ d'(A_{i+1}) = b_3 + c_3 = d(A_{i+1}) \geq \delta \]

Inequalities holds because \( \max\{b_2, c_2\} \leq \min\{b_3, c_3\} \), \( a_3 < c_2 \), and \( a_3 < b_2 \). Alternatively, we can replace the fourth inequality by comparing \( d'(C_i) = a_3 + a_4 + b_2 + c_2 > b_2 + c_2 = d(A_{i-2}) \geq \delta \). As the size of column \( L_{i-2} \) increased, no degrees in column \( L_{i-3} \) decreased.

\( (3.3.3) \) \( \min\{b_2, c_2\} < \max\{b_3, c_3\} \) and \( \max\{b_2, c_2\} > \min\{b_3, c_3\} \)

First modify the cardinalities on \( L_i - 1 \) and \( L_i \) as in Figure 2.11.

In Figure 2.11: Cardinalities on Case (3.3.3) before (left) and after (right).
Since $b_2 + c_2 = \left\lceil \frac{b_2 + c_2}{2} \right\rceil + \left\lceil \frac{b_2 + c_2}{2} \right\rceil$ and $b_3 + c_3 = \left\lceil \frac{b_3 + c_3}{2} \right\rceil + \left\lceil \frac{b_3 + c_3}{2} \right\rceil$, we can easily check some of the degrees conditions:

\[
d'(A_{i-2}) = b_2 + c_2 = d(A_{i-2}) \geq \delta
\]

\[
d'(A_i) = b_2 + b_3 + c_2 + c_3 = d(A_i) \geq \delta
\]

\[
d'(A_{i+1}) = b_3 + c_3 = d(A_{i+1}) \geq \delta
\]

Check the degree conditions for the remaining clumps:

\[
d'(B_{i-1}) = a_1 + a_3 + \left\lceil \frac{b_2 + c_2}{2} \right\rceil + \left\lceil \frac{b_3 + c_3}{2} \right\rceil
\]

\[
\geq a_1 + a_3 + \min\{b_2 + b_3, c_2 + c_3\} \quad \text{(by Lemma 2.8)}
\]

\[
> \min\{a_1 + a_3 + b_2 + b_3, a_1 + a_3 + c_2 + c_3\}
\]

\[
= \min\{d(C_{i-1}), d(B_{i-1})\} \geq \delta
\]

\[
d'(C_{i-1}) = a_1 + a_3 + \left\lceil \frac{b_2 + c_2}{2} \right\rceil + \left\lceil \frac{b_3 + c_3}{2} \right\rceil
\]

\[
\geq a_1 + a_3 + \min\{b_2 + b_3, c_2 + c_3\} \quad \text{(by Lemma 2.8)}
\]

\[
> \min\{a_1 + a_3 + b_2 + b_3, a_1 + a_3 + c_2 + c_3\}
\]

\[
= \min\{d(C_{i-1}), d(B_{i-1})\} \geq \delta
\]

\[
d'(B_i) = a_3 + a_4 + \left\lceil \frac{b_2 + c_2}{2} \right\rceil + \left\lceil \frac{b_3 + c_3}{2} \right\rceil
\]

\[
\geq a_3 + a_4 + \min\{b_2 + b_3, c_2 + c_3\} \quad \text{(by Lemma 2.8)}
\]

\[
> \min\{a_3 + a_4 + b_2 + b_3, a_3 + a_4 + c_2 + c_3\}
\]

\[
= \min\{d(C_i), d(B_i)\} \geq \delta
\]

\[
d'(C_i) = a_3 + a_4 + \left\lceil \frac{b_2 + c_2}{2} \right\rceil + \left\lceil \frac{b_3 + c_3}{2} \right\rceil
\]

\[
\geq a_3 + a_4 + \min\{b_2 + b_3, c_2 + c_3\} \quad \text{(by Lemma 2.8)}
\]

\[
> \min\{a_3 + a_4 + b_2 + b_3, a_3 + a_4 + c_2 + c_3\}
\]

\[
= \min\{d(C_i), d(B_i)\} \geq \delta
\]

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Next, we apply the \((ABC - A)\) pattern elimination procedure as described in Case (3.3.1). Applying the procedure to Figure 2.11 (right) gives Figure 2.12.

![Figure 2.12: Applying surgery as in Case (3.3.1) after cardinalities are modified.](image)

WLOG assume \(b_2 \leq c_2\) and \(b_3 \geq c_3\), verify the degree conditions:

\[
\begin{align*}
    d'(A_{i-2}) &= b_2 + c_2 = d(A_{i-2}) \geq \delta \\
    d'(B_{i-1}) &= a_1 + \left\lfloor \frac{b_2 + c_2}{2} \right\rfloor + b_3 + c_3 \geq a_1 + c_2 + b_3 + c_3 \\
    &> a_1 + a_3 + c_2 + c_3 = d(B_{i-1}) \geq \delta \\
    d'(C_{i-1}) &= a_1 + \left\lfloor \frac{b_2 + c_2}{2} \right\rfloor + b_3 + c_3 \geq a_1 + b_2 + b_3 + c_3 \\
    &> a_1 + a_3 + b_2 + b_3 = d(C_{i-1}) \geq \delta \\
    d'(A_i) &= a_3 + a_4 + b_2 + c_2 \geq b_2 + c_2 = d(A_{i-2}) \geq \delta \\
    d'(B_{i+1}) &= b_3 + c_3 = d(A_{i+1}) \geq \delta
\end{align*}
\]

Summarizing our cases analysis: assuming that \(#(ABC - A) > 0\), we can do surgery to eliminate the \((ABC - A)\) pattern without creating any new \((ABC - A)\), \((AB - AB)\), or \((AB - A)\) pattern. This contradicts minimality again. Hence \(#(ABC - A) = 0\), and thus complete our proof.
2.3 Counterexample

Let $G$ be a 3-colorable graph which clump graph is a repetition of the pattern shown in Figure 2.13. The weight of each clump represents the number of vertices inside it. Based on Conjecture 2.2, if $\delta$ is divisible by 8, then we should have $D \leq \frac{16n}{7\delta}$.

Figure 2.13: Repetitive block for the clump graph of $G$.

Suppose there are $R$ repetitions of the pattern shown in Figure 2.13, it is easy to see that the minimum degree of $G$ is $\delta$. Also, $D = 7R - 1$ and $n = (3\delta + 3)R$. Hence $D = \frac{7n}{3\delta + 3} + O(1)$. Since $\frac{7n}{3\delta + 3} \leq \frac{16n}{7\delta}$ only true when $\delta \leq 48$, any such construction with an even $\delta > 48$ is a counterexample for Conjecture 2.2.

2.4 Linear Programming Approach

In this section we use a linear programming approach to came up with a construction method that obviate the need of divisibility condition for $\delta$ to determine an upper bound for diameter of 3-colorable graphs.

Think of a different view of looking at our problem: Let $H$ be a clump graph of $G$ as described in Section 2.2. Assign non-negative real weights $w(x)$ to $x \in V(H)$. Consider optimization problem to

$$\text{Minimize } \sum_{x \in V(H)} w(x),$$
subject to condition

\[ \sum_{x \in V(H), xy \in E(H)} w(x) \geq \delta \quad \forall y \in V(H). \]

If \( w(x) \) are integers, this is an attempt to recreate \( G \) from \( H \) by finding out the cardinalities of the clumps.

Let \( c(i) \) denote the number of colors in \( L_i \), so \( c(i) \) = 1, 2, or 3. If \( c(i) = 1 \), we say \( i \in S \), i.e. \( L_i \) is single colored, otherwise \( i \notin S \). Let \( C_t \) denote the number of columns \( i \) with \( c(i) = t \). Any vertex in \( L_i \) can have neighbors only in \( L_{i-1}, L_i, L_{i+1} \). Let \( |L_i| = \ell_i \).

For a vertex \( v \) in this graph, we denote by \( N(v) \) the open neighborhood of vertex \( v \). From the minimum degree condition, for any \( v \) we have \( |N(v)| \geq \delta \).

By our assumption of maximality, any two different color classes in \( L_i \cup L_{i+1} \) induce a complete bipartite graph. Let the colors of the 3-coloration be \( A, B, C \), and let \( A_i, B_i, C_i \) denote the subsets of \( L_i \) colored by the corresponding color. We use \( A_i, B_i, C_i \) notation only for non-empty sets, i.e. when the color is present in \( L_i \). When we do not want to specify the color of a color class, we use the notation \( X_i, Y_i, Z_i \) for the color classes in \( L_i \), and insist that different letters indicate different classes.

Furthermore, by Theorem 2.9, if \( i + 1 \in S \) and \( L_{i+1} = X_{i+1} \), then color class \( X_i \) is not present in \( L_i \). Similarly, if \( c(i) = c(i + 1) = 2 \) and \( L_i = X_i \cup Y_i \), then color class \( Z_{i+1} \) is present in \( L_{i+1} \).

**Theorem 2.10 (Duality Theorem of Linear Programming).**

Minimize \( b^T y \) subject to \( Ay \geq c, y \geq 0 \) \( \equiv \) Maximize \( c^T x \) subject to \( A^T x \leq b, x \geq 0 \) (Primal problem \( \equiv \) Dual problem). If the primal problem has the optimal solution \( y^* \) then dual problem has the optimal solution \( x^* \) where \( b^T y^* = c^T x^* \).

We translate our problem into its dual and show that Theorem 2.10 is applicable. Given a connected graph \( G \) with diameter \( D \) and minimum degree \( \delta \), apply clump decomposition as described in Section 2.2.
Take $b^T = \begin{bmatrix} 1 & 1 & \ldots & 1 \end{bmatrix}$ and $c^T = \begin{bmatrix} \delta & \delta & \ldots & \delta \end{bmatrix}$, both are $1 \times (D+1)$ matrices. Let $L_i^j$ denotes the $j$th clump of layer $L_i$ for $j \in \{1, 2, 3\}$. Let $w(L_i^j)$ denotes the number of vertices in clump $L_i^j$, and take

$$y^T = \begin{bmatrix} w(L_1^1) & w(L_1^2) & w(L_2^1) & w(L_2^2) & w(L_3^1) & \ldots & w(L_D^1) \end{bmatrix}.$$ 

Let $A$ be the $(D+1) \times (D+1)$ adjacency matrix between $L_i^j$’s and themselves. So $A^T y \geq c$ corresponds to

$$\sum_{x \in V(H) : xy \in E(H)} w(x) \geq \delta \quad \forall y \in V(H).$$

Next we consider the dual problem. Assigning non-negative real weights $u(x)$ to $x \in V(H)$, the corresponding dual problem is

$$\text{Maximize } \delta \cdot \sum_{x \in V(H)} u(x),$$

subject to condition

$$\forall y \in V(H) \quad \sum_{x \in V(H) : xy \in E(H)} u(x) \leq 1. \quad (2.7)$$

Let $u(L_i^j)$ denotes the unknown dual label assignment for clump $L_i^j$, and take

$$x^T = \begin{bmatrix} u(L_1^1) & u(L_1^2) & u(L_2^1) & u(L_2^2) & u(L_3^1) & \ldots & u(L_D^2) & u(L_D^3) \end{bmatrix}.$$ 

Then $A^T x \leq b$ in the dual problem corresponds to

$$\sum_{x \in V(H) : xy \in E(H)} u(x) \leq 1 \quad \forall y \in V(H).$$

Letting $N^*(H) = \min \sum_{x \in V(H)} w(x)$, we have by Theorem 2.10:

$$b^T y^* = c^T x^*$$

$$N^*(H) = \delta \cdot \sum_{x \in V(H)} u^*(x).$$

This means if the RHS is not maximum, we have RHS \leq LHS. Thus any feasible solution $u$ for the dual problem provides a lower bound $\delta \cdot \sum_{x \in V(H)} u(x) \leq N^*(H)$. 

