Spring 2020

Connections Between Extremal Combinatorics, Probabilistic Methods, Ricci Curvature of Graphs, and Linear Algebra

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by

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Bachelor of Science
Carnegie Mellon University, 2015

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2020

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Dedication

To mom and dad.

Acknowledgments

First and foremost, I would like to thank my advisor Linyuan Lu for his guidance and support over my five years of Ph.D. study. Thank you for introducing me to a broad spectrum of mathematics and for always treating me like a family member. I hope one day my mathematical intuition and skills can be ϵ much of yours (for some *fixed* $\epsilon > 0$) and then I can rest in peace. Thank you for the many conferences and academic visits that you brought me along, the numerous hours that you put in working with me, and the miles that we ran together. I am forever indebted and grateful for having you as my advisor.

I also would like to thank Professors László Székely and Éva Czabarka for their tremendous mentoring and guidance on my career and research. Thank you for being my unofficial advisors and mentors, for the legal and illegal thinking that we did together, and for pushing me to do the right things at the right time. Thank you for taking care of me beyond your responsibility and for always having faith in me when I don't even have faith in myself.

I want to thank many professors who have helped me tremendously in my Ph.D journey: Professor Joshua Cooper for his wonderful teaching in probabilistic methods and spectral graph theory, Professor Ryan Martin for the collaboration and recommendation letter, Professor Sean Yee for his mentoring and recommendation letter, Professor Fan Chung for her guidance and mentoring, Professor Edsel Peña for his wonderful class on Probability theory, Professor Marco Valtorta for our collaboration, and Professors Gyula O. H. Katona, Shing-Tung Yau, Mark Ellingham for hosting my research visits.

I would like to thank many friends whose companionship and support made this journey so enjoyable: Inne Singgih, Josiah Reiswig for all our time doing maths and having fun; Ruth Luo for her continuous help and support starting from our CMU time; Michael Levet and Xiangcheng Zheng for being awesome roommates, Garner Cochran, Xiaofei Yi, Josiah Reiswig, Shuang Liu for being amazing office-mates; Xing Peng, Shuliang Bai for being wonderful role models and trailblazers; all my UofSC Department of Mathematics friends for our fun time together, all my GRWC/MRC friends and conference buddies for their companion and collaborations; and all my old friends for their support and encouragement.

I would like to extend my gratitude to all the funding agencies that partially supported my work and travel during my graduate study.

Finally, I would like to thank my parents who devoted their unwavering love and support to every journey I embarked on. Thank you for loving me no matter whom I am, where I am, and what I do.

Acknowledgement of co-authorships:

- Section 2.2 is a version of material appearing in "On the size-Ramsey number of tight path", *SIAM Journal on Discrete Mathematics*, 32(3) (2018) 2172-2179, co-authored with Linyuan Lu. The author was one of the primary investigators and authors of this paper.

- Section 2.3 is a version of material appearing in "Anti-Ramsey number of edge-disjoint rainbow spanning trees", co-authored with Linyuan Lu, which has been submitted for publication. The author was one of the primary investigators and authors of this paper.

- Section 2.4 is a version of material appearing in "Ramsey numbers of Berge-hypergraphs and related structures", *Electronic Journal of Combinatorics*, 26(4)

(2019) P4.40, co-authored with Nika Salia, Casey Tompkins and Oscar Zamora. The author was one of the primary investigators and authors of this paper.

- Section 2.5 is a version of material appearing in “On the cover Ramsey number of Berge hypergraphs”, co-authored with Linyuan Lu, which has been submitted for publication. The author was one of the primary investigators and authors of this paper.

- Section 2.6 is a version of material appearing in “Erdős-Szekeres theorem for cyclic permutations”, *Involve: A Journal of Mathematics, 12(2) (2019) 351-360*, co-authored with Éva Czabarka. The author was one of the primary investigators and authors of this paper.

- Section 3.1 is a version of material appearing in “On the cover Turán number of Berge hypergraphs”, co-authored with Linyuan Lu, which has been submitted for publication. The author was one of the primary investigators and authors of this paper.

- Section 3.2 is a version of material appearing in “On Hamiltonian Berge cycles in 3-uniform hypergraphs”, co-authored with Linyuan Lu, which has been submitted for publication. The author was one of the primary investigators and authors of this paper.

- Chapter 4 is a version of material appearing in “Concentration inequalities in spaces of random configurations with positive Ricci curvatures”, co-authored with Linyuan Lu, which has been submitted for publication. The author was one of the primary investigators and authors of this paper.

- Chapter 5 is a version of material appearing in “Maximum spectral radius of outerplanar 3-uniform hypergraphs”, co-authored with Mark Ellingham and

Linyuan Lu, which are in preparation. The author was one of the primary investigators and authors of this paper.

Abstract

This thesis studies some problems in extremal and probabilistic combinatorics, Ricci curvature of graphs, spectral hypergraph theory and the interplay between these areas. The first main focus of this thesis is to investigate several Ramsey-type problems on graphs, hypergraphs and sequences using probabilistic, combinatorial, algorithmic and spectral techniques:

- The *size-Ramsey* number $\hat{R}(\mathcal{G}, r)$ is defined as the minimum number of edges in a hypergraph $\mathcal{H}$ such that every r-edge-coloring of $\mathcal{H}$ contains a monochromatic copy of $\mathcal{G}$ in $\mathcal{H}$. We improved a result of Dudek, La Fleur, Mubayi and Rödl [*J. Graph Theory 2017*] on the size-Ramsey number of tight paths and extended it to more colors.

- An edge-colored graph G is called *rainbow* if every edge of G receives a different color. The *anti-Ramsey* number of t edge-disjoint rainbow spanning trees, denoted by $r(n, t)$, is defined as the maximum number of colors in an edge-coloring of K_n containing no t edge-disjoint rainbow spanning trees. Confirming a conjecture of Jahanbekam and West [*J. Graph Theory 2016*], we determine the anti-Ramsey number of t edge-disjoint rainbow spanning trees for all values of n and t.

- We study the extremal problems on Berge hypergraphs. Given a graph $G = (V, E)$, a hypergraph $\mathcal{H}$ is called a *Berge-G*, denoted by BG, if there exists an injection $i : V(G) \to V(\mathcal{H})$ and a bijection $f : E(G) \to E(\mathcal{H})$ such that for every $e = uv \in E(G)$, $(i(u), i(v)) \subseteq f(e)$. We investigate the hypergraph Ramsey

number of Berge cliques, the cover-Ramsey number of Berge hypergraphs, the cover-Turán desity of Berge hypergraphs as well as Hamiltonian Berge cycles in 3-uniform hypergraphs.

The second part of the thesis uses the 'geometry' of graphs to derive concentration inequalities in probabilities spaces. We prove an Azuma-Hoeffding-type inequality in several classical models of random configurations, including the Erdős-Rényi random graph models $G(n,p)$ and $G(n,M)$, the random d-out(in)-regular directed graphs, and the space of random permutations. The main idea is using Ollivier's work on the Ricci curvature of Markov chairs on metric spaces. We give a cleaner form of such concentration inequality in graphs. Namely, we show that for any Lipschitz function f on any graph (equipped with an ergodic random walk and thus an invariant distribution ν) with Ricci curvature at least $\kappa > 0$, we have

$$\nu\left(|f - E_\nu f| \geq t\right) \leq 2\exp\left(-\frac{t^2\kappa}{7}\right).$$

The third part of this thesis studies a problem in spectral hypergraph theory, which is the interplay between graph theory and linear algebra. In particular, we study the maximum spectral radius of outerplanar 3-uniform hypergraphs. Given a hypergraph $\mathcal{H}$, the shadow of $\mathcal{H}$ is a graph G with $V(G) = V(\mathcal{H})$ and $E(G) = \{uv : uv \in h \text{ for some } h \in E(\mathcal{H})\}$. A 3-uniform hypergraph $\mathcal{H}$ is called *outerplanar* if its shadow is outerplanar and all faces except the outer face are triangles, and the edge set of $\mathcal{H}$ is the set of triangle faces of its shadow. We show that the outerplanar 3-uniform hypergraph on n vertices of maximum spectral radius is the unique hypergraph with shadow $K_1 + P_{n-1}$.

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Chapter 1

Introduction

1.1 Overview

This thesis studies the interplay between extremal combinatorics, probabilistic methods, discrete geometry, and spectral graph theory. Given a combinatorial structure (e.g. graphs, sequences, poset, etc.), questions in *extremal combinatorics* are concerned about optimizing some graph parameter subject to a certain constraint. For example, the classical Turán's theorem studies the maximum number of edges in an n-vertex graph without a complete graph K_{r+1} as a subgraph. In recent decades, *probabilistic methods*, largely initiated by Paul Erdős, have been hugely successful in tackling challenging problems not only in combinatorics, but also in number theory, discrete geometry, etc. The basic approach is as follows: in order to show that some combinatorial structure satisfies certain property, one first defines an appropriate probability space, and then shows that a randomly chosen element in this probability space satisfies the desired property with positive probability. Part of this thesis (Chapter 2 and 3) will highlight some applications of probabilistic tools and random constructions in some Ramsey-type and Turán-type problems. Conversely, the 'geometry' of graphs also reveals nice properties of probability measures. In Riemannian geometry, manifolds with non-negative *Ricci curvature* enjoy many interesting properties, some with probabilistic interpretations. Ricci curvature can also be defined on the Markov chains of metric spaces [143], or more specifically graphs (see e.g., [155, 130, 129]). Given a Lipschitz function f on any graph G equipped with an ergodic random walk and thus an invariant distribution, one can obtain asymptotically sharp concentration results of f using the lower bound of the Ricci curvature of G. These concentration results can then be used to derive Azuma-Hoeffding-type inequalities in several classical graph models [132]. See Chapter 4 or [132, 143] for more details. Besides probability and geometry, linear algebra has also proven to be a very powerful tool in solving combinatorial problems. One of the main approaches in this area (which is called *spectral graph theory*) is to use the eigenvalues and eigenvectors of an appropriate matrix associated to a graph to deduce combinatorial properties of the graph (see e.g., [30]). Recently, spectral tools have been extended and intensively developed in *hypergraphs* as well. One of the main directions in this area is to find the largest spectral radius of a hypermatrix associated to a hypergraph that satisfies certain constraints and characterize the extremal hypergraphs. Similar to graph case, results in spectral hypergraph theory could potentially shed new lights on extremal hypergraph theory, which the mathematics community knows very little in general at this point. Chapter 5 will determine the maximum spectral radius of an n-vertex 3-uniform outerplanar hypergraph as well as the unique extremal hypergraph.

1.2 Terminology and Notations

We will list some basic definitions and notations that will be used throughout the thesis.

(1) **Interval Notation:** For integers n, m with $m \geq n \geq 1$, we use the notation $[n] = \{1, 2, \cdots, n\}$, and $[n, m] = \{n, n+1, \cdots, m-1, m\}$.

(2) **Set Notation:** For a discrete finite set A and integer $k \geq 1$, define $\binom{A}{k} = \{S \subset A : |S| = k\}$. We also use 2^A to denote the power set of A.

(3) **Asymptotic Notation:** Given two functions $f, g : \mathbb{Z}^+ \to \mathbb{R}$, we say $f = O(g)$ if there exist some constants C and n_0 such that for all $n \geq n_0$, $f(n) \leq Cg(n)$.

We say $f = o(g)$ if $\lim_{n\to\infty} \frac{f(n)}{g(n)} = 0$. We write $f = \Omega(g)$ if $g = O(f)$ and $f = \omega(g)$ if $g = o(f)$. Moreover, we say $f = \Theta(g)$ if $f = O(g)$ and $g = O(f)$.

(4) **Basic Hypergraph Terminology:** A hypergraph $\mathcal{H} = (V, E)$ is a pair (V, E) such that V is the vertex set and E is the edge set where each edge $h \in E$ is a subset of V. We use $V(\mathcal{H}), E(\mathcal{H})$ to denote the vertex set and edge set of $\mathcal{H}$ respectively. A hypergraph is sometimes considered as a collection of hyperedge. Thus $|\mathcal{H}|$ is commonly used to denote the number of hyperedges of $\mathcal{H}$. Sometimes we may also use $v(\mathcal{H})$ and $e(\mathcal{H})$ to denote the number of vertices and edges of $\mathcal{H}$ respectively. The *neighborhood* of a vertex v in $\mathcal{H}$, denoted by $N_{\mathcal{H}}(v)$ or $\Gamma_{\mathcal{H}}(v)$, is defined by $N_{\mathcal{H}}(v) = \{h : v \notin h, h \cup \{v\} \in E(\mathcal{H})\}$. The *degree* of a vertex, denoted by $d_{\mathcal{H}}(v)$, is $|N_{\mathcal{H}}(v)|$. We use $\delta(\mathcal{H})$ and $\Delta(\mathcal{H})$ to denote the minimum and maximum degree of $\mathcal{H}$ respectively. Moreover generally, given an R-graph $\mathcal{H} = (V, E)$ and a set $S \in \binom{V}{s}$, we use $\deg(S)$ (or simply $d(S)$) to denote the number of edges containing S and $\delta_s(\mathcal{H})$ be the minimum s-degree of $\mathcal{H}$, i.e., the minimum of $\deg(S)$ over all s-element sets $S \in \binom{V}{s}$. Given a graph $\mathcal{H}$ and $S \subseteq V(\mathcal{H})$, we use $\mathcal{H}[S]$ to denote the subgraph of $\mathcal{H}$ induced by S, i.e., $V(\mathcal{H}[S]) = S$ and $E(\mathcal{H}[S]) = \{h \in E(\mathcal{H}), h \subseteq S\}$. A hypergraph $\mathcal{H}$ is r-uniform if every edge has cardinality r. More generally, given a set of positive integers R, a hypergraph $\mathcal{H}$ is R-uniform if the cardinality of each edge of $\mathcal{H}$ belongs to R. We use K_n^r to denote the n-vertex r-uniform complete graph (or clique), i.e., every r-subset of the vertex set is a hyperedge.

(5) **Basic Graph Terminology:** A graph $G = (V, E)$ is simply a 2-uniform hypergraph. A graph G is *simple* if there is no *loop* (i.e., edge of the form (v, v) for some $v \in V(G)$) and no multiple edges between two vertices. Given a simple graph G, it is also common in the literature (in the absence of hypergraph) to use $|G|$ to denote $|V(G)|$ and $\|G\|$ to denote $|E(G)|$.

(6) **Berge hypergraph:** Given a graph G, a hypergraph $\mathcal{H}$ is called a *Berge*-G, denoted by BG, if there exists an injection $i : V(G) \to V(\mathcal{H})$ and a bijection $f : E(G) \to E(\mathcal{H})$ such that for every $e = uv \in E(G)$, $(i(u), i(v)) \subseteq f(e)$.

(7) **Shadow:** Given a hypergraph $\mathcal{H}$, the 2-*shadow*(or *shadow*) of $\mathcal{H}$, denoted by $\partial(\mathcal{H})$, is a simple 2-uniform graph $G = (V, E)$ such that $V(G) = V(\mathcal{H})$ and $uv \in E(G)$ if and only if $\{u, v\} \subseteq h$ for some $h \in E(\mathcal{H})$.

1.3 Summary of main results

We will briefly describe the main results in this thesis. Each chapter or section (if necessary) will also have its own introduction containing more definitions, historical backgrounds and prior results.

Size-Ramsey number of tight paths in hypergraphs

The *size-Ramsey* number $\hat{R}(G, r)$ is defined as the minimum number of edges in a graph H such that every r-edge-coloring of H contains a monochromatic copy of G in H. Size-Ramsey number was first studied by Erdős, Faudree, Rousseau and Schelp [64] in 1978. Answering a question of Erdős [62], Beck [13] showed by a probabilistic construction that the size-Ramsey number of a path on n vertices $\hat{R}(P_n, 2) = O(n)$. Dudek, La Fleur, Mubayi and Rödl [54] initiated the systematic study of the size-Ramsey number of hypergraphs. There are several ways to define a path in a hypergraph. An ℓ-path, denoted by $\mathcal{P}^{(k)}_{n,\ell}$, is a k-uniform hypergraph with vertex set $[n]$ and edge set containing intervals of length k in $[n]$ and consecutive edges intersect in exactly l vertices. When $\ell = k-1$, we call $\mathcal{P}^{(k)}_{n,k-1}$ a *tight path*. Dudek, La Fleur, Mubayi and Rödl [54] showed that $\hat{R}(\mathcal{P}^{(k)}_{n,k-1}, 2) = O(n^{k-1-\alpha}(\log n)^{1+\alpha})$ where $\alpha = \frac{k-2}{\binom{k-1}{2}+1}$. In Chapter 2.2, we improved their results and extended it to more colors:
Theorem. $\hat{R}(\mathcal{P}^{(k)}_{n,k-1}, r) = O(r^k(n\log n)^{k/2})$ for all $k \geq 3$ and $r \geq 2$.

Anti-Ramsey number of spanning trees

Given an edge-colored G, a subgraph H of G is *rainbow* if all the edges of H receive distinct colors. The general *anti-Ramsey problem* asks for the maximum number of colors in an edge-coloring of K_n having no rainbow copy of some graph in a class $\mathcal{G}$. Let $r(n,t)$ be the maximum number of colors in an edge-coloring of K_n not having t edge-disjoint rainbow spanning trees. Akbari and Alipour [2] showed that $r(n,2) = \binom{n-2}{2} + 2$ for $n \geq 6$. Jahanbekam and West [109] showed that

$$r(n,t) = \begin{cases} \binom{n-2}{2} + t & \text{for } n > 2t + \sqrt{6t - \frac{23}{4}} + \frac{5}{2} \\ \binom{n}{2} - t & \text{for } n = 2t, \end{cases}$$

and they conjectured that $r(n,t) = \binom{n-2}{2} + t$ whenever $n \geq 2t + 2 \geq 6$. In Chapter 2.3 we confirm their conjecture in the positive. In particular we showed that

Theorem. For all positive integers t,

$$r(n,t) = \begin{cases} \binom{n-2}{2} + t & \text{for } n \geq 2t + 2 \\ \binom{n-1}{2} & \text{for } n = 2t + 1 \\ \binom{n}{2} - t & \text{for } n = 2t, \end{cases}$$

Ramsey number of Berge hypergraphs

The Ramsey number of Berge hypergraphs, denoted as $R^r(BG, BG)$, is defined as the smallest integer n_0 such that for any 2-edge-coloring of a complete r-uniform hypergraph on $n \geq n_0$ vertices, there is a monochromatic Berge-G subhypergraph. In collaboration with Nika Salia, Casey Tompkins and Oscar Zamora, we completely determined the 2-color Ramsey number of Berge-K_t.

Theorem.

$$R^3(BK_s, BK_t) = \begin{cases} t+s-1 & \text{if } s=t=2,\ s=t=3 \text{ or } \{s,t\}=\{2,3\} \text{ or } \{s,t\}=\{2,4\}, \\ t+s-2 & \text{if } s=2, t\geq 5, \text{ or } s=3, t\geq 4 \text{ or } s=t=4, \\ t+s-3 & \text{if } s\geq 4 \text{ and } t\geq 5. \end{cases}$$

We also showed that $R^4(BK_t, BK_t) = t+1$ when $t \geq 6$ and $R^r(BK_t, BK_t) = t$ when $r \geq 5$ and $t \geq t_0(r)$ for some $t_0(r)$.

Cover-Ramsey and cover-Turán number of Berge hypergraphs

Following up on the research of the Ramsey number of Berge hypergraphs, We approach the study of Berge hypergraphs from the perspectives of the shadow graph. We define a new type of Ramsey number, namely the *cover Ramsey number*, denoted as $\hat{R}^R(BG_1, BG_2)$, as the smallest integer n_0 such that for every R-uniform hypergraph $\mathcal{H}$ on $n \geq n_0$ vertices whose shadow is a complete graph, and every 2-edge-coloring (blue and red) of $\mathcal{H}$, there is either a blue Berge-G_1 or a red Berge-G_2 sub-hypergraph. When $R = \{2\}$, $\hat{R}^R(BG_1, BG_2)$ is exactly the classical Ramsey number. This variant of Ramsey number of Berge hypergraphs more closely resembles the behavior of the classical Ramsey number, as exhibited by the following theorem.

Theorem. For every $k \geq 2$, there exists $c_k > 0$ such that for any two finite graphs G_1 and G_2,

$$R(G_1, G_2) \leq \hat{R}^{[k]}(BG_1, BG_2) \leq c_k \cdot R(G_1, G_2)^3.$$

Theorem.

1. For every $k \geq 2$ and sufficiently large t,

$$\hat{R}^{\{k\}}(BK_t, BK_t) > (1+o(1))\frac{\sqrt{2}}{e}t2^{t/2}.$$

2. For each positive integer d and k, there exists a constant $c = c(d,k)$ such that if G is a graph on n vertices with maximum degree at most d, then

$$\hat{R}^{[k]}(BG, BG) \leq cn.$$

Similarly, we also define a variant of the Turán number of Berge hypergraphs from the persepctives of the shadow. In particular, define the *R-cover Turán number* of G, denoted as $\hat{ex}_R(n,G)$, as the maximum number of edges in the shadow graph of a Berge-G-free R-graph on n vertices. We also define the *R-cover Turán density*, denoted as $\hat{\pi}_R(G)$, as $\hat{\pi}_R(G) = \limsup_{n\to\infty} \frac{\hat{\mathrm{ex}}_R(n,G)}{\binom{n}{2}}$. In Section 3.1, we showed an analogue of the Erdős-Stone-Simonovits theorem on the cover Turán number of Berge hypergraphs:

Theorem. For any fixed graph G and any fixed $\epsilon > 0$, there exists n_0 such that for any $n \geq n_0$,

$$\hat{\mathrm{ex}}_k(n,G) \leq \left(1 - \frac{1}{\chi(G)-1} + \epsilon\right)\binom{n}{2}.$$

Moreover, if $\chi(G) \geq k+1$, then $\hat{\pi}_k(G) = 1 - \frac{1}{\chi(G)-1}$.

For 3-uniform hypergraphs, we then completely determine the cover Turán density of all graphs:

Theorem. Given a simple graph G,

$$\hat{\pi}_3(G) = \begin{cases} 1 - \frac{1}{\chi(G)-1} & \text{if } \chi(G) \geq 4, \\ 0 & \text{if } G \text{ is a subgraph of one of the graphs in Figure 3.2,} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

Hamiltonian Berge cycles in covering hypergraph

Given a graph G, a spanning cycle of G is also called the *Hamiltonian cycle* of G. One of the earliest results on Hamiltonian cycle is the Dirac's Theorem which states that every n-vertex graph with minimum degree $\delta \geq n/2$ contains a Hamiltonian cycle. In Section 3.2, we study the minimum 2-degree threshold for Berge Hamiltonian cycles in hypergraphs. We call a hypergraph $\mathcal{H}$ *covering* if its shadow is a complete graph. The following theorem can be implied from a series of results on rainbow spanning structures in a k-bounded edge-colored graph [68, 101, 78, 4, 77, 21, 46]:

Theorem. For any fixed $r \geq 2$ and any set of integers $R \subseteq [r]$, any sufficiently large covering R-graph $\mathcal{H}$ is Berge-pancyclic, i.e., it contains a Berge cycle C_s for any $3 \leq s \leq n$.

In fact, [21] allows us to find Berge copies of general spanning graphs with maximum degree increasing with n while [46] only requires the shadow of $\mathcal{H}$ to have minimum degree at least $n/2$. All theorems above require $\mathcal{H}$ to have sufficiently large number of vertices. We show a more precise result when $r = 3$: every covering [3]-graph $\mathcal{H}$ on $n \geq 6$ vertices contains a Berge cycle C_s for any $3 \leq s \leq n$. Moreover, every covering [3]-graph $\mathcal{H}$ on $n \geq 6$ vertices contains a Hamiltonian Berge path. Using the theorems above, we determined the maximum *Lagrangian* λ of Berge-C_t-free and Berge-P_t-free hypergraphs respectively:

Theorem. For fixed $k \geq 2$ and sufficiently large $t = t(k)$ and $n \geq t$, let $\mathcal{H}$ be a k-uniform hypergraph on n vertices without a Berge-C_t (or Berge-P_t respectively). Then

$$\lambda(\mathcal{H}) \leq \lambda(K_{t-1}^k) = \frac{1}{(t-1)^k}\binom{t-1}{k}.$$

Ricci curvature and concentration inequalities

Consider a graph (loops allowed) $G = (V, E)$ equipped with a random work $m := \{m_v : v \in V\}$ where $m_v : N(v) \to [0, 1]$ is a distribution for each vertex v, i.e.,

$$\sum_{x \in N(v)} m_v(x) = 1.$$

Assume that this random walk is *ergodic* so that an invariant distribution ν exists. A function $f: V \to \mathbb{R}$ is called c-Lipschitz on G if $|f(u) - f(v)| \le c$ for any $uv \in E(G)$. Given a graph G (equipped with a random walk) with positive Ricci curvature at least $\kappa > 0$ (see Chapter 4 for definition), we can derive the following Azuma–Hoeffding-type concentration inequalities:

Theorem. Suppose that a graph $G = (V, E)$ equipped with an ergodic random walk m (and invariant distribution ν) has a positive Ricci curvature at least $\kappa > 0$. Then for any 1-Lipschitz function f and any $t \ge 1$, we have

$$\nu\left(f - E_\nu[f] > t\right) \le \exp\left(\frac{-t^2\kappa}{7}\right), \tag{1.1}$$

$$\nu\left(f - E_\nu[f] < -t\right) \le \exp\left(\frac{-t^2\kappa}{7}\right). \tag{1.2}$$

In Chapter 4, we will give applications of the above theorem in four classical models of random configurations, including the Erdős-Rényi random graph model $G(n,p)$ and $G(n,M)$, the random d-out(in)-regular directed graphs, and the space of random permutations.

Maximum spectral radius of 3-uniform planar hypergraph

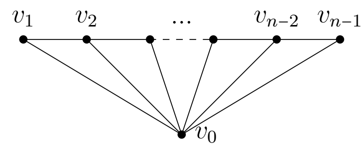
A graph is *outerplanar* if it can be embedded in the plane such that all vertices lie on the boundary of its outer face. We say a 3-uniform hypergraph $\mathcal{H}$ is *outerplanar* if $\partial(\mathcal{H})$ is outerplanar, all faces except the outer face are triangles, and the edge set of $\mathcal{H}$ is the set of triangle faces of its shadow. Given an r-uniform hypergraph $\mathcal{H}$ on n vertices, the polynomial form of $\mathcal{H}$ is a multi-linear function $P_{\mathcal{H}}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ defined for any vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ as

$$P_H(\boldsymbol{x}) = r \sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

The *spectral radius* λ of $\mathcal{H}$, introduced by Cooper and Dutle [45], is defined as

$$\lambda(\mathcal{H}) \coloneqq \max_{\|\boldsymbol{x}\|_r = 1} P_{\mathcal{H}}(\boldsymbol{x}) = \max_{\boldsymbol{x} \in \mathbb{R}} \frac{P_{\mathcal{H}}(\boldsymbol{x})}{\|\boldsymbol{x}\|_r^r},$$

where $\|\boldsymbol{x}\|_r := (|x_1|^r + |x_2|^r + \cdots + |x_n|^r)^{1/r}$. If $\boldsymbol{x} \in \mathbb{R}^n$ is a vector with $\|\boldsymbol{x}\|_r = 1$ and $P_H(\boldsymbol{x}) = \lambda(H)$, then $\boldsymbol{x}$ is called an *eigenvector* corresponding to $\lambda(H)$.



In Chapter 5, we show the hypergraph analogue of a conjecture by Cvetković and Rowlinson [47]:

Theorem. For large enough n, the outerplanar 3-uniform hypergraph graph $\mathcal{H}$ on n vertices of maximum spectral radius is the unique hypergraph whose shadow is $K_1 + P_{n-1}$.

Chapter 2

Ramsey-type problems

2.1 Introduction

Ramsey theory is among the oldest and most intensely investigated topics in combinatorics. The *Ramsey number* $R^k(m,n)$, is the minimum N such that every red-blue coloring of the edges of K_N^k contains a monochromatic red copy of K_m^k or a monochromatic blue copy of K_n^k. The existence of the Ramsey number $R^k(m,n)$ follows from the seminal result of Ramsey [146] from 1930. When restricted to (2)-graphs, we ignore the superscript and denote $R^2(m,n)$ as $R(m,n)$.

Determining the Ramsey number is a notoriously hard problem. Even $R(5,5)$ is unknown despite the advancement of our computing capabilities. The classical results of Erdős and Szekeres [73] and Erdős [63] gives that $\Omega(2^{n/2}) = R(n,n) = O(2^{2n})$. The best known lower and upper bounds for $R(n,n)$ are

$$(1+o(1))\frac{\sqrt{2}n}{e}(\sqrt{2})^n \leq R(n,n) \leq n^{-c\log n/(\log\log n)}4^n,$$

shown by Spencer [161] and Conlon [43] respectively. For hypergraph diagonal Ramsey number, a result of Erdős, Hajnal and Rado [66] established that $2^{c_1 n^2} < r^3(n,n) < 2^{2^{c_2 n}}$ for some absolute constants c_1 and c_2. Alternative proof of the lower bound above was also given by Conlon, Fox and Sudakov [44]. More generally, for $k \geq 4$, the best lower and upper bounds (see [73, 69, 65]) are

$$\mathrm{twr}_{k-1}(c_1 n^2) \leq R^k(n,n) \leq \mathrm{twr}_k(c_2 n),$$

where the *tower function* $\mathrm{twr}_k(x)$ is defined by $\mathrm{twr}_1(x) = x$ and $\mathrm{twr}_{i+1}(x) = 2^{\mathrm{twr}_i(x)}$.

For more results on hypergraph Ramsey numbers, see e.g., [140] for an excellent survey.

Generalizing the spirit of the Ramsey number, *Ramsey-type* problems embed the idea that in every partition of a sufficiently large structured object, one of the classes is guaranteed to contain a large structured sub-object. In this chapter, we discuss several Ramsey-type results on graphs, hypergraphs, and sequences.

2.2 Size-Ramsey number of tight paths

Given two simple graphs G and H and a positive integer r, say that $H \to (G)_r$ if every r-edge-coloring of H results in a monochromatic copy of G in H. In this notation, the *Ramsey number* $R(G)$ of G is the minimum n such that $K_n \to (G)_2$. The *size-Ramsey number* $\hat{R}(G,r)$ of G is defined as the minimum number of edges in a graph H such that $H \to (G)_r$, i.e.,

$$\hat{R}(G,r) = min\{|E(H)| : H \to (G)_r\}.$$

When $r = 2$, we ignore r and simply use $\hat{R}(G)$.

Size-Ramsey number was first studied by Erdős, Faudree, Rousseau and Schelp [64] in 1978. By the definition of $R(G)$, we have

$$\hat{R}(G) \le \binom{R(G)}{2}.$$

Chvátal (see, e.g.[64]) showed that this bound is tight for complete graphs, i.e. $\hat{R}(K_n) = \binom{R(K_n)}{2}$. Answering a question of Erdős [62], Beck [13] showed by a probabilistic construction that

$$\hat{R}(P_n) = O(n).$$

Alon and Chung [5] gave an explicit construction of a graph G with $O(n)$ edges such that $G \to P_n$. Recently, Dudek and Prałat [55] provided a simple alternative proof for this result (See also [127]). The best upper bound $\hat{R}(P_n) \le 74n$ is due to Dudek and Prałat [56] by considering a random 27-regular graph of a proper order.

Analogously, size-Ramsey number has also been studied in hypergraphs. A k-uniform hypergraph $\mathcal{G}$ on a vertex set $V(\mathcal{G})$ is a family of k-element subsets (called edges) of $V(\mathcal{G})$. We use $E(\mathcal{G})$ to denote the edge set. Given k-uniform hypergraphs $\mathcal{G}$ and $\mathcal{H}$, we say that $\mathcal{H} \to (\mathcal{G})_r$ if every r-edge-coloring of $\mathcal{H}$ results in a monochromatic copy of $\mathcal{G}$ in $\mathcal{H}$. Define the *size-Ramsey number* $\hat{R}(\mathcal{G}, r)$ of a k-uniform hypergraph $\mathcal{G}$ as

$$\hat{R}(\mathcal{G}, r) = min\{|E(\mathcal{H})| : \mathcal{H} \to (\mathcal{G})_r\}.$$

When $r = 2$, we simply use $\hat{R}(\mathcal{G})$ for the ease of reference.

Bielak and Gorgol, in [17], first investigated the size-Ramsey number of k-stars as well as the asymmetric size-Ramsey number of 3-uniform cliques and small 3-stars. Dudek, La Fleur, Mubayi, and Rődl [54] initiated the study of (symmetric) size-Ramsey number of cliques, paths, trees and bounded degree hypergraphs in k-uniform hypergraphs. In this section, we focus on the size-Ramsey number of paths.

Given integers $1 \leq l < k$ and $n \equiv l \pmod{k-l}$, an l-path $\mathcal{P}_{n,l}^{(k)}$ is a k-uniform hypergraph with vertex set $[n]$ and edge set $\{e_1, \cdots, e_m\}$, where $e_i = \{(i-1)(k-l) + 1, (i-1)(k-l)+2, \cdots, (i-1)(k-l)+k\}$ and $m = \frac{n-l}{k-l}$, i.e. the edges are intervals of length k in $[n]$ and consecutive edges intersect in exactly l vertices. A $\mathcal{P}_{n,1}^{(k)}$ is commonly referred as a *loose* path and a $\mathcal{P}_{n,k-1}^{(k)}$ is called a *tight* path.

Dudek, La Fleur, Mubayi and Rődl [54] showed that when $l \leq \frac{k}{2}$, the size-Ramsey number of a path $\mathcal{P}_{n,l}^{(k)}$ can be easily reduced to the graph case. In particular, they showed that if $1 \leq l \leq \frac{k}{2}$, then

$$\hat{R}\left(\mathcal{P}_{n,l}^{(k)}\right) \leq \hat{R}(P_n) = O(n).$$

For tight paths, they showed in the same paper that for fixed $k \geq 3$,

$$\hat{R}\left(\mathcal{P}_{n,k-1}^{(k)}\right) = O(n^{k-1-\alpha}(\log n)^{1+\alpha}),$$

where $\alpha = (k-2)/(\binom{k-1}{2} + 1)$. Observe that $\hat{R}\left(\mathcal{P}_{n,l}^{(k)}\right) \leq \hat{R}\left(\mathcal{P}_{n,k-1}^{(k)}\right)$. Thus any upper bound on the size-Ramsey number of tight paths is also an upper bound for other l-path $\mathcal{P}_{n,l}^{(k)}$.

Motivated by their approach, we use a different probabilistic construction and improve the upper bound to $O((n \log n)^{k/2})$. In particular, we show the following result on the multi-color size-Ramsey number of tight paths in hypergraphs:

Theorem 2.2.1. *For any fixed $k \geq 3$, any $r \geq 2$, and sufficiently large n, we have*

$$\hat{R}\left(\mathcal{P}_{n,k-1}^{(k)}, r\right) = O\left(r^k (n \log n)^{\frac{k}{2}}\right).$$

Remark 2.2.1. *For $k = 3$, our upper bound is the same as the upper bound by Dudek, La Fleur, Mubayi and Rődl. Very recently, Han, Kohayakawa, Mota and Parczyk [102] showed that $\hat{R}(\mathcal{P}_{n,2}^{(3)}, 2) = O(n)$. The case for general k is still open.*

2.2.1 Proof of Theorem 2.2.1

The approach of our proof is inspired by Dudek, La Fleur, Mubayi and Rődl's approach in their proof of Theorem 2.8 in [54]. In their proof, they constructed their hypergraph by setting edges to be the k-cliques of an Erdős-Rényi random graph. Then they use a greedy algorithm to show that the number of edges of each color is smaller than $\frac{1}{r}$ fraction of the total number of edges, which gives a contradiction. Motivated by their approach, we use the same greedy algorithm but a different probabilistic construction of the hypergraph. Instead of using k-cliques of an Erdős-Rényi random graph as edges, we use k-cycles of a random C_k-colorable graph (which will be defined later) as edges.

Throughout the section, we will use the following version of Chernoff inequalities for the binomial random variables $X \sim Bin(n, p)$ (for details, see, e.g. [28]):

$$Pr\left(X \leq E(X) - \lambda\right) \leq exp\left(-\frac{\lambda^2}{2E(X)}\right), \tag{2.1}$$

$$Pr\,(X \geq E(X) + \lambda) \leq exp\left(-\frac{\lambda^2}{2(E(X) + \lambda/3)}\right). \tag{2.2}$$

We follow a similar notation as [54]. A graph G is *C_k-colorable* if there is a graph homomorphism π mapping G to the cycle C_k. That is, $V(G)$ can be partitioned into k-parts $V_1 \cup V_2 \cup \cdots \cup V_k$ so that $E(G) \subseteq \bigcup_{i=1}^{k} E(V_i, V_{i+1})$ with $V_{k+1} = V_1$ and $E(V_i, V_{i+1})$ denoting the set of edges between a vertex in V_i and a vertex in V_{i+1}. For such a graph G, we say a k-cycle C in G is *proper* if it intersects each V_i by exactly one vertex. For $1 \leq l \leq k-1$, we say a path P_l of l vertices in G is *proper* if it intersects each V_i by at most one vertex. Let $\mathcal{T}_{k-1}(G)$ denote the set of all proper $(k-1)$-paths in G. Let $\mathcal{B} \subseteq \mathcal{T}_{k-1}$ be a family of pairwise vertex-disjoint proper $(k-1)$-paths. Let $t_{\mathcal{B}}$ be the total number of proper k-cycles in G that extend some $B \in \mathcal{B}$. For $A \subseteq V$, define $y_{A,\mathcal{B}}$ as the number of proper k-cycles in G that extend a proper $(k-1)$-path $B \in \mathcal{B}$ with a vertex $v \in A \cup \bigcup_{B \in \mathcal{B}} V(B)$. Given $C \subseteq V(G)$, we use z_C to denote the number of proper k-cycles in G that intersect C. We use t_k to denote the total number of proper k-cycles in G.

We say an event in a probability space holds *a.a.s.* (aka, *asymptotically almost surely*) if the probability that it holds tends to 1 as n goes to infinity. Finally, we use $\log n$ to denote natural logarithms.

Proposition 2.2.1. *For every $r \geq 2$, $k \geq 3$, and sufficiently large n, there exists a C_k-colorable graph $G = (V, E)$ of order $16k^3rn$ satisfying the following:*

(i) For every $\mathcal{B}$ consisting of n pairwise vertex-disjoint proper $(k-1)$-paths, and every $A \subseteq V \setminus \bigcup_{B \in \mathcal{B}} V(B)$ with $|A| \leq n$, we have

$$y_{A,\mathcal{B}} < \frac{1}{2kr} t_{\mathcal{B}}.$$

(ii) For every $C \subseteq V$ with $|C| \leq (k-1)n$, we have

$$z_C < \frac{t_k}{2r}.$$

(iii) The total number of proper k-cycles satisfies

$$t_k = O(r^k (n \log n)^{k/2}).$$

Proof. Set $c = 16k^2 r$ and $p = \frac{\sqrt{\log n}}{\sqrt{n}}$. Consider the following random C_k-colorable graph G. Let $V(G) = V_1 \cup V_2 \cup \cdots \cup V_k$ be the disjoint union of k sets. Each V_i (for $1 \le i \le k$) has the same size cn. For any pair of vertices $\{u, v\}$ in two consecutive parts, i.e., there is an $i \in [k]$, such that $u \in V_i$ and $v \in V_{i+1}$ (with the convention $V_{k+1} = V_1$), add uv as an edge of G with probability p independently. There is no edge inside each V_i or between two non-consecutive parts.

We will show that this random C_k-colorable graph G satisfies a.a.s. $(i) - (iii)$.

First we show that G a.a.s. satisfies (i). For a fixed family $\mathcal{B}$ of n pairwise vertex-disjoint proper $(k-1)$-paths, we would like to give a lower bound of $t_{\mathcal{B}}$. For each proper $(k-1)$-path $B \in \mathcal{B}$, there are cn vertices that can extend B into a proper k-cycle, each with probability p^2 independently. Thus, we have $t_{\mathcal{B}} \sim Bin(cn^2, p^2)$ with

$$E[t_{\mathcal{B}}] = cn^2 p^2 = cn \log n = 16k^2 rn \log n.$$

Applying Chernoff inequality, we have

$$\begin{aligned} Pr\left(t_{\mathcal{B}} \le \frac{E[t_{\mathcal{B}}])}{2}\right) &\le exp\left(-\frac{1}{8}E[t_{\mathcal{B}}]\right) \\ &= exp\left(-2k^2 rn \log n\right). \end{aligned}$$

Now for fixed $A \subseteq V \setminus \bigcup_{B \in \mathcal{B}} V(B)$, we estimate the upper bound of $y_{A,\mathcal{B}}$. Without loss of generality, we can assume that $|A| = n$. We have $y_{A,\mathcal{B}} \le Y \sim Bin(2n^2, p^2)$, thus

$$E[Y] = 2n^2 p^2 = 2n \log n.$$

Thus if we apply the Chernoff bound (2.2) with $\lambda = (2k-1)E[Y]$, then

$$Pr\left(Y \ge \frac{1}{4kr}E[t_{\mathcal{B}}]\right) = Pr\left(Y \ge 2kE[Y]\right)$$

$$
\begin{aligned}
&= Pr\left(Y \geq E[Y] + \lambda\right) \\
&\leq \exp\left(-\frac{\lambda^2}{2(\mathrm{E}[Y] + \lambda/3)}\right) \\
&\leq \exp\left(-\frac{3(2k-1)^2}{2k+2} n \log n\right).
\end{aligned}
$$

Note that since G is a C_k-colorable graph, every proper $(k-1)$-path in G contains at most one vertex from each V_i for $i \in [k]$. Thus $|(\bigcup_{B\in\mathcal{B}} V(B)) \cap V_i| \leq n$. For each V_i, there are at most $n!$ ways to assign the vertices in $\bigcup_{B\in\mathcal{B}} V(B) \cap V_i$ to the n paths in $\mathcal{B}$. It follows that the number of possible choices of $\mathcal{B}$ is upper bounded by $\left(\binom{cn}{n} \cdot n!\right)^k$. Similarly, the number of possible choices of A and $\mathcal{B}$ is upper bounded by $\left(\binom{cn}{n,n,(c-2)n} \cdot n!\right)^k$, where $\binom{cn}{n,n,(c-2)n}$ is the multinomial coefficient that counts the number of ways to choose n vertices (for A) and another n vertices (for $\mathcal{B}$) from each V_i. Stirling approximation of binomial coefficient gives us that

$$
\begin{aligned}
\log\left(\binom{cn}{n} \cdot n!\right)^k &= (1 + o(1))\left(kn \log n\right), \\
\log\left(\binom{cn}{n, n, (c-2)n} \cdot n!\right)^k &= (1 + o(1))\left(kn \log n\right).
\end{aligned}
$$

Therefore by the union bound, we have

$$
\begin{aligned}
Pr\left(\bigcup_{\mathcal{B}} \{t_{\mathcal{B}} \leq \frac{E[t_{\mathcal{B}}]}{2}\}\right) &\leq \left(\binom{cn}{n} \cdot n!\right)^k Pr\left(t_{\mathcal{B}} \leq \frac{E[t_{\mathcal{B}}]}{2}\right) \\
&\leq exp\left((1 + o(1))kn \log n - 2k^2 rn \log n\right) \\
&= o(1).
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
Pr\left(\bigcup_{A,\mathcal{B}} \{y_{A,\mathcal{B}} \geq \frac{1}{4kr} E[t_{\mathcal{B}}]\}\right) &\leq \left(\binom{cn}{n, n, (c-2)n} \cdot n!\right)^k Pr\left(Y \geq \frac{1}{4kr} E[t_{\mathcal{B}}]\right) \\
&\leq exp\left((1 + o(1))kn \log n - \frac{3(2k-1)^2}{2k+2} n \log n\right) \\
&= o(1).
\end{aligned}
$$

In the last step, we observe $\frac{3(2k-1)^2}{2k+2} > k$ for all $k \geq 3$. Therefore, combining previous inequalities, it follows that for all $A, \mathcal{B}$ satisfying the condition in (i), we have, a.a.s.,

$$y_{A,\mathcal{B}} < \frac{1}{4kr} E[t_{\mathcal{B}}] \leq \frac{1}{2kr} t_{\mathcal{B}}.$$

This finishes the proof of (i).

Now we will prove that G satisfies (ii) and (iii) a.a.s.

We will use the Kim-Vu inequality [119] stated as below:

> Let H be a (weighted) hypergraph with $V(H) = [n]$. Edge edge e has some weight $w(e)$. Suppose $\{t_i : i \in [n]\}$ is a set of Bernoulli independent random variables with probability p of being 1. Consider the polynomial
>
> $$Y_H = \sum_{e \in E(H)} w(e) \prod_{s \in e} t_s.$$
>
> Furthermore, for a subset A of $V(H)$, define
>
> $$Y_{H_A} = \sum_{e, A \subset e} w(e) \prod_{i \in e \setminus A} t_i.$$
>
> If we define
>
> $$E_i(H) = \max_{A \subset V(H), |A| = i} E(Y_{H_A}),$$
> $$E(H) = \max_{i \geq 0} E_i(H),$$
> $$E'(H) = \max_{i \geq 1} E_i(H),$$
>
> then
>
> $$Pr\left(|Y_H - E_0(H)| > a_k (E(H)E'(H))^{1/2} \lambda^k\right) = O\left(exp(-\lambda + (k-1)\log n)\right) \tag{2.3}$$
>
> for any positive number $\lambda > 1$ and $a_k = 8^k (k!)^{1/2}$.

In our context, for a fixed $v \in V(G)$, let H be the k-uniform hypergraph constructed by the proper k-cycles of G containing v. The edge set of H is the collection of all k-tuples $\{vv_1, v_1v_2, \cdots, v_{k-2}v_{k-1}, v_{k-1}v\}$ such that $vv_1v_2\cdots v_{k-1}v$ is a proper k-cycle in G and all edges have weight 1.

Fix $v \in V(G)$. we let X_v denote the number of proper k-cycles in G that contain v. Then it is not hard to see that

$$E_0(X_v) = E(X_v) = (cn)^{k-1}p^k = c^{k-1}n^{\frac{k-2}{2}}(\log n)^{\frac{k}{2}}.$$

$$E'(X_v) = (cn)^{k-2}p^{k-1} = c^{k-2}n^{\frac{k-3}{2}}(\log n)^{\frac{k-1}{2}}.$$

Applying Kim-Vu inequality with $\lambda = 2(k-1)\log n$, we get that for each $v \in V(G)$,

$$Pr\left(|X_v - E_0(X_v)| > a_k(E(X_v)E'(X_v))^{1/2}\lambda^k\right) = O\left(exp(-(k-1)\log n)\right).$$

Observe that $a_k(E(X_v)E'(X_v))^{1/2}\lambda^k = o(E_0(X_v))$. Applying union bound for all $v \in V(G)$, we obtain that a.a.s. that

$$X_v = (1 \pm o(1))(cn)^{k-1}p^k = (1 \pm o(1))c^{k-1}n^{\frac{k}{2}-1}(\log n)^{\frac{k}{2}}.$$

Recall that t_k denotes the total number of proper k-cycles in G and z_C denotes the number of proper k-cycles in G that intersect C. Suppose $|C| \le (k-1)n$. Then

$$z_C \le (1 + o(1))(k-1)nc^{k-1}n^{\frac{k}{2}-1}(\log n)^{\frac{k}{2}} = (1 + o(1))(k-1)c^{k-1}(n\log n)^{\frac{k}{2}}.$$

Note that $t_k = \frac{1}{k}\sum_{v\in V(G)} X_v$. Thus

$$\begin{aligned} t_k &\ge \frac{1}{k}(1 - o(1))kcn \cdot c^{k-1}n^{\frac{k}{2}-1}(\log n)^{\frac{k}{2}} \\ &\ge (1 - o(1))c^k(n\log n)^{\frac{k}{2}}. \end{aligned}$$

Since $c = 16k^2r$, we have that for n sufficiently large,

$$z_C < \frac{t_k}{2r}.$$

Moreover, similar to the above calculation, we have that a.a.s.,

$$t_k \le (1 + o(1))c^k(n\log n)^{\frac{k}{2}} = O(r^k(n\log n)^{\frac{k}{2}}).$$

□

Now we will prove the main result. We use the same greedy algorithm approach by Dudek, La Fleur, Mubayi and Rődl in [54].

Proof of Theorem 2.2.1: We show that there exists a k-uniform hypergraph $\mathcal{H}$ with $|E(\mathcal{H})| = O(r^k n^{\frac{k}{2}}(\log n)^{\frac{k}{2}})$ such that any r-coloring of the edges of $\mathcal{H}$ yields a monochromatic copy of $\mathcal{P}^{(k)}_{n,k-1}$.

Let G be the graph constructed from Proposition 2.2.1 for n sufficiently large. Let $\mathcal{H}$ be a k-uniform hypergraph such that $V(\mathcal{H}) = V(G)$ and $E(\mathcal{H})$ be the collection of all proper k-cycles in G.

Take an arbitrary r-coloring of the edges $\mathcal{H}_0 = \mathcal{H}$ and assume that there is no monochromatic $\mathcal{P}^{(k)}_{n,k-1}$. Without loss of generality, suppose the color class with the most number of edges is blue. We will consider the following greedy algorithm:

(1) Let $\mathcal{B} = \varnothing$ be a *trash set* of proper $(k-1)$-paths in G. Let A be a blue tight path in $\mathcal{H}$ that we will iteratively modify. Throughout the process, let $U = V(\mathcal{H})\setminus(V(A)\cup\bigcup_{B\in\mathcal{B}}V(B))$ be the set of *unused* vertices. If at any point $|\mathcal{B}| = n$, terminate.

(2) If possible, choose a blue edge $v_1v_2\cdots v_{k-1}v_k$ from U and put these vertices into A and set the pointer to v_k. Otherwise, if not possible, terminate.

(3) Suppose the pointer is at v_i and $v_{i-k+2},\cdots,v_{i-1},v_i$ are the last $k-1$ vertices of the constructed blue path A. There are two cases:

Case 1: If there exists a vertex $u\in U$ such that $v_{i-k+2},\cdots,v_{i-1},v_i,u$ form a blue edge in $\mathcal{H}$, then we *extend* P, i.e. add $v_{i+1} = u$ into A. Set the pointer to v_{i+1} and restart Step (3).

Case 2: Otherwise, remove the last $k-1$ vertices from A and set $\mathcal{B} = \mathcal{B} \cup \{\{v_{i-k+2}, \cdots, v_{i-1}, v_i\}\}$. Set the pointer to v_{i-k+1}. Now if $|A| < k$, then set $A = \varnothing$ and go to Step (2). Otherwise, restart Step (3).

Note that this procedure will terminate under two circumstances: either $|\mathcal{B}| = n$ or there is no blue edge in U.

Let us first consider the case when $|\mathcal{B}| = n$, i.e. there are n pairwise vertex-disjoint proper $(k-1)$-paths in $\mathcal{B}$. Moreover, $|A| \leq n$ since there is no blue path of n vertices. Applying Proposition 2.2.1 with sets A and $\mathcal{B}$, we obtain that

$$y_{A,\mathcal{B}} < \frac{1}{2kr} t_{\mathcal{B}}.$$

Observe that every edge of $\mathcal{H}$ that extends some $B \in \mathcal{B}$ with a vertex from $V(\mathcal{H}_0) \setminus \left(V(A) \cup \bigcup_{B \in \mathcal{B}_m} B \right)$ must be non-blue. Therefore, the number of blue edges of $\mathcal{H}$ that contain some $B \in \mathcal{B}$ as subgraph is at most $y_{A,\mathcal{B}}$.

