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# Windows and Generalized Drinfeld Kernels

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# WINDOWS AND GENERALIZED DRINFELD KERNELS

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# DEDICATION

I dedicate this work to Tillye

# Acknowledgments

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# Abstract

We develop a generalization of a construction of Drinfeld, first inspired by the $Q$-construction of Ballard, Diemer, and Favero. We use this construction to provide kernels for Grassmann flops over an arbitrary field of characteristic zero. In the case of Grassmann flops this generalization recovers the kernel for a Fourier-Mukai functor on the derived category of the associated global quotient stack studied by Buchweitz, Leuschke, and Van den Bergh. We show an idempotent property for this kernel, which after restriction, induces a derived equivalence over any twisted form of a Grassmann flop.

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# CHAPTER 1

# BACKGROUND AND INTRODUCTION

## 1.1 INTRODUCTION

Algebraic geometry, since the 1800's, has concerned itself with curves, surfaces, and other geometric objects which are carved out by polynomial equations, known as varieties. One inescapable concept in geometry is the remarkable structure of symmetries which arise from group actions. These symmetries known best as orbits, are employed by more than just mathematicians as a curious field of study but also by physicists, who use this mechanism to describe the motion of celestial bodies, their namesake, and even in quantum field theory, general relativity, and geometric PDE's with gauge groups. These orbits are often described using a modern generalization of a variety, or an Artin stack.

To better understand the intrinsic properties of orbits and stacks mathematicians take advantage of a diverse collection of tools. One of these tools, first thought to be a simple book keeping device for algebraic homology, has now come to be a powerful instrument in studying the hidden geometric properties of a space: the derived category of coherent sheaves. By lending the impressive machinery of homological algebra to the subject, it has been able to explain old phenomena, and has opened up previously unexpected directions of research, including exciting conjectures such as the homological mirror symmetry conjecture by Kontsevich 1995. The study of group action through the lens of algebraic geometry is known as Geometric Invariant Theory (GIT). Since the invention of GIT, it has been understood that the quotient it constructs is not entirely canonical, but depends on a choice: the choice of a linearization of the group action. This dependency, first thought to be a deficiency in the theory, has now been firmly established as a extraordinary feature for constructing new birational models of these quotients. These linearizations which parameterize the different quotients are built from an appropriate group action on an ample line bundle. It has become well established for normal projective varieties with the work of Dolgachev and Hu 1998 or Thaddeus 1996 that the ample cone is split into a finite number of convex subsets, known as chambers, and as one varies the line bundle from one chamber to an adjacent chamber (wall crossing) then the corresponding quotients are related by a birational transformation which is similar to a log flip. The study of these quotients while varying the line bundle is known as variations of geometric invariant theory (VGIT). What is truly remarkable is that conversely any birational map between smooth and projective varieties can be obtained through such GIT variations.

More recently ties to rationality and the derived category of coherent sheaves has been established, for example Kuznetsov 2016, Bernardara, Bolognesi, and Faenzi 2016, as well as Auel and Bernardara 2018. While VGIT has been the study of great interest it is a surprise that we know much less about the behavior of the corresponding derived categories as one varies the linearization, leading us to a natural question, "What happens to associated derived categories as one varies the chambers?" By focusing on VGIT it is the objective of this research to develop a collection of powerful tools which would allow researchers to better understand the derived categories behind birational equivalences via wall crossings.

While there are many interesting reasons to study the derived categories of quotients, this work is motivated by the following conjecture of A. Bondal and D. Orlov 1995 extended by Kawamata 2002.

**Conjecture** (Bondal-Orlov 1995)**.** *Assume that $Z$ and $Z'$ are smooth complex varieties. If $Z$ and $Z'$ are related by a flop, then there is a $\mathbb{C}$-linear triangulated equivalence of their bounded derived categories of coherent sheaves*

$$\mathsf{D}^{\mathsf{b}}(Z) \cong \mathsf{D}^{\mathsf{b}}(Z').$$

There has been major progress in this work due to M. Ballard, Diemer, and D. Favero 2017 who propose a kernel of an integral transform, the so called $Q$-construction, which should realize this conjecture. Given a variety $X$ with a $\mathbb{G}_m$-action, the authors constructed an idempotent kernel on the equivariant derived category $\mathsf{D}([X/\mathbb{G}_m])$. The kernel $Q$, being the identity on its essential image, fully-faithfully identifies an interesting component of the derived category $\mathsf{D}([X/\mathbb{G}_m])$. In fact, it always gives a two-term semi-orthogonal decomposition. This construction has some natural extensions.

Following Drinfeld 2013, we can recognize it as a piece of more general story. The inclusion $\mathbb{G}_m \subset \mathbb{A}^1$ can be viewed as a partial monodial compactification of $\mathbb{G}_m$. The fibers of the multiplication map $\mathbb{A}^1 \times \mathbb{A}^1 \to \mathbb{A}^1$ are a family of $\mathbb{G}_m$ orbits which degenerate over $0 \in \mathbb{A}^1$. From Drinfeld's perspective, the idempotent kernel constructed in M. Ballard, Diemer, and D. Favero 2017 is the structure sheaf of a variety that parameterizes such degenerations in $X$.

This viewpoint allows for an immediate generalization: we can replace $\mathbb{A}^1$ with $M$ where $M$ is a monoidal scheme. If we have a variety with an action of the units of $M$, we can produce a kernel for $X$. In this work we examine the case of the group $\mathrm{GL}(V)$ and the monoidal scheme $\mathrm{End}(V)$ for $V$ a finite dimensional vector space and the natural action of the units $\mathrm{GL}(V)$ on the vector space $\mathrm{Hom}(V, W) \times \mathrm{Hom}(W, V)$ for $W$ another finite dimesional vector space.

We show that the idempotent property still holds:

**Theorem** (A)**.** *There exists a morphism of kernels $Q \to \Delta$ inducing an isomorphism of $Q \circ Q \to Q$, where $\circ$ denotes convolution of kernels and $\Delta$ is the kernel of the identity.*

Further, using an approach similar to that in M. Ballard, Diemer, and D. Favero 2017 we show that our new construction for $Q$ realizes the derived equivalence for the Grassmann flops of Donovan and Segal 2014. That is for the two GIT quotients $X^+$ and $X^-$, with $\iota^{\pm} : X^{\pm} \hookrightarrow X$, and denoting $\mathcal{Q} := (\iota^+ \times \iota^-)^* Q$ we have the following main theorem.

**Theorem** (B)**.** *The following integral transform is an equivalence of categories*

$$\Phi_{\mathcal{Q}} : \mathsf{D}^{\mathrm{b}}\left(X^+\right) \to \mathsf{D}^{\mathrm{b}}\left(X^-\right).$$

We have another major theorem which shows an even tighter connection between the work of Donovan and Segal 2014 and our $Q$ construction. If we denote Ҡ as the exceptional collection developed by M. M. Kapranov 1988 for the Grassmannians.

**Theorem** (C)**.** *If we denote $\mathcal{Q}^+ := (\iota^+ \times \mathbb{1}_X)^* Q$ then we have*

$$\operatorname{Im}(\Phi_{\mathcal{Q}^+}) \cong \text{Ҡ}$$

*where* Im *is the essential image.*

This result is shown using techniques similar to a method developed by Kempf, see e.g. Weyman 2003.

The structure of the thesis is as follows. The remainder of this chapter is dedicated to background material including triangulated categories and Young's diagrams. In the second chapter we deliver the main example of the standard flop, that is the case covered in M. Ballard, Diemer, and D. Favero 2017. In the final chapter we consider our most general case and handle our main theorems.

### 1.1.1 Notational Conventions

All rings are commutative with unit, we will consider most objects over a fixed field $\Bbbk$ with. We denote the category of $\mathbb{Z}^n$-graded $\Bbbk$-algebras with graded morphisms of degree 0 as $\mathsf{Gr}_k^n$, and denote the category of affine schemes over $\Bbbk$ with a $\mathbb{G}_m^n$-action as $\mathsf{AffSch}_k^{\mathbb{G}_m^n}$, with $\mathbb{G}_m^n$ equivariant morphisms, for the reader versed in these objects, they will quickly note that these are opposite categories. The main category of interest for us is a full subcategory of these, the varieties in $\mathsf{AffSch}_k^{\mathbb{G}_m^n}$, which we will denote as $\mathsf{AffVar}_k^{\mathbb{G}_m^n}$.

## 1.2 Algebraic Groups and Monoids

In this subsection we give the basics of algebraic groups and monoids, the interested reader can see a more detailed treatment in Springer 2010; Humphreys 1975.

**Definition 1.1.** An affine algebraic monoid (AAM) is an affine algebraic variety $M$ over $\Bbbk$ (i.e. exists a morphism $s : M \to \operatorname{Spec} \Bbbk$), with a $\Bbbk$-morphism (multiplication) $\mu : M \times_{\Bbbk} M \to M$, satisfying the condition:

**A**ssociativity:

$$\begin{tikzcd}
M \times_{\Bbbk} M \times_{\Bbbk} M \arrow[r, "\mathrm{Id}_M \times_{\Bbbk} \mu"] \arrow[d, "\mu \times_{\Bbbk} \mathrm{Id}_M"] & M \times_{\Bbbk} M \arrow[d, "\mu"] \\
M \times_{\Bbbk} M \arrow[r, "\mu"] & M
\end{tikzcd}$$

commutes.

Further, the existence of a morphism $e : \mathrm{Spec}(\Bbbk) \to G$ so that

**L**aw of Identity:

$$\begin{tikzcd}
 & \mathrm{Spec}\,\Bbbk \times_{\Bbbk} M \arrow[dr, "e\times_{\Bbbk}\mathrm{Id}_M"] & & \\
M \arrow[ur, "\cong"'] \arrow[dr, "\cong"] \arrow[rrr, bend left=60, "\mathrm{Id}_M"] \arrow[rrr, bend right=60, "\mathrm{Id}_M"'] & & G \times_{\Bbbk} G \arrow[r, "\mu"] & M \\
 & M \times_{\Bbbk} \mathrm{Spec}\,\Bbbk \arrow[ur, "\mathrm{Id}_M \times_{\Bbbk} e"'] & &
\end{tikzcd}$$

commutes.

**Definition 1.2. (**The Matrix Monoid) We define the affine algebraic monoid referred to as the the matrix monoid of dimension $n$ over $\Bbbk$, as

$$M_n(\Bbbk) := \mathrm{Spec}\left(\Bbbk[u_{11}, ..., u_{ij}]\right)$$

with the natural structure sheaf. We define $\mu : M_n(\Bbbk) \to M_n(\Bbbk)$, by the co-multiplication $\mu^\sharp : \Bbbk[u_{11}, ..., u_{nn}] \otimes_{\Bbbk} \Bbbk[u_{11}, ..., u_{nn}]$, defined as $u_{ij} \mapsto \sum_{\ell=1}^{n} u_{ij} \otimes_{\Bbbk} u_{j\ell}$.

**Lemma 1.3.** *The matrix monoid over $\Bbbk$, $M_n(\Bbbk)$, is an AAM.*

*Proof.* First, we show the associativity law, by confirming it on the side of rings, that is we will verify that the following diagram commutes:

$$\begin{tikzcd}
\Bbbk[u_{11}, ..., u_{nn}]^{\otimes_{\Bbbk} 3} & \Bbbk[u_{11}, ..., u_{nn}]^{\otimes_{\Bbbk} 2} \arrow[l, "\mathrm{Id}_{\Bbbk[u_{11},...,u_{nn}]} \otimes_{\Bbbk} \mu^\sharp"'] \\
\Bbbk[u_{11}, ..., u_{nn}]^{\otimes_{\Bbbk} 2} \arrow[u, "\mu^\sharp \otimes_{\Bbbk} \mathrm{Id}_{\Bbbk[u_{11},...,u_{nn}]}"] & \Bbbk[u_{11}, ..., u_{nn}] \arrow[l, "\mu^\sharp"] \arrow[u, "\mu^\sharp"']
\end{tikzcd}$$

Since the morphisms are all linear with respect to the $\Bbbk$-algebra structure, we will simply chase the element $u_{ij}$ in the bottom right around the square to show commutativity.

$$\text{First, we go up} \quad \mu^\sharp(u_{ij}) = \sum_{\ell=1}^{n} u_{i\ell} \otimes_\Bbbk u_{\ell j} \quad \text{and then we go left}$$

$$\sum_{\ell=1}^{n} \mathrm{Id}_{\Bbbk[u,u^{-1}]}(u_{i\ell}) \otimes_\Bbbk \mu^\sharp(u_{\ell j}) = \sum_{\ell,m=1}^{n} u_{i\ell} \otimes_\Bbbk u_{\ell m} \otimes_\Bbbk u_{mj}$$

$$\text{now, first we go left} \quad \mu^\sharp(u) = \sum_{m=1}^{n} u_{im} \otimes_\Bbbk u_{mj} \quad \text{and next we go up}$$

$$\sum_{m=1}^{n} \mu^\sharp(u_{im}) \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u_{11},\ldots,u_{nn},d]}(u_{mj}) = \sum_{\ell,m=1}^{n} u_{i\ell} \otimes_\Bbbk u_{\ell m} \otimes_\Bbbk u_{mj}$$

and hence we have the desired commutativity. □

**Definition 1.4.** A linear algebraic group (LAG) is an AAM, $G$ over $\Bbbk$, with $\Bbbk$-morphisms (multiplication) $\mu : G\times_\Bbbk G \to G$, (inverse) $\beta : G \to G$, (identity) $e : \Bbbk \to G$ satisfying the group identities:

**L**aw of Inverse:

$$\begin{tikzcd}
 & & \operatorname{Spec}\Bbbk \arrow[drr, "e"] & & \\
G \arrow[r, "\Delta"] \arrow[urr, "s"] \arrow[drr, "s"] & G\times_\Bbbk G \arrow[rr, shift left, "\mathrm{Id}_G\times_\Bbbk\beta"] \arrow[rr, shift right, "\beta\times_\Bbbk\mathrm{Id}_G"'] & & G\times_\Bbbk G \arrow[r, "\mu"] & G \\
 & & \operatorname{Spec}\Bbbk \arrow[urr, "e"] & &
\end{tikzcd}$$

commutes (where $\Delta$ is the diagonal).

Let's take a moment to recall a few of the linear algebraic groups that will be of most interest to us.

**Definition 1.5.** (The $n$-Dimensional Torus) We define the linear algebraic group called <u>the torus</u>, over $\Bbbk$, as

$$\mathbb{G}_m := \operatorname{Spec} \Bbbk[u, u^{-1}],$$

with its natural structure sheaf. We further define

$$\mu : \mathbb{G}_m \times_{\Bbbk} \mathbb{G}_m \to \mathbb{G}_m$$

by the sheaf homomorphism induced by the ring morphism, (that we will refer to as) co-multiplication $\mu^\sharp : \Bbbk[u, u^{-1}] \to \Bbbk[u, u^{-1}] \otimes_{\Bbbk} \Bbbk[u, u^{-1}]$, defined as

$$u \mapsto u \otimes_{\Bbbk} u$$

In addition, we define the inverse map $\beta : \mathbb{G}_m \to \mathbb{G}_m$ by the induced map of the co-inverse $\beta^\sharp : \Bbbk[u, u^{-1}] \to \Bbbk[u, u^{-1}]$ as $u \mapsto u^{-1}$. Also, we define the identity map, $e : \operatorname{Spec} \Bbbk \to \mathbb{G}_m$, again by the co-identity map, $e^\sharp : \Bbbk[u, u^{-1}] \to k$, defined as $u \mapsto 1$.

Moreover, we define the <u>$n$-dimensional torus</u> over $\Bbbk$, as

$$\mathbb{G}_m^n := \mathbb{G}_m \times_{\Bbbk} \mathbb{G}_m \times_{\Bbbk} \ldots \times_{\Bbbk} \mathbb{G}_m$$

where we fiber product $n$-times, with all maps defined by the induced map given by the fiber product.

**Lemma 1.6.** *The $n$-dimensional torus over $\Bbbk$, $\mathbb{G}_m^n$, is a LAG.*

*Proof.* We show it in the case where $n = 1$, the general $n$ case follows similarly. First we show the associativity law, by confirming it on the side of rings, that is we will verify that the following diagram commutes:

$$\begin{tikzcd}
\Bbbk[u, u^{-1}] \otimes_{\Bbbk} \Bbbk[u, u^{-1}] \otimes_{\Bbbk} \Bbbk[u, u^{-1}] & \Bbbk[u, u^{-1}] \otimes_{\Bbbk} \Bbbk[u, u^{-1}] \arrow[l, "\mathrm{Id}_{\Bbbk[u,u^{-1}]} \otimes_{\Bbbk} \mu^\sharp"'] \\
\Bbbk[u, u^{-1}] \otimes_{\Bbbk} \Bbbk[u, u^{-1}] \arrow[u, "\mu^\sharp \otimes_{\Bbbk} \mathrm{Id}_{\Bbbk[u,u^{-1}]}"] & \Bbbk[u, u^{-1}] \arrow[l, "\mu^\sharp"] \arrow[u, "\mu^\sharp"']
\end{tikzcd}$$

Since the morphisms are all linear with respect to the $\Bbbk$-algebra structure, we will simply chase the element $u$ in the bottom right around the square to show commutativity.

$$\begin{aligned}
\text{First, we go up} \quad \mu^\sharp(u) &= u \otimes_\Bbbk u \quad \text{and then we go left} \\
\mathrm{Id}_{\Bbbk[u,u^{-1}]}(u) \otimes_\Bbbk \mu^\sharp(u) &= u \otimes_\Bbbk u \otimes_\Bbbk u \\
\text{now, first we go left} \quad \mu^\sharp(u) &= u \otimes_\Bbbk u \quad \text{and next we go up} \\
\mu^\sharp(u) \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u,u^{-1}]}(u) &= u \otimes_\Bbbk u \otimes_\Bbbk u
\end{aligned}$$

and hence we have the desired commutativity. Next we verify the law of inverse, again by looking on the side of rings, and hence we will verify that we have the following equalities:

$$\Delta^\sharp \circ (\mathrm{Id}_{\Bbbk[u,u^{-1}]} \otimes_\Bbbk \beta^\sharp) \circ \mu^\sharp = \iota \circ e^\sharp = \Delta^\sharp \circ (\beta^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u,u^{-1}]}) \circ \mu^\sharp$$

where $\iota : \Bbbk \to \Bbbk[u, u^{-1}]$ is the natural inclusion. So, again due to the linearity we need only check where $u \in \Bbbk[u, u^{-1}]$ is sent, that is notice:

$$\begin{aligned}
\Delta^\sharp \circ (\mathrm{Id}_{\Bbbk[u,u^{-1}]} \otimes_\Bbbk \beta^\sharp) \circ \mu^\sharp(u) &= \Delta^\sharp \circ (\mathrm{Id}_{\Bbbk[u,u^{-1}]} \otimes_\Bbbk \beta^\sharp)(u \otimes_\Bbbk u) \\
&= \Delta^\sharp(u \otimes_\Bbbk u^{-1}) = 1 \\
&= \iota(1) = \iota \circ e^\sharp(u), \quad \text{and} \\
\Delta^\sharp \circ (\beta^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u,u^{-1}]}) \circ \mu^\sharp(u) &= \Delta^\sharp \circ (\beta^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u,u^{-1}]})(u \otimes_\Bbbk u) \\
&= \Delta^\sharp(u^{-1} \otimes_\Bbbk u) = 1 \\
&= \iota(1) = \iota \circ e^\sharp(u)
\end{aligned}$$

as desired. Finally, we verify the law of identity, again on the side of rings, that is we will show that

$$\pi_r \circ (e^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u,u^{-1}]}) \circ \mu^\sharp = \mathrm{Id}_{\Bbbk[u,u^{-1}]} = \pi_\ell \circ (\mathrm{Id}_{\Bbbk[u,u^{-1}]} \otimes_\Bbbk e^\sharp) \circ \mu^\sharp$$

where, $\pi_\ell : \Bbbk[u, u^{-1}] \otimes_\Bbbk \Bbbk \to \Bbbk[u, u^{-1}]$ and $\pi_r : k \otimes_\Bbbk \Bbbk[u, u^{-1}] \to \Bbbk[u, u^{-1}]$ are the canonical isomorphisms. As, has become the trend, due to the linearity we will simply look at the element $u \in \Bbbk[u, u^{-1}]$, that is notice

$$\begin{aligned}
\pi_r \circ (e^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u,u^{-1}]}) \circ \mu^\sharp(u) &= \pi_r \circ (e^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u,u^{-1}]})(u \otimes_\Bbbk u) \\
&= \pi_r(1 \otimes_\Bbbk u) \\
&= u = \mathrm{Id}_{\Bbbk[u,u^{-1}]}(u) \\
\pi_\ell \circ (\mathrm{Id}_{\Bbbk[u,u^{-1}]} \otimes_\Bbbk e^\sharp) \circ \mu^\sharp(u) &= \pi_\ell \circ (\mathrm{Id}_{\Bbbk[u,u^{-1}]} \otimes_\Bbbk e^\sharp)(u \otimes_\Bbbk u) \\
&= \pi_\ell(u \otimes_\Bbbk 1) \\
&= u = \mathrm{Id}_{\Bbbk[u,u^{-1}]}(u)
\end{aligned}$$

as desired. □

**Definition 1.7.** (The General Linear Group) We define the linear algebraic group called the general linear group of dimension $n$ over $\Bbbk$, as

$$\mathrm{GL}_n(\Bbbk) := \operatorname{Spec} \Bbbk[u_{11}, ..., u_{nn}, d]/I$$

where,

$$I = \left( \left( \sum_{\tau \in \mathfrak{S}_n} (-1)^{\mathrm{sgn}(\tau)} \prod_{i=1}^{n} u_{i\tau(i)} \right) d - 1 \right)$$

with the natural structure sheaf. We define $\mu : \mathrm{GL}_n(\Bbbk) \to \mathrm{GL}_n(\Bbbk)$, by the co-multiplication $\mu^\sharp : \Bbbk[\mathrm{GL}_n(\Bbbk)] \to \Bbbk[\mathrm{GL}_n(\Bbbk)] \otimes_\Bbbk \Bbbk[\mathrm{GL}_n(\Bbbk)]$, defined as

$$u_{ij} \mapsto \sum_{\ell=1}^{n} u_{i\ell} \otimes_\Bbbk u_{\ell j}.$$

The identity $e : \operatorname{Spec} \Bbbk \to \mathrm{GL}_n(\Bbbk)$ is defined by the co-identity,

$$e^\sharp : \Bbbk[u_{11}, ..., u_{nn}, d]/I \to \Bbbk,$$

as $u_{ij} \mapsto 1$ if $i = j$ and 0 otherwise. Finally, the inverse $\beta : \mathrm{GL}_n(\Bbbk) \to \mathrm{GL}_n(\Bbbk)$ is as well defined by the co-inverse

$$\beta^\sharp : \Bbbk[u_{11}, ..., u_{nn}, d]/I \to \Bbbk[u_{11}, ..., u_{nn}, d]/I.$$

We define $\beta^\sharp$ as follows $u_{ij} \mapsto d \cdot U^{(ij)}$ where $U^{(ij)}$ is the cofactor (i.e. the appropriate $(n-1) \times (n-1)$ minor, corrected by signs) of the generic matrix

$$\begin{pmatrix} u_{11} & \dots & u_{1n} \\ \vdots & \ddots & \vdots \\ u_{n1} & \dots & u_{nn} \end{pmatrix}$$

**Lemma 1.8.** *The general linear group over* $\Bbbk$, $\mathrm{GL}_n(\Bbbk)$, *is a LAG*

*Proof.* Since the action of $d$ must follow from the action on each $u_{ij}$, we have already verified the associativity, we begin by verifying the law of inverse, again by looking on the side of rings, and hence we will verify that we have the following equalities:

$$\Delta^\sharp \circ (\mathrm{Id}_{\Bbbk[u_{11},\dots,u_{nn},d]/I} \otimes_\Bbbk \beta^\sharp) \circ \mu^\sharp = \iota \circ e^\sharp = \Delta^\sharp \circ (\beta^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u_{11},\dots,u_{nn},d]/I}) \circ \mu^\sharp$$

where $\iota : \Bbbk \to \Bbbk[u_{11}, ..., u_{nn}, d]/I$ is the natural inclusion. So, again due to the linearity we need only check where $u_{ij} \in \Bbbk[u_{11}, ..., u_{nn}, d]/I$ is sent, that is notice:

$$\begin{aligned} \Delta^\sharp \circ (\mathrm{Id}_{\Bbbk[u_{11},\dots,u_{nn},d]/I} \otimes_\Bbbk \beta^\sharp) \circ \mu^\sharp(u_{ij}) &= \Delta^\sharp \circ (\mathrm{Id}_{\Bbbk[u_{11},\dots,u_{nn},d]/I} \otimes_\Bbbk \beta^\sharp) \left( \sum_{\ell=1}^n u_{i\ell} \otimes_\Bbbk u_{\ell j} \right) \\ &= \sum_{\ell=1}^n \Delta^\sharp \left( u_{i\ell} \otimes_\Bbbk d \cdot U^{(\ell j)} \right) \\ &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \\ &= \iota \circ e^\sharp(u_{ij}), \quad \text{and} \\ \Delta^\sharp \circ (\beta^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u_{11},\dots,u_{nn},d]/I}) \circ \mu^\sharp(u) &= \Delta^\sharp \circ (\beta^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u_{11},\dots,u_{nn},d]/I}) \left( \sum_{\ell=1}^n u_{i\ell} \otimes_\Bbbk u_{\ell j} \right) \\ &= \sum_{\ell=1}^n \Delta^\sharp (d \cdot U^{(i\ell)} \otimes_\Bbbk u_{\ell j}) \\ &= \begin{cases} 1 & \text{if } i = j \\ 0 & \text{otherwise} \end{cases} \\ &= \iota \circ e^\sharp(u_{ij}) \text{ as desired.} \end{aligned}$$

Finally, we verify the law of identity, again on the side of rings, that is we will show that

$$\pi_r \circ (e^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u_{11},...,u_{nn},d]/I}) \circ \mu^\sharp = \mathrm{Id}_{\Bbbk[u_{11},...,u_{nn},d]/I} = \pi_\ell \circ (\mathrm{Id}_{\Bbbk[u_{11},...,u_{nn},d]/I} \otimes_\Bbbk e^\sharp) \circ \mu^\sharp$$

where,

$$\pi_\ell : \Bbbk[u_{11}, ..., u_{nn}, d]/I \otimes_\Bbbk \Bbbk \to \Bbbk[u_{11}, ..., u_{nn}, d]/I$$

and

$$\pi_r : \Bbbk \otimes_\Bbbk \Bbbk[u_{11}, ..., u_{nn}, d]/I \to \Bbbk[u_{11}, ..., u_{nn}, d]/I$$

are the canonical isomorphisms. As, has become the trend, due to the linearity we will simply look at the element $u_{ij} \in \Bbbk[u_{11}, ..., u_{nn}, d]/I$, that is notice

$$\begin{aligned}
\pi_r \circ (e^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u_{11},...,u_{nn},d]/I}) \circ \mu^\sharp(u) &= \pi_r \circ (e^\sharp \otimes_\Bbbk \mathrm{Id}_{\Bbbk[u_{11},...,u_{nn},d]/I}) \left( \sum_{m=1}^{n} u_{im} \otimes_\Bbbk u_{mj} \right) \\
&= \sum_{m=1}^{n} \pi_r(e^\sharp(u_{im}) \otimes_\Bbbk u_{mj}) \\
&\pi_r(1 \otimes_\Bbbk u_{ij}) \\
&= u_{ij} = \mathrm{Id}_{\Bbbk[u_{11},...,u_{nn},d]/I}(u_{ij}) \\
\pi_\ell \circ (\mathrm{Id}_{\Bbbk[u_{11},...,u_{nn},d]/I} \otimes_\Bbbk e^\sharp) \circ \mu^\sharp(u) &= \pi_\ell \circ (\mathrm{Id}_{\Bbbk[u_{11},...,u_{nn},d]/I} \otimes_\Bbbk e^\sharp) \left( \sum_{m=1} u_{im} \otimes_\Bbbk u_{mj} \right) \\
&= \sum_{m=1}^{n} \pi_\ell(u_{im} \otimes_\Bbbk e^\sharp(u_{mj})) \\
&= \pi_\ell(u_{ij} \otimes_\Bbbk 1) \\
&= u_{ij} = \mathrm{Id}_{\Bbbk[u_{11},...,u_{nn},d]/I}(u_{ij})
\end{aligned}$$

as desired. □

**Definition 1.9.** (Group Action) A Linear algebraic group $G$ over $\mathbb{k}$ <u>acts</u> on a scheme $X$ over $\mathbb{k}$ if a $\mathbb{k}$-morphism $\sigma : G \times_{\mathbb{k}} X \to X$ is given such that:

$$\begin{tikzcd}
G \times_{\mathbb{k}} G \times_{\mathbb{k}} X \arrow[r, "\mathrm{Id}_G \times_{\mathbb{k}} \sigma"] \arrow[d, "\mu \times_{\mathbb{k}} \mathrm{Id}_X"'] & G \times_{\mathbb{k}} X \arrow[d, "\sigma"] \\
G \times_{\mathbb{k}} X \arrow[r, "\sigma^\sharp"'] & X
\end{tikzcd}$$

commutes, (where $\mu$ is the group law of $G$).

$$\begin{tikzcd}
X \cong \operatorname{Spec} \mathbb{k} \times_{\mathbb{k}} X \arrow[r, "e \times_{\mathbb{k}} \mathrm{Id}_X"] \arrow[rr, bend right, "\mathrm{Id}_X"'] & G \times_{\mathbb{k}} X \arrow[r, "\sigma"] & X
\end{tikzcd}$$

commutes (where $e$ is the identity morphism for $G$). When $\mathbb{k}$ is understood we usually just refer to such an $X$ as a $G$-scheme. We will often use the notation of group actions and denote this $G \circlearrowright X$.