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If we find a weighting scheme \( u \) such that for a constant \( c > 0 \) we have
\[
\sum_{x \in V(H)} u(x) \geq cD - O(1) \tag{2.8}
\]
where \( O(1) \) might comes from the weight on the endpoints \( L_0 \) and \( L_{D+1} \). Hence:
\[
cD - O(1) \leq \sum_{x \in V(H)} u(x) \leq \max \sum_{x \in V(H)} u(x) = \frac{N^*=H}{\delta} \leq \frac{n}{\delta}
\]
\[
D \leq \frac{1}{c} \cdot \frac{n}{\delta} + O(1).
\]

Note that condition (2.8) means that in average, the total weight in each column is \( c \). This is automatically satisfied if the total weight in each column is \( c \).

### 2.5 Diameter of 3-colorable Graphs using Duality


**Theorem 2.11.** [4] *For every connected 4-colorable graph \( G \) of order \( n \) and \( \delta \geq 1 \),
\[
diam(G) \leq \frac{5n}{2\delta} - 1.
\]

The proof in [4] is nowhere trivial, involved tedious case analysis, and unexpectedly is not extendable to 3-colorable graphs. Using duality as described in Section 2.4, we provide an alternative proof to the 3-colorable part of Theorem 2.11.

**Theorem 2.12.** *For every connected 3-colorable graph \( G \) of order \( n \) and \( \delta \geq 1 \),
\[
diam(G) \leq \frac{5n}{2\delta} + O(1).
\]

**Proof.** Assume WLOG the clump graph that we want to handle with duality satisfies Preposition 2.6, Proposition 2.7, 7 and Theorem 2.9. We came up with a color scheme such the total weight of each column is \( \frac{2}{5} \) that will give the \( \frac{5n}{2\delta} \) bound. In single color layers, the vertex get weight \( \frac{2}{5} \). In 2-color layers, each vertex gets weight \( \frac{1}{5} \). These cases shown on Figure 2.14.
In 3-color layers, we consider 4 cases. Figure 2.15 and Figure 2.16 describe these cases. For clarity, connection between clumps are not included in the figures.

(a) If it is between two 3-color layers, each of the three vertices get $\frac{2}{15}$.
(b) If it is between a 2-color layer and a 3-color layer, assign $\frac{1}{5}$ to the color that is not present in the 2-color layer, and assign $\frac{1}{10}$ to each of the other two vertices.

c) If it is between two 2-color layers, the same two colors are present in the 2-color layers: assign $\frac{1}{10}$ to each of those colors and $\frac{1}{5}$ to the third color.
(d) If it is between two 2-color layers, different two colors are present in the 2-color layers: assign $\frac{2}{15}$ to all three vertices.

Figure 2.14: Weighting scheme for column with 1 clump and with 2 clumps

Figure 2.15: Weighting scheme for 3-color layers case (a) (left) and (b) (right).

Figure 2.16: Weighting scheme for 3-color layers case (c) (left) and (d) (right).
It follows immediately that no vertex is joined to vertices with cumulative weight exceeding 1, and every layer got weight $\frac{2}{5}$, so conditions (2.7) and (2.8) are satisfied, giving the claimed bound.

Recall our counterexample in Section 2.3. We try to use the same approach to prove Conjecture 2.13. Based on our counterexample of $\frac{7n}{3\delta+3} + O(1)$ upper bound, we were inclined to believe the conjecture is true.

**Conjecture 2.13.** For every connected 3-colorable graph $G$ of order $n$ and $\delta \geq 1$, 

$$diam(G) \leq \frac{7n}{3\delta} + O(1).$$

This upper bound is not tight.

However, using the same scheme one can not have the total weight in each column is $c = \frac{3}{7}$. Some cases does not satisfy condition (2.7) for $c \geq \frac{2}{5}$. Figure 2.17 show this particular example for 3-color layers case (c).

![Figure 2.17: Weighting scheme for 3-color layers case (c) for $c = 2a$.](image)

In Figure 2.17 we can see that the average column weight is $2a$. However the sum of the shaded clump’s neighbors’ weight is $4a + 2 \cdot \frac{a}{2} = 5a$, and the condition (2.7) is not satisfied as $5a \leq 1$ implies $c = 2a \leq \frac{2}{5}$.

We considered and wrote a code in attempt to find different weighting schemes that satisfy both conditions (2.7) and (2.8). Rather than forcing each column to have total weight $c = \frac{3}{7}$, we looked at the average weight of certain number of columns which total weight averaged to $c = \frac{3}{7}$, one example as described in Figure 2.18.
The code is given in Section A.1. We tried, with no useful result, several variations for average values $\frac{2}{5} \leq c \leq \frac{3}{7}$, and also several different number of columns into the code. Note that the length (number of columns) of the pattern we need to look at may not be a bounded. There is a possibility that this approach can be continued further by writing more useful code, but as for now we do not possess relevant background to do so. Instead, we utilize the inclusion-exclusion principle for a different optimization problem approach in Section 2.6.

2.6 Diameter of 3-colorable Graphs using Inclusion-Exclusion

In this section we consider a different linear programming approach. Given a connected graph $G$ with diameter $D$ and minimum degree $\delta$. We again apply clump decomposition as described in Section 2.2 and coloring as described in Section 2.4. In the LP problem, we want to maximize the objective function $\phi = \frac{D\delta}{n}$. The ideal goal is to have $\max \phi = \frac{7}{3}$, so that $\text{diam}(G) = D \leq \max \phi \cdot \frac{\delta}{\delta} = \frac{7n}{3\delta} + O(1)$ as speculated in Conjecture 2.13. However, later we show in Section 2.7 that we have not met this ideal goal albeit improved the bound from Theorem 2.12.

For the constraints, we are analyzing Sieve formula of the neighbourhood of a vertex in any layer $L_i$. Since a vertex in $L_i$ can have neighbors only in $L_{i-1}, L_i, L_{i+1}$, we consider Sieve formula by individual, two consecutive, and three consecutive layers. For the rest of this section, we use notation $x_i, y_i, z_i$ to represent a vertex in the clump with color $X_i, Y_i, Z_i$, respectively.
Theorem 2.14 (Sieve Formula). For finite sets $A_1, \ldots, A_n$:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{i=1}^{n} |A_i| - \sum_{1 \leq i < j \leq n} |A_i \cap A_j| + \sum_{1 \leq i < j < k \leq n} |A_i \cap A_j \cap A_k| - \ldots + (-1)^{n-1}|A_1 \cap \ldots \cap A_n|$$

2.6.1 Sieve by Individual Layer

Consider three cases:

- **Case 1.** $c(i) = 1$

  $$X_{i} \quad |N(x_{i})| \geq \delta \Rightarrow \ell_{i-1} + \ell_{i+1} \geq \delta$$

  For convenience we prefer to write this as

  $$2\ell_{i-1} + 2\ell_{i} + 2\ell_{i+1} \geq 2\delta + 2\ell_{i}.$$  \hfill (2.9)

- **Case 2.** $c(i) = 2$

  $$X_{i} \quad |N(x_{i})| \geq \delta \Rightarrow \ell_{i-1} + |Y_{i}| + \ell_{i+1} \geq \delta$$
  $$Y_{i} \quad |N(y_{i})| \geq \delta \Rightarrow \ell_{i-1} + |X_{i}| + \ell_{i+1} \geq \delta$$

  Adding both inequalities we have $2\ell_{i-1} + \ell_{i} + 2\ell_{i+1} \geq 2\delta$.

  We prefer to write this as

  $$2\ell_{i-1} + 2\ell_{i} + 2\ell_{i+1} \geq 2\delta + \ell_{i}.$$  \hfill (2.10)

- **Case 3.** $c(i) = 3$

  $$X_{i} \quad |N(x_{i}) \cup N(y_{i}) \cup N(z_{i})| \geq 3\delta - |X_{i}| - |X_{i-1}| - |X_{i+1}|$$
  $$Y_{i} \quad |N(y_{i})| \geq \delta \Rightarrow \ell_{i-1} + |X_{i}| + \ell_{i+1} \geq \delta$$
  $$Z_{i} \quad |N(z_{i})| \geq \delta \Rightarrow \ell_{i-1} + |Y_{i}| + \ell_{i+1} \geq \delta$$

  We prefer to write this as

  $$2\ell_{i-1} + 2\ell_{i} + 2\ell_{i+1} \geq 3\delta.$$  \hfill (2.11)

Adding up (2.9), (2.10), and (2.11) across $D+1$ layers, we obtain

$$6n = 6 \sum_{i=1}^{D+1} \ell_{i} \geq (2D + 2 + C_3)\delta + \sum_{i \in c(i) = 2} \ell_{i} + 2 \sum_{i \in c(i) = 1} \ell_{i} + O(\delta).$$  \hfill (2.12)
2.6.2 Sieve by Two Consecutive Layers

Consider the following cases:

- **Case 1.** \( i \in S, i + 1 \in S \)

\[
\begin{align*}
X_i & \quad Y_{i+1} \\
|N(x_i)| \geq \delta & \Rightarrow \ell_{i-1} + \ell_{i+1} \geq \delta \\
|N(y_{i+1})| \geq \delta & \Rightarrow \ell_i + \ell_{i+2} \geq \delta
\end{align*}
\]

Adding both inequalities we have

\[\ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2} \geq 2\delta. \tag{2.13}\]

- **Case 2.** \( i \notin S \) and \( i + 1 \notin S \) (dotted clumps might or might not exist)

\[
\begin{align*}
X_i & \quad X_{i+1} \\
|N(x_i)| \geq \delta & \Rightarrow \ell_{i-1} + |Y_i| + |Z_i| + |Y_{i+1}| + |Z_{i+1}| \geq \delta \\
Y_i & \quad Y_{i+1} \\
|N(x_{i+1})| \geq \delta & \Rightarrow \ell_{i+2} + |Y_i| + |Z_i| + |Y_{i+1}| + |Z_{i+1}| \geq \delta \\
Z_i & \quad Z_{i+1} \\
|N(y_i)| \geq \delta & \Rightarrow \ell_{i-1} + |X_i| + |Z_i| + |X_{i+1}| + |Z_{i+1}| \geq \delta \\
|N(z_{i+1})| \geq \delta & \Rightarrow \ell_{i+2} + |X_i| + |Z_i| + |X_{i+1}| + |Z_{i+1}| \geq \delta
\end{align*}
\]

Adding one-third of the first two inequalities with two-third of the last two inequalities we have

\[
\frac{1}{3} [|N(x_i)| + |N(x_{i+1})|] + \frac{2}{3} [|N(y_i)| + |N(z_{i+1})|] \geq 2\delta
\]

\[
\ell_{i-1} + \ell_{i+2} + \frac{4}{3} |Y_i| + \frac{4}{3} |Z_i| + \frac{4}{3} |X_i| + \frac{4}{3} |Y_{i+1}| + \frac{4}{3} |Z_{i+1}| + \frac{4}{3} |X_{i+1}| \geq 2\delta
\]

\[
\ell_{i-1} + \frac{4}{3} (\ell_i + \ell_{i+1}) + \ell_{i+2} \geq 2\delta \tag{2.14}
\]

- **Case 3a.** \( i \in S \) and \( i + 1 \notin S \)

\[
\begin{align*}
X_i & \quad Y_{i+1} \\
|N(x_i)| \cup N(y_i) \cup N(z_i) | & \geq 3\delta - \frac{l_{i+1}}{2} - \frac{l_{i+1}}{2} - (l_i + l_{i+2}) \\
Z_{i+1} &
\end{align*}
\]

\[
\ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2} \geq 3\delta - \ell_{i+1} - \ell_i - \ell_{i+2}
\]