Consider $A, \mathcal{B}$ as $A_0, \mathcal{B}_0$ respectively. Now remove all the blue edges from $\mathcal{H}_0$ that contain some $B \in \mathcal{B}_0$ as subgraph and denote the resulting hypergraph as $\mathcal{H}_1$. Perform the greedy procedure again on $\mathcal{H}_1$. This will generate a new A_1 and $\mathcal{B}_1$. Applying Proposition 2.2.1 again, we have $y_{A_1,\mathcal{B}_1} \leq \frac{1}{2kr} t_{\mathcal{B}_1}$. Keep repeating the procedure until it is no longer possible. Observe that for $i < j$, since we removed from $\mathcal{H}_i$ all the blue edges that contain some $B \in \mathcal{B}_i$, any $B \in \mathcal{B}_i$ does not appear as subset of a blue edge in $\mathcal{H}_j$. It follows that $\mathcal{B}_i \cap \mathcal{B}_j = \varnothing$.

When the above procedure can not be repeated anymore, we are in the case that $|\mathcal{B}_m| < n$ for some positive integer m and there are no more blue edges in $V(\mathcal{H}) \setminus \bigcup_{B \in \mathcal{B}_m} B$. In this case, $A_m = \varnothing$ and all the blue edges remaining in $\mathcal{H}_m$ have to intersect the set $C = \bigcup_{B \in \mathcal{B}_m} B$. By Proposition 2.2.1, it follows that

$$z_C < \frac{1}{2r} t_k.$$

Let $e_b(\mathcal{H})$ denote the total number of blue edges in $\mathcal{H}$. We have

$$\begin{aligned} e_b(\mathcal{H}) &\leq \sum_{i=0}^{m-1} y_{A_i,\mathcal{B}_i} + z_C \\ &< \sum_{i=0}^{m-1} \frac{1}{2kr} t_{\mathcal{B}_i} + \frac{1}{2r} t_k. \end{aligned}$$

Note that since G is C_k-colorable, every proper k-cycle intersects V_i at exactly one vertex for each $i \in [k]$. Moreover, we can obtain a proper $(k-1)$-path by deleting any of the k vertices from a proper k-cycle in G. Therefore every proper k-cycle can extend exactly k proper $(k-1)$-paths. We then have $\sum_{i=0}^{m-1} t_{\mathcal{B}_i} \leq k t_k$. Thus,

$$\begin{aligned} e_b(\mathcal{H}) &< \frac{1}{2kr} \sum_{i=0}^{m-1} t_{\mathcal{B}_i} + \frac{1}{2r} t_k \\ &\leq \frac{1}{2r} t_k + \frac{1}{2r} t_k \\ &= \frac{1}{r} |E(\mathcal{H})|. \end{aligned}$$

The conclusion is that the number of blue edges in $\mathcal{H}$ is strictly smaller than $\frac{1}{r}$ of the total number of edges in $\mathcal{H}$, which contradicts that blue is the color class with the most number of edges of $\mathcal{H}$. □

2.3 Anti-Ramsey number of edge-disjoint rainbow spanning trees

An edge-colored graph G is called *rainbow* if every edge of G receives a different color. The general *anti-Ramsey problem* asks for the maximum number of colors $AR(n, \mathcal{G})$ in an edge-coloring of K_n containing no rainbow copy of any graph in a class $\mathcal{G}$. For some earlier results when $\mathcal{G}$ consists of a single graph, see the survey [79]. In particular, Montellano-Baallesteros and Neumann-Lara [138] showed a conjecture of Erdős, Simonovits and Sós [71] by computing $AR(n, C_k)$. Jiang and West [113] determined the anti-Ramsey number of the family of trees with m edges.

Anti-Ramsey problems have also been investigated for rainbow spanning subgraphs. In particular, Hass and Young [99] showed that the anti-Ramsey number for perfect matchings (when n is even) is $\binom{n-3}{2}+2$ for $n \geq 14$. For spanning trees, Bialostocki and Voxman [16] showed that the maximum number of colors in an edge-coloring of K_n ($n \geq 4$) with no rainbow spanning tree is $\binom{n-2}{2}+1$. Jahanbekam and West [109] extended the investigations to finding the anti-Ramsey number of t edge-disjoint rainbow spanning subgraphs of certain types including matchings, cycles and trees. In particular, for rainbow spanning trees, let $r(n,t)$ be the maximum number of colors in an edge-coloring of K_n not having t edge-disjoint rainbow spanning trees. Akbari and Alipour [2] showed that $r(n,2) = \binom{n-2}{2}+2$ for $n \geq 6$. Jahanbekam and West [109] showed that

$$r(n,t) = \begin{cases} \binom{n-2}{2}+t & \text{for } n > 2t + \sqrt{6t - \frac{23}{4}} + \frac{5}{2} \\ \binom{n}{2}-t & \text{for } n = 2t, \end{cases}$$

and they made the following conjecture:

Conjecture 2.3.1. [109] $r(n,t) = \binom{n-2}{2}+t$ *whenever* $n \geq 2t+2 \geq 6$.

In this section, we show that the above conjecture holds and we also determine the value of $r(n,t)$ when $n = 2t+1$. Together with previous results ([16],[2],[109]), this gives the anti-Ramsey number of t edge-disjoint rainbow spanning trees for all values of n and t.

Theorem 2.3.1. *For all positive integers t,*

$$r(n,t) = \begin{cases} \binom{n-2}{2}+t & \textit{for } n \geq 2t+2 \\ \binom{n-1}{2} & \textit{for } n = 2t+1 \\ \binom{n}{2}-t & \textit{for } n = 2t, \end{cases}$$

Remark 2.3.1. *Note that if $n < 2t$, then K_n does not have enough edges for t edge-disjoint spanning trees.*

The main tools we use are two structure theorems that characterize the existence of t color-disjoint rainbow spanning trees or the existence of a *color-disjoint* extension of t edge-disjoint rainbow spanning forests into t edge-disjoint rainbow spanning trees. When $t = 1$, Broersma and Li [22] showed that determining the largest rainbow spanning forest of a graph can be solved by applying the Matroid Intersection Theorem. The following characterization was established by Schrijver [157] using matroid methods, and later given graph theoretical proofs by Suzuki [164] and also by Carraher and Hartke [24].

Theorem 2.3.2. ([157, 164, 24]) *An edge-colored connected graph G has a rainbow spanning tree if and only if for every $2 \leq k \leq n$ and every partition of G with k parts, at least $k-1$ different colors are represented in edges between partition classes.*

The above results can be generalized to t *color-disjoint* rainbow spanning trees using similar matroid methods by Schrijver [157]. For the sake of self-completeness, we reproduce the proof using matroid methods in Section 2.3.1. We also give a new graph theoretical proof of Theorem 2.3.3.

Theorem 2.3.3. [157] *An edge-colored multigraph G has t pairwise color-disjoint rainbow spanning trees if and only if for every partition P of $V(G)$ into $|P|$ parts, at least $t(|P|-1)$ distinct colors are represented in edges between partition classes.*

Remark 2.3.2. *Recall the famous Nash-Williams-Tutte Theorem (*[142, 169]*): A multigraph contains t edge-disjoint spanning trees if and only if for every partition P of its vertex set, it has at least $t(|P|-1)$ cross-edges. Theorem 2.3.3 implies the Nash-Williams-Tutte Theorem by assigning every edge of the multigraph a distinct color.*

Theorem 2.3.3 can be also generalized to extend edge-disjoint rainbow spanning forests to edge-disjiont rainbow spanning trees. Let G be an edge-colored multigraph. Let $F_1, \ldots, F_t$ be t edge-disjoint rainbow spanning forests. We are interested in whether $F_1, \ldots, F_t$ can be extended to t edge-disjoint rainbow spanning trees $T_1, \ldots, T_t$ in G, i.e., $E(F_i) \subset E(T_i)$ for each i. We say the extension is *color-disjoint* if all edges in $\cup_i (E(T_i) \smallsetminus E(F_i))$ have distinct colors and these colors are different from the colors appearing in the edges of $\cup_i E(F_i)$. Using similar matroid methods or graph theoretical arguments, we can also obtain a criterion that characterizes the existence of a color-disjoint extension of rainbow spanning forests into rainbow spanning trees.

Theorem 2.3.4. *A family of t edge-disjoint rainbow spanning forests $F_1, \ldots, F_t$ has a color-disjoint extension in G if and only if for every partition P of G into $|P|$ parts,*

$$|c(cr(P, G'))| + \sum_{i=1}^{t} |cr(P, F_i)| \geq t(|P| - 1). \tag{2.4}$$

Here G' is the spanning subgraph of G by removing all edges with colors appearing in some F_i, and $c(cr(P, G'))$ be the set of colors appearing in the edges of G' crossing the partition P.

It would be interesting to find a similar criterion for the existence of t edge-disjoint rainbow trees in a general graph since applications of Theorem 2.3.3 and Theorem 2.3.4 usually require large number of colors in the host graph.

2.3.1 Proof of Theorem 2.3.3

We first reproduce the proof of Theorem 2.3.3 using matroid methods. A matroid is defined as $M = (E, \mathcal{I})$ where E is the ground set and $\mathcal{I} \subseteq 2^E$ is a set containing subsets of E (called indepedent sets) that satisfy (i) if $A \subseteq B \subseteq E$, and $B \in \mathcal{I}$, then $A \in \mathcal{I}$; (ii) if $A \in \mathcal{I}$, $B \in \mathcal{I}$ and $|A| > |B|$, then $\exists\ a \in A \backslash B$ such that $B \cup \{a\} \in \mathcal{I}$. Given a matroid $M = (E, \mathcal{I})$, the rank function $r_M : 2^E \to \mathbb{N}$ is defined as $r_M(S) = \max\{|I| : I \subseteq S, I \in \mathcal{I}\}$. Thus $r_M(E)$ is the size of the maximum independent set of M. Two matroids of interests here are the *graphic matroid* and the *partition matroid*. Given an edge-colored graph G, the graphic matroid of G is the matroid $M = (E, \mathcal{I})$

where $E = E(G)$ and $\mathcal{I}$ is the set of forests in G. The partition matroid of G, is the matroid $M' = (E', \mathcal{I}')$ where $E' = E(G)$ and $\mathcal{I}$ is the set of rainbow subgraphs of G. Given k matroids $\{M_i = (E_i, \mathcal{I}_i)\}_{i \in [k]}$, one can define the *union* of the k matroids, $M_1 \vee \cdots \vee M_k = (E, \mathcal{I})$, by

$$E = \bigcup_{i=1}^{k} E_i,$$

and

$$\mathcal{I} = \{I_1 \cup \cdots \cup I_k : I_i \in \mathcal{I}_i \text{ for all } i \in [k]\}.$$

It is well known in matroid theory [57, 141] that $M_1 \vee \cdots \vee M_k$ is a matroid with rank function

$$r(S) = \min_{T \subseteq S} \left(|S \backslash T| + \sum_{i=1}^{k} r_{M_i}(T \cap E_i) \right).$$

Given two matroids $M_1 = (E, \mathcal{I}_1)$ and $M_2 = (E, \mathcal{I}_2)$ on the same ground set with rank functions r_1 and r_2 respectively, consider the family of independent sets common to both matroids, i.e., $\mathcal{I}_1 \cap \mathcal{I}_2$. The well-known Matroid Intersection Theorem [58] asserts that

$$\max_{I \in \mathcal{I}_1 \cap \mathcal{I}_2} |I| = \min_{U \subseteq E} \left(r_1(U) + r_2(E \backslash U) \right).$$

2.3.2 Proof of Theorem 2.3.3 using Matroid methods

Again we remark that the proof essentially follows the same approaches as Schrijver [157] and we only reproduce it here for the sake of completeness.

Proof of Theorem 2.3.3. The forward direction is clear. Thus it remains to show that if for every partition P of $V(G)$ into $|P|$ parts, at least $t(|P| - 1)$ distinct colors are represented in edges between partition classes, then there exist t edge-disjoint rainbow spanning trees in G.

Given an edge-colored graph G, let $M = (E, \mathcal{I})$ be the graphic matroid of G and $M' = (E, \mathcal{I}')$ be the partition matroid of G. Moreover, let $M^t = M \vee M \vee \cdots \vee M =$

$(E, \mathcal{I}^t)$, where we take t copies of M. By the matriod union theorem, we obtain that

$$r_{M^t}(S) = \min_{T \subseteq S} \left(|S \backslash T| + t \cdot r_M(T) \right).$$

By the Matroid Intersection Theorem,

$$\begin{aligned} \max_{I \in \mathcal{I}^t \cap \mathcal{I}'} |I| &= \min_{U \subseteq E} \left(r_{M^t}(U) + r_{M'}(E \backslash U) \right) \\ &= \min_{U \subseteq E} \left(\min_{T \subseteq U} \left(|U \backslash T| + t \cdot r_M(T) \right) + r_{M'}(E \backslash U) \right). \end{aligned}$$

Let $T, U \subseteq E$ be arbitrarily chosen such that $T \subseteq U$. Observe that $t \cdot r_M(T) = t(n - q(T))$, where $q(T)$ is the number of components of $G[T]$. Now we claim that

$$|U \backslash T| + r_{M'}(E \backslash U) \geq r_{M'}(E \backslash T) \geq t(q(T) - 1).$$

Indeed, for any color c appearing in some edge $e \in E \backslash T$, if $e \in E \backslash U$, then the color c is counted in $r_{M'}(E \backslash U)$; if $e \in U$, then that color is counted in $|U \backslash T|$. In particular, at least $t(q(T) - 1)$ distinct colors are represented in edges between connected components of T, thus in $E \smallsetminus T$. It follows that

$$|U \backslash T| + t \cdot r_M(T) + r_{M'}(E \backslash U) \geq t(q(T) - 1) + t(n - q(T)) \geq t(n - 1),$$

which implies that $\max_{I \in \mathcal{I}^t \cap \mathcal{I}'} |I| \geq t(n-1)$. By definition, we then have t edge-disjoint rainbow spanning trees.

$\square$

2.3.3 Proof of Theorem 2.3.3 using graph theoretical arguments

In this subsection, we give a new graph theoretical proof of Theorem 2.3.3. Given a graph G, we use $V(G), E(G)$ to denote its vertex set and edge set respectively. We use $\|G\|$ to denote the number of edges in G. Given a set of edges E, we use $c(E)$ to denote the set of colors that appear in E. For clarity, we abuse the notation to use $c(e)$ to denote the color of an edge e. We say a color c has *multiplicity* k in G if the number of edges with color c in G is k. The *color multiplicity* of an edge in G is the multiplicity of the color of the edge in G.

For any partition P of the vertex set $V(G)$ and a subgraph H of G, let $|P|$ denote the number of parts in the partition P and let $cr(P, H)$ denote the set of crossing edges in H whose end vertices belong to different parts in the partition P. When $H = G$, we also write $cr(P, G)$ as $cr(P)$. Given two partitions $P_1 \colon V = \cup_i V_i$ and $P_2 \colon V = \cup_j V_j'$, let the intersection $P_1 \cap P_2$ denote the partition given by $V = \bigcup\limits_{i,j} V_i \cap V_j'$. Given a spanning disconnected subgraph H, there is a natural partition P_H associated to H, which partitions V into its connected components. Without loss of generality, we abuse our notation $cr(H)$ to denote the crossing edges of G corresponding to this partition P_H. Recall we want to show that an edge-colored multigraph G has t color-disjoint rainbow spanning trees if and only if for any partition P of $V(G)$ (with $|P| \geq 2$),

$$|c(cr(P))| \geq t(|P| - 1). \tag{2.5}$$

Proof of Theorem 2.3.3. One direction is easy. Suppose that G contains t pairwise color-disjoint rainbow spanning trees $T_1, T_2, \ldots, T_t$. Then all edges in these trees have distinct colors. For any partition P of the vertex set V, each tree contributes at least $|P| - 1$ crossing edges, thus t trees contribute at least $t(|P| - 1)$ crossing edges and the colors of these edges are all distinct.

Now we prove the other direction. Assume that G satisfies inequality (2.5). We would like to prove G contains t pairwise color-disjoint rainbow spanning trees. We will prove by contradiction. Assume that G does not contain t pairwise color-disjoint rainbow spanning trees. Let $\mathcal{F}$ be the collection of all families of t color-disjoint rainbow spanning forests $\{F_1, \cdots, F_t\}$. Consider the following deterministic process:

Initially, set $C' := \bigcup\limits_{j=1}^{t} c(cr(F_j))$

while $C' \neq \varnothing$ **do**

for each color x in C', **d**o

for j from 1 to t, **d**o

if color x appears in F_j, **t**hen

delete the edge in color x from F_j

endif

endfor

endfor

set $C' := \bigcup_{j=1}^{t} c(cr(F_j)) - C'$

endwhile

For $i \geq 0$, $F_j^{(i)}$ denote the rainbow spanning forest F_j after i iterations of the while loop. In particular, $F_j^{(0)} = F_j$ for all $j \in [t]$ and $F_j^{(\infty)}$ is the resulting rainbow spanning forest of F_j after the process. Similarly, let C_i denote the set C' after the i-th iteration of the while loop. Note that C_i is the set of new colors crossing components of F_js after some edges are deleted in the i-th iteration.

Observe that since the procedure is deterministic, $\{F_j^{(i)} : j \in [t], i > 0\}$ is unique for a fixed family $\{F_1, \cdots, F_t\}$. We define a *preorder* on $\mathcal{F}$. We say a family $\{F_j\}_{j=1}^t$ is less than or equal to another family $\{F_j'\}_{j=1}^t$ if there is a positive integer l such that

1. For $1 \leq i < l$, $\sum_{j=1}^{t} \|F_j^{(i)}\| = \sum_{j=1}^{t} \|F'^{(i)}_j\|$.
2. $\sum_{j=1}^{t} \|F_j^{(l)}\| < \sum_{j=1}^{t} \|F'^{(l)}_j\|$.

Since G is finite, so is $\mathcal{F}$. There exists a maximal element $\{F_1, F_2, \cdots, F_t\} \in \mathcal{F}$. Run the deterministic process on $\{F_1, F_2, \cdots, F_t\}$.

The goal is to construct a common partition P by refining $cr(F_j)$ so that $|c(cr(P))| < t(|P| - 1)$. In particular, we will show that all forests in $\{F_j^{(\infty)} : j \in [t]\}$ admit the same partition P.

Claim (a): $\bigcup_{j=1}^{t} c\left(\mathrm{cr}(F_j^{(i)})\right) \subseteq \left(\bigcup_{j=1}^{t} c\left(\mathrm{cr}(F_j^{(i-1)})\right)\right) \cup \left(\bigcup_{j=1}^{t} c(F_j^{(i)})\right)$.

AFSOC that there is a color $x \in \bigcup_{j=1}^{t} c(cr(F_j^{(i)})) \setminus \bigcup_{j=1}^{t} c(cr(F_j^{(i-1)}))$ and there is no edge in color x in all forests $F_1^{(i)}, \ldots, F_t^{(i)}$. Let e be the edge such that $c(e) = x$ and $e \in cr(F_s^{(i)})$ for some $s \in [t]$. Observe that since $c(e) \notin \bigcup_{j=1}^{t} c(cr(F_j^{(i-1)}))$, it follows that $F_s^{(i-1)} + e$ contains a rainbow cycle, which passes through e and another edge $e' \in F_s^{(i-1)}$ joining two distinct components of $F_s^{(i)}$. Now let us consider a new family of rainbow spanning forests $\{F'_1, \cdots, F'_t\}$ where $F'_j = F_j$ for $j \neq s$ and $F'_s = F_s - e' + e$. The color-disjoint property is maintained since the color of edge e is not in any F_j. Observe that since $c(e) \notin \bigcup_{j=1}^{t} c(\mathrm{cr}(F_j^{(i-1)}))$, $F_s'^{(i)}$ will have one fewer component than $F_s^{(i)}$. Thus we have

$$\sum_{j=1}^{t} \|F_j^{(k)}\| = \sum_{j=1}^{t} \|F_j'^{(k)}\| \text{ for } k < i.$$

$$\sum_{j=1}^{t} \|F_j'^{(i)}\| > \sum_{j=1}^{t} \|F_j^{(i)}\|.$$

which contradicts our maximality assumption of $\{F_i : i \in [t]\}$. That finishes the proof of Claim (a).

Claim (a) implies that for each $x \in C_i$, there is an edge e of color x in exactly one of the forests in $\{F_j^{(i)} : j \in [t]\}$. Thus removing that edge in the next iteration will increase the sum of number of partitions exactly by 1. Thus we have that

$$\sum_{j=1}^{t} |P_{F_j^{(i+1)}}| = \sum_{j=1}^{t} |P_{F_j^{(i)}}| + |C_i|.$$

It then follows that

$$\begin{aligned}\sum_{j=1}^{t} |P_{F_j^{(\infty)}}| &= \sum_{j=1}^{t} |P_{F_j}| + \sum_{i} |C_i| \\ &= \sum_{j=1}^{t} |P_{F_j}| + |\bigcup_{j=1}^{t} c(\mathrm{cr}(F_j^{(\infty)}))|.\end{aligned}$$

Finally set the partition $P = \bigcap_{j=1}^{t} P_{F_j^{(\infty)}}$. We claim $P_{F_j^{(\infty)}} = P$ for all j. This is because all edges in $cr(P_{F_j^{(\infty)}}) \cap \bigcup_{k=1}^{t} E(F_k^{(\infty)})$ have been already removed. We then have that

$$
\begin{aligned}
t|P| &= \sum_{j=1}^{t} |P_{F_j^{(\infty)}}| \\
&= \sum_{j=1}^{t} |P_{F_j}| + |\bigcup_{j=1}^{t} c(\mathrm{cr}(F_j^{(\infty)}))| \\
&= \sum_{j=1}^{t} |P_{F_j}| + |c(\mathrm{cr}(P))| \\
&\geq t + 1 + |c(\mathrm{cr}(P))|.
\end{aligned}
$$

We obtain

$$|c(\mathrm{cr}(P))| \leq t(|P| - 1) - 1.$$

Contradiction.

$\square$

Corollary 2.3.1. *The edge-colored complete graph K_n has t color-disjoint rainbow spanning trees if the number of edges colored with any fixed color is at most $n/(2t)$.*

Proof. Suppose K_n does not have t color-disjoint rainbow spanning trees, then there exists a partition P of $V(K_n)$ into r parts $(2 \leq r \leq n)$ such that the number of distinct colors in the crossing edges of P is at most $t(r-1)-1$. Let m be the number of edges crossing the partition P. It follows that

$$m \leq (t(r-1)-1) \cdot \frac{n}{2t} \leq \frac{n}{2}(r-1) - \frac{n}{2t}.$$

On the other hand,

$$m \geq \binom{n}{2} - \binom{n-(r-1)}{2}.$$

Hence we have

$$\binom{n}{2} - \binom{n-(r-1)}{2} \leq \frac{n}{2}(r-1) - \frac{n}{2t}.$$

which implies

$$(n-r)(r-1) \leq -\frac{n}{t}.$$

which contradicts that $2 \leq r \leq n$. $\square$

Remark: This result is tight since the total number of colors used in K_n could be as small as $\binom{n}{2}/(n/(2t)) = t(n-1)$, but any t color-disjoint rainbow spanning trees need $t(n-1)$ colors. On the contrast, a result by Carraher, Hartke and Horn [25] implies there are $\Omega(n/\log n)$ edge-disjoint rainbow spanning trees.

2.3.4 Proof of Theorem 2.3.4

Recall we want to show that any t edge-disjoint rainbow spanning forests $F_1, \ldots, F_t$ have a color-disjoint extension to edge-disjoint rainbow spanning trees in G if and only if

$$|c(\mathrm{cr}(P, G'))| + \sum_{j=1}^{t} |cr(P, F_j)| \geq t(|P| - 1).$$

where G' is the spanning subgraph of G by removing all edges with colors appearing in some F_j.

Proof. Again, the forward direction is trivial. We only need to show that condition (2.4) implies there exists a color-disjoint extension to edge-disjoint rainbow spanning trees. The proof is similar to the proof of Theorem 2.3.3. Consider a set of edge-maximal forests $F_1^{(0)}, \ldots, F_t^{(0)}$ which is a color-disjoint extension of $F_1, \ldots, F_t$. From $\{F_j^{(0)}\}$ we delete all edges (in $\{F_j^{(0)}\}$) of some color c appearing in $\bigcup_{j=1}^t c(\mathrm{cr}(F_j^{(0)}, G'))$ to get a new set $\{F_j^{(1)}\}$. Repeat this process until we reach a stable set $\{F_j^{(\infty)}\}$. Since we only delete edges in G', we have $E(F_j) \subseteq E(F_j^{(\infty)})$ for each $1 \leq j \leq t$. The edges and colors in $\cup_{j=1}^t E(F_j)$ will not affect the process. A similar claim still holds:

$$\bigcup_{j=1}^{t} c(\mathrm{cr}(F_j^{(i)}, G')) \subseteq \left(\bigcup_{j=1}^{t} c(\mathrm{cr}(F_j^{(i-1)}, G'))\right) \cup \left(\bigcup_{j=1}^{t} c\left(E(F_j^{(i)}) \cap E(G')\right)\right).$$

In particular, let $C_i = \left(\bigcup_{j=1}^t c(\mathrm{cr}(F_j^{(i)}, G'))\right) \setminus \left(\bigcup_{j=1}^t c(\mathrm{cr}(F_j^{(i-1)}, G'))\right)$. Then we have

$$\sum_{j=1}^{t} |P_{F_j^{(i+1)}}| = \sum_{j=1}^{t} |P_{F_j^{(i)}}| + |C_i|.$$

It then follows that

$$\begin{aligned}\sum_{j=1}^{t} |P_{F_j^{(\infty)}}| &= \sum_{j=1}^{t} |P_{F_j^{(0)}}| + \sum_i |C_i| \\ &= \sum_{j=1}^{t} |P_{F_j^{(0)}}| + |\bigcup_{j=1}^{t} c(\mathrm{cr}(F_j^{(\infty)}, G'))|.\end{aligned}$$

Finally set the partition $P = \bigcap_{j=1}^{t} P_{F_j^{(\infty)} \setminus E(F_j)}$. Clearly all edges in $\mathrm{cr}(P, G')$ are removed. All possible edges remaining in G that cross the partition P are exactly the edges in $\bigcup_{j=1}^{t} \mathrm{cr}(P, F_j)$. We have

$$\begin{aligned}t|P| &= \sum_{j=1}^{t} |P_{F_j^{(\infty)}}| + \sum_{j=1}^{t} |\mathrm{cr}(P, F_j)| \\ &= \sum_{j=1}^{t} |P_{F_j^{(0)}}| + |\bigcup_{j=1}^{t} c(\mathrm{cr}(F_j^{(\infty)}, G'))| + \sum_{j=1}^{t} |\mathrm{cr}(P, F_j)| \\ &= \sum_{j=1}^{t} |P_{F_j^{(0)}}| + |c(\mathrm{cr}(P, G'))| + \sum_{j=1}^{t} |\mathrm{cr}(P, F_j)| \\ &\geq t + 1 + |c(\mathrm{cr}(P, G'))| + \sum_{j=1}^{t} |\mathrm{cr}(P, F_j)|.\end{aligned}$$

We obtain

$$|c(\mathrm{cr}(P, G'))| + \sum_{j=1}^{t} |\mathrm{cr}(P, F_j)| \leq t(|P| - 1) - 1.$$

Contradiction.

□

2.3.5 Proof of Theorem 2.3.1

Recall that $r(n,t)$ is the maximum number of colors in an edge-coloring of the complete graph K_n not having t edge-disjoint rainbow spanning trees.

Lower Bound: Jahanbekam and West (See Lemma 5.1 in [109]) showed the following lower bound for $r(n,t)$.

Proposition 2.3.1. *[109] For positive integers n and t such that $t \leq 2n-3$, there is an edge-coloring of K_n using $\binom{n-2}{2}+t$ colors that does not have t edge-disjoint rainbow* *spanning trees. When $n = 2t+1$, the construction improves to $\binom{n-1}{2}$ colors. When $n = 2t$, it improves to $\binom{n}{2} - t$.*

This matches the upper bounds in Theorem 2.3.1. Hence we will skip the proof of lower bounds in the subsequent theorems. Moreover, we only consider the case $t \geq 2$ since the case $t = 1$ was already resolved in Bialostocki and Voxman [16]. In Section 2.3.6, we prove a technical lemma that will be used in the proof of Theorem 2.3.1. In Section 2.3.7, 2.3.8,2.3.9, we show Theorem 2.3.1 when n is in different range of values with respect to t.

2.3.6 Technical lemma

Lemma 2.3.1. *Let G be an edge-colored graph with s colors $c_1, \cdots, c_s$ and $|V(G)| = n = 2t+2$ where $t \geq 3$. For color c_i, let m_i be the number of edges of color c_i. Suppose $\sum_{i=1}^{s}(m_i - 1) = 3t$ and $m_i \geq 2$ for all $i \in [s]$. Then we can construct t edge-disjoint rainbow forests $F_1, \ldots, F_t$ in G such that if we define $G_0 = G - \bigcup_{i=1}^{t} E(F_i)$, then*

$$|E(G_0)| \leq 2t+1. \tag{2.6}$$

and

$$\Delta(G_0) \leq t+1. \tag{2.7}$$

Proof. We consider two cases:

Case 1: $m_1 \geq 2t+2$. Note that

$$\sum_{i=2}^{s}(m_i - 1) = 3t - (m_1 - 1) \leq t - 1.$$

Thus, $s \leq t$. Let $d_i(v)$ be the number of edges in color c_i and incident to v in the current graph G. We construct the edge-disjoint rainbow forests $F_1, F_2, \ldots, F_t$ in two rounds: In the first round, we greedily extract edges only in color c_1. For $i = 1, \ldots, t$, at step i, pick a vertex v with maximum $d_1(v)$ (break tie arbitrarily). Pick an edge in color c_1 incident to v, assign it to F_i, and delete it from G.

We claim that after the first round $d_1(v) \le t+1$ for any vertex v. Suppose not, i.e., $d_1(v) \ge t+2$. Since $n-1-(t+2) < t$, it follows that there exists another vertex u with $d_1(u) \ge d_1(v)-1 \ge t+1$. This implies

$$m_1 \ge t + d_1(v) + d_1(u) - 1 \ge 3t + 2.$$

However,

$$m_1 - 1 \le \sum_{i=1}^{s}(m_i - 1) = 3t.$$

which gives us the contradiction.

In the second round, we greedily extract edges not in color c_1. For $i = 1, \ldots, t$, at step i, among all vertices with at least one neighboring edge not in color c_1, pick a vertex v with maximum vertex degree $d(v)$ (pick arbitrarily if tie). Pick an edge incident to v and not in color c_1, assign it to F_i, and delete it from G.

If we succeed with selecting t edges not in color c_1 in the second round, we claim $d(v) \le t+1$ for any vertex v. Suppose not, if $d(v) \ge t+2$. Then there is another vertex u with $d(u) \ge d(v) - 1 \ge t+1$. It implies

$$\sum_{i=1}^{s} m_i \ge 2t + d(u) + d(v) - 1 \ge 4t + 2.$$

However, since $s \le t$, we have

$$\sum_{i=1}^{s} m_i \le 3t + s \le 4t.$$

Contradiction. Therefore it follows that $d(v) \le t+1$. Moreover, $|E(G_0)| \le 4t - 2t \le 2t$.

If the process stops at step $i = l < t$, then all remaining edges in G_0 must be in color 1. Thus, by the previous claim, $\Delta(G_0) \le t+1$. Moreover,

$$|E(G_0)| \le m_1 - t \le (3t+1) - t = 2t + 1.$$

In both cases above, $F_1, \cdots F_t$ are edge-disjoint rainbow forests that satisfies inequality (2.6) and (2.7).

Case 2: $m_1 \le 2t+1$.

Claim: there exists t edge-disjoint rainbow forests $F_1, F_2, \cdots, F_t$ such that $\Delta(G_0) \le t+1$.

Proof of Claim. For $j = 1, 2, \ldots, t$, we will construct a rainbow forest F_j by selecting a rainbow set of edges such that after deleting these edges from G, $\Delta(G_0) \le 2t+1-j$. Notice that when $j = t$, we will have $\Delta(G_0) \le t+1$. Our procedure is as follows:

For step j, without loss of generality, let $v_1, v_2, \cdots, v_l$ be the vertices with degree $2t+2-j$ and let $c_1, c_2, \cdots, c_m$ be the set of colors of edges incident to $v_1, v_2, \cdots, v_l$ in G. If there is no such vertex, simply pick an edge incident to the max-degree vertex and assign it to F_j. Otherwise, we will construct an auxiliary bipartite graph $H = A \cup B$ where $A = \{v_1, \cdots, v_l\}$ and B $= \{c_1, c_2, \ldots, c_m\}$ and $v_x c_y \in E(H)$ if and only if there is an edge of color c_y incident to v_x. We claim that there exists a matching of A in H. Suppose not, then by Hall's theorem, there exists a set of vertices $A' = \{u_1, u_2, \cdots u_k\} \subseteq A$ such that $|N(A')| < |A'| = k$ where $k \ge 2$. Without loss of generality, suppose $N(A) = \{c'_1, c'_2, \cdots, c'_q\}$ where $q \le k-1$. Let m'_i be the number of edges of color c'_i remaining in G.

Note that $k \neq 2$ since otherwise we will have one color with at least $2 \cdot (2t+2-j) - 1 \ge 2t+3$ edges, which contradicts our assumption in this case.

Notice that for every $i \in [k]$, u_i has at least $(2t+2-j)$ edges incident to it. Moreover, at least $j-1$ edges are already deleted from G in previous steps. Therefore, we have

$$\frac{k(2t+2-j)}{2} \le \sum_{i=1}^{q} m'_i \le \left(\sum_{i=1}^{q} (m'_i - 1) \right) + (k-1) \le 3t - (j-1) + (k-1).$$

It follows that

$$k \le 2 + \frac{2t}{2t-j} \le 4.$$

Similarly, using another way of counting the edges incident to some u_i ($i \in [k]$), we have

$$k(2t+2-j) - \binom{k}{2} \le 3t - (j-1) + (k-1).$$

which implies that

$$t(2k-3) \le \frac{k(k-3)}{2} + j(k-1) \le \frac{k(k-3)}{2} + t(k-1).$$

It follows that $t \le \frac{k(k-3)}{2(k-2)}$. Since $k \le 4$ and $k > 2$, we obtain that $t \le 1$, which contradicts our assumption that $t \ge 2$. Thus by contradiction, there exists a matching of A in H. This implies that there exists a rainbow set of edges E_j that cover all vertices with degree $2t+2-j$ in step j. We can then find a maximally acyclic subset F_j of E_j such that F_j is a rainbow forest and every vertex of degree $2t+2-j$ is adjacent to some edge in F_j. Delete edges of F_j from G and we have $\Delta(G_0) \le 2t+1-j$. As a result, after t steps, we obtain t edge-disjoint rainbow forests $F_1, \cdots, F_t$ and $\Delta(G_0) \le t+1$. This finishes the proof of the claim. □

Now let $\{F_1, F_2, \cdots, F_t\}$ be an edge-maximal set of t edge-disjoint rainbow forests that satisfies $\Delta(G_0) \le t+1$. We claim that $|E(G_0)| \le 2t+1$. Suppose not, i.e., $|E(G_0)| \ge 2t+2$. It follows that $\sum_{i=1}^{t} |E(F_i)| \le 6t - (2t+2) < 4t$, i.e. there exists a $j \in [t]$ such that F_j has at most 3 edges. Since F_j is edge maximal, none of the edges in G_0 can be added to F_j. We have three cases:

Case 2a: $|E(F_j)| = 1$. It then follows that all edges in G_0 have the same color (call it c_1') as the single edge in F_j. Thus we have a color with multiplicity at least $2t+3$, which contradicts that $m_1 < 2t+2$.

Case 2b: $|E(F_j)| = 2$. Similarly, we have that at least $2t+1$ edges in G_0 that share the same colors (call them c_1', c_2') as edges in F_j. It follows that $m_1 + m_2 \ge 2t+3$. Similar to Case 1, in this case, we have that $s \le t+1$

and $|E(G)| = 3t + s \leq 4t + 1$. Since $|E(G_0)| \geq 2t + 2$, that implies that $\sum_{i=1}^{t} |E(F_i)| \leq (4t+1) - (2t+2) = 2t - 1$. Hence there exists some F_k such that $|E(F_k)| \leq 1$ and we are done by Case 2a.

Case 2c: $|E(F_j)| = 3$. Similarly, we have that at least $2t - 1$ edges in G_0 share the same colors (call them c'_1, c'_2, c'_3) as edges in F_j. It follows that $m_1 + m_2 + m_3 \geq 2t + 2$. By inequality (2.8), we have that $s \leq t + 4$ and $|E(G)| \leq 4t + 4$. Since $|E(G_0)| \geq 2t + 2$, that implies that $\sum_{i=1}^{t} |E(F_i)| \leq 2t + 2$. Since $t \geq 3$ by our assumption, there exists a $k \in [t]$ such that $|E(F_k)| \leq 2$ and we are done by Case 2b and Case 2c.

Therefore, by contradiction, we have that $|E(G_0)| \leq 2t + 1$ and we are done.

□

2.3.7 Proof of Theorem 2.3.1 where $n = 2t + 2$

Proposition 2.3.2. *For any $n = 2t + 2 \geq 6$, we have $r(n,t) = \binom{n-2}{2} + t = 2t^2$.*

Proof. Note that the lower bound is shown by Jahanbekam and West in Proposition 2.3.1. For the upper bound, we will assume that $t \geq 3$ since the case when $t = 2$ is implied by the result of Akbari and Alipour [2]. We will show that any coloring of K_{2t+2} with $2t^2 + 1$ distinct colors contains t edge-disjoint rainbow spanning trees. Call this edge-colored graph G. Let m_i be the multiplicity of the color c_i in G. Without loss of generality, say the first s colors have multiplicity at least 2, i.e.

$$m_1 \geq m_2 \geq \cdots \geq m_s \geq 2.$$

Let G_1 be the spanning subgraph of G consisting of all edges with color multiplicity greater than 1 in G. Let G_2 be the spanning subgraph consisting of the remaining edges. We have

$$\sum_{i=1}^{s} (m_i - 1) = \binom{n}{2} - (2t^2 + 1) = 3t. \tag{2.8}$$

In particular, we have

$$|E(G_1)| = \sum_{i=1}^{s} m_i = 3t + s \le 6t.$$

By Lemma 2.3.1, it follows that we can construct t edge-disjoint rainbow spanning forests $F_1, \ldots, F_t$ in G such that if we define $G_0 = E(G_1) - \bigcup_{i=1}^{t} E(F_i)$, then

$$|E(G_0)| \le 2t + 1.$$

and

$$\Delta(G_0) \le t + 1.$$

Now we show that $F_1, \ldots, F_t$ have a color-disjoint extension to t edge-disjoint rainbow spanning trees. Consider any partition P. We will verify

$$|c(\mathrm{c}r(P), G_2)| + \sum_{i=1}^{t} |\mathrm{c}r(P, F_i)| \ge t(|P| - 1). \tag{2.9}$$

We will first verify the case when $3 \le |P| \le n$. Note that

$$|c(\mathrm{c}r(P), G_2)| + \sum_{i=1}^{t} |\mathrm{c}r(P, F_i)| - t(|P| - 1) \ge \binom{n}{2} - (2t + 1) - \binom{n - |P| + 1}{2} - t(|P| - 1).$$

We want to show that the right hand side of the above inequality is nonnegative. Note that the function on the right hand side is concave downward with respect to $|P|$. Thus it is sufficient to verify it at $|P| = 3$ and $|P| = n$.

When $|P| = 3$, we have

$$\binom{n}{2} - (2t + 1) - \binom{n - 2}{2} - 2t = 0.$$

When $|P| = n$, we have

$$\binom{n}{2} - (2t + 1) - t(n - 1) = 0.$$

It remains to verify the inequality (2.9) for $|P| = 2$. By Theorem 2.3.4, we have $|E(G_0)| \le 2t + 1$. If each part of P contains at least 2 vertices, then we have

$$|c(\mathrm{c}r(P), G_2)| + \sum_{i=1}^{t} |\mathrm{c}r(P, F_i)| - t(|P| - 1)$$

$$
\begin{aligned}
&\geq \binom{n}{2} - |E(G_0)| - \left(\binom{n-2}{2} + 1\right) - t \\
&\geq \binom{n}{2} - (2t+1) - \left(\binom{n-2}{2} + 1\right) - t \\
&= t - 1 \geq 0.
\end{aligned}
$$

Otherwise, P is of the form $V(G) = \{v\} \cup B$ for some $v \in V(G)$ and $B = V(G)\backslash\{v\}$. By Lemma 2.3.1, we have $d_{G_0} \leq t+1$. Thus,

$$|c(cr(P), G_2)| + \sum_{i=1}^{t} |cr(P, F_i)| - t(|P|-1) \geq (n-1) - d_{G_0}(v) - t \geq 2t+1-(t+1)-t = 0.$$

Therefore, by Theorem 2.3.4, $F_1, \ldots, F_t$ have a color-disjoint extension to t edge-disjoint rainbow spanning trees. □

2.3.8 Proof of Theorem 2.3.1 where $n \geq 2t+3$

Proposition 2.3.3. *For any $n \geq 2t+2 \geq 6$, we have $r(n,t) = \binom{n-2}{2} + t$.*

Proof. Again, the lower bound is due to Proposition 2.3.1. For the upper bound, we will show that every edge-coloring of K_n with exactly $\binom{n-2}{2} + t + 1$ distinct colors has t edge-disjoint spanning trees. Call this edge-colored graph G.

Given a vertex v, we define $D(v)$ to be the set of colors C such that every edge with colors in C is incident to v. Given a vertex v and a set of colors C, define $\Gamma(v, C)$ as the set of edges incident to v with colors in C. For ease of notation, we let $\Gamma(v) = \Gamma(v, D(v))$.

For fixed t, we will prove the theorem by induction on n. The base case is when $n = 2t+2$, which is proven in Proposition 2.3.2. Let's now consider the theorem when $n \geq 2t+3$.

Case 1: there exists a vertex $v \in V(G)$ with $|\Gamma(v)| \geq t$ and $|D(v)| \leq n-3$.

In this case, we set $G' = G - \{v\}$. Note that G' is an edge-colored complete graph with at least $\binom{n-2}{2} + t + 1 - (n-3) = \binom{n-3}{2} + t + 1$ distinct colors. Moreover

$|G'| \geq 2t + 2$. Hence by induction, there exists t edge-disjoint rainbow spanning trees in G'. Note that by our definition of $D(v)$, none of the colors in $D(v)$ appear in $E(G')$. Moreover, since $|\Gamma(v)| \geq t$, we can extend the t edge-disjoint rainbow spanning trees in G' to G by adding one edge in $\Gamma(v)$ to each of the rainbow spanning trees in G'.

Case 2: Suppose we are not in Case 1. We first claim that there exists two vertices $v_1, v_2 \in V(G)$ such that $|\Gamma(v_1)| \leq t - 1$ and $|\Gamma(v_2)| \leq t - 1$.

Otherwise, there are at least $n - 1$ vertices u with $|\Gamma(u)| \geq t$. Since we are not in Case 1, it follows that all these vertices u also satisfy $|D(u)| \geq n - 2$. Hence by counting the number of distinct colors in G, we have that

$$\frac{(n-1)(n-2)}{2} \leq \binom{n-2}{2} + t + 1.$$

which implies that $n \leq t + 3$, giving us the contradiction.

Now suppose $|\Gamma(v_1)| \leq t - 1$ and $|\Gamma(v_2)| \leq t - 1$. Let $D = D(v_1) \cup D(v_2)$. Add new colors to D until $|\Gamma(v_1, D)| \geq t$, $|\Gamma(v_2, D)| \geq t + 1$ and $|D| \geq t + 1$. Call the resulting color set S. Note that

$$t + 1 \leq |S| \leq 2t + 1 \leq n - 2.$$

Now let $G' = G - \{v_1, v_2\}$ and delete all edges of colors in S from G'.

We claim that G' has t color-disjoint rainbow spanning trees. By Theorem 2.3.3, it is sufficient to verify the condition that for any partition P of $V(G')$,

$$|c(cr(P, G'))| \geq t(|P| - 1).$$

Observe

$$\begin{aligned}
&|c(cr(P, G'))| - t(|P| - 1) \\
&\geq |c(E(G')| - \binom{n-1-|P|}{2} - t(|P| - 1)
\end{aligned}$$

$$\geq \binom{n-2}{2} + t + 1 - |S| - \binom{n-1-|P|}{2} - t(|P|-1)$$
$$\geq \binom{n-2}{2} + t + 1 - (n-2) - \binom{n-1-|P|}{2} - t(|P|-1).$$

Note the expression above is concave downward as a function of $|P|$. It is sufficient to check the value at 2 and $n-2$. When $|P| = 2$, we have

$$|c(cr(P, G'))| - t(|P|-1) \geq \binom{n-2}{2} + t + 1 - (n-2) - \binom{n-3}{2} - t = 0.$$

When $|P| = n-2$, we have

$$\begin{aligned} |c(cr(P, G'))| - t(|P|-1) &\geq \binom{n-2}{2} + t + 1 - (n-2) - t(n-3) \\ &= \frac{(n-4)(n-2t-3)}{2} \\ &\geq 0. \end{aligned}$$

Here we use the assumption $n \geq 2t+3$ in the last step. Now it remains to extend the t color-disjoint spanning trees we found to G by using only the colors in S. Let $e_1, \cdots, e_k$ be the edges in G incident to v_1 with colors in S. Let $e'_1, \cdots e'_l$ be the edges in $G \backslash \{v_1\}$ incident to v_2 with colors in S. With our selection of S, it follows that $k, l \geq t$. Now construct an auxiliary bipartite graph H with partite sets $A = \{e_1, \cdots, e_k\}$ and $B = \{e'_1, \cdots, e'_l\}$ such that $e_i e'_j \in E(H)$ if and only if e_i, e'_j have different colors in G.

We claim that there is a matching of size t in H. Let M be the maximum matching in H. Without loss of generality, suppose $e_1 e'_1, \cdots, e_m e'_m \in M$ where $m < t$. It follows that $\{e_j : m < j \leq k\} \cup \{e'_j : m < j \leq l\}$ all have the same color (otherwise we can extend the matching). Without loss of generality, they all have color x. Now observe that for every matched edge $e_i e'_i$, exactly one of the two end vertices must be in color x. Otherwise, we can extend the matching by pairing e_i with e'_t and e_t with e'_i. This implies that H has at most t colors, which contradicts that $|S| \geq t+1$.

Hence there is a matching of size t in H. Since none of the edges in G' have colors in S, it follows that we can extend the t color-disjoint rainbow spanning trees in G' to t edge-disjoint rainbow spanning trees in G.

Hence in all of the three cases, we obtain that G has t edge-disjoint rainbow spanning trees.

□

2.3.9 Theorem 2.3.1 where $n = 2t + 1$

Proposition 2.3.4. *For positive integers $t \geq 1$ and $n = 2t+1$, we have $r(n,t) = \binom{n-1}{2} = 2t^2 - t$.*

Proof. Again, the lower bound is due to Proposition 2.3.1. Now we prove that any edge-coloring of K_{2t+1} with $2t^2 - t + 1$ distinct colors contains t edge-disjoint rainbow spanning trees. Call this edge-colored graph G. The proof approach is similar to the case when $n = 2t + 2$. Let m_i be the multiplicity of the color c_i in G. Without loss of generality, say the first s colors have multiplicity greater than or equal to 2:

$$m_1 \geq m_2 \geq \cdots \geq m_s \geq 2.$$

Let G_1 be the spanning subgraph consisting of all edges whose color multiplicity is greater than 1 in G. Let G_2 be the spanning subgraph consisting of the remaining edges. We have

$$\sum_{i=1}^{s}(m_i - 1) = \binom{n}{2} - (2t^2 - t + 1) = 2t - 1. \tag{2.10}$$

In particular, we have

$$|E(G_1)| = \sum_{i=1}^{s} m_i = 2t - 1 + s \leq 4t - 2.$$

Claim: we can construct t edge-disjoint rainbow forests $F_1, \ldots, F_t$ in G_1 such that if we let $G_0 = G_1 \setminus \bigcup_{i=1}^{t} E(F_i)$, then $|E(G_0)| \leq t$. Again, for the proof of the claim, we consider two cases:

Case 1: $m_1 \geq t+2$. By equation (2.10), we have that $s \leq (2t-1)-(t+1)+1 = t-1$. We construct t edge-disjoint rainbow forests $F_1, \cdots, F_t$ as follows: First take t edges of color c_1 and add one edge to each of $F_1, \cdots F_t$. Next, pick one edge from each of the remaining $s-1$ colors and add each of them to a distinct F_i.

Clearly, we can obtain t edge-disjoint rainbow forests in this way. Furthermore,

$$|E(G_0)| \leq 2t-1+s-(t+s-1) = t.$$

which proves the claim.

Case 2: $m_1 < t+2$. Let $F_1, \ldots, F_t$ be the edge-maximal family of rainbow spanning forests in G_1. Let $G_0 = G_1 \setminus \bigcup_{i=1}^{t} E(F_i)$. Suppose that $|E(G_0)| > t$. Then

$$\sum_{i=1}^{t} |E(F_i)| \leq 2t-1+s-(t+1) = t+s-2.$$

Since $s \leq 2t-1$, it follows that there exists some j such that $|E(F_j)| \leq 2$.

Case 2a: $|E(F_j)| = 1$. Since $\{F_1, \ldots, F_t\}$ is edge-maximal and $|E(G_0)| \geq t+1$, it follows that all edges in G_0 share the same color (call it c_1') as the single edge in F_j. Thus $m_1 \geq t+2$, which contradicts that $m_1 < t+2$ since we are in Case 2.

Case 2b: $|E(F_j)| = 2$. Similarly, at least t edges in G_0 share the same colors (call them c_1', c_2') as the two edges in F_j. It follows that $m_1 + m_2 \geq t+2$. Hence $s \leq t+1$.

Now since $|E(G_0)| \geq t+1$, it follows that

$$\sum_{i=1}^{t} |E(F_i)| \leq 2t-1+s-(t+1) = t+s-2 \leq 2t-1,$$

Hence there exists some forest with only one edge, in which case we are done by Case 2a.

Hence by contradiction, we obtain that $|E(G_0)| \leq t$, which completes the proof of the claim.

Now we show that $F_1, \ldots, F_t$ have a color-disjoint extension to t edge-disjoint rainbow spanning trees. Consider any partition P. We will verify

$$|c(\mathrm{cr}(P), G_2)| + \sum_{i=1}^{t} |\mathrm{c}r(P, F_i)| \geq t(|P| - 1).$$

We have

$$|c(cr(P), G_2)| + \sum_{i=1}^{t} |\mathrm{c}r(P, F_i)| - t(|P| - 1) \geq \binom{n}{2} - t - \binom{n - |P| + 1}{2} - t(|P| - 1).$$

Note that the function on right is concave downward on $|P|$. It is enough to verify it at $|P| = 2$ an $|P| = n$. When $|P| = 2$, we have

$$\binom{n}{2} - t - \binom{n-1}{2} - t = n - 1 - 2t \geq 0.$$

When $|P| = n$, we have

$$\binom{n}{2} - t - t(n-1) = 0.$$

By Theorem 2.3.4, $F_1, \ldots, F_t$ have a color-disjoint extension to t edge-disjoint rainbow spanning trees. □

2.4 Ramsey number of Berge hypergraphs

Let $\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_c$ be nonempty collections of r-uniform hypergraphs. The *hypergraph Ramsey number* $R_c^r(\mathcal{H}_1, \mathcal{H}_2, \ldots, \mathcal{H}_c)$ is defined to be the minimum integer N such that if the hyperedges of K_N^r are colored with c colors, then for some $1 \leq i \leq c$, there is a monochromatic copy of a member of $\mathcal{H}_i$. We omit c if it is clear from context. If some of the collections $\mathcal{H}_i$ consist of a single hypergraph G, then we write G in place of $\mathcal{H}_i = \{G\}$.

In the remaining sections and Chapter 3, we mainly study hypergraphs defined in a natural way from a given graph G. In the case when G is a path or a cycle, Berge [14] introduced a very general class of hypergraphs defined in terms of G. In particular, a *Berge path* of length t is a collection of t hyperedges $h_1, h_2, \ldots, h_t \in E$ and $t+1$ vertices $v_1, \ldots, v_{t+1}$ such that $\{v_i, v_{i+1}\} \subseteq h_i$ for each $i \in [t]$. Similarly, a k-graph $\mathcal{H} = (V, E)$ is called a *Berge* cycle of length t if E consists of t distinct edges $h_1, h_2, \ldots, h_t$ and V contains t distinct vertices $v_1, v_2, \ldots, v_t$ such that $\{v_i, v_{i+1}\} \subseteq h_i$ for every $i \in [t]$ where $v_{t+1} \equiv v_1$. Note that there may be other vertices than $v_1, \ldots, v_t$ in the edges of a Berge cycle or path.