**Lemma 1.10.** *There are two canonical* $\mathrm{GL}_n$ *actions on* $M_n$ *we denote the first as* $M_n^+$ *which is defined by its co-action* $\sigma_+ : \mathbb{k}[M_n] \to \mathbb{k}[M_n] \otimes_{\mathbb{k}} \mathbb{k}[\mathrm{GL}_n]$. *We also denote the second action as* $M_n^-$ *and define it as well by its co-action*

$$\sigma_- : \mathbb{k}[M_n] \to \mathbb{k}[M_n] \otimes_{\mathbb{k}} \mathbb{k}[\mathrm{GL}_n].$$

*We define the "positive" action as* $\sigma_+ := \mu^\sharp \circ \iota$, *where* $\iota : M_n \to \mathrm{GL}_n$ *is the inclusion. Finally, we define the "negative" action as* $\sigma_- := (\mathrm{Id} \otimes_{\mathbb{k}} \beta^\sharp) \circ \mu^\sharp \circ \iota$

*Proof.* Notice the positive map is well defined since $\mathrm{Im}(\mu \circ \iota) \subset \mathbb{k}[M_n] \otimes_{\mathbb{k}} \mathbb{k}[\mathrm{GL}_n]$, and since the negative map is $(\mathrm{Id} \otimes_{\mathbb{k}} \beta^\sharp) \circ \sigma_+$ we have that the negative map is also well defined. That the positive action clearly defines an action by the work done in Lemma 1.8, so we need only show that the negative action defines an action. So to start we show that the associativity condition is satisfied, as has been the trend we show that the following diagram commutes.

$$\begin{tikzcd}
\Bbbk[M_n] \otimes_{\Bbbk} \Bbbk[\mathrm{GL}_n] \otimes_{\Bbbk} \Bbbk[\mathrm{GL}_n] & \Bbbk[M_n] \otimes_{\Bbbk} \Bbbk[\mathrm{GL}_n] \arrow[l, "\sigma_{-} \otimes_{\Bbbk} \mathrm{Id}_{\Bbbk[\mathrm{GL}_n]}"'] \\
\Bbbk[M_n] \otimes_{\Bbbk} \Bbbk[\mathrm{GL}_n] \arrow[u, "\mathrm{Id}_{\Bbbk[M_n]} \otimes_{\Bbbk} \mu^{\sharp}"] & \Bbbk[M_n] \arrow[l, "\sigma_{-}"] \arrow[u, "\sigma_{-}"']
\end{tikzcd}$$

□

**Remark 1.** *Notice that geometrically the positive action corresponds to right multiplication by an invertible matrix, and the negative action corresponds to right multiplication by the inverse. This imitates the different actions on $\mathbb{A}^1$ by $\mathbb{G}_m$ used in the papers Drinfeld 2013; M. Ballard, Diemer, and D. Favero 2017 which we will use in section 2 to define one of the main objects of study for this work.*

**Definition 1.11.** **(**Equivariant Morphism of Schemes) We define the *G-equivariant morphisms of schemes* for an algebraic group $G$ between $\Bbbk$-schemes, $X$ and $Y$, both with a $G$-action, denoted $\mathrm{Hom}^G(X,Y)$, as $f \in \mathrm{Hom}(X,Y)$ such that the following commutes

$$\begin{tikzcd}
G \times_{\Bbbk} X \arrow[r, "\sigma_X"] \arrow[d, "\mathrm{Id} \times_{\Bbbk} f"'] & X \arrow[d, "f"] \\
G \times_{\Bbbk} Y \arrow[r, "\sigma_Y"'] & Y
\end{tikzcd}$$

where $\sigma_X$ is the $G$-action on $X$ and $\sigma_Y$ is the $G$-action on $Y$. For two $G$-schemes $X$ and $Y$ we denote the set of $G$-equivariant morphisms as $\mathrm{Hom}^G(X,Y)$.

For this article our main interest is with affine rings with the co-action of two dimensional torus, yet we present the most general of this result in the next lemma.

**Lemma 1.12.** *The cateogry of $\mathbb{Z}^n$-graded $\Bbbk$-algebras, and affine schemes with $\mathbb{G}_m^n$-actions are equivalent*

*Proof.* First, set out some notation that will be prevalent through most of this treatment.

For a $\mathbb{Z}^n$-graded algebra, will be denoted as the following:

$$A = \bigoplus_{(\ell_1,\ldots,\ell_n)\in\mathbb{Z}^n} A_{\ell_1,\ldots,\ell_n},$$

further for a homogenous element $a \in A_{\ell_1,..,\ell_n}$ we will denote $\ell_i := \deg_i a$. Also, we will denote $\bar{U}_n = \{u_1, u_1^{-1}, \ldots, u_n, u_n^{-1}\}$.

Now let $A$ be a $\mathbb{Z}^n$-graded $\Bbbk$-algebra, we will define an action of $\mathbb{G}_m^n$ on $\operatorname{Spec} A$. We do this by defining the co-action, that is a map

$$\sigma : A \to \Bbbk[\bar{U}_n] \otimes_{\Bbbk} A \cong A[\bar{U}_n].$$

We define this map, for $a \in A$, where $a$ is homogenous, as $\sigma(a) = a u_1^{\deg_1 a} \ldots u_n^{\deg_n a}$. Next, we will to need to show that this morphism indeed defines an action of $\mathbb{G}_m^n$ on $\operatorname{Spec} A$. That is we need to verify that

$$\begin{tikzcd}
\mathbb{G}_m^n \times_{\Bbbk} \mathbb{G}_m^n \times_{\Bbbk} \operatorname{Spec} A \arrow[r, "\mathrm{Id}_{\mathbb{G}_m^n} \times_{\Bbbk} \sigma^\sharp"] \arrow[d, "\mu \times_{\Bbbk} \mathrm{Id}_{\operatorname{Spec} A}"] & \mathbb{G}_m \times_{\Bbbk} \operatorname{Spec} A \arrow[d, "\sigma^\sharp"] \\
\mathbb{G}_m^n \times_{\Bbbk} \operatorname{Spec} A \arrow[r, "\sigma^\sharp"] & \operatorname{Spec} A
\end{tikzcd}$$

commutes, where $\sigma^\sharp$ is the morphism of $\Bbbk$-schemes induced by the co-action $\sigma$. Now, since $\mathbb{G}_m^n$ is a linear algebraic group (in particular affine), we can look at this at the level of rings:

$$\begin{tikzcd}
A \otimes_{\Bbbk} \Bbbk[\bar{U}_n] \otimes_{\Bbbk} \Bbbk[v_1, v_1^{-1}, \ldots, v_n, v_n^{-1}] & A \otimes_{\Bbbk} \Bbbk[\bar{U}_n] \arrow[l, "\sigma \otimes_{\Bbbk} \mathrm{Id}_{\Bbbk[\bar{U}_n]}"'] \\
A \otimes_{\Bbbk} \Bbbk[\bar{U}_n] \arrow[u, "\mathrm{Id}_A \otimes_{\Bbbk} \mu^\sharp"] & A \arrow[l, "\sigma"'] \arrow[u, "\sigma"']
\end{tikzcd}$$

and $\mu^\sharp$ is the co-multiplication defined by $u_i \mapsto u_i \otimes_{\Bbbk} v_i$. So to see that this commutes we will chase a homogenous element $a \in A$ from the bottom right. So note that

$$\text{first going up} \quad \sigma(a) = a \otimes_{\Bbbk} (u_1^{\deg_1 a} \ldots u_n^{\deg_n a}) \quad \text{then going left}$$

$$\sigma(a) \otimes_{\Bbbk} \mathrm{Id}_{\Bbbk[\bar{U}_n]}(u_1^{\deg_1 a} \ldots u_n^{\deg_n a}) = a \otimes_{\Bbbk} (u_1^{\deg_1 a} \ldots u_n^{\deg_n a}) \otimes_{\Bbbk} (v_1^{\deg_1 a} \ldots v_n^{\deg_n a})$$

$$\text{or first going left} \quad \sigma(a) = a \otimes_{\Bbbk} (u_1^{\deg_1 a \cdot} .. u_n^{\deg_n a}) \quad \text{then going up}$$

$$\mathrm{Id}_A(a) \otimes_{\Bbbk} \mu^\sharp(u^{\deg_1 a} \ldots u^{\deg_n a}) = a \otimes_{\Bbbk} (u_1^{\deg_1 a} \ldots u_n^{\deg_n a}) \otimes_{\Bbbk} (v_1^{\deg_1 a} \ldots v_n^{\deg_n a})$$

Hence we have our desired equality. To finish off this claim, we need to verify the following commutes:

$$\begin{tikzcd}
\operatorname{Spec} A \cong \operatorname{Spec} \mathbb{k} \times_{\mathbb{k}} \operatorname{Spec} A \arrow[r, "e \times_{\mathbb{k}} \mathrm{Id}_{\operatorname{Spec} A}"] \arrow[rr, bend right, "\mathrm{Id}_{\operatorname{Spec} A}"'] & \mathbb{G}_m^n \times_{\mathbb{k}} \operatorname{Spec} A \arrow[r, "\sigma^\sharp"] & \operatorname{Spec} A
\end{tikzcd}$$

Again, since we are affine this can be verified on the level of rings:

$$\begin{tikzcd}
A \cong \mathbb{k} \otimes_{\mathbb{k}} A & \mathbb{k}[\bar{U}_n] \otimes_{\mathbb{k}} A \arrow[l, "e^\sharp \otimes_{\mathbb{k}} \mathrm{Id}_A"] & A \arrow[l, "\sigma"] \arrow[ll, bend right, "\mathrm{Id}_A"']
\end{tikzcd}$$

where $e^\sharp$ is the co-identity, defined for $u_i \in \mathbb{k}[\bar{U}_n]$ as $u_i \mapsto 1$, and to verify the commutativity we just calculate, again for a homogenous element $a \in A$,

$$\left(e^\sharp \otimes_{\mathbb{k}} \mathrm{Id_A}\right)(\sigma(a)) = e^\sharp(u_1^{\deg_1 a} ... u_n^{\deg_n a}) \otimes_{\mathbb{k}} \mathrm{Id}_A(a) = 1 \otimes_{\mathbb{k}} a \cong a = \mathrm{Id}_A(a)$$

and hence we have our desired commutativity.

Now given a $\mathbb{G}_m^n$-affine scheme $\operatorname{Spec} R$ over $\mathbb{k}$, we notice that when we denote the action

$$\delta^\sharp : \mathbb{G}_m^n \times_{\mathbb{k}} \operatorname{Spec} R \to \operatorname{Spec} R$$

since we are in the affine case this is induced by a ring homomorphism (the co-action) $\delta : R \to \mathbb{k}[\bar{U}_n] \otimes_{\mathbb{k}} R$. We recall that if we denote the identity morphism $\epsilon^\sharp : \operatorname{Spec} \mathbb{k} \to \mathbb{G}_m^n$, and the co-identinty $\epsilon : \mathbb{k}[\bar{U}_n] \to k$, similar to the last step above, we have that

$$\begin{tikzcd}
\mathbb{G}_m^n \times_{\mathbb{k}} \operatorname{Spec} R \arrow[r, "\delta^\sharp"] & \operatorname{Spec} R \\
\operatorname{Spec} \mathbb{k} \times_{\mathbb{k}} \operatorname{Spec} R \arrow[u, "\epsilon^\sharp \times_{\mathbb{k}} \mathrm{Id}_{\operatorname{Spec} R}"] \arrow[r, "\cong"'] & \operatorname{Spec} R \arrow[u, "\mathrm{Id}_{\operatorname{Spec} R}"']
\end{tikzcd}$$

i.e. for our co-action

$$\begin{tikzcd}
R[\bar{U}_n] \arrow[d, "\epsilon \otimes_{\mathbb{k}} \mathrm{Id}_R"'] & R \arrow[l, "\delta"'] \arrow[d, "\mathrm{Id}_R"] \\
\mathbb{k} \otimes_{\mathbb{k}} R & R \arrow[l, "\cong"]
\end{tikzcd}$$

Therefore, we have $(\epsilon \otimes_{\Bbbk} \mathrm{Id}_R) \circ \delta \cong \mathrm{Id}_R$, and hence $\delta$ is injective. Next, noting that

$$\Bbbk[\bar{U}_n] = \bigoplus_{(\ell_1,\ldots,\ell_n)\in\mathbb{Z}^n} k u_1^{\ell_1}...u_n^{\ell_n}$$

we have that $R[\bar{U}_n] = \bigoplus_{(\ell_1,\ldots,\ell_n)\in\mathbb{Z}^n} R u_1^{\ell_1}...u_n^{\ell_n}$, it follows from the injectivity of $\delta$ that,

$$R = \delta^{-1}(R[\bar{U}_n]) = \bigoplus_{(\ell_1,\ldots,\ell_n)\in\mathbb{Z}^n} \delta^{-1}\left(R u_1^{\ell_1}...u_n^{\ell_n}\right),$$

as $\Bbbk$-modules. To see this first note that the containment

$$\bigoplus_{\bar{\ell}\in\mathbb{Z}^n} \delta^{-1}(R u_1^{\ell_1}...u_n^{\ell_n}) \subseteq R$$

is obvious. For the other direction we note that it will suffice to verify that for any $r \in R$ that if $\delta(r) = \sum_{\bar{\ell}\in\mathbb{Z}^n} r_\ell u_1^{\ell_1}...u_n^{\ell_n}$, then $\delta(r_\ell) = r_\ell u_1^{\ell_1}...u_n^{\ell_n}$. To see this we note that by Definition 1.9 we have

$$\begin{tikzcd}
R[\bar{U}_n, \bar{V}_n] & R[\bar{U}_n] \arrow[l, "\delta\otimes\mathrm{Id}"'] \\
R[\bar{U}_n] \arrow[u, "\mu\otimes\mathrm{Id}"] & R \arrow[l, "\delta"'] \arrow[u, "\delta"']
\end{tikzcd}$$

so when we chase our element around we see that

$$\sum_{\bar{\ell}\in\mathbb{Z}^n} r_\ell u_1^{\ell_1} v_1^{\ell_1}....u_n^{\ell_n} v_n^{\ell_n} = \sum_{\bar{\ell}\in\mathbb{Z}^n} \delta(r_\ell) u_1^{\ell_1}...u_n^{\ell_n}$$

and hence $\delta(r_\ell)$ equals what we desire. Next, to see that the sum is direct, note, for the sake of contradiction, that if there did exists $a_{\ell_1,..,\ell_n} \in \Bbbk$ (with only finitely many non-zero) and $x_{\ell_1,..,\ell_n} \in \delta^{-1}(R u_1^{\ell_1}...u_n^{\ell_n})$, for $\ell_i \in \mathbb{Z}$ such that $\sum_{(\ell_1,..,\ell_n)\in\mathbb{Z}^n} a_{\ell_1,..,\ell_n} x_{\ell_1,..,\ell_n} = 0$, then $\sum_{(\ell_1,..,\ell_n)\in\mathbb{Z}^n} a_{\ell_1,..,\ell_n} \delta(x_{\ell_1,..,\ell_n}) = 0$, yet since this sum in $R[\bar{U}_n]$ is indeed direct, this implies that each $a_{\ell_1,\ldots,\ell_n} \delta(x_{\ell_1,..,\ell_n}) = 0$ and hence $a_{\ell_1,..,\ell_n} x_{\ell_1,..,\ell_n} = 0$, (since injective) and thus our desired sum is direct. Next, we need to verify

$$\delta^{-1}(R u_1^{\ell_1}...u_n^{\ell_n}) \cdot \delta^{-1}(R u_1^{m_1}...u_n^{m_n}) \subset \delta^{-1}(R u_1^{\ell_1+m_1}, ..., u_n^{\ell_n+m_n})$$

for any $\ell_i, m_i \in \mathbb{Z}$.

To see this, we choose

$$x \in \delta^{-1}(Ru_1^{\ell_1}...u_n^{\ell_n}) \quad \text{and} \quad y \in \delta^{-1}(Ru_1^{m_1}...u_n^{m_n})$$

and since $\delta$ is a ring homomorphism we have that $\delta(xy) = \delta(x)\delta(y)$, and notice $\delta(x)\delta(y) \in Ru_1^{\ell_1+m_1}...u_n^{\ell_n+m_n}$ and hence $xy \in \delta^{-1}(Ru_1^{\ell_1+m_1}...u_n^{\ell_n+m_n})$ as desired.

To finish off the proof we just need to show that these processes give the same grading if we start with a $\mathbb{Z}^n$-graded $\Bbbk$-algebra $R$. That is if $R = \bigoplus_{(\ell_1,..,\ell_n)\in\mathbb{Z}^n} R_{\ell_1,..,\ell_n}$ then

$$R_{\ell_1,...,\ell_n} = \sigma^{-1}(Ru_1^{\ell_1}...u_n^{\ell_n}).$$

First notice that $R_{m_1,..,m_n} \subseteq \sigma^{-1}(Ru_1^{m_1}...u^{m_n})$, since if $x \in R_{m_1,...,m_n}$ then

$$\sigma(x) = xu_1^{m_1}...u_n^{m_n}.$$

Next, we will show that $\sigma^{-1}(Ru_1^{m_1}...u_n^{m_n}) \subseteq R_{m_1,...,m_n}$. Well, as $\sigma$ is defined above, notice that if $x \in \sigma^{-1}(Ru_1^{m_1}...u_n^{m_n})$ then $\sigma(x) = yu_1^{m_1}...u_n^{m_n}$ for some $y \in R$. In general

$$x = \sum_{(\ell_1,...,\ell_n)\in\mathbb{Z}^n} a_{\ell_1,...,\ell_n} z_{\ell_1,...,\ell_n}$$

for only finitely many $a_{\ell_1,...,\ell_n} \neq 0 \in \Bbbk$ and $z_{\ell_1,...,\ell_n} \in R_{\ell_1,...,\ell_n}$, and then

$$\sigma(x) = \sum_{(\ell_1,..,\ell_n)\in\mathbb{Z}^n} a_{\ell_1,...,\ell_n}\sigma(z_{\ell_1,...,\ell_n}) = \sum_{(\ell_1,...,\ell_n)\in\mathbb{Z}^n} a_{\ell_1,...,\ell_n} z_{\ell_1,...,\ell_n} u_1^{\ell_1}...u_n^{\ell_n} = yu^{m_1,..,m_n},$$

and since $R[\bar{U}_n]$ is graded we have that $a_{\ell_1,...,\ell_n} = 0$ for any $(\ell_1,..,\ell_n) \neq (m_1,..,m_n)$, and hence $y = a_{m_1,..,m_n} z_{m_1,...,m_n} \in R_{m_1,..,m_n}$ and hence $\sigma(y) = yu_1^{m_1}...u_n^{m_n}$ and as argued above $\sigma$ is injective and hence $x = y \in R_{m_1,..,m_n}$ as desired. □

**Remark 2.** *For those who study toric varieties, this previous lemma may look overly simple. For example in the study of toric varieties one uses characters to classify toric actions on affine toric varieties. From this lemma we can recover the characters by finding the smallest non-zero homongenous pieces.*

**Remark 3.** *Some interesting geometry comes from this grading, it is in part the main purpose of this paper, it is the generalization to the case for $n = 1$, shown in for example Drinfeld 2013; M. Ballard, Diemer, and D. Favero 2017. For a scheme $X$ with a $\mathbb{G}_m$-action, in these papers they study the several stable loci, including: the fixed point locus, the attracting locus, and the repelling locus. In the next section we will give a similar definition for these loci as given in Drinfeld 2013*

**Lemma 1.13.** *$\mathbb{G}_m^n$ equivariant maps between affine schemes with $\mathbb{G}_m^n$ actions are in 1-1 correspondence with $\mathbb{Z}^n$-graded morphisms of degree 0.*

*Proof.* We will prove this in the case when $n = 1$, for a larger $n$ is simple extension of this proof. For the remainder of the proof set $X = \operatorname{Spec} R$ and $Y = \operatorname{Spec} S$ as two affine $\Bbbk$-schemes with $\mathbb{G}_m$-action, and denote their co-action as $\sigma_R$ and $\sigma_S$ respectively. Thus by Lemma 1.12 we have that both $R$ and $S$ are $\mathbb{Z}$-graded $\Bbbk$-algebras. So let $\varphi : S \to R$ be a graded $\Bbbk$-linear morphism of degree 0, we will show this induces a $\mathbb{G}_m$-equivariant morphism of $X$ to $Y$. Recall, that a $\mathbb{G}_m$-equivariant morphism must satisfy the following commutative diagram:

$$\begin{tikzcd}
\mathbb{G}_m \times_{\Bbbk} X \arrow[r, "\sigma_R^\sharp"] \arrow[d, "\mathrm{Id} \times_{\Bbbk} \varphi^\sharp"'] & X \arrow[d, "\varphi^\sharp"] \\
\mathbb{G}_m \times_{\Bbbk} Y \arrow[r, "\sigma_S^\sharp"'] & Y
\end{tikzcd}$$

and hence we will verify that we have the following commutative diagram:

$$\begin{tikzcd}
R[u, u^{-1}] & R \arrow[l, "\sigma_R"'] \\
S[u, u^{-1}] \arrow[u, "\varphi \otimes_{\Bbbk} \mathrm{Id}"] & S \arrow[l, "\sigma_S"] \arrow[u, "\varphi"']
\end{tikzcd}$$

To achieve this goal let $s \in S_m$ be a homogenous element of $S$ of degree $m$, and denote $\varphi(s) = ar$ where $a \in \Bbbk$ and $r \in R_m$ (since $\varphi$ is graded of degree 0).

Thus

$$\sigma_R\left(\varphi\left(s\right)\right) = \sigma_R(ar) = aru^m$$

and

$$\varphi \otimes_{\Bbbk} \mathrm{Id}\left(\sigma_S(s)\right) = \varphi(s) \otimes_{\Bbbk} \mathrm{Id}(u^m) = ru^m$$

and hence we have our desired commutativity.

Next, let $f : X \to Y$ be an $\mathbb{G}_m$-equivariant morphism of affine schemes, and hence $f$ is induced by a morphism of rings $f^\sharp : S \to R$, we will show that this morphism must be graded of degree 0, to finish our proof. For a homogenous element $s \in S_m$, denote $f^\sharp(s) = \sum_{\ell \in \mathbb{Z}} a_\ell r_\ell$ where for each $\ell \in \mathbb{Z}$ $a_\ell \in \Bbbk$ and $r_\ell \in R_\ell$. Now, since $f$ is $\mathbb{G}_m$-equivariant we have the following commutativity.

$$\begin{tikzcd}
R[u, u^{-1}] & R \arrow[l, "\sigma_R"'] \\
S[u, u^{-1}] \arrow[u, "f^\sharp \otimes_{\Bbbk} \mathrm{Id}"] & S \arrow[l, "\sigma_S"] \arrow[u, "f^\sharp"']
\end{tikzcd}$$

Thus

$$\sigma_R\left(f^\sharp(s)\right) = \sigma_R\left(\sum_{\ell \in \mathbb{Z}} a_\ell r_\ell\right) = \sum_{\ell \in \mathbb{Z}} a_\ell r_\ell u^\ell$$

and

$$f^\sharp \otimes_{\Bbbk} \mathrm{Id}\left(\sigma_S(s)\right) = f^\sharp\left(s\right) \otimes_{\Bbbk} \mathrm{Id}(u^m) = \left(\sum_{\ell \in \mathbb{Z}} a_\ell r_\ell\right) u^m$$

and hence for these to be equal, since $R[u, u^{-1}]$ is graded we have that $a_\ell r_\ell = 0$ for any $\ell \neq m$ and hence $f^\sharp(s) = a_m r_m \in R_m$ as desired. □

## 1.3 $G$-SHEAVES

Another important concept to us is a sheaf of $\mathcal{O}_X$-modules for a $G$-scheme $X$, such that the action of $G$ "lifts" to an action on the sheaf. In the following definition we will, as usual, denote $\mu : G \to G$ as the multiplication map on an algebraic group $G$ over $k$.