Simplifying, we have

\[\ell_{i-1} + 2\ell_i + 2\ell_{i+1} + 2\ell_{i+2} \geq 3\delta \tag{2.15}\]
• Case 3b. Alternative of Case 3a

\[
\begin{align*}
X_i & \quad Y_{i+1} \\
Z_{i+1} & \quad |N(x_i)| \geq \delta \Rightarrow \ell_{i-1} + \ell_{i+1} \geq \delta \\
& \quad |N(y_{i+1})| \geq \delta \Rightarrow \ell_i + \ell_{i+2} + |Z_{i+1}| \geq \delta \\
& \quad |N(z_{i+1})| \geq \delta \Rightarrow \ell_i + \ell_{i+2} + |Y_{i+1}| \geq \delta
\end{align*}
\]

Adding the first inequality by half of the last two inequalities, we have

\[
\ell_{i-1} + \ell_i + \frac{3}{2}\ell_{i+1} + \ell_{i+2} \geq 2\delta \quad (2.16)
\]

• Case 4a. \(i \notin S\) and \(i + 1 \in S\)

\[
\begin{align*}
X_i & \quad Z_{i+1} \\
Y_{i+1} & \quad |N(x_i) \cup N(y_i) \cup N(z_i)| \geq 3\delta - \frac{l_i}{2} - \frac{l_i}{2} - (l_{i-1} + l_{i+1}) \\
& \quad \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2} \geq 3\delta - \ell_{i-1} - \ell_i - \ell_{i+1}
\end{align*}
\]

Simplifying, we have

\[
2\ell_{i-1} + 2\ell_i + 2\ell_{i+1} + \ell_{i+2} \geq 3\delta \quad (2.17)
\]

• Case 4b. Alternative of Case 4a

\[
\begin{align*}
X_i & \quad Z_{i+1} \\
Y_i & \quad |N(x_i)| \geq \delta \Rightarrow \ell_{i-1} + |Y_i| \geq \delta \\
& \quad |N(y_i)| \geq \delta \Rightarrow \ell_{i-1} + \ell_{i+1} + |X_i| \geq \delta \\
& \quad |N(z_{i+1})| \geq \delta \Rightarrow \ell_i + \ell_{i+2} \geq \delta
\end{align*}
\]

Adding the first inequality by half of the last two inequalities, we have

\[
\ell_{i-1} + \frac{3}{2}\ell_i + \ell_{i+1} + \ell_{i+2} \geq 2\delta \quad (2.18)
\]

Let \(J\) denote the number of changes from single to non-single and vice versa, parsing through the \(D + 1\) layers. Summing up (2.13), (2.14), and half of (2.15), (2.17), along consecutive pairs of the layers, one obtains that

\[
4n + \sum_{i,j \in S, i \neq j} \frac{1}{3} \ell_i - \sum_{i \in S, i \neq S} \frac{1}{2} \ell_i - \sum_{i \in S, i \neq S} \frac{1}{2} \ell_i \geq \left(2D - \frac{J}{2}\right)\delta + O(\delta). \quad (2.19)
\]

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Alternatively, adding up (2.13), (2.14), (2.16), (2.18) we obtain

\[ 4n + \sum_{i \in S, j \in S, i \neq j} \frac{1}{3} \ell_i + \sum_{i+1 \in S, i \notin S} \frac{1}{2} \ell_i + \sum_{i-1 \in S, i \notin S} \frac{1}{2} \ell_i \geq 2D\delta + O(\delta). \]  (2.20)

2.6.3 Sieve by Three Consecutive Layers

We are going to give lower bounds to

\[ 2(\ell_{i-2} + \ell_{i-1} + \ell_i + \ell_{i+1} + \ell_{i+2}) = 2|L_{i-2} \cup L_{i-1} \cup L_i \cup L_{i+1} \cup L_{i+2}| \]  (2.21)

using inclusion-exclusion, based on a case analysis of the color content of \( L_{i-1}, L_i, L_{i+1} \).

Consider the following cases (dotted clumps might or might not exist):

- **Case 1a.** \( i - 1 \notin S, i \notin S, i + 1 \notin S \), same colors in \( L_{i-1} \) and \( L_{i+1} \)

  \[ |N(y_{i-1}) \cup N(z_i) \cup N(x_{i+1})| \geq 3\delta - \frac{\ell_{i-1}}{2} - \ell_i - \frac{\ell_{i+1}}{2} \]

  \[ |N(x_{i-1}) \cup N(z_i) \cup N(y_{i+1})| \geq 3\delta - \frac{\ell_{i-1}}{2} - \ell_i - \frac{\ell_{i+1}}{2} \]

  Combining two inequalities above we have

  \[ (2.21) \geq 6\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1}. \]  (2.22)

- **Case 1b.** \( i - 1 \notin S, i \notin S, i + 1 \notin S \), distinct colors between \( L_{i-1} \) and \( L_{i+1} \)

  \[ |N(x_{i-1}) \cup N(z_i) \cup N(y_{i+1})| \geq 3\delta - \frac{\ell_{i-1}}{2} - \frac{\ell_i}{2} - \frac{\ell_{i+1}}{2} \]

  \[ |N(y_{i-1}) \cup N(x_i) \cup N(z_{i+1})| \geq 3\delta - \frac{\ell_{i-1}}{2} - \frac{\ell_i}{2} - \frac{\ell_{i+1}}{2} \]

  Combining two inequalities above we have

  \[ (2.21) \geq 6\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1}. \]  (2.23)
• Case 2. Special case of Case 1a

\[ X_{i-1} \quad X_{i+1} \]
\[ Y_{i-1} \quad Y_{i+1} \]
\[ Z_i \]

|N(y_{i-1}) \cup N(z_i) \cup N(x_{i+1})| \geq 3\delta - \frac{\ell_i - 1}{2} - \ell_i - \frac{\ell_{i+1}}{2}

|N(x_{i-1}) \cup N(z_i) \cup N(y_{i+1})| \geq 3\delta - \frac{\ell_{i-1}}{2} - \ell_i - \frac{\ell_{i+1}}{2}

Combining two inequalities above we have

\[ (2.21) \geq 6\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1}. \] \hspace{1cm} (2.24)

• Case 3.

\[ X_{i-1} \]
\[ Y_i \]
\[ Z_{i+1} \]

|N(x_{i-1}) \cup N(z_i) \cup N(z_{i+1})| \geq 3\delta - \ell_i

Multiplying the inequality by two we have

\[ (2.21) \geq 6\delta - 2\ell_i. \] \hspace{1cm} (2.25)

• Case 4.

\[ X_{i-1} \quad X_{i+1} \]
\[ Y_i \]
\[ Z_{i+1} \]

|N(x_{i-1}) \cup N(y_i) \cup N(x_{i+1})| \geq 3\delta - \ell_i - |Z_{i+1}|

|N(x_{i-1}) \cup N(y_i) \cup N(z_{i+1})| \geq 3\delta - \ell_i - |X_{i+1}|

Combining two inequalities above we have

\[ (2.21) \geq 6\delta - 2\ell_i - (|X_{i+1}| + |Z_{i+1}|) \]
\[ \geq 6\delta - 2\ell_i - (|X_{i+1}| + |Y_{i+1}| + |Z_{i+1}|) \]
\[ \geq 6\delta - 2\ell_i - \ell_{i+1}. \] \hspace{1cm} (2.26)
• Case 5.

\[ \begin{align*}
X_{i-1} & \quad X_{i+1} \\
Y_i & \\
Z_{i-1}
\end{align*} \]

\(|N(x_{i-1}) \cup N(y_i) \cup N(x_{i+1})| \geq 3\delta - \ell_i - |Z_{i-1}|

|N(z_{i-1}) \cup N(y_i) \cup N(x_{i+1})| \geq 3\delta - \ell_i - |X_{i-1}|

Combining two inequalities above we have

\[(2.21) \geq 6\delta - 2\ell_i - (|X_{i-1}| + |Z_{i-1}|) \geq 6\delta - 2\ell_i - (|X_{i-1}| + |Y_{i-1}| + |Z_{i-1}|) \geq 6\delta - \ell_{i-1} - 2\ell_i. \quad (2.27)\]

• Case 6a.

\[ \begin{align*}
X_{i-1} & \quad X_{i+1} \\
Y_i & \\
Z_i
\end{align*} \]

\(|N(x_{i-1}) \cup N(y_i) \cup N(x_{i+1})| \geq 3\delta - 2|Z_i| - |Y_i|

|N(x_{i-1}) \cup N(z_i) \cup N(x_{i+1})| \geq 3\delta - |Z_i| - 2|Y_i|

Combining two inequalities above we have

\[(2.21) \geq 6\delta - 3(|Y_i| + |Z_i|) \geq 6\delta - 3(|X_i| + |Y_i| + |Z_i|) \geq 6\delta - 3\ell_i. \quad (2.28)\]

• Case 6b. Alternative of Case 6a

\[ \begin{align*}
X_{i-1} & \quad X_{i+1} \\
Y_i & \\
Z_i
\end{align*} \]

\[ \left| \bigcup_{j=i-2}^{i+2} L_j \right| \geq |N(x_{i-1}) \cup N(x_{i+1}) \cup N(y_i) \cup N(z_i)| \geq |N(x_{i-1}) \cup N(x_{i+1})| + |N(y_i) \cup N(z_i)| - |(N(x_{i-1}) \cup N(x_{i+1})) \cap (N(y_i) \cup N(z_i))| \geq (2\delta - \ell_i) + (2\delta - \ell_{i-1} - \ell_{i+1}) - \ell_i = 4\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1} \]

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Multiplying the inequality by two we have

\[
8 \delta - 4 \ell_i - 2 \ell_{i-1} - 2 \ell_{i+1} \leq \frac{2}{2}\]

\[
= (6 \delta - 2 \ell_i - \ell_{i-1} - \ell_{i+1}) + (2 \delta - 2 \ell_i - \ell_{i-1} - \ell_{i+1}) . \tag{2.29}
\]

- **Case 7a.**

\[
\begin{align*}
X_{i-1} & \quad X_{i+1} \\
Y_{i-1} & \quad Y_{i+1} \quad |N(x_{i-1}) \cup N(y_i) \cup N(x_{i+1})| \geq 3 \delta - 2|Z_i| - |Y_i| - |Z_{i+1}| \\
Z_{i-1} & \quad Z_{i+1} \quad |N(x_{i-1}) \cup N(z_i) \cup N(x_{i+1})| \geq 3 \delta - 2|Y_i| - |Z_i| - |Y_{i+1}|
\end{align*}
\]

Combining two inequalities above we have

\[
\frac{2}{2}\leq 6 \delta - 3(|Y_i| + |Z_i|) - (|Y_{i+1}| + |Z_{i+1}|) \geq 6 \delta - 3(|X_i| + |Y_i| + |Z_i|) - (|X_{i+1}| + |Y_{i+1}| + |Z_{i+1}|) \]

\[
= 6 \delta - 3 \ell_i - \ell_{i+1} \tag{2.30}
\]

- **Case 7b. Alternative of Case 7a**

\[
\begin{align*}
\left| \bigcup_{j=i-2}^{i+2} L_j \right| \geq |N(x_{i-1}) \cup N(x_{i+1}) \cup N(y_i) \cup N(z_i)| \\
& \geq |N(x_{i-1}) \cup N(x_{i+1})| + |N(y_i) \cup N(z_i)| \\
& \quad - |(N(x_{i-1}) \cup N(x_{i+1})) \cap (N(y_i) \cup N(z_i))| \\
& \geq (2 \delta - \ell_i) + (2 \delta - \ell_{i-1} - |X_{i+1}|) \\
& \quad - (\ell_i + |L_{i+1} \setminus X_{i+1}|) \\
& = 4 \delta - 2 \ell_i - \ell_{i-1} - \ell_{i+1}
\end{align*}
\]