The extremal problems for Berge-paths and cycles have received a lot of attention. For Ramsey-type results, Gyárfás and Sárközy [93] showed that the 3-color Ramsey number of a 3-uniform Berge-cycle of length n is asymptotic to $\frac{5n}{4}$ (the 2-color case was settled exactly in [92]). For Turán-type results, let $ex_k(n, G)$ denote the maximum number of hyperedges in a k-uniform Berge-G-free hypergraph. Győri, Katona and Lemons [95] showed that for a k-graph $\mathcal{H}$ containing no Berge path of length t, if $t \geq k+2 \geq 5$, then $e(\mathcal{H}) \leq \frac{n}{t}\binom{t}{k}$; if $3 \leq t \leq k$, then $e(\mathcal{H}) \leq \frac{n(t-1)}{k+1}$. Both bounds are sharp. The remaining case of $t = k+1$ was settled by Davoodi, Győri, Methuku and Tompkins [48]. For cycles of a given length, Győri and Lemons [96, 97] showed that $ex_k(n, C_{2t}) = \Theta(n^{1+1/t})$. The same asymptotic upper bound holds for odd cycles of length $2t+1$ as well. The problem of avoiding all Berge cycles of length at least k has been investigated in a series of papers [123, 80, 81, 74, 98]. The general definition of a Berge-G for an arbitrary graph G was introduced by Gerbner and Palmer in [86]. For Turán-type results on Berge-G-free hypergraphs for an arbitrary graph G, see for example [8, 85, 88, 144]. For Turán-type results on Berge cliques, see for example [94, 135, 91, 85, 82].

In this section, we investigate the analogous Ramsey problems for Berge hypergraphs and determine the 2-color Ramsey number of Berge-cliques for all uniformities.

Let us recall the definition of a Berge hypergraph. In fact, we will give a more general definition in which rather than starting with a graph G we may start with any uniform hypergraph.

Definition 2.4.1. *Given a k-uniform hypergraph $\mathcal{H} = (V, E)$ and an integer $r \geq k$, we use $B^r\mathcal{H}$ to denote the set of r-uniform Berge-copies of $\mathcal{H}$, i.e., the set of r-uniform hypergraphs $\mathcal{H}' = (W, F)$ such that there exist $U \subseteq W$ and bijections $\phi : V \to U$, $\psi : E \to F$ such that for all $h = \{v_1, v_2, \ldots, v_k\} \in E$, $\{\phi(v_1), \phi(v_2), \cdots, \phi(v_k)\} \subseteq \psi(h)$. In this case, we call U the core of $\mathcal{H}'$.*

For simplicity, we will often (when it cannot lead to confusion) say that a r-uniform hypergraph is a $B\mathcal{H}$ if it is an element of $B^r\mathcal{H}$. For example we may, in an edge-colored hypergraph, say that a certain r-uniform hypergraph is a red BK_t, meaning that it is an element of the set B^rK_t with all its edges colored red.

In this paper, we show that the 2-color Ramsey number of BK_t versus BK_s is linear. In particular, we prove the following theorem:

Theorem 2.4.1.

$$R^3(BK_s, BK_t) = \begin{cases} t+s-1 & \text{if } \{s,t\} = \{2\}, \{3\}, \{2,3\} \text{ or } \{2,4\}, \\ t+s-2 & \text{if } s=2,\ t \geq 5, \text{ or } s=3,\ t\geq 4 \text{ or } s=t=4, \\ t+s-3 & \text{if } s \geq 4 \text{ and } t \geq 5. \end{cases}$$

For higher uniformity, we will show the following theorem.

Theorem 2.4.2.

$$R^4(BK_t, BK_t) = \begin{cases} t+2 & \text{if } 2 \leq t \leq 5, \\ t+1 & \text{Otherwise.} \end{cases}$$

Moreover, for general uniformity k we prove

Theorem 2.4.3. *For $k \geq 5$ and $t \geq t_0(k)$ (for $k=5$, $t_0 = 23$ suffices),*

$$R^k(BK_t, BK_t) = t.$$

Remark 2.4.1. *We remark that a similar direction (but with mostly non-overlapping results) has been pursued by two other groups independently [10, 84]. In particular, Gerbner, Methuku, Omidi and Vizer [84] showed that $R_c^k(BK_n) = n$ if $k > 2c$; $R_c^k(BK_n) = n+1$ if $k = 2c$ and obtained bounded on $R_c^k(BK_n)$ when $k < 2c$. They also determined the exact value of $R_2^3(BT_1, BT_2)$ for every pair of trees. Similar investigations have also been started independently by Axenovich and Gyárfás [10] who focus on the Ramsey number of small fixed graphs where the number of colors may go to infinity.*

To avoid tedious case analysis, some of the small cases are verified by computer. The code is available at `https://github.com/wzy3210/berge_Ramsey`. We list below the results verified by the computer.

Proposition 2.4.1. *We have*

(1) $R^3(BK_3, BK_4) = 5$.

(2) $R^3(BK_4, BK_5) = 6$.

(3) $R^4(BK_t, BK_t) \le t + 2$ *for* $2 \le t \le 5$.

(4) $R^4(BK_6, BK_6) \le 7$.

2.4.1 Proof of Theorem 2.4.1

Recall that the number $R^3(BK_s, BK_t)$ is the smallest number N such that any 2-edge-colored complete 3-uniform hypergraph (with colors blue and red) on $n \ge N$ vertices either contains a blue Berge K_s or a red Berge K_t. In this subsection, we will show that

$$R^3(BK_s, BK_t) = \begin{cases} t+s-1 & \text{if } \{s,t\} = \{2\}, \{3\}, \{2,3\} \text{ or } \{2,4\}, \\ t+s-2 & \text{if } s = 2, t \ge s+3, \text{ or } s = 3, t \ge s+1 \text{ or } s = t = 4, \\ t+s-3 & \text{if } s \ge 4 \text{ and } t \ge 5. \end{cases}$$

Let us first deal with the cases when one of s or t is small. In particular, we prove them in the following proposition.

Proposition 2.4.2. *We have*

(1) $R^3(BK_2, BK_2) = 3$.

(2) $R^3(BK_2, BK_3) = 4$.

(3) $R^3(BK_3, BK_3) = 5$.

(4) $R^3(BK_2, BK_4) = 5$.

(5) $R^3(BK_4, BK_4) = 6$.

(6) $R^3(BK_2, BK_t) = t$ *when* $t \geq 5$.

(7) $R^3(BK_3, BK_t) = t + 1$ *when* $t \geq 4$.

Proof. (1) is trivial since any non-trivial edge-colored 3-uniform hypergraph contains at least 3 vertices and any edge is a BK_2. For (2), $R^3(BK_2, BK_3) > 3$ since a single red edge is a complete $K_3^{(3)}$ and is not a red BK_3. For the upper bound, suppose we have an edge-colored $K_4^{(3)}$. If it has a blue edge, we get a blue BK_2. Otherwise all of the 4 edges are red, in which case we have a red BK_3. Similar reasoning gives (4) and (6). For (3), $R^3(BK_3, BK_3) > 4$ since an edge-colored $K_4^{(3)}$ with two red and two blue edges does not have a monochromatic BK_3. Similar reasoning gives the lower bound of (5). The upper bounds of (3) and (5) follow from Lemma 2.4.1. For (7), we first show that $R^3(BK_3, BK_t) > t$. Let $\mathcal{H}$ be an edge-color $K_t^{(3)}$ with two special vertices v_1, v_2 such that any hyperedge containing both v_1, v_2 is blue and all other hyperedges are colored red. Observe that any blue Berge clique or red Berge clique cannot contain both v_1 and v_2. Therefore, there is no blue BK_3 or red BK_t in $\mathcal{H}$. For the upper bound, it is checked by computer that $R^3(BK_3, BK_4) = 5$ and the bound $R^3(BK_3, BK_t) \le t+1$ ($t \ge 5$) follows from Lemma 2.4.1, which will be proven later. $\square$

Next we show the lower bound in the following proposition.

Proposition 2.4.3. *Suppose $s, t \ge 3$. We then have*

$$R^3(BK_t, BK_s) \ge t+s-3.$$

Proof. We will construct a 2-edge-colored complete 3-uniform hypergraph $\mathcal{H}$ on $t+s-4$ vertices without a blue BK_t and red BK_s. Let $V(\mathcal{H}) = A \sqcup B$ where $|A| = t-2$ and $|B| = s-2$. For all $a, a' \in A$, $b \in B$, color the hyperedge $\{a, a', b\}$ blue. For all $a \in A$, $b, b' \in B$, color the hyperedge $\{a, b, b'\}$ red. Moreover, color all triples in A blue and all triples in B red. Observe that any blue Berge clique contains at most one vertex from B and any red Berge clique contains at most one vertex from A. It follows that $\mathcal{H}$ does not contain a blue BK_t or a red BK_s. Hence $R^3(BK_t, BK_s) \ge t+s-3$. $\square$

Before we present the proof of Theorem 2.4.1, we will prove the following lemma.

Lemma 2.4.1. *Suppose $t, s \ge 3$. Then*

$$R^3(BK_t, BK_s) \le \max\{R^3(BK_{t-1}, BK_s), R^3(BK_t, BK_{s-1})\} + 1.$$

Proof. Without loss of generality, assume $t \ge s$. Let $\mathcal{H}$ be a 2-edge-colored complete 3-uniform hypergraph with $N := \max\{R^3(BK_{t-1}, BK_s), R^3(BK_t, BK_{s-1})\} + 1$ vertices, and let V be the set of vertices. We want to show that $\mathcal{H}$ contains either a blue BK_t or a red BK_s as a sub-hypergraph.

Take $v \in V$ and let $\mathcal{H}'$ be the hypergraph induced by the vertices $V' := V \backslash \{v\}$. Since $|V'| \ge R^3(BK_{t-1}, BK_s)$, it follows by definition that $\mathcal{H}'$ contains a blue BK_{t-1} or a red BK_s. If there is a red BK_s we are done. Otherwise suppose we have a blue BK_{t-1}, with the vertex set Y as its core. Now let us consider G, the blue trace of v in $\mathcal{H}$, i.e., G is a 2-edge-colored complete graph with vertex set V' and there exists an edge $\{x, y\}$ in G if and only if the hyperedge $\{x, y, v\}$ in $\mathcal{H}$ is colored blue.

Claim 2.4.1. *Either we can extend Y using v to obtain a blue BK_t or there exists a vertex $u \in Y$ with $d_G(u) \le 1$. Moreover if $d_G(u) = 1$ and $\{u, w\}$ is the only edge containing u, then $d_G(w) < N - 2$.*

Proof. Consider the incidence graph of G, i.e., the bipartite graph $I = Y \cup E(G)$ such that for every $u \in Y$, $e \in E(G)$, u is incident to e if and only if $u \in e$. Observe that Y is the core of a blue BK_{t-1} with none of its hyperedges containing v. Therefore, by our definition of G (the blue trace of v in $\mathcal{H}$), if there is a matching of Y in I, then we can obtain a blue BK_t with $Y \cup \{v\}$ as its core.

Now assume I does not contain a matching of Y. We first claim that there exists a vertex $u \in Y$ with $d_G(u) \le 1$. Note that the degree of each $e \in E(G)$ is at most 2. Thus, if $d_I(u) \ge 2$ for all $u \in Y$, then it follows that for every $S \subseteq Y$, $|N_I(S)| \ge |S|$, which gives us a matching on Y by Hall's condition. Thus by contradiction, we have a vertex in Y of degree at most 1 in G.

Suppose now $d_G(u) = 1$ for some u in Y and $e = \{u, w\}$ is the unique edge containing u. We claim that $d_G(w) < N - 2$. Suppose not, i.e., $d_G(w) \ge N - 2$. This implies that $\{v, w, z\}$ is a blue edge for every $z \in V(\mathcal{H}) \setminus \{v, w\}$. Moreover, by our lower bound in Proposition 2.4.2 (when s, t are small) and Proposition 2.4.3, there exists another vertex $y \in V' \setminus Y$. It follows that we can extend Y into the core of a blue BK_t with the following embedding: for each $z \in Y \setminus \{w\}$, embed $\{v, z\}$ to the hyperedge $\{v, z, w\}$. Then embed $\{v, w\}$ to $\{v, w, y\}$. Thus if we do not have a blue BK_t with $Y \cup v$ as its core, then we have $d_G(w) < N - 2$. $\square$

This claim says that either there exists $u \in Y$ such that $\{v, u, x\}$ is red for every $x \in V' \setminus \{u\}$, or there exists $u, w \in V'$ such that $\{v, u, x\}$ is red for every $x \neq w$ and there exists w_x such that $\{v, w, w_x\}$ is red. Note that the second case covers the first case by taking $w_x = u$. So it suffices to assume the second case. Now since $N - 1 \ge R^3(BK_t, BK_{s-1})$, it follows that $\mathcal{H}'$ either contains a blue BK_t or a red BK_{s-1}.

We are done in the former case. Otherwise, suppose that $\mathcal{H}'$ contains a red BK_{s-1}. We will show that we can extend this BK_{s-1} by adding the vertex v into its core. Let X be the core of the Berge-K_{s-1}. Now for every $x \in X$ with $x \notin \{u, w\}$, we know that the edge $\{v, u, x\}$ is colored red. Hence we can embed $\{v, x\}$ into the red hyperedge $\{v, u, x\}$. It follows that we have an embedding of the edges from v to all but at most two vertices of X, namely u, w. In the case that $w \in X$, we can embed $\{v, w\}$ into the hyperedge $\{v, w, w_x\}$, which is red. Now if $u \notin X$, we are done. Otherwise, assume $u \in X$. Note that by the lower bounds in Proposition 2.4.2 (when s, t are small) and Proposition 2.4.3, $|V'| = N - 1 \geq \max\{R^3(BK_{t-1}, BK_s), R^3(BK_t, BK_{s-1})\} \geq s + 1$. Hence it follows that there exists another vertex $y \in V(\mathcal{H}')\backslash(X \cup \{w\})$. Note that by our choice of u, $\{v, u, y\}$ is red. Thus we can embed $\{v, u\}$ into $\{v, u, y\}$. The above embedding extends X into the core of a red BK_s and we are done. □

Lemma 2.4.2. *$R^3(BK_4, BK_t) = t + 1$ for $t \geq 5$.*

Proof. We will proceed by induction on t. The base case that $R^3(BK_4, BK_5) = 6$ is verified by computer. Suppose now that Lemma 2.4.2 is true for all $5 \leq t' < t$. Let $\mathcal{H}$ be a 2-edge-colored complete 3-uniform hypergraph on $t + 1$ vertices. Note that by Proposition 2.4.2, we have $R^3(BK_3, BK_t) = t + 1$. Hence $\mathcal{H}$ either contains a blue BK_3 or a red BK_t. If the latter happens, we are done. So suppose $\mathcal{H}$ contains a blue BK_3, with the vertex set Y as its core. Note that $t + 1 \geq 7$ and a Berge-triangle contains at most 6 vertices. Hence there exists a vertex v that is not used by any hyperedge in the blue BK_3. Similar to Lemma 2.4.1, let G be the blue trace of v in $\mathcal{H}$. Again by Claim 2.4.1, either we can extend Y using to v to obtain a blue BK_4 or there exists a vertex $u \in Y$ with $d_G(u) \leq 1$. Moreover, if $d_G(u) = 1$ and $\{u, w\}$ is the only edge containing u, then $d_G(w) < t - 1$. In the former case, we are done. Otherwise, WLOG, assume that there exists a $u \in Y$ and $w \in V(\mathcal{H})\backslash\{v, u\}$ such that $\{v, u, x\}$ is red for every $x \neq w$ and there exists some vertex w_x such that $\{v, w, w_x\}$ is red. By induction, $\mathcal{H}[V(\mathcal{H})\backslash\{v\}]$ contains either a blue BK_4 or a red BK_t. In the former case, we are done. In the latter case, we can extend the red BK_t to a red BK_{t+1} in the same way as in Lemma 2.4.1. □

Now this result together with Lemma 2.4.1 allows us to show the following proposition.

Proposition 2.4.4. *$R^3(BK_t, BK_s) \leq t+s-3$, for $t, s \geq 4$ and $\max\{s,t\} \geq 5$.*

Proof. We already know this is true if one of t or s is 4, and so for $t, s \geq 5$ the result follows from induction on $t+s$, using Lemma 2.4.1. □

Theorem 2.4.1 follows from Proposition 2.4.2, 2.4.3 and 2.4.4.

2.4.2 Proof of Theorem 2.4.2

In this section, for ease of reference, sometimes we use the notation $h \to e$ to denote that the hyperedge $h \in E(\mathcal{H})$ is mapped to the vertex pair $e \in E(G)$ when constructing the embedding of $E(G)$ in $E(\mathcal{H})$.

Let us first deal with Theorem 2.4.2 for small values of t.

Proposition 2.4.5. *For $2 \leq t \leq 5$, $R^4(BK_t, BK_t) = t+2$.*

Proof. For the lower bound, we use the fact that if $R^4(BK_t, BK_t) = n$, for some t, then $\binom{n}{4} \geq 2\binom{t}{2} - 1$. For $2 \leq t \leq 5$, this shows that $R^4(BK_t, BK_t) \geq t+2$. The upper bound that $R^4(BK_t, BK_t) \leq t+2$ for $2 \leq t \leq 5$ is verified by computer. □

Now we want to show that $R^4(BK_t, BK_t) = t+1$ for all $t \geq 6$. Again we start with the lower bound by showing the following proposition.

Proposition 2.4.6. *$R^4(BK_t, BK_t) \geq t+1$ for all $t \geq 6$.*

Proof. We want to construct a 2-edge-coloring of a complete 4-uniform hypergraph on t vertices without a monochromatic BK_t. Let $\mathcal{H}$ be a $K_t^{(4)}$ with two special vertices v_1, v_2. Any hyperedge containing both v_1, v_2 is colored blue. All other hyperedges are colored red. We claim that there is no monochromatic BK_t in $\mathcal{H}$. Indeed, there is no red BK_t since only one of v_1, v_2 can be in any red BK_t. For blue BK_t, note that by our coloring there are only $\binom{t-2}{2}$ blue edges, which are fewer than the $\binom{t}{2}$ edges needed for BK_t. □

Now let us move on to the upper bound.

Lemma 2.4.3. *For $t \geq 6$, we have that*

$$R^4(BK_t, BK_t) \leq t + 1.$$

Proof. We prove the lemma by inducting on t. The base case that $R^4(BK_6, BK_6) \leq 7$ is verified by computer. Now assume that $t \geq 7$ and the lemma is true for all $t' < t$.

Let $\mathcal{H}$ be a 2-edge-colored complete 4-uniform hypergraph on a vertex set V of size $t + 1$. For ease of reference, given a set of vertices S, let $d_b(S)$ and $d_r(S)$ denote the number of blue and red hyperedges containing S as subset, respectively.

Claim 2.4.2. *Suppose $\mathcal{H}$ does not contain a monochromatic BK_t. Let v be a fixed vertex in $\mathcal{H}$. If there is a monochromatic BK_{t-1} (without loss of generality, assume it is blue) without using any hyperedge containing v, then there exists another vertex u such that $d_b(\{v, u\}) \leq 2$, i.e., all hyperedges containing both v, u are red except for at most two.*

Proof. Let $\mathcal{H}_b$ be the blue Berge-K_{t-1} hypergraph not using any hyperedge containing v. Let $\{u_1, u_2, \ldots u_{t-1}\}$ be the core of $\mathcal{H}_b$. Construct a bipartite graph $G = A \cup B$ where $A = \{u_1, \ldots, u_{t-1}\}$ and $B = \binom{V \setminus \{v\}}{3}$. For $u_i \in A$, $S \in B$, u_i is adjacent to S in G if and only if $u_i \in S$ and $\{v\} \cup S$ is a blue edge in $\mathcal{H}$. Note that for every $S \in B$, $d_G(S) \leq 3$. Therefore, if $d_G(u_i) \geq 3$ for every $u_i \in A$, then there exists a matching of A in G by Hall's theorem, which implies that we can extend $\mathcal{H}_b$ to a blue BK_t by adding v into the core of $\mathcal{H}_b$. This contradicts our assumption that $\mathcal{H}$ does not have a monochromatic BK_t, and the proof of Claim 2.4.2 is complete. □

Now for every $v \in V$, there exists a monochromatic BK_{t-1} in $\mathcal{H}[V \backslash \{v\}]$ by induction. Hence by Claim 2.4.2, for every vertex v, there exists another vertex u in V, such that $d_c(\{v,u\}) \geq \binom{t-1}{2} - 2$, for some $c \in \{\text{blue, red}\}$. We then call the pair $\{v,u\}$ a c *couple* where $c \in \{\text{blue, red}\}$. Moreover, call $\{a,b\}$ a 'bad pair' of $\{v,u\}$ if the hyperedge $\{a,b,v,u\}$ is not in color c.

By Claim 2.4.2, every vertex is contained in a couple. It follows that we have at least $(t+1)/2 \geq 4$ couples so at least two of them are of the same color. Without loss of generality, let $\{v_1,u_1\}$ and $\{v_2,u_2\}$ be two red couples. Our goal is to obtain a red embedding of a BK_t using mostly edges containing $\{v_1,u_1\}$ and $\{v_2,u_2\}$. We assume that $\{v_1,u_1\} \cap \{v_2,u_2\} = \emptyset$ and remark that the other case is similar and simpler. Let $\{a_1,b_1\},\{a_2,b_2\}$ be the two possible bad pairs of $\{v_1,v_2\}$. Let $\{c_1,d_1\}$, $\{c_2,d_2\}$ be two possible bad pairs of $\{v_2,u_2\}$. If $\{v_1,u_1\}$ has exactly two bad pairs, we can assume that for at least one of them (with loss of generality the pair $\{a_2,b_2\}$) there is a red edge h containing it. Otherwise $\{a_1,b_1\}$ and $\{a_2,b_2\}$ are blue couples with no bad pairs and it is easy to find a blue BK_t by only using the blue edges containing $\{a_1,b_1\}$ and $\{a_2,b_2\}$.

If $\{v_1,u_1\}$ has exactly one bad pair, let $\{a_1,b_1\}$ be that pair and pick $\{a_2,b_2\}$ arbitrarily. Note that $\{a_2,b_2\}$ is contained in some red edge h. If $\{v_1,u_1\}$ has no bad pair, then pick $\{a_1,b_1\}$ and $\{a_2,b_2\}$ arbitrarily. Moreover, we assume that $\{v_1,u_1,v_2,u_2\}$ is a red edge and observe that otherwise constructing the embedding is easier.

Suppose $\{a_1,b_1\}$ and $\{a_2,b_2\}$ have a common vertex u. If $u \notin \{v_2,u_2\}$, relabel a_1,b_1 such that $a_1 = u$, and if $u \in \{v_2,u_2\}$ relabel u_2,v_2,a_1,b_1 such that $b_1 = u_2 = u$. Otherwise just relabel a_1,b_1 such that $a_1 \notin \{v_2,u_2\}$. Let $x_1,x_2,\ldots,x_{t-4}$ be an enumeration of $V' := V \setminus \{v_1,v_2,u_1,u_2,a_1\}$. If $b_1 \notin \{v_2,u_2\}$, assume $x_1 = b_1$. Otherwise WLOG that $b_1 = u_2$. We are going to construct the embedding in three phases:

Phase 1: Embed all vertex pairs in V'. Consider the following embedding: For $i,j \in \{1,\ldots,t-4\}$, embed $\{x_i,x_j\}$ in $\{u_1,v_1,x_i,x_j\}$ if $i+j$ is odd otherwise in

$\{u_2, v_2, x_i, x_j\}$. We have a red BK_{t-4} except possibly for at most three missing edges. Without loss of generality, let $\{x_{i_1}, x_{j_1}\}$, $\{x_{i_2}, x_{j_2}\}$, $\{x_{i_3}, x_{j_3}\}$ be the three possible bad pairs where $i_1 + j_1$ is odd and both $i_2 + j_2$ and $i_3 + j_3$ are even. If $\{x_{i_1}, x_{j_1}\}$ is indeed a bad pair of $\{v_1, u_1\}$, then it follows that $\{x_{i_1}, x_{j_1}\} = \{a_2, b_2\}$. Then we can embed $\{x_{i_2}, x_{j_2}\}$ in $\{v_1, u_1, x_{i_2}, x_{j_2}\}$, embed $\{x_{i_3}, x_{j_3}\}$ in $\{v_1, u_1, x_{i_3}, x_{j_3}\}$ and embed $\{x_{i_1}, x_{j_1}\}$ in h. Otherwise, $\{x_{i_1}, x_{j_1}\}$ does not exist and the above embedding still works except when one of $\{x_{i_2}, x_{j_2}\}, \{x_{i_3}, x_{j_3}\}$ is the pair $\{a_2, b_2\}$. We can then use h to embed $\{a_2, b_2\}$.

Phase 2: Embed all edges from $\{v_1, u_1, v_2, u_2\}$ to vertices in V'. Consider the following embedding:

$$\{v_1, u_1, a_1, x_i\} \to \{x_i, u_1\} \text{ for } i \neq 1.$$

$$\{v_1, u_1, v_2, x_i\} \to \{x_i, v_1\} \text{ for } i \neq 1.$$

$$\{v_2, u_2, a_1, x_i\} \to \{x_i, u_2\}.$$

$$\{v_1, v_2, u_2 x_i\} \to \{x_i, v_2\}.$$

Note that x_1 can only be contained in one bad pair otherwise we would have picked x_1 to be a_1. Hence among the three edges $\{v_1, u_1, x_1, v_2\}$, $\{v_1, u_1, x_1, u_2\}$, $\{v_1, u_1, a_1, x_1\}$, at least two of them are red. Embed $\{x_1, v_1\}$, $\{x_1, u_1\}$ into those two red edges. If all three are red, do not use $\{v_1, u_1, u_2, x_1\}$ in this part of the embedding.

Now let us analyze the potential bad cases. There are at most 3 of these edges in Phase 2 that are not red.

If $\{u_1, v_1, a_1, x_i,\}, i \neq 1$ is blue, then use the edge $\{v_1, u_1, u_2, x_i\}$ to embed $\{u_1, x_i\}$.

If $\{v_1, u_1, v_2, x_i\}, i \neq 1$ is blue, then use the edge $\{v_1, u_1, u_2, x_i\}$ to embed $\{v_1, x_i\}$.

If there are two different indexes i, j such that $h_1 \in \{\{v_2, u_2, a_1, x_i\}, \{v_1, v_2, u_2, x_i\}\}$ and $h_2 \in \{\{v_2, u_2, a_1, x_j\}, \{v_1, v_2, u_2, x_j\}\}$ are both blue, then we can replace h_1 with $\{u_1, v_2, u_2, x_i\}$ and replace h_2 with $\{u_1, v_2, u_2, x_j\}$. The same embedding works if there is only one bad pair of $\{v_2, u_2\}$ in this phase.

If for some i both edges $\{v_1, v_2, u_2, x_i\}, \{v_2, u_2, a_1, x_i\}$ are blue, then it follows that the edge $\{v_2, u_2, x_i, y\}$ is red for every vertex y, with $y \notin \{v_1, a_1, v_2, u_2, x_i\}$. Consider the set of edges $E_i = \{\{v_2, u_2, x_i, y\} : y \notin \{v_1, v_2, u_2, a_1, x_i\}\}$. Note that $|E_i| = t-4$. In Phase 1, at most $\lceil (t-6)/2 \rceil$ edges in E_i are used except when t is even and i is odd, in which case $\lfloor (t-6)/2 \rfloor$ edges in E_i are used. If t is even and i is odd, we have at least $t-4-\lfloor (t-6)/2 \rfloor \geq 3$ edges in E_i still available. In other cases, we have at least $t-4-\lceil (t-6)/2 \rceil \geq 2$ edges in E_i still available. Either there exist two edges in E_i that can be used to embed $\{v_2, x_i\}$ and $\{u_2, x_i\}$, or in Phase 1 there exists some j such that $\{v_1, u_1, x_i, x_j\}$ is blue and $\{v_2, u_2, x_i, x_j\}$ is used to embed $\{x_i, x_j\}$. In this case, there exists some $k \in \{1, \ldots t-4\} \backslash \{i\}$ such that $i+k$ is even and $\{v_1, u_1, x_i, x_k\}$ is red. Embed $\{x_i, x_k\}$ into $\{v_1, u_1, x_i, x_k\}$. It follows that we again have two available red edges containing x_i, v_2, u_2 to embed $\{v_2, x_i\}$, $\{u_2, x_i\}$.

Phase 3: Embed the edges in $\binom{\{u_1, v_1, u_2, v_2\}}{2}$. If the edge $\{u_1, v_1, v_2, a_1\}$ is red, then use it to embed $\{v_1, v_2\}$. Otherwise we know that $\{v_2, a_1\}$ and $\{u_2, a_1\}$ are the two bad pairs of $\{v_1, u_1\}$. It follows that the edge $\{v_1, u_1, u_2, x_1\}$ is still available and the edge $\{v_1, u_1, v_2, x_1\}$ was used to embed x_1 with one of v_1 or u_1 (without loss of generality, assume v_1). In this case, embed $\{v_1, x_1\}$ in $\{v_1, u_1, u_2, x_1\}$ instead and use the edge $\{v_1, u_1, v_2, x_1\}$ to embed $\{v_1, v_2\}$. Now we will embed $\{v_1, u_2\}$ and $\{u_1, u_2\}$. Let $E_{u_2} = \{\{v_1, u_1, u_2, y\} : y \notin \{v_1, u_1, v_2, u_2\}\}$. Note that $|E_{u_2}| = t-3$ and at most 2 edges in E_{u_2} are blue. Hence at least $(t-3)-2 \geq 2$ of the edges in E_{u_2} are red. For each red edge in E_{u_2}, if it was used, it was because there exists some bad pair of $\{v_1, u_1\}$ which did not use u_2. That in turn implies that there are still at least 2 edges in E_{u_2} that are red and available. Hence we can embed $\{v_1, u_2\}$ and $\{u_1, u_2\}$ into these two edges. Similarly we can find an edge of the form $\{v_2, u_1, u_2, y\}$ to embed $\{u_1, v_2\}$.

Finally, by counting the edges used, it is easy to check that there are still red edges of the form $\{v_1, u_1, x, y\}$ and $\{v_2, u_2, x, y\}$ available to embed both $\{v_1, u_1\}$ and $\{v_2, u_2\}$, since each pair is in at least $\binom{t-1}{2} - 2$ red edges. □

In the case of cliques of different sizes we have the following bounds which are trivial from Theorem 2.4.2.

Proposition 2.4.7. *Suppose* $t \geq s \geq 2$ *and* $t \geq 6$, *then*

$$t \leq R^4(BK_t, BK_s) \leq t + 1.$$

Proof. The construction is trivial, we just take a clique on $t - 1$ vertices. The upper bound follows since $s \leq t$ implies $R^4(BK_t, BK_s) \leq R^4(BK_t, BK_t)$. □

For $s = t - 1$ we obtain the same bound as the case $s = t$.

Proposition 2.4.8. $R^4(BK_t, BK_{t-1}) = t + 1$ *for* $t \geq 6$.

Proof. The same construction works as the $R^4(BK_t, BK_t)$ case, and the upper bound follows from $R^4(BK_t, BK_{t-1}) \leq R^4(BK_t, BK_t)$. □

Theorem 2.4.4. *Assume* $2 \leq s \leq t - 2$, *and* $t \geq 34$, *then* $R^4(BK_t, BK_s) = t$.

Proof. In a red-blue coloring of a hypergraph $\mathcal{H}$, given a pair of vertices $\{v, u\}$, we define its blue degree to be $d_B(\{v, u\}) = |h \in E(\mathcal{H}) : \{v, u\} \subseteq h \text{ and } h \text{ is blue}\}|$. The red degree $d_R(\{v, u\})$ is defined analogously. Let

$$\delta_B^2 = \min_{\{v,u\} \in \binom{V(\mathcal{H})}{2}} d_B(\{v, u\}),$$

and define δ_R^2 similarly.

Call $\{v,u\}$ a c couple, $c \in \{blue, red\}$, if all but at most 5 of the hyperedges $\{v,u,x,y\}$ are c colored, and also call a pair $\{x,y\}$ a bad pair of the c couple $\{v,u\}$ if the hyperedge $\{v,u,x,y\}$ is not colored c.

Note that if $\delta_B^2 = 0$ then we can find a pair $\{v,u\}$ such that $\{v,u,x,y\}$ is red for all x,y, and therefore there is a red BK_{t-2}. So we can assume $\delta_B^2 \geq 1$.

Claim 2.4.3. *Suppose there are two blue couples, then either we can find a blue BK_t or we can find two red couples such that each have at most 4 bad pairs.*

Proof. Assume we have two disjoint blue couples $\{u_1,v_1\}$ and $\{u_2,v_2\}$, the case where these pairs are not disjoint is similar and simpler, and enumerate the other $t-4$ vertices as $x_1,x_2,\dots,x_{t-4}$. Now let us do a preliminary embedding, for $i,j \in [t-4]$ use $\{u_1,v_1,x_i,x_j\}$ to embed $\{x_i,x_j\}$ when $i+j$ is odd and $\{u_2,v_2,x_i,x_j\}$ otherwise. If $i+j$ is odd and in this part of the embedding we used a red edge $\{u_1,v_1,x_i,x_j\}$ to embed $\{x_i,x_j\}$, but the edge $\{u_2,v_2,x_i,x_j\}$ is blue, then use the edge $\{u_2,v_2,x_i,x_j\}$ instead. If $i+j$ is even and in this part of the embedding we used a red edge $\{u_2,v_2,x_i,x_j\}$ to embed $\{x_i,x_j\}$, but the edge $\{u_1,v_1,x_i,x_j\}$ is blue, then use the edge $\{u_1,v_1,x_i,x_j\}$ instead. Let us call such a change to the embedding a swap. If both edges $\{u_1,v_1,x_i,x_j\}$ and $\{u_2,v_2,x_i,x_j\}$ are red or blue, then we do not change anything.

Note that at this point we have embedded a BK_{t-4} such that every edge is blue except at most five edges, in particular the possible pairs which are simultaneously bad pairs of $\{u_1,v_1\}$ and $\{u_2,v_2\}$.

Let $e_1,e_2,\dots,e_k$ be these common bad pairs, $k \leq 5$. We begin with a simple observation which we will use again later.

Observation 2.4.1. *If $k \leq 1$ we could complete the embedding in such a way that each pair is contained in at least 1 blue edge.*

If $k \geq 2$ and all but at most one e_i is in at least 5 blue edges, then we can greedily embed the edges, starting from the one that is in less than 5 blue edges, since each is in at least one unused blue edge. So we can either find two of the e_i which are in at most 4 blue edges and the claim is proven or we complete the embedding of a blue BK_{t-4}, and if that is the case we will see we can complete this embedding to a blue BK_t.

Since for any fixed i, there are at most $\lceil \frac{t-4}{2} \rceil$ indices j such that $i + j$ is odd and also x_i can be in at most 10 bad pairs of $\{u_1, v_1\}$ or $\{u_2, v_2\}$, it follows that for every $i \in [t-4]$ there are at least $t - 5 - \lceil \frac{t-4}{2} \rceil - 10 \geq 4$ values of $j \in [t-4]$ not used in the previous steps of the embedding such that the edge $\{u_1, v_1, x_i, x_j\}$ is blue. Then again by Hall's Theorem in the incidence graph with components $X = \{\{x_i, v_2\} : i \in [t-4]\} \cup \{\{x_i, u_2\} : i \in [t-4]\}$ and Y the set of blue edges in $\{\{x_i, x_j, u_2, v_2\} : 1 \leq i < j \leq t-4\}$, we can find an embedding of the edges $\{x_i, v_2\}$ and $\{x_i, u_2\}$ for $i \in [t-4]$, and similarly we can find an embedding of the edges $\{x_i, v_1\}$ and $\{x_i, u_1\}$ for $i \in [t-4]$.

We have not yet used the hyperedges of the form $\{v_1, u_1, v_2, y\}$; there are at least $t - 8 \geq 26$ of these which are blue, and we can use them to embed $\{v_1, u_1\}, \{v_1, v_2\}$ and $\{u_1, v_2\}$. Similarly we can embed $\{v_2, u_2\}, \{u_1, u_2\}$ and $\{u_1, u_2\}$. Therefore either we can complete the matching or we find two pairs e_1, e_2 which are red couples, with at most 4 bad pairs. This completes the proof of Claim 2.4.3. □

Claim 2.4.4. *Suppose there are two red couples such that at least one has at most 4 bad pairs, then either we can find a red BK_{t-2} or we can find two blue couples such that each have at most 1 bad pair.*

Proof. Again we assume the red couples are disjoint. Let $\{u_1, v_1\}$ and $\{u_2, v_2\}$ be couples such that $\{u_1, v_1\}$ has at most 4 bad pairs, and let $\{a_1, b_1\}, \{a_2, b_2\}, \{a_3, b_3\}, \{a_4, b_4\}$ be the bad pairs of $\{u_1, v_1\}$. Suppose these pairs are arranged by their red degree in increasing order. Now let $x_1, x_2, \dots, x_{t-6}$ be an enumeration of the set $V' = V \backslash \{v_1, v_2, u_1, u_2, a_1, a_2\}$. Let us consider the following embedding which is similar to the one used in the previous claim: For $i, j \in [t-6]$ use $\{u_1, v_1, x_i, x_j\}$ to embed $\{x_i, x_j\}$ when $i + j$ is odd and $\{u_2, v_2, x_i, x_j\}$ otherwise. Similarly as in Claim 2.4.3, if we encounter a bad pair of one couple but not the other, then we can change the embedding to use more red edges, and at the end we have an embedding of a BK_{t-6} with almost every edge red, the only possible exceptions are the common bad pairs of $\{u_1, v_1\}$ and $\{u_2, v_2\}$ in V'. Hence here we have at most two ($\{a_3, b_3\}$ and $\{a_4, b_4\}$). If the red degree of these edges is at least 2, then we can greedily embed these two in these pairs to complete a red clique on V'. Otherwise one of these, and by the ordering also $\{a_1, b_1\}$ and $\{a_2, b_2\}$, will be in at most 1 red pair.

Similarly as in the proof of Claim 2.4.3, we use Hall's theorem to embed $\{x_i, v_2\}$, $\{x_i, u_2\}$, $\{x_i, v_1\}$ and $\{x_i, u_1\}$ for $i \in [t-6]$ (here the number $t - 5 - \lceil \frac{t-4}{2} \rceil - 10$ is replaced by $t - 7 - \lceil \frac{t-6}{2} \rceil - 8$, which is at least 5). Since $\{v_1, u_1, v_2, y\}$ is red for at least $t - 7 \geq 29$, and these hyperedges have not been used yet, it follows that we have enough hyperedges to embed $\{v_1, u_1\}, \{v_1, v_2\}$ and $\{u_1, v_2\}$ and similarly we can embed $\{v_2, u_2\}, \{v_1, u_2\}$ and $\{u_1, u_2\}$. □

Note that if there is at most one blue couple, say $\{v, u\}$, we may put $V' = V \backslash \{u\}$ and for every pair $x, y \in V'$ the red degree of $\{x, y\}$ is at least 6. Then by Hall's Theorem, we can find a red BK_{t-1}. So we can assume there are at least two blue couples. Thus, by Claim 2.4.3 either we find a blue BK_t or we have two red couples such that at least one has at most 4 bad pairs, the conditions of Claim 2.4.4. From here we either find a red BK_{t-2} or satisfy conditions stronger than those of Claim 2.4.3. In this case, there is at most one shared bad pair and so we would be able to find a blue BK_t by Observation 2.4.1. □

2.4.3 Proof of Theorem 2.4.3

In this short section, we will show that $R^k(BK_t, BK_t) = t$ when t is sufficiently large.

Claim 2.4.5. *If for all $v, u \in V$, there are at least $\binom{k}{2}$ red distinct hyperedges containing both v and u, then $\mathcal{H}$ contains a red BK_t.*

Proof. Consider the bipartite graph G with vertex set $V(G) = A \sqcup B$, where $A = \binom{V(\mathcal{H})}{2}$ and B is the set of all hyperedges of $\mathcal{H}$. For $a \in A$, $h \in B$, a is adjacent to h in G if and only if $a \subset h$ and h is colored red in $\mathcal{H}$. Note that for every $h \in B$, $d_G(h) \leq \binom{k}{2}$. Hence, if for all $\{v, u\} \in A$, $d_G(\{v, u\}) \geq \binom{k}{2}$, then by Hall's theorem we have a matching of A in G, which implies the existence of a red BK_t in $\mathcal{H}$. □

Claim 2.4.6. *If $\binom{t-4}{k-4} \geq 2\binom{k}{2} - 1$, then $R^k(BK_t, BK_t) \leq t$.*

Proof. If the condition in Claim 2.4.5 does not hold, then there exist two vertices $v, u \in V(\mathcal{H})$ such that all but at most $\binom{k}{2} - 1$ hyperedges containing both v and u are blue. We claim that there exists a copy of a blue BK_t in $\mathcal{H}$ using only blue hyperedges containing both v and u. Consider again the bipartite graph G with vertex set $V(G) = A \sqcup B$, where $A = \binom{V(\mathcal{H})}{2}$ and B is the set of blue hyperedges of $\mathcal{H}$ containing both v and u. Note that for every $a \in A$ there are at least $\binom{t-4}{k-4} - \binom{k}{2} + 1 \geq \binom{k}{2}$ blue hyperedges containing a, and again by Hall's theorem we have a blue BK_t. □

Using Claim 2.4.6, we show that $R^k(BK_t, BK_t) = t$ when $k \geq 5$ and t sufficiently large. We did not make an attempt to find the best possible constant.

Corollary 2.4.1. *We have*

(1) $R^5(BK_t, BK_t) = t$ *when* $t \geq 23$.

(2) $R^6(BK_t, BK_t) = t$ *when* $t \geq 13$.

(3) $R^7(BK_t, BK_t) = t$ *when* $t \geq 12$.

(4) $R^k(BK_t, BK_t) = t$ *when* $k \in \{8, 9, 10\}$ *and* $t \geq k + 4$.

(5) $R^k(BK_t, BK_t) = t$ *when* $k \geq 11$ *and* $t \geq k + 3$.

Remark 2.4.2. *Note that for* $k \geq 11$, *this result is sharp since for* $t = k + 2$ *we have that* $\binom{t}{r} \leq 2\binom{t}{2} - 2$. *Hence* $R^k(BK_t, BK_t) \geq r + 3$.

Superlinear lower bounds for sufficiently many colors

In this subsection we show that for all uniformities and for sufficiently many colors, the Ramsey number for a Berge t-clique is superlinear. We start with the case $r = 3$.

Claim 2.4.7. *For any* $\epsilon < 1$ *we have* $R_3^3(BK_t, BK_t, BK_t) \geq (t-1)t^{\epsilon}$ *for* t *sufficiently large.*

Proof. Let $\epsilon < 1$. Take a vertex set consisting of the disjoint union of $t - 1$ sets of vertices, $V_1, V_2, \ldots, V_{t-1}$, each of size t^{ϵ}. If a hyperedge contains vertices from three different V_i, then color it green. By the well-known lower bound on the diagonal Ramsey number $R(K_{t^{1-\epsilon}}, K_{t^{1-\epsilon}}) = \Omega(2^{t^{1-\epsilon}/2})$, we can find a coloring of K_{t-1} containing no clique of size $t^{1-\epsilon}$ when t is sufficiently large. Given such a red-blue coloring on the complete graph with vertex set $\{1, 2, \ldots, t-1\}$ we color the hyperedges consisting of two vertices from V_i and one from V_j by the color of $\{i, j\}$ in the graph. We color every hyperedge completely contained in some V_i red. Observe that the core of any red or blue BK_t may contain vertices in less than $t^{1-\epsilon}$ different classes and so has a total of less than t vertices. □

Remark 2.4.3. *This proof can give a slightly better bound on the order of* $\frac{t^2}{\log(t)}$ *but we write the bound in terms of* ϵ *for a simpler presentation.*

Theorem 2.4.5. *For any uniformity* $r \geq 4$, *and sufficiently large* c *and* t, *we have*

$$R_c^r(BK_t, BK_t, \ldots, BK_t) > t^{1 + \left(\frac{r-3}{r-2}\right)^{r-3} - \left(\frac{r-3}{r-2}\right)^{r-2}}.$$

Remark 2.4.4. *The lower bound above was subsequently improved in [84] and [83]. The best bound (when c is sufficiently large) is due to Pálvölgyi [145], who established the first exponential lower bound:* $R^r_c(BK_t) > (1+\frac{1}{r^2})^{t-1}$ *if* $c > \binom{r}{2}$.

Theorem 2.4.5 will follow from the following claim which we will prove by induction on r by choosing the optimal ϵ.

Claim 2.4.8. *For any uniformity* $r \geq 3$*, and for any* ϵ *where* $\epsilon < 1$*, for sufficiently large* c *and* t*, we have*

$$R^r_c(BK_t, BK_t, \ldots, BK_t) > t^{1+(1-\epsilon)^{r-3}-(1-\epsilon)^{r-2}}.$$

Proof. The base case follows from Claim 2.4.7. Now assume that $r \geq 4$. Let $\epsilon < 1$. Let c_s be the number of colors required for Claim 2.4.8 to hold for an s-uniform hypergraph for $2 \leq s \leq r-1$. Let M be the lower bound we obtain by induction for the function $R^{r-1}_{c_{r-1}}(BK_{t^{1-\epsilon}}, BK_{t^{1-\epsilon}}, \ldots, BK_{t^{1-\epsilon}})$. We will show

$$R^r_{c_r}(BK_t, BK_t, \ldots, BK_t) > M \cdot t^{\epsilon}.$$

for some constant c_r depending on r.

Take the complete r-uniform hypergraph $\mathcal{H}$ on $N = M \cdot t^{\epsilon}$ vertices. Partition the vertex set into sets $V_1, V_2, \ldots, V_M$ each consisting of t^{ϵ} vertices. We consider s-uniform complete hypergraphs $\mathcal{H}_s$ defined on the vertex set $\{1, 2, \ldots, M\}$ for $2 \leq s \leq r-1$. Since the lower bounds in Claim 2.4.8 are decreasing (in r), we have for c_s colors a coloring of $\mathcal{H}_s$ with no Berge clique of size $t^{1-\epsilon}$ provided t is sufficiently large. Assume, indeed, that t is at least the maximum required for any s.

Now, given the colorings of $\mathcal{H}_i$ with c_i colors, for $2 \leq i \leq r-1$, we define a coloring on $\mathcal{H}$ with $c_r = \sum_{s=2}^{r-1} c_s + 2$ colors and no monochromatic BK_t. For $2 \leq s \leq r-1$ we color all hyperedges containing elements of the vertex sets $V_{i_1}, V_{i_2}, \ldots, V_{i_s}$ with the same color as $\{i_1, i_2, \ldots, i_s\}$ in the coloring of $\mathcal{H}_s$. Observe that the core of a monochromatic BK_t in $\mathcal{H}$ can contain vertices from fewer than $t^{1-\epsilon}$ classes. Since $\mathcal{H}_s$

has no monochromatic $BK_{t^{1-\epsilon}}$, and each class has t^{ϵ} vertices, it follows that $\mathcal{H}$ has no monochromatic BK_t using hyperedges containing vertices from between 2 and $r-1$ classes. Finally, we may color the hyperedges contained in each V_i with any color used so far and the hyperedges containing vertices from r classes with a new color.

It remains to verify that $M \cdot t^{\epsilon}$ yields the required bound. Indeed,

$$M \cdot t^{\epsilon} = t^{(1-\epsilon)\left(1+(1-\epsilon)^{r-4}-(1-\epsilon)^{r-3}\right)} \cdot t^{\epsilon} = t^{1+(1-\epsilon)^{r-3}-(1-\epsilon)^{r-2}}. \qquad \square$$

We now discuss briefly the case of forbidding Berge-cliques of higher uniformity. First we collect some basic lemmas about the Ramsey number for Berge cliques in different uniformities.

Lemma 2.4.4. *For any r, c, a, b, where $a < b$ and for t sufficiently large, we have*

$$R_c^r(BK_t^{(b)}, BK_t^{(b)}, \ldots, BK_t^{(b)}) \geq R_c^r(BK_t^{(a)}, BK_t^{(a)}, \ldots, BK_t^{(a)}).$$

Proof. It is sufficient to prove that for sufficiently large t, there is an injection from $\binom{[t]}{a}$ to $\binom{[t]}{b}$ mapping sets to one of their supersets. Let $S \subset \binom{[t]}{a}$ and $\phi(S)$ be the elements of $\binom{[t]}{b}$ which contain some element from S. We have $|S|\binom{t-a}{b-a} \leq |\phi(S)|\binom{b}{a}$ by double-counting the relations between the two levels. Then $|\phi(S)| \geq |S|$ is obvious for sufficiently large t, and we have the desired injection by Hall's theorem. $\square$

Corollary 2.4.2. *For any uniformity r, $a < r$, and sufficiently large c and t, we have*

$$R_c^r(BK_t^{(a)}, BK_t^{(a)}, \ldots, BK_t^{(a)}) \geq t^{1+\left(\frac{r-3}{r-2}\right)^{r-3}-\left(\frac{r-3}{r-2}\right)^{r-2}}.$$

Proof. The result is immediate from Lemma 2.4.4 and Theorem 2.4.5. $\square$

2.5 cover-Ramsey number of Berge hypergraphs

A hypergraph is a pair $\mathcal{H} = (V, E)$ where V is a vertex set and $E \subseteq 2^V$ is an edge set. For a fixed set of positive integers R, we say $\mathcal{H}$ is an R-uniform hypergraph, or R-graph for short, if the cardinality of each edge belongs to R. If $R = \{k\}$, then an R-graph is simply a k-uniform hypergraph or a k-graph. Given an R-graph $\mathcal{H} = (V, E)$ and a set $S \in \binom{V}{s}$, let $\deg(S)$ denote the number of edges containing S and $\delta_s(\mathcal{H})$ be the minimum s-degree of $\mathcal{H}$, i.e., the minimum of $\deg(S)$ over all s-element sets $S \in \binom{V}{s}$. When $s = 2$, $\delta_2(\mathcal{H})$ is also called the minimum *co-degree* of $\mathcal{H}$. Given a hypergraph $\mathcal{H}$, the 2-*shadow*(or *shadow*) of $\mathcal{H}$, denoted by $\partial(\mathcal{H})$, is a simple 2-uniform graph $G = (V, E)$ such that $V(G) = V(\mathcal{H})$ and $uv \in E(G)$ if and only if $\{u, v\} \subseteq h$ for some $h \in E(\mathcal{H})$. Note that $\delta_2(\mathcal{H}) \geq 1$ if and only if $\partial(\mathcal{H})$ is a complete graph. In this case, we say $\mathcal{H}$ is *covering*.

There are several notions of a path or a cycle in hypergraphs. A *Berge path* of length t is a collection of t hyperedges $h_1, h_2, \ldots, h_t \in E$ and $t+1$ vertices $v_1, \ldots, v_{t+1}$ such that $\{v_i, v_{i+1}\} \subseteq h_i$ for each $i \in [t]$. Similarly, a k-graph $\mathcal{H} = (V, E)$ is called a *Berge* cycle of length t if E consists of t distinct edges $h_1, h_2, \ldots, h_t$ and V contains t distinct vertices $v_1, v_2, \ldots, v_t$ such that $\{v_i, v_{i+1}\} \subseteq h_i$ for every $i \in [t]$ where $v_{t+1} \equiv v_1$. Note that there may be other vertices than $v_1, \ldots, v_t$ in the edges of a Berge cycle or path. Gerbner and Palmer [86] extended the definition of Berge paths and Berge cycles to general graphs. In particular, given a simple graph G, a hypergraph $\mathcal{H}$ is called *Berge-G*, denoted by BG, if there is an injection $i: V(G) \to V(\mathcal{H})$ and a bijection $f: E(G) \to E(\mathcal{H})$ such that for all $e = uv \in E(G)$, we have $\{i(u), i(v)\} \subseteq f(e)$.