**Definition 1.14.** Let $G$, $X$, $E$, $\sigma$ be a linear algebraic group, a $G$-scheme, an $\mathcal{O}_X$-module, and an action of $G$ on $X$ respectively. Then a G-linearization of $E$ consists of an isomorphism:

$$\varphi : \sigma^* E \xrightarrow{\cong} \pi^* E$$

of sheaves on $G \times X$, satisfying the co-cycle condition: that is the commutativity of the following,

$$\begin{tikzcd}
\left[\sigma \circ (\mathrm{Id}_G \times \sigma)\right]^* E \arrow[r, "(\mathrm{Id}_G \times\sigma)^*\varphi"] \arrow[ddd, "="] & \left[\pi \circ (\mathrm{Id}_G \times \sigma)\right]^* E \arrow[d, "="] \\
 & \left[\sigma \circ p_{23}\right]^* E \arrow[d, "p_{23}^*\varphi"] \\
 & \left[\pi \circ p_{23}\right]^* E \arrow[d, "="] \\
\left[\sigma \circ (\mu \times \mathrm{Id}_X)\right]^* E \arrow[r, "(\mu\times\mathrm{Id}_X)^*\varphi"] & \left[\pi \circ (\mu \times \mathrm{Id}_X)\right]^* E
\end{tikzcd}$$

where $p_{23}$, $\mu \times \mathrm{Id}_X$, and $\mathrm{Id}_G \times \sigma$ all map $G \times G \times X$ to $G \times X$, and specifically $p_{23}$ is the projection of the second and third component of $G \times G \times X$ to the $G \times X$. We will call the pair $(E, \varphi)$ a G-sheaf or a G-equivariant sheaf. A G-equivariant morphism of G-sheaves is a morphism of sheaves $E \to E'$, for $G$-sheaves $(E, \varphi)$ and $(E', \varphi')$ such that the following diagram commutes:

$$\begin{tikzcd}
\sigma^* E \arrow[r, "\varphi"] \arrow[d, "\sigma^* f"'] & \pi^* E \arrow[d, "\pi^* f"] \\
\sigma^* E' \arrow[r, "\varphi'"] & \pi^* E'
\end{tikzcd}$$

**Remark 4.** *When $\mathcal{E}$ is a line bundle we will refer to $G$-linearization as a $G$-linearized line bundle.*

**Definition 1.15.** Let $\mathrm{Qcoh}_G X$ be the Abelian category of quasi-coherent $G$-equivariant sheaves on $X$. Analogously, we let $\mathrm{Coh}_G X$ denote the Abelian category of coherent $G$-equivariant sheaves.

**Definition 1.16.** Let $\mathcal{E}$ and $\mathcal{F}$ be quasi-coherent $G$-equivariant sheaves on $X$. The tensor product of $\mathcal{E}$ and $\mathcal{F}$ is the quasi-coherent sheaf $\mathcal{E} \otimes_{\mathcal{O}_X} \mathcal{F}$ together with the equivariant structure, $\varphi^{\mathcal{E}} \otimes_{\mathcal{O}_{G\times X}} \varphi^{\mathcal{F}}$. The sheaf of homomorphisms from $\mathcal{E}$ to $\mathcal{F}$ is the quasi-coherent $\mathcal{H}om_X(\mathcal{E}, \mathcal{F})$ together with the equivariant structure

$$\varphi^{\mathcal{F}} \circ (\underline{\quad}) \circ (\varphi^{\mathcal{E}})^{-1}.$$

Let $X$ and $Y$ be separated, finite-type schemes equipped with actions $\sigma_X$ and $\sigma_Y$, of $G$ and projections $\pi_X$, $\pi_Y$. Let $f : X \to Y$ be a $G$-equivariant morphism of schemes, we then get an adjoint pair of functors,

$$\begin{aligned} f^* : \operatorname{Qcoh}_G Y &\to \operatorname{Qcoh}_G X \\ (\mathcal{F}, \varphi) &\mapsto (f^*\mathcal{F}, (1 \times f)^*\varphi), \\ f_* : \operatorname{Qcoh}_G X &\to \operatorname{Qcoh}_G Y \\ (\mathcal{F}, \varphi) &\mapsto (f_*\mathcal{F}, (1 \times f)_*\varphi). \end{aligned}$$

**Remark 5.** *The definition of $f_*$ and $f^*$ are sensible, since $\sigma_X$ and $\pi_X$ are flat and the following commute:*

$$\begin{tikzcd} G \times X \arrow[r, "1\times f"] \arrow[d, "\sigma_X"] & G \times Y \arrow[d, "\sigma_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd} \qquad \begin{tikzcd} G \times X \arrow[r, "1\times f"] \arrow[d, "\pi_X"] & G \times Y \arrow[d, "\pi_Y"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

**Definition 1.17.** Given an affine algebraic group, $G$, we let

$$\hat{G} := \operatorname{Hom}_{\text{alg grp}}(G, \mathbb{G}_m).$$

We will refer to $\hat{G}$, as the group of characters of $G$. As $\hat{G}$ is abelian, we shall use additive notation for the group structure on $\hat{G}$. For a character, $\chi \in \hat{G}$, we let $K_\chi$ denote the kernel of $\chi$. We also get an auto-equivalence

$$\begin{aligned} (\chi) : \operatorname{Qcoh}_G X &\to \operatorname{Qcoh}_G X \\ \mathcal{E} &\mapsto \mathcal{E} \otimes_{\mathcal{O}_X} t^*\mathcal{L}_\chi \end{aligned}$$

In the previous equation, we have $t : X \to \operatorname{Spec} k$ is the structure map and $\mathcal{L}_\chi$ is the object of $\operatorname{Qcoh}_G(\operatorname{Spec} k)$ corresponding to $\chi$, which we will call the twist of $\mathcal{E}$ by $\chi$.

**Remark 6.** *The object of* $\operatorname{Qcoh}_G(\operatorname{Spec} k)$ *corresponding to* $\chi$*, more specifically is the coherent sheaf corresponding to a 1-dimensional vector space where the k-points of* $G$ *act on this vector space via the character* $\chi$*. Later in Example 1.26 we will show how this can be realized as graded modules, when* $X$ *is affine and* $G = \mathbb{G}_m$*.*

**Lemma 1.18.** *Let* $G$ *act on* $X$ *and* $Y$*. Assume we have an equivariant morphism,* $f : X \to Y$*. For* $\mathcal{E} \in \operatorname{Qcoh}_G Y$ *locally-free and* $\mathcal{F} \in \operatorname{Qcoh}_G X$*, there is a natural isomorphism*

$$f_*\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{E} \cong f_*(\mathcal{F} \otimes_{\mathcal{O}_Y} f^*\mathcal{E})$$

This Lemma follows from the usual projection formula applied to both $\mathcal{E}$ and $\varphi$.

While these constructions are great we would like to compare the quasi-coherent sheaves from spaces with different group actions, to do this we will need some more machinery.

**Definition 1.19.** Let $H$ and $G$ be affine algebraic groups and let $X$ and $Y$ be separated schemes of finite type equipped with actions, $\sigma_{H,X} : H \times X \to X$ and $\sigma_{G,Y} : G \times Y \to Y$. Let $\psi : H \to G$ be a homomorphism of algebraic groups. A $\psi$-equivariant morphism, or a morphism equivariant with respect to $\psi$, is a morphism of schemes, $f : X \to Y$, such that the following diagram commutes

$$\begin{tikzcd}
H \times X \arrow[r, "\psi \times f"] \arrow[d, "\sigma_{H,X}"] & G \times Y \arrow[d, "\sigma_{G,Y}"] \\
X \arrow[r, "f"] & Y
\end{tikzcd}$$

Given a $\psi$-equivariant morphism, $f$, we can define the pull-back functor

$$\begin{aligned} f^* : \operatorname{Qcoh}_G Y &\to \operatorname{Qcoh}_H X \\ (\mathcal{F}, \varphi) &\mapsto (f^*\mathcal{F}, (\psi \times f)^*\varphi). \end{aligned}$$

In the case $X = Y$, and $f = 1$, we will denote this functor $\mathrm{Res}_\psi$. If, in addition, $\psi : H \to G$ is a closed subgroup, the pull-back is called the <u>restriction functor</u> and will be denoted by $\mathrm{Res}^G_H$.

**Remark 7.** *Notice that we have chosen to not change the notation of this pull-back, and hope that the context will always be clear.*

**Example 1.20.** We will take a moment to analyze this in a case that is of great importance to us. Let $R$ be a $\mathbb{Z}^n$-graded $k$-algebra, and let $M$ be a graded $\mathbb{Z}^n$ $R$-module, that is $M = \bigoplus_{\bar{\ell}\in\mathbb{Z}^n} M_{\bar{\ell}}$, such that for any homogenous element, $r \in R_{\bar{\ell}}$, we have that $r_{\bar{\ell}} M_{\bar{t}} \subset M_{\bar{\ell}+\bar{t}}$. We will denote $\tilde{M}$ by the induced quasi-coherent sheaf of $\mathcal{O}_{\mathrm{Spec}\, R}$-modules. First notice that $\sigma^*\tilde{M}$ is identified with the sheaf associated to the module $(k[\mathbb{G}^n_m] \otimes_{\Bbbk} R)\,_\sigma{\otimes_{\mathrm{Id}}}\, M$, while $p_2^*\tilde{M}$ can be identified with the sheaf associated to the module $(k[\mathbb{G}^n_m] \otimes_{\Bbbk} R)\,_\pi{\otimes_{\mathrm{Id}}} M \cong k[\mathbb{G}^n_m] \otimes_{\Bbbk} M := M[u_1, u_1^{-1}, ..., u_n, u_n^{-1}]$. Thus an isomorphism $\varphi : \sigma^*\tilde{M} \xrightarrow{\sim} \pi^*\tilde{M}$ corresponds to a $(k[\mathbb{G}^n_m] \otimes_{\Bbbk} R)$-module homomorphism, $\tilde{\varphi} : (k[\mathbb{G}^n_m] \otimes_{\Bbbk} R)\,_\sigma{\otimes_{\mathrm{Id}}} M \xrightarrow{\sim} M[u_1, u_1^{-1}, .., u_n, u_n^{-1}]$. The next Lemma shows that in this case the graded modules are the quasi-coherent $\mathbb{G}^n_m$-sheaves.

**Lemma 1.21.** *First define* $X := \mathrm{Spec}\, R$*, now let* $R$ *be a* $\mathbb{Z}^n$*-graded* $k$*-algebra, the category of quasi-coherent* $\mathbb{G}^n_m$*-sheaves is equivalent to the category of* $\mathbb{Z}^n$*-graded* $R$*-modules*

*Proof.* Let $M = \bigoplus_{\bar{\ell}\in\mathbb{Z}^n} M_{\bar{\ell}}$ be a $\mathbb{Z}^n$-graded $R$-module. We will define

$$\tilde{\varphi} : (k[\mathbb{G}^n_m] \otimes_{\Bbbk} R)\,_\sigma{\otimes_{\mathrm{Id}}}\, M \xrightarrow{\sim} M[u_1, ..., u_n^{-1}],$$

an isomorphism of $(k[\mathbb{G}^n_m] \otimes_{\Bbbk} R)$-modules, first note for a pure tensor with $a \in \Bbbk[\mathbb{G}^n_m]$, $b \in R_{\bar{t}}$, and $m \in M$, we have the following relation in

$$a \otimes b \otimes m \sim au^{-\bar{t}} \otimes 1 \otimes bm \tag{1.1}$$

$(\Bbbk[\mathbb{G}^n_m] \otimes R)_\sigma \otimes M$. Now we define $\tilde{\varphi}$ as follows

$$\tilde{\varphi}(a \otimes 1 \otimes m) = au^{-\bar{\ell}} m.$$

In the previous equation, we have $\bar{\ell} \in \mathbb{Z}^n$, and $m \in M_{\bar{\ell}}$ a homogenous element of $M$. By the definition it is clear that this morphism is indeed an isomorphism. Next, we will show that this indeed defines a $\mathbb{G}_m^n$-linearization of the sheaf $\tilde{M}$, by showing it satisfies the co-cycle condition. To see this let $a, b \in \Bbbk[\mathbb{G}_m^n]$, $c \in R_{\bar{t}}$, and $m \in M_{\bar{\ell}}$ then we again have the following relation in $\Gamma(X, [\sigma \circ (\mathrm{Id}_{\mathbb{G}_m^n} \times \sigma)]^* \tilde{M}) = \Gamma(X, [\sigma \circ (\mu \times \mathrm{Id}_X]^* \tilde{M})$,

$$a \otimes b \otimes c \otimes m \sim av^{-\bar{t}} \otimes bu^{-\bar{t}} \otimes 1 \otimes cm.$$

Thus we chase the diagram for the co-cycle condition using a pure tensor of the form $a \otimes b \otimes 1 \otimes m$, first the top morphism we have

$$a \otimes b \otimes 1 \otimes m \mapsto a \otimes 1 \otimes bu^{-\bar{\ell}} m$$

then down on the right

$$a \otimes 1 \otimes bu^{-\bar{\ell}} m \mapsto av^{-\bar{\ell}} bu^{-\bar{\ell}} m$$

and finally the bottom morphism just gives us $a \otimes b \otimes 1 \otimes m \mapsto av^{-\bar{\ell}} bu^{-\bar{\ell}} m$.

Now we will verify that if $\tilde{M}$ is a $\mathbb{G}_m^n$-coherent sheaf, then $M$ is a $\mathbb{Z}^n$-graded module. To do this we define for $\bar{\ell} \in \mathbb{Z}^n$

$$M_{\bar{\ell}} := \left\{ m \in M \,\middle|\, \tilde{\varphi}\left(u^{\bar{\ell}} \otimes 1 \otimes m\right) = m \right\} = \left\{ m \in M \,\middle|\, \tilde{\varphi}\left(1 \otimes 1 \otimes m\right) = u^{-\bar{\ell}} m \right\}$$

our claim is that $M = \bigoplus_{\bar{\ell} \in \mathbb{Z}^n} M_{\bar{\ell}}$, from the relation pointed out in Equation 1.1 we see that this would indeed define a grading. Now we see that the sum is direct, note that if there exists $\alpha_{\bar{\ell}} \in \Bbbk$, finitely many non-zero, such that $\sum_{\bar{\ell} \in \mathbb{Z}^n} \alpha_{\bar{\ell}} m_{\bar{\ell}} = 0$ with $m_{\bar{\ell}} \in M_{\bar{\ell}}$ then as $\tilde{\varphi}$ is an isomorphism this implies $\sum_{\bar{\ell} \in \mathbb{Z}^n} \alpha_{\bar{\ell}} u^{\bar{\ell}} \otimes 1 \otimes m_{\bar{\ell}} = 0$ and as the $u^{\bar{\ell}}$ are linearly independent we have that this implies $\alpha_{\bar{\ell}} = 0$ for all $\bar{\ell} \in \mathbb{Z}^n$ as desired. Next we show that we have the desired equality, the inclusion $\bigoplus_{\bar{\ell} \in \mathbb{Z}^n} M_{\bar{\ell}} \subseteq M$ is clear so we will show the opposite inclusion. To see this we will use the co-cycle condition, let $m \in M$ then $\tilde{\varphi}(1 \otimes 1 \otimes m) = \sum_{\bar{\ell} \in \mathbb{Z}^n} \alpha_{\bar{\ell}} u^{\bar{\ell}} m_{\bar{\ell}}$ for some $\alpha_{\bar{\ell}} \in \Bbbk$, only finitely many non-zero, our claim is that $m_{\bar{\ell}} \in M_{\bar{\ell}}$ for all $\bar{\ell} \in \mathbb{Z}^n$. Now we will chase the element

$1 \otimes 1 \otimes 1 \otimes m$ around the co-cycle diagram, first by through the top then the right and we get

$$1 \otimes 1 \otimes 1 \otimes m \mapsto \sum_{\bar{\ell} \in \mathbb{Z}^n} \alpha_{\bar{\ell}} v^{\bar{\ell}} \tilde{\varphi}(m_{\bar{\ell}})$$

and then on the bottom we get

$$1 \otimes 1 \otimes 1 \otimes m \mapsto \sum_{\bar{\ell} \in \mathbb{Z}^n} \alpha_{\bar{\ell}} v^{\bar{\ell}} u^{\bar{\ell}} m_{\bar{\ell}}$$

and by the commutivity of the co-cycle since the $u^{\bar{\ell}}$ are linearly independent we have that $\tilde{\varphi}(m_{\bar{\ell}}) = u^{\bar{\ell}} m_{\bar{\ell}}$ and thus $m_{\bar{\ell}} \in M_{\bar{\ell}}$ as desired. From the definition of our original $\varphi$ we see that this grading defines the same as the first map, and thus we have established our correspondence. □

**Remark 8.** *For a LAG $G$, and an algebraic variety $X$ with an action of $G$. The (quasi)coherent sheaves on $[X/G]$ are equivalent the $G$-equivariant sheaves on $X$. The standard reference for this equivalence is Vistoli 1989.*

### 1.4 GIT Quotients

This section will serve as a brief reminder of the quotients which arise from Geometric Invariant Theory, for more complete and thourough treatments see Mumford, Fogarty, and Kirwan 1994; Vistoli 1989. For an algebraic variety, $X$, acted on by an LAG, $G$, in general the orbit space $X/G$ does not exist in the category of algebraic varieties. A common reason is the existence of non-closed orbits. A way to overcome this problem is the use of Geometric Invariant Theory (GIT). While another is the use of quotient stacks. At first glance these two approaches may seem very different yet when the algebraic quotient represents (in the sense of Yoneda) the corresponding stack.

**Definition 1.22.** Given a $G$-linearized line bundle, $\mathcal{L}$, we define the semi-stable locus of $X$ as

$$X^{\mathrm{ss}}(\mathcal{L}) = \left\{ x \in X \;\middle|\; \exists s \in \Gamma(X, \mathcal{L}^{\otimes m})^G,\ s(x) \neq 0 \right\}.$$

Where $(\_)^G$ denotes the invariant ring.

**Definition 1.23.** Let $\mathcal{L}$ be a $G$-line bundle. The GIT quotient of $X$ by $G$ with respect to $\mathcal{L}$ is the quotient stack, $[X^{\mathrm{ss}}(\mathcal{L})/G]$. We will often denote this quotient by $X//_{\mathcal{L}}G$.

**Lemma 1.24.** *For a $G$-equivariant line bundle, $\mathcal{L}$. If we denote*

$$\mathcal{A}^{\mathcal{L}} := \bigoplus_{d=0}^{\infty} \Gamma(X, \mathcal{L}^d)$$

*then the rational points of the variety $X \backslash \mathcal{V}\left( (\mathcal{A}_+^{\mathcal{L}})^G \right)$ are the semi-stable points of $X$ corresponding to $\mathcal{L}$, where $\mathcal{V}(\_\_)$ denotes the vanishing locus, and $\mathcal{A}_+^{\mathcal{L}}$ denotes the irrelevant ideal. Hence we see the semi-stable locus is an open subscheme*

$$X^{\mathrm{ss}}(\mathcal{L}) := X \backslash \mathcal{V}\left( (\mathcal{A}_+^{\mathcal{L}})^G \right).$$

The proof follows directly from the definition.

**Definition 1.25.** Let $\mathcal{L}$ be a $G$-equivariant line bundle, we define

$$X^{\mathrm{us}}(\mathcal{L}) := \mathcal{V}\left( (\mathcal{A}_+^{\mathcal{L}})^G \right).$$

We will refer to the rational points of this locus as the unstable points with respect to $\mathcal{L}$, and refer to the scheme itself as the unstable locus of $X$ with respect to $\mathcal{L}$.

Now we will give some examples of the GIT quotient.

**Example 1.26.** Consider the standard action of $\mathbb{G}_m$ on $\mathbb{A}_{\Bbbk}^{n+1}$, as seen in Lemma 1.12 this is equivalent to choosing a basis for $\Gamma(\mathbb{A}^n, \mathcal{O}) = \Bbbk[x_0, ..., x_n] := R$ and giving the ring $\Bbbk[x_0, ..., x_n]$ the $\mathbb{Z}$-grading $|x_i| = 1$.

Next we see that for $\mathcal{O}(-1)$, where $\_(-1)$ is the twist from Definition 1.17 and we use the short hand to indicate the 1-dimensional $\Bbbk$-vector space $V$ with an action of $\mathbb{G}_m$ defined for $\alpha \in \mathbb{G}_m$ and $v \in V$ as $\alpha \cdot v = \alpha^{-1} v$. Next, notice that $\left(\mathbb{A}_{\Bbbk}^{n+1}\right)^{\text{us}} (\mathcal{O}(-1)) = \mathcal{V}\left(\bigoplus_{i>0} R_i\right)$, where $R_i$ is the $i^{\text{th}}$-homogeneous component of the ring $R$. Therefore $\left(\mathbb{A}_{\Bbbk}^{n+1}\right)^{\text{ss}} (\mathcal{O}(-1)) = \bigcup_{i=0}^{n} U_i$ where $U_i = \operatorname{Spec}\{x_i^{\ell}\}^{-1} R$. Finally we note that $\mathbb{A}_{\Bbbk}^{n+1} //_{\mathcal{O}(-1)} \mathbb{G}_m \cong \mathbb{P}_{\Bbbk}^{n}$.

**Example 1.27.** Consider an $n$-dimensional $\Bbbk$-vector space $V$ with the standard action of $\mathrm{GL}(V)$ and an $m$-dimensional $\Bbbk$-vector space with $m \geqslant n$ this induces an action of $\mathrm{GL}(V)$ on $\operatorname{End}(V, W) = \operatorname{Spec}(\Bbbk[x_1, .., x_{n \times m}])$ via pre-composition. We see that $\mathbb{A}_{\Bbbk}^{n} //_{\mathcal{O}(V^\vee)} \mathrm{GL}(V) = \mathbb{Gr}(n, m)$.

## 1.5 TRIANGULATED CATEGORIES

In this subsection we recall some basics of triangulated categories, we will not recall all of category theory and we suggest the reader refer to Mac Lane 2013 for the necessary background. For the objects we do recall the interested reader can see Huybrechts 2006 for a more detailed description, and for the reader looking for the original text of triangulated categories see Verdier 1977.

Before we give the definition of the abstract object that will be most helpful for us, triangulated categories, we give a couple of definitions which help build terminology that we will use in the definition.

**Remark 9.** *We will be considering abelian categories $\mathcal{A}$, with an endo-functor $T_{\mathcal{A}} : \mathcal{A} \to \mathcal{A}$ which we will call a shift automorphism, which is an equivalence. Further for a shift automorphism for any object $A, B \in \mathcal{A}$ we will denote $T_{\mathcal{A}}^{i}(A) := A[i]$, and any morphism $f \in \operatorname{Hom}_{\mathcal{A}}(A, B)$ we will denote $T_{\mathcal{A}}^{i}(f) := f[i]$, for any $i \in \mathbb{Z}$. As well, we will write*

$$\operatorname{Hom}_{\mathscr{A}}^{i}(A, B) = \operatorname{Hom}_{\mathscr{A}}(A, B[i]).$$

*We are most concerned will chains of morphisms of the type*

$$A \longrightarrow B \longrightarrow C \longrightarrow A[1]$$

*which we will call* <u>*triangles*</u>*. A* <u>*morphism of triangles*</u> *can be understood as a commutative diagram*

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \arrow[r] \arrow[d] & C \arrow[r] \arrow[d] & A[1] \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

*an* <u>*isomorphism of triangles*</u> *is when each vertical arrow is an isomorphism.*

**Definition 1.28.** A <u>triangulated category</u> is an additive category $\mathscr{A}$ equipped with a *shift automorphism* $T_{\mathscr{A}} : \mathscr{A} \to \mathscr{A}$, and a collection of <u>distinguished triangles</u>

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

of morphisms of $\mathscr{A}$ satisfying the following axioms,

[**T**R1]

(i) Any triangle of the form

$$A \xrightarrow{\text{Id}} A \longrightarrow 0 \longrightarrow A[1]$$

is distinguished.

(ii) Any triangle isomorphic to a distinguished triangle is distinguished.

(iii) Any morphism $f : A \to B$ can be completed to a distinguished triangle

$$A \xrightarrow{f} B \longrightarrow C \longrightarrow A[1]$$

where we call $C$ the <u>mapping cone</u> of $A \xrightarrow{f} B$.

[**T**R 2] The triangle

$$A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} A[1]$$

is a distinguished triangle if and only if

$$B \xrightarrow{g} C \xrightarrow{h} A[1] \xrightarrow{-f[1]} B[1]$$

is a distinguished triangle.

[**T**R 3] Suppose there exists a commutative diagram of distinguished triangles with vertical arrows $f$ and $g$:

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "f"'] & B \arrow[r] \arrow[d, "g"'] & C \arrow[r] \arrow[d, dashed, "h"'] & A[1] \arrow[d] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & A'[1]
\end{tikzcd}$$

Then the diagram can be completed to a commutative diagram, i.e. to a morphism of triangles, by a (not necessarily unique) morphism $h : C \to C'$.

[**T**R 4] (The octahedral axiom) Suppose we have morphism $u : A \to B$ and $v : B \to C$, so that we also have a composed morphism $v \circ u : A \to C$, by **T**R 1 we can form distinguished triangle for each of these three morphisms, that is:

$$A \xrightarrow{u} B \xrightarrow{j} C' \xrightarrow{k} A[1]$$

$$B \xrightarrow{v} C \xrightarrow{l} A' \xrightarrow{i} B[1]$$

$$A \xrightarrow{v \circ u} C \xrightarrow{m} B' \xrightarrow{n} C[1]$$

then there exists a distinguished triangle

$$C' \xrightarrow{f} B' \xrightarrow{g} A' \xrightarrow{h} C'[1]$$

such that

$$l = g \circ m, \;\; k = n \circ f, \;\; h = j[1] \circ i, \;\; i \circ g = u[1] \circ n, \;\; f \circ j = m \circ v$$

which can be viewed as the commutativity of the following diagram:

$$\begin{tikzcd}
 & & B' \arrow[rr, dashed, "g"] \arrow[ddll] & & \\
C' \arrow[urr, dashed, "f"] \arrow[d] & & & & A' \arrow[llll, "h"] \arrow[ddll, "i"] \\
A \arrow[rrrr, "v\circ u"] \arrow[drr, "u"] & & & & C \arrow[u, "l"] \arrow[uull, "m"] \\
 & & B \arrow[urr, "v"] \arrow[uull, "j"] & &
\end{tikzcd}$$

or better yet the top:

$$\begin{tikzcd}
A' \arrow[dd] & & & & C \arrow[llll] \arrow[dl] \\
 & d & B' \arrow[ull, "g"] \arrow[drr] & d & \\
C' \arrow[urr, "f"] \arrow[rrrr] & & & & A \arrow[uu, "v\circ u"]
\end{tikzcd}$$

and the bottom:

$$\begin{tikzcd}
A' \arrow[dddd] \arrow[ddr, "i"] & & C \arrow[ll] \\
 & d & \\
 & B \arrow[uur, "v"] \arrow[ddl, "j"] & \\
 & d & \\
C' \arrow[rr] & & A \arrow[uuuu, "v\circ u"] \arrow[uul, "u"]
\end{tikzcd}$$

where the triangles labeled with $d$ are distinguished.