Multiplying the inequality by two we have

\[
\frac{2}{2}\leq 8 \delta - 4 \ell_i - 2 \ell_{i-1} - 2 \ell_{i+1} \leq \frac{2}{2}\]

\[
= (6 \delta - 2 \ell_i - \ell_{i-1} - \ell_{i+1}) + (2 \delta - 2 \ell_i - \ell_{i-1} - \ell_{i+1}) . \tag{2.31}
\]
Case 8a.

\[ |N(x_{i-1}) \cup N(y_i) \cup N(x_{i+1})| \geq 3\delta - |Z_{i-1}| - |Y_i| - 2|Z_i| \]

\[ |N(x_{i-1}) \cup N(z_i) \cup N(x_{i+1})| \geq 3\delta - |Y_{i-1}| - |Z_i| - 2|Y_i| \]

Combining two inequalities above we have

\[ (2.21) \geq 6\delta - 3(|Y_i| + |Z_i|) - (|Y_{i-1}| + |Z_{i-1}|) \]

\[ \geq 6\delta - 3(|X_i| + |Y_i| + |Z_i|) - (|X_{i-1}| + |Y_{i-1}| + |Z_{i-1}|) \]

\[ = 6\delta - 3\ell_i - \ell_{i-1} \quad (2.32) \]

Case 8b. Alternative of Case 8a

\[ \left| \bigcup_{j=i-2}^{i+2} L_j \right| \geq |N(x_{i-1}) \cup N(x_{i+1}) \cup N(y_i) \cup N(z_i)| \]

\[ \geq |N(x_{i-1}) \cup N(x_{i+1})| + |N(y_i) \cup N(z_i)| - |(N(x_{i-1}) \cup N(x_{i+1})) \cap (N(y_i) \cup N(z_i))| \]

\[ \geq (2\delta - \ell_i) + (2\delta - \ell_{i+1} - |X_{i-1}|) \]

\[ - (\ell_i + |L_{i-1} \setminus X_{i-1}|) \]

\[ = 4\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1} \]

Multiplying the inequality by two we have

\[ (2.21) \geq 8\delta - 4\ell_i - 2\ell_{i-1} - 2\ell_{i+1} \]

\[ = (6\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1}) + (2\delta - 2\ell_i - \ell_{i-1} - \ell_{i+1}). \quad (2.33) \]

Summing (2.22), (2.23), (2.24), (2.25), (2.26), (2.27), (2.28), (2.30), (2.32) across \( D + 1 \) layers, one obtains that

\[ 10n \geq 6\delta(D - 1) - 4n + O(\delta) - \sum_{i \in S} \ell_i \]

\[ + \sum_{i \in S} \ell_i \times [\#\text{single neighbors of } i + \#\text{singular triplets containing } i] \quad (2.34) \]
Alternatively, summing (2.22), (2.23), (2.24), (2.25), (2.26), (2.27), (2.29), (2.31), (2.33) across $D + 1$ layers, with $s$ being the number of layers containing exactly one clump, one obtains that:

$$10n \geq (6D + 2s)\delta - 4n - \sum_{i \in S} 2\ell_i - \sum_{i,i+1 \not\in S} \ell_i - \sum_{i,i+2 \not\in S} \ell_i$$

$$- \sum_{i \in S} \ell_i \times [\# \text{non-single neighbors of } i - \# \text{single neighbors of } i] \quad (2.35)$$

The inequalities (2.12), (2.19), (2.20), (2.34), and (2.35) are the building blocks of the constraints for our linear programming problem. We first define the decision variables by grouping together patterns, which details are given in Section 2.7.

### 2.7 Optimization

Recall that $\phi = \frac{D\delta}{n}$. For convenience we define $\psi = \frac{\delta s}{n}$ and $\gamma = \frac{\delta J}{n}$. Next we introduce variables for the ratio of the number of vertices that belong to certain type of layer:

- $\mu := \frac{1}{n} \sum_{i : c(i) = 1} \ell_i$
- $\nu := \frac{1}{n} \sum_{i : c(i) = 2, \ (i-1,i+1) \not\in S \neq \emptyset} \ell_i$
- $\alpha_1 := \frac{1}{n} \sum_{i : c(i) = 2, c(i+2), c(i+1) = 1} \ell_i$
- $\alpha_2 := \frac{1}{n} \sum_{i : c(i) = 2, c(i-1) = 1, c(i+2) = 1} \ell_i + \frac{1}{n \ c(i-1) = 2, c(i-2) = 1} \ell_i$
- $\alpha_3 := \frac{1}{n} \sum_{i : c(i) = 2, c(i-1) = 1, c(i+1) = 2, c(i+2) = 2} \ell_i + \frac{1}{n \ c(i-1) = 2, c(i-2) = 1} \ell_i$
Furthermore, \( \alpha_1 = \alpha_1' + \alpha_1'' + \alpha_1''' \), where:

\[
\alpha_1 := \frac{1}{n} \sum_{c(i) = 2, c(i-1) = c(i+1) = 1} \ell_i \\
\text{or} \\
\alpha_1'' := \frac{1}{n} \sum_{c(i) = 2, c(i-1) = c(i+1) = 1} \ell_i + \frac{1}{n} \sum_{c(i) = 2, c(i-2) = c(i+2) = 1} \ell_i \\
\alpha_1''' := \frac{1}{n} \sum_{c(i) = 2, c(i-2) = c(i+2) = 1} \ell_i
\]

By definition, all variables are non-negative and clearly \( \nu = \alpha_1 + \alpha_2 + \alpha_3 \).

Now we finally build constraints from (2.12), (2.19), (2.20), (2.34), and (2.35).

From (2.12):

\[
6n \geq 2D\delta + \sum_{c(i) = 2} \ell_i + \sum_{c(i) = 1} \ell_i + O(\delta)
\]

\[
6n \geq 2D\delta + n\nu + 2n\mu + O(\delta)
\]

\[
3 \geq \frac{D\delta}{n} + \frac{\nu}{2} + \mu + O\left(\frac{\delta}{n}\right) = \phi + \frac{\nu}{2} + \mu + O\left(\frac{\delta}{n}\right)
\]

\[
\phi \leq 3 - \frac{\nu}{2} - \mu + O\left(\frac{\delta}{n}\right)
\]

Combining with \( \nu = \alpha_1 + \alpha_2 + \alpha_3 \) and \( \alpha_1 = \alpha_1' + \alpha_1'' + \alpha_1''' \) we have

\[
2\phi + \nu + 2\mu \leq 6 + O\left(\frac{\delta}{n}\right)
\]

\[
2\phi + 2\mu + \alpha_1' + \alpha_1'' + \alpha_1''' + \alpha_2 + \alpha_3 \leq 6 + O\left(\frac{\delta}{n}\right)
\]  

(2.36)

From (2.34):

\[
10n \geq 6\delta(D - 1) - 4n + O(\delta) - \sum_{c(i) = 1} \ell_i + \sum_{c(i) = 1} \ell_i \times \left[\ldots\right] \\
14n \geq 6D\delta - n\nu - 2n\mu + O(\delta)
\]

\[
\frac{14}{6} \geq \phi - \nu - 2\mu + O\left(\frac{\delta}{n}\right)
\]

\[
\phi \leq \frac{1}{6} \left(14 + \nu - 2\mu\right) + O\left(\frac{\delta}{n}\right)
\]

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Combining with $\nu = \alpha_1 + \alpha_2 + \alpha_3$ and $\alpha_1 = \alpha'_1 + \alpha''_1 + \alpha'''_1$ we have

$$6\phi - \nu + 2\mu \leq 14 + O\left(\frac{\delta}{n}\right)$$

$$6\phi + 2\mu - \alpha'_1 - \alpha''_1 - \alpha'''_1 - \alpha_2 - \alpha_3 \leq 14 + O\left(\frac{\delta}{n}\right) \quad (2.37)$$

For constraints derived from (2.19), (2.20), and (2.35), first observe that

$$\frac{1}{n} \sum_i \ell_i = 1 \quad \text{so} \quad (1 - \mu - \nu) = \sum_{c(i) \geq 3} \ell_i$$

From (2.35):

$$10n \geq 6D\delta + 2s\delta - 4n - \sum_{c(i) = 1} \ell_i \times \ldots - \sum_{c(i), c(i+1) = 1} \ell_i - \sum_{c(i), c(i-1) = 1} \ell_i + O(\delta)$$

$$14n \geq 6D\delta + 2s\delta - 2n\mu - 2n\mu - 2n\alpha_2 - 2n(1 - \mu - \nu) + O(\delta)$$

$$14 \geq 6\phi + 2\psi - 2\nu - 2\mu - \alpha_2 - 2(1 - \mu - \nu) + O\left(\frac{\delta}{n}\right)$$

For convenience write it as

$$6\phi + 2\psi - \alpha_2 \leq 16 + O\left(\frac{\delta}{n}\right) \quad (2.38)$$

From (2.20):

$$4n + \sum_{c(i) = 1} \frac{1}{3} \ell_i + \sum_{c(i), c(i+1) = 1} \frac{1}{2} \ell_i + \sum_{c(i), c(i-1) = 1} \frac{1}{2} \ell_i \geq 2D\delta + O(\delta)$$

$$4n + \frac{1}{3} n(\alpha_2 + \alpha_3) + \frac{2}{3} n(1 - \mu - \nu) + \frac{1}{2} n(\alpha_2 + \alpha_3) + n\alpha_1 \geq 2D\delta + O(\delta)$$

$$4n + \frac{5}{6} n(\alpha_2 + \alpha_3) + \frac{2}{3} n(1 - \mu - \nu) + n\alpha_1 \geq \phi + O(\delta)$$

$$2 + \frac{5}{12}(\alpha_2 + \alpha_3) + \frac{1}{3}(1 - \mu - \nu) + \frac{\alpha_1}{2} \geq \phi + O\left(\frac{\delta}{n}\right)$$

Multiplying the last inequality by six, then combining with $\nu = \alpha_1 + \alpha_2 + \alpha_3$ and $\alpha_1 = \alpha'_1 + \alpha''_1 + \alpha'''_1$ we have

$$24 + 5(\alpha_2 + \alpha_3) + 6(\alpha'_1 + \alpha''_1 + \alpha'''_1) + 4 - 4(\alpha_1 + \alpha_2 + \alpha_3) - 4\mu \geq 12\phi + O\left(\frac{\delta}{n}\right)$$

$$28 - 4\mu + 2(\alpha'_1 + \alpha''_1 + \alpha'''_1) + (+\alpha_2 + \alpha_3) \geq 12\phi + O\left(\frac{\delta}{n}\right)$$

$$12\phi + 4\mu - 2\alpha'_1 - 2\alpha''_1 - 2\alpha'''_1 - \alpha_2 - \alpha_3 \leq 28 + O\left(\frac{\delta}{n}\right) \quad (2.39)$$
From (2.19):

\[
4n + \sum_{\substack{(i, j) \in S \\ j \neq S, |i - j| = 1}} \frac{1}{3} \ell_i - \sum_{i \in 1 + kS} \frac{1}{2} \ell_i - \sum_{i \in 2 + kS} \frac{1}{2} \ell_i \geq \left(2D - \frac{J}{2}\right) \delta + O(\delta) \tag{2.40}
\]

where

\[
A = \frac{1}{n} \left( \sum_{(i, j) \in S} \frac{1}{3} \ell_i \right) \leq \frac{2}{3} (1 - \nu - \mu) + \frac{1}{3} (\alpha_2 + \alpha_3). \tag{2.41}
\]

\[
B = \frac{1}{n} \left( - \sum_{i \in kS} \frac{1}{2} \ell_i - \sum_{i \in 1 + kS} \frac{1}{2} \ell_i \right) \geq -\mu - \sum_{i \in 1 + kS} \frac{1}{2} \ell_i - \sum_{i \in 2 + kS} \frac{1}{2} \ell_i = -\mu - \sum_{i \in kS} \frac{1}{2} \ell_i - \sum_{i \in 1 + kS} \frac{1}{2} \ell_i + \sum_{i \in 1 + kS} \frac{1}{2} \ell_i + \sum_{i \in 2 + kS} \frac{1}{2} \ell_i \tag{2.42}
\]

\[
\frac{B}{n} = -\mu - \frac{\nu + \alpha_1}{2} + \alpha_1'' + \frac{1}{2} \alpha_1'''.
\]