Extremal problems related to Berge hypergraphs have been receiving increasing attention lately. For Turán-type results, let $ex_k(n, G)$ denote the maximum number of hyperedges in a k-uniform Berge-G-free hypergraph. Győri, Katona and Lemons [95] showed that for a k-graph $\mathcal{H}$ containing no Berge path of length t, if $t \geq k+2 \geq 5$, then $e(\mathcal{H}) \leq \frac{n}{t}\binom{t}{k}$; if $3 \leq t \leq k$, then $e(\mathcal{H}) \leq \frac{n(t-1)}{k+1}$. Both bounds are sharp. The remaining case of $t = k+1$ was settled by Davoodi, Győri, Methuku and Tompkins [48]. For cycles of a given length, Győri and Lemons [96, 97] showed that $ex_k(n, C_{2t}) = \Theta(n^{1+1/t})$. The same asymptotic upper bound holds for odd cycles of length $2t+1$ as well. The problem of avoiding all Berge cycles of length at least k has been investigated in a series of papers [123, 80, 81, 74, 98]. For general results on the maximum size of a Berge-G-free hypergraph for an arbitrary graph G, see for example [85, 88, 144].

For Ramsey-type results, define $R_c^k(BG_1, \dots, BG_c)$ as the smallest integer n such that for any c-edge-coloring of a complete k-uniform hypergraph on n vertices, there exists a Berge-G_i subhypergraph with color i for some i. For convenience, we use $R_c^k(BG)$ to denote $R_c^k(BG_1, \dots, BG_c)$ when $G_1 = \cdots = G_c = G$. The study of Ramsey problems for Berge hypergraphs was initiated by three groups of authors independently [10, 84, 154]. Salia, Tompkins, Wang and Zamora [154] showed that $R_2^3(BK_s, BK_t) = s + t - 3$ for $s, t \geq 4$ and $\max(s,t) \geq 5$; $R_2^4(BK_t, BK_t) = t + 1$ for $t \geq 6$ and $R_2^k(BK_t, BK_t) = t$ for $k \geq 5$ and t sufficiently large. Independently and more generally, Gerbner, Methuku, Omidi and Vizer [84] showed that the Ramsey number of Berge cliques is linear when the number of colors is less than the uniformity (of the host complete hypergraph). In particular, they showed that $R_c^k(BK_t) = t$ if $c < k/2$; $R_c^k(BK_t) = t + 1$ if $c = k/2$ and $R_c^k(BK_t) \leq ct$ when $k/2 < c < k$. When $c \geq k$, a superlinear lower bound was shown in [154] for $c = k = 3$ and for every other r for large enough c. This was improved in [82] to $R_c^k(BK_t) = \Omega(t^d)$ for $c > (d-1)\binom{k}{2}$ and $R_k^k(BK_t) = \Omega(t^{1+1/(k-2)}/\log t)$. Pálvölgyi [145] further improved it and gave the first exponential lower bound by showing $R_c^k(BK_t) > (1 + \frac{1}{k^2})^{t-1}$ when $c > \binom{k}{2}$. Similar investigations have also been started independently by Axenovich and Gyárfás [10] who focus on the Ramsey number of small fixed graphs where the number of colors may go to infinity.

Although it is pleasant to see that the Ramsey number of Berge cliques is linear when the number of colors is less than the uniformity of the host hypergraph, the result is also not surprising partially because K_t^k has much more edges than BK_t (for large k and t). This motivates us to define a new type of Ramsey number such that the host hypergraph has relatively small number of edges. In particular, given a collection of families of R-uniform hypergraphs, $\mathcal{H}_1, \mathcal{H}_2, \cdots, \mathcal{H}_c$, we define the *cover Ramsey number*, denoted as $\hat{R}^R(\mathcal{H}_1, \cdots, \mathcal{H}_c)$, as the smallest integer n_0 such that for every covering R-uniform hypergraph $\mathcal{H}$ on $n \geq n_0$ vertices and every c-edge-coloring of $\mathcal{H}$ with colors in $[c]$, $\mathcal{H}$ contains a monochromatic copy of some member of $\mathcal{H}_i$ in color i. For convenience, when $\mathcal{H}_1 = \cdots = \mathcal{H}_c$, we simply use $\hat{R}^R_c(\mathcal{H}_1)$ to denote $\hat{R}^R(\mathcal{H}_1, \cdots, \mathcal{H}_c)$. Moreover, we use $\hat{R}^k(\mathcal{H}_1, \cdots, \mathcal{H}_c)$ to denote $\hat{R}^{\{k\}}(\mathcal{H}_1, \cdots, \mathcal{H}_c)$. It is easy to see that $\hat{R}^k(\mathcal{H}_1, \cdots, \mathcal{H}_c) \leq \hat{R}^{[k]}(\mathcal{H}_1, \cdots, \mathcal{H}_c)$. Note that when $R = \{2\}$, $\hat{R}^R(BG_1, BG_2)$ is exactly the classical Ramsey number. Let $R_c(G_1, \ldots, G_c)$ denote the classical multi-color Ramsey number, i.e., the smallest integer n such that any c-edge-coloring of K_n contains a monochromatic G_i in the i-th color for some $i \in [c]$. When $c = 2$, we simply write $R_2(G_1, G_2)$ as $R(G_1, G_2)$. We first show the following theorem.

Theorem 2.5.1. *For every $k \geq 2$, there exists some constant c_k such that for any two finite graphs G_1 and G_2,*

$$R(G_1, G_2) \leq \hat{R}^{[k]}(BG_1, BG_2) \leq c_k \cdot R(G_1, G_2)^3.$$

Theorem 2.5.1 implies $\hat{R}^R(BG_1, BG_2)$ is always finite, thus well-defined. In fact, let k be the greatest integer in R. We have $R \subseteq [k]$ and

$$\hat{R}^R(BG_1, BG_2) \leq \hat{R}^{[k]}(BG_1, BG_2) \leq c_k \cdot R(G_1, G_2)^3.$$

Note that Theorem 2.5.1 does not give a lower bound for $\hat{R}^k(BG_1, BG_2)$. For complete graphs K_t, we show that the cover Ramsey number of Berge cliques is at least exponential in t. Note that this is very different from the 2-color hypergraph Ramsey number of Berge cliques (see [154] and [84]), which is linear when the uniformity is at least 3.

Theorem 2.5.2. *For every $k \geq 2$ and sufficiently large t, we have that*

$$\hat{R}^k(BK_t, BK_t) > (1 + o(1))\frac{\sqrt{2}}{e} t 2^{t/2}.$$

Remark 2.5.1. *For a fixed t and $R \subseteq [k]$, let $N(t)$ be the set of integers n such that for every covering R-uniform hypergraph $\mathcal{H}$ on n vertices and every 2-edge-coloring of $\mathcal{H}$, there is a monochromatic Berge-K_t. We remark that $N(t)$ may not be a single interval. However, by Theorem 2.5.1, there exists some n_0 such that $[n_0, \infty) \subseteq N(t)$.*

For a graph G with bounded maximum degree, Chvátal, Rödl, Szemerédi and Trotter showed in [42] that for each positive integer d, there exists a constant $C = C(d)$ such that if G is a graph on n vertices with $\Delta(G) \le d$, then $R(G,G) \le Cn$. In this note, we show that the cover Ramsey number of Berge bounded-degree graphs is also linear. The proof uses a modification of the proof of Chvátal, Rödl, Szemerédi and Trotter in [42] that allows for more than two colors.

Theorem 2.5.3. *For each positive integer c, d and k, there exists a constant $C = C(c,d,k)$ such that if G is a graph on n vertices with maximum degree at most d, then*

$$\hat{R}_c^{[k]}(BG) \le Cn.$$

Theorem 2.5.3 implies that for fixed positive integers c, d and k, there is a constant $C' := C'(c,d,k)$ such that $\hat{R}_c^{[k]}(BG) \le C' \cdot R(G,G)$ holds for any graph G with maximum degree at most d. It is an interesting question whether $\lim_{t\to\infty} \dfrac{\hat{R}^{[k]}(BK_t, BK_t)}{R(K_t,K_t)} = \infty$ for all $k \ge 3$.

2.5.1 Proof of Theorem 2.5.1

Proof of Theorem 2.5.1. The lower bound that $\hat{R}^{[k]}(BG_1, BG_2) \ge R(G_1, G_2)$ is clear from the definition since $R(G_1,G_2) = \hat{R}^{\{2\}}(BG_1, BG_2) \le \hat{R}^{[k]}(BG_1, BG_2)$.

For the upper bound, given $k \ge 2$, set $c_k = k^3/12$. Let $\mathcal{H} = (V, E)$ be a 2-edge-colored R-graph on $n = c_k R(G_1,G_2)^3$ vertices. Assume further that $\mathcal{H}$ is edge-minimal with respect to the covering property. Suppose $E = \{h_1, h_2, \ldots, h_m\}$ where $m = |E|$. Since $\mathcal{H}$ is edge-minimal and covering, it follows that $\binom{n}{2}/\binom{k}{2} \le m \le \binom{n}{2}$.

Now let $S \subseteq V$ be a uniformly and randomly chosen subset of V of size $s = R(G_1, G_2)$. For each $i \in [m]$, let B_i be the event that $|h_i \cap S| \geq 3$. It is not hard to see that

$$\Pr(B_i) \leq \binom{k}{3} \frac{\binom{n-3}{s-3}}{\binom{n}{s}}.$$

Taking a union bound over all B_i, we have that

$$\begin{aligned}\Pr(B_1 \vee \ldots \vee B_m) &\leq \binom{n}{2}\binom{k}{3}\frac{\binom{n-3}{s-3}}{\binom{n}{s}} \\ &= 3\frac{\binom{k}{3}\binom{s}{3}}{n-2} \\ &< 1.\end{aligned}$$

The last step is due to the following inequality:

$$n = \frac{k^3}{12}s^3 \geq 3\left(\binom{k}{3}+1\right)\left(\binom{s}{3}+1\right) > 3 + 3\binom{k}{3}\binom{s}{3}$$

for any $k \geq 2$ and $s \geq 2$. Hence with positive probability, a uniformly and randomly chosen $S \subseteq V$ of size $R(G_1, G_2)$ intersects every hyperedge in at most 2 points. Hence there exists such an S. Now consider the trace of $\mathcal{H}$ on S, denoted by $G = \mathcal{H}_S$, i.e., $E(\mathcal{H}_S) = \{h \cap S : h \in E(\mathcal{H})\}$. By the covering property and the choice of S, G is a complete graph (ignoring edges of cardinality 1). Moreover, for each edge $e \in G$, there exists some $h = \phi(e) \in E(\mathcal{H})$ such that $e = h \cap S$. Note that for $e_1 \neq e_2$, $\phi(e_1) \neq \phi(e_2)$ due to the choice of S. Now for each edge $e \in E(G)$, color the edge e with the same color of $\phi(e)$ in $\mathcal{H}$. Since $|S| = R(G_1, G_2)$, it follows that there exists either a blue G_1 or a red G_2 in G, which corresponds to a blue Berge G_1 or a red Berge G_2 in $\mathcal{H}$. This shows that $\hat{R}^{[k]}(BG_1, BG_2) \leq k^3/12 \cdot R(G_1, G_2)^3$. $\square$

In fact, the proof of Theorem 2.5.1 implies a stronger statement than Theorem 2.5.1. Given a simple graph G, a hypergraph $\mathcal{H}$ is called a *trace-G*, denoted by $\mathcal{T}G$, if there is an injection $i: V(G) \to V(\mathcal{H})$ and a bijection $f: E(G) \to E(\mathcal{H})$ such that for all $e = uv \in E(G)$, $f(e) \cap i(V(G)) = \{i(u), i(v)\}$. Note that $\mathcal{T}G$ is a subset of BG.

The Ramsey number of $\mathcal{T}G$ was investigated in [154]. The proof of Theorem 2.5.1 in fact implies that

$$\hat{R}^{[k]}(BG_1, BG_2) \le \hat{R}^{[k]}(\mathcal{T}G_1, \mathcal{T}G_2) \le c_k \cdot R(G_1, G_2)^3.$$

2.5.2 Proof of Theorem 2.5.2

The construction comes from a random 2-edge-coloring of a covering k-uniform hypergraph that is obtained from a combinatorial design.

A *resolvable BIBD* (balanced incomplete block design), denoted as BIBD(n, k, λ), is a collection $P_1, \ldots, P_m$ of partitions of an underlying n-element set into k-element subsets such that every 2-element subset of the n-element set is contained by exactly λ of the $\frac{mn}{k}$ k-element sets listed in the partitions. We restrict ourselves to $\lambda = 1$, that is, each 2-element subset of the n-element set is contained in one and only one of the k-element sets listed in the partitions.

Note that the existence of such a design implies that $|P_i| = \frac{n}{k}$ and $m\frac{n}{k}\binom{k}{2} = \binom{n}{2}$, i.e. $m = \frac{n-1}{k-1}$, which gives the well known necessary condition that $n \equiv k \pmod{k(k-1)}$ for the existence of such a resolvable BIBD. For the $k = 3$ case (which is commonly called a Kirkman triple system to honor Kirkman [120] who posed the problem), it is also a sufficient condition [147], and for $k = 4$ the corresponding $n \equiv 4 \pmod{12}$ is also a sufficient condition [104]. For every k, the congruence is also a sufficient condition for all $n > n_0(k)$ [148]. Also, for every even $k \ge 4$, the congruence implies existence for $n > \exp\{\exp\{k^{18k^2}\}\}$ [26].

Proof of Theorem 2.5.2. For a fixed $k \ge 2$, let t_0 be sufficiently large such that for all $n \ge (1 + o(1))\frac{t_0\sqrt{2}}{e}2^{t_0/2}$ and $n \equiv k \pmod{k(k-1)}$, a resolvable BIBD $(n, k, 1)$ exists.

Let $t \ge t_0$. Choose an integer n such that a resolvable BIBD $(n, k, 1)$ exists and $n = (1 + o(1))\frac{t\sqrt{2}}{e}2^{t/2}$. Let $\mathcal{H} = (V, E)$ be a k-uniform hypergraph such that V is the underlying n-element set of the resolvable BIBD $(n, k, 1)$ and E is the collection of k-element sets listed in the partitions $P_1, \ldots, P_m$. Note that by the definition of $(n, k, 1)$, $\mathcal{H}$ is a covering k-graph with $\binom{n}{2}/\binom{k}{2}$ edges and every vertex pair of $\mathcal{H}$ is contained in exactly one hyperedge. Our goal is to construct a coloring of $\mathcal{H}$ with no monochromatic BK_t as subhypergraph. Color each hyperedge of $\mathcal{H}$ in blue and red uniformly and randomly with probability 1/2. For any set S of t vertices, let A_S be the bad event that S induces a monochromatic BK_t. We will apply the Lovász Local Lemma [67, 160] to show that we can avoid all bad events $\{A_S : S \subseteq V \text{ and } |S| = t\}$.

Note that by the definition of $(n, k, 1)$, for each vertex pair of S, there exists a unique hyperedge containing that vertex pair. Hence there is at most one Berge-K_t with S as the underlying vertex set. Furthermore, if there is a Berge-K_t with S as the underlying vertex set, then the hyperedges containing the vertex pairs of S are all distinct. Hence

$$\Pr(A_S) = \begin{cases} 2^{1-\binom{t}{2}} & \text{if there is no } h \in E(\mathcal{H}) \text{ such that } |h \cap S| \geq 3, \\ 0 & \text{otherwise.} \end{cases}$$

Two bad events A_S and A_T are independent if there is no edge f intersecting both S and T on exactly two vertices. For a fixed event A_S, the number d of bad events A_T dependent on A_S satisfies

$$d \leq \binom{t}{2}\binom{k}{2}\binom{n-2}{t-2} - 1.$$

Applying the symmetric version of the Lovász Local Lemma [67, 160], if $e(d + 1)\Pr(A_S) < 1$ for all S, then $\Pr\left(\wedge_S \overline{A_S}\right) > 0$.

It suffices to have

$$e \cdot \binom{t}{2}\binom{k}{2}\binom{n-2}{t-2} 2^{1-\binom{t}{2}} < 1,$$

which is satisfied if we choose $n = (1 + o(1))\frac{\sqrt{2}}{e} t 2^{t/2}$. Hence there exists a coloring of $\mathcal{H}$ with no monochromatic Berge K_t as subhypergraph. It follows by definition that $\hat{R}^k(BK_t, BK_t) > (1 + o(1))\frac{\sqrt{2}}{e} t 2^{t/2}$.

$\square$

2.5.3 Proof of Theorem 2.5.3

The proof of Theorem 2.5.3 uses a modification of the proof Chvátal, Rödl, Szemerédi and Trotter in [42] to allow for more than two colors. For the reason of self-completeness, we state and give the details in this section. Let $R_c(G)$ denote the multicolor Ramsey number $R_c(G, G, \ldots, G)$.

Theorem 2.5.4. *[42] For each positive integer c and d, there exists a constant $C = C(c, d)$ such that if G is a graph on n vertices with maximum degree at most d, then $R_c(G) \leq Cn$.*

We first show how Theorem 2.5.4 implies Theorem 2.5.3.

Proof of Theorem 2.5.3. For fixed positive integers c, d and k, let $C = C(c\binom{k}{2}, d)$ be the constant obtained from Theorem 2.5.4. We will show that if G is a graph on n vertices with maximum degree at most d, then $\hat{R}_c^{[k]}(BG) \leq Cn$.

Let $\mathcal{H} = (V, E)$ be a c-edge-colored covering $[k]$-graph on $N = Cn$ vertices with colors in $[c]$. Suppose $E = \{h_1, \ldots, h_m\}$. For each h_i, give each vertex pair $uv \subseteq h_i$ a unique label $\phi_{h_i}(uv)$ in $[\binom{k}{2}]$. Now consider a $c\binom{k}{2}$-edge coloring of K_N: for each $uv \in E(K_N)$, pick an arbitrary hyperedge $h \in E(\mathcal{H})$ such that $\{u, v\} \subseteq h$. Such h exists since $\mathcal{H}$ is covering. If h is colored with the i-th color in $\mathcal{H}$, then color $uv \in E(K_N)$ with a color represented by the ordered pair $(i, \phi_h(uv))$. Note that K_N is indeed a $c\binom{k}{2}$-edge-colored graph. Since $N = Cn$, by the definition of multi-color Ramsey number, it follows that if G is a graph on n vertices with maximum degree at most d, then K_N contains a monochromatic G as subgraph. WLOG, suppose G is colored $(1, r)$ where $1 \leq r \leq \binom{k}{2}$. Now by our construction, for each $e \in E(G)$, there exists hyperedge $h = h(e)$ such that h is colored 1 in $\mathcal{H}$. Moreover we claim that for $e_1 \neq e_2 \in E(G)$, $h(e_1) \neq h(e_2)$. Suppose not, i.e., h contains both e_1 and e_2. Then $\phi_h(e_1) \neq \phi_h(e_2)$, which contradicts that e_1, e_2 receives the same color in K_N. Hence, it follows that we can find a monochromatic Berge copy of G in $\mathcal{H}$. $\square$

In the remaining of this section, we will give a proof of Theorem 2.5.4. We remark again that the proof follows along the same line of [42] and we are only giving the details here for the sake of self-completeness.

As suggested by [42], the proof requires a generalization of the regularity lemma, which is an easy modification of the original proof in [165]. Given a graph G, let $V(G) = A_1 \cup A_2 \cup \cdots \cup A_k$ be a partition of $V(G)$ into disjoint subsets. We call such partition *equipartite* if $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$. Moreover, given two disjoint sets $X, Y \subseteq V(G)$, the *edge density* of (X, Y), denoted as $d(X, Y)$, is defined as $d(X, Y) = |e(X, Y)|/|X||Y|$ where $e(X, Y) = \{xy \in E(G) : x \in X, y \in Y\}$.

Lemma 2.5.1. *[122] For every $\epsilon > 0$ and integers c, m, there exists an M and N_0 such that if the edges of a graph G on $n \geq N_0$ vertices are c-colored, then there exists an equipartite partition $V(G) = A_1 \cup A_2 \cup \ldots \cup A_k$ for some $m \leq k \leq M$, such that all but at most ϵk^2 pairs (A_i, A_j) are ϵ-regular: for every $X \subseteq A_i$ and $Y \subseteq A_j$ with $|X| \geq \epsilon |A_i|$, $|Y| \geq \epsilon |A_j|$, we have*

$$|d_s(X, Y) - d_s(X, Y)| < \epsilon$$

for each $s \in [c]$ where d_s is the edge-density in the s-th color.

Proof of Theorem 2.5.4. Let d be any positive integer. Let N be large enough so that if we define $\epsilon = 1/N$, then $\frac{1}{c \log(2c)} \log\left(\frac{1}{2\epsilon}\right) \geq d + 1$. Observe that with this choice of N, we also have $1/(2c)^d > 2d^2\epsilon$. Let M, N_0 be the constants given by Lemma 2.5.1 when c is the number of colors and $m = 1/\epsilon$. Set $C = C(c, d) = \max\{N_0, M/d^2\epsilon\}$.

Now let G be a graph on n vertices $x_1, \ldots, x_n$ with maximum degree at most d. Consider an arbitrary c-coloring of K_{Cn}. Let $H_1, \ldots, H_c$ denote the subgraphs of G induced by each of the c colors respectively. By Lemma 2.5.1, there exists an equipartite partition $V(K_{Cn}) = A_1 \cup A_2 \cup \ldots \cup A_k$ that satisfies the regularity condition for each color class, i.e., for each $i \in [c]$, $V(H_i) = A_1 \cup A_2 \cup \ldots \cup A_k$ gives an equipartite ϵ-regular partition.

Let H^* denote the graph whose vertex set is $\{A_i : i \in [k]\}$ and A_iA_j is an edge if and only (A_i, A_j) is ϵ-regular in H. By Lemma 2.5.1, $|E(H)| \geq (1-\epsilon)\binom{k}{2}$. Hence by Turán's theorem, there exists a complete subgraph H^{**} of H^* of size at least $1/2\epsilon$. WLOG (with relabeling), assume that $V(H^{**}) = \{A_i : 1 \leq i \leq 1/2\epsilon\}$. Now for each $A_i, A_j \in V(H^{**})$, color the edge A_iA_j with color s if $d_s(A_i, A_j)$ is the largest among all colors in $[c]$ (break arbitrarily if the same). Recall that $R_c(K_t) \leq c^{ct}$ by an easy extension of the Erdős-Szekeres argument and $\frac{1}{c\log(2c)}\log\left(\frac{1}{2\epsilon}\right) \geq d+1$ by our assumption. Hence we have that $1/2\epsilon \geq R_c(K_{d+1})$. Then it follows from Ramsey's theorem that there is a monochromatic complete subgraph H^{***} with $d+1$ vertices. WLOG, H^{***} is in color 1. Then we can relabel the sets in the partition so that

(i) (A_i, A_j) is ϵ-regular, and

(ii) $d_1(A_i, A_j) \geq \frac{1}{c}$

for all i, j with $1 \leq i < j \leq d+1$. We then claim that H_1 contains a copy of G. Suppose that $V(G) = \{x_i : i \in [n]\}$. We will choose $y_1, y_2, \ldots, y_n \in V(H_1)$ inductively so that the map $\phi : x_i \to y_i$ is an embedding of G in H_1. In particular, the points are chosen so that for each $i \in [n]$, the followings are satisfied:

(a) $y_t \in A_j$ for some $j \in [d+1]$ for each $t \in [i]$.

(b) For $t_1, t_2 \in [i]$, if $x_{t_1}x_{t_2} \in E(G)$, then y_{t_1}, y_{t_2} are adjacent in H_1 and are in different partition.

(c) For $i < t \leq n$, define $V(t,i) = \{y_j : j \in [i], x_jx_t \in E(G)\}$. For each $r \in [d+1]$ such that $A_r \cap V(t,i) = \varnothing$, A_r contains a subset A'_r having at least $|A'_r|/(2c)^{|V(t,i)|}$ vertices so that every point in A'_r is adjacent to every point in $V(t,i)$.

Suppose that for some $i \in [n]$, the points $\{y_t : t \leq [i]\}$ are already chosen so that the conditions (a)-(c) above are satisfied. We will then pick y_{i+1} so that conditions (a)-(c) remain true.

First pick some $r_0 \in [d+1]$ so that $A_{r_0} \cap V(i+1,i) = \varnothing$. This is possible since the degree of x_{i+1} is at most d. By condition (c), there exists $A'_{r_0} \subseteq A_{r_0}$ such that $|A'_{r_0}| \geq |A_{r_0}|/(2c)^\ell$ where $\ell = |V(i+1,i)|$. Moreover, each vertex of A'_{r_0} is adjacent to every vertex of $V(i+1,i)$. It's easy to see that with any choice of y_{i+1} from A'_{r_0}, condition (a) and (b) are clearly satisfied. For condition (c), observe we only need to handle the values of $i+1 < t \leq n$ such that $x_t x_{i+1} \in E(G)$. There are at most d such values since $d(x_{i+1}) \leq d$. Pick one such t arbitrarily. Now pick an arbitrary $r \neq r_0$ such that $A_r \cap V(t,i) = \varnothing$. Observe $\ell' = |V(t,i+1)| = |V(t,i)|+1$. By condition (c), we already know that there exists some $A'_r \subseteq A_r$ such that $|A'_r| \geq |A_r|/(2c)^{\ell'-1} \geq \epsilon |A_r|$ and every vertex of A'_r is adjacent to every vertex of $V(t,i)$. Now since (A_r, A_{r_0}) is ϵ-regular and $d_1(A_r, A_{r_0}) \geq \frac{1}{c}$, it follows that at most $\epsilon|A_{r_0}|$ of the points in A'_{r_0} are adjacent to less than $\frac{1}{2c}$ of the points in A'_r. Fixing t and proceeding through all values of r, we would eliminate at most $d\epsilon|A_{r_0}|$ candidates for y_{i+1} in A'_{r_0}. Ranging over all of the d possible values of t, we then eliminate at most $d^2\epsilon|A_{r_0}|$ candidates of y_{i+1} in A'_{r_0}. Moreover, there are at most n points in A'_{r_0} that may have been selected previously already. Since the number of partitions $k \leq M$ and $C \geq M/d^2\epsilon$, we have that $|A_{r_0}| \geq Cn/M$, which implies that $n \leq d^2\epsilon|A_{r_0}|$.

In order to be able to pick y_{i+1}, it suffices to show that $|A'_{r_0}| > 2d^2\epsilon|A_{r_0}|$. This holds because $|A'_{r_0}|/|A_{r_0}| > 1/(2c)^d > 2d^2\epsilon$. This completes the proof of the theorem. □

2.6 Erdős-Szekerem theorem for cyclic permutations

The study of the longest monotone subsequence of a finite sequence of numbers has inspired a body of research in mathematics, bioinformatics, and computer science. In 1935, Erdős and Szekeres [73] showed in their namesake theorem that any permutation of $\{1,2,...,k\ell+1\}$ has an increasing subsequence of length $k+1$ or a decreasing subsequence of length $\ell+1$. As a sequence $(a_1,\ldots,a_n)$ can be represented by a set of n points of the form (i,a_i) in the plane, the Erdős-Szekeres theorem can be interpreted geometrically in the following way: for any set of $k\ell + 1$ points in the plane, no two of which are on the same horizontal or vertical line, there exists a polygonal path of either k positive-slope edges or ℓ negative-slope edges. It follows immediately from the Erdős-Szekeres theorem that the expected length of a longest increasing subsequence (LIS) in a random permutation of length n is at least $\frac{1}{2}\sqrt{n}$. Moreover, computing LIS is also used in MUMmer systems for aligning whole genomes [49]. A natural extension of the well-known Erdős-Szekeres theorem is to consider its analogue to cyclic sub-permutations.

Definition 2.6.1. *A cyclic sub-permutation τ of a cyclic permutation σ is the restriction of σ on τ, i.e. remove all elements not in τ from σ.*

For example, (135) is a cyclic sub-permutation of the cyclic permutation (12345).

Definition 2.6.2. *A cyclic permutation is increasing if it can be written in the form $(j_1, j_2, \ldots, j_n)$ with $j_1 < j_2 < \ldots < j_n$. Similarly, a cyclic permutation is decreasing if it can be written in the form $(j_1, j_2, \ldots, j_n)$ with $j_1 > j_2 > \ldots > j_n$.*

For example, $(6, 1, 4, 2, 7, 3, 5)$ is a cyclic permutation for which the longest increasing cyclic sub-permutation is $(1, 2, 3, 5, 6)$ and the longest decreasing cyclic sub-permutations are $(7, 5, 4, 2)$ or $(7, 6, 4, 2)$.

Cyclic permutations can be viewed as circular lists, which arise naturally in the field of phylogenetics since the genomes of bacteria are considered to be circular. Geometrically, an increasing/decreasing cyclic subsequence of a circular list corresponds to a polygonal path of positive/negative-slope edges when the points are drawn on the side of a cylinder. Albert et al. in [3] give a Monte Carlo algorithm to compute the longest increasing circular subsequence with worst case run-time $O(n^{3/2} \log n)$ and also showed that the expected length $\mu(n)$ of the longest increasing circular subsequence satisfies $\lim_{n\to\infty} \dfrac{\mu(n)}{2\sqrt{n}} = 1$. We extend the Erdős-Szekeres theorem to cyclic permutations and examine the structures of the extremal constructions achieving the lower bound for our theorem.

Definition 2.6.3. *Given positive integers k and ℓ, let $\alpha(k,\ell)$ be the smallest positive integer n, such that for any cyclic permutation of length n, there exists either an increasing cyclic sub-permutation of length $k+1$, or a decreasing cyclic sub-permutation of length $\ell+1$.*

We show in Section 2.6.1 that

Theorem 2.6.1. *For $k,\ell \geq 1$,*

$$\alpha(k,\ell) = (k-1)(\ell-1)+2.$$

Definition 2.6.4. *Given positive integers k and ℓ, let $\mathbb{C}_{k,\ell}$ be the set of cyclic permutations of length $(k-1)(\ell-1)+1$ that contain no increasing cyclic sub-permutation of length $k+1$, or decreasing cyclic sub-permutation of length $\ell+1$; let $\mathbb{S}_{k,\ell}$ be the set of linear permutations of length $k\ell$ that contain no increasing linear sub-permutation of length $k+1$, or decreasing linear sub-permutation of length $\ell+1$; and let $\mathbb{Y}_{\ell,k}$ be the set of standard Young tableaux on a $\ell \times k$ rectangular diagram, i.e. the set of $\ell \times k$ matrices where the set of entries is $\{1,2,\dots,k\ell\}$ and each row and column forms an increasing sequence.*

It was noted by Knuth [[121], Exercise 5.1.4.9] (see also [[162], Example 7.23.19(b)]) that the permutations in $\mathbb{S}_{k,\ell}$ are in bijection with $\mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$ via the Robinson-Schensted correspondence. The hook-length formula [75] expresses the number of standard Young tableaux and allows us to directly compute $|\mathbb{S}_{k,\ell}|$, which increases rapidly as k,ℓ increase (see sequence A060854 in the On-Line Encyclopedia of Integer Sequences). In particular, WLOG, assuming $k \leq l$ (since $|\mathbb{S}_{k,\ell}| = |\mathbb{S}_{\ell,k}|$), we have that

$$|\mathbb{S}_{k,\ell}| = \left(\frac{(\ell k)!}{1^1 2^2 \dots k^k (k+1)^k \dots \ell^k (\ell+1)^{k-1} \dots (k+\ell-1)}\right)^2.$$

Although the Robinson-Schensted correspondence establishes the bijection between $\mathbb{S}_{k,l}$ and $\mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$, it is an algorithmic procedure which can be difficult to analyze. Romik, in [152], gave a simple description of the mapping from pairs of square Young Tableaux to elements of $\mathbb{S}_{k,k}$. Before we state the theorem, let us introduce a few definitions.

Definition 2.6.5. *The* grid-function *of an* $\vec{a} = [a_1, \ldots, a_{k\ell}] \in \mathbb{S}_{k,\ell}$ *is* $\gamma_{\vec{a}} : [k\ell] \to [\ell] \times [k]$, *defined by* $\gamma_{\vec{a}}(t) = (i,j)$ *where* i *is the length of the longest decreasing subsequence of* $\vec{a}$ *ending at* a_t *and* j *is the length of the longest increasing subsequence of* $\vec{a}$ *ending at* a_t.

Definition 2.6.6. *The* grid-ranking $R_{\vec{a}} = (r_{ij})$ *and* grid-valuation $V_{\vec{i,j}} = (v_{ij})$ *are* $\ell \times k$ *matrices defined by* $r_{ij} = \gamma_{\vec{a}}^{-1}(i,j)$, *and* $v_{ij} = a_{\gamma^{-1}(\ell+1-i,j)}$.

Note that the Erdős-Szekeres theorem implies that for a linear permutation $\vec{a} \in \mathbb{S}_{k,\ell}$, the longest increasing subsequence has length k and the longest decreasing subsequence has length ℓ (as both $k(\ell-1)+1$ and $(k-1)\ell+1$ are at most $k\ell$), which means that $\gamma_{\vec{a}}$ indeed defines a function.

Working towards our characterization of $\mathbb{C}_{k,\ell}$, Section 2.6.2 reproves the following result of [152], partially for the sake of self-containment and partially for its use in thse proof of Theorem 2.6.3.

Theorem 2.6.2. *For positive integers* k, ℓ, $\mathbb{S}_{k,\ell}$ *is isomorphic to* $\mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$. *In particular,* $\phi : \mathbb{S}_{k,\ell} \to \mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$ *defined by* $\phi(\vec{a}) = (R_{\vec{a}}, V_{\vec{a}})$ *is a bijection.*

In contrast to the exponential size of $\mathbb{S}_{k,l}$, $\mathbb{C}_{k,l}$ has at most 2 elements and we can characterize them precisely. In particular, in Section 2.6.3, we show the following theorem:

Theorem 2.6.3. *For $k, \ell \geq 1$, let $\mathbb{C}_{k,\ell}$ denote the set of cyclic permutations of $[(k-1)(\ell-1)+1]$ that contain no increasing cyclic sub-permutation of length $k+1$, or decreasing cyclic sub-permutation of length $\ell+1$. Then we have:*

(1) If $\min(k,\ell) \leq 2$ then $|\mathbb{C}_{k,\ell}| = 1$ and the single element of $\mathbb{C}_{k,\ell}$ is the decreasing cyclic permutation when $k \leq 2$ and the increasing cyclic permutation when $k \geq 3$.

(2) If $\min(k,\ell) \geq 3$ then $|\mathbb{C}_{k,\ell}| = 2$, and $(1, a_1, \ldots, a_{(k-1)(\ell-1)}) \in \mathbb{C}_{k,\ell}$ precisely when the sequence it satisfies one of the following:

(i) For each $(i,j) \in [\ell-1] \times [k-1]$,

$$a_{(j-1)(\ell-1)+i} = (\ell-1-i)(k-1)+j+1.$$

(ii) For each $(i,j) \in [\ell-1] \times [k-1]$,

$$a_{(i-1)(k-1)+j} = (j-1)(\ell-1)+(\ell-i)+1.$$

Note that when $\min(k,\ell) = 2$, the structures described in parts (2) (i) and (ii) are the same and coincide with the single structure described in part (1). Figure 2.1 illustrates the structures in parts (2) (i) and (ii) for $k = 4$ and $\ell = 5$. The two extremal examples are $(1, 11, 8, 5, 2, 12, 9, 6, 3, 13, 10, 7, 4)$ and $(1, 5, 9, 13, 4, 8, 12, 3, 7, 11, 2, 6, 10)$ respectively.

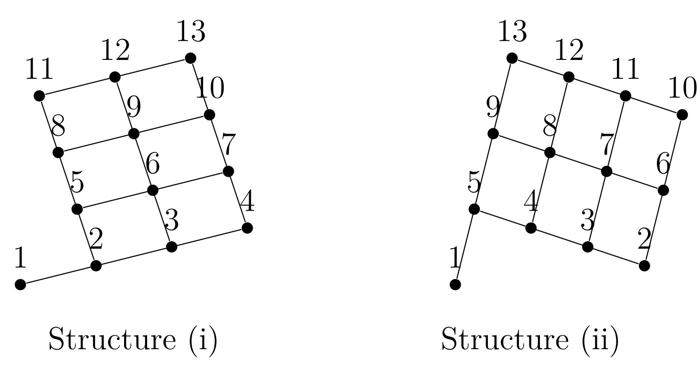


Figure 2.1: Extremal examples for $k = 4$ and $\ell = 5$.

2.6.1 Proof of Theorem 2.6.1

Lemma 2.6.1. *For $k, \ell \geq 1$,*

$$\alpha(k, \ell) \leq (k-1)(\ell-1) + 2.$$

Proof. The statement is obviously true when $\min(k, \ell) = 1$, so assume that $\min(k, \ell) \geq 2$. Without loss of generality $\pi = (1, a_1, a_2, ..., a_{(k-1)(\ell-1)+1})$. Consider the sequence $[a_1, a_2, ..., a_{(k-1)(\ell-1)+1}]$. By the Erdős-Szekeres theorem, it has either an increasing subsequence of length k or a decreasing subsequence of length ℓ. If there is an increasing subsequence $[a_{i_1}, a_{i_2}, ..., a_{i_k}]$, then $(1, a_{i_1}, a_{i_2}, ..., a_{i_k})$ would form an increasing cyclic sub-permutation of π of length $k+1$. Otherwise, if there is a decreasing subsequence $[a_{i_1}, a_{i_2}, ..., a_{i_\ell}]$, then $(a_{i_1}, a_{i_2}, ..., a_{i_\ell}, 1)$ would form a decreasing cyclic sub-permutation of π of length $\ell + 1$. □

Lemma 2.6.2. *For $k, \ell \geq 1$,*

$$\alpha(k, \ell) > (k-1)(\ell-1) + 1.$$

In particular, if $\min(k, \ell) \geq 2$, $\pi = (1, a_1, \ldots, a_{(k-1)(\ell-1)})$ where the sequence a_i is given by one of the formulas in Theorem 2.6.3 part (2) (i) or (ii), then π does not have an increasing cyclic sub-permutation of length $k+1$ or a decreasing cyclic sub-permutation of length $\ell + 1$.

Proof. The lemma is trivial when $\min(k, \ell) = 1$. Assume $\min(k, \ell) \geq 2$ and $\pi = (1, a_1 \ldots, a_{(k-1)(\ell-1)})$, where $[a_1, \ldots, a_{(k-1)(\ell-1)}]$ is given by Theorem 2.6.3 part (2) (i), i.e. for each $(i, j) \in [\ell-1] \times [k-1]$ $a_{(j-1)(\ell-1)+i} = (\ell-1-i)(k-1) + j + 1$. (The example given in Figure 2.1 for $k = 4$ and $\ell = 5$ is $\pi = (1, 11, 8, 5, 2, 12, 9, 6, 3, 13, 10, 7, 4)$.) The other case can be handled analogously.

We claim π does not have an increasing cyclic sub-permutation of length $k+1$ nor does it have a cyclic sub-permutation of length $\ell+1$. Starting from a_1, we can partition the sequence $A = [a_1, \ldots, a_{(k-1)(\ell-1)}]$ into $(k-1)$ decreasing sub-sequences $D_1, \ldots, D_{k-1}$, each consisting of $(\ell-1)$ consecutive elements of the original sequence. In particular, $D_i = [a_t, a_{t+1}, \ldots, a_{t+\ell-2}]$ where $t = (i-1)(\ell-1)+1$. In Figure 2.1, this partition corresponds to $[11,8,5,2]$, $[12,9,6,3]$,$[13,10,7,4]$. Let L be the longest increasing cyclic sub-permutation of π. Suppose $L = (a_{i_1}, a_{i_2}, \ldots, a_{i_t})$ where $a_{i_1} < a_{i_2} < \ldots < a_{i_t}$. L and D_i has at most 2 common elements for each i, as the elements in D_i are decreasing in A. If $a_{i_1} = 1$, then L can contain at most one element from each of the D_is. Since there are at most $k-1$ D_is, it follows that L has length at most k. If $a_{i_1} \neq 1$, then $a_{i_1} \in D_j$ for some $j \in [k-1]$. In this case, $1 \notin L$. Furthermore, L can have at most 2 elements from D_j, and at most one element from D_i for each $i \in [k-1]\backslash\{j\}$. Thus L has length at most k.

We can also partition A into $(\ell-1)$ increasing subsequences $C_1, \ldots, C_{\ell-1}$ of length $(k-1)$. In particular, let $C_i = [c_i, c_i+1, \ldots, c_i+k-2]$ where $c_i = 2+(i-1)(k-1)$. In the example above, C_1, C_2, C_3, C_4 would correspond to $[2,3,4]$, $[5,6,7]$, $[8,9,10]$ and $[11,12,13]$. Similar to the analysis above, let L be the longest decreasing cyclic sub-permutation of π. Suppose $L = (a_{i_1}, a_{i_2}, \ldots, a_{i_t})$ where $a_{i_1} > a_{i_2} > \ldots > a_{i_t}$. As before, L can have at most 2 common elements with each C_i. If $a_{i_t} = 1$, then L can contain at most one element from each of the C_is. Since there are at most $\ell-1$ C_is, it follows that L has length at most ℓ. If $a_{i_t} \neq 1$, observe that if for some j L has 2 common elements with C_j, then every other C_i $(i \neq j)$ can contain at most one element from L since numbers in C_t are strictly larger than all numbers in C_s for $s < t$. Thus L has length at most ℓ. $\square$

Theorem 2.6.1 follows from Lemma 2.6.1 and 2.6.2.

2.6.2 The structure of the extremal examples in the linear Erdős-Szekeres problem

We will first consider the linear problem, i.e. sub-permutations will be linear sub-permutations. We will emphasize this by using the vector notation $\vec{a} = [a_1, \ldots, a_n]$

when talking about linear permutations. Recall the definition of $\gamma_{\vec{a}}, R_{\vec{a}}, V_{\vec{a}}$ in Definition 2.6.5 and 2.6.6. It is easy to see that $\gamma_{\vec{a}}$ is an injective (and therefore bijective) function, since for $t_1 < t_2$ we have $a_{t_1} \neq a_{t_2}$ and either every increasing sequence ending at a_{t_1} can be extended to an increasing sequence ending at a_{t_2}, or every decreasing sequence. The following are immediate from the definitions and prior statements in the lemma:

Lemma 2.6.3. *Let $\vec{a} \in \mathbb{S}_{k,\ell}$. The following are true:*

(1) Let $t_1, t_2 \in [k\ell]$ such that $t_1 < t_2$, and define i_1, i_2, j_1, j_2 by $\gamma_{\vec{a}}(t_q) = (i_q, j_q)$ for $q \in [2]$. If $a_{t_1} < a_{t_2}$ then $j_1 < j_2$ and if $a_{t_1} > a_{t_2}$ then $i_1 < i_2$.

(2) Let $i_2 \leq i_1$, $j_2 \leq j_1$ and $\gamma_{\vec{a}}(t_q) = (i_q, j_q)$ where $q \in [2]$. Then $t_2 \leq t_1$.

(3) $R_{\vec{a}} \in \mathbb{Y}_{\ell,k}$.

(4) For any $i \in [\ell], j \in [k]$ the sequence $[a_{\gamma_{\vec{a}}^{-1}(i,1)}, \ldots, a_{\gamma_{\vec{a}}^{-1}(i,k)}]$ is an increasing subsequence of $\vec{a}$ and the sequence $[a_{\gamma_{\vec{a}}^{-1}(1,j)}, \ldots, a_{\gamma_{\vec{a}}^{-1}(\ell,j)}]$ is a decreasing subsequence of $\vec{a}$.

(5) $V_{\vec{a}} \in \mathbb{Y}_{\ell,k}$.

(6) $\phi : \mathbb{S}_{k,\ell} \to \mathbb{Y}_{\ell,k} \times \mathbb{Y}_{\ell,k}$ defined by $\phi(\vec{a}) = (R_{\vec{a}}, V_{\vec{a}})$ is an injective function

Proof. (1) follows from the fact that if $a_{t_1} < a_{t_2}$ ($a_{t_1} > a_{t_2}$) then any increasing (decreasing) subsequence of $\vec{a}$ ending on a_{t_1} can be extended to a longer increasing (decreasing) subsequence ending at a_{t_2}. This in turn implies (2), which gives (3). (2) implies that for any $i \in [\ell], j \in [k]$ the sequences $[\gamma^{-1}(i,1), \gamma^{-1}(i,2), \ldots, \gamma^{-1}(i,k)]$ and $[\gamma^{-1}(1,j), \gamma^{-1}(2,j), \ldots, \gamma^{-1}(\ell,j)]$ are increasing, and this together with (1) gives (4). (5) follows from (4). (3) and (5) gives that ϕ is a well-defined function, and it follows from the definitions that ϕ must be injective, so (6) is true.

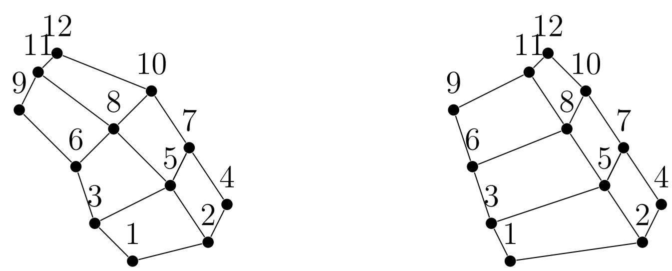
□

The proof of Theorem 2.6.2 is finished by showing that

Lemma 2.6.4. *Let $R = (r_{ij}), V = (v_{ij}) \in \mathbb{Y}_{\ell,k}$ and define the sequence $\vec{a} = [a_1, \ldots, a_{k\ell}]$ by $a_t = v_{ij}$ if and only if $t = r_{\ell+1-i,j}$. Then $\vec{a} \in \mathbb{S}_{k,\ell}$, $R = R_{\vec{a}}$ and $V = V_{\vec{a}}$. Consequently, the function ϕ defined in Lemma 2.6.3 is a bijection.*

Proof. From the fact that the entries of V (and also the entries of R) are unique, it follows that $\vec{a}$ is a well-defined permutation of $[k\ell]$. To show, $\vec{a} \in \mathbb{S}_{k,\ell}$, it is enough to show that $\vec{a}$ does not have an increasing subsequence of length $k+1$ or a decreasing subsequence of length $\ell+1$. Assume to the contrary that $[a_{t_1} \ldots, a_{t_{k+1}}]$ is an increasing subsequence of length $k+1$ of $\vec{a}$. For each $q \in [k+1]$ define (i_q, j_q) by $a_{t_q} = v_{i_q j_q}$. By the pigeonhole principle there is a $q_1 < q_2$ such that $j_{q_1} = j_{q_2}$. Since $V \in \mathbb{Y}_{\ell,k}$, $t_{q_1} < t_{q_2}$ and $a_{t_1} < a_{t_2}$, this implies $i_{q_1} < i_{q_2}$, so $\ell + 1 - i_{q_1} > \ell + 1 - i_{q_2}$, which together with $R \in \mathbb{Y}_{k,\ell}$ gives $t_{q_1} > t_{q_2}$, a contradiction. The statement that $\vec{a}$ does not have a decreasing subsequence of length $\ell + 1$ follows similarly, so $\vec{a} \in \mathbb{S}_{k,\ell}$. Fix an $i \in [\ell]$ and define the sequence $\vec{t} = [t_1, \ldots, t_k]$ by $t_q = r_{i,q}$. Since $R \in \mathbb{Y}_{\ell,k}$, $\vec{t}$ is an increasing sequence. Moreover, since $a_{t_q} = v_{\ell+1-i,q}$ and $V \in \mathbb{Y}_{\ell,k}$, $[a_{t_1}, \ldots, a_{t_k}]$ is an increasing subsequence of $\vec{a}$. Similarly for any $j \in [k]$ define $\vec{w} = [w_1, \ldots, w_\ell]$ by $w_q = r_{q,j}$, then $\vec{w}$ is increasing and $[a_{w_1}, \ldots, a_{w_\ell}]$ is a decreasing subsequence of $\vec{a}$. This implies that for each $i \in [\ell]$ and $j \in [k]$, $\gamma_{\vec{a}}(r_{i,j}) = (i', j')$ where $i' \geq i$ and $j' \geq j$. Since both $\gamma_{\vec{a}}$ and γ are bijections from $[k\ell]$ to $[\ell]\times[k]$, we get that $\gamma_{\vec{a}}(r_{i,j}) = (i,j)$ and so $r_{ij} = \gamma_{\vec{a}}^{-1}(i,j)$. Thus we obtain $R = R_{\vec{a}}$. Since for $V_{\vec{a}} = (v^\star_{ij})$ we have by definition that $v^\star_{ij} = a_{\gamma_{\vec{a}}^{-1}(\ell+1-i,j)} = a_{r_{\ell+1-i,j}} = v_{ij}$, we obtain $V = V_{\vec{a}}$. So $\phi(\vec{a}) = (R, V)$, therefore ϕ is surjective, which together with Lemma 2.6.3 part (6) gives that ϕ is a bijection. □

We remark that similar ideas appear in [9] to find the longest increasing subsequence of a sequence. Fix $k, \ell \geq 1$ and set $n = k\ell$. Note that the above results imply that if we represent the sequence $\vec{a} = [a_1, \ldots, a_n]$ as the set of n points (t, a_t) and connect two points (t_1, a_{t_1}) and (t_2, a_{t_2}) precisely when $\gamma_{\vec{a}}(t_1)$ and $\gamma_{\vec{a}}(t_1)$ agree in one of the coordinates and differ by 1 on the other, then we get a (potentially somewhat distorted) $\ell \times k$ grid where the slope of the line from t_1 to t_2 is positive exactly when $\gamma_{\vec{a}}(t_2)$ agrees with $\gamma_{\vec{a}}(t_1)$ on the first coordinate, and negative otherwise. The grid may be distorted in the sense that it is formed by quadrangles that are not necessarily rectangles and are not necessarily isomorphic, and the grid "balances on one of its corners"; in fact it balances on the grid-point indexed $(\ell+1,1)$ with sequence value 1. Indeed, any sequence $[a_1,\ldots,a_n]$ that is a permutation of $[n]$ is in $\mathbb{S}_{k,\ell}$ precisely when such a grid can be fit on its n-point representation in the plane (where the corner on which the distorted grid balances is the grid-point $(\ell+1,1)$ and has height 1). Figure 2.2 shows two examples of extremal sequences for the linear Erdős-Szekeres theorem for $k=4$ and $\ell=5$ with distorted grid representation. Note that they have the same valuation but different ranking.



$(9,11,12,6,3,8,1,10,5,7,2,4)$ $(9,6,3,1,11,12,8,10,5,7,2,4)$

Figure 2.2: Extremal sequences for $k=4$ and $\ell=5$ with distorted grid representation.

2.6.3 The structure of the extremal examples in the circular Erdős-Szekeres problem

We devote this section to the proof of Theorem 2.6.3. The statement is obvious when $\min(k,\ell)=1$, so we assume that $\min(k,\ell)\geq 2$. For this case we have shown in Lemma 2.6.2 that the structures described in Theorem 2.6.3 are all in $\mathbb{C}_{k,\ell}$, the proof of Theorem 2.6.3 is finished by showing that these structures are the only elements od $\mathbb{C}_{k,\ell}$. Moreover, since any cyclic permutation of length at least 3 that is not the increasing (decreasing) permutation contains a decreasing (increasing) sub-permutation of length at least 3, the statement follows when $\min(k,\ell) = 2$. So it is enough to focus on the case when $\min(k,\ell) \geq 3$.

We will define $\mathbb{C}^{\star}_{k,\ell}$ as the set of those sequences in $\mathbb{S}_{k-1,\ell-1}$ that, taken as as cyclic permutations have no increasing cyclic sub-permutation of length $k+1$, and no decreasing cyclic sub-permutations of length $\ell+1$. For the ease of reference, given a sequence $\vec{\rho} \in \mathbb{C}^{\star}_{k,\ell}$ we will use ρ to denote the cyclic permutation corresponding to $\vec{\rho}$.