**Remark 10.** *It is easy to see that by combining TR1 and TR3 that for a distinguished triangle*

$$A \xrightarrow{f} B \xrightarrow{g} C \longrightarrow A[1]$$

*that* $A \to B \to C$ *is zero, since TR3 guarantees the commutativity of the following*

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[r] & C \arrow[r] & A[1] \\
A \arrow[u, "\mathrm{Id}"] \arrow[r, "\mathrm{Id}"] & A \arrow[u, "f"] \arrow[r] & 0 \arrow[u, dashed] \arrow[r] & A[1] \arrow[u]
\end{tikzcd}$$

*that is* $g \circ f = 0 \circ 0 = 0$ *as desired. Also, using TR2 and the same trick we see that* $B \to C \to A[1]$ *is also zero. As well, applying TR2 iteratively and TR1 (iii) gives us that distinguished triangles are closed under shift. Another immediate property is that for* $f \in \mathrm{Hom}_{\mathcal{A}}(A, B)$*, the mapping cone of* $f$ *is unique up to (not necessarily unique) isomorphism. To see this we just assume that there exists another object* $C'$ *and morphisms* $B \xrightarrow{g} C \xrightarrow{h} A[1]$ *and* $B \xrightarrow{g'} C' \xrightarrow{h'} A[1]$ *making* $C$ *and* $C'$ *a mapping cone, then by TR 3 we have the commutativity of the following*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & C \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "\mathrm{Id}"] \\
A \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & C' \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "\mathrm{Id}"] \\
A \arrow[r, "f"] \arrow[d, "\mathrm{Id}"'] & B \arrow[r] \arrow[d, "\mathrm{Id}"] & C \arrow[r] \arrow[d, dashed] & A[1] \arrow[d, "\mathrm{Id}"] \\
A \arrow[r, "f"] & B \arrow[r] & C' \arrow[r] & A[1]
\end{tikzcd}$$

*which shows that* $C \cong C'$*, yet the isomorphism is not necessarily unique since the maps from* $C$ *to* $C'$ *and* $C'$ *to* $C$ *are not necessarily unique.*

**Definition 1.29.** A triangulated category $\mathscr{A}$ is trivial if every object is a zero object. A functor $F : \mathscr{A} \to \mathscr{B}$ between triangulated categories is exact if it commutes with the shift automorphisms and takes distinguished triangles of $\mathscr{A}$ to distinguished triangles of $\mathscr{B}$.

We will see later that the derived categories we are most concerned with are indeed triangulated categories. Yet, the real power of triangulated categories comes from the utility of decomposing these categories, and using this decomposition to show functors are fully faithful, and at times equivalences.

**Definition 1.30.** Let $\mathcal{T}$ be a traingulated category. A full additive subcategory $\mathcal{S}$ in $\mathcal{T}$ is called a triangulated sub-category whenever every object is isomorphic to an object of $\mathcal{S}$ is in $\mathcal{S}$, and $T_{\mathcal{S}}(\mathcal{S}) = \mathcal{S}$, as well as for any distinguished triangle

$$X \longrightarrow Y \longrightarrow Z \longrightarrow X[1]$$

such that the objects $X$ and $Y$ are in $\mathcal{S}$, the object $Z$ is also in $\mathcal{S}$. We will call $\mathcal{S}$ a thick triangulated sub-category whenever $\mathcal{S}$ is a triangulated sub-category which is closed under extensions.

**Definition 1.31.** A spanning class of a triangulated category $\mathcal{A}$ is a subclass $\Omega$ of the objects of $\mathcal{A}$ such that for any object $A \in \mathcal{A}$

$$\mathrm{Hom}^i_{\mathcal{A}}(A, B) = 0, \quad \text{for all } B \in \Omega,\ i \in \mathbb{Z},\ \text{implies } A \cong 0$$

and

$$\mathrm{Hom}^i_{\mathcal{A}}(B, A) = 0, \quad \text{for all } B \in \Omega, i \in \mathbb{Z},\ \text{implies } A \cong 0.$$

A triangulated category $\mathcal{A}$ is decomposable as an orthogonal direct sum of two full subcategories $\mathcal{A}_1$ and $\mathcal{A}_2$ when every object of $\mathcal{A}$ is isomorphic to a direct sum $A_1 \oplus A_2$ with $A_j \in \mathcal{A}_j$, such that

$$\mathrm{Hom}^i_{\mathcal{A}}(A_1, A_2) = \mathrm{Hom}^i(A_2, A_1) = 0$$

for any pair of objects $A_j \in \mathcal{A}_j$ and all integers $i$. The category $\mathcal{A}$ is called indecomposable when for any such decomposition we have that one of the two categories $\mathcal{A}_j$ is trivial.

**Example 1.32.** A simple example of a triangulated category is the category of vector spaces over a field $\Bbbk$, $\mathsf{Vec}_{\Bbbk}$, when we consider the trivial shift, i.e. for a vector space $V$, we have $V[1] = V$. Finally the distinguished triangles are simply the exact triangles.

We will see later more important examples.

## 1.6 The Derived Category

Now we will discuss an important triangulated category: the derived category.

### 1.6.1 Definitons and Basic Results

For an abelian category $\mathcal{A}$, do build the derived category of $\mathcal{A}$ is classically presented in two steps, first defining the category of complexes and then the homotopy category (in the model category section we show an alternate process to build the derived category using weak equivalences). So first we will recall a few concepts about the first piece of this puzzle.

To define the category $\mathrm{Kom}(\mathcal{A})$, the category of complexes, we will first describe the objects.

**Definition 1.33.** A complex in an abelian category $\mathcal{A}$ consists of a diagram of objects and morphisms in $\mathcal{A}$ of the form

$$\ldots \longrightarrow A^{i-1} \xrightarrow{d^{i-1}} A^i \xrightarrow{d^i} A^{i+1} \longrightarrow \ldots$$

satsifying $d^i \circ d^{i-1} = 0$, equivalently, $\mathrm{Im}(d^{i-1}) \subset \ker(d^i)$, for all $i \in \mathbb{Z}$. We denote this complex as $A^\bullet$, and refer to the morphisms $d^i$ associated to a complex $A^\bullet$ as the differentials. Further, a morphism $f : A^\bullet \to B^\bullet$ between two complexes $A^\bullet$ and $B^\bullet$ is the data of a collection of morphisms $\{f^i : A^i \to B^i\}$, such that the following diagram commutes in $\mathcal{A}$.

$$\begin{tikzcd}
\dots \arrow[r] & A^{i-1} \arrow[r, "d_A^{i-1}"] \arrow[d, "f^{i-1}"] & A^i \arrow[r, "d_A^i"] \arrow[d, "f^i"] & A^{i+1} \arrow[r] \arrow[d, "f^{i-1}"] & \dots \\
\dots \arrow[r] & B^{i-1} \arrow[r, "d_B^{i-1}"] & B^i \arrow[r, "d_B^i"] & B^{i+1} \arrow[r] & \dots
\end{tikzcd}$$

Thus we may define the category of complexes $\mathrm{Kom}(\mathcal{A})$ of an abelian category $\mathcal{A}$ is the category whose objects are complexes $A^\bullet$ in $\mathcal{A}$ and whose morhpisms are morphisms of complexes.

**Remark 11.** *Notice that* $\mathrm{Kom}(\mathcal{A})$ *of an abelian category* $\mathcal{A}$ *is again abelian with the obvious zero, kernel and cokernel objects. Further notice that* $\mathcal{A}$ *is canonically identified with a full subcategory of* $\mathrm{Kom}(\mathcal{A})$ *in many different (equivalent by the shift functor defined below) ways, by choosing an index and putting an object in this positon. Yet this new category has extra structure which we will be taking advantage of, namely the following.*

**Definition 1.34.** Let $A^\bullet \in \mathrm{Kom}(\mathcal{A})$ be a given complex, in an abelian category $\mathcal{A}$. Then we define shift $A^\bullet[n]$, for $n \in \mathbb{Z}$, as the complex with $(A^\bullet[n])^i := A^{i+n}$ and differential $d^i_{A[n]} := (-1)^n d_A^{i+n}$. The shift of a morphism $f : A^\bullet \to B^\bullet$, denoted $f[n]$ as the complex morhism $f[n] : A^\bullet[n] \to B^\bullet[n]$ given by $f[n]^i := f^{i+n}$. This lays out the properties to define a functor specifically we define the shift functor $T : \mathrm{Kom}(\mathcal{A}) \to \mathrm{Kom}(\mathcal{A})$, as $A^\bullet \mapsto A^\bullet[1]$.

**Remark 12.** *It is easy to see that the shift functor* $T$ *defines an equivalence of abelian categories for any abelian category* $\mathcal{A}$*. The inverse functor* $T^{-1}$ *has an easy interpretation as* $A^\bullet \mapsto A^\bullet[-1]$*. With these pieces the first time reader may try to realize* $\mathrm{Kom}(\mathcal{A})$ *as a triangulated category, yet no triangles satisfy the the conditions on distinguished triangles.*

**Definition 1.35.** For a complex $A^\bullet$ in $\mathrm{Kom}(\mathcal{A})$ for an abelian category $\mathcal{A}$ , we define the cohomology $H^i(A^\bullet)$ as the quotient

$$H^i(\mathcal{A}) := \frac{\ker(d^i)}{\mathrm{im}(d^{i-1})} \in \mathcal{A}$$

Therefore, $H^i(A^\bullet) = \mathrm{Coker}(\mathrm{Im}(d^{i-1}) \to \ker(d^i))$. We call a complex acyclic when $H^i(A^\bullet) = 0$ for all $i \in \mathbb{Z}$. Any complex morphism $f : A^\bullet \to B^\bullet$ induces natural homomorphisms

$$H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$$

**Remark 13.** *Notice, that for any morphism of complex* $f : A^\bullet \to B^\bullet$, *we get the induced morphism*

$$H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet).$$

**Definition 1.36.** A morphism of complexes $f : A^\bullet \to B^\bullet$ is a quasi-isomorphism (qis) if for all $i \in \mathbb{Z}$ the induced map $H^i(f) : H^i(A^\bullet) \to H^i(B^\bullet)$ is an isomorphism.

The motivation of the derived category is to define the appropriate setting for a category where quasi-isomorphic complexes become isomorphic objects. The next theorem, which guarantees the existence of the derived category, is left without proof, yet the interested reader may refer to Huybrechts 2006 for a detailed discussion of the proof.

**Theorem 1.37.** *Let* $\mathcal{A}$ *be an abelian category and let* $\mathrm{Kom}(\mathcal{A})$ *be its category of complexes. Then there exists a category* $\mathsf{D}(\mathcal{A})$, *the* derived category *of* $\mathcal{A}$, *and a functor*

$$Q : \mathrm{Kom}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$$

*such that:*

*(i) If* $f : A^\bullet \to B^\bullet$ *is a quasi-isomorphism, then* $Q(f)$ *is an isomorphism in* $\mathsf{D}(\mathcal{A})$

*(ii)* *Any functor* $F : \mathrm{Kom}(\mathcal{A}) \to C$ *satisfying property (i) factorizes uniquely over* $Q : \mathrm{Kom}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$*, i.e. there exists a unique functor (up to isomorphism)* $G : \mathsf{D}(\mathcal{A}) \to C$ *with* $F \cong G \circ Q$*:*

$$\begin{tikzcd}
\mathrm{Kom}(\mathcal{A}) \arrow[rr, "Q"] \arrow[dr, "F"'] & & \mathsf{D}(\mathcal{A}) \arrow[dl, dashed, "G"] \\
& C &
\end{tikzcd}$$

This Theorem gives the existence of the derived category for any abelian category $\mathcal{A}$, and thus defines the category by the above universal property. In order to get our hands dirty with some calculations it is useful to also note the following, again without proof and the reader is referred to Huybrechts 2006.

**Corollary 1.38.** *(i) Under the functor* $Q : \mathrm{Kom}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ *the objects of the two categories* $\mathrm{Kom}(\mathcal{A})$ *and* $\mathsf{D}(\mathcal{A})$ *are identified.*

*(ii) The cohomology objects* $H^i(A^\bullet)$ *of an object* $A^\bullet \in \mathsf{D}(\mathcal{A})$ *are well-defined objects of the abelian category* $\mathcal{A}$*.*

*(iii) Viewing any object in* $\mathcal{A}$ *as a complex concentrated in degree zero yields an equivalence between* $\mathcal{A}$ *and a full subcategory of* $\mathsf{D}(\mathcal{A})$ *that consists of all complexes* $A^\bullet$ *with* $H^i(A^\bullet) = 0$ *for* $i \neq 0$*.*

Unlike $\mathrm{Kom}(\mathcal{A})$, in general, the derived category $\mathsf{D}(\mathcal{A})$ is not abelian, yet it is always triangulated, as we will note shortly, the reader is again referred to Huybrechts 2006 for a detailed discussion. Furthermore, as one of the main object of study will be derived functors, for the sake of calculation we will use another intermediate object, the homotopy category, which we will see is also a triangulated category and that there is an exact functor relating it to the derived category.

**Definition 1.39.** Two morphisms of complexes

$$f, g : A^\bullet \to B^\bullet$$

are called homotopically equivalent, denoted $f \sim g$, if there exists a collection of homomorphisms $h^i : A^i \to B^{i-1}$, $i \in \mathbb{Z}$ such that

$$f^i - g^i = h^{i+1} \circ d_A^i + d_b^{i-1} \circ h^i$$

We will denote the Homotopy Category of complexes, $\mathrm{K}(\mathcal{A})$, as the category whose objects are the objects of $\mathrm{Kom}(\mathcal{A})$, and morphisms

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(A^\bullet, B^\bullet) := \mathrm{Hom}_{\mathrm{Kom}(\mathcal{A})}(A^\bullet, B^\bullet)/\sim .$$

That the category is well defined and that homotopy equivalence is for example an equivalence relation, can be found in any standard text, for example Weibel 1994. We do note that the functor given in Theorem 1.37 does induce a natural functor from $\mathrm{K}(\mathcal{A})$ to $\mathsf{D}(\mathcal{A})$, which we will also denote as $Q_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$.

We now give a description of the derived category, $D(\mathcal{A})$, for an abelian category $\mathcal{A}$, as follows: The objects of $D(\mathcal{A})$ are the same ojects of $K(\mathcal{A})$ which are the same objects of $\mathrm{Kom}(\mathcal{A})$. The morphisms, $\mathrm{Hom}_{D(\mathcal{A})}(A^\bullet, B^\bullet)$, for complexes $A^\bullet$ and $B^\bullet$ viewed as objects of $D(\mathcal{A})$, is the set of equivalence classes of diagrams (in $K(\mathcal{A})$) of the form

$$\begin{tikzcd}
& C^\bullet \arrow[dl, "\overset{\mathrm{qis}}{f}"'] \arrow[dr] & \\
A^\bullet & & B^\bullet
\end{tikzcd}$$

where $f : C^\bullet \to A^\bullet$ is a quasi-isomorphism.

Where, two such diagrams:

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "\overset{\text{qis}}{f_1}"'] \arrow[dr] & & \text{and} & & C_2^\bullet \arrow[dl, "\overset{\text{qis}}{f_2}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & A^\bullet & & B^\bullet \end{tikzcd}$$

are equivalent when there exists a commutative diagram in K($A$) of the form

$$\begin{tikzcd} & & C^\bullet \arrow[dl] \arrow[dr] \arrow[ddll, dashed, bend right, "\text{qis}"'] & & \\ & C_1^\bullet \arrow[dl, "\overset{\text{qis}}{f_1}"'] \arrow[drrr] & & C_2^\bullet \arrow[dlll, "\overset{\text{qis}}{f_2}"] \arrow[dr] & \\ A^\bullet & & & & B^\bullet \end{tikzcd}$$

by the above commutation we see that $C^\bullet \to C_1^\bullet \to A^\bullet$ and $C^\bullet \to C_2^\bullet \to A^\bullet$ are homotopy equivalent, and hence both quasi-isomorphisms. The requirement for these diagrams may seem mysterious, yet in our next description of composition in this category, without this condition the composition would not be well defined. So next, we will describe composition in this category. Let the following be representatives, in $K(\mathcal{A})$, of morphisms from $A^\bullet \to B^\bullet$ and $B^\bullet \to C^\bullet$ from $D(\mathcal{A})$,

$$\begin{tikzcd} & C_1^\bullet \arrow[dl, "\overset{\text{qis}}{f_1}"'] \arrow[dr] & & \text{and} & & C_2^\bullet \arrow[dl, "\overset{\text{qis}}{f_2}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & B^\bullet & & C^\bullet \end{tikzcd}$$

To define another morphism $A^\bullet \to C^\bullet$, we would naively like an object $C_0^\bullet$ such that we have the following diagram

$$\begin{tikzcd} & & C_0^\bullet \arrow[dl, "\text{qis}"'] \arrow[dr] & & \\ & C_1^\bullet \arrow[dl, "\overset{\text{qis}}{f_1}"'] \arrow[dr] & & C_2^\bullet \arrow[dl, "\overset{\text{qis}}{f_2}"'] \arrow[dr] & \\ A^\bullet & & B^\bullet & & C^\bullet \end{tikzcd}$$

The existence and uniqueness of such a diagram follows from the existence of the mapping cone, which we define next. For the details the reader is referred to Huybrechts 2006.

**Definition 1.40.** Let $f : A^\bullet \to B^\bullet$ be a morphism of complexes. Its mapping cone is the complex $C(f)$, defined as:

$$C(f)^i = A^{i+1} \oplus B^i \quad \text{and} \quad d^i_{C(f)} := \begin{pmatrix} -d_A^{i+1} & 0 \\ f^{i+1} & d_B^i \end{pmatrix}.$$

This is clearly a complex, furthermore, there exists two natural morphisms

$$\tau : B^\bullet \to C(f) \quad \text{and} \quad \pi : C(f) \to A^\bullet[1]$$

given by the universal properties of co-product and product, since in an abelian category these are equivalent when finite.

**Definition 1.41.** A triangle

$$A_1^\bullet \longrightarrow A_2^\bullet \longrightarrow A_3^\bullet \longrightarrow A_1^\bullet[1]$$

in $\mathsf{D}(\mathcal{A})$ is called distinguished when it is isomorphic in $\mathsf{D}(A)$ (resp. $\mathrm{K}(\mathcal{A})$) to a triangle of the form

$$A^\bullet \xrightarrow{f} B^\bullet \xrightarrow{\tau} C(f) \xrightarrow{\pi} A^\bullet[1]$$

with $f$ a morphism of complexes.

**Proposition 1.** *Distinguished triangles given in Definition 1.41 and the natural shift functor for complexes $A^\bullet \mapsto A^\bullet[1]$ make the derived category $\mathsf{D}(\mathcal{A})$ (resp. $\mathrm{K}(\mathcal{A})$) of an abelian category into a triangulated category. Moreover, the natural functor $Q_{\mathcal{A}} : \mathrm{K}(\mathcal{A}) \to \mathsf{D}(\mathcal{A})$ is an exact functor of triangulated categories.*

## 1.7 INTEGRAL TRANSFORMS

In this section we recall some of the basic structure of integral transformations. For a more details the reader is suggested to look at Huybrechts 2006. Throughout this section we will let $X$ and $Y$ be algebraic varieties over a field $\Bbbk$, as well we will denote the two projections by

$$\pi_X : X \times Y \to X \quad \text{and} \quad \pi_Y : X \times Y \to Y.$$

**Definition 1.42.** Let $\mathcal{K} \in \mathsf{D}^{\mathrm{b}}\left(X \times Y\right)$. The induced <u>integral transform</u> is defined as the functor

$$\begin{aligned} \Phi_{\mathcal{K}} : \mathsf{D}^{\mathrm{b}}(X) &\to \mathsf{D}^{\mathrm{b}}(Y) \\ \mathcal{E}^{\bullet} &\mapsto (\pi_Y)_* \Big( (\pi_X)^* \mathcal{E}^{\bullet} \otimes \mathcal{K} \Big). \end{aligned}$$

The object $\mathcal{K}$ is referred to as the <u>kernel</u> of the integral transformation $\Phi_{\mathcal{K}}$ and $(\pi_Y)_*, (\pi_X)^*$, and $\otimes$ are all derived functors.

**Remark 14.** *The literature seems split on what to name this functor. In some sources e.g. Huybrechts 2006, they call this functor a Fourier-Mukai transform. We will reserve that terminology for when the above functor is an equivalence. Further, one notices that by switching the roles of $\pi_X$ and $\pi_Y$ we could have just as easily created a functor in the opposite direction, we will do little to differentiate these two very similar functors and try to make it clear in the exposition.*

We now give a few standard examples of integral kernels.

**Example 1.43.** We will see later on that the integral kernel for the identity on the $G$-equivariant derived categories will be slightly different. Yet, for $X$ an algebraic variety the kernel for the identity functor

$$\mathbb{1}_X : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(X)$$

is naturally isomorphic to the integral transform $\Phi_{\mathcal{O}_\Delta}$ where $\mathcal{O}_\Delta$ is the structure sheaf of the diagonal $\Delta \subset X \times X$. To see this first denote the natural projections.

$$\begin{tikzcd} & X \times X \arrow[dl, "p"'] \arrow[dr, "q"] & \\ X & & X \end{tikzcd}$$

Also, for $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$ denoting the diagonal morphism we see

$$\begin{aligned} \Phi_{\mathcal{O}_\Delta}(\mathcal{E}^\bullet) &= p_*\left(q^*\mathcal{E}^\bullet \otimes \mathcal{O}_\Delta\right) = p_*\left(q^*\mathcal{E}^\bullet \otimes \iota_*\mathcal{O}_X\right) \\ &\cong p_*\left(\iota_*\left(\iota^*q^*\mathcal{E}^\bullet \otimes \mathcal{O}_X\right)\right) && \text{(projection formula)} \\ &\cong (p \circ \iota)_*(q \circ \iota)^*\mathcal{E}^\bullet \cong \mathcal{E}^\bullet && (\text{as } p \circ \iota = \mathbb{1} = q \circ \iota) \end{aligned}$$

**Example 1.44.** For this example we will let $f : X \to Y$ be a morphism of algebraic projective varieties, then we have that

$$f_* \cong \Phi_{\mathcal{O}_{\Gamma_f}} : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(Y),$$

where $\Gamma_f \subset X \times X$ is the graph of $f$. A specific example that will arise for us later we can consider the cohomology $H^*(X, \_\_)$ as the integral transform

$$\Phi_{\mathcal{O}_X} : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(\mathsf{Vec}_f(\mathbb{k})),$$

where $X \subset X \times \mathrm{Spec}(\mathbb{k})$ is considered as the graph of the structure morphism

$$X \to \mathrm{Spec}(\mathbb{k}).$$

**Example 1.45.** Let $\mathcal{L}$ be a line bundle on $X$, an algebraic variety; then $\mathcal{E}^\bullet \mapsto \mathcal{E}^\bullet \otimes \mathcal{L}$ defines a *twist* auto-equivalence $\mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(X)$. This functor is isomorphic to the integral transform with kernel $\iota_*(\mathcal{L})$, where again $\iota : X \xrightarrow{\sim} \Delta \subset X \times X$ is the diagonal morphism for $X$.

The following properties will be used throughout with very little reference so we take the time to list them here. Again all of these properties can be found in any standard text including Huybrechts 2006. The first of these is known as the convolution of two kernels.

Before giving this definition we label the following morphisms:

$$\begin{tikzcd}
 & & & X \times Y \times Z \arrow[dll, "\pi_{XY}"] \arrow[dd, "\pi_{XZ}"] \arrow[drr, "\pi_{YZ}"] & & & \\
 & X \times Y \arrow[dl, "q"] \arrow[dr, "p"] & & & & Y \times Z \arrow[dl, "u"] \arrow[dr, "t"] & \\
X & & Y & X \times Z \arrow[dl, "s"] \arrow[dr, "r"] & Y & & Z \\
 & & X & & Z & &
\end{tikzcd}$$

**Definition 1.46.** Let $\mathcal{P} \in \mathsf{D}^{\mathrm{b}}(X \times Y)$, and $\mathcal{Q} \in \mathsf{D}^{\mathrm{b}}(Y \times Z)$, we define the convolution of $\mathcal{P}$ and $\mathcal{Q}$, denoted by $\mathcal{P} \star \mathcal{Q}$ as

$$\mathcal{P} \star \mathcal{Q} := (\pi_{XZ})_* \left((\pi_{XY})^* \mathcal{P} \otimes (\pi_{YZ})^* \mathcal{Q}\right)$$

The following is Proposition 5.10 from Huybrechts 2006.

**Proposition 1.** *The composition*

$$\mathsf{D}^{\mathrm{b}}(X) \xrightarrow{\Phi_{\mathcal{P}}} \mathsf{D}^{\mathrm{b}}(Y) \xrightarrow{\Phi_{\mathcal{Q}}} \mathsf{D}^{\mathrm{b}}(Z)$$

*is isomorphic to the integral transform*

$$\Phi_{\mathcal{P} \star \mathcal{Q}} : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(Z)$$

The following is Exercise 5.12 from Huybrechts 2006, for brevity we will only provide the proof of the first equivalence.

**Lemma 1.47.** *Let* $\mathcal{P} \in \mathsf{D}^{\mathrm{b}}(X \times Y)$ *and* $\Phi := \Phi_{\mathcal{P}} : \mathsf{D}^{\mathrm{b}}(X) \to \mathsf{D}^{\mathrm{b}}(Y)$ *be the associated integral transform, then the following hold:*

*i) For* $f : Y \to Z$ *the composition* $f_* \circ \Phi$ *is isomorphic to the integral transform with kernel* $(\mathbb{1}_X \times f)_* \mathcal{P} \in \mathsf{D}^{\mathrm{b}}(X \times Z)$

*ii) For* $f : Z \to Y$ *the composition* $f^* \circ \Phi$ *is isomorphic to the integral transform with kernel* $(\mathbb{1}_X \times f)^* \mathcal{P} \in \mathsf{D}^{\mathrm{b}}(X \times Z)$

iii) *For* $g : W \to X$ *the composition* $\Phi \circ g_*$ *is isomorphic to the integral transform with kernel* $(g \times \mathbb{1}_Y)^* \mathcal{P} \in \mathsf{D}^{\mathrm{b}}(W \times Y)$

iv) *For* $g : X \to W$ *the composition* $\Phi \circ g^*$ *is isomorphic to the integral transform with kernel* $(g \times \mathbb{1}_Y)_* \mathcal{P} \in \mathsf{D}^{\mathrm{b}}(W \times Y)$

*Proof.* As mentioned above we will only prove i) here, the others follow in the same manner. First we denote a the following morphisms:

$$\begin{tikzcd}
 & X \times Y \arrow[ld, "\pi_X"'] \arrow[rd, "\pi_Y"] & & & X \times Z \arrow[ld, "p"'] \arrow[rd, "\pi_Z"] & \\
X & & Y & X & & Z
\end{tikzcd}$$

we also note the following commutative diagrams:

$$\begin{tikzcd}
X \times Y \arrow[r, "\mathbb{1}_X \times f"] \arrow[d, "\pi_Y"] & X \times Z \arrow[r, "\pi_Z"] & Z & X \times Y \arrow[r, "\mathbb{1}_X \times f"] \arrow[d, "\pi_X"] & X \times Z \arrow[r, "p"] & X \\
Y \arrow[rru, "f"'] & & & X \arrow[rru, "\mathbb{1}_X"'] & &
\end{tikzcd}$$

Now we have the following calculation, for $\mathcal{E}^\bullet$:

$$\begin{aligned}
(\pi_Z)^* \left( p^* \mathcal{E}^\bullet \otimes (\mathbb{1}_X \times f)_* \mathcal{P} \right) &\cong (\pi_Z)_* (\mathbb{1}_X \times f)_* \left( (\mathbb{1} \times f)^* p^* \mathcal{E}^\bullet \otimes \mathcal{P} \right) \text{ (projection formula)} \\
&\cong (\pi_Z \circ (\mathbb{1}_X \times f))_* \left( (p \circ (\mathbb{1}_X \times f))^* \mathcal{E}^\bullet \otimes \mathcal{P} \right) \\
&\cong (f \circ \pi_Y)_* (\pi_X^* \mathcal{E}^\bullet \otimes \mathcal{P}) \text{ (commutative diagrams shown above)} \\
&\cong f_* (\pi_Y)_* (\pi_X^* \mathcal{E}^\bullet \otimes \mathcal{P}) \\
&= f_* \circ \Phi(\mathcal{E}^\bullet)
\end{aligned}$$

□

## 1.8 Decompositions of the Derived Category

In this section we recall some basics decomposition tools for derived categories, for more details the reader is directed towards A. I. Bondal and Mikhail Mikhailovich Kapranov 1990; Alexei Bondal and Dmitri Orlov 1995; Huybrechts 2006. throughout $\mathcal{T}$ be a triangulated category.