Substituting (2.41) and (2.42) into \(\frac{1}{n}\) times (2.40) we have

\[
4 + \frac{2}{3} (1 - \nu - \mu) + \frac{1}{3} (\alpha_2 + \alpha_3) - \mu - \frac{\nu + \alpha_1}{2} + \alpha_1'' + \frac{1}{2} \alpha_1''' + O \left(\frac{\delta}{n}\right) \geq 2\phi + \frac{\gamma}{2}. \tag{2.43}
\]

Multiplying the last inequality by six, then combining with \(\nu = \alpha_1 + \alpha_2 + \alpha_3\) and \(\alpha_1 = \alpha_1' + \alpha_1'' + \alpha_1'''\) we have

\[
24 + 4 - 4\mu + 2(\alpha_2 + \alpha_3) - 6\mu - 3\nu - 3\alpha_1 + 6\alpha_1'' + 3\alpha_1''' \geq 12\phi - 3\gamma + O \left(\frac{\delta}{n}\right) \tag{2.44}
\]
Summarizing, we have our LP problem:

Maximize \( \phi = \frac{D\delta}{n} \)
subject to

\[
\begin{align*}
2\phi + 2\mu + \alpha_1' + \alpha_1'' + \alpha_1''' + \alpha_2 + \alpha_3 & \leq 6 + O\left(\frac{\delta}{n}\right) \\
6\phi + 2\mu - \alpha_1' - \alpha_1'' - \alpha_1''' - \alpha_2 - \alpha_3 & \leq 14 + O\left(\frac{\delta}{n}\right) \\
6\phi + 2\psi - \alpha_2 & \leq 16 + O\left(\frac{\delta}{n}\right) \\
12\phi + 4\mu - 2\alpha_1' - 2\alpha_1'' - 2\alpha_1''' - \alpha_2 - \alpha_3 & \leq 28 + O\left(\frac{\delta}{n}\right) \\
12\phi + 10\mu - 3\gamma + 10\alpha_1' + 7\alpha_1'' + 4\alpha_1''' + 5\alpha_2 + 5\alpha_3 & \leq 28 + O\left(\frac{\delta}{n}\right) \\
\phi, \mu, \nu, \gamma, \psi, \alpha_1', \alpha_1'', \alpha_1''', \alpha_2, \alpha_3 & \geq 0
\end{align*}
\]

For convenience rename variables:

\[
\begin{align*}
x_1 & = \phi & x_4 & = \gamma & x_7 & = \alpha_1''' \\
x_2 & = \mu & x_5 & = \alpha_1' & x_8 & = \alpha_2 \\
x_3 & = \psi & x_6 & = \alpha_1'' & x_9 & = \alpha_3
\end{align*}
\]

We first ignore the error term \( O\left(\frac{\delta}{n}\right) \), and rewrite the LP problem before running the simplex tables.

Maximize \( x_1 \)
Subject to

\[
\begin{align*}
2x_1 + 2x_2 + x_5 + x_6 + x_7 + x_8 + x_9 & \leq 6 \\
6x_1 + 2x_2 - x_5 - x_6 - x_7 - x_8 - x_9 & \leq 14 \\
6x_1 + 2x_3 - x_8 & \leq 16 \\
12x_1 + 4x_2 - 2x_5 - 2x_6 - 2x_7 - x_8 - x_9 & \leq 28 \\
12x_1 + 10x_2 - 3x_4 + 10x_5 + 7x_6 + 4x_7 + 5x_8 + 5x_9 & \leq 28 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 & \geq 0
\end{align*}
\]
The optimal solution is \( x_1 = \frac{5}{2} \), obtained when

\[
\begin{align*}
x_1 &= \frac{5}{2} & x_4 &= 2 & x_7 &= 1 \\
x_2 &= 0 & x_5 &= 0 & x_8 &= 0 \\
x_3 &= 0 & x_6 &= 0 & x_9 &= 0
\end{align*}
\]

This result does not give us a better bound, as for a 3-colorable graph \( G \), the bound \( \text{diam}(G) \leq \frac{5}{2} n + O(1) \) is already proved in Theorem 2.12. Hence we need more constraints.

As a different approach to obtain more constraints, first fix an \( \varepsilon \) where \( 0 < \varepsilon < 2\delta \). Let \( \alpha_1 \) is as previously defined. Let \( D_1 \) be the number of columns of type \( \alpha_1 \) such that \( \ell_i \geq 2\delta - \varepsilon \), \( D_2 \) be the number of columns of type \( \alpha_1 \) such that \( \ell_i < 2\delta - \varepsilon \), \( D_3 \) be the number of columns that is not \( D_1 \) or \( D_2 \), with its neighbours. By definitions:

\[
\text{diam}(G) \leq 3(D_1 + D_2) + D_3 \tag{2.45}
\]

Also, the \( D_3 \) columns and their neighbours satisfy minimum degree condition:

\[
\delta \leq \ell_{i-1} + \ell_i + \ell_{i+1}
\]

\[
\delta D_3 \leq \sum_{D_3} (\ell_{i-1} + \ell_i + \ell_{i+1}) \leq 3n(1 - \alpha_1)
\]

\[
D_3 \leq 3(1 - \alpha_1) \cdot \frac{n}{\delta} \tag{2.46}
\]

Relate \( D_1, D_2, D_3 \) with \( \psi \) and \( \gamma \), we can observe that

\[
\psi = \frac{\delta s}{n} \geq (D_1 + D_2) \cdot \frac{\delta}{n} \quad \text{and} \quad \gamma = \frac{\delta J}{n} = 2(D_1 + D_2) \cdot \frac{\delta}{n} \tag{2.47}
\]

Combining (2.45), (2.46), and (2.47) we have:

\[
\begin{align*}
D &\leq 3(D_1 + D_2) + D_3 \\
\frac{n}{\delta} &\leq \frac{3}{2} \gamma + 3(1 - \alpha_1) \frac{n}{\delta} \\
\phi &\leq \frac{3}{2} \gamma + 3(1 - \alpha_1) \\
2\phi &\leq 3\gamma + 6 - 6\alpha_1
\end{align*}
\]
These two inequalities add to the constraints (2.44). Combining the constraints we have the updated linear programming problem:

Maximize $\phi = \frac{D\delta}{n}$

subject to

\[
\begin{align*}
2\phi + 2\mu + \alpha_1' + \alpha_1'' + \alpha_1''' + \alpha_2 + \alpha_3 & \leq 6 + O\left(\frac{\delta}{n}\right) \\
6\phi + 2\mu - \alpha_1' - \alpha_1'' - \alpha_1''' - \alpha_2 - \alpha_3 & \leq 14 + O\left(\frac{\delta}{n}\right) \\
6\phi + 2\psi - \alpha_2 & \leq 16 + O\left(\frac{\delta}{n}\right) \\
12\phi + 10\mu - 3\gamma - 10\alpha_1' + 7\alpha_1'' + 4\alpha_1''' + 5\alpha_2 + 5\alpha_3 & \leq 28 + O\left(\frac{\delta}{n}\right) \\
2\phi - 3\gamma + 6\alpha_1' + 6\alpha_1'' + 6\alpha_1''' & \leq 6 + O\left(\frac{\delta}{n}\right) \\
\phi - 3\psi + 3\alpha_1' + 3\alpha_1'' + 3\alpha_1''' & \leq 3 + O\left(\frac{\delta}{n}\right) \\
\phi, \mu, \nu, \gamma, \psi, \alpha_1', \alpha_1'', \alpha_1''', \alpha_2, \alpha_3 & \geq 0
\end{align*}
\]

Before compiling the LP problem, we checked using Maple the importance of each constraint and found that some constraints are redundant. The code is given in Section \[A.2\]. Removing all redundant constraints, we finalized our LP problem as follows:

Maximize $\phi = \frac{D\delta}{n}$

subject to

\[
\begin{align*}
2\phi + 2\mu + \alpha_1' + \alpha_1'' + \alpha_1''' + \alpha_2 + \alpha_3 & \leq 6 + O\left(\frac{\delta}{n}\right) \\
6\phi + 2\psi - \alpha_2 & \leq 16 + O\left(\frac{\delta}{n}\right) \\
12\phi + 4\mu - 2\alpha_1' - 2\alpha_1'' - 2\alpha_1''' - \alpha_2 - \alpha_3 & \leq 28 + O\left(\frac{\delta}{n}\right) \\
\phi - 3\psi + 3\alpha_1' + 3\alpha_1'' + 3\alpha_1''' & \leq 3 + O\left(\frac{\delta}{n}\right) \\
\phi, \mu, \nu, \gamma, \psi, \alpha_1', \alpha_1'', \alpha_1''', \alpha_2, \alpha_3 & \geq 0
\end{align*}
\]
Renaming the variables and ignoring the error term $O\left(\frac{\delta}{n}\right)$ as before, we have:

Maximize $x_1$

Subject to

\begin{align*}
2x_1 + 2x_2 + x_5 + x_6 + x_7 + x_8 + x_9 & \leq 6 \\
6x_1 + 2x_3 - x_8 & \leq 16 \\
12x_1 + 4x_2 - 2x_5 - 2x_6 - 2x_7 - x_8 - x_9 & \leq 28 \\
x_1 - 3x_3 + 3x_5 + 3x_6 + 3x_7 & \leq 3 \\
x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9 & \geq 0
\end{align*}

Adding slack variables $s_1, s_2, s_3, s_4$, we obtain the standard form:

Maximize $x_1$

Subject to

\begin{align*}
2x_1 + 2x_2 + x_5 + x_6 + x_7 + x_8 + x_9 + s_1 & = 6 \\
6x_1 + 2x_3 - x_8 + s_2 & = 16 \\
12x_1 + 4x_2 - 2x_5 - 2x_6 - 2x_7 - x_8 - x_9 + s_3 & = 28 \\
x_1 - 3x_3 + 3x_5 + 3x_6 + 3x_7 + s_4 & = 3
\end{align*}

Utilizing the open source online tool “PHPSimplex” (http://www.phpsimplex.com/en/) the process is described in Table 2.4 – Table 2.8

Table 2.4: Tableau 1: $x_1$ entering, $s_3$ leaving.

<table>
<thead>
<tr>
<th>T.1</th>
<th>$c_i$</th>
<th>$b$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_3$</th>
<th>$s_4$</th>
<th>$s_1$</th>
</tr>
</thead>
<tbody>
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<td>0</td>
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<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
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<td>6</td>
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<td>2</td>
<td>0</td>
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<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
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<td>0</td>
</tr>
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<td>12</td>
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<td>0</td>
<td>0</td>
<td>-2</td>
<td>-2</td>
<td>-2</td>
<td>-1</td>
<td>-1</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_4$</td>
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<td>0</td>
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<td>0</td>
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<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
</tr>
<tr>
<td>$z$</td>
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<td>-1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
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<td>0</td>
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<td>0</td>
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<td>0</td>
</tr>
</tbody>
</table>

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Table 2.5: Tableau 2, after 1 pivoting: $x_5$ entering, $s_4$ leaving.