As an increasing (decreasing) cyclic sub-permutation of a cyclic permutation either starts (ends) with 1 or does not contain 1, the following is obvious:

Lemma 2.6.5. *Let $k,\ell \in \mathbb{Z}$ with $\min(k,\ell) \geq 2$. $(1, a_1, \ldots, a_{(k-1)(\ell-1)}) \in \mathbb{C}_{k,\ell}$ if and only if $[a_1 - 1, a_2 - 1, \ldots, a_{(k-1)(\ell-1)} - 1] \in \mathbb{C}^{\star}_{k,\ell}$.*

By the above Lemma, to characterize the extremal examples in the cyclic Erdős-Szekeres theorem it is enough to determine $\mathbb{C}^{\star}_{k,\ell}$. The proof of Theorem 2.6.3 is concluded by showing that

Lemma 2.6.6. *Let $k,\ell \in \mathbb{Z}$ with $\min(k,\ell) \geq 3$ and $\vec{\rho} = [a_1, \ldots, a_{(k-1)(\ell-1)}] \in \mathbb{C}^{\star}_{k,\ell}$. Then we have one of the following:*

(i) For each $i \in [\ell-1]$ and $j \in [k-1]$ $a_{(j-1)(\ell-1)+i} = (\ell-1-i)(k-1)+j$.

(ii) For $i \in [\ell-1]$ and $j \in [k-1]$ $a_{(i-1)(k-1)+j} = (j-1)(\ell-1)+(\ell-i)$.

Proof. Let $\vec{\rho} = [a_1, \ldots, a_{(k-1)(\ell-1)}] \in \mathbb{C}^{\star}_{k,\ell} \subseteq \mathbb{S}_{k-1,\ell-1}$. For shortness, we will use γ for $\gamma_{\vec{\rho}}$. For each $i \in [\ell-1]$, define the sequences $C_i = [c_{i,1}, \ldots, c_{i,k-1}]$ by $c_{i,j} = a_{\gamma^{-1}(i,j)}$ and for each $j \in [k-1]$, let $D_j = [c_{1,j}, c_{2,j}, \ldots, c_{\ell-1,j}]$. Clearly, $C_1, \ldots, C_{\ell-1}$ and $D_1, \ldots, D_{k-1}$ partition the elements of $\vec{\rho}$. By Lemma 2.6.3 part (4) the C_is are increasing and the D_js are decreasing subsequences of $\vec{\rho}$. As $\vec{\rho} \in \mathbb{C}^{\star}_{k,\ell}$, the cyclic permutation ρ does not have an increasing cyclic sub-permutation of length $k+1$ or decreasing cyclic sub-permutation of length $\ell+1$. We have two possibilities

Case 1: $\gamma^{-1}(\ell-1,1) < \gamma^{-1}(1,k-1)$. As for each $j \in [k-1]$, D_j is an decreasing subsequence of $\vec{\rho}$ we get

$$a_{\gamma^{-1}(1,j)} > a_{\gamma^{-1}(2,j)} > \cdots > a_{\gamma^{-1}(\ell-1,j)}$$

Using this for $j \in \{1, k-1\}$ and the condition, for each $i \in [\ell-2]$ we have

$$(c_{1,k-1}, c_{2,k-1}, \ldots, c_{\ell-i,k-1}, c_{\ell-i-1,1}, c_{\ell-i,1}, \cdots, c_{k-1,1})$$

is a cyclic sub-permutation of length $\ell+1$ of the cyclic permutation ρ. Since this can not be an decreasing sub-permutation, we must have $c_{\ell-i,k-1} < c_{\ell-i-1,1}$. Let $i^\star \in [\ell-i-1]$ and $j \in [k-1]$. As D_1 is decreasing and $C_{i^\star}$ is increasing, we have $c_{\ell-i,k-1} < c_{\ell-i-1,1} \le c_{i^\star,1} \le c_{i^\star,j}$ and consequently $c_{\ell-i,k-1} \le (k-1)i$. Using that $C_{\ell-i}$ is increasing, induction on i gives that $c_{\ell-i,j} = a_{\gamma^{-1}(\ell-i,j)} = (i-1)(k-1)+j$.

Since C_1 and $C_{\ell-1}$ are both increasing subsequences of $\vec{\rho}$ and $C_{\ell-1}$ contains the smallest $(k-1)$ elements of $[(k-1)(\ell-1)]$, we must have that for each $j \in [k-2]$ that $\gamma^{-1}(1,j+1) > \gamma^{-1}(\ell-1,j)$, otherwise

$$(c_{\ell-1,j}, c_{\ell-1,j+1}, \ldots, c_{\ell-1,k-1}, c_{1,1}, c_{1,2}, \ldots, c_{1,j+1})$$

would form an increasing cyclic sub-permutation of length $k+1$ of ρ. Using the fact that D_j is a sub-permutation and induction on j, for each $j \in [k-1]$ we get $\gamma^{-1}(\ell-i,j) = (j-1)(\ell-1)+\ell-i$.

Combining these we must have that for $i \in [\ell-1]$ and $j \in [k-1]$ $a_{(j-1)(\ell-1)+i} = (\ell-1-i)(k-1)+j$, giving case (i) of this lemma.

Case 2: $\gamma^{-1}(l-1,1) > \gamma^{-1}(1,k-1)$.

As before, we get that for each $j \in [k-2]$ the sequence

$$(c_{l-1,1}, c_{l-1,2}, \ldots, c_{l-1,k-j}, c_{1,k-j-1}, c_{1,k-j}, \ldots, c_{1,k-1})$$

is a cyclic sub-permutation of length $k+1$ of ρ, and as it can not be increasing, we have $c_{l-1,k-j} > c_{1,k-j-1}$. Using the same logic as in Case 1 we obtain for each $j \in [k-1]$ and $i \in [\ell-1]$ $a_{\gamma^{-1}(i,j)} = (j-1)(\ell-1) + (\ell-i)$.

Again, for each $i \in [\ell-1]$ we have $\gamma^{-1}(i+1,1) > \gamma^{-1}(i,k-1)$, otherwise

$$(c_{i,k-1}, c_{i+1,k-1}, \dots, c_{\ell-1,k-1}, c_{1,1}, c_{2,1}, \dots, c_{i+1,1})$$

forms decreasing cyclic sub-permutation of length $\ell+1$ of ρ. We obtain that $\gamma^{-1}(i,j) = (i-1)(k-1)+j$. Combining these we must have that for $i \in [\ell-1]$ and $j \in [k-1]$ $a_{(i-1)(\ell-1)+j} = (j-1)(\ell-1)+(\ell-i)$, giving case (ii) of this lemma.

□

For $k, \ell \geq 2$, set $n = (k-1)(\ell-1)$. and consider the sequence $\vec{\rho} = [1, a_1, \dots, a_n]$; i.e. use the sequence representation of the cyclic permutation or ρ that starts with 1. It is worth noting that $\rho \in \mathbb{C}_{k,\ell}$ precisely when taking the $n+1$ points representing $\vec{\rho}$ in the plane and putting in the grid lines corresponding to $[a_1-1, \dots, a_n-1]$ described in the end of the previous section to the n points of the form (i, a_i), they form a non-distorted grid, i.e. a grid with rectangles (and not just quadrangles) that are of the same size (in fact, the ratio of the side length of each rectangle is $\frac{k-1}{\ell-1}$), and the point $(1,1)$ lies on either the first or the last line with positive slope, as in Figure 2.1.

Chapter 3

Turán-type and Dirac-type problems

3.1 cover-Turán number of Berge hypergraphs

In Section 2.5, we defined a new type of Ramsey number, namely the *cover Ramsey number*, which behaves more like the classical Ramsey number than the Ramsey number of Berge hypergraphs defined in Section 2.4. Motivated this phenomenon, we extend the investigations to the analogous *cover Turán number* for Berge hypergraphs. In particular, given a fixed graph G and a finite set of positive integers $R \subseteq [k]$, we define the *R-cover Turán number* of G, denoted as $\hat{ex}_R(n,G)$, as the maximum number of edges in the shadow graph of a Berge-G-free R-graph on n vertices. The *R-cover Turán density*, denoted as $\hat{\pi}_R(G)$, is defined as

$$\hat{\pi}_R(G) = \limsup_{n\to\infty} \frac{\hat{\mathrm{ex}}_R(n,G)}{\binom{n}{2}}.$$

When R is clear from the context, we ignore R and use *cover Turán number* and *cover Turán density* for short. A graph is called *R-degenerate* if $\hat{\pi}_R(G) = 0$. For the ease of reference, when $R = \{k\}$, we simply denote $\hat{\pi}_R(G)$ as $\hat{\pi}_k(G)$ and call G *k-degenerate* if $\hat{\pi}_{\{k\}}(G) = 0$. We remark that the Turán number of graphs only differ by a constant factor when the host hypergraph is uniform compared to non-uniform. In particular, we show the following proposition.

Proposition 3.1.1. *If R is a finite set of positive integers such that $min(R) = m \geq 2$ and $max(R) = M$. Then given a fixed graph G,*

$$\max_{r\in R} \hat{ex}_r(n,G) \leq \hat{ex}_R(n,G) \leq \frac{\binom{M}{2}}{\binom{m}{2}} \hat{ex}_m(n,G).$$

Indeed, the first inequality is clear from definition. For the second inequality, suppose we have an R-graph $\mathcal{H}$ with more than $\binom{M}{2}/\binom{m}{2} \cdot \hat{\mathrm{ex}}_m(n, G)$ edges in its shadow. For each hyperedge h in $\mathcal{H}$, shrink it to a hyperedge of size m by uniformly and randomly picking m vertices in h. Call the resulting hypergraph $\mathcal{H}'$. It is easy to see that for any edge $e \in E(\partial(\mathcal{H}))$, $\Pr(e \in E(\partial(\mathcal{H}'))) \geq \binom{m}{2}/\binom{M}{2}$. Hence by linearity of expectation, the expected number of edges in $\partial(\mathcal{H}')$ is more than $\hat{\mathrm{ex}}_m(n, G)$. It follows that there exists a way to shrink $\mathcal{H}$ to a m-graph with at least $(\hat{\mathrm{ex}}_m(n, G) + 1)$ edges in its shadow. Thus, by definition of the cover Turán number, $\mathcal{H}'$ contains a Berge copy of G, which corresponds to a Berge-G in $\mathcal{H}$.

Remark 3.1.1. *Note that Proposition 3.1.1 implies that if a graph G is k-degenerate (where $k \geq 2$), then it is R-degenerate for any finite set R satisfying $min(R) \geq k$. In particular, a bipartite graph is k-degenerate for all $k \geq 2$.*

In this paper, we determine the cover Turán density of all graphs when the uniformity of the host graph equals to 3. We first establish a general upper bound for the cover Turán density of graphs.

Theorem 3.1.1. *For any fixed graph G and any fixed $\epsilon > 0$, there exists n_0 such that for any $n \geq n_0$,*

$$\hat{ex}_k(n, G) \leq \left(1 - \frac{1}{\chi(G) - 1} + \epsilon\right)\binom{n}{2}.$$

We remark that Theorem 3.1.1 holds when the host hypergraph is non-uniform as well, i.e. we can replace k with any fixed finite set of positive integers R. If $\chi(G) > k$, there is a construction giving the matching lower bound. Partition the vertex set into $t := \chi(G) - 1$ equitable parts $V = V_1 \cup V_2 \cup \cdots \cup V_t$. Let $\mathcal{H}$ be the k-uniform hypergraph on the vertex set V consisting of all k-tuples intersecting each V_i on at most one vertex. The shadow graph is simply the Turán graph with $(1 - \frac{1}{\chi(G)-1} + o(1))\binom{n}{2}$ edges. The shadow graph is K_{t+1}-free, thus contains no subgraph G. It follows that $\mathcal{H}$ is Berge-G-free. Therefore, we have the following theorem:

Theorem 3.1.2. *For any $k \geq 2$, and any fixed graph G with $\chi(G) \geq k+1$, we have*

$$\hat{\pi}_k(G) = 1 - \frac{1}{\chi(G)-1}.$$

Given a simple graph G on n vertices $v_1, \ldots, v_n$ and a sequence of n positive integers $s_1, \ldots, s_n$, we denote $B = G(s_1, \ldots, s_n)$ the $(s_1, \ldots, s_n)$-*blowup* of G obtained by replacing every vertex $v_i \in G$ with an independent set I_i of s_i vertices, and by replacing every edge (v_i, v_j) of G with a complete bipartite graph connecting the independent sets I_i and I_j. If $s = s_1 = s_2 = \cdots = s_n$, we simply write $G(s_1, \ldots, s_n)$ as $G(s)$ where s is called the *blowup factor*. We also define a *generalized blowup* of G, denoted by $G(s_1, \ldots, s_n; M)$ where $M \subseteq E(G) \subseteq \binom{[n]}{2}$, as the graph obtained by replacing every vertex $v_i \in G$ with an independent set I_i of s_i vertices, and by replacing every edge (v_i, v_j) of $E(G)\backslash M$ with a complete bipartite graph connecting I_i and I_j and replacing every edge $(v_i, v_j) \in M$ with a maximal matching connecting I_i and I_j. When $M = \varnothing$, we simply write $G(s_1, \ldots, s_n; M)$ as the standard blowup $G(s_1, \ldots, s_n)$.

We first want to characterize the class of degenerate graphs when the host hypergraph is 3-uniform. Observe that $\hat{\mathrm{ex}}_k(n, G) \leq \binom{k}{2} ex_k(n, G)$. This implies that any graph G satisfying $ex_k(n, G) = o(n^2)$ is k-degenerate. In particular, by results of [96, 97, 86, 144], any cycles of fixed length at least 4 and $K_{2,t}$ are 3-degenerate. For triangles, Grósz, Methuku and Tompkins [88] showed that the uniformity threshold of a triangle is 5, which implies that C_3 is 5-degenerate. Moreover, there are constructions which show that C_3 is not 3-degenerate or 4-degenerate. For $K_{s,t}$ where $s, t \geq 3$, it is shown [144, 88, 6] that $ex_r(n, K_{s,t}) = \Theta(n^{r - \frac{r(r-1)}{2s}})$. Thus in this case, the corresponding results on Berge Turán number do not imply the degeneracy of $K_{s,t}$ in the cover Turán density.

In this paper, we classify all degenerate graphs when the host hypergraph is 3-uniform.

Theorem 3.1.3. *Given a simple graph G, $\hat{\pi}_3(G) = 0$ if and only if G satisfies both of the following conditions:*

(1) G is triangle-free, and there exists an induced bipartite subgraph $B \subseteq G$ such that $V(G) - V(B)$ is a single vertex.

(2) There exists a bipartite subgraph $B \subseteq G$ such that $E(G) - E(B)$ is a matching (possibly empty) in one of the partitions of B.

Corollary 3.1.1. *Given a simple graph G, $\hat{\pi}_3(G) = 0$ if and only if G is contained in both $C_5(1, s, s, s, s)$ and $C_3(s, s, s; \{\{1, 2\}\})$ for some positive integer s.*

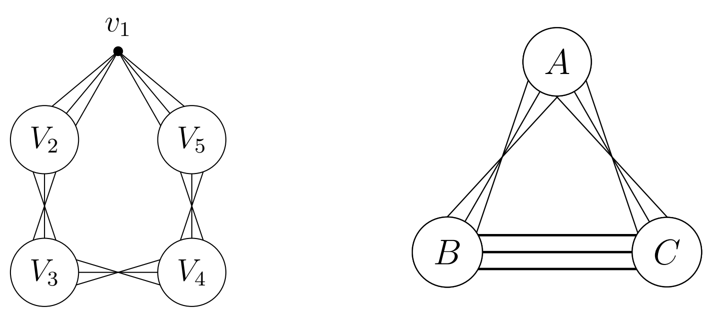


Figure 3.1: $C_5(1, s, s, s, s)$ and $C_3(s, s, s; \{\{1, 2\}\})$

Corollary 3.1.2. *Given a simple graph G, $\hat{\pi}_3(G) = 0$ if and only if G is a subgraph of one of the graphs in Figure 3.2.*

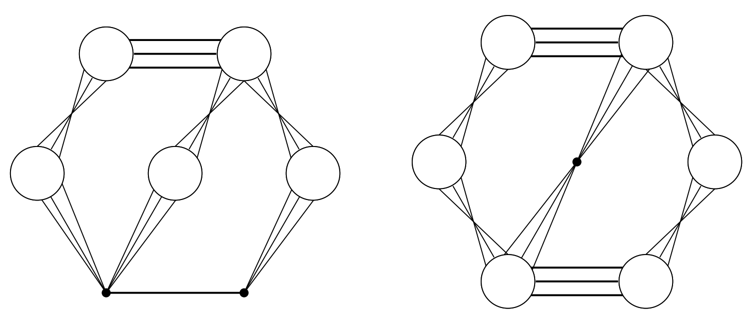


Figure 3.2: Characterization of 3-degenerate graphs.

With Theorem 3.1.1 and Theorem 3.1.3, we can then determine the cover Turán density of all graphs when $k = 3$. The results are summarized in the following theorem.

Theorem 3.1.4. *Given a simple graph G,*

$$\hat{\pi}_3(G) = \begin{cases} 1 - \frac{1}{\chi(G)-1} & \text{if } \chi(G) \geq 4, \\ 0 & \text{if } G \text{ satisfies the condition in Theorem 3.1.3,} \\ \frac{1}{2} & \text{otherwise.} \end{cases}$$

For 3-cover Turán number, we also show the following proposition:

Proposition 3.1.2. *Let G be a connected bipartite graph such that every edge is contained in a C_4 and every two vertices in the same part have a common neighbor. Then*

$$\hat{ex}_3(n, G) = \Theta(ex(n, G)).$$

Proof. The fact that $\hat{\mathrm{ex}}_3(n, G) = O(ex(n, G))$ is a consequence of Proposition 3.1.1. For the lower bound, consider an extremal G-free graph H with $ex(n, G)$ edges. It follows that there is a bipartite subgraph $H' = A \cup B$ of H which is G-free and contains at least $\frac{1}{2}ex(n, G)$ edges. We then construct a 3-graph $\mathcal{H}$ as follows. For each $a \in A$, replace a with two new vertices a_1, a_2. The vertex set B remains the same. For each $e = \{a, b\} \in E(H')$ with $a \in A$, $b \in B$, we have a hyperedge $\{a_1, a_2, b\}$ in $\mathcal{H}$. We claim that $\mathcal{H}$ contains no Berge-G. Indeed, if there is any Berge-G in $\mathcal{H}$, then one of the following two cases must happen:

Case 1: An edge in G is mapped to $\{a_1, a_2\}$ for some $a \in A$. However, note that there is no C_4 containing a_1a_2 in $\partial(\mathcal{H})$ while every edge of G is contained in a C_4. This gives us a contradiction.

Case 2: Two vertices of G from the same part are mapped to $\{a_1, a_2\}$ for some $a \in A$. In this case, by our assumption, a_1, a_2 have a common neighbor w in

G. However, there are no two distinct hyperedges embedding a_1w, a_2w by our construction. Contradiction.

Hence it follows that $\mathcal{H}$ is Berge-G-free and has $\Omega(ex(n,G))$ hyperedges. $\square$

Remark 3.1.2. *We give a class of graphs satisfying the conditions in Proposition 3.1.2. Let $B = B_1 \cup B_2$ be an arbitrary connected bipartite graph with minimum degree 2 such that each part has a vertex that is adjacent to all the vertices in the other part. It's easy to check that B satisfies the conditions in Proposition 3.1.2.*

Using Proposition 3.1.2, we have the following corollary on the asymptotics of the cover Turán number of $K_{s,t}$.

Corollary 3.1.3. *For positive integers $t \geq s \geq 2$, we have*

$$\hat{ex}_3(n, K_{s,t}) = \Theta(ex(n, K_{s,t})).$$

The following questions would be interesting for further investigations:

Question 3.1.1. *Characterize all k-degenerate graphs or determine the $\{k\}$-cover Turán density of all graphs for $k \geq 4$.*

Question 3.1.2. *Determine the asymptotics of the cover Turán number of the 3-degenerate graphs in Theorem 3.1.3.*

3.1.1 Proof of Theorem 3.1.1

Proof of Theorem 3.1.1. Let $k \geq 2$ and G be a fixed graph with $\chi(G) \geq 2$. Let $\epsilon > 0$. Suppose $\mathcal{H}$ is an edge-minimal k-uniform hypergraph on sufficiently large n vertices such that

$$|E(\partial(\mathcal{H}))| \geq \left(1 - \frac{1}{\chi(G)-1} + \epsilon\right)\binom{n}{2}.$$

Our goal is to show that $\mathcal{H}$ contains a Berge copy of G. For ease of reference, set $H = \partial(\mathcal{H})$. Let $M = k^2/\epsilon$. Let H' be the subgraph of H obtained by deleting all the edges uv from H with co-degree $d(\{u,v\}) \geq M$ in $\mathcal{H}$.

Claim 3.1.1. $|E(H')| \geq \left(1 - \frac{1}{\chi(G)-1} + \epsilon/2\right)\binom{n}{2}$.

Proof. Let $L = E(H) \backslash E(H')$. By double counting, the number of hyperedges containing some edge in L is at least $LM/\binom{k}{2}$. Since $\mathcal{H}$ is assumed to be edge-minimal, it follows that every hyperedge h contains a vertex pair that is only contained in h. Hence $|E(\mathcal{H})| \leq \binom{n}{2}$. It follows that

$$LM/\binom{k}{2} \leq |E(\mathcal{H})| \leq \binom{n}{2},$$

which implies that

$$L \leq \frac{k^2}{2M}\binom{n}{2} \leq \frac{\epsilon}{2}\binom{n}{2}.$$

This completes the proof of the claim. □

Let G' be the blowup of G by a factor of $b = Mv(G)^2 k$, i.e., $G' = G(b)$. Suppose $V(G) = \{v_1, \ldots, v_s\}$ and V_i is the blowed-up independent set in G' that corresponds to v_i. Recall the celebrated Erdős-Stone-Simonovits theorem [70, 72], which states that for a fixed simple graph F, $ex(n, F) = \left(1 - \frac{1}{\chi(F)-1} + o(1)\right)\binom{n}{2}$. Since $\chi(G') = \chi(G)$, it follows by the Erdős-Stone-Simonovits theorem that for sufficiently large n, H' contains G' as a subgraph.

Our goal is to give an embedding f of G into G' so that $f(v_i) \in V_i$ for all $1 \leq i \leq s$ and every edge of G is embedded in a distinct hyperedge in $\mathcal{H}$. For ease of reference, set $L_j = \{v_1, \ldots, v_j\}$. For $1 \leq t \leq s$ and $v \in V(G)$, set $N_t(v) = N_G(v) \cap L_t$. For $i = 1$, just embed v_1 to an arbitrary vertex in V_1. Suppose that $v_1, \ldots, v_t$ are already embedded and edges in $G[L_t]$ are already embedded in distinct hyperedges. We now want to embed v_{t+1} into an appropriate vertex in V_{t+1}, i.e., we want to find a vertex $u \in V_{t+1}$ such that there are distinct unused hyperedges embedding the edges from u to $f(N_t(v_{t+1}))$. Note that each vertex u in V_{t+1} is adjacent to all vertices in $f(N_t(v_{t+1}))$ in G'. Let $S_t(u) = \{u\} \times f(N_t(v_{t+1}))$, i.e., $S_t(u)$ is the set of vertex pairs which contain u and another vertex in $f(N_t(v_{t+1}))$.

Recall that $|V_{t+1}| = Mv(G)^2k$. At most $e(G)(k-2)$ vertices in V_{t+1} are contained in hyperedges that are already used. For any of the remaining vertices $u \in V_{t+1}$, if there are no distinct hyperedges embedding all vertex pairs in $S_t(u)$, that means some hyperedge contains at least two vertex pairs uw_1, uw_2 in $S_t(u)$. Note that $d_{H'}(\{w_1, w_2\}) \leq M$ by the definition of H'. Thus the number of vertices $u \in V_{t+1}$ such that there exists some hyperedge containing at least two vertex pairs in $S_t(u)$ is at most

$$\binom{t}{2}M(k-2) \leq \frac{Mv(G)^2k}{2}.$$

Since $|V_{t+1}| = Mv(G)^2k$, it follows that there exists some $u \in V_{t+1}$ such that u is not contained in any hyperedge already used and there is no hyperedge containing at least two vertex pairs in $S_t(u)$. It follows that there are distinct unused hyperedges containing all vertex pairs in $S_t(u)$. Set $f(v_{t+1})$ to be this u.

By induction, we can then conclude that $\mathcal{H}$ contains a Berge copy of G. This completes the proof of Theorem 3.1.1. □

3.1.2 Proof of Theorem 3.1.3

Regularity Lemma

The proof of Theorem 3.1.3 uses the Szemerédi Regularity Lemma. Given a graph G, and two disjoint vertex sets $X, Y \subseteq V(G)$, let $e(X,Y)$ denote the number of edges intersecting both X and Y. Define $d(X,Y) = e(X,Y)/|X||Y|$ as the *edge density* between X and Y. (X,Y) is called ϵ-regular if for all $X' \subseteq X$, $Y' \subseteq Y$ with $|X'| \geq \epsilon|X|$ and $|Y'| \geq \epsilon|Y|$, we have $|d(X,Y) - d(X',Y')| \leq \epsilon$. We say a vertex partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$ *equipartite* (with the *exceptional set* V_0) if $|V_i| = |V_j|$ for all $i, j \in [k]$. The vertex partition $V = V_0 \cup V_1 \cup \cdots \cup V_k$ is said to be ϵ-regular if all but at most ϵk^2 pairs (V_j, V_j) with $1 \leq i < j \leq k$ are ϵ-regular and $|V_0| \leq \epsilon n$. The extremely powerful Szemerédi's regularity lemma states the following:

Theorem 3.1.5. *[165] For every ϵ and m, there exists N_0 and M such that every graph G on $n \geq N_0$ admits an ϵ-regular partition $V_0 \cup V_1 \cup \cdots \cup V_k$ satisfying that $m \leq k \leq M$.*

A ϵ-regular pair satisfies the following simple lemma.

Lemma 3.1.1. *Suppose (X, Y) is an ϵ-regular pair of density d. Then for every $Y' \subseteq Y$ of size $|Y'| \geq \epsilon|Y|$, there exists less than $\epsilon|X|$ vertices in X that have less than $(d-\epsilon)|Y'|$ neighbors in Y'.*

Proof. Let $Y' \subseteq Y$ with $|Y'| \geq \epsilon|Y|$. Let X' be the set of vertices of X that have less than $(d-\epsilon)|Y'|$ neighbors in Y'. Note that $d(X', Y') < (d-\epsilon)$, which can only happen if $|X'| < \epsilon|X|$. □

Using Lemma 3.1.1, we will show the following lemma using the standard embedding technique.

Lemma 3.1.2. *Fix a positive integer s. Suppose (X, Y) is an ϵ-regular pair of density d such that $\epsilon \leq 1/4s$, $(d-\epsilon)^s \geq 4\epsilon$ and $|X|, |Y| \geq 4s/(d-\epsilon)^s$. Then there exist disjoint subsets $A, C \subseteq X$ and $B, D \subseteq Y$ such that $|A| = |B| = s$, $|C| \geq \epsilon|X|$, $|D| \geq \epsilon|Y|$, and there is a complete bipartite graph connecting A and D, B and C as well as A and B.*

Proof. Denote $A = \{a_1, \ldots, a_s\}$ and $B = \{b_1, \ldots, b_s\}$. For each $i \in [s]$, we will first embed a_i to X one vertex at a time. After embedding the k^{th}-vertex, we will show that the following condition is satisfied:

$$\left| Y \cap \bigcap_{i=1}^{k} N(a_i) \right| \geq (d-\epsilon)^k |Y|.$$

The condition is trivially satisfied when $k = 0$. Suppose that we already embedded the vertices $a_1, \ldots, a_t$ for some $t > 0$. Let $Y'_t = Y \cap \bigcap_{i=1}^{t} N(a_i)$. By induction, $|Y'_t| \geq (d-\epsilon)^t|Y| > \epsilon|Y|$. Hence by Lemma 3.1.1, at least $((1-\epsilon)|X| - s)$ vertices in X have at least $(d-\epsilon)|Y'_t|$ neighbors in Y'_t. Embed a_{t+1} to one of these $((1-\epsilon)|X|-s)$ vertices and it's easy to see that

$$\left|Y \cap \bigcap_{i=1}^{t+1} N(a_i)\right| \geq (d-\epsilon)|Y'_t| \geq (d-\epsilon)^{t+1}|Y|.$$

Now we want to embed b_i to Y'_s one vertex at a time. The process is entirely the same as long as

$$(d-\epsilon)^s(|X|-s) \geq \epsilon|X|$$

and

$$(d-\epsilon)^s|Y| - \epsilon|Y| - s \geq 1,$$

which are satisfied by our assumption on d, $|X|$ and $|Y|$.

□

CONSTRUCTIONS FOR THEOREM 3.1.3

Before we prove Theorem 3.1.3, we first give two constructions and show that if G does not satisfy the conditions (1) and (2) in Theorem 3.1.3, then at least one of the constructions do not contain a Berge copy of G. In particular, suppose A, B are two disjoint set of vertices enumerated as $A = \{a_1, \dots, a_{n/2}\}$ and $B = \{b_1, \dots, b_{n/2}\}$. Let $\mathcal{H}_1$ be a 3-uniform hypergraph such that $V(\mathcal{H}_1) = A \cup B$ and $E(\mathcal{H}_1) = \{\{a_i, b_j, b_{j+1}\} : j \text{ is odd}\}$. Let $\mathcal{H}_2$ be a 3-uniform hypergraph such that $V(\mathcal{H}_2) = A \cup B$ and $E(\mathcal{H}_2) = \{\{b_1, a_i, b_j\} : a_i \in A, b_j \in B\backslash\{b_1\}\}$. Observe that

$$\lim_{n\to\infty} \frac{|E(\partial(\mathcal{H}_1))|}{\binom{n}{2}} = \lim_{n\to\infty} \frac{|E(\partial(\mathcal{H}_2))|}{\binom{n}{2}} = \frac{1}{2}.$$

Claim 3.1.2. *If $\hat{\pi}_3(G) = 0$, then condition (1) and (2) of Theorem 3.1.3 must hold.*

Proof. Suppose that $\hat{\pi}_3(G) = 0$. We claim that (1) and (2) must hold. First observe that $\mathcal{H}_1$ contains no Berge triangle. Hence G must be triangle-free otherwise $\mathcal{H}_1$ is Berge-G-free. Now note that given a hypergraph $\mathcal{H}$, if $\partial(\mathcal{H})$ is G-free, then $\mathcal{H}$ must be Berge-G-free. Observe that $\partial(\mathcal{H}_1)$ contains a bipartite subgraph $B \subseteq \partial(\mathcal{H}_1)$ such that $E(\partial(\mathcal{H}_1)) - E(B)$ is a matching (possibly empty) in one of the partition of B. Hence if there is no such bipartite subgraph in G, then $\partial(\mathcal{H}_1)$ is G-free, implying that $\mathcal{H}_1$ is Berge-G-free. Since $\hat{\pi}_3(G) = 0$, it follows that G must satisfy condition (2). Similarly, observe that $\partial(\mathcal{H}_2)$ satisfies condition (1). Hence if G doesn't satisfy condition (1), then $\mathcal{H}_2$ is Berge-G-free, which contradicts that $\hat{\pi}_3(G) = 0$. Therefore we can conclude that (1) and (2) must hold for G. □

Proof of Theorem 3.1.3

The forward direction is proved in Claim 3.1.2. It remains to show that if G satisfies the conditions (1) and (2) in Theorem 3.1.3, then $\hat{\pi}_3(G) = 0$. Suppose not, i.e., $\hat{\pi}_3(G) \geq d$ for some $d > 0$. Our goal is to show that for every 3-graph $\mathcal{H}$ on (sufficiently large) n vertices and at least $d\binom{n}{2}$ edges in $\partial(\mathcal{H})$, $\mathcal{H}$ contains a Berge copy of G.

Assume first that $\mathcal{H}$ is edge-minimal while maintaining the same shadow. It follows that in every hyperedge h of $\mathcal{H}$, there exists some $e \in \binom{h}{2}$ such that e is contained only in h. Moreover, note that since each hyperedge covers at most 3 edges in $\partial(\mathcal{H})$, we have that

$$|E(\mathcal{H})| \geq \frac{1}{3}|E(\partial(\mathcal{H}))| \geq \frac{d}{3}\binom{n}{2}.$$

Call an edge $e \in \partial(\mathcal{H})$ *uniquely embedded* if there exists a unique hyperedge $h \in E(\mathcal{H})$ containing e. Now randomly partition $V(\mathcal{H})$ into three sets X, Y, Z of the same size. Let $e(X, Y, Z)$ denote the number of hyperedges of $\mathcal{H}$ intersecting each of the sets X, Y, Z on at most one vertex. It's easy to see that $\mathrm{E}[e(X, Y, Z)] = \frac{2}{9}|E(\mathcal{H})|$. Hence there exists a 3-partite subhypergraph $\mathcal{H}_1 = X \cup Y \cup Z$ of $\mathcal{H}$ such that $|E(\mathcal{H}_1)| \geq \frac{2}{9}|E(\mathcal{H})|$. Note that each hyperedge h of $\mathcal{H}_1$ contains some $e \in \binom{h}{2}$ that is uniquely embedded. Hence there are at least $\frac{2}{9}|E(\mathcal{H})|$ uniquely embedded edges in $\partial(\mathcal{H}_1)$. Without loss of generality, assume that there are at least $\frac{2}{27}|E(\mathcal{H})|$ uniquely embedded edges between the vertex sets X and Y in $\partial(\mathcal{H}_1)$. Let $\mathcal{H}'$ be the subhypergraph of $\mathcal{H}_1$ with only hyperedges containing a uniquely embedded edge between X and Y.

For ease of reference, let $H' = \partial(\mathcal{H}')$ and let $H'[X \cup Y]$ be the subgraph of $\partial(\mathcal{H}')$ induced by $X \cup Y$. Note that $H'[X \cup Y]$ is bipartite with at least $\frac{2}{27}|E(\mathcal{H})| \geq \frac{2d}{81}\binom{n}{2} = d'\binom{n}{2}$ edges.

Let $\epsilon = \epsilon(s, d'/2)$ be small enough so that ϵ satisfies the assumptions in Lemma 3.1.2. Applying the regularity lemma on $H'[X \cup Y]$, we can find an ϵ-regular partition in which there exist two parts $X' \subseteq X, Y' \subseteq Y$ such that (X', Y') is an ϵ-regular pair with edge density at least $d'/2$. Moreover, $|X'|, |Y'| \geq n/M$ for some constant $M > 0$. By Lemma 3.1.2, we can find disjoint subsets $A, C \subseteq X'$ and $B, D \subseteq Y'$ such that $|A| = |B| = 2s$, $|C| \geq \epsilon|X'|$, $|D| \geq \epsilon|Y'|$, and there is a complete bipartite graph connecting A and D, B and C as well as A and B.

Now consider the subhypergraph $\hat{\mathcal{H}} = \mathcal{H}'[C \cup D \cup Z]$ of $\mathcal{H}'$ induced by the vertex set $C \cup D \cup Z$, i.e., all hyperedges in $\hat{\mathcal{H}}$ contain vertices only in $C \cup D \cup Z$. Given a vertex set $S \subseteq V(\hat{\mathcal{H}})$, define $\hat{d}_S(v)$ as the number of neighbors of v in S in $\partial(\hat{\mathcal{H}})$.

Claim 3.1.3. *If there exists some $z \in Z$ such that $\hat{d}_C(v) \geq 2s$ and $\hat{d}_D(v) \geq 2s$, then $\mathcal{H}'$ contains a Berge-$C_5(1, s, s, s, s)$ as subhypergraph.*

Proof. Denote the $C_5(1, s, s, s, s)$ that we wish to embed as $\{v_1\} \cup V_2 \cup V_3 \cup V_4 \cup V_5$. Let $v_1 = z$. Let C_z, D_z be the set of neighbors of z in C and D respectively in $\partial(\hat{\mathcal{H}})$. We wish to embed V_2 in C_z, V_3 in B, V_4 in A and V_5 in D_z. Note that $|C_z|, |D_z| \geq 2s$ by our assumption. Pick arbitrary s of them to be V_2. For each vertex pair $\{z, w\}$ where $w \in V_2$, there exists a hyperedge $h \subseteq C \cup D \cup Z$ containing $\{z, w\}$. Use h to embed $\{z, w\}$. Observe that at most s vertices in D_z or B are contained in hyperedges embedding the edges from z to V_2. Since $|D_z| \geq 2s$, we can set V_5 to be arbitrary s vertices among vertices in D_z that are not contained in any hyperedge embedding the edges from z to V_2. Similarly, since $|A|, |B| \geq 2s$, we can set V_3 and V_4 to be arbitrary s vertices among vertices in B and A that are not contained in any hyperedge embedding the edges from z to V_2 and from z to V_5 respectively. We then have distinct hyperedges (in $\hat{\mathcal{H}}$ only) embedding the edges from z to V_2 and

z to V_5, V_2 to V_3 and V_4 to V_5 respectively. Moreover, recall that by our choice of X' and Y', vertex pairs between V_4 and V_5 are uniquely embedded (with the third vertex in Z), i.e., there exist distinct hyperedges embedding them. Hence, we obtain a Berge-$C_5(1,s,s,s,s)$ in $\mathcal{H}'$. $\square$

Now observe that $|C| \geq \epsilon|X'|$, $|D| \geq \epsilon|Y'|$. Hence by the ϵ-regularity of (X', Y'), the number of edges $e(C,D)$ in $\partial(\hat{\mathcal{H}})$ satisfies that

$$e(C,D) \geq (\frac{d'}{2} - \epsilon)|C||D| \geq (\frac{d'}{2} - \epsilon)\epsilon^2|X'||Y'| \geq (\frac{d'}{2} - \epsilon)\epsilon^2\frac{n^2}{M^2} = cn^2$$

where c is a constant depending on ϵ and d'.

Claim 3.1.4. *If $\mathcal{H}'$ contains no Berge-$C_5(1,s,s,s,s)$ as subhypergraph, it must contain a Berge-F where F is any triangle-free subgraph of $C_3(s,s,s;\{\{1,2\}\})$.*

Proof. By claim 3.1.3, since $\mathcal{H}'$ contains no Berge-$C_5(1,s,s,s,s)$ as subhypergraph, it follows that given any $v \in Z$, one of $\hat{d}_C(v)$, $\hat{d}_D(v)$ must be smaller than $2s$. Let Z_1 be the set of vertices $z \in Z$ with $\hat{d}_C(v) < 2s$, and Z_2 be the set of vertices $z \in Z$ with $\hat{d}_D(v) < 2s$. Let $e(Z_1, D)$ and $e(Z_2, C)$ denote the number of edges between Z_1 and D, Z_2 and C respectively in $\partial(\hat{\mathcal{H}})$. Since $e(C,D) \geq cn^2$ and all hyperedges in $\hat{\mathcal{H}}$ contains a vertex in Z, it follows that at least one of $e(Z_1, D)$ and $e(Z_2, C)$ must be at least $\Omega(n^2)$. WLOG, suppose $e(Z_1, D) \geq c'n^2$ for some $c' > 0$. Recall the classical result of Kővári, Sós and Turán [125], who showed that $ex(n, K_{r,t}) = O(n^{2-1/r})$ where $r \leq t$. By the Turán number of complete bipartite graphs, we have that for sufficiently large n, $\partial(\hat{\mathcal{H}})[D \cup Z_1]$ contains a complete bipartite graph $K_{(2s)^{s+1},(2s)^{s+1}}$. For ease of reference, call this complete bipartite graph K.

Let F be an arbitrary triangle-free subgraph of $C_3(s,s,s;\{\{1,2\}\})$. We now show that $\hat{\mathcal{H}}$ contains a Berge-F subhypergraph. Let C_1 be the collection of vertices v in C such that there is some hyperedge containing v and one of the edges in K. Observe that for each $v \in C_1$, $\hat{d}_{Z_1 \cap K}(v) \leq s$, otherwise we obtain a Berge-$C_5(1,s,s,s,s)$ in $\mathcal{H}'$.

Moreover, recall that for every $v \in Z_1$, $\hat{d}_C(v) < 2s$. It follows that there must be an edge $x_1y_1 \in \partial(\hat{\mathcal{H}})$ with $x_1 \in C_1, y_1 \in Z_1$ such that at least $(2s)^s$ vertices in $D \cap K$ form a hyperedge containing x_1y_1. Now consider the subgraph K' of K induced by these $(2s)^s$ vertices in $D \cap K$ as well as the non-neighbors of x_1 in $Z_1 \cap K$. Observe that K' is also a complete bipartite graph with at least $(2s)^s$ vertices in each partition. Hence by the same logic, we can find another edge $x_2y_2 \in \partial(\hat{\mathcal{H}})$ with $x_2 \in C_1, y_2 \in Z_1$ such that at least $(2s)^{s-1}$ vertices in $D \cap K'$ form hyperedges containing x_1y_1 and x_2y_2 respectively. Continuing this process s steps, it is not hard to see that we can find a Berge-F subhypergraph in $\hat{\mathcal{H}}$.

In summary, if $\mathcal{H}$ is 3-graph with at least $d\binom{n}{2}$ edges in $\partial(\mathcal{H})$ for some $d > 0$ and n sufficiently large, then $\mathcal{H}$ contains either a Berge-$C_5(1,s,s,s,s)$ or a Berge-F where F is any triangle-free subgraph of $C_3(s,s,s;\{\{1,2\}\})$. Moreover, observe that if G satisfies the conditions (1) and (2) in Theorem 3.1.3, then G is a subgraph of both $C_5(1,s,s,s,s)$ and $C_3(s,s,s;\{\{1,2\}\})$. Hence it follows that $\hat{\pi}_3(G) = 0$. This completes the proof of the theorem.

$\square$

It is easy to see that Theorem 3.1.3 implies Corollary 3.1.1. In the remaining of this section, we show that Corollary 3.1.1 and Corollary 3.1.2 are indeed equivalent.

Proof of Corollary 3.1.2. It suffices to show that a graph G is contained in both $C_5(1,s,s,s,s)$ and $C_3(s,s,s;\ \{\{1,2\}\})$ (for some s) if and only if G is a subgraph of one of the graphs in Figure 3.2. We follow the labelling in Figure 3.1. The backward direction is easy. For the forward direction, there are two cases:

Case 1: With loss of generality, v_1 is in B. Let $v_2 \in C$ be the vertex matched to v_1. Let $B' = B \setminus \{v_1\}$, and $C' = C \setminus \{v_2\}$. Note that $G - v_1$ is a bipartite graph, i.e., $V(G) - v_1 = U_1 \cup U_2$. With loss of generality, we can assume $B' \subseteq U_1$, $C' \subseteq U_2$

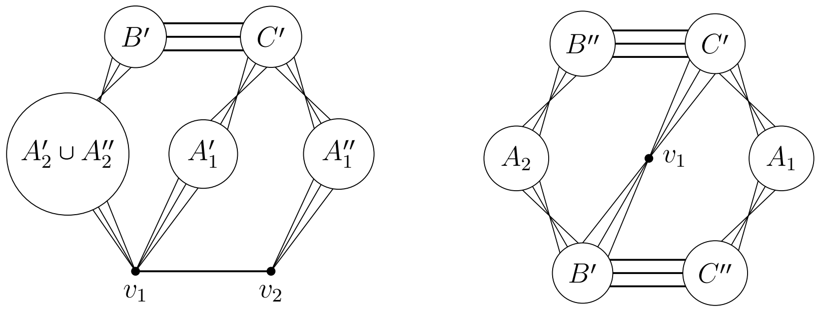


Figure 3.3: Equivalence of characterizations in Corollary 3.1.1 and 3.1.2.

and $v_2 \in U_2$ by properly swapping two ends of the matching edges between B and C if needed.

Since $G - v_1$ is bipartite, the vertex set A is partitioned into two parts $A_1 \subseteq U_1, A_2 \subseteq U_2$. Let A_1', A_2' be the neighbors of v in A_1, A_2 respectively, A_1'', A_2'' be the non-neighbors of v in A_1, A_2 respectively. Recall that $v_2 \in U_2$. It follows that v_2 is independent with $A_2' \cup A_2''$. Moreover, since G is triangle-free, v_2 is also independent with A_1'.

It then follows that G can be embedded into the first graph of Figure 3.2 in the same way labelled in Figure 3.3 (note that there are no edges between v_1 and A_2'').

Case 2: v_1 is in A. Since $G - v_1$ is bipartite, we can write $V(G) - v_1 = U_1 \cup U_2$. WLOG, assume that $B \subseteq U_1$ and $C \subseteq U_2$ by properly swapping two ends of the matching edges between B and C if needed. Moreover, write $A = A_1 \cup A_2 \cup \{v\}$ where $A_1 \in U_1$ and $A_2 \in U_2$. Write $B = B' \cup B''$, $C = C' \cup C''$ such that B' and C' are the neighbors of v_1 in B and C respectively. Since G is triangle-free, it follows that v_1 is independent with B'' and C''.

It then follows that G can be embedded into the second graph of Figure 3.2 in the same way labelled in Figure 3.3.

$\square$

3.1.3 Proof of Theorem 3.1.4

If $\chi(G) \geq 4$, we are done by Theorem 3.1.2. If $\chi(G) \leq 3$ and G is not degenerate, the two hypergraphs we constructed in Section 3.1.2 provide the lower bound 1/2, which is also an upper bound by Theorem 3.1.1. Theorem 3.1.3 resolves the case when G is degenerate.

3.2 On Hamiltonian Berge cycles in 3-uniform hypergraphs

A *hypergraph* is a pair $\mathcal{H} = (V, E)$ where V is a vertex set and every *hyperedge* $h \in E$ is a subset of V. For a fixed set of positive integers R, we say $\mathcal{H}$ is an R-*uniform* hypergraph, or R-*graph* for short, if the cardinality of each hyperedge belongs to R. If $R = \{k\}$, then an R-graph is simply a k-uniform hypergraph or a k-graph. Given an R-graph $\mathcal{H} = (V, E)$ and a set $S \in \binom{V}{s}$, let $\deg(S)$ denote the number of edges containing S and $\delta_s(\mathcal{H})$ be the minimum s-degree of $\mathcal{H}$, i.e., the minimum of $\deg(S)$ over all s-element sets $S \in \binom{V}{s}$. Given a hypergraph $\mathcal{H}$, the 2-*shadow* of $\mathcal{H}$, denoted by $\partial(\mathcal{H})$, is a simple 2-uniform graph $G = (V, E)$ such that $V(G) = V(\mathcal{H})$ and $uv \in E(G)$ if and only if $\{u, v\} \subseteq h$ for some $h \in E(\mathcal{H})$. In this paper, since we are dealing with 3-uniform hypergraphs, for convenience we will simply use the term *shadow* instead of 2-shadow. we say $\mathcal{H}$ is *covering* if the shadow of $\mathcal{H}$ is a complete graph. Note that $\mathcal{H}$ is covering if and only if $\delta_2(\mathcal{H}) \geq 1$.

There are several notions of a path or a cycle in hypergraphs. A *Berge path* of length t is a collection of t distinct hyperedges $h_1, h_2, \ldots, h_t$ and $t+1$ vertices $v_1, \ldots, v_{t+1}$ such that $\{v_i, v_{i+1}\} \subseteq h_i$ for each $i \in [t]$. Similarly, a k-graph $\mathcal{H} = (V, E)$ is called a *Berge* cycle of length t if E consists of t distinct edges $h_1, h_2, \ldots, h_t$ and V contains t distinct vertices $v_1, v_2, \ldots, v_t$ such that $\{v_i, v_{i+1}\} \subseteq h_i$ for every $i \in [t]$ where $v_{t+1} \equiv v_1$. Note that there may be other vertices than $v_1, \ldots, v_t$ in the edges of a Berge cycle or path. We say an R-graph $\mathcal{H}$ on n vertices contains a *Hamiltonian* Berge cycle (path) if it contains a Berge cycle (path) of length n (or $n-1$).

For k-uniform hypergraphs, there are more structured notions of Berge cycles as well. Given $1 \leq \ell < k$, a k-graph C is called an ℓ-cycle if its vertices can be ordered cyclically such that each of its edges consists of k consecutive vertices and every two consecutive edges (in the natural order of the edges) share exactly ℓ vertices. In particular, in a k-graph, a $(k-1)$-cycle is often called a *tight cycle* while a 1-cycle is often called a *loose cycle*. A k-graph contains a *Hamiltonian ℓ-cycle* if it contains an ℓ-cycle as a spanning subhypergraph.

The problem of finding Hamiltonian cycles has been widely studied. In 1952, Dirac [52] showed that for $n \geq 3$, every n-vertex graph with minimum degree at least $n/2$ contains a Hamiltonian cycle. Since then, problems that relate the minimum degree (or minimum s-degree in hypergraphs) to the structure of the (hyper)graphs are often referred to as *Dirac-type problems*. In the setting of hypergraphs, define the threshold $h_s^\ell(k,n)$ as the smallest integer m such that every k-graph $\mathcal{H}$ on n vertices with $\delta_s(\mathcal{H}) \geq m$ contains a Hamiltonian ℓ-cycle, provided that $k-\ell$ divides n. These thresholds for different values of s, ℓ and k have been intensively studied in a series of papers (e.g., [117, 149, 150, 151, 168, 136, 126, 103], see [171] for a recent survey). For Berge cycles, Bermond, Germa, Heydemann, and Sotteau [15] showed a Dirac-type theorem for Berge cycles. Kostochka, Luo and Zirlin [124] showed Dirac-type conditions for a hypergraph with few edges to be Hamiltonian.

The problem of finding Hamiltonian Berge cycles in a hypergraph is closely related to the problem of finding rainbow Hamiltonian cycles in an edge-colored complete graph K_n. An edge-colored graph G is *rainbow* (or *multicolored*) if each edge is of a different color. An edge-colored graph G is *k-bounded* if no color appears in more than k edges. Observe that given any covering k-graph $\mathcal{H}$ with hyperedges $h_1, \cdots, h_m$, we can construct an edge-colored complete graph G (using colors $\{c_1, \cdots, c_m\}$) on $|V(\mathcal{H})|$ vertices by assigning any edge $uv \in E(G)$ color c_i if $uv \in h_i$ for some i (pick arbitrarily if uv is contained in multiple hyperedges). Notice that G is $\binom{k}{2}$-bounded. Moreover, any rainbow subgraph G' of G corresponds to a Berge-G' in $\mathcal{H}$ by embedding $uv \in E(G')$ into the hyperedge h_i if uv is colored c_i.

There have been intensive investigations on the largest k (compared to n) such that any k-bounded edge-coloring of K_n contains a rainbow Hamiltonian path or cycle. In this framework, Hahn [100] conjected that any $(n/2)$-bounded coloring of K_n contains a rainbow Hamiltonian path. Hahn's conture was disproved by Maamoun and Meyniel [133] who showed that the conjecture is not true for proper colorings of K_{2^t} for integers $t \geq 2$. The problem for rainbow Hamilton cycles was first mentioned in Erdős, Nesdtril and Rödl [68] as an Erdős-Stein problem and show that k can be any constant. Hahn and Thomassen [101] showed that k could grow as fast as $n^{1/3}$ and conjectured that the growth rate of k can be linear. Rödl and Winkler later in an unpublished work improved it to $n^{1/2}$. Frieze and Reed [78] improved it to $O(n/\ln n)$. Albert, Frieze and Reed [4] confirmed the conjecture of Hahn and Thomassen by showing that if n is sufficiently large and k is at most $\lceil cn \rceil$ where $c < \frac{1}{32}$, then any k-bounded edge-coloring of K_n contains a rainbow Hamiltonian cycle. Frieze and Krivelevich [77] showed that there exists absolute constant $c > 0$ such that if an edge-coloring of K_n is cn-bounded, then there exists rainbow cycles of all sizes $3 \leq \ell \leq n$. In the context of Berge Hamiltonian cycles, the results above imply the following theorem:

Theorem 3.2.1. *[68, 101, 78, 4] For any fixed set of integers $R \subseteq [k]$ where $k \geq 2$, there is an integer $n_0 := n_0(k)$ such that every covering R-graph $\mathcal{H}$ on at least n_0 vertices contains Berge cycles of all sizes $3 \leq \ell \leq n$.*

Corollary 3.2.1. *For any fixed set of integers $R \subseteq [k]$ where $k \geq 2$, there is an integer $n_0 := n_0(k)$ such that every covering R-graph $\mathcal{H}$ on at least n_0 vertices contains a Berge Hamiltonian path.*

Further results on the rainbow spanning subgraphs lead to results that are even stronger than Theorem 3.2.1. In particular, Böttcher, Kohayakawa and Procacci [21] showed that for $c \le n/(51\Delta^2)$ every cn-bounded K_n contains a rainbow copy of every graph with maximum degree Δ. Recently, Coulson and Perarnau [46] showed that there exists $c > 0$ such that if G is a Dirac graph (i.e. minimum degree at least $n/2$) on n vertices (for sufficiently large n), then any cn-bounded coloring of G contains a rainbow Hamiltonian cycle.