**Definition 1.48.** A collection $\Omega$ of objects in $\mathcal{T}$ is a spanning class of $\mathcal{T}$ (or spans $\mathcal{T}$) if for all $B \in \mathcal{T}$ the following two conditions hold:

1. If $\mathrm{Hom}(A, B[i]) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.
2. If $\mathrm{Hom}(B[i], A) = 0$ for all $A \in \Omega$ and all $i \in \mathbb{Z}$, then $B \cong 0$.

**Definition 1.49.** For a collection $\Omega$ of objects in $\mathcal{T}$ we denote $\langle \Omega \rangle$ as the smallest thick full triangulated sub-category of $\mathcal{T}$ containing the elements of $\Omega$, and refer to it as the span of $\Omega$ or $\mathcal{T}$ is generated by $\Omega$.

**Definition 1.50.** Let $\mathcal{N} \subseteq \mathcal{T}$ be a full triangulated sub-category. We define the right orthogonal to $\mathcal{N}$ is the full subcategory denoted $\mathcal{N}^{\perp} \subseteq \mathcal{T}$ consisting of objects $\mathcal{E}$ such that $\mathrm{Hom}(N, \mathcal{E}) = 0$ for any $N \in \mathcal{N}$. The left orthogonal, ${}^{\perp}\mathcal{N}$ is defined similarly.

**Remark 15.** *Note that each orthogonal is also a triangulated sub-category, see Alexei Bondal and Dmitri Orlov 1995.*

**Definition 1.51.** Let $I : \mathcal{N} \hookrightarrow \mathcal{T}$ be an embedding of a full triangulated sub-category $\mathcal{N}$. We say that $\mathcal{N}$ is right admissible (resp. left admissible) if there is a right (resp. left) adjoint functor $Q : \mathcal{T} \to \mathcal{N}$. The subcategory $\mathcal{N}$ will be called admissible if it is right and left admissible.

**Remark 16.** *If $\mathcal{N} \subseteq \mathcal{T}$ is a right admissible subcategory, then we say that the category $\mathcal{T}$ has a weak semiorthogonal decomposition $\left\langle \mathcal{N}^{\perp}, \mathcal{N} \right\rangle$. We say the same when $\mathcal{N} \subseteq \mathcal{T}$ is a left admissible sub-category, we say that $\mathcal{T}$ has a weak semiorthogonal decomposition $\left\langle \mathcal{N}, {}^{\perp}\mathcal{N} \right\rangle$.*

**Definition 1.52.** A sequence of full triangulated sub-categories $(\mathcal{N}_1, ..., \mathcal{N}_n)$ in $\mathcal{T}$ will be called a weak semiorthogonal decomposition of $\mathcal{T}$ when there exists a sequence of left admissible subcategories $\mathcal{T}_1 = \mathcal{N}_1 \subset \mathcal{T}_2 \subset ... \subset \mathcal{T}_n = \mathcal{T}$ such that $\mathcal{N}_\ell$ is left orthogonal to $\mathcal{T}_{\ell-1}$ in $\mathcal{T}_{\ell}$. When we have such categories we write $\mathcal{T} = \langle \mathcal{N}_1, ..., \mathcal{N}_n \rangle$. If all $\mathcal{N}_\ell$ are admissible in $\mathcal{T}$ then we call this decomposition semiorthogonal.

For what follows let $X$ be an algebraic variety, we will denote the category of coherent sheaves of $X$ as $\mathsf{Coh}(X)$.

**Definition 1.53.** We will denote the full triangulated sub-category of $\mathsf{D}(\mathsf{Coh}(X))$ generated by the the complexes with finitely many non-zero positions, which we will call the bounded derived category of coherent sheaves, and denote this category $\mathsf{D}^b(X)$.

**Remark 17.** *There is a natural inclusion of* $\mathsf{Coh}(X)$ *into the category* $\mathsf{D}^b(X)$ *by realizing a coherent sheaf* $M$ *as a complex with only one non-zero entry in the* $0^{th}$ *position.*

## 1.9 BOUSFIELD LOCALIZATIONS

This section recalls Bousfield (co)-localizations which will be a major process in showing fullness of functors later on. We recall that the existence of a Bousfield triangle produces a semi-orthogonal decomposition, and we show that the essential image of our functor is an inclusion into one of these pieces. We refer the reader to Krause 2010 for a more detailed treatment of these concepts. While the proofs of the statements refer to M. Ballard, Diemer, and D. Favero 2017 we recall all of the statements here for ease of reference.

**Definition 1.54.** Let $\mathcal{T}$ be a triangulated category. A Bousfield localization is an exact endofunctor $L : \mathcal{T} \to \mathcal{T}$ equipped with a natural transformation $\delta : 1_{\mathcal{T}} \to L$ such that:

a) $L\delta = \delta L$ and

b) $L\delta : L \to L^2$ is invertible.

A Bousfield co-localization is given by an endofunctor $C : \mathcal{T} \to \mathcal{T}$ equipped with a natural transformation $\epsilon : C \to 1_{\mathcal{T}}$ such that:

a) $C\epsilon = \epsilon C$ and

b) $C\epsilon : C^2 \to C$ is invertible.

**Definition 1.55.** Assume there are natural transformations of endofunctors

$$C \xrightarrow{\epsilon} 1_{\mathcal{T}} \xrightarrow{\delta} L$$

of a triangulated category $\mathcal{T}$ such that

$$Cx \xrightarrow{\epsilon_{Cx}} x \xrightarrow{\delta_x} Lx$$

is an exact triangle for any object $x$ of $\mathcal{T}$. Then we refer to $C \to 1_{\mathcal{T}} \to L$ as a Bousfield triangle for $\mathcal{T}$ when any of the following equivalent conditions are satisfied:

1) $L$ is a Bousfield localization and $C(\epsilon_x) = \epsilon_{C_x}$

2) $C$ is a Bousfield co-localization and $L(\delta_x) = \delta_{L_x}$

3) $L$ is a Bousfield localization and $C$ is a Bousfield co-localization.

For a proof that the above properties are indeed equivalent, we refer the reader to M. Ballard, Diemer, and D. Favero 2017, Definition 3.33. The following Lemma can be found in M. Ballard, Diemer, and D. Favero 2017 as Lemmas 3.3.4 and 3.3.5.

**Lemma 1.56.** *Let $C \to \mathbb{1}_{\mathcal{T}} \to L$ be a Bousfield triangle for a triangulated category $\mathcal{T}$. Then there is a weak Semi-orthogonal decomposition*

$$\mathcal{T} = \langle \operatorname{Im} L, \operatorname{Im} C \rangle$$

*where* Im *denotes the essential image. Further let $C' \to \mathbb{1}_{\mathcal{T}} \to L'$ be another Bousfield triangle for $\mathcal{T}$ such that $LC' \xrightarrow{L(\epsilon')} L$ is an isomorphism, then there is a weak semi-orthogonal decomposition*

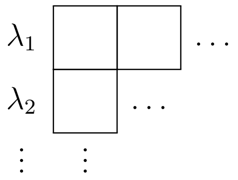
$$\mathcal{T} = \langle \operatorname{Im} L', \operatorname{Im} C' \circ L, \operatorname{Im} C' \circ C \rangle$$

*which induces a fully-faithful functor $F : \mathcal{T}/\operatorname{Im} C \to \mathcal{T}$.*

## 1.10 Kapranov's Collection

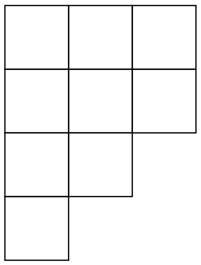
One of the major results in this work is to show the essential image of the associated integral transform for our generalized $Q$-construction, a generalization of the construction from M. Ballard, Diemer, and D. Favero 2017, is the same as an exceptional collection first introduced by M. M. Kapranov 1988. Originally Kapranov arrived at this collection by resolving the diagonal over the Grassmanians, instead of recapping this work we will instead give a combinatorial description of this collection.

Before we jump into our description, first recall that all irreducible modules of $\mathrm{GL}(V)$ are categorized by Young diagrams. Young diagrams are visualizations of partitions of positive integers. Specifically, for a partition $\lambda = (\lambda_1, ..., \lambda_n)$ (we will always follow the convention that $\lambda_1 \geqslant \lambda_2 \geqslant ... \geqslant \lambda_n \geqslant 0$), where $\sum_{i=1}^{d} \lambda_i = n$, we may represent this partition as a Young diagram,



with $\lambda_i$ boxes on the $i$th row aligned to the left. We refer to the number of rows in a partition as the *height* of the partition, and denote $\lambda_1$ as the *width* of the partition. For two partitions $\lambda = (\lambda_1, ..., \lambda_n)$ and $\alpha = (\alpha_1, ..., \alpha_n)$, we write $\lambda \subset \alpha$ when $\lambda_i \leqslant \alpha_i$ for all $i$. If $\lambda_i = \lambda_j$ for all $i$ and $j$, we write $\lambda = (\lambda_1)^n$.

**Example 1.57.** The partition $(3, 3, 2, 1)$ of a 9-element set will be drawn



This partition has height 4 and width 3.

We now recall Kapranov's collection. Let $\mathcal{S}$ be the tautological $d$-dimensional vector bundle over $\mathbb{Gr}(d, W)$, and denote $Z(d, m)$ as the set of sheaves $M_\alpha V$ on $\mathbb{Gr}(d, W)$ where $\alpha$ runs over Young diagrams with no more than $n$ rows and and no more than $m - n$ columns. Furthermore, $M_\alpha$ is the irreducible representation of $\mathrm{GL}(V)$ of highest weight $|\alpha|$. It follows from Weyman 2003, Theorem 2.2.10 that $M_\alpha V = L_{\alpha'} V$, where $\alpha'$ is the conjugate Young diagram of $\alpha$ and $L_{\alpha'} V$ is the Schur module of $V$. As $V$ is naturally a vector bundle on $Z$, so is $L_\alpha V$, and similarly for their duals. We denote

$$\delta_{m,d} := \{\text{Young diagrams } \gamma \text{ with height} \leqslant m - d \text{ and width} \leqslant d\},$$

so that Kapranov's collection is given by

$$\mathsf{K} := \left\{ L_\alpha V^\vee \;\middle|\; \alpha \in \delta_{m,d} \right\}.$$

It will become important in Chapter 2 to note that when $\Bbbk = \mathbb{C}$, this exactly the zero$^{\text{th}}$ window $W_0$ from Donovan and Segal 2014, Section 3.1.

# Chapter 2

# Examples and Further Motivation

One of the main objectives of this work is to outline a procedure which conjecturally should have an appropriate generalization in many occasions that come up "naturally" in mirror symmetry. The standard example which motivates this program is the standard 3-fold flop which we discuss in the so titled section. Yet before we walk through the favorites of the mirror symmetry fan club, we will first take an in depth look at the most used and abused example of all of group actions (in many different contexts of the word) namely projective space, to have as a comparison for our more exotic examples.

## 2.1 Projective Space

Throughout my education I have not yet met a mathematical object which is held in such high regards across so many different fields of mathematics, nor have I met a space which has so many different constructions than that of projective space. The main focus of this work is that of action by algebraic groups, so with that in mind we will choose to build this mythical space using these group actions. First in the classical case of complex manifolds and Lie groups (in the exposition we do not mention the Lie group, yet it is there) then in our case of algebraic geometry. As we have delivered the desired background in the previous chapter we can now jump directly into our basic constructions.

### 2.1.1 The Construction

We now provide examples in the classical manifold case in a small dimension (specifically 2) instead of a more general approach first, with hopes the reader will readily be able to use their imagination and build the more general case. So, we begin with a 2-dimensional complex vector space $V$. Any complex vector space (specifically $V$ in our case), has a natural action by $\mathbb{C}^*$, specifically that induced by scalar multiplication, that is $\mathbb{C}^* \circlearrowright V$. To construct the classical manifold of 1-dimensional projective space (which we will denote $\mathbb{C}P^1$) we subtract the zero vector, $\vec{0}$ (a common term for this vector is the origin) from $V$ and then quotient out by the left group action.

$$ {}_{\mathbb{C}^*}\backslash^{\left(V\backslash\{\vec{0}\}\right)} := \mathbb{C}P^1 $$

At this point the trained differential geometer would next build an appropriate atlas. Yet, for us this is not the story we choose to tell, as these familiar vector spaces are considered by algebraic geometers (and some physicists) as having a much different structure, an algebraic structure, that is we do not simply consider them as vector spaces, yet we consider them as schemes, specifically we consider not $V$ but its analogue from algebraic geometry, that is $\mathbb{A}^2_{\mathbb{C}}$ (while in some of the literature it seems to be common to still continue to use the classical vector space notation). So, using notation of section 1.2 we are considering an action

$$ \mathbb{G}_m \circlearrowright \mathbb{A}^2_{\mathbb{C}} $$

which is equivalent to a co-action of $\Gamma(\mathbb{G}_m, \mathcal{O}_{\mathbb{G}_m}) = \mathbb{C}[x, x^{-1}]$ on $\Gamma(\mathbb{A}^4_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^4_{\mathbb{C}}}) = \mathbb{C}[x_0, x_1] := R$ which is equivalent to a $\mathbb{Z}$-grading by Lemma 1.12. So to summarize we will consider the $\mathbb{Z}$-graded ring

$$ R := \mathbb{C}[x_0, x_1] $$

where $\deg(x_i) = 1$.

The origin in this setting is thought of as the zero locus of the ideal (which we name a little funny to help show the symmetry of our situation) as $I^+ := (x_0, x_1)$. Now, analogously we can define our space sans the origin, i.e. we may consider the space

$$U^+ := \mathbb{A}^2_{\mathbb{C}} \backslash V(I^+) = \operatorname{Spec}(R_{x_0}) \cup \operatorname{Spec}(R_{x_1})$$

(where the appropriate gluing takes place) notice the action of $\mathbb{G}_m$ stays closed on this open set, thus we may take the affine GIT quotient and get

$$U^+ // \mathbb{G}_m = \operatorname{Spec}(R_{x_0}^{\mathbb{G}_m}) \cup \operatorname{Spec}(R_{x_1}^{\mathbb{G}_m}) := \mathbb{P}^1_{\mathbb{C}}$$

## 2.2 The Standard 3-Fold Flop

A further generalization is one of the prototypical flop, that is the 3-fold flop (or even more generally the Atiyah Flop). Most importantly for us (besides the obvious beautiful story) is this example motivates for this entire work. With this at heart the following sections will layout the idea for most generalizations of our main object "The BFD kernel" leaving an exact definition and general consideration for much later in the exposition. In what follows we will define the exact structures we need when we need them.

### 2.2.1 The Construction

First to build this 3-fold flop, again, we begin in the familiar linear algebra setting (the setting first encountered by physicists) and denote $V$ as a 2-dimensional vector space, and denoting $V^\vee$ as the dual space. From the standard action of $\mathbb{C}^*$ on $V$, it is common to induce an action on the dual space by acting by the multiplicative inverse, that is for $a \in \mathbb{C}^*$ and $w \in V^\vee$ we have $a \cdot w = a^{-1}w$ (using the standard notation of $\cdot$ for group action, and concatenation as scalar multiplication). With this setup we have an induced action

$$\mathbb{C}^* \circlearrowright V \oplus V^\vee$$

In the example of projective space we took away the origin and then considered a quotient of the action. In this case we have two different quotients which we will consider:

$$X_- :=_{\mathbb{C}^*} \backslash^{V\oplus\left(V^\vee\backslash\{\vec{0}\}\right)}$$

and

$$X_+ :=_{\mathbb{C}^*} \backslash^{\left(V\backslash\{\vec{0}\}\right)\oplus V^\vee}$$

Again we consider these vector spaces with the additional algebraic information, admittedly this is less intuitive than the previous examples. So, to again use the notation of Section 1.2, we are considering an action

$$\mathbb{G}_m \circlearrowleft \mathbb{A}^4_{\mathbb{C}}$$

We can also consider the algebraic side of this problem by considering the co-action $\Gamma(\mathbb{G}_m, \mathcal{O}_{\mathbb{G}_m}) = \mathbb{C}[x, x^{-1}]$ on
$\Gamma(\mathbb{A}^4_{\mathbb{C}}, \mathcal{O}_{\mathbb{A}^4_{\mathbb{C}}}) = \mathbb{C}[x_0, x_1, y_0, y_1] := R$ which is equivalent to $\mathbb{Z}$-grading by Lemma 1.12. So to summarize we will consider the $\mathbb{Z}$-graded ring

$$R := \mathbb{C}[x_0, x_1, y_0, y_1]$$

where $\deg(x_i) = 1$ and $\deg(y_i) = -1$. In this example since there are two points closed under the action of $\mathbb{G}_m$, there are two important ideals of this action, namely those generated by the positive weights, which we will denote $I^+ := (x_0, x_1)$ and those with the negative weights $I^- := (y_0, y_1)$. As in the example of projective space we would like to consider the open sets associated to the deletion of each of these points.

$$U^+ := \mathbb{A}^4_{\mathbb{C}}\backslash V(I^+) = \operatorname{Spec}(R_{x_0}) \cup \operatorname{Spec}(R_{x_1})$$
$$U^- := \mathbb{A}^4_{\mathbb{C}}\backslash V(I^-) = \operatorname{Spec}(R_{y_0}) \cup \operatorname{Spec}(R_{y_1})$$

Of course again we will need to consider the appropriate gluing, yet unlike the familiar case of projective space we will take a moment to examine the gluing here.

To do so, first we consider the intersection of these two open sets $u_1 := \mathrm{Spec}(R_{x_0})$ and $u_2 := \mathrm{Spec}(R_{x_1})$, their intersection $u_1 \cap u_2 = \mathrm{Spec}(R_{x_0 x_1})$, and the ring morphisms corresponding to this gluing are the obvious localizations maps. Also, there are two natural inclusions which we will denote as the following

$$\begin{tikzcd} & \mathbb{A}^4_{\mathbb{C}} & \\ U^+ \arrow[ur, "i^+"] & & U^- \arrow[ul, "i^-"'] \end{tikzcd}$$

So once again we take the affine GIT quotients and get

$$X_+ := U^+//\mathbb{G}_m = \mathrm{Spec}(R_{x_0}^{\mathbb{G}_m}) \cup \mathrm{Spec}(R_{x_1}^{\mathbb{G}_m})$$
$$X_- := U^-//\mathbb{G}_m = \mathrm{Spec}(R_{y_0}^{\mathbb{G}_m}) \cup \mathrm{Spec}(R_{y_1}^{\mathbb{G}_m})$$

Here the gluing morphisms are induced from the previous morphism, yet lets take a closer look at the rings to gain some insight.

$$R_{x_0}^{\mathbb{G}_m} := \mathbb{C}[x_1 \cdot x_0^{-1}, x_0 \cdot y_0, x_0 \cdot y_1, x_1 \cdot y_0, x_1 \cdot y_1] \cong \mathbb{C}[x, y, z]$$
$$R_{x_1}^{\mathbb{G}_m} := \mathbb{C}[x_0 \cdot x_1^{-1}, x_0 \cdot y_0, x_0 \cdot y_1, x_1 \cdot y_0, x_1 \cdot y_1] \cong \mathbb{C}[x, y, z]$$
$$R_{x_0 x_1}^{\mathbb{G}_m} := \mathbb{C}[x_0 \cdot x_1^{-1}, x_1 \cdot x_0^{-1}, x_0 \cdot y_0, x_0 \cdot y_1, x_1 \cdot y_0, x_1 \cdot y_1] \cong \mathbb{C}[x, x^{-1}, y, z]$$

So here the gluing morphisms are $\varphi_1 : R_{x_0}^{\mathbb{G}_m} \to R_{x_0 x_1}^{\mathbb{G}_m}$ where $\varphi_1(x) = x$, $\varphi_1(y) = y$, and $\varphi_1(z) = z$, as well as $\varphi_2 : R_{x_0}^{\mathbb{G}_m} \to R_{x_0 x_1}^{\mathbb{G}_m}$ where $\varphi_2(x) = x^{-1}$, $\varphi_2(y) = xy$, and $\varphi_2(z) = xz$.

### 2.2.2 The BFD Kernel

One of our objectives is to build a derived equivalence between $D^b_{\mathbb{G}_m}(U^+)$ and $D^b_{\mathbb{G}_m}(U^-)$ (recall Remark 8 so equivalently an equivalence of $D^b(X_+)$ and $D^b(X_-)$). In the following sections we will provide new Fourier-Mukai kernels for this equivalence first shown in A. Bondal and D. Orlov 1995 and then shown in many different fashions by authors like Donovan and Segal 2014 and Kawamata 2002.

So with out further delay, lets introduce one of the main stars of our show. We will denote the BFD compactification of the standard 3-fold flop, as the spectrum of the following ring:

$$Q := \mathbb{C}[x_0, x_1, y_0, y_1, u]$$

with two natural maps from $R$, (at this point these maps may seem mysterious, yet when we explain how $Q$ arises from a partial compactification of the action by $\mathbb{G}_m$, these maps will seem more natural)

$$p : R \to Q$$

$$s : R \to Q$$

defined as

$$\begin{array}{llll} p(x_0) = x_0, & p(x_1) = x_1, & p(y_0) = uy_0, & p(y_1) = uy_1, \\ s(x_0) = ux_0, & s(x_1) = ux_1, & s(y_0) = y_0, & s(y_1) = y_1. \end{array}$$

With this new ring in hand we are ready to explore it's deep connection with these well studied flops. To do so we will consider the sheaf $\mathcal{O}_Q := \mathcal{O}_{\operatorname{Spec} Q}$. Using the maps, $p$ and $s$, we may consider this sheaf as a kernel of an integral transform. Specifically, by considering the integral transform with kernel $(p \times s)_* \mathcal{O}_Q$, which is an endo-functor of the $\mathbb{G}_m$-equivariant derived category $D^b(\mathbb{A}^4_{\mathbb{C}})$, which we will denote

$$\Phi_Q : D^b_{\mathbb{G}_m}(\mathbb{A}^4) \to D^b_{\mathbb{G}_m}(\mathbb{A}^4).$$

The following lemma gives us a realization of this functor.

**Lemma 2.1.** *For any $E \in D^b_{\mathbb{G}_m}(\mathbb{A}^4_{\mathbb{C}})$, we have that*

$$\Phi_Q(E) = s_* p^*(E)$$

*where $p_*$ and $s^*$ are left and right $\mathbb{G}_m$-equivariantley derived respectively.*

*Proof.* To see this, we first note that $Q$ is a flat $R$-module via either the $s$ or $p$. Thus if we denote the projections $\pi_i : \mathbb{A}^4 \times \mathbb{A}^4 \to \mathbb{A}^4$, for $i \in \{1, 2\}$, then from the projection formula it follows that:

$$\begin{aligned}\Phi_Q(E) &= \pi_{2*}\left[\Big((p \times s)_*\mathcal{O}_Q\Big) \otimes^{\mathbb{L}}_{\mathcal{O}_{\mathbb{A}^4}} \Big(\pi_1^* E\Big)\right] \\ &\cong (p \times s)_*\pi_{2*}\left[\mathcal{O}_Q \otimes_{\mathcal{O}_Q} \Big((p \times s)^*\pi_1^* E\Big)\right] \\ &\cong s_* p^* E\end{aligned}$$

as desired. □

We recall here that each $p_*$ and $s^*$ are $\mathbb{G}_m$-equivariant derived functors, that is they are equipped with a restriction of action and extension of action respectively. We recall this, since it will play a significant role in the subsequent sections.

### 2.2.3 Bondal-Orlov Equivalence

For the general case of an Atiyah flop, A. Bondal and D. Orlov 1995 proved there is a derived equivalence between $X_+$ and $X_-$ abstractly, by considering the common blow-up resolving the birational map, that is if we name this common blow-up $\tilde{X}$, we have the following diagram

$$\begin{tikzcd} & \tilde{X} \arrow[dl, "p_+"'] \arrow[dr, "p_-"] & \\ X_+ \arrow[rr, dashed] & & X_- \end{tikzcd}$$

these authors then showed (of course consider the left and right derived functors of the following) that

$$(p_-)_*(p_+)^* : D^b(X_+) \to D^b(X_-)$$

is an equivalence. So again by Remark 8 we have that

$$D^b(\operatorname{coh} X_\pm) \cong D^b(\operatorname{coh}^{\mathbb{G}_m} U^\pm).$$

With the previous identification we will realize this Bondal-Orlov flop equivalence as an equivalence of equivariant derived categories, by using an appropriate restriction of $Q$, and hence providing an explicit kernel.

To achieve this goal we again consider an additional space

$$X_0 := \mathrm{Spec}(R^{\mathbb{G}_m}) = \mathrm{Spec}\left(\mathbb{C}[x_0 \cdot y_0, x_1 \cdot y_1, x_1 \cdot y_0, x_1 \cdot y_1]\right) \cong \mathrm{Spec}(\mathbb{C}[x, y, z, w])$$

as an intermediate space giving us the following diagram:

$$\begin{tikzcd}
X_+ \arrow[rr, dotted] \arrow[dr, "q_+"'] & & X_- \arrow[dl, "q_-"] \\
& X_0 &
\end{tikzcd}$$

and note that in Proposition 5.5 Kawamata 2002 showed that the functor $(p_-)_*(p_+)^*$ is equivalent to taking a fiber product along $q_\pm$ as the Fourier-Mukai kernel of the functor, that is we may consider the Fourier-Mukai functor with kernel $X_+ \times_{X_0} X_-$ for our equivalence.

Now we will see that the fiber product agrees with an appropriate restriction of $Q$, first observed by the authors in Proposition 2.2.3 of M. Ballard, Diemer, and D. Favero 2017.