<table>
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<tr>
<th>$T.2$</th>
<th>$c_i$</th>
<th>$b$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
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<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_1$</th>
<th>$s_4$</th>
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<tbody>
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<td>$s_1$</td>
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<td>0</td>
<td>4/3</td>
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<td>0</td>
<td>4/3</td>
<td>4/3</td>
<td>7/6</td>
<td>7/6</td>
<td>1</td>
<td>0</td>
<td>-1/6</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>$s_2$</td>
<td>0</td>
<td>2</td>
<td>0</td>
<td>-2</td>
<td>2</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
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<td>-1/2</td>
<td>0</td>
<td>1</td>
<td>-1/2</td>
<td>0</td>
</tr>
<tr>
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<td>1/3</td>
<td>0</td>
<td>0</td>
<td>-1/6</td>
<td>-1/6</td>
<td>-1/6</td>
<td>-1/12</td>
<td>-1/12</td>
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<td>0</td>
<td>1/12</td>
<td>0</td>
</tr>
<tr>
<td>$s_4$</td>
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<td>-1/3</td>
<td>-3</td>
<td>0</td>
<td>19/6</td>
<td>19/6</td>
<td>19/6</td>
<td>1/12</td>
<td>1/12</td>
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<td>0</td>
<td>-1/12</td>
<td>1</td>
</tr>
<tr>
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<td>-1/6</td>
<td>-1/6</td>
<td>-1/6</td>
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<td>-1/12</td>
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<td>1/12</td>
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</tr>
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</table>

Table 2.6: Tableau 3, after 2 pivoting: $x_3$ entering, $s_2$ leaving.

<table>
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<th>$c_i$</th>
<th>$b$</th>
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<th>$x_9$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_1$</th>
<th>$s_4$</th>
</tr>
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<td>1</td>
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<td>-5/38</td>
<td>-8/19</td>
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<tr>
<td>$s_2$</td>
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<td>-1/38</td>
<td>6/19</td>
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<td>6/19</td>
<td>-3/19</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3/38</td>
<td>-3/38</td>
<td>0</td>
<td>0</td>
<td>3/38</td>
<td>1/19</td>
<td></td>
</tr>
</tbody>
</table>

Table 2.7: Tableau 4, after 3 pivoting: $x_8$ entering, $s_1$ leaving.

<table>
<thead>
<tr>
<th>$T.4$</th>
<th>$c_i$</th>
<th>$b$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$x_9$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_1$</th>
<th>$s_4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$s_1$</td>
<td>0</td>
<td>2/7</td>
<td>0</td>
<td>16/7</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>19/14</td>
<td>13/14</td>
<td>1</td>
<td>-3/7</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>17/28</td>
<td>0</td>
<td>-9/4</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-5/28</td>
<td>9/56</td>
<td>0</td>
<td>19/56</td>
<td>-9/56</td>
<td>-3/28</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>69/28</td>
<td>1</td>
<td>3/14</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3/28</td>
<td>-3/56</td>
<td>0</td>
<td>3/56</td>
<td>3/56</td>
<td>1/28</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>11/14</td>
<td>0</td>
<td>-5/7</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>-7/7</td>
<td>5/28</td>
<td>0</td>
<td>9/28</td>
<td>-5/28</td>
<td>3/14</td>
</tr>
<tr>
<td>$z$</td>
<td>69/28</td>
<td>0</td>
<td>3/14</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-3/28</td>
<td>-3/56</td>
<td>0</td>
<td>3/56</td>
<td>3/56</td>
<td>1/28</td>
<td></td>
</tr>
</tbody>
</table>
Table 2.8: Tableau 5, after 4 pivoting: An optimal solution is obtained.

<table>
<thead>
<tr>
<th>T.5</th>
<th>$c_i$</th>
<th>$b$</th>
<th>$x_1$</th>
<th>$x_2$</th>
<th>$x_3$</th>
<th>$x_4$</th>
<th>$x_5$</th>
<th>$x_6$</th>
<th>$x_7$</th>
<th>$x_8$</th>
<th>$s_1$</th>
<th>$s_2$</th>
<th>$s_1$</th>
<th>$s_1$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x_8$</td>
<td>0</td>
<td>$\frac{4}{19}$</td>
<td>0</td>
<td>$\frac{32}{19}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>$\frac{13}{19}$</td>
<td>$\frac{14}{19}$</td>
<td>$\frac{6}{19}$</td>
<td>$\frac{1}{19}$</td>
<td>$-\frac{4}{19}$</td>
</tr>
<tr>
<td>$x_3$</td>
<td>0</td>
<td>$\frac{49}{76}$</td>
<td>0</td>
<td>$-\frac{13}{38}$</td>
<td>1</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{43}{152}$</td>
<td>$\frac{5}{38}$</td>
<td>$\frac{13}{152}$</td>
<td>$-\frac{23}{152}$</td>
</tr>
<tr>
<td>$x_1$</td>
<td>1</td>
<td>$\frac{189}{76}$</td>
<td>1</td>
<td>$\frac{15}{38}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{3}{152}$</td>
<td>$\frac{3}{38}$</td>
<td>$\frac{3}{152}$</td>
<td>$\frac{9}{152}$</td>
</tr>
<tr>
<td>$x_5$</td>
<td>0</td>
<td>$\frac{31}{38}$</td>
<td>0</td>
<td>$-\frac{9}{19}$</td>
<td>0</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>0</td>
<td>$\frac{21}{76}$</td>
<td>$\frac{2}{19}$</td>
<td>$\frac{21}{76}$</td>
<td>$-\frac{13}{76}$</td>
</tr>
<tr>
<td>$z$</td>
<td>$\frac{189}{76}$</td>
<td>0</td>
<td>$\frac{15}{38}$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$\frac{3}{152}$</td>
<td>$\frac{3}{38}$</td>
<td>$\frac{3}{152}$</td>
<td>$\frac{9}{152}$</td>
<td>$\frac{1}{76}$</td>
</tr>
</tbody>
</table>

From Table 2.8, we have the optimal solution is $x_1 = \frac{189}{76}$, obtained when

$$
\begin{align*}
  x_1 &= \frac{189}{76} \\
  x_4 &= 0 \\
  x_2 &= 0 \\
  x_5 &= \frac{31}{38} \\
  x_3 &= \frac{49}{76} \\
  x_6 &= 0 \\
  x_7 &= 0 \\
  x_8 &= \frac{4}{19} \\
  x_9 &= 0
\end{align*}
$$

Thus we have $\max \phi = \frac{D\delta}{n} = \frac{189}{76}$, which implies

$$
D = \text{diam}(G) \leq \max \phi \cdot \frac{n}{\delta} = \frac{189}{76} \cdot \frac{n}{\delta}
$$

Since $\frac{189}{76} \approx 2.48684... < 2.5 = \frac{5}{2}$, this optimal solution is an improvement from Theorem 2.12 toward Conjecture 2.13. We restate our main result in Theorem 2.15.

**Theorem 2.15.** For every connected 3-colorable graph $G$ of order $n$ and $\delta \geq 1$,

$$
\text{diam}(G) \leq \frac{189n}{76\delta} + O(1).
$$

Note that $\frac{189}{76}$ is closer to $\frac{5}{2}$ than it is to $\frac{7}{3}$. It is highly likely that one can consider some additional inequalities in order to tighten the bound on Theorem 2.15 toward proving Conjecture 2.13. However, there are enormous number of inequalities that can be obtained from all known information, and determining relevant inequalities among them is not easy. We leave this possibility for further investigation.
CHAPTER 3

SOME REMARKS ON MIDRANGE CROSSING CONSTANT

3.1 History and Progress

Conjecture 3.1 (Erdős-Guy Conjecture). \[5\] Let \(G\) be a graph on \(n\) vertices and \(m\) edges. If \(n \ll m \ll n^2\) then

\[
\min_G \left( \frac{\text{cr}(G) n^2}{m^3} \right)
\]

converges to a positive constant.

Theorem 3.2. \[11\] Let \(\kappa(n,m)\) denotes the minimum crossing number of graphs that have \(n\) vertices and at least \(m\) edges. There exists a constant \(\gamma > 0\), called the midrange crossing constant, such that

\[
\lim_{n \to \infty} \kappa(n,m) \frac{n^2}{m^3}
\]

under the constrains \(\frac{m}{n} \to \infty\) and \(m = o(n^2)\), exists and is equal to \(\gamma\).

The first step towards proved the Erdős-Guy conjecture was the discovery of the Crossing Lemma, whose discovery occurred in the absence of awareness of the Erdős-Guy conjecture. The Crossing Lemma gives a lower bound on the minimum number of crossings of a given graph, as a function of the number of edges and vertices of the graph.

Lemma 3.3 (The Crossing Lemma). \[2\] For a simple graph \(G\) on \(n\) vertices and \(m\) edges such that \(m > 4n\), we have \(\text{cr}(G) \geq \frac{1}{64} \frac{m^3}{n^2}\).
In 2015 Ackerman proved a better lower bound on the cost of strengthen the condition \( m > 4n \) into \( m > 7n \).

**Theorem 3.4.** \([1]\) For a simple graph \( G \) on \( n \) vertices and \( m \) edges such that \( m > 7n \), we have \( \text{cr}(G) \geq \frac{1}{29} \frac{m^3}{n^2} \).

The Crossing Lemma asserted that for \( m > 4n \) we have \( \gamma \geq \frac{1}{64} \). Ackerman’s result, the current best, gives \( \gamma \geq \frac{1}{29} \).

In 1965, Moon \([10]\) observed that selecting \( n \) points on the unit sphere independently according to the uniform distribution, and for any two points, connecting them on the shorter arc of their great circle, the expected number of crossings is \( \left( \frac{1}{m^4} + o(1) \right) n^4 \), which is asymptotically the same as the conjectured crossing number of the complete graph in the Harary-Hill conjecture. This result is truly surprising.

In 1999, Pach, Spencer and Tóth \([11]\) showed that the Erdős-Guy conjecture is true with the additional (and needed) assumption that \( m \) is subquadratic. In 2006, Pach, Radoičić, Tardos and Tóth gave a construction yielding \( \gamma \leq \frac{8}{9n^2} \approx 0.0900633 \) for the rectilinear midrange crossing constant. Their construction was a \( \sqrt{n} \times \sqrt{n} \) grid, with the points slightly moved into general position, so that no three of them are collinear, and they joined the pairs of points with straight line segments if their distance did not exceed some number \( d \). Details of neither of these calculations, which are said to be long and unpleasant, are not available to the public.

In Section 3.2 we provide a simple alternative calculation that yields the same \( \frac{8}{9n^2} \) bound as Pach and Tóth construction. Our calculation uses two ideas. The first idea is that the construction of Pach and Tóth is an imitation of a uniformly distributed large point set, the second is that calculations on the sphere are simpler than calculations on the plane. We restrict the Moon construction by connecting only pairs of points with distance at most \( d \) for some fixed but very small \( d \).
3.2 Calculation

Take two points $P$ and $Q$ independently from the uniform distribution on the unit sphere. The area of the spherical cap with polar angle $\alpha$, WLOG taking point $P$ as the north pole is $2\pi(1 - \cos \alpha)$. Consider the uniform distribution on the sphere. The probability of a set will be the area of the set divided by $4\pi$, the area of the sphere. Hence the distribution of arc length

$$\Pr[\overline{PQ} \leq \alpha] = \frac{2\pi(1 - \cos \alpha)}{4\pi} = \frac{1 - \cos \alpha}{2}.$$  

With differentiation with respect to $\alpha$, we obtain that the density function of the length $\alpha$ of the shorter arc connecting $P$ and $Q$ on their great circle is

$$\frac{1}{2} \sin \alpha \ (0 < \alpha < \pi).$$

Select $R$ and $S$ as well independently from the uniform distribution on the unit sphere. Fixing the great circle of $P$ and $Q$, the probability that $R$ and $S$ fall into different hemispheres is $\frac{1}{2}$. If they fall in the same hemisphere, then $PQ$ and $RS$ arcs do not cross. If they fall in different hemispheres, then the shorter arc in their great circle intersect to a point the great circle of $PQ$. Since the length of the shorter $PQ$ arc is $\alpha$ and the perimeter of the great circle of $PQ$ is $2\pi$, then we have

$$\Pr[PQ \text{ arc crosses } RS \text{ arc} \mid \text{ length of } PQ = \alpha] = \frac{\alpha}{2\pi} = \frac{\alpha}{4\pi}.$$  

Moon [10] shows from here that

$$\Pr[PQ \text{ arc crosses } RS \text{ arc}] = \int_0^\pi \frac{\alpha}{4\pi} \left(\frac{1}{2} \sin \alpha\right) d\alpha = \frac{1}{8},$$

showing that the expected number of crossings in his drawing of the complete graph is at most $\frac{1}{16}\left(\frac{n}{2}\right)^{n-2} = \left(\frac{1}{16} + o(1)\right)n^4$ as he claimed. We generalize these arguments.