The results above assume n is sufficiently large. In this note, we prove more precise results and focus on the Hamiltonian Berge paths and cycle problems in [3]-uniform hypergraphs (i.e., all hyperedges have cardinality at most 3). In particular, we show the following theorems:

Theorem 3.2.2. *Every covering* $[3]$*-graph* $\mathcal{H}$ *on* $n \ge 3$ *vertices with at least* $n-1$ *hyperedges contains a Hamiltonian Berge path.*

Note that for $n \ge 6$, the fact that $\mathcal{H}$ is covering implies that $\mathcal{H}$ has at least $n-1$ edges.

Theorem 3.2.3. *Every covering* $[3]$*-graph* $\mathcal{H}$ *on* $n \ge 6$ *vertices contains a Berge cycle* C_s *for any* $3 \le s \le n$*.*

Note that every covering [3]-graph on $n \ge 6$ vertices has at least n hyperedges. On the other hand, there exists a covering 3-graph on 5 vertices with 4 edges, thus without a Hamiltonian Berge cycle. Hence the condition $n \ge 6$ is necessary.

In general, in order for a $[k]$-graph to have a Hamiltonian Berge cycle or path, we need $\lceil \binom{n}{2} / \binom{k}{2} \rceil$ to be at least n or $n-1$ respectively (to simply have enough hyperedges). Thus we conjecture the following:

Conjecture 3.2.1. *For* $k \ge 2$*, every covering* $[k]$*-graph on* $n \ge k(k-1)+1$ *vertices contains a Hamiltonian Berge cycle.*

Conjecture 3.2.2. *For $k \geq 2$, every covering $[k]$-graph on $n \geq k(k-1)$ vertices contains a Hamiltonian Berge path.*

Remark 3.2.1. *Theorem 3.2.1 confirms Conjecture 3.2.2 and 3.2.1 for all $k \geq 2$ but with sufficiently large n. Theorem 3.2.2 and Theorem 3.2.3 confirms Conjecture 3.2.2 and 3.2.1 for $k = 3$.*

As an application, using Theorem 3.2.1, 3.2.2 and 3.2.3, we determine the maximum *Lagrangian* of Berge-P_t-free and Berge-C_t-free k-graphs when t is sufficiently large. Given a k-uniform hypergraph $\mathcal{H}$ on n vertices, the polynomial form $P_{\mathcal{H}}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ is defined for any vector $\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as

$$P_{\mathcal{H}}(\boldsymbol{x}) = \sum_{\{i_1, i_2, \cdots, i_k\} \in E(\mathcal{H})} x_{i_1} \cdots x_{i_k}.$$

For $k \geq 2$, the *Lagrangian* of a k-uniform hypergraph $\mathcal{H} = (V, E)$ on n vertices is defined to be

$$\lambda(\mathcal{H}) = \max_{\boldsymbol{x} \in \mathbb{R}^n : \|\boldsymbol{x}\|_1 = 1} P_{\mathcal{H}}(\boldsymbol{x}).$$

where the $\|\boldsymbol{x}\|_1 = \sum_{i=1}^{n} |x_i|$ is the *1-norm* of $\boldsymbol{x} \in \mathbb{R}^n$. Note that $P_H(\boldsymbol{x})$ can always reach its maximum at some nonnegative vectors.

Lagrangians for graphs (i.e., 2-graphs) were introduced by Motzkin and Straus in 1965 [139]. They showed $\lambda(G) = \frac{1}{2}(1 - \frac{1}{\omega(G)})$, where $\omega(G)$ is the clique number of G. The Lagrangian of a k-graph $\mathcal{H}$ is closely related to the maximum edge density of the blow-up of $\mathcal{H}$, which is very useful in the Turán theory [167, 118].

Extremal problems on Berge hypergraphs have been intensively investigated. The Turán number of a Berge-G, denoted by $ex_k(n, G)$, is the maximum number of hyperedges in k-uniform Berge-G-free hypergraph. Turán numbers for Berge paths and cycles have been studied in a series of papers [95, 48, 96, 97, 123, 80, 81, 74, 98]. For general results on the Turán number of arbitrary graphs, see for example [85, 88, 144]. Regarding the maximum Lagrangian of Berge-C_t-free and Berge-P_t-free hypergraphs, we show the following:

Theorem 3.2.4. *For fixed $k \geq 2$ and sufficiently large $t = t(k)$ and $n \geq t-1$, let $\mathcal{H}$ be a k-uniform hypergraph on n vertices without a Berge cycle of length t. Then*

$$\lambda(\mathcal{H}) \leq \lambda(K_{t-1}^k) = \frac{1}{(t-1)^k}\binom{t-1}{k}.$$

As a corollary, we obtain the same results for the Berge-P_t-free hypergraphs as well.

Corollary 3.2.2. *For fixed $k \geq 2$ and sufficiently large $t = t(k)$ and $n \geq t-1$, let $\mathcal{H}$ be a k-uniform hypergraph on n vertices without a Berge-P_t. Then*

$$\lambda(\mathcal{H}) \leq \lambda(K_{t-1}^k) = \frac{1}{(t-1)^k}\binom{t-1}{k}.$$

Both the bounds in Theorem 3.2.4 and Corollary 3.2.2 are tight. Indeed, let $\mathcal{H}$ be a k-graph obtained from K_{t-1}^k by adding $(n-t+1)$ isolated vertices. Clearly $\mathcal{H}$ is Berge-C_t-free and Berge-P_t-free and $\lambda(\mathcal{H}) = \binom{t-1}{k}/(t-1)^k$. For $k = 3$, due to Theorem 3.2.2 and Theorem 3.2.3, we obtain more precise results.

Corollary 3.2.3. *Let $\mathcal{H}$ be a 3-uniform hypergraph on n vertices without a Berge-C_t where $n \geq t \geq 6$. Then*

$$\lambda(\mathcal{H}) \leq \lambda(K_{t-1}^3) = \frac{1}{(t-1)^3}\binom{t-1}{3}.$$

Corollary 3.2.4. *Let $\mathcal{H}$ be a 3-uniform hypergraph on n vertices without a Berge-P_t where $n \geq t \geq 6$. Then*

$$\lambda(\mathcal{H}) \leq \lambda(K_{t-1}^3) = \frac{1}{(t-1)^3}\binom{t-1}{3}.$$

3.2.1 Proof of Theorem 3.2.2 and Theorem 3.2.3

Proof of Theorem 3.2.2. Let $\mathcal{H} = (V, E)$ be a covering [3]-uniform hypergraph on $n \geq 4$ vertices with at least $n-1$ hyperedges. Let $P = v_1 v_2 \ldots v_t$ be a maximum-length Berge path in $\mathcal{H}$. If $t = n$, we are done. Otherwise assume that $t < n$ and let u

be a vertex not in P. Observe that by the maximality of P, we have $t \geq 3$. Call a hyperedge h *used* if h is an edge in the Berge path P, otherwise call it *free*. Since $\mathcal{H}$ is covering, there exists a hyperedge h_1 containing $\{u, v_1\}$. The edge h_1 must be used in P since otherwise we can extend P by embedding $\{u, v_1\}$ in h_1. Since $\mathcal{H}$ is [3]-uniform, the only way that h_1 can be used in P is to embed $\{v_1, v_2\}$. Similarly, there exists a hyperedge h_t that contains $\{u, v_t\}$ and is used to embed $\{v_{t-1}, v_t\}$. Now consider a hyperedge h' containing $\{v_1, v_t\}$. Note that h' is free since both $\{v_1, v_2\}$ and $\{v_{t-1}, v_t\}$ have already been embedded. Now consider the path

$$P' = v_2 \cdots v_t v_1 u$$

such that $\{v_t, v_1\}$ is embedded in h', $\{v_1, u\}$ is embedded in h_1 and other edges in P' are embedded in the same hyperedges as in P. Notice that P' is a Berge hyperpath in $\mathcal{H}$ that is longer than P. This gives us the contradiction. Hence $t = n$ and P is a Hamiltonian Berge path in $\mathcal{H}$. □

Lemma 3.2.1. *Let $\mathcal{H} = (V, E)$ be a covering $[3]$-graph on $n \geq 6$ vertices. Then $\mathcal{H}$ contains a Hamiltonian Berge cycle.*

Proof of Lemma 3.2.1. Let $\mathcal{H} = (V, E)$ be a covering [3]-graph on $n \geq 6$ vertices. Suppose otherwise that $\mathcal{H}$ does not contain a Hamiltonian Berge cycle.

We first claim that there exists a Berge cycle on $n-1$ vertices. By Theorem 3.2.2, there is a Hamiltonian Berge path $P = u_1 u_2 \dots u_n$ in $\mathcal{H}$. Since $\mathcal{H}$ is covering, if follows that there exists an edge $h \in E(\mathcal{H})$ such that $\{u_1, u_n\} \subseteq h$. If h is not an edge in P, then we embed $u_1 u_n$ in h and obtain a Hamiltonian Berge cycle. Otherwise, h is used to embed either $u_1 u_2$ or $u_{n-1} u_n$. WLOG, h embeds $u_{n-1} u_n$. Then $h = \{u_1, u_{n-1}, u_n\}$. If we embed $u_1 u_{n-1}$ in h, we then obtain a Berge cycle $C = u_1 u_2 \dots u_{n-1}$ on $n - 1$ vertices.

Let $C = v_1 v_2 \dots v_{n-1}$ be a Berge cycle in $\mathcal{H}$ on $n-1$ vertices and call the remaining vertex w. For ease of reference, consider $v_n \equiv v_1$ and $v_0 \equiv v_{n-1}$. For a 2-edge $e = v_i v_{i+1}$, we use $\phi(e)$ to denote the hyperedge in C that embeds e. Consider a two-edge-coloring on $\{v_iv_{i+1} : i \in [n-1]\}$: color v_iv_{i+1} red if the hyperedge that embeds v_iv_{i+1} also contains w; otherwise color it blue. Assume that C is picked among all Berge cycles on $n-1$ vertices such that C has the most number of red edges (when viewed as a 2-uniform cycle).

Again, from now on, we call a hyperedge h *used* if h is a hyperedge in C, otherwise call it *free*. Moreover, when we say 2-edges of C, we mean the 2-uniform edges of C when $C = v_1v_2\dots v_{n-1}$ is viewed as a 2-uniform cycle. Otherwise, C is considered a [3]-graph.

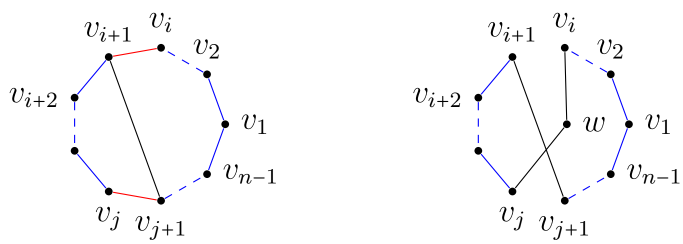


Figure 3.4: Using a bridge to extend the cycle.

Claim 3.2.1. *If there exist two disjoint red pairs v_iv_{i+1}, v_jv_{j+1} such that there is a free edge h containing either v_iv_j or $v_{i+1}v_{j+1}$, then we have a Hamiltonian Berge cycle.*

Proof. Recall that $\phi(v_kv_{k+1})$ denotes the hyperedge in C that embeds v_kv_{k+1}. Suppose there is a free edge h containing $v_{i+1}v_{j+1}$ (as shown in Figure 3.4). Consider the cycle

$$C' = v_iwv_j \dots v_{i+1}v_{j+1} \dots v_i.$$

Embed v_iw in $\phi(v_iv_{i+1})$; embed wv_j in $\phi(v_jv_{j+1})$; embed $v_{i+1}v_{j+1}$ in h. For any other edge e of C', embed e in $\phi(e)$. We then obtain a Hamiltonian Berge cycle. □

Observe that given two disjoint red pairs v_iv_{i+1}, v_jv_{j+1}, if the hyperedge h containing v_iv_j is not free, then it must be used to embed either $v_{i-1}v_i$ or v_jv_{j-1}. Similarly, if the hyperedge containing $v_{i+1}v_{j+1}$ is not free, then it must be used to embed either

$v_{i+1}v_{i+2}$ or $v_{j+1}v_{j+2}$. Given disjoint vertex pairs v_iv_{i+1}, v_jv_{j+1}, call the vertex pair v_iv_j or $v_{i+1}v_{j+1}$ a *bridge* if v_iv_{i+1}, v_jv_{j+1} are both red. By Claim 3.2.1, if a bridge is free, then we are done. Otherwise by the above observation, a bridge must be used to embed a blue 2-edge in C that intersects the bridge. Call a sequence of vertices a *segment* if they are consecutive in C. A segment is red (or blue) if the 2-edges in C (viewed as a 2-uniform cycle) induced by the vertices in the segment are all red (or blue). By Claim 3.2.1, it is easy to derive the following consequence:

(i) There are no four pairwise disjoint red segments. This is because, for any four pairwise disjoint red segments, there are at least $2\binom{4}{2} = 12$ bridges but only at most 8 blue edges that intersects the four red segments. Hence one of the bridges must be free. Then we are done by Claim 3.2.1.

(ii) If there are three pairwise disjoint red segments, there must be at least two blue edges (in both directions) between every two red segments. Moreover, each of the red segments has length 1. This is because, three pairwise disjoint red segments have at least six bridges. If there is only one blue edge between two of the red segments, then there are at most five blue edges intersecting the red segments. Hence one of the bridges must be free and we are done by Claim 3.2.1.

(iii) There can be only one red segment of length at least 2. Moreover, if there is any other red segment, then there must be at least two blue edges (in both directions) between the two red segments. The logic is the same as the above two cases.

(iv) If there is a red segment of length 3, there is no other red segment.

(v) There is no red segment of length at least 4.

Claim 3.2.2. *If there exist three consecutive blue edges in C, i.e., $v_i, v_{i+1}, v_{i+2}, v_{i+3}$ such that $v_k v_{k+1}$ is blue for $k \in \{i, i+1, i+2\}$, then we have a Hamiltonian Berge cycle.*

Proof. Since $\mathcal{H}$ is covering, it follows that there exist free edges h_1, h_2 such that h_1 contains wv_{i+1} and h_2 contains wv_{i+2}. Note that $h_1 \neq h_2$ otherwise we have a free $h = \{w, v_{i+1}, v_{i+2}\}$, which contradicts our assumption that C is picked such that it has the maximum number of red edges. Now consider the cycle

$$C' = v_1 \dots v_{i+1} w v_{i+2} \dots v_{n-1}.$$

Embed $v_{i+1}w$ in h_1; embed wv_{i+2} in h_2; embed any other edge e the same way it is embedded in C. We then obtain a Hamiltonian Berge cycle. □

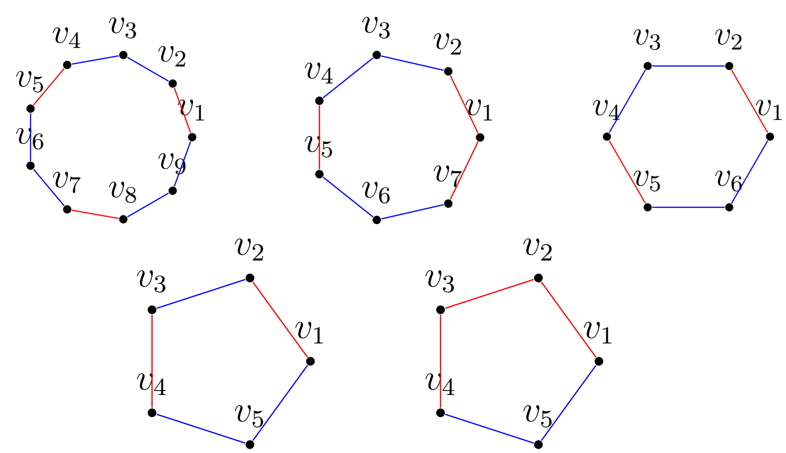


Figure 3.5: Remaining five cases: (a): $n = 10$; (b): $n = 8$; (c): $n = 7$; (d),(e): $n = 6$.

Combining Claim 3.2.2, the consequences (i)–(v) above and the fact that $n \geq 6$, it is easy to deduce that there are only 5 cases left. Let us analyze them one by one:

Case 1: $n = 10$. In this case, observe there must be a free hyperedge containing each of wv_3, wv_6 and wv_9. Moreover, the free hyperedges containing wv_3, wv_6 and wv_9 cannot be the same hyperedge. Hence, WLOG, let h_1 be the free edge containing wv_3 and h_2 be the free hyperedge containing wv_9. Now observe that v_2v_8 is bridge. Let h be an hyperedge containing v_2v_8. If h is free, we are done by Claim 3.2.1. Otherwise, WLOG, h is used to embed v_2v_3, i.e., $h = \{v_2, v_3, v_8\}$. Then consider the cycle

$$C' = v_2v_8v_7 \dots v_3wv_9v_1v_2$$

where v_2v_8 is embedded in h; v_3w is embedded in h_1; wv_9 is embedded in h_2; and any other 2-edge of C' is embedded in the same way as in C.

Case 2: $n = 8$. Note that v_4v_1 is a bridge. Hence if the edge h containing v_4v_1 is free, then we are done by Claim 3.2.1. Otherwise, WLOG, suppose h is used to embed v_3v_4, i.e. $h = \{v_1, v_3, v_4\}$. Moreover there is another free edge h' that contains wv_3. Now consider the cycle

$$C' = v_1v_4v_5v_6v_7wv_3v_2v_1$$

such as v_1v_4 is embedded in h, v_7w is embedded in $\phi(v_7v_1)$, wv_3 is embedded in h', and every other 2-edge of C' is embedded in the same way as in C.

Case 3: $n = 7$. Note that v_4v_1 is a bridge. Hence if the edge h containing v_4v_1 is free, then we are done by Claim 3.2.1. Otherwise, WLOG, suppose h is used to embed v_3v_4, i.e., $h = \{v_1, v_3, v_4\}$. Moreover there are free edges h_1, h_2 (may be the same) such that $\{w, v_3\} \subseteq h_1$ and $\{w, v_6\} \subseteq h_2$. If $h_1 \neq h_2$, then consider the cycle

$$v_1v_4v_5v_6wv_3v_2v_1$$

such that v_1v_4 is embedded in h, v_6w is embedded in h_2, wv_3 is embedded in h_1 and all other edges are embedded in the same way as before. We then obtain a Hamiltonian Berge cycle. On the other hand, suppose $h_1 = h_2$, then it follows that $h' = \{v_3, v_6, w\}$ is a free edge. Now consider the cycle

$$v_1v_2v_3v_6v_5v_4$$

such as v_3v_6 is embedded in h', v_4v_1 is embedded in h and all other edges are embedded in the same way as before. Observe that this cycle, using the same coloring scheme as before, has three red edges, which contradicts our assumption that the cycle in Figure 3.5 has the maximal number of red edges.

Case 4: $n = 6$. There are two possible coloring for $n = 6$ (see Figure 3.5$(d)(e)$). Let us first look at the case (Figure 3.5(d)) when there are two disjoint red segments of length 1. Let h_0 be the hyperedge embedding wv_5. Observe h_0 must be free, since otherwise it must be embedding v_1v_5 or v_4v_5, which contradicts that v_1v_5 and v_4v_5 are blue (recall that the cycle C is picked to have as many red edges as possible). Let h_1, h_2 be the hyperedges embedding v_1v_3 and v_2v_4 respectively. Note since v_1v_3 and v_2v_4 are bridges, if either of h_1, h_2 is free, then we are done by Claim 3.2.1. Otherwise, there are two subcases:

Case 4(a): $h_1 = \{v_1, v_3, v_5\}$ and $h_2 = \{v_2, v_4, v_5\}$. In this case, the hyperedge h_3 embedding v_1v_4 must be free. Hence consider the cycle

$$wv_5v_1v_4v_3v_2w$$

such that wv_5 is embedded in h_0, v_1v_4 is embedded in h_3, wv_2 is embedded in $\phi(v_1v_2)$, and any other 2-edge embedded in the same way as in C.

Case 4(b): WLOG, $h_1 = \{v_1, v_2, v_3\}$ and $h_2 = \{v_2, v_4, v_5\}$. Then consider the cycle

$$wv_5v_1v_3v_4v_2w$$

such that wv_5 is embedded in h_0, v_1v_3 is embedded in h_1, v_4v_2 is embedded in h_2, wv_2 is embedded in $\phi(v_1v_2)$, and any other 2-edge embedded in the same way as in C.

In both cases, we obtain a Hamiltonian Berge cycle. Hence we are done with the case in Figure 3.5(d). The case in Figure 3.5(e) is the same as Case 4(a).

□

Before we proceed to the proof of Theorem 3.2.3, let us state an easy observation on trace hypergraph and Berge cycles. Given an $[k]$-graph $\mathcal{H} = (V, E)$ and a subset $S \subseteq V$, the *trace* of $\mathcal{H}$ on S is defined to be the $[k]$-graph $\mathcal{H}_S = (S, E')$ with the vertex set S and the edge set $E' := \{F \cap S : F \in E(\mathcal{H})\}$. Traces of hypergraphs are very useful in extremal problems involving (non-uniform) hypergraphs. For some examples of results on trace functions and applications, see [156, 159, 170, 114]. Regarding the trace of covering hypergraphs, the following observations can be easily verified by definition.

Proposition 3.2.1. *Let $\mathcal{H}$ be a $[k]$-graph and $S \subseteq V(\mathcal{H})$ be any subset of vertices. Then the following statements hold:*

1. *If $\mathcal{H}$ is covering, so is $\mathcal{H}_S$.*
2. *Every Berge-cycle (or Berge-path) in $\mathcal{H}_S$ can be lifted to a Berge-cycle (or Berge-path) in $\mathcal{H}$ of the same length.*

Proof of Theorem 3.2.3. Let $\mathcal{H}$ be a covering $[3]$-graph on $n \geq 6$ vertices. We want to show that $\mathcal{H}$ contains all Berge cycles of length $3 \leq s \leq n$. Observe that given any $S \subseteq V(\mathcal{H})$ with $|S| \geq 6$, Lemma 3.2.1 implies that $\mathcal{H}_S$ contains a Hamiltonian Berge cycle, which by Proposition 3.2.1, can be lifted to a Berge cycle of length $|S|$ in $\mathcal{H}$. Hence $\mathcal{H}$ contains Berge cycles of length $6 \leq s \leq n$.

Claim 3.2.3. *$\mathcal{H}$ contains a Berge cycle of length* 5.

Proof. We know that $\mathcal{H}$ contains a Berge cycle C of length 6. Let $C = \{v_1, v_2, \cdots, v_6\}$. For convenience assume $v_i \equiv v_{i \bmod s}$. Again call an hyperedge h free if h is not a hyperedge of the Berge cycle C. Now for each $i \in [6]$, if the hyperedge h_i embedding $v_i v_{i+2}$ is free or $h_i = \{v_i, v_{i+1}, v_{i+2}\}$, then we are done since we can obtain a Berge cycle $C' = v_1 \cdots v_i v_{i+2} \cdots v_6 v_1$ of length 5 by embedding $v_i v_{i+2}$ in h_i and every other 2-edge with the same hyperedge in C. Otherwise, for each $i \in [6]$, the hyperedge h_i must be either $\{v_i, v_{i+2}, v_{i+3}\}$ or $\{v_{i-1}v_iv_{i+2}\}$.

Case 1: there is some i such that both $\{v_i, v_{i+2}, v_{i+3}\}$ or $\{v_{i+1}v_{i+3}v_{i+4}\}$ are both hyperedges of C. WLOG, $i = 1$, i.e., $\{v_1, v_3, v_4\}$ and $\{v_2, v_4, v_5\}$ are both in C, then consider the cycle

$$v_1v_3v_2v_5v_6v_1$$

such as v_1v_3 is embedded in $\{v_1, v_3, v_4\}$, v_2v_5 is embedded in $\{v_2, v_4, v_5\}$ and every other 2-edge is embedded the same way in C. We then obtain a Berge cycle of length 5. Similarly, if there is some i such that both $\{v_iv_{i+1}v_{i+3}\}$ and $\{v_{i-1}v_iv_{i+2}\}$ are hyperedges of C, then we are done too.

Case 2: WLOG, assume that the vertex pair v_2v_4 is embedded in $\{v_1, v_2, v_4\}$. Since we are not in Case 1, then v_3v_5 must be embedded in $\{v_3, v_5, v_6\}$, v_4v_6 must be embedded in $\{v_3, v_4, v_6\}$, etc. With this logic, we then obtain a hypergraph on 6 vertices with at least the following hyperedges: $h_1 = \{v_1, v_2, v_4\}$, $h_2 = \{v_3, v_5, v_6\}$, $h_3 = \{v_3, v_4, v_6\}$, $h_4 = \{v_1, v_2, v_5\}$, $h_5 = \{v_2, v_5, v_6\}$, $h_6 = \{v_1, v_3, v_4\}$. Now consider the cycle

$$v_2v_5v_6v_3v_4$$

by using the hyperedges h_4, h_5, h_2, h_3, h_1 respectively.

In both cases, we obtain a Berge cycle of length 5. □

The fact that $\mathcal{H}$ contains a Berge cycle of length 4 follows from similar logic in the above claim. A Berge triangle can be easily found by greedily embedding the edges of the triangle. We will leave the details to the readers. This completes the proof of Theorem 3.2.3.

□

3.2.2 Proof of Theorem 3.2.4

Before we show the proof of Theorem 3.2.4, we need a few definitions and lemmas. For a vector $\boldsymbol{x} = (x_1, \ldots, x_n)$ of real numbers, the *support* of $\boldsymbol{x}$ is defined as $Supp(\boldsymbol{x}) := \{1 \le i \le n : x_i \ne 0\}$. Given a family of subsets of $[n]$ and $I \subseteq [n]$, we say $\mathcal{F}$ *covers pairs with respect to* I if for every $i, j \in I$, there exists some $h \in \mathcal{F}$ such that $\{i, j\} \subseteq h$. Moreover, we define $\mathcal{F}[I] = \{h \in \mathcal{F} : h \subseteq I\}$.

Lemma 3.2.2 ([76]). *Let $\mathcal{F}$ be a family of k-subsets of $[n]$. Suppose $\boldsymbol{x} = (x_1, \ldots, x_n)$ with $x_i \ge 0$ such that $\sum_{i=1}^n x_i = 1$. Moreover, suppose that $P_{\mathcal{F}}(x) = \lambda(\mathcal{F})$ and $I = Supp(x)$ is minimal. Then $\mathcal{F}[I]$ covers pairs with respect to I.*

Proof of Theorem 3.2.4. Let $\mathcal{H}$ be a Berge-C_t-free k-uniform hypergraph on n vertices that achieves the maximum Lagrangian where $t \ge n_0(\{k\})$ in Theorem 3.2.1. Suppose that $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ such that $x_i \ge 0$, $\sum_{i=1}^n x_i = 1$ and $P_{\mathcal{H}}(\boldsymbol{x}) = \lambda(\mathcal{H})$. Further assume that $I = Supp(\boldsymbol{x})$ is minimal. By Lemma 3.2.2, we have that $\mathcal{H}[I]$ covers pairs with respect to I. Since $\mathcal{H}$ is Berge-C_t-free, it follows by Theorem 3.2.1 that $|I| \le t-1$. Hence

$$\begin{aligned}
\lambda(\mathcal{H}) &= P_{\mathcal{H}}(x) \\
&= \sum_{\{i_1, i_2, \cdots, i_k\} \in E(\mathcal{H})} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&= \sum_{\substack{\{i_1, i_2, \cdots, i_k\} \in E(\mathcal{H}) \\ i_1, i_2, \cdots, i_k \in I}} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&\le \sum_{\{i_1, i_2, \cdots, i_k\} \in \binom{I}{k}} x_{i_1} x_{i_2} \cdots x_{i_k} \\
&\le \frac{1}{(t-1)^k} \binom{t-1}{k} \\
&= \lambda(K_{t-1}^k).
\end{aligned}$$

$\square$

For $k = 3$, due to Theorem 3.2.3, we obtain, by the same logic, Corollary 3.2.3 and 3.2.4: if $\mathcal{H}$ is a 3-uniform hypergraph on n vertices without a Berge-C_t (or Berge-P_t) where $n \geq t \geq 6$, then

$$\lambda(\mathcal{H}) \leq \lambda(K_{t-1}^3) = \frac{1}{(t-1)^3}\binom{t-1}{3}.$$

Chapter 4

Ricci curvature of graphs and concentration inequalities

4.1 Introduction

One of the main tools in probabilistic analysis and random graph theory is the concentration inequalities, which are meant to bound the probability that a random variable deviates from its expectation. Many of the classical concentration inequalities (such as those for binomial distributions) provide best possible deviation results with exponentially small probabilistic bounds. Such concentration inequalities usually require certain independence assumptions (e.g., the random variable is a sum of independent random variables). For concentration inequalities without the independence assumptions, one popular approach is the martingale method. A martingale is a sequence of random variables $X_0, X_1, \ldots, X_n$ with finite means such that $E[X_{i+1}|X_i, X_{i-1}, \ldots, X_0] = X_i$ for all $0 \leq i < n$. For $\mathbf{c} = (c_1, c_2, \ldots, c_n)$ with positive entries, a martingale X is said to be $\mathbf{c}$-Lipschitz if $|X_i - X_{i-1}| \leq c_i$ for $i \in [n]$. A powerful tool for controlling martingales is the Azuma-Hoeffding inequality [11, 105]: if a martingale is $\mathbf{c}$-Lipschitz, then

$$\Pr\left(|X - E[X]| \geq t\right) \leq 2\exp\left(-\frac{t^2}{2\sum_{i=1}^{n} c_i^2}\right).$$

For more general versions of martingale inequalities as well as applications of martingale inequalities, we refer the readers to [7, 38].

A graph $G = (V, E)$ is a pair of the vertex set V and the edge set E where each edge is an unordered pair of two vertices. Given a vertex $v \in V$, we use $\Gamma(v)$ to denote the set of *open neighbors* of v in G, i.e., $\Gamma(v) = \{u \in V : vu \in E\}$. Moreover, let $N(v) = \Gamma(v) \cup \{v\}$ be the *closed neighbors* of v. A graph parameter/function X is called *vertex-Lipschitz* if $|X(G_1) - X(G_2)| \leq 1$ whenever G_1 and G_2 can be made isomorphic by deleting one vertex from each. A graph parameter X is called *edge-Lipschitz* if $|X(G_1) - X(G_2)| \leq 1$ whenver G_1 and G_2 differs by an edge. Many graph parameters are vertex(edge)-Lipschitz, e.g., the independence number $\alpha(G)$, the chromatic number $\chi(G)$, the clique number $\omega(G)$, the domination number $\gamma(G)$, the matching number $\beta(G)$, etc.

Concentration inequalities are among the most important tools in the probabilistic analysis of random graphs. The classical binomial random graph model, denoted by $G(n,p)$, is a random graph model in which a graph with n vertices is constructed by connecting the vertices randomly such that each vertex pair appears as an edge with probability p independently from every other edge. The Erdős-Rényi random graph model $G(n,M)$ is the model, in which a graph is chosen uniformly at random from the collection of all graphs with n vertices and m edges. A standard application of the Azuma-Hoeffding inequality gives us that for any vertex-Lipschitz function X defined on a vertex-exposure martingale (see e.g. [7] for definition), we have

$$\Pr(|X - \mathrm{E}(X)| \geq t) \leq 2\exp\left(-\frac{t^2}{2n}\right). \tag{4.1}$$

Similar concentration results can be obtained for edge-exposure martingale as well.

In this chapter, we will take an alternative approach for such an inequality. The main idea is using Ollivier's work [143] on the Ricci curvature of Markov chairs on metric spaces. Although the Ricci curvature of graphs has been introduced since 2009, it has not been widely used by the communities of combinatorists and graph theorists. In this chapter, we prove a clean concentration result (Theorem 4.1.1) on graphs with positive Ricci curvature. Then we show that it can be applied to some classical models of random configurations including the Erdős-Rényi random graph model $G(n,p)$ and $G(n,M)$, the random d-out(in)-regular directed graphs, and the space of random permutations, through a geometrization process.

Consider a graph (loops allowed) $G = (V, E)$ equipped with a random work $m := \{m_v : v \in V\}$. Here for each vertex v, $m_v : N(v) \to [0,1]$ is a distribution, i.e., $\sum_{x \in N(v)} m_v(x) = 1$. Assume that this random walk is *ergodic* so that an invariant distribution ν exists. In the context of random walks on graphs, in order for the random walk to be ergodic, it is sufficient that the underlying graph G is connected and non-bipartite. Note that ν is a probability measure on V. It turns V into a probability space. A function $f : V \to \mathbb{R}$ is called c-Lipschitz on G if

$$|f(u) - f(v)| \le c \quad \text{for any } uv \in E(G). \tag{4.2}$$

We have the following theorem on the concentration result of f. All we need is that the graph G (equipped with a random walk) has positive Ricci curvature at least $\kappa > 0$. (See the definition of Ricci curvature (in Ollivier's notion) in next section.)

Theorem 4.1.1. *Suppose that a graph $G = (V, E)$ equipped with an ergodic random walk m (and invariant distribution ν) has a positive Ricci curvature at least $\kappa > 0$. Then for any 1-Lipschitz function f and any $t \ge 1$, we have*

$$\nu\left(f - E_\nu[f] > t\right) \le \exp\left(\frac{-t^2\kappa}{7}\right), \tag{4.3}$$

$$\nu\left(f - E_\nu[f] < -t\right) \le \exp\left(\frac{-t^2\kappa}{7}\right). \tag{4.4}$$

Remark 4.1.1. *The constant 7 can be improved to 5 if $\kappa \to 0$ as $|V(G)| \to \infty$. It can be improved to $1 + o(1)$ if we further assume $t\kappa \to 0$ as $|V(G)| \to \infty$.*

Remark 4.1.2. *Ollivier [143] proved a concentration inequality for any random walk on a metric space with positive Ricci curvature at least $\kappa > 0$ and unique invariant distribution ν. His result is more general but more technical to apply in the context of graphs. In particular, he defined two quantities related to the local behavior of the random walk: the* diffusion constant *$\sigma(x)$ and the* local dimension *n_x at vertex x.*

Moreover, define $D_x^2 = \frac{\sigma(x)^2}{n_x \kappa}$, $D^2 = E_\nu[D_x^2]$, $t_{max} = \frac{D^2}{\max(\sigma_\infty, 2C/3)}$ *where* C *satisfies that the function* $x \to D_x^2$ *is* C*-Lipschitz. He proved ([143] Theorem 33, on page 834) for any* 1*-Lipschitz function* f *and for any* $t \leq t_{max}$*, we have*

$$\nu\left(f - E_\nu[f] > t\right) \leq \exp\left(\frac{-t^2}{6D^2}\right). \tag{4.5}$$

and for $t \geq t_{max}$*,*

$$\nu\left(f - E_\nu[f] > t\right) \leq \exp\left(\frac{-t^2}{6D^2} - \frac{t - t_{max}}{\max(3\sigma_\infty, 2C)}\right). \tag{4.6}$$

Remark 4.1.3. *Note in Ollivier's result for graphs, we have* $D^2 = O(\kappa^{-1})$ *and* $\sigma_\infty \approx 1$*. Inequality* (4.3) *has about the same power as Inequalities* (4.5) *and* (4.6)*, but cleaner; thus is easier to apply in the context of graphs.*

Besides Ollivier's definition of Ricci curvature, another notion of Ricci curvature on discrete spaces, via geodesic convexity of the entropy (in the spirit of Sturm [163], Lott and Villani [131]), was proposed in [134] and systematically studied in [60] and [137]. Similar Gaussian-type concentration inequalities (as ones in Theorem 4.1.1) in this notion of Ricci curvature was proven in [60]. Erbar, Maas, and Tetali [61] recently calculated the Ricci curvature lower bound of some classical random walks, e.g., the Bernoulli-Laplace model and the random transposition model of permutations.

In this chapter, we adopt Ollivier's notion of coarse Ricci curvature as it does not require the reversibility of the random walk on graphs. The chapter is organized as follows. In Section 4.2, we will give the history and definitions of Ricci curvature. The proof of Theorem 4.1.1 will be given in Section 4.3. In last section, we will give applications of Theorem 4.1.1 in four classical models of random configurations, including the Erdős-Rényi random graph model $G(n,p)$ and $G(n,M)$, the random d-out(in)-regular directed graphs, and the space of random permutations.

4.2 Ricci Curvatures of graphs

In Riemannian geometry, spaces with positive Ricci curvature enjoy very nice properties, some of them with probabilistic interpretations. Many interesting properties are found on manifolds with non-negative Ricci curvature or on manifolds with Ricci curvature bounded below. The definition of the Ricci curvature on metric spaces first came from the Bakry and Emery notation [12] who defined the “lower Ricci curvature bound" through the heat semigroup $(P_t)_{t\geq 0}$ on a metric measure space. Ollivier [143] defined the coarse Ricci curvature of metric spaces in terms of how much small balls are closer (in Wasserstein transportation distance) then their centers are. This notion of coarse Ricci curvature on discrete spaces was also made explicit in the Ph.D. thesis of Sammer [155]. Under the assumption of positive curvature in a metric space, Gaussian-like or Poisson-like concentration inequalities can be obtained. Such concentration inequalities have been investigated in [115] for time-continuous Markov jump processes and in [143, 116] in metric spaces.

Graphs and manifolds share some similar properties through Laplace operators, heat kernels and random walks, etc. A series of work in this area were done by Chung, Yau and their coauthors [32, 34, 35, 36, 33, 31, 40, 29, 39, 37, 41]. The first definition of Ricci curvature on graphs was introduced by Chung and Yau in [35]. For a more general definition of Ricci curvature, Lin and Yau [130] gave a generalization of lower Ricci curvature bound in the framework of graphs. Lin, Lu, and Yau [129] defined a new kind of Ricci curvature on graphs, which is based on Ollivier's work [143].

In this chapter, we will use the same notation as in [129]. A probability distribution (over the vertex set $V(G)$) is a mapping $m : V \to [0,1]$ satisfying $\sum_{x\in V} m(x) = 1$. Suppose two probability distributions m_1 and m_2 have finite support. A coupling between m_1 and m_2 is a mapping $A : V \times V \to [0,1]$ with finite support so that

$$\sum_{y\in V} A(x,y) = m_1(x) \text{ and } \sum_{x\in V} A(x,y) = m_2(y).$$

Let $d(x,y)$ be the graph distance between two vertices x and y. The *transportation distance* between two probability distributions m_1 and m_2 is defined as follows:

$$W(m_1, m_2) = \inf_{A} \sum_{x,y \in V} A(x,y) d(x,y).$$

where the infimum is taken over all coupling A between m_1 and m_2. By the duality theorem of a linear optimization problem, the transportation distance can also be written as follows:

$$W(m_1, m_2) = \sup_{f} \sum_{x \in V} f(x) \left(m_1(x) - m_2(x)\right)$$

where the supremum is taken over all 1-Lipschitz functions f.

A random walk m on $G = (V, E)$ is defined as a family of probability measures $\{m_v(\cdot)\}_{v \in V}$ such that $m_v(u) = 0$ for all $\{v, u\} \notin E$. It follows that $m_v(u) \geq 0$ for all $v, u \in V$ and $\sum_{u \in N(v)} m_v(u) = 1$. The Ricci cuvature κ of G can then be defined as follows:

Definition 4.2.1. *Given $G = (V, E)$, a random walk $m = \{m_v(\cdot)\}_{v \in V}$ on G and two vertices $x, y \in V$,*

$$\kappa(x,y) = 1 - \frac{W(m_x, m_y)}{d(x,y)}.$$

Remark 4.2.1. *We say a graph G equipped with a random walk m has Ricci curvature at least κ_0 if $\kappa(x,y) \geq \kappa_0$ for all $x, y \in V$.*

For $0 \leq \alpha < 1$, the *α-lazy random walk* m_x^{α} (for any vertex x), is defined as

$$m_x^{\alpha}(v) = \begin{cases} \alpha & \text{if } v = x, \\ (1-\alpha)/d_x & \text{if } v \in \Gamma(x), \\ 0 & \text{otherwise.} \end{cases}$$

In [129], Lin, Lu and Yao defined the Ricci curvature of graphs based on the α-lazy random walk as α goes to 1. More precisely, for any $x, y \in V$, they defined the

α-Ricci-curvature $\kappa_\alpha(x,y)$ to be

$$\kappa_\alpha(x,y) = 1 - \frac{W(m_x^\alpha, m_y^\alpha)}{d(x,y)}$$

and the Ricci curvaure κ_{LLY} of G to be

$$\kappa_{\text{LLY}}(x,y) = \lim_{\alpha\to 1} \frac{\kappa_\alpha(x,y)}{(1-\alpha)}.$$

They showed [129] that κ_α is concave in $\alpha \in [0,1]$ for any two vertices x, y. Moreover,

$$\kappa_\alpha(x,y) \le (1-\alpha)\frac{2}{d(x,y)}.$$

for any $\alpha \in [0,1]$ and any two vertices x and y.

In the context of graphs, the following lemma shows that it is enough to consider only $\kappa(x,y)$ for $xy \in E(G)$.

Lemma 4.2.1. *[143, 129] If $\kappa(x,y) \ge \kappa_0$ for any edge $xy \in E(G)$, then $\kappa(x,y) \ge \kappa_0$ for any pair of vertices (x,y).*

4.3 Proof of Theorem 4.1.1

We first define an averaging operator associated to the random walk.

Definition 4.3.1 (Discrete averaging operator)**.** *Given a function $f : X \to \mathbb{R}$, let the averaging operator M be defined as*

$$Mf(x) := \sum_{y\in V} f(y)\cdot m_x(y).$$

The following proposition shows a Lipschitz contraction property in the metric measure space. We include its proof here for the sake of completeness.

Proposition 4.3.1 (Lipschitz contraction)**.** *[143, 53] Let (G,d,m) be a random walk on a simple graph G. Let $\kappa \in \mathbb{R}$. Then the Ricci curvature of G is at least κ, if and only if, for every k-Lipschitz function $f : X \to \mathbb{R}$, the function Mf is $k(1-\kappa)$-Lipschitz.*

Proof. Suppose that the Ricci curvature of G is at least κ. For $x, y \in V$, let $A : V \times V \to [0,1]$ be the optimal coupling measure of m_x and m_y.

$$\begin{aligned}
Mf(y) - Mf(x) &= \sum_{u \in V} f(u) m_y(u) - \sum_{u \in V} f(u) m_x(u) \\
&= \sum_{u \in V} f(u) \sum_{v \in V} A(v,u) - \sum_{u \in V} f(u) \sum_{v \in V} A(u,v) \\
&= \sum_{u,v} \left(f(v) - f(u)\right) A(u,v) \\
&\le k \sum_{u,v} d(u,v) A(u,v) \\
&= kW(m_x, m_y) \\
&= k(1 - \kappa(x,y)) d(x,y)
\end{aligned}$$

Conversely, suppose that whenever f is 1-Lipschitz, Mf is $(1-\kappa)$-Lipschitz. Then by the duality theorem for the transportation distance, we have that for all $x, y \in V(G)$,

$$\begin{aligned}
W(m_x, m_y) &= \sup_{f \text{ 1-Lipschitz}} \sum_{z \in V} f(z) \left(m_x(z) - m_y(z)\right) \\
&= \sup_{f \text{ 1-Lipschitz}} Mf(x) - Mf(y) \\
&\le (1 - \kappa) d(x,y).
\end{aligned}$$

It follows that

$$\kappa(x,y) = 1 - \frac{W(m_x, m_y)}{d(x,y)} \ge \kappa.$$

□

Remark 4.3.1. *Note that for any constant c,*

$$Var(f) = E[(f-c)^2] - \left(E[f] - c\right)^2. \tag{4.7}$$

Thus for any $x \in V$ and an α-Lipschitz function $f : Supp\ m_x \to \mathbb{R}$,

$$\begin{aligned}
Var_{m_x} f &\le E_{m_x}[(f - f(x))^2] \\
&\le \sum_{y \in Supp\ m_x} (f(y) - f(x))^2 m_x(y) \\
&\le \alpha^2.
\end{aligned}$$

Lemma 4.3.1. *[129, 143] Let G be a finite graph with Ricci curvature at least $\kappa > 0$. Then*

$$\kappa \leq \frac{2}{diam(G)}.$$

Moreover, if $m_x(x) = \alpha$ for all $x \in V(G)$, then $\kappa \leq (1-\alpha)\frac{2}{diam(G)}$.

The following lemma is similar to Lemma 38 in [143].

Lemma 4.3.2. *Let $\phi : V(G) \to \mathbb{R}$ be an α-Lipschitz function with $\alpha \leq 1$. Then for $x \in V(G)$, we have*

$$\left(Me^{\lambda\phi}\right)(x) \leq e^{\lambda M\phi(x) + \frac{1}{2}\lambda^2 e^{2\lambda}\alpha^2}.$$

Proof. For any smooth function g and any real-valued random variable Y, a Taylor expansion with Lagrange remainder gives

$$Eg(Y) \leq g(EY) + \frac{1}{2}(\sup g'')\mathrm{Var}Y.$$

Applying this with $g(Y) = e^{\lambda Y}$, we get

$$(Me^{\lambda\phi})(x) = E_{m_x}e^{\lambda\phi} \leq e^{\lambda M\phi(x)} + \frac{\lambda^2}{2}\left(\sup_{\mathrm{Supp}\ m_x} e^{\lambda\phi}\right)\mathrm{Var}_{m_x}\phi.$$

Note that diam Supp $m_x \leq 2$ and ϕ is α-Lipschitz, it follows that

$$\sup_{\mathrm{Supp}\ m_x} \phi \leq E_{m_x}\phi + \alpha \cdot (\mathrm{diam\ Supp}\ m_x) \leq E_{m_x}\phi + 2\alpha.$$

Moreover, by Remark 4.3.1, $\mathrm{Var}_{m_x}\phi \leq \alpha^2$. Hence we have that

$$\begin{aligned}
\left(Me^{\lambda\phi}\right)(x) &\leq e^{\lambda M\phi(x)} + \frac{\lambda^2}{2}(\alpha^2)e^{\lambda M\phi(x)+2\lambda\alpha} \\
&\leq e^{\lambda M\phi(x)}\left(1 + \frac{\lambda^2}{2}\alpha^2 e^{2\lambda\alpha}\right) \\
&\leq \exp\left(\lambda M\phi(x) + \frac{1}{2}\lambda^2\alpha^2 e^{2\lambda\alpha}\right).
\end{aligned}$$

□

Proof of Theorem 4.1.1. Note that since f is 1-Lipschitz, it follows that $|f(x) - f(y)| \le diam(G)$ for any $x, y \in V(G)$. Hence if $t > \frac{2}{\kappa}$, then

$$\Pr\left(|f - E_\nu[f]| \ge t\right) \le \Pr\left(|f - E_\nu[f]| > \frac{2}{\kappa}\right) \le \Pr\left(\mathrm{diam}(G) > \frac{2}{\kappa}\right) = 0,$$

in which case we are done. So from now on, assume $t \le 2/\kappa$.

Apply Lemma 4.3.2 iteratively and use Proposition 4.3.1, we obtain that for any $i \ge 1$,

$$\begin{aligned} M^i(e^{\lambda f}) &\le e^{\lambda M^i f} \cdot \prod_{j=0}^{i-1} \exp\left(\frac{1}{2}\lambda^2(1-\kappa)^{2j} e^{2\lambda}\right) \\ &\le \exp\left(\lambda M^i f + \frac{1}{2}\lambda^2 e^{2\lambda} \sum_{j=0}^{i-1} (1-\kappa)^{2j}\right). \end{aligned}$$

Meanwhile, $(M^i e^{\lambda f})(x)$ tends to $E_\nu e^{\lambda f}$. Hence

$$\begin{aligned} E_\nu e^{\lambda f} &\le \lim_{i\to\infty} \exp\left(\lambda M^i f + \frac{1}{2}\lambda^2 e^{2\lambda} \sum_{j=0}^{i-1}(1-\kappa)^{2j}\right) \\ &\le \exp\left(\lambda E_\nu f + \frac{\lambda^2 e^{2\lambda}}{2\kappa(2-\kappa)}\right). \end{aligned}$$

Let λ_0 be the root of the equation $x \cdot e^{2x} = 2(2-\kappa)$ and set $\lambda = \frac{t\kappa\lambda_0}{2}$. Note that since $t \le \frac{2}{\kappa}$, we have $\lambda \le \lambda_0$. Now, we have

$$\begin{aligned} \Pr\left(f - E_\nu f \ge t\right) &\le \Pr\left(e^{\lambda f} \ge e^{t\lambda + \lambda E_\nu f}\right) \\ &\le E_\nu e^{\lambda f} \cdot e^{-t\lambda - \lambda E_\nu f} \\ &\le \exp\left(-t\lambda + \frac{\lambda^2 e^{2\lambda}}{2\kappa(2-\kappa)}\right) \\ &\le \exp\left(-t\lambda + \frac{\lambda t \lambda_0 e^{2\lambda}}{4(2-\kappa)}\right) \qquad (4.8) \\ &\le \exp\left(-t\lambda + \frac{\lambda t \lambda_0 e^{2\lambda_0}}{4(2-\kappa)}\right) \\ &= \exp\left(-\frac{1}{2}t\lambda\right) \\ &\le \exp\left(-\frac{t^2\kappa\lambda_0}{4}\right) \end{aligned}$$

where λ_0 is the solution to $x \cdot e^{2x} = 2(2-\kappa)$. If G is the complete graph, then $|f - E_\nu(f)| \leq 1$ holds for all vertices. Inequality 4.3 holds. If G is not the complete graph, then we must have $\kappa \leq 1$ (otherwise, contradiction to $diam(G) \leq \frac{2}{\kappa}$). Thus $\lambda_0 \leq 0.60108...$, which is the root of $x \cdot e^{2x} = 2$. We have $\frac{\lambda_0}{4} > \frac{1}{7}$. Hence we obtain that

$$\Pr\left(f - E_\nu f \geq t\right) \leq \exp\left(-\frac{t^2\kappa}{7}\right).$$

If $\kappa \to 0$ as $|V(G)| \to \infty$ (which is true in all the examples in Section 4.4), then we have $\lambda_0 \to 0.80290...$ which is the root of $x \cdot e^{2x} = 4$. We have $\frac{\lambda_0}{4} > \frac{1}{5}$. We have

$$\Pr\left(f - E_\nu f \geq t\right) \leq \exp\left(-\frac{t^2\kappa}{5}\right).$$

Furthermore, if $\kappa \to 0$ and $t\kappa \to 0$ as $|V(G)| \to \infty$, then continuing from inequality (4.8), we have that $e^{2\lambda} \to 1$ and $(2-\kappa) \to 2$ (as $|V(G)| \to \infty$). By setting $\lambda_0 = 4$, we have

$$\begin{aligned}
\Pr\left(f - E_\nu f \geq t\right) &\leq \exp\left(-t\lambda + \frac{\lambda t \lambda_0 e^{2\lambda}}{4(2-\kappa)}\right) \\
&\leq \exp\left(-\left(\frac{1}{2} + o(1)\right)t\lambda\right) \\
&\leq \exp\left(-\left(\frac{1}{4} + o(1)\right)t^2\kappa\lambda_0\right) \\
&\leq \exp\left((1+o(1))t^2\kappa\right).
\end{aligned}$$

The lower tail can be obtained from the upper tail by changing f to $-f$ since $-f$ is also 1-Lipschitz. □

4.4 Applications to random models of configurations

In order to apply Theorem 4.1.1 to a finite probability space (Ω, μ), we will construct a graph H with the vertex set Ω such that μ is the invariant distribution over a proper random walk m on H. We call the pair (H, m) a *geometrization* of (Ω, μ). In this section, we will give geometrization of four popular random model of configuarations.