**Proposition 2.** *With $Q$ as in section 2.2.2, let*

$$Y := Spec(Q) \times_{X \times X} (U^+ \times U^-) \tag{2.1}$$

*denote the restriction $Q$ to the open subset $U^+ \times U^- \subset X \times X$. Then there is a natural isomorphism*

$$Y \cong U^+ \times_{X_0} U^-. \tag{2.2}$$

*That is if we regard* $Q^f = \Gamma(Y, \mathcal{O})$ *as an object of* $D^b(\mathrm{coh}^{\mathbb{G}_m^2} U^- \times U^+)$*, then the Fourier-Mukai functor*

$$\Phi_{Q^f} : D^b(\mathrm{coh}^{\mathbb{G}_m}) \to D^b(\mathrm{coh}^{\mathbb{G}_m} U^+)$$

*is an equivalence and* $\Phi_{Q^f}$ *is naturally isomorhpic to the Bondal/Orlov equivalence as discussed above.*

*Proof.* Let's take a closer look at $Y$. First to build this fiber product we need a morphism $\psi : Q \to X \times X$, this morphism will be induced by the morphisms $p : R \to Q$ and $s : R \to Q$ from section 2.2.2, namely $p \otimes_{\mathbb{C}} s : R \otimes R \to Q$, defined by $r_1 \otimes r_2 \mapsto s(r_1)p(r_2)$. With this morphism in hand there is an open cover for this space, namely that which uses the open cover for $U^+$ and $U^-$, that is for $m, n \in \{0, 1\}$ we consider the collection of open sets:

$$W_{mn} := \mathrm{Spec}(Q) \times (\mathrm{Spec}(R_{x_m}) \times \mathrm{Spec}(R_{y_n}))$$

which has a simple description, using the notation of section 2.2.2:

$$W_{mn} \cong \mathrm{Spec}(Q_{x_m y_n}) = \mathrm{Spec}\left(\mathbb{C}[x_0, x_1, y_0, y_1, u, x_m^{-1}, y_m^{-1}]\right).$$

Now, lets take a deeper look at $U^+ \times_{X_0} U^-$, to do so first we realize this as an open set of the spectrum of $R \otimes_{R^{\mathbb{G}_m}} R$, which this ring, has an easy description:

$$R \otimes_{R^{\mathbb{G}_m}} R = \mathbb{C}[x_{00}, x_{01}, y_{00}, y_{01}, x_{10}, x_{11}, y_{10}, y_{11}]/(x_{0i}y_{0j} - x_{1m}y_{1n} : m, n \in \{0, 1\})$$

Again, we may find an open cover, for $U^+ \times_{X_0} U^-$, which is the collection of the following open sets for $m, n \in \{0, 1\}$

$$V_{mn} := \mathrm{Spec}(R_{x_m}) \times_{X_0} \mathrm{Spec}(R_{y_n}) = \mathrm{Spec}((R \otimes_{R^{\mathbb{G}_m}} R)_{x_{0m}y_{1n}}).$$

and note that

$$(R \otimes_{R^{\mathbb{G}_m}} R)_{x_{0m}y_{1n}} \cong \mathbb{C}[x_{00}, x_{01}, x_{0m}^{-1}, y_{0j}, y_{10}, y_{11}, y_{1n}^{-1}].$$

Next, by using the maps $p : R \to Q$ and $s : R \to Q$ defined in section 2.2.2, we can define a morphism, $p \otimes_{R^{\mathbb{G}_m}} s : R \otimes_{R_0} R \to Q$, again, for $r_1 \otimes r_2 \to p(r_1)s(r_2)$. We will use this morphism to find our desired isomorphism. Specifically this morphism induces a morphism $\varphi_{ij} : W_{ij} \to V_{ij}$, defined by the following the morphism on rings:

$$\begin{aligned} (p \otimes_{R^{\mathbb{G}_m}} s)(x_{00}) &= x_0, & (p \otimes_{R^{\mathbb{G}_m}} s)(x_{01}) &= x_1, \\ (p \otimes_{R^{\mathbb{G}_m}} s)(x_{0i}^{-1}) &= x_m^{-1}, & (p \otimes_{R^{\mathbb{G}_m}} s)(y_{0n}) &= y_n u, \\ (p \otimes_{R^{\mathbb{G}_m}} s)(y_{10}) &= y_0, & (p \otimes_{R^{\mathbb{G}_m}} s)(y_{11}) &= y_1 \\ (p \otimes_{R^{\mathbb{G}_m}} s)(y_{1n}^{-1}) &= y_n^{-1} \end{aligned}$$

hence to see that this morphism is surjective the only thing missing from above is that $(p \otimes_{R^{\mathbb{G}_m}} s)(y_{0n} y_{1n}^{-1}) = u$, and by the definition above it is clear that this morphism is injective, showing our desired isomorphism. $\square$

### 2.2.4 WINDOW EQUIVALENCES

While this classic equivalence is realized by our BFD kernel, to show the further utility of this remarkable object we will next see that the window equivalences of Donovan and Segal 2014 can also be realized as the correct restriction of our BDF compactification. First we recall that in Segal 2011 on $\mathcal{X} := [X//\mathbb{G}_m]$, has a tautological line bundle $\mathcal{O}(-1)$. We will consider the subcategory

$$\mathcal{W}_{-1} := (\mathcal{O}, \mathcal{O}(-1)) \subset D^b(\mathcal{X})$$

That is the smallest triangulated subcategory containing $\mathcal{O}, \mathcal{O}(-1)$ and all finite summands (in some of the literature this is referred to as split generated). In Donovan and Segal 2014 they refer to this subcategory as a *window.* The significance of this window is shown in Proposition 2.1 of Donovan and Segal 2014 using properties of tilting objects in the derived category that $(j_\pm)^* : \mathcal{W}_{-1} \to D^b(X_\pm)$ is an equivalence, where we defined $j_\pm$ are the restrictions of the natural inclusions $j : X_\pm \to \mathcal{X}$. Hence, we may define an equivalence of $D^b(X_+)$ and $D^b(X_-)$ by the following composition.

$$\begin{tikzcd}
& \mathcal{W}_0 \arrow[dr, "(j_-)^*"] & \\
D^b_{\mathbb{G}_m}(X_+) \arrow[ur, "((j_+)^*)^{-1}"] \arrow[rr, dotted, "\cong"] & & D^b(X_-)
\end{tikzcd}$$

Now, with the same ideas of Remark 8 used in Section 2.2.3 we will use the BFD compactification to construct a kernel on the corresponding equivalence of $D^b(\mathrm{coh}^{\mathbb{G}_m}(U^+))$ and $D^b(\mathrm{coh}^{\mathbb{G}_m}(U^-))$. To do so we will first look at the equivalence

$$j_\pm^* : \mathcal{W}_{-1} \to D^b(X_+),$$

and build equivalences, using our BFD compactification, $\Phi_{Q^+} : \overline{\mathcal{W}}_{-1} \to D^b(\mathrm{coh}^{\mathbb{G}_m} U^+)$, and $(i_-)^* : \overline{\mathcal{W}}_{-1} \to D^b_{\mathbb{G}_m}(\mathrm{coh}(U^-))$ where $\overline{\mathcal{W}}_{-1}$ is the appropriate window in $\mathbb{A}^4_{\mathbb{C}}$. That is we will have the following:

$$\begin{tikzcd}
& \overline{\mathcal{W}}_{-1} \arrow[dr, "(i_-)^*"] & \\
D^b_{\mathbb{G}_m}(U^+) \arrow[ur, "\Phi_{Q^+}"] \arrow[rr, dashed, "\cong"] & & D^b_{\mathbb{G}_m}(U^-)
\end{tikzcd}$$

Specifically we may consider the window

$$\overline{\mathcal{W}}_{-1} := (\mathcal{O}_{\mathbb{A}^4}, \mathcal{O}_{\mathbb{A}^4}(-1)) \subset D^b_{\mathbb{G}_m}(\mathbb{A}^4)$$

and defining the morphisms:

$$\Phi_{Q^+} : D^b_{\mathbb{G}_m}(U^+) \to D^b_{\mathbb{G}_m}(\mathbb{A}^4)$$
$$(i_-)^* : D^b_{\mathbb{G}_m}(\mathbb{A}^4) \to D^b_{\mathbb{G}_m}(U^-)$$

as $\Phi_{Q^+} = \Phi_Q \circ (i_+)_*$, where $\Phi_Q$ as in section 2.2.3, and $i_\pm : U^\pm \to \mathbb{A}^4_{\mathbb{C}}$ are the inclusions as defined in section 2.2.1. So, it if we can show that

$$\Phi_{Q^+} \cong \Big( (i_+)^* |_{\overline{\mathcal{W}}_{-1}} \Big)^{-1}, \tag{2.3}$$

it is an immediate consequence that our composition is just the equivalence in Proposition 2.1 Donovan and Segal 2014 with an appropriate shift, which once we explore the geometric meaning of $Q$ gives a geometric understanding of this equivalence, absent from Donovan and Segal 2014.

To prove that we have the equivalence in (2.3), we will first show that if we pre-compose with $(i_+)^*|_{\overline{\mathcal{W}}_{-1}}$, then

$$\Phi_{Q^+} \circ (i_+)^*|_{\overline{\mathcal{W}}_{-1}} \cong 1$$

To see this we will look at the effect of this functor on $(i_+)^*(\mathcal{O}(\ell))$ for $\ell \in \mathbb{Z}$ and then infer it's action on $(i_+)^*(\mathcal{O})$ and $(i_+)^*(\mathcal{O}(-1))$. To achieve this, first note that $(i_+)^*$ is flat and hence $(i_+)_*(i_+)^*(\mathcal{O}(\ell)) = (i_+)^*\mathcal{O}_{U^+}$. To calculate the action of the right derived functor of $(i_+)^*$ on $\mathcal{O}_{U^+}$, we will replace sheaf with the čech complex, since each of the sheaves in said complex are flabby, once taking the right derived functor of $(i_+)^*$, to the resolution we obtain:

$$\mathcal{O}|_{U_0}(\ell) \oplus \mathcal{O}|_{U_1}(\ell) \to \mathcal{O}|_{U_0 \cap U_1}(\ell)$$

Where, as is standard, $U_0 := \mathbb{A}^4_{\mathbb{C}} \backslash V(x_0) \subset U^+$, and $U_1 := \mathbb{A}^4_{\mathbb{C}} \backslash V(x_1) \subset U^+$, and we use a non-standard notation and write, for an open set $W \subset X$, $\mathcal{O}|_W := \iota_{W*}\mathcal{O}_W$, where $\iota_W : W \to \mathbb{A}^4_{\mathbb{C}}$ is the natural inclusion. Now since we are in the derived category of coherent $\mathcal{O}$-modules, this is equivalent to looking at the chain of $\mathbb{C}[x_0, x_1, y_0, y_1]$-modules:

$$R_{x_0}(\ell) \oplus R_{x_1}(\ell) \to R_{x_0 x_1}(\ell)$$

Next, we would like to apply the functor $\Phi_Q$ to this chain. To do this we recall from Lemma 2.1 that this is equivalent to applying the functor, $s_* p^*$, so first we apply the functor $p^*$ to the complex to obtain:

$$(Q_p \otimes R_{x_0}) \begin{pmatrix} \ell \\ 0 \end{pmatrix} \oplus (Q_p \otimes R_{x_1}) \begin{pmatrix} \ell \\ 0 \end{pmatrix} \to (Q_p \otimes R_{x_0 x_1}) \begin{pmatrix} \ell \\ 0 \end{pmatrix}$$

Next, by noting the following degrees:

$$[x_i] = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad [y_i] = \begin{pmatrix} 0 \\ -1 \end{pmatrix}, \quad [u] = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

then since taking invariants (i.e. applying $s^*$) is exact we can just look at $H^0$ and $H^1$ of our complex to see interesting behavior of this map, specifically $H^0$ of the above complex is

$$\bigoplus_{m,n\geqslant 1} k[u] \left(\frac{1}{x_0^m x_1^n}\right) \begin{pmatrix} \ell \\ 0 \end{pmatrix}$$

thus taking the invariant $\begin{pmatrix} 0 \\ \star \end{pmatrix}$ we see that

$$\left(\bigoplus_{m,n\geqslant 1} k[u] \left(\frac{1}{x_0^m x_1^n}\right) \begin{pmatrix} \ell \\ 0 \end{pmatrix}\right)_{\begin{pmatrix} 0 \\ \star \end{pmatrix}} \cong \begin{cases} 0 & \ell \geqslant -1 \\ \left(\bigoplus_{m,n\geqslant 1} k\left(\frac{1}{x_0^m x_1^n}\right)\right)(\ell) & \text{otherwise} \end{cases}$$

Next, $H^1$ of our complex is

$$Q \begin{pmatrix} \ell \\ 0 \end{pmatrix}$$

again taking invariants we see that

$$\left(Q \begin{pmatrix} \ell \\ 0 \end{pmatrix}\right)_{\begin{pmatrix} 0 \\ \star \end{pmatrix}} = \begin{cases} R(\ell) & \ell \leqslant 0 \\ (x_0, x_1)^\ell \otimes_k k[y_0, y_1] & \text{otherwise} \end{cases}$$

So analyzing these two we see that $\Phi_{Q_+}(\mathcal{O}(\ell)) = \mathcal{O}(\ell)$ when $-1 \leqslant \ell \leqslant 0$, which is exactly our window $\overline{\mathcal{W}}_{-1}$. That is this calculation shows that we have our desired equivalence.

# CHAPTER 3

# GRASSMANNIAN FLOPS

In this chapter we explore an application for our generalization of the construction from Drinfeld 2013 mentioned in the introduction. Specifically we show that for Grassmannian Flops, named in Donovan and Segal 2014, that the generalized Drinfeld kernel induces an equivalence. This work appears in M. R. Ballard et al. 2019, where we show that the essential image of this functor aligned with an exceptional collection from M. M. Kapranov 1988. This will first show that our kernel from M. R. Ballard et al. 2019 is indeed a generalized Drinfeld kernel and shows that this kernel is equivalent to the one in Buchweitz, Leuschke, and Van den Bergh 2011 which verifies the equivalence. Then dives into the exceptional collection of M. M. Kapranov 1988.

## 3.1 BASICS

Throughout, fix a $d$-dimensional $\Bbbk$-vector space $V$. Now we will establish further notational conventions. Given a collection $A = \{a_{ij}\}_{i \in [m], j \in [d]}$ of indeterminates, for any subsets $J \subseteq [d]$, $I \subseteq [m]$, we let $A_{I,J}$ denote the collection of variables

$$A_{I,J} := (a_{ij})_{i \in I,\, j \in J}.$$

For $I \subseteq [m]$ with $|I| = d$, we may list this set in increasing order, and denote the corresponding ordered set by $I := \{\ell_1, .., \ell_d\}$. We then write

$$\det\left(A_{I,[d]}\right) := \sum_{\sigma \in \mathfrak{S}_d} \operatorname{sgn}(\sigma) \left( \prod_{1 \leqslant i \leqslant d} A_{\ell_i, \sigma(i)} \right).$$

Given two collections $A = \{a_{ij}\}_{i\in[m],j\in[d]}$ and $B = \{b_{ij}\}_{i\in[d],j\in[m]}$, we use $BA$ to denote the collection of polynomials

$$\left\{\sum_{\ell=1}^{n} b_{i\ell}a_{\ell j}\right\}_{i\in[d],j\in[d]}.$$

Lastly, given two $\Bbbk$-algebras $R$ and $S$ and elements $r \in R$ and $s \in S$, we let $r^L$ and $s^R$ denote the elements $r \otimes 1$ and $1 \otimes s$ in $R \otimes_{\Bbbk} S$.

### 3.2 The PC-kernel

Recall that in M. Ballard, Diemer, and D. Favero 2017 the authors exhibit a kernel using a partial compactification of a certain $\mathbb{G}_m$-action. We follow a similar line of reasoning, and begin by defining our main categories of interest. Let $W$ and $W'$ be arbitrary finite dimensional $\Bbbk$-vector spaces. Next, consider the vector spaces

$$\mathrm{Hom}_{\mathsf{Vec}_{\Bbbk}}(V, W) \oplus \mathrm{Hom}_{\mathsf{Vec}_{\Bbbk}}(W', V),$$

which carry a natural action, for $\varsigma \in \mathrm{GL}(V)$ (a point), $\varphi \in \mathrm{Hom}_{\mathsf{Vec}_{\Bbbk}}(V, W)$, and $\vartheta \in \mathrm{Hom}_{\mathsf{Vec}_{\Bbbk}}(W', V)$ as $\varsigma \cdot (\varphi \circ \varsigma, \varsigma^{-1} \circ \vartheta)$. The category of all such geometric bundles of the above type will be denoted as $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, whose morphisms are morphisms of schemes which are $\mathrm{GL}(V)$-equivariant relative to the above action. Let us provide a few more specifics concerning the action of $\mathrm{GL}(V)$ on arbitrary objects of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$. For such an object $Z$, the induced action

$$\sigma_Z : \mathrm{GL}(V) \times_{\Bbbk} Z \to Z$$

is equivalent to the co-action as Hopf algebra modules. Choosing bases for $V$, $W$ and $W'$, we may write

$$Z = \mathrm{Spec}\left(\Bbbk\left[\{a_{ij}\}_{i\in[d],j\in[m_0]}, \{b_{ij}\}_{i\in[m_1],j\in[d]}\right]\right),$$

where $m_1 = \dim W$ and $m_0 = \dim W'$. Letting $B := (b_{ij})$ and $A := (a_{ij})$, we have

$$Z = \mathrm{Spec}\left(\Bbbk\left[A, B\right]\right). \tag{3.1}$$

The co-action on the global sections

$$\sigma_Z^\sharp : \Bbbk[Z] \to \Bbbk\left[\mathrm{GL}(V)\right] \otimes_\Bbbk \Bbbk[Z],$$

is defined on the generators as

$$\begin{aligned} b_{ij} &\mapsto (\det(C))^{-1} \sum_{r=1}^{d} \mathrm{Adj}(C)_{rj} \otimes_\Bbbk b_{ir} \\ a_{ij} &\mapsto \sum_{r=1}^{d} c_{ir} \otimes_\Bbbk a_{rj}, \end{aligned}$$

where $\mathrm{Adj}(C)$ is the adjoint matrix of $C$.

To build the readers intuition further let $\varphi \in \mathrm{Hom}(V, W)$, and choose a basis of $V$ and $W$ such that $\varphi = (\varphi_{ij})$ with $i \in [m_0]$ and $j \in [d]$, and $\varsigma \in \mathrm{GL}(V)$ such that under the chose of basis for $V$ then $\varsigma = (\varsigma_{s\ell})$ for $s, \ell \in [d]$. Therefore the action of $\mathrm{GL}(V)$ on $\mathrm{Hom}(V, W)$ can be defined as

$$\begin{aligned} (\varsigma, \varphi) &\mapsto \varphi \circ \varsigma \\ \varphi_{ij} &\mapsto \sum_{\ell=1}^{d} \varphi_{i\ell}\varsigma_{\ell j} \end{aligned}$$

Thus we can define a co-action on the dual $(\mathrm{Hom}(V, W))^\vee$ by:

$$\varphi_{ji} = (\varphi_{ij})^\vee \mapsto \left(\sum_{\ell=1}^{d} \varphi_{i\ell}\varsigma_{\ell j}\right)^\vee = \sum_{\ell=1}^{d} \varsigma_{j\ell}\varphi_{\ell i}$$

Therefore if we identify $(\mathrm{Hom}(V, W))^\vee$ with $\mathrm{Sym}^1\left((\mathrm{Hom}(V, W)^\vee\right)$ we get an action on $\mathrm{Spec}\left(\mathrm{Sym}\left((\mathrm{Hom}(V, W))^\vee\right)\right)$, i.e. the geometric bundle of $\mathrm{Hom}(V, W)$ over $\mathrm{Spec}(\Bbbk)$. With a similar calculation on the regular functions of $\mathrm{Hom}(W', V)$ we derive part of the action on $Z$ described above.

Furthermore, the projection

$$\pi_Z : \mathrm{GL}(V) \times_\Bbbk Z \to Z$$

induces the map

$$\pi_Z^\sharp : \Bbbk[Z] \to \Bbbk\left[\mathrm{GL}(V)\right] \otimes_\Bbbk \Bbbk[Z].$$

Further, let $\mathsf{HP}_{\Bbbk}^{\mathrm{GL}(V)}$ denote the full subcategory of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$ consisting of objects of the form

$$\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V)$$

such that $\dim(W), \dim(W') \geqslant \dim(V)$.

**Lemma 3.1.** *For $Z = \mathrm{Spec}(R)$ an object of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, the scheme $\mathrm{GL}(V) \times_{\Bbbk} Z = \Delta_Z$ has a natural $(\mathrm{GL}(V) \times_{\Bbbk} \mathrm{GL}(V))$-action*

$$\sigma_{\Delta_Z} : \mathrm{GL}(V) \times_{\Bbbk} \mathrm{GL}(V) \times_{\Bbbk} \Delta_Z \to \Delta_Z$$

*uniquely determined by the co-action*

$$\sigma^{\sharp}_{\Delta_Z} : \Bbbk[\Delta_Z] \to \Bbbk\left[\mathrm{GL}(V)\right] \otimes_k \Bbbk\left[\mathrm{GL}(V)\right] \otimes_{\Bbbk} \Bbbk[\Delta_Z]$$

*defined by*

$$\begin{aligned} \sigma^{\sharp}_{\Delta_Z}\left(1 \otimes r\right) &= \Big(\iota_1 \otimes 1_{\Delta_Z}\Big) \circ \sigma^{\sharp}_{Z}(r) \\ \sigma^{\sharp}_{\Delta_Z}\left(t \otimes 1\right) &= \Big(\big(1 \otimes \mu^{\sharp}\big) \circ \Big(\beta^{\sharp} \otimes 1\Big) \circ s^{\sharp} \circ \mu^{\sharp}(t)\Big) \otimes 1_R. \end{aligned}$$

*Here $r \in R$, $t \in \Bbbk[\mathrm{GL}(V)]$ and $\iota_1 : \Bbbk\left[\mathrm{GL}(V)\right] \to \Bbbk\left[\mathrm{GL}(V)\right] \otimes_{\Bbbk} \Bbbk\left[\mathrm{GL}(V)\right]$ is the natural inclusion into the first component; $\beta^{\sharp} : \Bbbk\left[\mathrm{GL}(V)\right] \to \Bbbk\left[\mathrm{GL}(V)\right]$ is the co-inverse, $\mu^{\sharp} : \Bbbk\left[\mathrm{GL}(V)\right] \to \Bbbk\left[\mathrm{GL}(V)\right] \otimes \Bbbk\left[\mathrm{GL}(V)\right]$ is the group co-multiplication and $s^{\sharp} : \Bbbk\left[\mathrm{GL}(V)\right] \otimes \Bbbk\left[\mathrm{GL}(V)\right] \to \Bbbk\left[\mathrm{GL}(V)\right] \otimes \Bbbk\left[\mathrm{GL}(V)\right]$ switches the factors in the tensor product.*

*Moreover, the map $\pi_Z \times_{\Bbbk} \sigma_Z : \mathrm{GL}(V) \times_{\Bbbk} Z \to Z \times_{\Bbbk} Z$ is equivariant with respect to this $\mathrm{GL}(V) \times_{\Bbbk} \mathrm{GL}(V)$ action.*

The proof is a straight-forward diagram chase and is left to the reader.

### 3.2.1 THE FUNCTOR

Before turning attention to our functor $\mathbf{Q}$, we introduce the functor $\Delta$, which gives the kernel of the identity functor.

**Notation 1.** *If* $Z = \operatorname{Spec}(R)$ *is an element of* $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, *we define the following scheme*

$$\Delta_Z := Z \times_{\Bbbk} \mathrm{GL}(V).$$

*The assignment* $Z \mapsto \Delta_Z$ *defines a functor* $\Delta : \mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)} \to \mathsf{Aff}_{\Bbbk}^{\mathrm{GL}(V) \times_{\Bbbk} \mathrm{GL}(V)}$.

We aim to use $\Delta_Z$ to produce the Fourier-Mukai kernel for the identity functor on the bounded $\mathrm{GL}(V)$-equivariant derived category $\mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}^{\mathrm{GL}(V)} Z)$ by associating to it an object of $\mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}^{\mathrm{GL}(V) \times_{\Bbbk} \mathrm{GL}(V)} Z \times_{\Bbbk} Z)$. To achieve this, consider the morphism

$$\pi_Z \times_{\Bbbk} \sigma_Z : Z \times_{\Bbbk} \mathrm{GL}(V) \to Z \times_{\Bbbk} Z.$$

We define a sheaf of modules over $Z \times_{\Bbbk} Z$ associated to $\Delta_Z$ as

$$\widetilde{\Delta}_Z := (\pi_Z \times_{\Bbbk} \sigma_Z)_* \mathcal{O}_{\Delta_Z},$$

where $\mathcal{O}_{\Delta_Z}$ denotes the structure sheaf of the affine scheme $\Delta_Z$. It remains to define an action that realizes this sheaf as a $(\mathrm{GL}(V) \times_{\Bbbk} \mathrm{GL}(V))$-equivariant sheaf over $Z \times_{\Bbbk} Z$, which is provided in the next lemma.

**Lemma 3.2.** *Let* $Z = \operatorname{Spec}(R)$ *be an object of* $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$ *and* $M$ *an object of* $\mathrm{Mod}^{\mathrm{GL}(V)}(R)$. *Then there is a* $\Bbbk[\mathrm{GL}(V)]$*-co-module isomorphism*

$$\Big(\Bbbk[\mathrm{GL}(V)] \otimes_{\Bbbk} M\Big)^{\mathrm{GL}(V)} \cong M,$$

*where* $\Bbbk[\mathrm{GL}(V)] \otimes_{\Bbbk} M$ *is given the left* $\Bbbk[\mathrm{GL}(V)]$*-co-action as a* $\Bbbk[\Delta_Z]$*-module.*

*Proof.* Note that there is a natural morphism

$$M \to \Big(\Bbbk[\mathrm{GL}(V)] \otimes_{\Bbbk} M\Big)^{\mathrm{GL}(V)}$$

given by the equivariant structure of $M$. Since the extension $\overline{\Bbbk}/\Bbbk$ is faithfully-flat, it suffices to show that this map is an isomorphism over $\overline{\Bbbk}$. Assume that $\overline{\Bbbk} = \Bbbk$.

By the Peter-Weyl Theorem, there is a decomposition $\Bbbk[\mathrm{GL}(V)] = \bigoplus S_i \otimes S_i^{\vee}$, where $S_i$ runs over every irreducible representation of $\mathrm{GL}(V)$.

Furthermore, since $\mathrm{GL}(V)$ is linearly reductive, we have a decomposition $M = \bigoplus M_i$ into irreducible components. Thus, we have

$$\Big(\Bbbk[\mathrm{GL}(V)] \otimes_{\Bbbk} M\Big)^{\mathrm{GL}(V)} \cong \bigoplus S_i \otimes (S_i^{\vee} \otimes M_i)^{\mathrm{GL}(V)} \cong \bigoplus S_i \otimes \mathrm{Hom}_{\Bbbk}^{\mathrm{GL}(V)}(S_i, M_i),$$

and our result follows from Schur's Lemma. □

**Lemma 3.3.** *For any object $Z$ of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, the object*

$$\widetilde{\Delta}_Z \in \mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}^{\mathrm{GL}(V)\times_{\Bbbk}\mathrm{GL}(V)}(Z \times_{\Bbbk} Z))$$

*is the Fourier-Mukai kernel of the identity functor on $\mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}^{\mathrm{GL}(V)} Z)$.*

*Proof.* First note that since $\Delta_Z$ is flat via either module structure and the Reynolds operator is flat, it is sufficient to prove this on the level of $R$-modules. For an $R$-module $M$, the integral transform associated to $\widetilde{\Delta}_Z$ is given by

$$\Phi_{\widetilde{\Delta}_Z}(\widetilde{M}) := \left[\mathbf{R}\pi_{2*}\left(\widetilde{\Delta}_Z \otimes^{\mathbf{L}} \mathbf{L}\pi_1^*\widetilde{M}\right)\right]^{\mathrm{GL}(V)},$$

where $\pi_i$ are the natural $\mathrm{GL}(V)$-equivariant projections $Z \times_{\Bbbk} Z \to Z$ (see Matthew Ballard, David Favero, and Katzarkov 2014, Section 2 for background). Our desired result is a consequence of the following calculation:

$$\begin{aligned}
\Phi_{\widetilde{\Delta}_Z}(\widetilde{M}) &= \left[\mathbf{R}\pi_{2*}\left[((\pi_Z \times_{\Bbbk} \sigma_Z)_*\mathcal{O}_{\Delta_Z}) \otimes^{\mathbf{L}}_{\mathcal{O}_{Z\times_{\Bbbk}Z}} (\mathbf{L}\pi_1^*\widetilde{M})\right]\right]^{\mathrm{GL}(V)} \\
&\cong \left[\pi_{2*}(\pi_Z \times_{\Bbbk} \sigma_Z)_*\left[\mathcal{O}_{\Delta_Z} \otimes_{\mathcal{O}_{\Delta_Z}} \left((\pi \times_{\Bbbk} \sigma_Z)^*\pi_1^*\widetilde{M}\right)\right]\right]^{\mathrm{GL}(V)} \\
&\cong \left[\sigma_{Z*}\pi_Z^*\widetilde{M}\right]^{\mathrm{GL}(V)} \\
&\cong \left[\mathcal{O}_{\Delta_Z} \otimes_{\mathcal{O}_Z} \widetilde{M}\right]^{\mathrm{GL}(V)} \\
&\cong \left[\left(\mathcal{O}_{\mathrm{GL}(V)} \otimes_{\Bbbk} \mathcal{O}_Z\right) \otimes_{\mathcal{O}_Z} \widetilde{M}\right]^{\mathrm{GL}(V)} \\
&\cong \left[\mathcal{O}_{\mathrm{GL}(V)} \otimes_{\Bbbk} \widetilde{M}\right]^{\mathrm{GL}(V)} \\
&\cong \widetilde{M},
\end{aligned}$$

where the first isomorphism follows from the projection formula, and the last follows from Lemma 3.2.