Consider the great circles of $P$ and $Q$, and of $R$ and $S$. With probability 1 these two great circles coincide, and hence have two intersection points, $T$ and $U$. 
Furthermore, the probability of the \( PQ \) arc crossing the \( RS \) arc does not depend on conditioning on two fixed great circles. Indeed, fixing two great circles, the length of the \( PQ \) arc as \( \alpha \) and the length of the \( RS \) arc as \( \beta \), the probability that the \( PQ \) arc crosses the \( RS \) arc is

\[
2 \cdot \frac{\alpha}{2\pi} \cdot \frac{\beta}{2\pi}.
\]

The first factor of \( 2 \) comes from deciding whether \( T \) or \( U \) will be the crossing point. Integrating out over arc length up to \( d \), we obtain

\[
\Pr[\text{\( PQ \) arc crosses \( RS \) arc and length of \( PQ \) \( \leq d \) and length of \( RS \) \( \leq d \)}] = \int_0^d \left( \int_0^d \left( 2 \cdot \frac{\alpha}{2\pi} \cdot \frac{\beta}{2\pi} \right) \cdot \frac{1}{2} \sin \alpha \ d\alpha \right) \cdot \frac{1}{2} \sin \beta \ d\beta = \frac{1}{8\pi^2} (\sin d - d \cos d)^2.
\]  

(3.1)

Define now a random graph drawn on the sphere in the following way: The vertices are \( n \) randomly and independently selected samples from the uniform distribution on the unit sphere. Join vertices \( P \) and \( Q \) if the shorter of their great circle arc has length at most \( d \), and represent the edge between them by this arc. Based on (3.1), the expected number of crossings in this drawn graphs is

\[
\frac{1}{8\pi^2} (\sin d - d \cos d)^2 \cdot \frac{1}{2} \binom{n}{2} \binom{n-2}{2}.
\]

Compute the expected number of edges in this graph. Recall that the formula for the area of a cap of radius \( d \) (measured on the surface) in the unit sphere is \( 2\pi(1 - \cos d) \). Therefore the expected number of neighbors of a vertex in our graph is

\[
2(n - 1) \cdot \frac{1 - \cos d}{4\pi},
\]

and the expected number of edges in the graph is

\[
n(n - 1) \cdot \frac{1 - \cos d}{4}.
\]

Note that our random graph drawn on the sphere has size and crossing number concentrated around their respective expected values. In fact, Moon [10] showed the
concentration of the crossing number in the case $d = \pi$, i.e. for the complete graph.

Summing up our results for the drawing $\mathcal{D}$ of the graph, we obtain

$$\text{cr}(\mathcal{D}) \cdot \frac{n^2}{m^3} = \frac{1}{8\pi^2} (\sin d - \cos d)^2 \cdot \frac{1}{2} \left( \frac{n}{2} \right) \left( \frac{n - 2}{2} \right) \cdot \frac{n^2}{\left[ n(n-1) \left( \frac{1-\cos d}{4} \right) \right]^3} \cdot (1 + o(1))$$

$$= \frac{(\sin d - \cos d)^2}{8\pi^2} \cdot \frac{n(n-1)(n-2)(n-3) \cdot 4^3 n^2}{2 \cdot 2 \cdot 2 \cdot n^3 (n-1)^3 (1-\cos d)^3} \cdot (1 + o(1))$$

$$= \frac{1}{\pi^2} \frac{(\sin d - \cos d)^2}{(1-\cos d)^3} \cdot \frac{n(n-1)(n-2)(n-3)}{n^3} \cdot (1 + o(1))$$

$$= \frac{1}{\pi^2} \frac{(\sin d - \cos d)^2}{(1-\cos d)^3} \cdot (1 + o(1))$$

The function $\frac{(\sin d - \cos d)^2}{(1-\cos d)^3}$ is increasing for $0 < d < \pi$. Hence, the smaller $d$ we take, the better upper bound we have. Taking the limit for $d \to 0^+$, we obtain

$$\lim_{d \to 0^+} \frac{1}{\pi^2} \frac{(\sin d - \cos d)^2}{(1-\cos d)^3} = \frac{8}{9\pi^2}.$$

To formalize the graph construction, for any $\epsilon > 0$, select a $d > 0$ such that

$$\text{cr}(\mathcal{D}) \cdot \frac{n^2}{m(\mathcal{D})^3} < \frac{1}{\pi^2} \frac{(\sin d - \cos d)^2}{(1-\cos d)^3} + \frac{\epsilon}{2}.$$

To handle the quadratic size of $\mathcal{D}$, take a sufficiently large $N$ such that $n$ divides $N$, and take $\frac{N}{n}$ copies of $\mathcal{D}$ redrawn in the plane using stereographic projection, such that edges of different copies do not cross each other. Call this drawing $\mathcal{D}'$. Clearly

$$\text{cr}(\mathcal{D}) \cdot \frac{n^2}{m(\mathcal{D})^3} = \text{cr}(\mathcal{D}') \cdot \frac{N^2}{m(\mathcal{D})^3},$$

and the size of $\mathcal{D}'$ satisfies the required conditions with the appropriate choice of $N$. 


Bibliography


Appendix A

Codes

A.1 Attempt to find different weighting schemes on Section 2.5

The code in this Section is written in sage (https://cocalc.com). The idea is to analyze a $3 \times L$ matrix, where $L$ is the number of columns in which we want the average weight to be $c$. The column of the matrix represent the column of the clump graph. The first, second, and third column of the matrix represent clump colored $A, B,$ and $C$, respectively. The entry of the matrix will 1 if the corresponding clump exist, and 0 otherwise. The code runs for a chunk of $L$ columns, each chunk is randomly generated. Our initial aim is to find required weighting scheme as explained in Section 2.5 in each chunk, then figure out the surgeries needed (if any) in the connection parts between each chunk (the leftmost and rightmost columns) in order to concatenate the required chunks to build the given clump graph $H$.

In this section we include our trial for executing the program with $L = 7$ and $c = \frac{3}{7}$. We include commentaries in the code (starting with the hashtag #) to explain some useful steps in the code.

```python
#This is a code for maximizing sum of each 7 columns
#The function 'conv' converting the number into bases b
def conv(num,b):
    convStr = "0123456789abcdefghijklmnopqrstuvwxyz"
    if num<b:
        return convStr[num]
    else:
        return conv(num//b,b) + convStr[num%b]

#The function 'OutputMatrix' create a 'existency matrix',
each row being the length 7 binary array
```
def OutputMatrix(num):
    A = [[0] * 7 for i in (1..3)]
    matrixProfile = conv(num,7).zfill(7)
    profileList = [int(s)+1 for s in list(matrixProfile)]
    for j in (0..6):
        colNums = [int(s) for s in list(format(profileList[j],'03b'))]
        for row in (0..2):
            A[row][j] = colNums[row]
    return A

def isLegalMatrix(A):
    # check whether a column with 3 clumps colors followed
    a column with a single clump
    num_row, num_col = len(A), len(A[0])
    for col in (0..num_col-2):
        curr_col_sum = sum([A[i][col] for i in (0..num_row-1)])
        next_col_sum = sum([A[i][col+1] for i in (0..num_row-1)])
        if curr_col_sum == num_row and next_col_sum == 1:
            return False
        elif next_col_sum == num_row and curr_col_sum == 1:
            return False
    return True

# The function 'FindWeights' will output the sum of the weights obtained
# from mixed linear programming applied to each existency matrix A
def FindWeights(A):
    p = MixedIntegerLinearProgram()
    X = p.new_variable(real=True, nonnegative=True)
    X.set_max(1) # maximum weight for each clump
    p.set_objective(sum(X[(i,j)] for i in (0..2) for j in (0..6)
        if A[i][j] == 1)) # our objective function is to maximize
    the sum of weights from column 0 to 6
    # To do next: translate the variables into the clumps

    # Handle the first column (L0): affected by neighbors in L0 and L1
    for r in (0..2):
        Neighbors = []
        if A[r][0] != 1: continue
        for rn in (0..2):
            if rn == r: continue
            if A[rn][1] == 1: Neighbors.append((rn,1))
            if A[rn][0] == 1: Neighbors.append((rn,0))
        if Neighbors:
            p.add_constraint(sum(X[couple] for couple in Neighbors) <= 1)
            # weight sum from neighbors should <= 1

    # Handle the last column: affected by neighbors in L6 and L5
    for r in (0..2):
        Neighbors = []
        if A[r][6] != 1: continue
        for rn in (0..2):
            if rn == r: continue
            if A[rn][5] == 1: Neighbors.append((rn,1))
            if A[rn][6] == 1: Neighbors.append((rn,0))
        if Neighbors:
            p.add_constraint(sum(X[couple] for couple in Neighbors) <= 1)
            # weight sum from neighbors should <= 1

    p.solve()
    weights = [p.get_value(X[(i,j)]) for i in (0..2) for j in (0..6)
        if A[i][j] == 1]
    return weights
if A[rn][6] == 1: Neighbors.append((rn,6))
if A[rn][5] == 1: Neighbors.append((rn,5))
if Neighbors:
p.add_constraint(sum(X[couple] for couple in Neighbors) <= 1)
#weight sum from neighbors should <=1