4.4.1 Vertex-Lipschitz functions on $G(n,p)$

Let H be the graph such that $V(H)$ is the set of all labeled graphs with n vertices. Moreover, two graphs $G_1, G_2 \in V(H)$ are adjacent in H if and only if there exists some v such that $G_1 - v = G_2 - v$. Now define a random walk m on H as follows: Let $G \in V(H)$. Define

$$m_G(G') = \begin{cases} \frac{1}{n} \sum\limits_{\substack{v \in V(G) \\ G-v=G'-v}} p^{d_{G'}(v)}(1-p)^{n-1-d_{G'}(v)} & \text{if } G' \in N_H(G), \\ \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 4.4.1. *Let ν be the unique invariant distribution of the random walk defined above. A random graph G picked according to ν, satisfies that $\nu(G) = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}$.*

Proof. Observe that H is not bipartite thus the random walk is ergodic. It suffices to show that the distribution $\nu'(G) = p^{e(G)}(1-p)^{\binom{n}{2}-e(G)}$ for every G is an invariant distribution for the random walk. Indeed, for every fixed $G \in V(H)$,

$$\begin{aligned}
&\sum_{G' \in H} \nu'(G') m_{G'}(G) \\
=&\sum_{v \in V} \sum_{G'-v=G-v} \nu'(G') \frac{1}{n} p^{d_G(v)}(1-p)^{n-1-d_G(v)} \\
=&\sum_{v \in V} \frac{1}{n} p^{d_G(v)}(1-p)^{n-1-d_G(v)} \sum_{G'-v=G-v} \nu'(G') \\
=&\sum_{v \in V} \frac{1}{n} p^{d_G(v)}(1-p)^{n-1-d_G(v)}. \\
&\left(p^{e(G)-d_G(v)}(1-p)^{\binom{n-1}{2}-(e(G)-d_G(v))} \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \right) \\
=&\frac{1}{n} p^{e(G)}(1-p)^{\binom{n}{2}-e(G)} \sum_{v \in V} \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \\
=&p^{e(G)}(1-p)^{\binom{n}{2}-e(G)} \\
=&\nu'(G).
\end{aligned}$$

□

Lemma 4.4.1. *Let H and the random walk m be defined as above. Then*

$$\kappa(G_1, G_2) \geq \frac{1}{n}$$

for all $G_1, G_2 \in V(H)$.

Proof. Again, by Lemma 4.2.1, we can assume that G_1, G_2 are neighbors in H. It then follows from definition that

$$\kappa(G_1, G_2) = 1 - W(m_{G_1}, m_{G_2}).$$

Assume that v is the unique vertex such that $G_1 - v = G_2 - v$. When G_1 and G_2 differ by an edge, it is possible that there are two vertices v satisfying $G_1 - v = G_2 - v$. We remark that the analysis is similar. Consider the support of m_{G_1}. For each $G_1' \in \Gamma(G_1) \backslash \{G_2\}$, we will match G_1' with a distinct graph $\phi(G_1') \in N(G_2)$. There are two possible cases:

Case 1: $G_1 - v = G_1' - v$. Then it follows that $G_1' - v = G_2 - v$ and we let $\phi(G_1') = G_1'$.

Case 2: $G_1 - u = G_1' - u$ for some $u \neq v$. In this case, we claim that for each G_1' such that $G_1 - u = G_1' - u$, there exists a unique $G_2' = \phi(G_1')$ such that $G_2' - u = G_2 - u$ and $G_1' - v = G_2' - v$. Indeed, let G_2' be obtained from G_2 by replacing the neighbors of u in G_2 by the neighbors of u in G_1'. It's not hard to see that $G_2' - u = G_2 - u$ and $G_1' - v = G_2' - v$.

Let us now define a coupling A (not necessarily optimal) between m_{G_1} and m_{G_2}. Define $A : V(H) \times V(H) \to \mathbb{R}$ as follows:

$$A(G_1', G_2') = \begin{cases} \frac{1}{n} \sum\limits_{u \neq v} p^{d_{G_1}(u)} (1-p)^{n-1-d_{G_1}(u)} & \text{if } G_2' = G_2, G_1' = G_1, \\ m_{G_1}(G_1') & \text{if } G_1' \in \Gamma(G_1) \backslash \{G_2\} \text{ and } G_2' = \phi(G_1'), \\ 0 & \text{otherwise.} \end{cases} \tag{4.9}$$

It follows that

$$\begin{aligned}
W(m_{G_1}, m_{G_2}) &\leq \sum_{G'_1, G'_2} A(G'_1, G'_2) d(G'_1, G'_2) \\
&\leq \frac{1}{n} \sum_{u \neq v} \sum_{G'-u=G_1-u} \frac{1}{n} p^{d_{G'}(u)} (1-p)^{n-1-d_{G'}(u)} \\
&\leq \frac{1}{n} \sum_{u \neq v} \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \\
&\leq \frac{n-1}{n}.
\end{aligned}$$

Thus

$$\kappa(G_1, G_2) \geq 1 - W(m_{G_1}, m_{G_2}) \geq \frac{1}{n}.$$

□

It follows by Theorem 4.1.1 that for any vertex-Lipschitz function f on graphs, we have that

$$\Pr\left(|f - E[f]| \geq t\right) \leq 2 \exp\left(-\frac{t^2}{5n}\right),$$

which in this context has the same strength as the Azuma67-Hoeffding inequality on vertex-exposure martingale.

4.4.2 Edge-Lipschitz functions on $G(n, M)$

Let $G \sim G(n, M)$ be a random graph with n vertices and M edges. Let H be the graph such that $V(H)$ is the set of all labeled graphs with n vertices and M edges. Moreover, two graphs $G_1, G_2 \in V(H)$ are adjacent in H if and only if there exist two distinct vertex pairs e_1, e_2 such that $e_1 \in E(G_1) \backslash E(G_2)$, $e_2 \in E(G_2) \backslash E(G_1)$ and $G_1 - e_1 = G_2 - e_2$. In other words, G_1, G_2 are adjacent in H if one can be obtained from the other by swapping an edge with a non-edge. It is easy to see that H is a connected regular graph. Moreover, for every $G \in V(H)$, $d_H(G) = M\left(\binom{n}{2} - M\right)$. The following proposition is clear from the definition of H.

Proposition 4.4.2. *If G_1, G_2 are adjacent in H, then there exists a unique pair of distinct vertex pairs e_1, e_2 such that $e_1 \in E(G_1) \backslash E(G_2)$, $e_2 \in E(G_2) \backslash E(G_1)$ and $G_1 - e_1 = G_2 - e_2$.*

Now define a random walk m on H as follows: Let $G \in V(H)$. Define

$$m_G(G') = \begin{cases} \frac{1}{M(\binom{n}{2} - M) + 1} & \text{if } G' \in N_H(G), \\ 0 & \text{otherwise.} \end{cases}$$

It's easy to see that for a fixed G, $\sum_{G'} m_G(G') = 1$.

Proposition 4.4.3. *Let ν be the unique invariant distribution of the random walk defined above. A random graph G picked according to ν, is equally likely to be one of the $\binom{\binom{n}{2}}{M}$ graphs that have M edges.*

Proof. Observe that H is not bipartite thus the random walk is ergodic. It suffices to show that $\nu'(G) = \binom{\binom{n}{2}}{M}^{-1}$ for every G is an invariant distribution for the random walk. Indeed, for every fixed $G \in V(H)$,

$$\begin{aligned} \sum_{G' \in H} \nu'(G') m_{G'}(G) &= \binom{\binom{n}{2}}{M}^{-1} \sum_{G' \in N(G)} m_{G'}(G) \\ &= \binom{\binom{n}{2}}{M}^{-1} \sum_{G' \in N(G)} m_G(G') \\ &= \binom{\binom{n}{2}}{M}^{-1} \\ &= \nu'(G). \end{aligned}$$

Since ν is the unique invariant distribution, it follows then that $\nu = \nu'$. □

Lemma 4.4.2. *Let H and the random walk m be defined as above. Then*

$$\kappa(G_1, G_2) \geq \frac{\binom{n}{2}}{M\left(\binom{n}{2} - M\right) + 1}$$

for all $G_1, G_2 \in H$.

Proof. By Lemma 4.2.1, we can assume that G_1, G_2 are neighbors in H. It then follows from definition that

$$\kappa(G_1, G_2) = 1 - W(m_{G_1}, m_{G_2}).$$

Suppose e_1, e_2 are the unique vertex pairs with $e_1 \in E(G_1), e_2 \notin E(G_1)$ such that $G_2 = G_1 - e_1 + e_2$. Consider the support of m_{G_1}, i.e., $N(G_1)$. For each $G_1' \in N(G_1)$, we will match G_1' with a distinct graph $\phi(G_1') \in N(G_2)$. First, let $\phi(G_1) = G_1$ and $\phi(G_2) = G_2$. For other neighbors $G_1' \in N(G_1)$, there are three types:

Type 1: $G_1 - e_1 = G_1' - e_3$ for some $e_3 \neq e_2$. Then it follows that $G_1' - e_3 = G_2 - e_2$ and we let $\phi(G_1') = G_1'$.

Type 2: $G_1 - e_3 = G_1' - e_2$ for some $e_3 \neq e_1$. Then it follows that $G_1' - e_1 = G_2 - e_3$ and we let $\phi(G_1') = G_1'$.

Type 3: $G_1 - e_3 = G_1' - e_4$ for some $e_3, e_4 \notin \{e_1, e_2\}$. In this case, we claim that there exists a unique $G_2' = \phi(G_1') \in N(G_2)$ such that $G_1' - e_1 = G_2' - e_2$. Indeed, $G_2' = G_2 - e_3 + e_4$ will satisfy the aforementioned property.

Let us now define a coupling A (not necessarily optimal) between m_{G_1} and m_{G_2}. Define $A : V(H) \times V(H) \to \mathbb{R}$ as follows:

$$A(G_1', G_2') = \begin{cases} \frac{1}{M(\binom{n}{2} - M) + 1} & \text{if } G_1' \in N(G_1) \text{ and } G_2' = \phi(G_1'), \\ 0 & \text{otherwise.} \end{cases} \tag{4.10}$$

Let us verify that A is a coupling of m_{G_1} and m_{G_2}. Indeed, for each fixed G_1', if $G_1' = G_1$, then $\sum_{G_2'} A(G_1', G_2') = A(G_1, G_1) = m_{G_1}(G_1)$; if $G_1' \neq G_1$, then $\sum_{G_2'} A(G_1', G_2') = A(G_1', \phi(G_1')) = m_{G_1}(G_1')$. Similarly, $\sum_{G_1'} A(G_1', G_2') = m_{G_2}(G_2')$. Now by definition,

$$\begin{aligned} W(m_{G_1}, m_{G_2}) &\leq \sum_{G_1', G_2'} A(G_1', G_2') d(G_1', G_2') \\ &\leq \sum_{G_1' \in N(G_1)} A(G_1', \phi(G_1')) d(G_1', \phi(G_1')) \end{aligned}$$

$$
\begin{aligned}
&= \sum_{\substack{G_1' \in N(G_1) \\ G_1' \text{is Type 3}}} A(G_1', \phi(G_1')) \\
&\leq \left((M-1)\left(\binom{n}{2} - M - 1 \right) \right) \cdot \frac{1}{M\left(\binom{n}{2} - M \right) + 1}.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
\kappa(G_1, G_2) &= 1 - W(m_{G_1}, m_{G_2}) \\
&\geq \frac{\binom{n}{2}}{M\left(\binom{n}{2} - M \right) + 1}.
\end{aligned}
$$

□

Let $G(n,M)$ be an Erdős-Rényi random graph with M edges. Let F be a fixed graph and X_F be the number of copies of F in the random graph $G(n,M)$. Denote the number of vertices and edges of F by $v(F)$ and $e(F)$ respectively. Let $p = M/\binom{n}{2}$ and $\mathrm{Aut}(F)$ denote the set of automorphisms of F. Then

$$
E[X_F] = (1+o(1))\frac{v(F)!}{|\mathrm{Aut}(F)|}\binom{n}{v(F)}p^{e(F)} = \Theta\left(n^{v(F)}p^{e(F)}\right).
$$

For a series of results on the upper tail of X_F using different techniques, we refer the readers to the survey [112] and the paper [111, 27, 50, 51, 1]. For $G(n,M)$ in particular, Janson, Oleszkiewicz, Ruciński [111] showed the following theorem:

Theorem 4.4.1. *[111] For every graph F and for every $t > 1$, there exist constants $c(t,F) > 0$ such that for all $n \geq v(F)$ and $e(F) \leq M \leq \binom{n}{2}$, with $p := M/\binom{n}{2}$,*

$$
\Pr\left(X_F \geq tE[X_F]\right) \leq \exp\left(-c(t,F)M_F^*(n,p)\right),
$$

where $M_F^*(n,p) \leq n^2 p = O(M)$, $M_{C_k}^*(n,p) = \Theta(n^2p^2)$ and $M_{K_k}^*(n,p) = \Theta(n^2p^{k-1})$.

Let us now apply Theorem 4.1.1 to obtain the concentration results from the perspective of the Ricci curvature. Recall that H is defined as the graph such that $V(H)$ is the set of all labeled graphs with n vertices and M edges. Moreover, two graphs $G_1, G_2 \in V(H)$ are adjacent in H if and only if there exist two distinct vertex pairs e_1, e_2 such that $e_1 \in E(G_1) \backslash E(G_2)$, $e_2 \in E(G_2) \backslash E(G_1)$ such that $G_1 - e_1 = G_2 - e_2$.

Again let X_F be the random variable denoting the number of copies of F in $G(n, M)$. For ease of reference, let $k = v(F)$. Observe that X_F is $\binom{n}{k-2}$-Lipschitz on H, i.e., if G_1, G_2 are adjacent in H, then $|X_F(G_1) - X_F(G_2)| \le \binom{n}{k-2}$. Thus by Theorem 4.1.1,

$$\Pr\left(\frac{X_F}{\binom{n}{k-2}} > \frac{E[X_F]}{\binom{n}{k-2}} + \frac{t}{\binom{n}{k-2}}\right) \le \exp\left(-\frac{t^2\kappa}{5\binom{n}{k-2}^2}\right).$$

It follows that

$$\Pr\left(X_F > E[X_F] + t\right) \le \exp\left(-\frac{t^2\kappa}{5\binom{n}{k-2}^2}\right).$$

Let $p = M/\binom{n}{2}$. We then obtain that

$$\Pr\left(X_F \ge tE[X_F]\right) \le \exp\left(-\frac{\left((t-1)E[X_F]\right)^2\kappa}{5\binom{n}{k-2}^2}\right) \le \exp\left(-C_k(t-1)^2 n^2 p^{2e(F)-1}\right). \tag{4.11}$$

Note that when $p = \Theta(1)$, i.e., $M = \Theta\left(\binom{n}{2}\right)$, the concentration inequalities obtained from Theorem 4.1.1 has the same asymptotic exponent as Theorem 4.4.1. For other ranges of p with $n^2p \to \infty$, the asymptotic exponent in (4.11) is worse than the bound in Theorem 4.4.1. Nonetheless, let us compare the bounds obtained from the Ricci curvature method with those obtained from other concentration inequalities. Janson and Ruciński [112] surveyed the existing techniques on estimating the exponents for upper tails in the small subgraphs problem in $G(n,p)$ (ignoring logarithmic factors). Please see Figure 4.1 for the summary.

Although we are mainly dealing with $G(n, M)$ in this section, it is well known that $G(n, M)$ and $G(n,p)$ with $p = M/\binom{n}{2}$ behaves similarly when $n^2p \to \infty$. Applying the inequalities in (4.11) to K_3, K_4, C_4 respectively, we have that the exponents (ignoring constant) obtained from the Ricci curvature method are n^2p^5, n^2p^{11} and n^2p^7 respectively. In this context, the concentration we obtained from Theorem 4.1.1

	K_3	K_4	C_4
1) Azuma	n^2p^6	n^2p^{12}	n^2p^8
2) Talagrand	n^2p^5	n^2p^{11}	n^2p^7
3A) Kim–Vu A	$\min(n^{1/3}p^{1/6},\ n^{1/2}p^{1/2})$	$\min(n^{1/6}p^{1/12},\ n^{1/3}p^{1/2})$	$\min(n^{1/4}p^{1/8},\ n^{1/2}p^{1/2})$
3B) Kim–Vu B [20]	$n^{3/2}p^{3/2}$	$n^{4/3}p^2$	$n^{4/3}p^{4/3}$
3C) (Kim–)Vu C	np	$n^{2/3}p$	np
4) Complement	—	—	—
5) Break-up	n^2p^3	n^2p^6	n^2p^4
6A) Deletion A [10]	$\max(n^{3/2}p^{3/2},\ n^2p^3)$	$\min(n^2p^3,\ n^{4/3}p^{5/3})$	$\min(n^2p^2,\ n^{4/3}p)$
6B) Deletion B	n^2p^3	n^2p^6	n^2p^4
6C) Deletion C	np	$n^{2/3}p$	np
7) Approximation	$\min(n^{3/2}p^{3/2},\ 1/(np^2))$	—	$\min(n^2p^2,\ 1/n^2p^3)$
Vu's lower bound	n^2p^2	n^2p^3	n^2p^2

Figure 4.1: Exponents for upper tails in the small subgraphs problem [112]

has the same strength as Talagrand inequality and slightly stronger than Azuma's inequality.

4.4.3 Edge-Lipschitz functions on random hypergraphs

Let $\mathcal{H} \sim \mathcal{H}^k(n, M)$ be a random k-uniform hypergraph with n vertices and M edges. Let H be a graph such that $V(H)$ is the set of all labeled k-uniform hypergraphs with n vertices and M edges. Moreover, two hypergraphs $\mathcal{H}_1, \mathcal{H}_2 \in V(H)$ are adjacent in H if and only if there exist two distinct k-sets h_1, h_2 such that $h_1 \in E(\mathcal{H}_1)\backslash E(\mathcal{H}_2)$, $h_2 \in E(\mathcal{H}_2)\backslash E(\mathcal{H}_1)$ and $\mathcal{H}_1 - h_1 = \mathcal{H}_2 - h_2$. In other words, $\mathcal{H}_1, \mathcal{H}_2$ are adjacent in H if one can be obtained from the other by swapping a hyperedge with a non-hyperedge. It is easy to see that H is a connected regular graph. Moreover, for every $\mathcal{H} \in V(H)$, $d_H(\mathcal{H}) = M\left(\binom{n}{k} - M\right)$. Now define a random walk m on H as follows: Let $\mathcal{H} \in V(H)$. Define

$$m_{\mathcal{H}}(\mathcal{H}') = \begin{cases} \frac{1}{M(\binom{n}{k}-M)+1} & \text{if } \mathcal{H}' \in \Gamma(\mathcal{H}), \\ 0 & \text{otherwise.} \end{cases}$$

By the same logic in Section 4.4.2, we can obtain a lower bound for the Ricci curvature of H, i.e., for all $\mathcal{H}_1, \mathcal{H}_2 \in V(H)$,

$$\kappa(\mathcal{H}_1, \mathcal{H}_2) \geq \frac{\binom{n}{k}}{M\left(\binom{n}{k} - M\right) + 1}.$$

Similar to before, we can also apply Theorem 4.1.1 to obtain concentration results for the number of copies of fixed sub-hypergraphs in a uniformly random hypergraph on n vertices and M edges. The idea is similar to Section 4.4.2 and we leave the details to the readers.

4.4.4 Vertex-Lipschitz functions on random d-out(in)-regular graphs

Given a directed graph G and a vertex v, we use $\delta^+(v)$ and $\delta^-(v)$ to denote the outdegree and indegree, respectively, of a vertex v. A *d-out-regular graph* G is a directed graph in which $\delta^+(v) = d$ for every $v \in V(G)$. Similarly, a *d-in-regular graph* G is a directed graph in which $\delta^-(v) = d$ for every $v \in V(G)$. Moreover, let $\Gamma^+(v) = \{u \in V(G) : vu \in E(G)\}$, $\Gamma^-(v) = \{u \in V(G) : uv \in E(G)\}$, $N^+(v) = \Gamma^+(v) \cup \{v\}$ and $N^-(v) = \Gamma^-(v) \cup \{v\}$.

Let H be a graph such that $V(H)$ is the set of all labeled d-out-regular graphs on n vertices. Two graphs $G_1, G_2 \in V(H)$ are adjacent in H if and only if there exists some vertex $v \in V(G_1) = V(G_2)$ such that one can be obtained from the other by changing $\Gamma^+(v)$. It is not hard to see that H is a connected graph with $diam(H) \leq n$. Moreover, it is also clear that if G_1, G_2 are adjacent in H, there is a unique vertex v such that one can be obtained from the other by changing $\Gamma^+(v)$.

Now define a random walk m on H as follows: let $G \in V(H)$ and define

$$m_G(G') = \begin{cases} \frac{1}{n\left(\binom{n-1}{d} - 1\right) + 1} & \text{if } G' \in N^+(G), \\ 0 & \text{otherwise.} \end{cases}$$

It's easy to see that for a fixed G, $\sum_{G'} m_G(G') = 1$.

Proposition 4.4.4. *Let ν be the unique invariant distribution of the random walk defined above. A random graph G picked according to ν, is equally likely to be one of the d-out-regular graphs on n vertices.*

Proof. Observe that H is not bipartite thus the random walk is ergodic. There are $\binom{n-1}{d}^n$ many d-out-regular graphs in total. Hence, it suffices to show that $\nu'(G) = \binom{n-1}{d}^{-n}$ for every G is an invariant distribution for the random walk. Indeed, for every fixed $G \in V(H)$,

$$\begin{aligned}\sum_{G' \in H} \nu'(G') m_{G'}(G) &= \binom{n-1}{d}^{-n} \sum_{G' \in H} m_{G'}(G) \\ &= \binom{n-1}{d}^{-n} \sum_{G' \in H} m_G(G') \\ &= \binom{n-1}{d}^{-n} \\ &= \nu'(G).\end{aligned}$$

Since ν is the unique invariant distribution, it follows then that $\nu = \nu'$. □

Lemma 4.4.3. *Let H and the random walk m be defined as above. Then*

$$\kappa(G_1, G_2) \geq \frac{1}{n}$$

for all $G_1, G_2 \in V(H)$.

Proof. Again, by Lemma 4.2.1, we can assume that G_1, G_2 are neighbors in H. It then follows from definition that

$$\kappa(G_1, G_2) = 1 - W(m_{G_1}, m_{G_2}).$$

Suppose v is the unique vertex such that G_2 can be obtained from G_1 by changing $\Gamma^+(v)$. Consider the support of m_{G_1}. For each $G_1' \in N(G_1)$, we will match G_1' with a distinct graph $\phi(G_1') \in N(G_2)$. Again, let $\phi(G_1) = G_1$ and $\phi(G_2) = G_2$. For other neighbors G_1' of G_1, there are two possible cases:

Case 1: $G_1 - v = G'_1 - v$. Then it follows that $G'_1 - v = G_2 - v$ and we let $\phi(G'_1) = G'_1$.

Case 2: $G_1 - u = G'_1 - u$ for some $u \neq v$. In this case, we claim that for each G'_1 such that $G_1 - u = G'_1 - u$, there exists a unique $G'_2 = \phi(G'_1)$ such that $G'_2 - u = G_2 - u$ and $G'_1 - v = G'_2 - v$. Indeed, let G'_2 be obtained from G_2 by replacing the out-neighbors of u in G_2 by the out-neighbors of u in G'_1. It's not hard to see that $G'_2 - u = G_2 - u$ and $G'_1 - v = G'_2 - v$.

Let us now define a coupling A (not necessarily optimal) between m_{G_1} and m_{G_2}. Define $A : V(H) \times V(H) \to \mathbb{R}$ as follows:

$$A(G'_1, G'_2) = \begin{cases} \frac{1}{n(\binom{n-1}{d}-1)+1} & \text{if } G'_1 \in N(G_1) \text{ and } G'_2 = \phi(G'_1), \\ 0 & \text{otherwise.} \end{cases} \tag{4.12}$$

It is not hard to verify that A is a coupling of m_{G_1} and m_{G_2}. Now by definition,

$$\begin{aligned} W(m_{G_1}, m_{G_2}) &\leq \sum_{G'_1, G'_2} A(G'_1, G'_2) d(G'_1, G'_2) \\ &\leq \sum_{u \neq v} \sum_{\substack{G'_1 \in N(G_1) \\ G'_1 - u = G_1 - u}} A(G'_1, \phi(G'_1)) d(G'_1, \phi(G'_1)) \\ &\leq (n-1)\left(\binom{n-1}{d} - 1\right) \frac{1}{n\left(\binom{n-1}{d} - 1\right) + 1} \end{aligned}$$

It follows that

$$\kappa(G_1, G_2) = 1 - W(m_{G_1}, m_{G_2}) \geq \frac{\binom{n-1}{d}}{n\left(\binom{n-1}{d} - 1\right) + 1} \geq \frac{1}{n}.$$

This completes the proof of the lemma. □

Let G be a uniformly random d-out-regular graph. A *directed triangle* is a cycle of length 3 with vertices u, v, w such that uv, vw and wu are all directed edges. Let $X_{n,d} := X(G)$ be the random variable denoting the number of directed triangle in G. It is not hard to see that

$$E[X_{n,d}] \approx 2\binom{n}{3}\left(\frac{d}{n-1}\right)^3.$$

We will now use Theorem 4.1.1 to derive the concentration behavior of $X_{n,d}$. Note that $X_{n,d}$ is (d^2)-Lipschitz. Hence by Theorem 4.1.1, we have that

$$\Pr\left(\left|\frac{X_{n,d}}{d^2} - \frac{E[X_{n,d}]}{d^2}\right| > \frac{t}{d^2}\right) \le 2\exp\left(-\frac{t^2\kappa}{5d^4}\right).$$

It follows that

$$\Pr\left(|X_{n,d} - E[X_{n,d}]| > t\right) \le 2\exp\left(-\frac{t^2\kappa}{5d^4}\right) \le 2\exp\left(-\frac{t^2}{5nd^4}\right).$$

4.4.5 Lipschitz functions on random linear permutations

We will denote a linear permutation σ by $\sigma = [a_1 a_2 \ldots a_n]$ such that $a_i \in [n]$ for all i and $\sigma(i) = a_i$. A linear permutation on $[n]$ can be viewed as a sequence of n distinct numbers from $[n]$. Thus, WLOG, $\{a_1, a_2, \ldots, a_n\} = [n]$. Given two permutations σ_1, σ_2 where $\sigma_1 = [a_1 a_2 \ldots a_n]$, we say σ_1 is (i,j)-*alike* to σ_2 if σ_2 can be obtained from σ_1 by moving the number i to the position after the number j in σ_1; moreover, σ_1 is $(i,0)$-alike to σ_2 if σ_2 can be obtained from σ_1 by moving the number i to the first position of σ_1. For example, $\sigma_1 = [12345]$ is $(2,4)$-alike to $\sigma_2 = [13425]$ and is $(4,0)$-alike to $\sigma_3 = [41235]$. Two *distinct* linear permutations σ_1, σ_2 are *insertion-alike* if one is (i,j)-alike to the other for some $i \ne j$.

Let H be the graph such that $V(H)$ is the set of all linear permutations of $[n]$ and two linear permutation σ_1, σ_2 are adjacent in H if and only if they are insertion-alike. Clearly H is a connected graph with diameter at most n. Moreover, every vertex (which is a linear permutation) in H has $(n-1)^2$ neighbors in H.

Now define a random walk m_α on H as follows: let $\sigma \in V(H)$ and define

$$m_\sigma(\sigma') = \begin{cases} \frac{1}{(n-1)^2+1} & \text{if } \sigma = \sigma' \text{ or } \sigma \text{ is insertion-alike to } \sigma', \\ 0 & \text{otherwise.} \end{cases}$$

It's not hard to see that for a fixed σ, $\sum_{\sigma'} m_\sigma(\sigma') = 1$. Moreover, $m_\sigma(\sigma') = m_{\sigma'}(\sigma)$ for every pair of σ, σ'.

Proposition 4.4.5. *Let ν be the unique invariant distribution of the random walk defined above. A random permutations σ picked according to ν, is equally likely to be one of the $n!$ permutations.*

Proof. Observe that H is not bipartite thus the random walk is ergodic. There are $n!$ permutations in total. Hence, it suffices to show that $\nu'(\sigma) = (n!)^{-1}$ for every σ is an invariant distribution for the random walk.

$$\begin{aligned}\sum_{\sigma' \in H} \nu'(\sigma') m_{\sigma'}(\sigma) &= \frac{1}{n!} \sum_{\sigma' \in V(H)} m_{\sigma'}(\sigma) \\ &= \frac{1}{n!} \sum_{\sigma' \in V(H)} m_{\sigma}(\sigma') \\ &= \frac{1}{n!} \\ &= \nu'(\sigma).\end{aligned}$$

Since ν is the unique invariant distribution, it follows then that $\nu = \nu'$. □

Lemma 4.4.4. *Let H and the random walk m be defined as above. If $\sigma_1, \sigma_2 \in V(H)$ are neighbors in H, then $\kappa(\sigma_1, \sigma_2) \geq \frac{1}{n}$.*

Proof. WLOG, suppose that σ_1 is (i,j)-alike to σ_2 (with $\sigma_2 \neq \sigma_1$). Consider the support of m_{σ_1}. For each $\sigma_1' \in N(\sigma_1)$, we will match σ_1' with a distinct permutation $\phi(\sigma_1') \in N(\sigma_2)$. First let $\phi(\sigma_1) = \sigma_1$ and $\phi(\sigma_2) = \sigma_2$. For other neighbors σ_1' of σ_1, there are two cases:

Case 1: σ_1 is (i,k)-alike to σ_1' where $k \neq j$. Then it follows that σ_1' is also (i,j)-alike to σ_2 and we let $\phi(\sigma_1') = \sigma_1'$.

Case 2: σ_1 is (i',j')-alike to σ_1' where $i' \neq i$ and σ_1 is not (i,k)-alike to σ_1' for any k. In this case, let σ_2' be the permutation such that σ_2 is (i',j')-alike to σ_2'. It follows easily that σ_1' is also (i,j)-alike to σ_2'. We then define $\phi(\sigma_1') = \sigma_2'$.

Let us now define a coupling A (not necessarily optimal) between m_{σ_1} and m_{σ_2}. Define $A : V(H) \times V(H) \to \mathbb{R}$ as follows:

$$A(\sigma_1', \sigma_2') = \begin{cases} \frac{1}{(n-1)^2+1} & \text{if } \sigma_1' \in N(\sigma_1) \text{ and } \sigma_2' = \phi(\sigma_1'), \\ 0 & \text{otherwise.} \end{cases} \tag{4.13}$$

It is not hard to verify that A is a coupling of m_{σ_1} and m_{σ_2}. Now by definition,

$$\begin{aligned} W(m_{\sigma_1}, m_{\sigma_2}) &\leq \sum_{\sigma_1', \sigma_2'} A(\sigma_1', \sigma_2') d(\sigma_1', \sigma_2') \\ &\leq \sum_{\sigma' \in N(\sigma_1)} A(\sigma_1', \phi(\sigma_1')) d(\sigma_1', \phi(\sigma_1')) \\ &\leq 1 - \frac{n}{(n-1)^2 + 1}. \end{aligned}$$

It follows that

$$\kappa(\sigma_1, \sigma_2) = 1 - W(m_{\sigma_1}, m_{\sigma_2}) \geq \frac{n}{(n-1)^2 + 1} \geq \frac{1}{n}.$$

This completes the proof of the lemma. □

Now we give an example of concentration results on the space of random linear permutations. In particular, we discuss the number of occurrences of certain patterns in random permutations. Denote the set of length n linear permutations by $\mathcal{S}_n$. Given a permutation pattern $\tau \in \mathcal{S}_k$, we say that a permutation $\pi = [\pi_1 \dots \pi_n] \in \mathcal{S}_n$ contains the pattern τ if there exists $1 \leq i_1 < i_2 < \dots < i_k \leq n$ such that the $\pi_{i_s} < \pi_{i_t}$ if and only if $\tau_s < \tau_t$ for every pair s, t. Each such subsequence in π is called an *occurrence* of the pattern τ. Let τ be a random permutation in $\mathcal{S}_n$ and let the random variable $X_{\tau,n} := X_\tau(\pi)$ be the number of copies of τ in π. We consider asymptotics as $n \to \infty$ for (one or several) fixed τ.

The (joint) distribution of the $X_{\tau,n}$ has been investigated in a series of paper [19, 18, 110]. In particular, Bona [19] showed that for every $\tau \in \mathcal{S}_k$, as $n \to \infty$,

$$\frac{X_{\tau,n} - E[X_{\tau,n}]}{n^{k-\frac{1}{2}}} \to N(0, Z_\tau) \tag{4.14}$$

for some $Z_\tau > 0$. Janson, Nakamura and Zeilberger [110] showed that the above holds jointly for any finite family of patterns τ.

Note that as a consequence of the convergence in (4.14), we obtain the following concentration inequality:

$$\Pr\left(|X_{\tau,n} - E[X_{\tau,n}]| > t\right) \le 2\exp\left(-\frac{t^2}{2n^{2k-1}Z_\tau}\right) \tag{4.15}$$

which is sharp up to a polynomial factor.

On the other hand, consider the graph H defined at the beginning of this subsection, where $V(H)$ is the set of all linear permutations of $[n]$. It is not hard to see that the function $X_{\tau,n} : V(H) \to \mathbb{Z}$ is $\binom{n-1}{k-1}$-Lipschitz. It follows by Theorem 4.1.1 that

$$\begin{aligned}\Pr\left(\left|\frac{X_{\tau,n}}{\binom{n-1}{k-1}} - \frac{E[X_{\tau,n}]}{\binom{n-1}{k-1}}\right| > \frac{t}{\binom{n-1}{k-1}}\right) &\le 2\exp\left(-\frac{t^2\kappa}{5\binom{n-1}{k-1}^2}\right)\\ &\le 2\exp\left(-\frac{t^2}{C_k n^{2k-1}}\right)\end{aligned}$$

for some $C_k > 0$. Hence the concentration result in Theorem 4.1.1 is in fact asymptotically optimal in the case of counting occurrences of patterns in random permutations.

Remark 4.4.1. *Similar Ricci curvature and concentration results can be obtained for the space of cyclic permutations as well.*

Remark 4.4.2. *Another possible way to geometrize the space of linear permutations is the random transposition model (see, e.g., [61]) as follows: let $V(H) = \mathcal{S}_n$ and two permutations $\sigma_1, \sigma_2 \in V(H)$ are adjacent in H if $\sigma_2 = \tau \circ \sigma_1$ for some transposition τ. Define a random walk m on H by*

$$m_\sigma(\sigma') = \begin{cases} \frac{2}{n(n-1)} & \text{if } \sigma \text{ and } \sigma' \text{ are adjacent in } H,\\ 0 & \text{otherwise.}\end{cases}$$

The invariant distribution is the uniform measure on $\mathcal{S}_n$. The Ricci curvature of this graph is $\Theta(n^{-2})$, as observed by Gozlan et al [87].

Chapter 5

Maximum spectral radius of outerplanar 3-uniform hypergraphs

5.1 Introduction

A graph G is *planar* if it can be embedded in the plane, i.e., it can be drawn on the plane in such a way that edges intersect only at their endpoints. A graph is *outerplanar* if it can be embedded in the plane such that all vertices lie on the boundary of its outer face. The study of the spectral radius of (outer)planar graphs has a long history, dating back to Schwenk and Wilson [158]. Given a graph G, the *spectral radius* λ of G is the largest eigenvalue of the adjacency matrix of G. The spectral radius of planar graphs is useful in geography as a measure of the overall connectivity of a planar graph [20, 47]. It is therefore of interest to geographers to find the maximum spectral radius of a planar graph as a theoretical upper bound for the connectivity of networks. Boots and Royle [20], and independently Cao and Vince [23] conjectured that the extremal planar graph achieving the maximum spectral radius is $P_2 + P_{n-2}$. Hong [106] first showed that for an n-vertex plananr graph G, $\lambda(G) \leq \sqrt{5n-11}$. This was subsequently improved in a seiries of papers [23, 107, 89, 108, 59]. Guiduli and Hayes [90] showed in an unpublished preprint that the Boots-Royle-Cao-Vince conjecture is true for sufficiently large n. For outerplanar graphs, it is conjectured by Cvetković and Rowlinson [47] that among all outerplanar graph on n vertices, $K_1 + P_{n-1}$ attains the maximum spectral radius. Partial progress has been made by Rowlinson [153], Cao and Vince [23], and Guiduli and Hayes [90].

Recently, Tait and Tobin [166] proved the Boots-Royle-Cao-Vince conjecture and the Cvetković-Rowlinson conjecture for large enough n. Lin and Ning [128] showed that the Cvetković-Rowlinson conjecture holds for all $n \geq 17$.

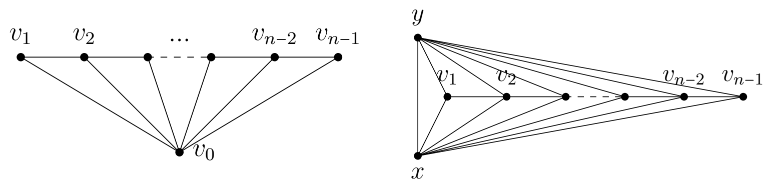


Figure 5.1: The graph $P_1 + P_{n-1}$ (left) and $P_2 + P_{n-1}$ (right).

In this paper, we extend the investigations into the maximum spectral radius of outerplanar 3-uniform hypergraphs. Given a 3-uniform hypergraph $\mathcal{H}$, the *shadow* of $\mathcal{H}$, denoted by $\partial(\mathcal{H})$, is a 2-uniform graph G with $V(G) = V(\mathcal{H})$ and $E(G) = \{uv : uv \in h \text{ for some } h \in E(\mathcal{H})\}$. A 3-uniform hypergraph $\mathcal{H}$ is called *planar* if $\partial(\mathcal{H})$ is a triangulation of the sphere. The edge set of such $\mathcal{H}$ is the set of faces of the triangulation. A 3-uniform hypergraph $\mathcal{H}$ is called *outerplanar* if $\partial(\mathcal{H})$ is outerplanar and all faces except the outer face are triangles. The edge set of $\mathcal{H}$ is the set of triangle faces of its shadow (except the outer face). Note that an n-vertex planar 3-uniform hypergraph has $2n-4$ hyperedges and $3n-6$ edges in its shadow. Similarly, an n-vertex outerplanar 3-uniform hypergraph has $n-2$ hyperedges and $2n-3$ edges in its shadow.

Now we define the spectral radius of an r-uniform hypergraph. Given an r-uniform hypergraph $\mathcal{H}$ on n vertices, the polynomial form of $\mathcal{H}$ is a multi-linear function $P_{\mathcal{H}}(\boldsymbol{x}) : \mathbb{R}^n \to \mathbb{R}$ defined for any vector $\boldsymbol{x} = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n$ as

$$P_H(\boldsymbol{x}) = r \sum_{\{i_1, i_2, \ldots, i_r\} \in E(G)} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

The *spectral radius* λ of $\mathcal{H}$, introduced by Cooper and Dutle [45], is defined as

$$\lambda(\mathcal{H}) := \max_{\|\boldsymbol{x}\|_r = 1} P_{\mathcal{H}}(\boldsymbol{x}) = \max_{\boldsymbol{x} \in \mathbb{R}} \frac{P_{\mathcal{H}}(\boldsymbol{x})}{\|\boldsymbol{x}\|_r^r}, \tag{5.1}$$

where $\|\boldsymbol{x}\|_r := (|x_1|^r + |x_2|^r + \cdots + |x_n|^r)^{1/r}$. If $\boldsymbol{x} \in \mathbb{R}^n$ is a vector with $\|\boldsymbol{x}\|_r = 1$ and $P_H(\boldsymbol{x}) = \lambda(H)$, then $\boldsymbol{x}$ is called an *eigenvector* corresponding to $\lambda(H)$. Note that $P_H(\boldsymbol{x})$ can always reach its maximum at some nonnegative vectors. By Lagrange's method, we have the *eigenequations* for $\lambda(H)$ and an eigenvector $\boldsymbol{x}$ corresponding to $\lambda(H)$:

$$\lambda(H)x_i^{r-1} = \sum_{\{i,i_2,\ldots,i_r\}\in E(H)} x_{i_2}\cdots x_{i_r} \quad \text{for } x_i > 0. \tag{5.2}$$

It was shown by Cooper and Dutle [45] that for any non-empty k-uniform hypergraph $\mathcal{H}$, the spectral radius of $\mathcal{H}$ is always a positive real number. Moreover, if $\mathcal{H}$ is connected, then a corresponding eigenvector can be chosen to be strictly positive.

Now we can state our main theorems.

Theorem 5.1.1. *For large enough n, the n-vertex outerplanar 3-uniform hypergraph of maximum spectral radius is the unique hypergraph whose shadow is $K_1 + P_{n-1}$.*

The shadow of the extremal hypergraph attaining the maximum spectral radius among all outerplane 3-uniform hypergraphs is exactly the extremal graph attaining the maximum spectral radius among all outplanar graphs. This motivates us to make the following analogous conjecture for planar 3-uniform hypergraphs:

Conjecture 5.1.1. *For large enough n, the n-vertex planar 3-uniform hypergraph graph $\mathcal{H}$ of maximum spectral radius is the unique hypergraph whose shadow is $P_2 + P_{n-1}$.*

5.2 Proof of Theorem 5.1.1

Given a graph G and $v \in V(G)$, we use $N_G(v)$ to denote the set of neighbors of v, i.e., $N_G(v) = \{u : vw \in E(G)\}$. The closed neighborhood of v, denoted by $N_G[v]$, is defined as $N_G[v] = N_G(v) \cup \{v\}$. Given a 3-uniform hypergraph $\mathcal{H}$ and $v \in V(\mathcal{H})$, we define $\Gamma_{\mathcal{H}}(v) = \{uw : vuw \in E(\mathcal{H})\}$. Moreover, set $d_G(v) = |N_G(v)|$ and $d_{\mathcal{H}}(v) = |\Gamma_{\mathcal{H}}(v)|$. In all the definitions above, we may ignore the subscript if the underlying (hyper)graph is clear from the context.

Let $\mathcal{H}$ be an n-vertex outerplanar 3-uniform hypergraph of maximum spectral radius. It's easy to see that we may assume $\partial(\mathcal{H})$ is connected. Throughout this section, let G be the shadow of $\mathcal{H}$, i.e., $V(G) = V(\mathcal{H})$ and $E(G) = \{vu : \{v, u\} \subseteq h \text{ for some } h \in E(\mathcal{H})\}$. It follows by definition that G is outerplanar, thus does not contain a $K_{2,3}$ minor.

Lemma 5.2.1. $\lambda(\mathcal{H}) \geq \sqrt[3]{4(n-1)}\left(1 - \frac{1}{n-1}\right)$.

Proof. Let G_0 be the wheel graph W_{n-1} with w being the vertex with degree $n-1$, and $\{v_1, \cdots, v_{n-1}\}$ being the vertices of degree 3. Let G' be the graph obtained from G_0 by deleting the edge v_1v_2. Let $\mathcal{H}'$ be the 3-uniform hypergraph with $E(\mathcal{H}')$ being the set of triangle faces of G'. Clearly $\mathcal{H}'$ is outerplanar. Consider the vector $\boldsymbol{x} \in \mathbb{R}^n$ with $x_w = 1/\sqrt[3]{3}$ and $x_{v_i} = \left(\frac{2}{3(n-1)}\right)^{1/3}$. Note that $\|\boldsymbol{x}\|_3 = 1$. It follows that

$$\lambda(\mathcal{H}) \geq \lambda(\mathcal{H}') \geq P_{\mathcal{H}'}(\boldsymbol{x}) = 3(n-2) \cdot \frac{1}{\sqrt[3]{3}} \cdot \left(\frac{2}{3(n-1)}\right)^{2/3} = \sqrt[3]{4(n-1)}\left(1 - \frac{1}{n-1}\right).$$

□

Note that since $\mathcal{H}$ is connected, there exists an eigenvector corresponding to $\lambda(\mathcal{H})$ such that all its entries are strictly positive. In the rest of this section, for convenience we assume that the eigenvector of the adjacency matrix of $\mathcal{H}$ corresponding to $\lambda(\mathcal{H})$ is re-normalized so that the maximum eigenvector entry is 1. Let v_0 be the vertex with the maximum eigenvector entry, i.e., $x_{v_0} = 1$.

Lemma 5.2.2. $d_G(v_0) > n - O(n^{2/3})$. *Moreover, for any other vertex* $u \neq v_0$, $x_u = O(n^{-1/3})$.

We first show a weaker version of Lemma 5.2.2. In particular, we show the following claim.

Claim 5.2.1. $d_G(v_0) > n - O(n^{5/6})$.

Proof of Claim 5.2.1. Recall that $x_{v_0} = 1$ where v_0 is the vertex with the maximum eigenvector entry of the Perron-Frobenius eigenvector of $\mathcal{H}$. Let $d = d_G(v_0)$. Let $\{v_1, v_2, \cdots, v_d\}$ be the neighbors of v_0 in the clockwise order of some outerplanar drawing of G. Observe that we can relabel them in such a way that $\{v_i, v_{i+1}, v_0\} \in E(\mathcal{H})$ for each $i \in [d-1]$. This is because if for some $j \neq d$ such that $\{v_j, v_{j+1}, v_0\} \notin E(\mathcal{H})$, then we can add the hyperedge $\{v_j, v_{j+1}, v_0\}$ to $\mathcal{H}$ and obtain an outerplanar hypergraph with larger spectral radius.

Now by the eigenequation on v_0, we have

$$\lambda = \lambda x_{v_0}^2 = \sum_{i=1}^{d-1} x_{v_i} x_{v_{i+1}} \leq \sum_{i=1}^{d} x_{v_i}^2,$$

using the fact $ab \leq \frac{a^2+b^2}{2}$. Set $z = \sum_{i=1}^{d} x_{v_i}^2$. We have $\lambda \leq z$. It again follows from the eigenequation expansion that

$$\begin{aligned}
\lambda z &\leq \sum_{i=1}^{d} \lambda x_{v_i}^2 \\
&\leq 2x_{v_0} \sum_{i=1}^{d} x_{v_i} + \sum_{i=1}^{d} \sum_{\substack{vw \in \Gamma(v_i) \\ v,w \neq v_0}} x_v x_w \\
&= 2 \sum_{i=1}^{d} x_{v_i} + \sum_{i=1}^{d} \sum_{\substack{vw \in \Gamma(v_i) \\ v,w \neq v_0}} x_v x_w \\
&\leq 2\sqrt{dz} + \sum_{i=1}^{d} \sum_{\substack{vw \in \Gamma(v_i) \\ v,w \neq v_0}} x_v x_w, \qquad (5.3)
\end{aligned}$$

where the last inequality is by the Cauchy-Schwarz inequality.

For ease of reference, set $R = \sum_{i=1}^{d} \sum_{vw \in \Gamma(v_i), v,w \neq v_0} x_v x_w$. In Figure 5.2, all the edges $vw \in E(G)$ corresponding to the summands $x_v x_w$ in R are colored red. Dividing both sides of the inequality above by λ, we then have $z - \frac{2\sqrt{dz}}{\lambda} \leq \frac{R}{\lambda}$. By completing the

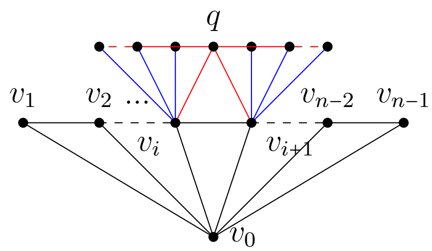


Figure 5.2: Neighborhood of v_0

square and rearranging the terms of the inequality, we obtain that

$$\begin{aligned} z &\le \left(\frac{\sqrt{d}}{\lambda} + \sqrt{\frac{d}{\lambda^2} + \frac{R}{\lambda}}\right)^2 \\ &= \frac{4d}{\lambda^2} + \frac{2R}{\lambda} - \left(\sqrt{\frac{d}{\lambda^2} + \frac{R}{\lambda}} - \frac{\sqrt{d}}{\lambda}\right)^2. \end{aligned} \tag{5.4}$$

It follows that

$$\lambda^3 \le \lambda^2 z \le 4d + 2\lambda R - \left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2. \tag{5.5}$$

By Lemma 5.2.1, we obtain that $\lambda^3 \ge 4n - 16$ when n is large enough. Let's now give a bound on $2\lambda R$. Observe that since G is an outerplanar graph, the neighborhood around an edge $v_i v_{i+1}$ will have the same structure as shown in Figure 5.2. The edges vw for which $x_v x_w$ appears in the summands of R are colored red. Let E_r be the collection of these red edges. Again using the fact that $2ab \le a^2 + b^2$, we replace all $2x_v x_w$ in R by $x_v^2 + x_w^2$. We then use the eigenequation on x_v and x_w to expand $\lambda(x_v^2 + x_w^2)$.

To make the analysis easier, we partition the vertices into three classes and pay attention to their multiplicity in the summation. Note that we only need to consider the vertices that are the endpoints of red edges. The first class of vertices (denoted by V_1) are the ones that are adjacent to v_0. It's easy to see that

$$\sum_{h \in E_r} \sum_{u \in V_1 \cup h} x_u^2 \le 2 \sum_{i=1}^{d} x_{v_i}^2.$$

Hence we have

$$\lambda \sum_{h \in E_r} \sum_{u \in V_1 \cup h} x_u^2 \le 2 \sum_{i=1}^{d} \lambda x_{v_i}^2 = 2\lambda z.$$

The next class of vertices (denoted by V_2) consists of the ones that form a hyperedge with two adjacent neighbors of v_0 (labeled as q in Figure 5.2). The set of the remaining vertices are denoted by V_3. Now using eigenequation equalities, we have

$$\lambda \sum_{h\in E_r} \sum_{u\in V_2\cup h} x_u^2 + \sum_{h\in E_r} \sum_{u\in V_3\cup h} x_u^2 = \sum_{h\in E_r} \sum_{\substack{u\in h \\ u\notin N(v_0)}} \sum_{vw\in\Gamma(u)} x_v x_w. \tag{5.6}$$

Let E' be the set of edges vw in G for which $x_v x_w$ appears as summands in the summation above. Note that none of the edges in E' contain v_0. For edges $vw \in E'$, we need to count the multiplicity of $x_v x_w$ in the summation above. For edges vw in E' such that $vwv_0 \in E(\mathcal{H})$, it's easy to see that $x_v x_w$ has multiplicity at most 4 since these terms come from the eigenequation expansion on some vertex of V_2, which is incident to at most 4 red edges. Moreover, by the eigenequation on x_{v_0}, we have

$$\sum_{h\in E_r} \sum_{u\in h} \sum_{vw\in\Gamma(v_0)\cap\Gamma(u)} x_v x_w \le 4\lambda x_{v_0}^2 = 4\lambda.$$

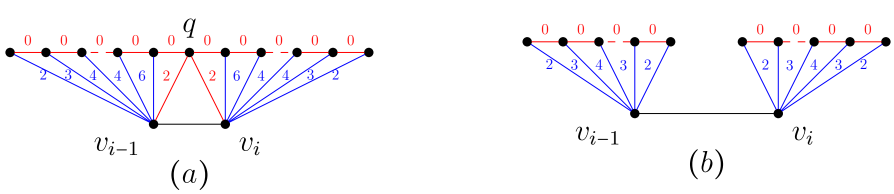


Figure 5.3: Neighborhood of edges $v_i v_{i+1}$.