Furthermore, on the second isomorphism we may forego the process of deriving these functors as they are either exact or remain an adapted class (as discussed above). $\square$

We now define the natural generalization of the functor $Q$ from M. Ballard, Diemer, and D. Favero 2017, Defn 2.1.6.

**Definition 3.4.** Given an object $Z = \mathrm{Spec}(R)$ of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, define

$$Q_Z := \left(\pi_Y^{\sharp}(R), \sigma_Y^{\sharp}(R), C\right) \subseteq \Bbbk\left[\mathrm{GL}(V) \times_{\Bbbk} Z\right].$$

that is the $\Bbbk$-subalgebra of $\Bbbk[\mathrm{GL}(V) \times Z]$ generated by the images of $\sigma_Z^{\sharp}, \pi_Z^{\sharp}$ and the image of the inclusion $\Bbbk[\mathrm{End}(V)] \hookrightarrow \Bbbk[\mathrm{GL}(V) \times_{\Bbbk} Z]$. For ease of notation we denote $\mathbf{Q}_Z := \mathrm{Spec}(Q_Z)$.

**Remark 18.** *Similar to the functor $Q$ in M. Ballard, Diemer, and D. Favero 2017, Def 2.1.6 our definition provides a partial compactification of the action of* $\mathrm{GL}(V)$ *on* $Z$*. For ease of reference we recall the definition of a partial compactification next.*

**Definition 3.5.** Let $G$ be an algebraic group and $Z$ a $\Bbbk$-scheme with $G$-action. Let $\tilde{Z}$ be a $\Bbbk$-scheme together an action of $G \times_{\Bbbk} G$ which is equipped with a $(G \times_{\Bbbk} G)$-equivariant open immersion

$$i : G \times_{\Bbbk} Z \hookrightarrow \widetilde{Z},$$

as well as a $(G \times_{\Bbbk} G)$-equivariant morphism

$$(p, s) : \widetilde{Z} \to Z \times_{\Bbbk} Z$$

such that the following diagram commutes

$$\begin{tikzcd}
 & \widetilde{Z} \arrow[d, shift right, "p"'] \arrow[d, shift left, "s"] \\
G \times_{\Bbbk} Z \arrow[ur, "i"] \arrow[r, shift left, "\pi"] \arrow[r, shift right, "\sigma"'] & Z
\end{tikzcd}$$

In the previous diagram, as usual, $\sigma$ is the action of $G$ on $Z$ and $\pi$ is the projection to $Z$. In this case, we refer to $\widetilde{Z}$, with the maps $p, s, i$, as a *partial compactification of the action of $G$ on $Z$.*

**Example 3.6.** If $\dim V = 1$, the category $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)} = \mathsf{AM}_{\Bbbk}^{\mathbb{G}_m}$ is a subcategory of $\mathsf{CR}_{\Bbbk}^{\mathbb{G}_m}$ as studied in M. Ballard, Diemer, and D. Favero 2017. In this case, the definition of $Q$ given here recovers that found in loc. cit.

**Lemma 3.7.** *Let $Z = \operatorname{Spec}(R)$ be an object of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$. Then there are morphisms*

$$\begin{tikzcd} \mathbf{Q}_Z \arrow[r, shift left, "p"] \arrow[r, shift right, "s"'] & Z. \end{tikzcd}$$

*which precompose with the open immersion $\Delta_Z \to \mathbf{Q}_Z$ to give the morphisms $\pi_Z$ and $\sigma_Z$.*

*Proof.* By definition, the maps $\pi_Z^\sharp$ and $\sigma_Z^\sharp$ both have images which lie in $Q_Z$. $\square$

**Lemma 3.8.** *For any object $Z$ of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, we have an isomorphism*

$$Q_Z \cong \Bbbk\left[A^L, B^L, A^R, B^R, C\right] / \left(B^L - B^R C, A^R - C A^L\right) \cong \Bbbk[A^L, B^R, C]. \tag{3.2}$$

*Proof.* We provide the reader with an easily verifiable isomorphism defined on the generators by

$$c_{ij} \mapsto c_{ij}, \qquad \pi_Z^\sharp(a_{ij}) \mapsto a_{ij}^L, \qquad \pi_Z^\sharp(b_{ij}) \mapsto b_{ij}^L,$$
$$\sigma_Z^\sharp(a_{ij}) \mapsto a_{ij}^R, \qquad \sigma_Z^\sharp(b_{ij}) \mapsto b_{ij}^R.$$

$\square$

**Remark 19.** *It follows from Equation* (3.2) *that $\mathbf{Q}_Z$ is isomorphic to the closed subvariety of $(Z \times_{\Bbbk} Z) \times_{\Bbbk} \operatorname{End}(V)$, consisting of the following points*

$$\{(\psi_1, \psi_2, \psi_3, \psi_4, \varphi) \mid \psi_1 = \psi_3 \circ \varphi,\ \psi_4 = \varphi \circ \psi_2\}.$$

**Lemma 3.9.** *For any object $Z$ of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, the scheme $\mathbf{Q}_Z$ admits a $(\mathrm{GL}(V) \times_{\Bbbk} \mathrm{GL}(V))$-action, denoted $\sigma_{\mathbf{Q}_Z}$, which is uniquely defined by the co-action*

$$\sigma^{\sharp}_{\mathbf{Q}_Z} : \Bbbk[A^L, B^R, C] \to \Bbbk[D^L, (\det D^L)^{-1}] \otimes \Bbbk[D^R, (\det D^R)^{-1}] \otimes \Bbbk[A^L, B^R, C],$$

*which maps the generators*

$$\begin{aligned} b^R_{ij} &\mapsto \big(\det(D^R)\big)^{-1} \sum_{r=1}^{n} \mathrm{Adj}(D^R)_{rj} \otimes_{\Bbbk} b^R_{ir}, \\ a^L_{ij} &\mapsto \sum_{r=1}^{n} {d^L}_{ir} \otimes_{\Bbbk} a^L_{rj}, \\ c_{ij} &\mapsto \big(\det(D^L)\big)^{-1} \sum_{r=1}^{n} \mathrm{Adj}(D^L)_{sj} \otimes_{\Bbbk} {d^R}_{ir} \otimes_{\Bbbk} c_{rs}, \end{aligned}$$

*where $\mathrm{Adj}(D)$ is the adjoint of the matrix $D$.*

*Proof.* This follows by restricting the action of $(\mathrm{GL}(V) \times_{\Bbbk} \mathrm{GL}(V))$ on $\Delta_Z$ that was defined in Lemma 3.1. □

The next lemma gives explicit descriptions of the two module structures that $Q_Z$ possesses.

**Lemma 3.10.** *For $Z = \mathrm{Spec}(\Bbbk[A, B])$ an object of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, we have the following two $\Bbbk[A, B]$-module structures on $Q_Z$ given by $p^{\sharp}$ and $s^{\sharp}$, respectively:*

$$p^{\sharp} : \Bbbk[A, B] \to \Bbbk[A^L, B^R, C]$$

$$B \mapsto B^R C$$

$$A \mapsto A^L$$

$$s^{\sharp} : \Bbbk[A, B] \to \Bbbk[A^L, B^R, C]$$

$$B \mapsto B^R$$

$$A \mapsto C A^L$$

*Proof.* These are just the maps induced by the description of $Q_Z$ from Lemma 3.8 under the identification

$$Q_Z = \Bbbk[A^L, B^R, C].$$

□

**Proposition 2.** *For any object $Z$ of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$ the assignment $Z \mapsto \mathbf{Q}_Z$ defines a functor $\mathbf{Q} : \mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)} \to \mathsf{Aff}_{\Bbbk[\mathrm{End}(V)]}^{\mathrm{GL}(V)\times\mathrm{GL}(V)}$.*

*Proof.* Let $X = \mathrm{Spec}(R)$ and $Y = \mathrm{Spec}(S)$ be objects of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$ with

$$f \in \mathrm{Hom}_{\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}}(X, Y).$$

Note that $\mathbf{Q}f : \mathbf{Q}_X \to \mathbf{Q}_Y$ is defined as the restriction of

$$f \otimes 1 : X \times_{\Bbbk} \mathrm{GL}(V) \to Y \times_{\Bbbk} \mathrm{GL}(V),$$

which is well defined since $f$ is assumed to be $\mathrm{GL}(V)$-equivariant. With this description it is readily verified that indeed $\mathbf{Q}$ is functorial. □

**Remark 20.** *It follows immediately from the definition that $\Delta$ is a subfunctor of $\mathbf{Q}$. Furthermore, this definition easily extends to any affine variety with a $\mathrm{GL}(V)$-action; yet this level of generalization is outside the scope of this paper. We note that our choice of subcategory $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)} \subset \mathsf{Aff}_{\Bbbk}^{\mathrm{GL}(V)}$, is intended to give an appropriate generalization of the varieties considered in Donovan and Segal 2014 while not having to encounter any unnecessary technical difficulties in the statements of this preliminary section.*

Now, we prove some properties of $Q$ that will be used in a later section to prove the fullness of a Fourier-Mukai transform constructed using $Q$.

**Lemma 3.11.** *For an object $Z = \mathrm{Spec}(R)$ of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, we have*

$$\mathrm{Tor}_i({}_pQ_Z, {}_sQ_Z) = 0$$

*for all $i > 0$, where the subscripts preceding $Q_Z$ denote the $R$-module structures given by $p^\sharp$ or $s^\sharp$, respectively.*

*Proof.* Let $R := \Bbbk[A, B]$ as in Equation (3.1). By Lemma 3.10, we have

$$
\begin{aligned}
{}_pQ_Z &\cong \Bbbk[A, B, B', C]/(B - B'C) \\
{}_sQ_Z &\cong \Bbbk[A, B, A', C]/(A - CA')
\end{aligned}
$$

Let us compute $Q_Z\ {}_s\otimes^{\mathbf{L}}_p Q_Z$ using the above expressions:

$$
\begin{aligned}
Q_Z\ {}_s\otimes^{\mathbf{L}}_p Q_Z &= \Bbbk[A, B, A', C]/(A - CA') \otimes^{\mathbf{L}}_{\Bbbk[A,B]} \Bbbk[A, B, B', C]/(B - B'C) \\
&\cong \Bbbk[A, B, A', C]/(A - CA') \otimes_{\Bbbk[A,B]} \mathcal{K}_{\Bbbk[A,B,B',C]}(B - B'C) \\
&\cong \mathcal{K}_{\Bbbk[A,B,A',B',C_1,C_2]/(A - C_2A')}(B - B'C_1) \\
&\cong \mathcal{K}_{\Bbbk[B,A',B',C_1,C_2]}(B - B'C_1),
\end{aligned}
$$

where we resolved the regular sequence $(B - B'C)$ by the Kozsul complex, denoted by $\mathcal{K}$, on the second line.

Finally, we see that the sequence $(B - B'C_1)$ is still regular in the ring $\Bbbk[B, A', B', C_1, C_2]$ and hence all the higher homologies vanish. ☐

**Notation 2.** *Similar to an observation of $\Bbbk[\Delta_Z]$, the ring $Q_Z$ is naturally associated to a sheaf of modules over $Z$, with its module structure defined via $s$ or $p$. We may thus realize $Q_Z$ as a sheaf of modules over $Z \times_\Bbbk Z$, and we denote this module by*

$$\widehat{Q}_Z := (p \times_\Bbbk s)_* \mathcal{O}_{\mathbf{Q}_Z}.$$

*We will use the same notation in the derived setting (see Section 3.2.2, particularly Remark 21). Furthermore, as $\Delta_Z$ is an open subset of $\mathbf{Q}_Z$ we will denote the natural open immersion as*

$$\eta : \mathrm{GL}(V) \times_\Bbbk Z \to \mathbf{Q}_Z.$$

We now specialize to the case where $Z = \mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V)$ is an arbitrary object of $\mathsf{HP}^{\mathrm{GL}(V)}_\Bbbk$, so that $\dim W, \dim W' \geqslant \dim V = d$. We also recall that we denote

$\dim W = m_0$ and $\dim W' = m_1$. Now consider the two open sets

$$U^+ := \Big(\operatorname{Hom}(V, W) \backslash \{\varphi : \operatorname{rank}(\varphi) \leqslant (d-1)\}\Big) \oplus \operatorname{Hom}(W', V)$$

$$U^- := \operatorname{Hom}(V, W) \oplus \Big(\operatorname{Hom}(W', V) \backslash \{\vartheta : \operatorname{rank}(\vartheta) \leqslant (d-1)\}\Big).$$

It will be useful to denote the following open covers of these quasi-affine sets. Let

$$U^+ = \bigcup_{J \subseteq [m_0],\ |J| = d} U_J^+, \tag{3.3}$$

$$U^- = \bigcup_{I \subseteq [m_1],\ |I| = d} U_I^-, \tag{3.4}$$

where

$$U_I^+ := \operatorname{Spec}\left(\Bbbk\left[A, B, \left(\det(A_{[d],I}\right)^{-1}\right]\right)$$

$$U_J^- := \operatorname{Spec}\left(\Bbbk\left[A, B, \left(\det(B_{J,[d]}\right)^{-1}\right]\right)$$

and (for example) $\det\left(A_{[d],I}\right)$ denotes the $(d \times d)$ minor of $A$ consisting of the rows indexed by $I$. Therefore, we have the following affine open covers:

$$U^+ \times_{Z/\!\!/0} U^- = \bigcup_{I \subseteq [m_0],\ J \subseteq [m_1],\ |I| = |J| = d} U_I^+ \times_{Z/\!\!/0} U_J^-, \tag{3.5}$$

$$U^+ \times_{\Bbbk} U^- = \bigcup_{I \subseteq [m_0],\ J \subset [m_1],\ |I| = |J| = d} U_I^+ \times_{\Bbbk} U_J^-, \tag{3.6}$$

where $Z/\!\!/0 := \operatorname{Spec}(\Bbbk[A, B]^{\mathrm{GL}(V)})$ denotes the invariant theoretic quotient of $Z$.

**Lemma 3.12.** *Let $Z$ be an object of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$. There is an isomorphism*

$$\Bbbk\left[Z \times_{Z/\!\!/0} Z\right] \cong \Bbbk\left[A^L, B^L, A^R, B^R\right] / \left(B^L A^L - B^R A^R\right),$$

*where the generators and relations are as in Definition 3.4.*

*Proof.* From Weyl's fundamental theorems for the action of $\mathrm{GL}(V)$ (for example see Kraft and Procesi 1996, Chapter 2.1 or the original text Weyl 1946) we have

$$Z/\!\!/0 = \{D \in \operatorname{Hom}_{\Bbbk}(W, W) \mid \operatorname{rank} D \leqslant \dim V\}.$$

The map $Z \to Z /\!\!/ 0$ is thus given by the homomorphism

$$\Bbbk[Z/\!\!/0] \to k[A, B]$$
$$D \mapsto BA.$$

Hence,

$$\Bbbk\left[Z \times_{Z/\!\!/0} Z\right] = \Bbbk[A^L, B^L] \otimes_{\Bbbk[Z/\!\!/0]} \Bbbk[A^R, B^R] \cong \Bbbk[A^L, B^L, A^R, B^R]/(B^L A^L - B^R A^R).$$

□

**Lemma 3.13.** *There exists a morphism*

$$\kappa := p^\# \otimes s^\# : \Bbbk[A^L, B^L] \otimes_{\Bbbk[Z/\!\!/0]} \Bbbk[A^R, B^R] \to Q_Z.$$

*Proof.* This follows since $p^\#$ and $s^\#$ are equal on $\Bbbk[Z/\!\!/0]$, by definition. □

**Lemma 3.14.** *With the conventions above we have the following containment of ideals in the ring* $\Bbbk[A^L, B^L, A^R, B^R, C]$*:*

$$\left(B^L A^L - B^R A^R\right) \subset \left(B^L - B^R C, A^R - C A^L\right).$$

*Proof.* This follows from

$$(B^L - B^R C)A^L + B^R(CA^L - A^R) = B^L A^L - B^R A^R.$$

□

**Proposition 3.** *Let* $Z = \operatorname{Spec}(\Bbbk[A, B])$ *be an object of* $\mathsf{HP}_{\Bbbk}^{\mathrm{GL}(V)}$ *and* $Q_Z$ *as in Equation* (3.2)*. Let* $\widehat{Q}_Z|_{U^+ \times_{\Bbbk} U^-}$ *be the restriction of* $\widehat{Q}_Z$ *to the open subset* $U^+ \times_{\Bbbk} U^- \subset Z \times_{\Bbbk} Z$*. Then* $\kappa$ *restricts to an isomorphism*

$$\kappa|_{U^+ \times_{\Bbbk} U^-} : \widehat{Q}_Z\Big|_{U^+ \times_{\Bbbk} U^-} \xrightarrow{\sim} \mathcal{O}_{U^+ \times_{Z/0} U^-}.$$

*Proof.* We look affine-locally using the covers of Equations 3.5 and 3.6. We need only show that under the above localization the map $\kappa : \Bbbk[Z \times_{Z/\!\!/0} Z] \to Q_Z$ becomes an isomorphism. For surjectivity, it suffices to show that there is an element (we find two such) which map to $C$. Indeed, we have

$$\begin{aligned} \Big((B^R)_{J[d]}\Big)^{-1} A^L &\mapsto C \\ B^R\Big((A^L)_{[d]I}\Big)^{-1} &\mapsto C, \end{aligned}$$

easily verified by the relations $B^L - B^R C$ and $A^R - CA^L$ in $Q(\Bbbk[A,B])$ given in Definition 3.4.

For injectivity, it suffices to check that under this localization we have the containment

$$\left(B^L - B^R C, A^R - CA^L\right) \subset \left(B^L A^L - B^R A^R\right),$$

since the opposite containment is Lemma 3. To see this, simply note that by multiplying by the appropriate elements in the above identification, we have

$$(B^L)_{J[d]} = (B^R)_{J[d]} C \qquad \text{and} \qquad (A^R)_{[d]I} = C(A^L)_{[d]I}.$$

Hence, multiplying by the appropriate units in our localization, we have

$$\left(B^L - B^R C, A^R - CA^L\right) = \left((B^R)_{J[d]}\left(A^R - CA^L\right), \left(B^L - B^R C\right)(A^L)_{[d]I}\right).$$

For example, by Equation (3.2.1), we have

$$(B^R)_{J[d]}\left(A^R - CA^L\right) = \left((B^R)_{J[d]} A^R - (B^L)_{J[d]} A^L\right) \in \left(B^L A^L - B^R A^R\right),$$

while the other relation follows similarly. This gives our desired isomorphism. □

Consider the restriction $Q|_{U^+ \times U^-}$. By descent, we have a corresponding object $P$ on the quotient $Z^+ \times Z^-$.

**Theorem 3.15.** *For an object $Z$ of $\mathsf{HP}_{\Bbbk}^{\mathrm{GL}(V)}$ we have an isomorphism*

$$P \cong \mathcal{O}_{Z^+ \times_{Z_0} Z^-}$$

*Proof.* This follows immediately by passing to the quotient in Proposition 3. □

We now examine a useful invariant when studying kernels in the next subsection. Note that for $Z$, an object of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$, the tensor product $Q_Z \, {}_s\!\otimes_p \, Q_Z$ is equipped with a natural $\mathrm{GL}(V)^{\times 4}$-action. This induces a $\mathrm{GL}(V)^{\times 3}$-action, which we denote

$$\sigma_3 : \mathrm{GL}(V)^{\times 3} \times_{\Bbbk} \mathbf{Q}_Z^{\times 2} \to \mathbf{Q}_Z^{\times 2}$$

and is defined as the product of the following compositions

$$\begin{tikzcd}
& \mathrm{GL}(V)^{\times 3} \times_{\Bbbk} \mathbf{Q}_Z^{\times 2} \arrow[dl, "\pi_{1,2,4}"'] \arrow[dr, "\pi_{2,3,5}"] & \\
\mathrm{GL}(V)^{\times 2} \times_{\Bbbk} \mathbf{Q}_Z \arrow[d, "\sigma_{\mathbf{Q}_Z}"'] & & \mathrm{GL}(V)^{\times 2} \times_{\Bbbk} \mathbf{Q}_Z \arrow[d, "\sigma_{\mathbf{Q}_Z}"] \\
\mathbf{Q}_Z & & \mathbf{Q}_Z
\end{tikzcd}$$

Here $\pi_{i,j,k} : \mathrm{GL}(V)^{\times 3} \times_{\Bbbk} \mathbf{Q}_Z^{\times 2} \to \mathrm{GL}(V)^{\times 2} \times_{\Bbbk} \mathbf{Q}_Z$ is the projection onto the $i^{\mathrm{th}}$, $j^{\mathrm{th}}$ and $k^{\mathrm{th}}$ components. For any ring $T$ with $\mathrm{GL}(V)^{\times 3}$-action, we will denote the invariant subring associated to the action corresponding to the middle component of $\mathrm{GL}(V)^{\times 3}$ by $T^{\bowtie}$. The notation $(-)^{\bowtie}$ is suggestive of pinching a module in the middle. Since taking invariants is functorial for equivariant morphisms, we obtain the following:

**Lemma 3.16.** *The following diagram commutes*

$$\begin{tikzcd}
& (Q_Z \, {}_p\!\otimes_\sigma \, \Delta_Z)^{\bowtie} \arrow[dr, "\sim"] & \\
(Q_Z \, {}_p\!\otimes_s \, Q_Z)^{\bowtie} \arrow[ur, "(1\otimes\eta)^{\bowtie}"] \arrow[dr, "(\eta\otimes 1)^{\bowtie}"'] & & Q_Z \\
& (\Delta_Z \, {}_\pi\!\otimes_s \, Q_Z)^{\bowtie} \arrow[ur, "\sim"'] &
\end{tikzcd}$$

*Furthermore, the morphism $\rho_Z : (Q_Z \, {}_p\!\otimes_s \, Q_Z)^{\bowtie} \to Q_Z$ is an isomorphism.*

*Proof.* First recall that we have a presentation from Lemma 3.10 of ${}_sQ_Z$ and ${}_pQ_Z$, which for ease of calculation we set the following simplified notation, with the hope that no confusion arises:

$$\begin{aligned}
{}_pQ_Z &\cong \Bbbk[A, B^L, B^R, C]/\left(B^L - B^R C\right) \cong \Bbbk[A, B^R, C] \\
&:= \Bbbk[A, B, C] \\
{}_sQ_Z &\cong \Bbbk[A^L, A^R, B, C]/\left(A^R - CA^L\right) \cong \Bbbk[A^L, B, C] \\
&:= \Bbbk[A, B, C]
\end{aligned}$$

Further we recall the notational preference that for $\Bbbk$-algebras $R, S$ and $r \in R$, $s \in S$ that the following pure tensors will be denoted: $r \otimes 1 := r^L$ and $1 \otimes s := s^R$. With these conventions we have the following presentations of rings:

$$\begin{aligned}
Q_Z \;{}_p{\otimes}_s\; Q_Z &\cong \Bbbk[A^L, A^R, B^L, B^R, C^L, C^R]\,/\big(B^L C^L - B^R, A^L - C^R A^R\big) \\
&\cong \Bbbk[A^R, B^L, C^L, C^R] \\
\Bbbk[\Delta_Z] \;{}_\pi{\otimes}_s\; Q_Z &\cong \Bbbk[A^L, A^R, B^L, B^R, C^L, C^R, \det(C^L)^{-1}]\,/\big(B^L - B^R, A^L - C^R A^R\big) \\
&\cong \Bbbk[A^R, B^L, C^L, C^R, \det(C^L)^{-1}] \\
Q_Z \;{}_p{\otimes}_\sigma\; \Bbbk[\Delta_Z] &\cong \Bbbk[A^L, A^R, B^L, B^R, C^L, C^R, \det(C^R)^{-1}]\Big/\Big(B^L - B^R \left(C^R\right)^{-1}, A^L - C^R A^R\Big) \\
&\cong \Bbbk[A^R, B^L, C^L, C^R, \det(C^R)^{-1}]
\end{aligned}$$

Hence, commutativity of the above diagram is clear. Furthermore, one verifies that we have an isomorphism $\Bbbk[\Delta_Z] \;{}_\pi{\otimes}_s\; Q_Z \cong Q_Z \;{}_p{\otimes}_\sigma\; \Bbbk[\Delta_Z]$, and thus

$$\left(\Bbbk[\Delta_Z] \;{}_\pi{\otimes}_s\; Q_Z\right)^{\bowtie} \cong \left(Q_Z \;{}_p{\otimes}_\sigma\; \Bbbk[\Delta_Z]\right)^{\bowtie}.$$

It is clear that the maps on the right-hand side of the diagram are isomorphisms since $\Bbbk[\Delta_Z]$ is the kernel of the identity by Lemma 3.3. We claim that

$$\begin{aligned}
\left(Q_Z \;{}_p{\otimes}_s\; Q_Z\right)^{\bowtie} &= \Bbbk[A^R, B^L, C^L \cdot C^R] \\
\left(Q_Z \;{}_p{\otimes}_\sigma\; \Bbbk[\Delta_Z]\right)^{\bowtie} &= \Bbbk[A^R, B^L, C^L \cdot C^R]
\end{aligned}$$

from which it follows that these rings are isomorphic. This claim is simply Weyl's Theorem for the invariants of $\Bbbk[V \otimes V^\vee]$.

□

### 3.2.2 THE INTEGRAL KERNEL

We now use $\mathbf{Q}$ to construct Fourier-Mukai kernels. We begin by recalling the following from M. Ballard, Diemer, and D. Favero 2017, Definition 3.1.4.