#Handle the middle columns L(i):
#affected by neighbors in L(i), L(i-1) and L(i+1)
for c in (1..5):
    for r in (0..2):
        Neighbors = []
        if A[r][c] != 1: continue
        for rn in (0..2):
            if rn == r: continue
            if A[rn][c+1] == 1: Neighbors.append((rn,c+1))
            if A[rn][c] == 1: Neighbors.append((rn,c))
            if A[rn][c-1] == 1: Neighbors.append((rn,c-1))
        if Neighbors:
p.add_constraint(sum(X[couple] for couple in Neighbors) <= 1)
#weight sum from neighbors should <=1
return p.solve()
def conv(num, b):
    convStr = "0123456789abcdefghijklmnopqrstuvwxyz"
    if num<b:
        return convStr[num]
    else:
        return conv(num//b, b) + convStr[num%b]

def outputAdjacencyMatrix(num):
    A = [[0] * 7 for i in (1..3)]
    matrixProfile = conv(num, 7).zfill(7)
    profileList = [int(s)+1 for s in list(matrixProfile)]
    for j in (0..6):
        colNums = [int(s) for s in list(format(profileList[j], '03b'))]
        for row in (0..2):
            A[row][j] = colNums[row]
    return A

def FindWeights(A):
    #this will output the sum of the weights obtained from
    #mixed linear programming
    p = MixedIntegerLinearProgram()
    X = p.new_variable(real=True, nonnegative=True)
    X.set_max(10)
    p.set_objective(sum(X[(i,j)] for i in (0..2) for j in (0..6)
                        if A[i][j] == 1))
    #To do: translate the variables into the clumps
    #Handle the first column and the last column
    for r in (0..2):
        Neighbors = []
        if A[r][0] != 1: continue
        for rn in (0..2):
            if rn == r: continue
            if A[rn][1] == 1: Neighbors.append((rn,1))
            if A[rn][0] == 1: Neighbors.append((rn,0))
        if Neighbors:
            p.add_constraint(sum(X[couple] for couple in Neighbors) <= 1)

    for r in (0..2):
        Neighbors = []
        if A[r][6] != 1: continue
        for rn in (0..2):
            if rn == r: continue
            if A[rn][6] == 1: Neighbors.append((rn,6))
            if A[rn][5] == 1: Neighbors.append((rn,5))
        if Neighbors:
            p.add_constraint(sum(X[couple] for couple in Neighbors) <= 1)

    for c in (1..5):
        for r in (0..2):
            Neighbors = []
            if A[r][c] != 1: continue
            for cn in (0..2):
                if cn == r: continue
                if A[cn][c] == 1: Neighbors.append((cn,c))
            if Neighbors:
                p.add_constraint(sum(X[couple] for couple in Neighbors) <= 1)
if A[r][c] != 1: continue
for rn in (0..2):
    if rn == r: continue
    if A[rn][c+1] == 1: Neighbors.append((rn,c+1))
    if A[rn][c] == 1: Neighbors.append((rn,c))
    if A[rn][c-1] == 1: Neighbors.append((rn,c-1))
if Neighbors:
    p.add_constraint(sum(X[couple] for couple in Neighbors) <= 1)

return "total weight is": p.solve(),
"and the weights are": p.get_values(X)

A = outputAdjacencyMatrix(54321) #number between 0 and 7^7-1
print A
print FindWeights(A)

Matrix #54321 is one of all possible matrices that satisfies clump graph assumption described in Theorem 2.9. The output of is as shown in Figure A.1. The total weight is 3.5 ≥ 3, and it is easy to see that the dual degree condition (2.7) holds.

[['θ', 'θ', 'θ', 'θ', 'θ', 'θ', 'θ'], ['θ', 'θ', 'θ', 'θ', 'θ', 'θ', 'θ'], ['θ', 'θ', 'θ', 'θ', 'θ', 'θ', 'θ'], ['θ', 'θ', 'θ', 'θ', 'θ', 'θ', 'θ'], ['θ', 'θ', 'θ', 'θ', 'θ', 'θ', 'θ'], ['θ', 'θ', 'θ', 'θ', 'θ', 'θ', 'θ'], ['θ', 'θ', 'θ', 'θ', 'θ', 'θ', 'θ']]
('total weight is:','3.5', 'and the weights are:','{('θ', 1): 1.0, (1, 2): 0.5, (1, 3): 0.0, (1, 4): 0.5, (2, 4): 0.0, (2, 0): 0.5, (θ, 5): 0.5, (θ, 3): 0.0})

Figure A.1: Output from matrix #54321.

The row in the matrix corresponds to the color where 0,1,2 represents color A,B,C, respectively. The column in the matrix represent the 7 columns, numbered 0 to 6 from left to right. For example, the weight (1,2) : 0.5 in the output means the clump colored B in column 2 (the 3rd column) has weight 0.5, the weight (0,1) : 1.0 in the output means the clump colored A in column 1 (the 2nd column) has weight δ, and so on. The output in Figure A.1 corresponds to the weighting scheme of the clump graph shown in Figure A.2.

Locally, this weighting scheme works. However, not in the connection parts when we concatenate two chunks together. For example, Figure A.3 gives the output from
of Matrix #54322. Again, the total weight is $3.6 \geq 3$, and it is easy to see that the dual degree condition (2.7) holds.

$$[[\emptyset, 1, \emptyset, 1, \emptyset, 1, \emptyset], [\emptyset, 1, \emptyset, 1, \emptyset, 1, \emptyset], [1, \emptyset, 1, \emptyset, 1, \emptyset, 1]]$$

('total weight is:', 3.666666666666667, 'and the weights are:', { (\emptyset, 1): 1.0, (1, 2): 0.0, (2, 3): 0.3333333333333337, (1, 6): 0.3333333333333337, (2, 4): 0.3333333333333337, (2, 0): 1.0, (2, 5): 0.6666666666666666, (2, 3): 0.0, (2, 5): 0.0, (2, 3): 0.0})

Figure A.3: Output from matrix #54322.

The corresponding clump graph is given in Figure A.4.

Figure A.4: The weighting scheme corresponds to matrix #54322.

If Matrix #54321 and #54322 concatenated together, we can see now that the clump colored $B$ on the rightmost column of Figure A.1 will now have extra neighbor colored $C$ from the leftmost column of Figure A.3 and so the clump colored $B$ no longer satisfy dual degree condition (2.7).

Moreover, we currently have no idea whether the length (the number of columns) that need to be considered is bounded or not. To continue on this direction, a deeper coding knowledge is needed.
A.2 Finding redundant constraints on Section 2.7

restart;
with(simplex):

We are using the following variables:
\[ x[1] = \phi \quad x[4] = \gamma \quad x[7] = \alpha_1^{\prime} \]
\[ x[2] = \mu \quad x[5] = \alpha_1 \quad x[8] = \alpha_2 \]
\[ x[3] = \psi \quad x[6] = \alpha_1^{\prime \prime} \quad x[9] = \alpha_3 \]

List all constraints:
\[
\text{constraints} := \{ \\
6 \times x[1] + 2 \times x[3] - x[8] \leq 16, \\
\};
\]

\[
\text{constraints} := \{ 6 \times x_1 + 2 \times x_3 - x_8 \leq 16, x_1 - 3 \times x_3 + 3 \times x_5 + 3 \times x_6 + 3 \times x_7 \leq 3, 2 \times x_1 \\
-3 \times x_4 + 6 \times x_5 + 6 \times x_6 + 6 \times x_7 \leq 6, 2 \times x_1 + 2 \times x_2 + x_3 + x_6 + x_7 + x_8 + x_9 \leq 6, 6 \times x_1 \\
+2 \times x_2 - x_5 - x_6 - x_7 - x_8 - x_9 \leq 14, 12 \times x_1 + 4 \times x_2 - 2 \times x_5 - 2 \times x_6 - 2 \times x_7 - x_8 \\
-x_9 \leq 28, 12 \times x_1 + 10 \times x_2 - 3 \times x_4 + 10 \times x_5 + 7 \times x_6 + 4 \times x_7 + 5 \times x_8 + 5 \times x_9 \leq 28 \}
\] (1)

The objective is to maximize \( x[1] \). Executing the maximize command will give the optimal solution:
\[
\text{maximize}[x[1], \text{constraints}, \text{NONNEGATIVE}];
\]
\[
\left\{ x_1 = \frac{189}{76}, x_2 = 0, x_3 = \frac{49}{76}, x_4 = \frac{70}{19}, x_5 = \frac{31}{38}, x_6 = 0, x_7 = 0, x_8 = \frac{4}{19}, x_9 = 0 \right\}
\] (2)

This verify our result in Section 2.7.
To check the important of each constraint, we modify the constraints set and execute the maximize command for each case.
If a constraint is redundant, then removing it will not change the optimal solution.
Without constraint 1: (no opt solution produced - Maple doesnot like this)

\[
\text{constraints1 := \{}
6 \times x[1] + 2 \times x[3] - x[8] \leq 16, \\
\}
\]

\[
\text{maximize(x[1], constraints1, NONNEGATIVE)};
\]

Without constraint 2: (redundant)

\[
\text{constraints2 := \{}
6 \times x[1] + 2 \times x[3] - x[8] \leq 16, \\
\}
\]

\[
\text{maximize(x[1], constraints2, NONNEGATIVE)};
\]

\[
\left\{ x_1 = \frac{189}{76}, \ x_2 = 0, \ x_3 = \frac{49}{76}, \ x_4 = \frac{70}{19}, \ x_5 = \frac{31}{38}, \ x_6 = 0, \ x_7 = 0, \ x_8 = \frac{4}{19}, \ x_9 = 0 \right\}
\] (3)

Without constraint 3: (not redundant)

\[
\text{constraints3 := \{}
\}
\]

\[
\text{maximize(x[1], constraints3, NONNEGATIVE)};
\]

\[
\left\{ x_1 = \frac{5}{2}, \ x_2 = 0, \ x_3 = \frac{5}{6}, \ x_4 = 4, \ x_5 = 1, \ x_6 = 0, \ x_7 = 0, \ x_8 = 0, \ x_9 = 0 \right\}
\] (4)
Without constraint 4: (not redundant)

```latex
\text{maximize}(\mathbf{x}[1], \text{ constraints4, \ NONNEGATIVE});
\begin{align*}
\{ \mathbf{x}_1 = \frac{5}{2}, \mathbf{x}_2 = 0, \mathbf{x}_3 & = \frac{11}{18}, \mathbf{x}_4 = \frac{98}{27}, \mathbf{x}_5 = \frac{7}{9}, \mathbf{x}_6 = 0, \mathbf{x}_7 = 0, \mathbf{x}_8 = \frac{2}{9}, \mathbf{x}_9 = 0 \} \tag{5}
\end{align*}
```

Without constraint 5: (redundant)

```latex
\text{maximize}(\mathbf{x}[1], \text{ constraints5, \ NONNEGATIVE});
\begin{align*}
\{ \mathbf{x}_1 = \frac{189}{76}, \mathbf{x}_2 = 0, \mathbf{x}_3 & = \frac{49}{76}, \mathbf{x}_4 = \frac{49}{38}, \mathbf{x}_5 = \frac{31}{38}, \mathbf{x}_6 = 0, \mathbf{x}_7 = 0, \mathbf{x}_8 = \frac{4}{19}, \mathbf{x}_9 = 0 \} \tag{6}
\end{align*}
```

Without constraint 6: (redundant)

```latex
\text{maximize}(\mathbf{x}[1], \text{ constraints6, \ NONNEGATIVE});
\begin{align*}
\{ \mathbf{x}_1 = \frac{189}{76}, \mathbf{x}_2 = 0, \mathbf{x}_3 & = \frac{49}{76}, \mathbf{x}_4 = \frac{70}{19}, \mathbf{x}_5 = \frac{31}{38}, \mathbf{x}_6 = 0, \mathbf{x}_7 = 0, \mathbf{x}_8 = \frac{4}{19}, \mathbf{x}_9 = 0 \} \tag{7}
\end{align*}
```
Without constraint 7: (not redundant)

\[
\begin{align*}
\text{constraints}7 & : = \\
6 \cdot x[1] + 2 \cdot x[3] - x[8] & \leq 16, \\
g \end{align*}
\]

Thus our reduced constraints become:

\[
\begin{align*}
\text{reducedconstrants} & : = \\
6 \cdot x[1] + 2 \cdot x[3] - x[8] & \leq 16, \\
\end{align*}
\]

\[
\begin{align*}
\text{maximize}(x[1], \text{reducedconstrants}, \text{NONNEGATIVE}); \\
\begin{cases} 
  x_1 = \frac{5}{2}, & x_2 = 0, x_3 = 0, x_4 = 4, x_5 = 1, x_6 = 0, x_7 = 0, x_8 = 0, x_9 = 0
\end{cases}
\end{align*}
\] (8)

\[
\begin{align*}
\text{maximize}(x[1], \text{reducedconstrants}, \text{NONNEGATIVE}); \\
\begin{cases} 
  x_1 = \frac{189}{76}, & x_2 = 0, x_3 = \frac{49}{76}, x_5 = \frac{31}{38}, x_6 = 0, x_7 = 0, x_8 = \frac{4}{19}, x_9 = 0
\end{cases}
\end{align*}
\] (9)