Next we analyze the average number of times that edges in $E'\backslash\Gamma(v_0)$ appear in the summands of (5.6). We do this by first considering the the structure of the neighborhood around each $v_i \in N_G(v_0)$. Observe that since G is outerplananr, the neighborhood around each vertex $v_i \in N_G(v_0)$ is a subgraph of the structures in Figure 5.3 (depending on whether it intersects with the neighborhood of another vertex v_j). Moreover, the neighborhood of v_i cannot intersect with both the neighborhoods of v_{i-1} and v_{i+1} (except at v_0). The multiplicities of the edges of $E'\backslash\Gamma(v_0)$ that is either incident to some v_i or forms a hyperedge with some v_i are labelled in Figure 5.3. It is easy to compute that the average multiplicity of such edges is at most 2.

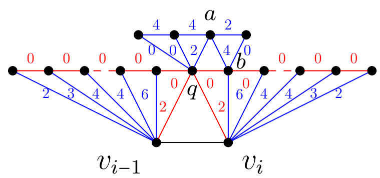


Figure 5.4: Average multiplicity of edges in $E'\backslash\Gamma(v_0)$

For the edges of $E'\backslash\Gamma(v_0)$ that is not incident to some v_i or forms some hyperedge with some v_i, we analyze their multiplicities similarly. It is easy to see from Figure 5.4 that in worst case the average multiplicities of the edges incident to or forms an edge with a vertex $q \in N_G(v_i)$ is at most 2 if we can subtract 2 from the total multiplicities (due to the hyperedege qab). Moreover, notice there are at most d such vertices q. To solve this, we count the multiplicities of the edges of qa into the multiplicities of the edges in qv_i and use the fact that $x_q x_{v_i} \le x_{v_0} x_{v_i} \le x_{v_i}$.

Moreover, $|E'\backslash\Gamma(v_0)| \le E(G) - (2d-1) \le 2n - 2d - 2$ since G is outerplanar. It follows that in (5.6) that

$$\sum_{h \in E_r} \sum_{\substack{u \in h \\ u \notin N(v_0)}} \sum_{vw \in \Gamma(u)} x_v x_w \le 4\lambda x_{v_0}^2 + 4\sum_{i \in [d]} x_{v_i} + 2\big(E(G) - (2d-1)\big) \max_{x_v,\ v \neq v_0} x_v^2$$
$$\le 4\lambda + 4\sqrt{dz} + (4n - 4d - 4) \max_{x_v,\ v \neq v_0} x_v^2.$$

Hence in summary, we have

$$\begin{aligned} 2\lambda R &= 2\lambda \sum_{i=1}^{d} \sum_{\substack{vw \in \Gamma(v_i) \\ v,w \neq v_0}} x_v x_w \\ &\le \lambda \sum_{h \in E_r} \sum_{u \in V_1 \cap h} x_u^2 + \lambda \sum_{h \in E_r} \sum_{u \in V_2 \cap h} x_u^2 + \lambda \sum_{h \in E_r} \sum_{u \in V_3 \cap h} x_u^2 \\ &\le 2\lambda z + 4\lambda + 4\sqrt{dz} + (4n - 4d - 4) \max_{x_v,\ v \neq v_0} x_v^2. \end{aligned} \tag{5.7}$$

Substitute $2\lambda R$ into (5.5), it follows that when n is large enough,

$$4n - 16 \le \lambda^3 \le \lambda^2 z \le 4d + \left(2\lambda z + 4\lambda + 4\sqrt{dz} + 4n - 4d - 4\right) - \left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2. \tag{5.8}$$

Cancelling terms and rearranging the inequality, we obtain that

$$\left(\sqrt{d + R\lambda} - \sqrt{d}\right)^2 \le 2\lambda(z+2) + 4\sqrt{dz} + 12,$$

which can be written as

$$\frac{(\lambda R)^2}{\left(\sqrt{d+\lambda R}+\sqrt{d}\right)^2} \le 2\lambda(z+2)+4\sqrt{dz}+12. \tag{5.9}$$

From here, we want to give an upper bound on λR. Note that from (5.8), we also have

$$\begin{aligned}\lambda^2 z &\le 4d+\left(2\lambda z+4\lambda+4\sqrt{dz}+4n-4d-4\right)\\ &\le 4n+2\lambda z+4\lambda+4\sqrt{dz}.\end{aligned}$$

Thus by the fact that $\lambda^3 \ge 4n-16$, we obtain that

$$z \le \frac{4n+4\lambda}{\lambda^2-2\lambda-4\sqrt{d}} \le (1+o(1))\,\lambda.$$

Since $\lambda^3 \le \lambda^2 z \le 4n+2\lambda z+4\lambda+4\sqrt{dz}$, we also have

$$\lambda = O(n^{1/3}). \tag{5.10}$$

Recall that $\lambda \le z$. Hence we have $z=(1+o(1))\lambda=\Theta(n^{1/3})$. Consequently we obtain from (5.7) that $\lambda R = O(n)$, which implies that $\left(\sqrt{d+\lambda R}+\sqrt{d}\right)^2 = O(n)$. Now it follows from (5.9) that

$$\lambda R = O(\sqrt{n\lambda z+n\sqrt{dz}}) = O(\sqrt{n\lambda^2+n^{3/2}\lambda^{1/2}}) = O(n^{5/6}).$$

Substitute λR into (5.5) and use the fact that $\lambda^3 \ge 4n-16$, we obtain that

$$4n-16 \le 4d+O(n^{5/6}),$$

which implies that $d \ge n-O(n^{5/6})$. This completes the proof of Claim 5.2.1.

□

In order to further improve the lower bound of d (as claimed in Lemma 5.2.2), we need to give a non-trivial upper bound on $\max_{v\ne v_0} x_v^2$. Let u_0 be a vertex attaining the second largest Perron-Frobenius eigenvector entry of the adjacency matrix of $\mathcal{H}$. We claim $x_{u_0} = O(n^{-1/3})$.

Let $d' = d_G(u_0)$ and $\{u_1, u_2, \cdots, u_{d'}\}$ be the neighbors of u_0 in G. Moreover, let $\Delta' = \max_{w \neq v_0} d_G(w)$. Note that since $d_G(v_0) \geq n - O(n^{5/6})$, it follows that $d' \leq \Delta' = O(n^{5/6})$. Otherwise by pigeonhole principle G has a $K_{2,3}$, which contradicts that G is outerplanar.

Most of the inequalities shown in Claim 5.2.1 hold in similar forms. In particular, by the eigenequation expansion on x_{u_0}, we have

$$\lambda x_{u_0}^2 = \sum_{i=1}^{d'-1} x_{u_i} x_{u_{i+1}} \leq 2x_{v_0} x_{u_0} + \sum_{u \in N_G(u_0), u \neq v_0} x_u^2.$$

Let $z' = \sum_{u \in N_G(u_0), u \neq v_0} x_u^2$. Similar to (5.3), if we apply the eigenequations on z', we have

$$\lambda z' \leq 2x_{u_0} + 2x_{u_0}\sqrt{d' z'} + R',$$

where

$$R' = \sum_{u \in N_G(u_0) \setminus \{v_0\}} \sum_{\substack{vw \in \Gamma(u) \\ v,w \neq u_0}} x_v x_w \leq \sum_{u \in N_G(u_0) \setminus \{v_0\}} \sum_{\substack{vw \in \Gamma(u) \\ v,w \neq u_0}} \frac{x_v^2 + x_w^2}{2}.$$

It follows from the same logic in (5.4) that

$$\begin{aligned} z' &\leq \left(\frac{\sqrt{d'} x_{u_0}}{\lambda} + \sqrt{\frac{d' x_{u_0^2}}{\lambda^2} + \frac{R' + 2x_{u_0}}{\lambda}} \right)^2 \\ &\leq \frac{4d' x_{u_0}^2}{\lambda^2} + \frac{2(R' + 2x_{u_0})}{\lambda} - \left(\sqrt{\frac{d' x_{u_0}^2}{\lambda^2} + \frac{R' + 2x_{u_0}}{\lambda}} - \frac{\sqrt{d'} x_{u_0}}{\lambda} \right)^2. \end{aligned}$$

Then it follows that

$$\begin{aligned} \lambda^2(z' + 2x_{u_0}) \leq & 4d' x_{u_0}^2 + 2\lambda(R' + 2x_{u_0}) - \left(\sqrt{d' x_{u_0}^2 + \lambda(R' + 2x_{u_0})} - \sqrt{d'} x_{u_0} \right)^2 + 2\lambda^2 x_{u_0} \\ \leq & 4d' x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda) x_{u_0}. \end{aligned}$$

Hence we have

$$(4n - 16) x_{u_0}^2 \leq \lambda^3 x_{u_0}^2 \leq 4d' x_{u_0}^2 + 2\lambda R' + (2\lambda^2 + 4\lambda) x_{u_0}. \tag{5.11}$$

We will use inequality similar to (5.7) to bound $2\lambda R'$. Let $E(R') = \{vw \in \Gamma(u) : v, w \neq u_0, u \in N_G(u_0)\backslash\{v_0\}\}$. Now,

$$\begin{aligned}
2\lambda R' &\leq \sum_{vw \in E(R')} \lambda x_v^2 + \lambda x_w^2 \\
&\leq 2\lambda \left(x_{v_0}^2 + \sum_{\substack{u \in N_G(u_0) \\ u \neq v_0}} x_u^2 \right) + 4 \sum_{i=1}^{d-1} x_{u_i} x_{u_{i+1}} + \sum_{h \in E(R')} \sum_{\substack{w \in h \\ w \notin N(u_0)}} \sum_{\substack{pq \in \Gamma(w) \\ pq \notin \Gamma(u_0)}} x_p x_q \\
&\leq 2\lambda(z' + 1) + 4\lambda x_{u_0}^2 + \sum_{h \in E(R')} \sum_{\substack{w \in h \\ w \notin N(u_0)}} \sum_{\substack{pq \in \Gamma(w) \\ pq \notin \Gamma(u_0)}} x_p x_q. \qquad (5.12)
\end{aligned}$$

We bound $x_p x_q$ by $x_{u_0}^2$ if neither p nor q is equal to v_0; else by x_{u_0}. So again it's important to bound the multiplicities of the terms $x_p x_q$ in the summation above. For convenience, let E'' be the collection of edges $pq \in E(G)$ with $x_p x_q$ appearing in the summation above.

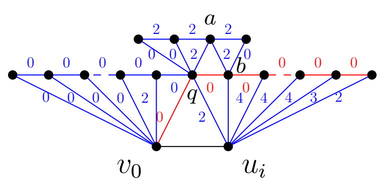


Figure 5.5: Average multiplicity of edges in $E'\backslash\Gamma(u_0)$

It's easy to see from Figure 5.4 that due to outerplanarity of G the multiplicity of each $pq \in E''$ is at most 6. Thus by eigenequation on v_0, we can bound the sum of all $x_p x_q$ (including multiplicities) for which p, q forms a hyperedge together with v_0:

$$\sum_{h \in E(R')} \sum_{\substack{w \in h \\ w \notin N(u_0)}} \sum_{\substack{pq \in \Gamma(w) \cap \Gamma(v_0) \\ pq \notin \Gamma(u_0)}} x_p x_q \leq 6\lambda v_0^2 = 6\lambda.$$

Moreover, note that $vw \notin \Gamma(v_0)$ for all edges $vw \in E(R')$. It easily follows from the outerplanarity of G that there are at most 2 edges $pq \in E''$ for which the term $x_p x_q$ contains x_{v_0} (otherwise we will see a $K_{2,3}$ minor). Hence there are at most $O(1)$ terms $x_p x_q$ (including multiplicities) containing x_{v_0} in the sums in (5.12). We bound each such term $x_p x_q$ by $x_{u_0} x_{v_0} = x_{u_0}$.

As a result, there are at most $(E(G) - (2d(v_0) - 1) + O(1)) = O(n^{5/6})$ edges in E'' that is not incident to v_0 and not in $\Gamma(v_0)$. For such edges pq, we bound $x_p x_q$ by $x_{u_0}^2$. Analogous to (5.7), we have the following inequality:

$$\begin{aligned} 2\lambda R' &\leq 2\lambda(z' + 1) + 6\lambda + O(x_{u_0}) + O(n^{5/6})x_{u_0}^2 \\ &\leq 2\lambda z' + 8\lambda + O(x_{u_0}) + O(n^{5/6})x_{u_0}^2. \end{aligned} \tag{5.13}$$

Substituting (5.13) into (5.11) and use the fact that $z' = \sum_{u \in N_G(u_0), u \neq v_0} x_u^2 \leq d' x_{u_0}^2$, we have

$$\begin{aligned} (4n - 16)x_{u_0}^2 &\leq \lambda^2(z' + 2x_{u_0}) \\ &\leq 4d' x_{u_0}^2 + \left(2\lambda z' + 8\lambda + O(x_{u_0}) + O(n^{5/6})x_{u_0}^2\right) + (2\lambda^2 + 4\lambda)x_{u_0} \\ &\leq 2\lambda z' + O(n^{5/6})x_{u_0}^2 + 8\lambda + (2\lambda^2 + 4\lambda + O(1))x_{u_0}. \end{aligned} \tag{5.14}$$

Rearranging the inequality in (5.14), we first obtain an upper bound on z':

$$z' \leq \frac{O(n^{5/6})x_{u_0}^2 + (4\lambda + O(1))x_{u_0} + 8\lambda}{\lambda^2 - 2\lambda} = O\left(n^{1/6} x_{u_0}^2 + \frac{4x_{u_0}}{\lambda} + \frac{1}{\lambda}\right).$$

Now using the upper bound on z' and (5.14), we have the following inequality:

$$\begin{aligned} (4n - 16)x_{u_0}^2 &\leq 2\lambda(z' + 1) + O(n^{5/6})x_{u_0}^2 + 8\lambda + (2\lambda^2 + 4\lambda + O(1))x_{u_0} \\ &= O\left(n^{5/6} x_{u_0}^2 + \lambda^2 x_{u_0} + \lambda\right). \end{aligned}$$

It follows from the fact that $\lambda = O(n^{1/3})$ that

$$x_{u_0} = O(n^{-1/3}).$$

Now use the bound $x_{u_0} = O(n^{-1/3})$ in (5.7), we obtain a better bound of $d = d_G(v_0)$ in Claim 5.2.1:

$$4n - 16 \leq \lambda^3 \leq 4d + 2\lambda z + 4\lambda + 4\sqrt{dz} + \left(4(n - d) \cdot O((n^{-1/3})^2)\right), \tag{5.15}$$

which gives us

$$d \geq n - O(n^{2/3}).$$

This completes the proof of Lemma 5.2.1.

Lemma 5.2.3. $d_{\mathcal{H}}(v_1) = 1$. *Moreover,* $x_{v_2} \geq x_{v_1}$.

Proof. Assume for the sake of contradiction that $d_{\mathcal{H}}(v_1) \geq 2$. We claim there must exist another hyperedge $\{v_1, v_2, t\}$ such that $t \neq v_0$. Suppose not, then there exists $w_1, w_2, \cdots, w_s \notin N(v_0)$ (for some s) such that $w_i w_{i+1} v_1 \in E(\mathcal{H})$ for $i \in [s-1]$ and $w_s v_1 v_2 \notin E(\mathcal{H})$. However, it's easy to see that if we add the hyperedge $w_s v_1 v_2$ into $\mathcal{H}$, the resulting hypergraph is still outerplanar, which contradicts that $\mathcal{H}$ attains the maximum spectral radius and is edge-maximal. Hence there must exist some vertex t such that $\{v_2, v_2, t\}$ is a hyperedge.

Consider now the hypergraph $\mathcal{H}'$ obtained from $\mathcal{H}$ by by removing the hyperedge $\{v_1, v_2, t\}$, adding the hyperedge $\{v_1, v_0, t\}$, and if needed replacing some hyperedges $h = \{v_2, u, w\}$ to $\{v_0, u, w\}$ to maintain the outerplanarity. Suppose $\boldsymbol{x}$ is the Perron-Frobenius eigenvector of $\mathcal{H}$. Then it is not hard to see that

$$\sum_{\{i_1,i_2,i_3\} \in E(\mathcal{H}')} x_{i_1} x_{i_2} x_{i_3} - \sum_{\{i_1,i_2,i_3\} \in E(\mathcal{H})} x_{i_1} x_{i_2} x_{i_3} \geq x_{v_1} x_t (x_{v_0} - x_{v_2}) > 0.$$

This implies that $\lambda(\mathcal{H}') > \lambda(\mathcal{H})$, which contradicts that $\lambda(\mathcal{H})$ is the extremal hypergraph of maximum spectral radius.

It remains to show that $x_{v_2} \geq x_{v_1}$. If $x_{v_2} < x_{v_1}$, then let $\boldsymbol{x}'$ be obtained from $\boldsymbol{x}$ by setting $x'_{v_1} = x_{v_2}$, $x'_{v_2} = x_{v_1}$ and every other entry the same. Since $d_{\mathcal{H}}(v_1) = 1$, it follows that $P_{\mathcal{H}}(\boldsymbol{x}') > P_{\mathcal{H}}(\boldsymbol{x})$, which contradicts that $\boldsymbol{x}$ is the Perron-Frobenius eigenvector of $\mathcal{H}$.

$\square$

Proof of Theorem 5.1.1. Let $\mathcal{H}$ be an outerplanar 3-uniform hypergraph on n vertices with maximum spectral radius. Let G be the shadow of $\mathcal{H}$. Suppose the Perron–Frobenius eigenvector $\boldsymbol{x}$ of the adjacency matrix of $\mathcal{H}$ is normalized so that the maximum eigenvector entry is 1. Let v_0 be the vertex with the maximum eigenvector entry and $\{v_1, v_2, \cdots, v_d\}$ be the neighbors of v_0 in the clockwise order of the planar drawing of G.

By Lemma 5.2.1, we have that $d(v_0) \geq n - O(n^{2/3})$ and for every other vertex $u \neq v_0$, $x_u = O(n^{-1/3})$. Now we claim that $x_{v_1} = \Omega(n^{-1/3})$. By Lemma 5.2.3, we have that $d_{\mathcal{H}}(v_1) = 1$, i.e., $v_1 v_2 v_0$ is the unique hyperedge containing v_1. It follows by Lemma 5.2.3 and the eigenequation on v_1 that

$$\lambda x_{v_1}^2 = x_{v_0} x_{v_2} = x_{v_2} \geq x_{v_1}.$$

Together with (5.10), this implies that

$$x_{v_1} \geq \frac{1}{\lambda} = \Omega(n^{-1/3}).$$

Now we claim that for every vertex $u \in V(G) \backslash \{v_0\}$, u is a neighbor of v_0 in G. Suppose not, then it follows from the outerplanarity of G that there exists some vertex w not adjacent to v_0 such that w is contained in a unique hyperedge $\{w, s, t\}$ $(s, t \neq v_0)$. Now similar to Lemma 5.2.3, consider the hypergraph $\mathcal{H}'$ obtained from $\mathcal{H}$ by by removing the hyperedge $\{w, s, t\}$ and adding the hyperedge $\{w, v_0, v_1\}$. It follows that

$$\sum_{\{i_1,i_2,i_3\} \in E(\mathcal{H}')} x_{i_1} x_{i_2} x_{i_3} - \sum_{\{i_1,i_2,i_3\} \in E(\mathcal{H})} x_{i_1} x_{i_2} x_{i_3} \geq x_w x_{v_0} x_{v_1} - x_w x_s x_t.$$

Note that $x_s x_t = O(n^{-2/3})$ while $x_{v_0} x_{v_1} = \Omega(n^{-1/3})$. It follows that $x_w x_{v_0} x_{v_1} > x_w x_s x_t$, which implies that $\lambda(\mathcal{H}') > \lambda(\mathcal{H})$, contradicting that $\mathcal{H}$ is the extremal hypergraph of maximum spectral radius. Hence by contradiction, every vertex $u \in V(G) \backslash \{v_0\}$ is a neighbor of v_0 in G. Again by the fact that $\mathcal{H}$ attains the maximum spectral radius, it follows that $\mathcal{H}$ is the unique 3-uniform hypergraph with $K_1 + P_{n-1}$ as it shadow. $\square$

Bibliography

[1] R. Adamczak and P. Wolff. "Concentration inequalities for non-Lipschitz functions with bounded derivatives of higher order". In: *Probab. Theory Related Fields* 162.3-4 (2015), pp. 531–586.

[2] S. Akbari and A. Alipour. "Multicolored trees in complete graphs". In: *J. Graph Theory* 54 (2007), pp. 221–232.

[3] M. H. Albert, M. D. Atkinson, D. Nussbaum, J. Sack, and N. Santoro. "On the longest increasing subsequence of a circular list". In: *Inform. Process. Lett.* 101.2 (2007), pp. 55–59.

[4] M. Albert, A. Frieze, and B. Reed. "Multicoloured Hamilton cycles". In: *Electron. J. Combin.* 2 (1995), Research Paper 10, approx. 13.

[5] N. Alon and F. R. K. Chung. "Explicit construction of linear sized tolerant networks". In: *Discrete Math.* 72 (1988), pp. 15–19.

[6] N. Alon and C. Shikhelman. "Many T copies in H-free graphs". In: *J. Combin. Theory Ser. B* 121 (2016), pp. 146–172.

[7] N. Alon and J. H. Spencer. *The probabilistic method*. Third. Wiley-Interscience Series in Discrete Mathematics and Optimization. With an appendix on the life and work of Paul Erdős. John Wiley & Sons, Inc., Hoboken, NJ, 2008, pp. xviii+352. ISBN: 978-0-470-17020-5.

[8] R. P. Anstee and S. Salazar. "Forbidden Berge hypergraphs". In: *Electron. J. Combin.* 24.1 (2017), Paper 1.59, 21.

[9] J. Aube, E.L. Stitzinger, S.B. Suanmali, and L.M. Zack. *Elementary Applications of Young diagrams and Young tableaux*. 2007. eprint: `http://math.highpoint.edu/\%7Elzack/ModuleChapter1-5.pdf`.

[10] M. Axenovich and A. Gyárfás. "A note on Ramsey numbers for Berge-G hypergraphs". In: *Discrete Math.* 342.5 (2019), pp. 1245–1252.

[11] K. Azuma. "Weighted sums of certain dependent random variables". In: *Tohoku Math. J. (2)* 19 (1967), pp. 357–367.

[12] D. Bakry and M. Émery. "Diffusions hypercontractives". In: *Séminaire de probabilités, XIX, 1983/84*. Vol. 1123. Lecture Notes in Math. Springer, Berlin, 1985, pp. 177–206.

[13] J. Beck. "On size Ramsey number of paths, trees, and circuits. I". In: *J. Graph Theory* 7.1 (1983), pp. 115–129.

[14] C. Berge. *Graphs and hypergraphs*. revised. Translated from the French by Edward Minieka, North-Holland Mathematical Library, Vol. 6. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York., 1976, ix+528pp.

[15] J.-C. Bermond, A. Germa, M.-C. Heydemann, and D. Sotteau. "Hypergraphes hamiltoniens". In: *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976)*. Vol. 260. Colloq. Internat. CNRS. CNRS, Paris, 1978, pp. 39–43.

[16] A. Bialostocki and W. Voxman. "On the anti-Ramsey numbers for spanning trees". In: *Bull. Inst. Combin. Appl.* 32 (2001), pp. 23–26.

[17] H. Bielak and I. Gorgol. "On size Ramsey numbers for certain hypergraphs". In: *Discuss. Math.* 13 (1993), pp. 5–14.

[18] M. Bóna. "On three different notions of monotone subsequences". In: *Permutation patterns*. Vol. 376. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2010, pp. 89–114.

[19] M. Bóna. *The copies of any permutation pattern are asymptotically normal*. 2007. eprint: `arXiv:0712.2792`.

[20] B. N. Boots and G. F. Royle. "A conjecture on the maximum value of the principal eigenvalue of a planar graph". In: *Geographical analysis* 23.3 (1991), pp. 276–282.

[21] J. Böttcher, Y. Kohayakawa, and A. Procacci. "Properly coloured copies and rainbow copies of large graphs with small maximum degree". In: *Random Structures Algorithms* 40.4 (2012), pp. 425–436.

[22] H. Broersma and X. Li. "Spanning trees with many or few colors in edge-colored graphs". In: *Discuss. Math. Graph Theory* 17.2 (1997), pp. 259–269.

[23] D. Cao and A. Vince. “The spectral radius of a planar graph”. In: *Linear Algebra Appl.* 187 (1993), pp. 251–257.

[24] J. M. Carraher and S. G. Hartke. “Eulerian circuits with no monochromatic transitions in edge-colored digraphs with all vertices of outdegree three”. In: *SIAM J. Discrete Math.* 31.1 (2017), pp. 190–209.

[25] J. Carraher, S. Hartke, and P. Horn. “Edge-disjoint rainbow spanning trees in complete graphs”. In: *European J. Combin.* 57 (2016), pp. 71–84.

[26] Y. Chang. “The existence of resolvable BIBD with k even and $\lambda = 1$”. In: *Discrete Math.* 218.1-3 (2000), pp. 9–23.

[27] S. Chatterjee. “The missing log in large deviations for triangle counts”. In: *Random Structures Algorithms* 40.4 (2012), pp. 437–451.

[28] H. Chernoff. “A note on an inequality involving the normal distribution”. In: *Ann. Probab.* 9.3 (1981), pp. 533–535.

[29] F. Chung. “Spanning trees in subgraphs of lattices”. In: *Applications of curves over finite fields (Seattle, WA, 1997)*. Vol. 245. Contemp. Math. Amer. Math. Soc., Providence, RI, 1999, pp. 201–219.

[30] F. R. K. Chung. *Spectral graph theory*. Vol. 92. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997, pp. xii+207. ISBN: 0-8218-0315-8.

[31] F. R. K. Chung, A. Grigor’yan, and S.-T. Yau. “Eigenvalues and diameters for manifolds and graphs”. In: *Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991)*. Int. Press, Cambridge, MA, 1997, pp. 79–105.

[32] F. R. K. Chung, A. Grigor’yan, and S.-T. Yau. “Upper bounds for eigenvalues of the discrete and continuous Laplace operators”. In: *Adv. Math.* 117.2 (1996), pp. 165–178.

[33] F. R. K. Chung and S.-T. Yau. “A combinatorial trace formula”. In: *Tsing Hua lectures on geometry & analysis (Hsinchu, 1990–1991)*. Int. Press, Cambridge, MA, 1997, pp. 107–116.

[34] F. R. K. Chung and S.-T. Yau. “A Harnack inequality for homogeneous graphs and subgraphs”. In: *Comm. Anal. Geom.* 2.4 (1994), pp. 627–640.

[35] F. R. K. Chung and S.-T. Yau. “Eigenvalues of graphs and Sobolev inequalities”. In: *Combin. Probab. Comput.* 4.1 (1995), pp. 11–25.

[36] F. R. K. Chung and S.-T. Yau. “Logarithmic Harnack inequalities”. In: *Math. Res. Lett.* 3.6 (1996), pp. 793–812.

[37] F. Chung, A. Grigor’yan, and S. T. Yau. “Higher eigenvalues and isoperimetric inequalities on Riemannian manifolds and graphs”. In: *Comm. Anal. Geom.* 8.5 (2000), pp. 969–1026.

[38] F. Chung and L. Lu. “Concentration inequalities and martingale inequalities: a survey”. In: *Internet Math.* 3.1 (2006), pp. 79–127.

[39] F. Chung and S.-T. Yau. “A Harnack inequality for Dirichlet eigenvalues”. In: *J. Graph Theory* 34.4 (2000), pp. 247–257.

[40] F. Chung and S.-T. Yau. “Coverings, heat kernels and spanning trees”. In: *Electron. J. Combin.* 6 (1999), Research Paper 12, 21.

[41] F. Chung and S.-T. Yau. “Discrete Green’s functions”. In: vol. 91. 1-2. In memory of Gian-Carlo Rota. 2000, pp. 191–214.

[42] C. Chvatál, V. Rödl, E. Szemerédi, and W. T. Trotter Jr. “The Ramsey number of a graph with bounded maximum degree”. In: *J. Combin. Theory Ser. B* 34.3 (1983), pp. 239–243.

[43] D. Conlon. “A new upper bound for diagonal Ramsey numbers”. In: *Ann. of Math. (2)* 170.2 (2009), pp. 941–960.

[44] D. Conlon, J. Fox, and B. Sudakov. “Hypergraph Ramsey numbers”. In: *J. Amer. Math. Soc.* 23.1 (2010), pp. 247–266.

[45] J. Cooper and A. Dutle. “Spectra of uniform hypergraphs”. In: *Linear Algebra Appl.* 436.9 (2012), pp. 3268–3292.

[46] M. Coulson and G. Perarnau. *A Rainbow Dirac’s Theorem.* 2018. eprint: `arXiv:1809.06392`.

[47] D. Cvetković and P. Rowlinson. “The largest eigenvalue of a graph: a survey”. In: *Linear and Multilinear Algebra* 28.1-2 (1990), pp. 3–33.

[48] A. Davoodi, E. Győri, A. Methuku, and C. Tompkins. “An Erdős-Gallai type theorem for uniform hypergraphs”. In: *European J. Combin.* 69 (2018), pp. 159–162.

[49] A. L. Delcher, S. Kasif, R. D. Fleischmann, J. Paterson, O. White, and S. L. Salzberg. “Alignment of whole genomes”. In: *Nucl. Acis Res.* 27 (1999), pp. 2369–2376.

[50] B. DeMarco and J. Kahn. "Upper tails for triangles". In: *Random Structures Algorithms* 40.4 (2012), pp. 452–459.

[51] B. Demarco and J. Kahn. "Tight upper tail bounds for cliques". In: *Random Structures Algorithms* 41.4 (2012), pp. 469–487.

[52] G. A. Dirac. "Some theorems on abstract graphs". In: *Proc. London Math. Soc. (3)* 2 (1952), pp. 69–81.

[53] H. Djellout, A. Guillin, and L. Wu. "Transportation cost-information inequalities and applications to random dynamical systems and diffusions". In: *Ann. Probab.* 32.3B (2004), pp. 2702–2732.

[54] A. Dudek, S. La Fleur, D. Mubayi, and V. Rődl. "On the size-Ramsey number of hypergraphs". In: *J. Graph Theory* 86.1 (2017), pp. 104–121.

[55] A. Dudek and P. Prałat. "An alternative proof of the linearity of the size-Ramsey number of paths". In: *Combin. Probab. Comput.* 24.3 (2015), pp. 551–555.

[56] A. Dudek and P. Prałat. "On some multicolour Ramsey properties of random graphs". In: *SIAM J. Discrete Math.* 31.3 (2017), pp. 2079–2092.

[57] J. Edmonds. "Matroid partition". In: *Mathematics of the Decision Sciences, Part I (Seminar, Stanford, Calif., 1967)*. Amer. Math. Soc., Providence, R.I., 1968, pp. 335–345.

[58] J. Edmonds. "Submodular functions, matroids, and certain polyhedra". In: *Combinatorial Structures and their Applications (Proc. Calgary Internat. Conf., Calgary, Alta., 1969)*. Gordon and Breach, New York, 1970, pp. 69–87.

[59] M. N. Ellingham and X. Zha. "The spectral radius of graphs on surfaces". In: *J. Combin. Theory Ser. B* 78.1 (2000), pp. 45–56.

[60] M. Erbar and J. Maas. "Ricci curvature of finite Markov chains via convexity of the entropy". In: *Arch. Ration. Mech. Anal.* 206.3 (2012), pp. 997–1038.

[61] M. Erbar, J. Maas, and P. Tetali. "Discrete Ricci curvature bounds for Bernoulli-Laplace and random transposition models". In: *Ann. Fac. Sci. Toulouse Math. (6)* 24.4 (2015), pp. 781–800.

[62] P. Erdős. "On the combinatorial problems which I would most like to see solved". In: *Combinatorica* 1.1 (1981), pp. 25–42.

[63] P. Erdős. “Some remarks on the theory of graphs”. In: *Bull. Amer. Math. Soc.* 53 (1947), pp. 292–294.

[64] P. Erdős, R. Faudree, C. Rousseau, and R. Schelp. “The size Ramsey number”. In: *Periodica Mathematica Hungarica* 9 (1978), pp. 145–161.

[65] P. Erdős and A. Hajnal. “On Ramsey like theorems. Problems and results”. In: *Combinatorics (Proc. Conf. Combinatorial Math., Math. Inst., Oxford, 1972).* 1972, pp. 123–140.

[66] P. Erdős, A. Hajnal, and R. Rado. “Partition relations for cardinal numbers”. In: *Acta Math. Acad. Sci. Hungar.* 16 (1965), pp. 93–196.

[67] P. Erdős and L. Lovász. “Problems and results on 3-chromatic hypergraphs and some related questions”. In: *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II.* 1975, 609–627. Colloq. Math. Soc. János Bolyai, Vol. 10.

[68] P. Erdős, J. Nešetřil, and V. Rödl. “On some problems related to partitions of edges of a graph”. In: *Graphs and other combinatorial topics (Prague, 1982).* Vol. 59. Teubner-Texte Math. Teubner, Leipzig, 1983, pp. 54–63.

[69] P. Erdős and R. Rado. “Combinatorial theorems on classifications of subsets of a given set”. In: *Proc. London Math. Soc. (3)* 2 (1952), pp. 417–439.

[70] P. Erdős and M. Simonovits. “A limit theorem in graph theory”. In: *Studia Sci. Math. Hungar.* 1 (1966), pp. 51–57.

[71] P. Erdős, M. Simonovits, and V. T. Sós. “Anti-Ramsey theorems”. In: *Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdős on his 60th birthday), Vol. II.* 1975, 633–643. Colloq. Math. Soc. János Bolyai, Vol. 10.

[72] P. Erdős and A. H. Stone. “On the structure of linear graphs”. In: *Bull. Amer. Math. Soc.* 52 (1946), pp. 1087–1091.

[73] P. Erdős and G. Szekeres. “A combinatorial problem in geometry”. In: *Compositio Math.* 2 (1935), pp. 463–470.

[74] B. Ergemlidze, E. Győri, A. Methuku, N. Salia, C. Tompkins, and O. Zamora. “Avoiding long Berge cycles: the missing cases $k = r + 1$ and $k = r + 2$”. In: *Combin. Probab. Comput.* (2019), pp. 1–13.

[75] J. S. Frame, G. de B. Robinson, and R. M. Thrall. “The hook graphs of the symmetric groups”. In: *Canad. J. Math.* 6 (1954), pp. 316–324.

[76] P. Frankl and V. Rödl. “Hypergraphs do not jump”. In: *Combinatorica* 4.2-3 (June 1984), pp. 149–159.

[77] A. Frieze and M. Krivelevich. “On rainbow trees and cycles”. In: *Electron. J. Combin.* 15.1 (2008), Research paper 59, 9.

[78] A. Frieze and B. Reed. “Polychromatic Hamilton cycles”. In: *Discrete Math.* 118.1-3 (1993), pp. 69–74.

[79] S. Fujita, C. Magnant, and K. Ozeki. “Rainbow generalizations of Ramsey theory: a survey”. In: *Graphs Combin.* 26.1 (2010), pp. 1–30.

[80] Z. Füredi, A. Kostochka, and R. Luo. “Avoiding long Berge cycles”. In: *J. Combin. Theory Ser. B* 137 (2019), pp. 55–64.

[81] Z. Füredi, A. Kostochka, and R. Luo. *Avoiding long Berge cycles II, exact bounds for all n*. 2018. eprint: `arXiv:1807.06119`.

[82] D. Gerbner. *A note on the Turán number of a Berge odd cycle*. 2019. eprint: `arXiv:1903.01002`.

[83] D. Gerbner. *On Berge-Ramsey problems*. 2019. eprint: `arXiv:1906.02465`.

[84] D. Gerbner, A. Methuku, G. Omidi, and M. Vizer. *Ramsey problems for Berge hypergraphs*. 2018. eprint: `arXiv:1808.10434`.

[85] D. Gerbner, A. Methuku, and C. Palmer. “General lemmas for Berge–Turán hypergraph problems”. In: *European J. Combin.* 86 (2020), p. 103082.

[86] D. Gerbner and C. Palmer. “Extremal results for Berge hypergraphs”. In: *SIAM J. Discrete Math.* 31.4 (2017), pp. 2314–2327.

[87] N. Gozlan, J. Melbourne, W. Perkins, C. Roberto, P-M. Samson, and P. Tetali. “Working Group in New directions in mass transport: discrete versus continuous”. In: *AIM SQuaRE report* (2013).

[88] D. Grósz, A. Methuku, and C. Tompkins. *Uniformity thresholds for the asymptotic size of extremal Berge-F-free hypergraphs*. 2018. eprint: `arXiv:1803.01953`.

[89] B. D. Guiduli. “Spectral Extrema for Graphs”. In: *Ph.D. Dissertation, University of Chicago* (1996).

[90] B. D. Guiduli and T. P. Hayes. “On the spectral radius of graphs of given genus”. In: *Preprint. Institut IMAG, Grenoble, France, and University of Chicago* (1998).

[91] A. Gyárfás. “The Turán number of Berge-K_4 in triple systems”. In: *SIAM J. Discrete Math.* 33.1 (2019), pp. 383–392.

[92] A. Gyárfás, J. Lehel, G. N. Sárközy, and R. H. Schelp. “Monochromatic Hamiltonian Berge-cycles in colored complete uniform hypergraphs”. In: *J. Combin. Theory Ser. B* 98.2 (2008), pp. 342–358.

[93] A. Gyárfás and G. N. Sárközy. “The 3-colour Ramsey number of a 3-uniform Berge cycle”. In: *Combin. Probab. Comput.* 20.1 (2011), pp. 53–71.

[94] E. Győri. “Triangle-free hypergraphs”. In: *Combin. Probab. Comput.* 15.1-2 (2006), pp. 185–191. URL: `https://doi.org/10.1017/S0963548305007108`.

[95] E. Győri, G. Y. Katona, and N. Lemons. “Hypergraph extensions of the Erdős-Gallai theorem”. In: *European J. Combin.* 58 (2016), pp. 238–246.

[96] E. Győri and N. Lemons. “3-uniform hypergraphs avoiding a given odd cycle”. In: *Combinatorica* 32.2 (2012), pp. 187–203.

[97] E. Győri and N. Lemons. “Hypergraphs with no cycle of a given length”. In: *Combin. Probab. Comput.* 21.1-2 (2012), pp. 193–201.

[98] E. Győri, N. Lemons, N. Salia, and O. Zamora. “The structure of hypergraphs without long Berge cycles”. In: *Acta Math. Univ. Comenian. (N.S.)* 88.3 (2019), pp. 761–771.

[99] R. Haas and M. Young. “The anti-Ramsey number of perfect matching”. In: *Discrete Math.* 312.5 (2012), pp. 933–937.

[100] G. Hahn. “Un jeu de colouration”. In: *Actes du Colloque de Cerisy* 12 (1980), pp. 18–18.

[101] G. Hahn and C. Thomassen. “Path and cycle sub-Ramsey numbers and an edge-colouring conjecture”. In: *Discrete Math.* 62.1 (1986), pp. 29–33.

[102] J. Han, Y. Kohayakawa, G. O. Mota, and O. Parczyk. *The size-Ramsey number of 3-uniform tight paths.* 2019. eprint: `arXiv:1907.08086`.

[103] J. Han and Y. Zhao. “Minimum codegree threshold for Hamilton ℓ-cycles in k-uniform hypergraphs”. In: *J. Combin. Theory Ser. A* 132 (2015), pp. 194–223.

[104] H. Hanani, D. K. Ray-Chaudhuri, and R. M. Wilson. "On resolvable designs". In: *Discrete Math.* 3 (1972), pp. 343–357.

[105] W. Hoeffding. "Probability inequalities for sums of bounded random variables". In: *J. Amer. Statist. Assoc.* 58 (1963), pp. 13–30.

[106] Y. Hong. "A bound on the spectral radius of graphs". In: *Linear Algebra Appl.* 108 (1988), pp. 135–139.

[107] Y. Hong. "On the spectral radius and the genus of graphs". In: *J. Combin. Theory Ser. B* 65.2 (1995), pp. 262–268.

[108] Y. Hong. "Upper bounds of the spectral radius of graphs in terms of genus". In: *J. Combin. Theory Ser. B* 74.2 (1998), pp. 153–159.

[109] S. Jahanbekam and D.B. West. "Anti-Ramsey problems for t edge-disjoint rainbow spanning subgraphs: cycles, matchings, or trees". In: *J. Graph Theory* 82.1 (2016), pp. 75–89.

[110] S. Janson, B. Nakamura, and D. Zeilberger. "On the asymptotic statistics of the number of occurrences of multiple permutation patterns". In: *J. Comb.* 6.1-2 (2015), pp. 117–143.

[111] S. Janson, K. Oleszkiewicz, and A. Ruciński. "Upper tails for subgraph counts in random graphs". In: *Israel J. Math.* 142 (2004), pp. 61–92.

[112] S. Janson and A. Ruciński. "The infamous upper tail". In: vol. 20. 3. Probabilistic methods in combinatorial optimization. 2002, pp. 317–342.

[113] T. Jiang and D. B. West. "Edge-colorings of complete graphs that avoid polychromatic trees". In: *Discrete Math.* 274.1-3 (2004), pp. 137–145.

[114] T. Johnston and L. Lu. "Turán problems on non-uniform hypergraphs". In: *Electron. J. Combin.* 21.4 (2014), Paper 4.22, 34.

[115] A. Joulin. "A new Poisson-type deviation inequality for Markov jump processes with positive Wasserstein curvature". In: *Bernoulli* 15.2 (2009), pp. 532–549.

[116] A. Joulin and Y. Ollivier. "Curvature, concentration and error estimates for Markov chain Monte Carlo". In: *Ann. Probab.* 38.6 (2010), pp. 2418–2442.

[117] G. Y. Katona and H. A. Kierstead. "Hamiltonian chains in hypergraphs". In: *J. Graph Theory* 30.3 (1999), pp. 205–212.

[118] P. Keevash. “Hypergraph Turán problems”. In: *Surveys in combinatorics 2011*. Vol. 392. London Math. Soc. Lecture Note Ser. Cambridge Univ. Press, Cambridge, 2011, pp. 83–139.

[119] J. H. Kim and V. H. Vu. “Concentration of multivariate polynomials and its applications”. In: *Combinatorica* 20 (2000), pp. 417–434.

[120] T. P. Kirkman. “On a problem in combinations”. In: *Cambridge and Dublin Math. J.* 2 (1847), pp. 191–204.

[121] D. E. Knuth. *The art of computer programming. Vol. 3.* Sorting and searching, Second edition [of MR0445948]. Addison-Wesley, Reading, MA, 1998, pp. xiv+780. ISBN: 0-201-89685-0.

[122] J. Komlós and M. Simonovits. “Szemerédi’s regularity lemma and its applications in graph theory”. In: *Combinatorics, Paul Erdős is eighty, Vol. 2 (Keszthely, 1993).* Vol. 2. Bolyai Soc. Math. Stud. János Bolyai Math. Soc., Budapest, 1996, pp. 295–352.

[123] A. Kostochka and R. Luo. “On r-uniform hypergraphs with circumference less than r”. In: *Discrete Applied Mathematics* (Aug. 2019).

[124] A. Kostochka, R. Luo, and D. Zirlin. *Super-pancyclic hypergraphs and bipartite graphs.* 2019. eprint: `arXiv:1905.03758`.

[125] T. Kövari, V. T. Sós, and P. Turán. “On a problem of K. Zarankiewicz”. In: *Colloq. Math.* 3 (1954), pp. 50–57.

[126] D. Kühn, R. Mycroft, and D. Osthus. “Hamilton ℓ-cycles in uniform hypergraphs”. In: *J. Combin. Theory Ser. A* 117.7 (2010), pp. 910–927.

[127] S. Letzter. “Path Ramsey number for random graphs”. In: *Combin. Probab. Comput.* 25.4 (2016), pp. 612–622.

[128] H. Lin and B. Ning. *The maximum spectral radius of outerplanar graphs.* 2019. eprint: `arXiv:1912.11627`.

[129] Y. Lin, L. Lu, and S.-T. Yau. “Ricci curvature of graphs”. In: *Tohoku Math. J. (2)* 63.4 (2011), pp. 605–627.

[130] Y. Lin and S.-T. Yau. “Ricci curvature and eigenvalue estimate on locally finite graphs”. In: *Math. Res. Lett.* 17.2 (2010), pp. 343–356.

[131] J. Lott and C. Villani. “Ricci curvature for metric-measure spaces via optimal transport”. In: *Ann. of Math. (2)* 169.3 (2009), pp. 903–991.

[132] L. Lu and Z. Wang. *On Hamiltonian Berge cycles in* 3*-uniform hypergraphs.* 2019. eprint: `arXiv:1901.06042`.

[133] M. Maamoun and H. Meyniel. “On a problem of G. Hahn about coloured Hamiltonian paths in K_{2n}”. In: *Discrete Math.* 51.2 (1984), pp. 213–214.

[134] J. Maas. “Gradient flows of the entropy for finite Markov chains”. In: *J. Funct. Anal.* 261.8 (2011), pp. 2250–2292.

[135] L. Maherani and M. Shahsiah. “Turán numbers of complete 3-uniform Berge-hypergraphs”. In: *Graphs Combin.* 34.4 (2018), pp. 619–632.

[136] K. Markström and A. Ruciński. “Perfect matchings (and Hamilton cycles) in hypergraphs with large degrees”. In: *European J. Combin.* 32.5 (2011), pp. 677–687.

[137] A. Mielke. “Geodesic convexity of the relative entropy in reversible Markov chains”. In: *Calc. Var. Partial Differential Equations* 48.1-2 (2013), pp. 1–31.

[138] J. J. Montellano-Ballesteros and V. Neumann-Lara. “An anti-Ramsey theorem on cycles”. In: *Graphs Combin.* 21.3 (2005), pp. 343–354.

[139] T. S. Motzkin and E. G. Straus. “Maxima for graphs and a new proof of a theorem of Turán”. In: *Canadian J. Math.* 17 (1965), pp. 533–540.

[140] D. Mubayi and A. Suk. “A survey of hypergraph Ramsey problems”. In: *Discrete Mathematics and Applications (edited by A. Raigorodskii and M. T. Rassias).* Springer, to appear.

[141] C. St. J. A. Nash-Williams. “An application of matroids to graph theory”. In: *Theory of Graphs - International Symposium (Rome, 1966; P. Rosenstiehl, ed.)* Gordon and Breach, New York, 1967, pp. 263–265.

[142] C. St. J. A. Nash-Williams. “Edge-disjoint spanning trees of finite graphs”. In: *J. London Math. Soc.* 36 (1961), pp. 445–450.

[143] Y. Ollivier. “Ricci curvature of Markov chains on metric spaces”. In: *J. Funct. Anal.* 256.3 (2009), pp. 810–864.

[144] C. Palmer, M. Tait, C. Timmons, and A. Wagner. “Turán numbers for Berge-hypergraphs and related extremal problems”. In: *Discrete Math.* 342.6 (2019), pp. 1553–1563.

[145] D. Pálvölgyi. *Exponential lower bound for Berge-Ramsey problems.* 2019. eprint: `arXiv:1906.04288`.

[146] F. P. Ramsey. “On a Problem of Formal Logic”. In: *Proc. London Math. Soc. (2)* 30.4 (1929), pp. 264–286.

[147] D. K. Ray-Chaudhuri and R. M. Wilson. “Solution of Kirkman’s schoolgirl problem”. In: *Combinatorics (Proc. Sympos. Pure Math., Vol. XIX, Univ. California, Los Angeles, Calif., 1968).* 1971, pp. 187–203.

[148] D. K. Ray-Chaudhuri and R. M. Wilson. “The existence of resolvable block designs”. In: *Survey of combinatorial theory (Proc. Internat. Sympos., Colorado State Univ., Fort Collins, Colo., 1971).* 1973, pp. 361–375.

[149] V. Rödl, A. Ruciński, and E. Szemerédi. “A Dirac-type theorem for 3-uniform hypergraphs”. In: *Combin. Probab. Comput.* 15.1-2 (2006), pp. 229–251.

[150] V. Rödl, A. Ruciński, and E. Szemerédi. “An approximate Dirac-type theorem for k-uniform hypergraphs”. In: *Combinatorica* 28.2 (2008), pp. 229–260.

[151] V. Rödl, A. Ruciński, and E. Szemerédi. “Dirac-type conditions for Hamiltonian paths and cycles in 3-uniform hypergraphs”. In: *Adv. Math.* 227.3 (2011), pp. 1225–1299.

[152] D. Romik. “Permutations with short monotone subsequences”. In: *Adv. in Appl. Math.* 37.4 (2006), pp. 501–510.

[153] P. Rowlinson. “On the index of certain outerplanar graphs”. In: vol. 29. C. Twelfth British Combinatorial Conference (Norwich, 1989). 1990, pp. 221–225.

[154] N. Salia, C. Tompkins, Z. Wang, and O. Zamora. “Ramsey numbers of Berge-hypergraphs and related structures”. In: *Electron. J. Combin.* 26.4 (2019), Paper 4.40, 25.

[155] M. D. Sammer. “Aspects of mass transportation in discrete concentration inequalities”. In: *Ph.D. Dissertation, Georgia Institute of Technology* (2005).

[156] N. Sauer. “On the density of families of sets”. In: *J. Combinatorial Theory Ser. A* 13 (1972), pp. 145–147.

[157] A. Schrijver. *Combinatorial optimization. Polyhedra and efficiency. Vol. B.* Vol. 24. Algorithms and Combinatorics. Matroids, trees, stable sets, Chapters 39–69. Springer-Verlag, Berlin, 2003, i–xxxiv and 649–1217.

[158] A. J. Schwenk and R. J. Wilson. “On the eigenvalues of a graph”. In: *Selected Topics in Graph Theory (L. W. Beineke and R. J. Wilson, Eds.)* Academic Press, London, 1978. Chap. 11, pp. 307–336.

[159] S. Shelah. "A combinatorial problem; stability and order for models and theories in infinitary languages". In: *Pacific J. Math.* 41 (1972), pp. 247–261.

[160] J. Spencer. "Asymptotic lower bounds for Ramsey functions". In: *Discrete Math.* 20.1 (1977), pp. 69–76.

[161] J. Spencer. "Ramsey's theorem—a new lower bound". In: *J. Combinatorial Theory Ser. A* 18 (1975), pp. 108–115.

[162] R. P. Stanley. *Enumerative combinatorics. Vol. 2.* Vol. 62. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 1999, pp. xii+581. ISBN: 0-521-56069-1; 0-521-78987-7.

[163] K. Sturm. "On the geometry of metric measure spaces. I & II". In: *Acta Math.* 196.1 (2006), pp. 65–131, 133–177.

[164] K. Suzuki. "A necessary and sufficient condition for the existence of a heterochromatic spanning tree in a graph". In: *Graphs Combin.* 22.2 (2006), pp. 261–269.

[165] E. Szemerédi. "Regular partitions of graphs". In: *Problèmes combinatoires et théorie des graphes (Colloq. Internat. CNRS, Univ. Orsay, Orsay, 1976).* Vol. 260. Colloq. Internat. CNRS. CNRS, Paris, 1978, pp. 399–401.

[166] M. Tait and J. Tobin. "Three conjectures in extremal spectral graph theory". In: *J. Combin. Theory Ser. B* 126 (2017), pp. 137–161.

[167] J. Talbot. "Lagrangians of hypergraphs". In: *Combin. Probab. Comput.* 11.2 (2002), pp. 199–216.

[168] A. Treglown and Y. Zhao. "Exact minimum degree thresholds for perfect matchings in uniform hypergraphs". In: *J. Combin. Theory Ser. A* 119.7 (2012), pp. 1500–1522.

[169] W. T. Tutte. "On the problem of decomposing a graph into n connected factors". In: *J. London Math. Soc.* 36 (1961), pp. 221–230.

[170] V. N. Vapnik and A. Ya. Chervonenkis. "On the uniform convergence of relative frequencies of events to their probabilities". In: *Measures of complexity.* Reprint of Theor. Probability Appl. **1**6 (1971), 264–280. Springer, Cham, 2015, pp. 11–30.

[171] Y. Zhao. "Recent advances on Dirac-type problems for hypergraphs". In: *Recent trends in combinatorics.* Vol. 159. IMA Vol. Math. Appl. Springer, [Cham], 2016, pp. 145–165.