**Definition 3.17.** For an object $Z$ of $\mathsf{HP}_{\Bbbk}^{\mathrm{GL}(V)}$, we let

$$\widehat{Q}_Z := (p \times s)_* \mathcal{O}_{\mathbf{Q}_Z} \in \mathsf{D}^{\mathsf{b}}\left(\mathrm{Qcoh}^{\mathrm{GL}(V) \times_{\Bbbk} \mathrm{GL}(V)}\, Z \times_{\Bbbk} Z\right),$$

where the pushforward is understood to be derived. We denote by $\widehat{Q}_Z^+$ the quasi-coherent sheaf on $Z_s^{\mathrm{ss}} \times_{\Bbbk} Z$ realized by restricting $\widehat{Q}_Z$ from $Z \times_{\Bbbk} Z$. That is,

$$\widehat{Q}_Z^+ = (j \times 1_Z)^* \widehat{Q}_Z,$$

where $j : Z_s^{\mathrm{ss}} \to Z$ is the inclusion. Finally, taking $\widehat{Q}_Z^+$ as the Fourier-Mukai kernel, we have the functor

$$\Phi_{\hat{Q}_Z^+} : \mathsf{D}^{\mathsf{b}}\left(\mathrm{Qcoh}^{\mathrm{GL}(V)}\, Z_s^{\mathrm{ss}}\right) \to \mathsf{D}^{\mathsf{b}}\left(\mathrm{Qcoh}^{\mathrm{GL}(V)}\, Z\right). \tag{3.7}$$

**Remark 21.** *Since the functor $(p \times s)_*$ is exact, $\widehat{Q}_Z$ is just the* $\mathrm{GL}(V)$*-linearized sheaf associated to $Q_Z$ with its $(p, s)$-bimodule structure given in Lemma 3.1. This justifies our use of $\widehat{Q}_Z$ in Notation 2.*

**Lemma 3.18.** *Let $Z$ be an object of $\mathsf{HP}_{\Bbbk}^{\mathrm{GL}(V)}$. Then $\Phi_{\hat{Q}_Z^+}$ is faithful.*

*Proof.* Our proof follows from the fact that the functor

$$i^* : \mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}_{\mathrm{GL}(V)}(Z_s^{ss})) \to \mathsf{D}^{\mathsf{b}}(\mathrm{Qcoh}_{\mathrm{GL}(V)}(Z))$$

is the left inverse of $\Phi_{\hat{Q}_Z^+}$. To see this, note that for any maximal minor $m$ of $B$, we have $R_m \otimes_s Q_Z \cong \Bbbk[\mathrm{GL}(V)] \otimes_{\Bbbk} R = \Bbbk[\Delta_Z]$. Indeed, inverting a minor on the left amounts to inverting the determinant of $C$. Since $\Delta_Z$ is the kernel of the identity, we obtain the desired result. $\square$

**Theorem 3.19.** *The functor*

$$\Phi_P : \mathsf{D}^{\mathsf{b}}(Z^+) \to \mathsf{D}^{\mathsf{b}}(Z^-)$$

*is an equivalence.*

*Proof.* This follows from Theorem 3.15 and Theorem D in Buchweitz, Leuschke, and Van den Bergh 2011 □

## 3.3 GENERALIZED DRINFELD FOR GRASSMANNIAN FLOPS

It was shown in M. Ballard, Diemer, and D. Favero 2017 that the functor $Q$ recovers a construction of Drinfeld 2013, at least in the affine case. In this section, we show that the kernel presented in this chapter is the same the generalized Drinfeld as mentioned in the introduction.

### 3.3.1 PRELIMINARIES

We now consider the following scheme of $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$ which we will denote as

$$\mathbb{X} = \mathrm{End}(V) \oplus \mathrm{End}(V),$$

and will often write $\mathbb{M}_n := \mathrm{End}(V)$. We equip $\mathbb{X}$ with the $\mathrm{GL}(V)$-action as above, which we denote $\sigma_{\mathbb{X}}$; recall that this action is given by

$$\varsigma \cdot (\varphi, \vartheta) := (\varphi \circ \varsigma^{-1}, \varsigma \circ \vartheta),$$

for $\varsigma \in \mathrm{GL}(V)$ and $\varphi$, $\vartheta \in \mathrm{End}(V)$. We give $\mathbb{X}$ the structure of a scheme over $\mathbb{M}_n$ defined by the morphism

$$\begin{aligned} \mathrm{End}(V) \oplus \mathrm{End}(V) &\to \mathbb{M}_n, \\ (\varphi, \vartheta) &\mapsto \varphi \circ \vartheta. \end{aligned}$$

Notice that this morphism is $\mathrm{GL}(V)$-equivariant with respect to the trivial action on $\mathbb{M}_n$ and the action on $\mathbb{X}$. For an $\mathbb{M}_n$-scheme $S$, we denote

$$\mathbb{X}_S := \mathbb{X} \times_{\mathbb{M}_n} S,$$

and in the case that $S = \mathrm{Spec}(R)$, we write $\mathbb{X}_R$. For any $\mathbb{M}_n$-scheme $S$, one obtains an action of $\mathrm{GL}(V)$ on $\mathbb{X}_S$, via $\sigma_{\mathbb{X}} \times 1$.

**Definition 3.20.** For a $\mathbb{k}$-scheme $Z$ equipped with a $\mathrm{GL}(V)$-action, we define $\mathbb{d}(Z)$ to be the space over $\mathbb{M}_n$ whose value on $S$ is

$$\mathrm{Hom}_{\mathsf{Sch}_{\mathbb{M}_n}}(S, \mathbb{d}(Z)) := \mathrm{Hom}^{\mathrm{GL}(V)}_{\mathsf{Sch}_{\mathbb{k}}}(\mathbb{X}_S, Z).$$

That is, for any $\mathbb{M}_n$-scheme $S$, an $S$-point of $\mathbb{d}(Z)$ is completely defined by the structure morphism $S \to \mathbb{M}_n$ and a $\mathrm{GL}(V)$-equivariant morphism $\mathbb{X}_S \to Z$.

It is not immediately obvious that $\mathbb{d}$ is a representable functor. The main result of the next section is that the functor $\mathbf{Q}$ may be identified with $\mathbb{d}$ for objects of $\mathsf{AM}^{\mathrm{GL}(V)}_{\mathbb{k}}$. While its existence should hold more generally, this fact is outside of the scope of this paper, and we forego any such formal statement or proof.

### 3.3.2 The Drinfeld equivalence

In this section we show that this generalized Drinfeld construction is equivalent to our functor $\mathbf{Q}$ for affine schemes. Notice by Definition 3.4 we have a natural inclusion $\mathbb{k}[\mathbb{M}_n] \hookrightarrow Q_Z$ and hence $\mathbf{Q}_Z$ is naturally an $\mathbb{M}_n$-scheme. This can described on global sections as

$$\begin{aligned} \mathbb{k}[\mathbb{M}_n] := \mathbb{k}[D] &\to Q_Z \\ D &\mapsto C, \end{aligned}$$

where $D = (d_{ij})_{i,j\in[d]}$ is a collection of indeterminants.

**Theorem 3.21.** *Let* $Z = \operatorname{Spec}(\Bbbk[a_{ij}, b_{ij}])$ *be an object of* $\mathsf{AM}_{\Bbbk}^{\mathrm{GL}(V)}$. *Then*

$$\mathbb{d}(Z) \cong \mathcal{F}\left(\mathbf{Q}_Z\right),$$

*where* $\mathcal{F} : \mathsf{Aff}_{\Bbbk[\mathbb{M}_n]}^{\mathrm{GL}(V)\times \mathrm{GL}(V)} \to \mathsf{Aff}_{\Bbbk[\mathbb{M}_n]}^{\mathrm{GL}(V)}$ *is the functor, which forgets the right action of* $\mathrm{GL}(V)$.

*Proof.* Let $R$ be a $\Bbbk$-algebra together with a morphism $\varphi : \operatorname{Spec}(R) \to \mathbb{M}_n$. Such a morphism guarantees the existence of a collection $\mathscr{R} := (r_{ij}^c)_{i,j\in[d]}$ of elements of $R$ such that

$$\Bbbk[\mathbb{X}_R] = R[D^-, D^+]/(D^- D^+ - \mathscr{R}), \tag{3.8}$$

that is, $\varphi^\sharp(d_{ij}) = r_{ij}^c$. The notation we chose will become obvious later in the proof. Furthermore, $\Bbbk[\mathbb{X}_R]$ is a free $R$-module. To see this note that the following is a basis for $\Bbbk[\mathbb{X}_R]$ as an $R$-module

$$\prod_{(\bar{n},\bar{t}\,)\in\mathfrak{A}} \left(d_{ij}^-\right)^{n_{ij}} \left(d_{ij}^+\right)^{t_{ij}},$$

where $\mathfrak{A} \subset \mathbb{N}^{2d^2}$ defined by

$$\mathfrak{A} := \left\{ \left(\bar{n}, \bar{t}\,\right) \in \mathbb{N}^{2d^2} \;\middle|\; \forall k, q \in [d],\;\; n_{k1} = 0 \;\text{ or }\; t_{1q} = 0 \right\}.$$

To briefly see this is indeed true consider the monomial associated to $(\bar{n}, \bar{m}\,) \in \mathbb{N}^{2d^2}$, that is

$$\prod_{(i,j)\in[d]^2} \left(d_{ij}^-\right)^{n_{ij}} \left(d_{ij}^+\right)^{m_{ij}}$$

so that there exists a $k, s \in [d]$ such that $n_{1s} \neq 0$ and $m_{k1} \neq 0$. Then using the relation

$$\sum_{p=1}^{d} d_{sp}^+ d_{pk}^+ = r_{sk} \tag{3.9}$$

so we have that

$$d_{s1}^- d_{1k}^+ = r_{sk} - \sum_{p\neq 1} d_{sp}^- d_{pk}^+$$

Now we assume that without loss of generality $m_{k1} > n_{1s}$, thus

$$\begin{aligned}\left(d^-_{1s}\right)^{n_{1s}}\left(d^+_{k1}\right)^{m_{k1}} &= \left(d^+_{k1}\right)^{m_{k1}-n_{1s}}\left(d^-_{1s}d^+_{k1}\right)^{n_{1s}} \\ &= \left(d^+_{k1}\right)^{m_{k1}-n_{1s}}\left(r_{sk} - \sum_{p\neq 1} d^-_{sp}d^+_{pk}\right)^{n_{1s}}\end{aligned}$$

hence our collection generates. We see that this set is free from Equation 3.9 that any relation on $\Bbbk[\mathbb{X}_R]$ would involve at lease one variable with an index 1.

Now to complete our proof, by Yoneda's Lemma we need only provide an isomorphism of Hom-sets

$$\gamma : \mathrm{Hom}_{\mathsf{Sch}_{\mathbb{M}_n}}(\mathrm{Spec}(R), \mathbf{Q}_Z) \to \mathrm{Hom}^{\mathrm{GL}(V)}_{\mathsf{Sch}_{\Bbbk}}(\mathbb{X}_R, Z).$$

For $f \in \mathrm{Hom}_{\mathsf{Sch}_{\mathbb{M}_n}}(\mathrm{Spec}(R), \mathbf{Q}_Z)$, we define $\gamma(f)$ on global sections as

$$\begin{aligned}\gamma(f)^\sharp(a_{ij}) &:= \sum_{\ell=1}^{n} f^\sharp(a_{i\ell})d^-_{\ell j} \\ \gamma(f)^\sharp(b_{ij}) &:= \sum_{\ell=1}^{n} d^+_{i\ell}f^\sharp(b_{\ell j}).\end{aligned}$$

From this definition one may verify that the above assignment provides a $\mathrm{GL}(V)$-equivariant morphism.

Next, we define an inverse to $\gamma$. Given $g \in \mathrm{Hom}^{\mathrm{GL}(V)}_{\mathsf{Sch}_{\Bbbk}}(\mathbb{X}_R, Z)$, we claim that $g^\sharp(a_{ij}) \in \mathrm{Span}_R\left\{d^-_{\ell j}\right\}_{\ell=1}^{n}$ and $g^\sharp(a_{ij}) \in \mathrm{Span}_R\left\{d^+_{i\ell}\right\}_{\ell=1}^{n}$, that is there exists $r^a_{ij}, r^b_{ij} \in R$ such that

$$\begin{aligned}g^\sharp(a_{ij}) &= \sum_{\ell=1}^{n} r^a_{i\ell}d^-_{\ell j} \\ g^\sharp(b_{ij}) &= \sum_{\ell=1}^{n} r^b_{\ell j}d^+_{i\ell}.\end{aligned}$$

Notice, as $\Bbbk[\mathbb{X}_R]$ is a free $R$-module, these assignments of $r^a_{ij}$ and $r^b_{ij}$ are unique.

Once we verify this claim, we may set

$$\gamma^{-1}(g)^{\sharp}(a_{ij}) := r^{a}_{ij}$$
$$\gamma^{-1}(g)^{\sharp}(b_{ij}) := r^{b}_{ij}$$
$$\gamma^{-1}(g)^{\sharp}(c_{ij}) := r^{c}_{ij},$$

which is obviously the inverse to $\gamma$.

We will now verify that $g^{\sharp}(a_{ij}) \in \mathrm{Span}_R \left\{d^{-}_{\ell j}\right\}_{\ell=1}^{n}$, since the other condition follows similarly. When we denote $\Lambda_{jj} := \mathrm{Span}_{\Bbbk} \left\{a_{ij}\right\}_{i=1}^{n} \cong V^{\vee}$, we see that by Schur's Lemma $g|_{\Lambda_{ij}}$ must either be 0 or an inclusion as it is irreducible. Thus we are looking for the $V^{\vee}$ isotypical piece with a right action of $\mathrm{GL}(V)$ of $\Bbbk[\mathbb{X}_R]$; as $R$ has a trivial action this is equivalent to finding the $V^{\vee}$ isotypical pieces of $\Big(\Bbbk[\mathrm{End}(V)] \otimes_{\Bbbk} \Bbbk[\mathrm{End}(V)]\Big)$. This is done by first recalling that we have the following decomposition into irreducible representations

$$\Bbbk[\mathrm{End}(V)] \cong \bigoplus_{N_i \text{ polynomial}} N_i \otimes N_i^{\vee},$$

where a polynomial representation is one which can be extended to an action by $\mathrm{End}(V)$. Then by Schur's lemma we may calculate the $V^{\vee}$ isotypical piece by the following calculation:

$$\begin{aligned}\Big(\Bbbk[\mathrm{End}(V)] \otimes \Bbbk[\mathrm{End}(V)] \otimes V\Big)^{\mathrm{GL}(V)} &\cong \left(\left(\bigoplus_i N_i \otimes N_i^{\vee}\right) \otimes \left(\bigoplus_j N_j \otimes N_j^{\vee}\right) \otimes V\right)^{\mathrm{GL}(V)} \\ &\cong \left(\bigoplus N_i \otimes N_i^{\vee} \otimes N_j \otimes N_j^{\vee} \otimes V\right)^{\mathrm{GL}(V)} \\ &\cong \bigoplus (N_i \otimes N_j) \otimes \left(N_j^{\vee} \otimes N_i^{\vee} \otimes V\right)^{\mathrm{GL}(V)} \\ &\cong \bigoplus (N_i \otimes N_j) \otimes \left(\bigoplus \left(N_j^{\vee} \otimes N_i^{\vee} \otimes V\right)^{\mathrm{GL}(V)}\right) \\ &\cong \bigoplus (N_i \otimes N_j) \otimes \left(\bigoplus \mathrm{Hom}_{\mathrm{GL}(V)}(V^{\vee}, N_i^{\vee} \otimes N_j^{\vee})\right).\end{aligned}$$

Next, note that

$$\mathrm{Hom}_{\mathrm{GL}(V)}(V^{\vee}, N_i^{\vee} \otimes N_j) \cong \bigoplus \mathrm{Hom}_{\mathrm{GL}(V)}(V^{\vee}, M_i).$$

Where in the previous equation the $M_i$ are the irreducible components of $N_i^\vee \otimes N_j$. It follows from Schur's Lemma that $\mathrm{Hom}_{\mathrm{GL}(V)}(V^\vee, M_i)$ is non-zero only when $M_i = V^\vee$. Hence, from the Littlewood Richardson rule, this can only occur when $N_i^\vee = V^\vee$ and $N_j$ is trivial or vice-versa. To see this recall that the Littlewood Richardson rule is

$$L_\lambda V \otimes L_\lambda V = \bigoplus_{|\nu|=|\lambda|+|\mu|} c(\lambda, \mu; \nu) L_\nu V$$

for some constant $c(\lambda, \mu; \nu)$. Also noting the equality $L_\lambda V^\vee = (L_\lambda V)^\vee$, we see that our desired equality happens when $|\lambda| = 1$ and thus $|\mu| = 0$. Thus the above calculation will only be non-zero when it comes from one factor of $\Bbbk[\mathrm{End}(V)]$. It is well-known that the $V^\vee$ isotypical pieces of $\Bbbk[\mathrm{End}(V)]$ are exactly the $\mathrm{Span}_\Bbbk\{d_{ij}\}_{i=1}^n$, and since our morphism is $\mathrm{GL}(V)$ equivariant it must come from the right side as desired.

□

## 3.4 A GEOMETRIC RESOLUTION

For the remainder of this chapter instead of presenting the alternative approach to the proof of Theorem 3.19 as seen in M. R. Ballard et al. 2019 we instead show that the essential image of our functor $\Phi_Q$ aligns with an exceptional collection of M. M. Kapranov 1988. In this section, we will denote $Z$ as the scheme

$$\mathrm{Hom}(V, W) \oplus \mathrm{Hom}(W', V).$$

Having established that $\Phi_{\hat{Q}^+}$ is fully faithful, the remaining objective of this work is to examine the essential image of the functor $\Phi_{\hat{Q}^+}$. We will show that this image is generated by an exceptional collection first discovered by Kapranov in M. M. Kapranov 1988. The method which we use is based on the underlying techniques of the well known 'geometric technique' of Kempf (see e.g. Weyman 2003).

### 3.4.1 A SKETCH OF KEMPF

The objective of the method of Kempf is to provide a free resolution of special modules by pulling back to a trivial geometric bundle over a projective variety.

Consider an algebraic variety $Y$. The total space of the sheaf $\mathcal{O}_Y^{\oplus n}$ is the scheme $Y \times \mathbb{A}^n$. Now let $X$ be the total space of a locally free sheaf $\mathcal{F} \subset \mathcal{O}_Y^{\oplus n}$ on $Y$. Let $\pi$ denote the projection $Y \times \mathbb{A}^n \to Y$.

We have the exact sequence of locally free sheaves on $Y \times \mathbb{A}^n$

$$0 \longrightarrow \pi^*\mathcal{F} \longrightarrow \pi^*\mathcal{O}_Y^{\oplus n} \xrightarrow{f} \pi^*\mathcal{T} \longrightarrow 0,$$

where $\mathcal{T}$ is the quotient sheaf.

Consider the section $s := f \circ \mathrm{taut} : \mathcal{O}_{Y\times\mathbb{A}^n} \to \pi^*\mathcal{T}$, where taut denotes the tautological section of $\pi^*\mathcal{O}_Y^{\oplus n}$ on $Y \times \mathbb{A}^n$. Then, we have the following statement.

**Proposition 4.** *With the above notation, a locally free resolution of the sheaf $\mathcal{O}_X$ as a $\mathcal{O}_{Y\times\mathbb{A}^n}$-module is given by the Koszul complex*

$$\mathcal{K}(s)_\bullet : 0 \to \bigwedge^{\mathrm{rnk}(\mathcal{T})} (\pi^*\mathcal{T}^\vee) \to \ldots \to \bigwedge^{2} (\pi^*\mathcal{T}^\vee) \to \pi^*\mathcal{T}^\vee \to \mathcal{O}_{Y\times\mathbb{A}^n}$$

*Proof.* On the vanishing locus $Z(s)$, the tautolgical section taut factors through $\pi^*\mathcal{F}$. Hence, the vanishing locus is the total space of the sheaf $\mathcal{F}$, which is $X$. We see that the section is regular as the codimension of $Z(s)$ equals the rank of the sheaf $\pi^*\mathcal{T}$; and the Kozsul complex resolves $\mathcal{O}_X$. For more details, see Weyman 2003, Proposition 3.3.2. ☐

### 3.4.2 THE RESOLUTION

Now we are ready to present a resolution which will open a window to view $\mathrm{Im}(\Phi_{\hat{Q}^+})$. First recall that we set $\dim(V) := d$. We define $\mathbf{Q}_Z^+$ as the base change:

$$\begin{tikzcd}
\mathbf{Q}_Z^+ \arrow[r] \arrow[d] & \mathbf{Q}_Z \arrow[d, "p\times s"] \\
Z_s^{\mathrm{ss}} \times Z \arrow[r, hook] & Z \times Z
\end{tikzcd}$$

Let $\mathcal{S}$ be the tautological bundle on $\mathrm{Gr}(d, W)$ i.e. the locally free sheaf on $\mathrm{Gr}(d, W) = [\mathrm{Hom}(V, W)^{\mathrm{ss}}_{s} / \mathrm{GL}(V)^L]$ corresponding to the $\mathrm{GL}(V)^L$-representation $V$. Then, we have the Euler sequence for the Grassmannian $\mathrm{Gr}(d, W)$:

$$0 \to \mathcal{S} \to W \to \mathcal{Q} \to 0.$$

Consider the pullback of the above sequence to $\mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$ along $q$ and apply $\mathscr{H}om(t^*V, -)$, where $q : \mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V) \to \mathrm{Gr}(d, W)$ and

$$t : \mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V) \to \mathrm{Hom}(W', V)$$

are projections

$$0 \to \mathscr{H}om(t^*V, q^*\mathcal{S}) \xrightarrow{\rho} \mathscr{H}om(t^*V, q^*W) \xrightarrow{\Xi} \mathscr{H}om(t^*V, q^*\mathcal{Q}) \to 0. \tag{3.10}$$

Let us denote $\mathcal{T} := \mathscr{H}om(t^*V, q^*\mathcal{Q})$. We denote the total space of the locally free sheaf $\mathscr{H}om(A, B)$ as $\mathbf{Hom}(A, B)$. From the discussion in the previous subsection, we get the following result:

**Lemma 3.22.** *The following Koszul complex is a free resolution for $\mathcal{O}_{\mathbf{Hom}(t^*V, q^*\mathcal{S})}$ as an $\mathcal{O}_{\mathrm{Gr}(d,W) \times Z}$-module.*

$$\mathcal{K}(s)_{\bullet} : \bigwedge^{d(m-d)} \pi^*\mathcal{T}^{\vee} \to \ldots \to \bigwedge^{2} \pi^*\mathcal{T}^{\vee} \to \pi^*\mathcal{T}^{\vee} \to \mathcal{O}_{\mathrm{Gr}(d,W) \times Z} \tag{3.11}$$

*where $\pi : \mathrm{Gr}(d, W) \times \mathrm{Hom}(V, W) \times \mathrm{Hom}(W', V) \to \mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$ is the projection morphism.*

*Proof.* We choose $Y = \mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$, and $\mathcal{F} = \mathscr{H}om(t^*V, q^*\mathcal{S})$, and apply Proposition 4. Notice that the total space of $\mathscr{H}om(t^*V, q^*W)$ on
$\mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$ is $\mathrm{Gr}(d, W) \times Z$. □

Now, we can identify $[\mathbf{Q}^{+}_{Z} / \mathrm{GL}(V)^L]$ as the total space $\mathbf{Hom}(t^*V, q^*\mathcal{S})$.

**Lemma 3.23.** *The quotient space $[\mathbf{Q}^{+}_{Z} / \mathrm{GL}(V)^L]$ is $\mathrm{GL}(V)^R$-equivariantly isomorphic to the total space $\mathbf{Hom}(t^*V, q^*\mathcal{S})$ as schemes over $\mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$.*

*Proof.* Recall from Equation (3.2), that $\mathbf{Q}_Z$ is associated to the module

$$\Bbbk[A^L, B^R, C]$$

Geometrically, we may view $\mathbf{Q}_Z$ as the total space of the locally free sheaf $\operatorname{End}(V)$ over $\operatorname{Spec}\Bbbk[A^L, B^R]$. Once we base change to the semistable locus and take the quotient with respect to the $\mathrm{GL}(V)^L$ action, we get that $[\mathbf{Q}_Z^+/\mathrm{GL}(V)^L]$ is isomorphic to the total space

$$\mathbf{Hom}(t^*V, q^*\mathcal{S}) \to \mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V).$$

Moreover, the inclusion, $\mathscr{H}om(t^*V, q^*\mathcal{S}) \to \mathscr{H}om(t^*V, q^*W)$ realizes it as a subspace of the total space $\mathbf{Hom}(t^*V, q^*W)$ over $\mathrm{Gr}(d, W) \times \mathrm{Hom}(W', V)$ which is $Z \times \mathrm{Gr}(d, W)$.

This inclusion $\mathscr{H}om(t^*V, q^*\mathcal{S}) \to \mathscr{H}om(t^*V, q^*W)$ is induced by the ring homomorphisn

$$\begin{gathered} k[A^L, A^R, B^R] \to k[A^L, B^R, C] \\ A^L \mapsto A^L \\ A^R \mapsto CA^L \\ B^R \mapsto B^R. \end{gathered}$$

which is equivariant with respect to the remaining $\mathrm{GL}(V)^R$-action. ☐

We denote $\pi_1 : [Z_s^{\mathrm{ss}}/\mathrm{GL}(V)^L] \to [\mathrm{Hom}(V, W)_s^{\mathrm{ss}}/\mathrm{GL}(V)^L]$ as the projection. Putting Lemma 3.22 and Lemma 3.23 together, we get a resolution of the sheaf $(\pi_1 \times \mathrm{Id}_Z)_* \widehat{Q}_Z^+$.

**Corollary 3.24.** *The Koszul complex (3.11) is a locally free resolution of the sheaf* $(\pi_1 \times \mathrm{Id}_Z)_* \widehat{Q}_Z^+$ *of* $\mathcal{O}_{\mathrm{Gr}(d,W)\times Z}$*-modules.*

**Remark 22.** *We note that we could also have constructed a locally free resolution of* $\widehat{Q}_Z^+$ *on* $Z^{\mathrm{ss}} \times Z$ *by the same method, and this will also lead to a similar proof as in the remainder of this paper.*

## 3.5 Analyzing the integral transform

In this section, we show that the kernel $\widehat{Q}_Z^+$ induces a derived equivalence for a Grassmann flop. We begin by showing that the essential image of this functor coincides with the 'window' description studied by Donovan and Segal in Donovan and Segal 2014, Section 3.1. Recall that $Z$ is the scheme

$$\operatorname{Hom}(V, W) \oplus \operatorname{Hom}(W', V)$$

with $\dim(W) =: m, \dim(W') =: m' \geqslant d := \dim(V)$. Specifically, we will show that the image of $\Phi_{\hat{Q}_Z^+}$ is generated by a collection of vector bundles corresponding to representations identified by Kapranov M. M. Kapranov 1988.

Let us recall Kapranov's collection. Consider the standard $\mathrm{GL}(V)$ representation $V$, where $GL(V)$ acts by left multiplication. Consider the Schur modules of $V$ associated to a Young diagram (or equivalently, partition) $\alpha$, and denote them by $L_\alpha V$. Kapranov's collection is defined by

$$\mathcal{K}_{d,m} := \left\{ L_\alpha V \;\middle|\; \alpha \in \text{Young diagrams of height} \leqslant m-d \text{ and width} \leqslant d \right\}.$$

We also consider pull backs of these representations to $\mathrm{Gr}(d, W)$ along the structure morphism. As $V$ pulls back to the tautological bundle $\mathcal{S}$, the Schur functors $L_\alpha V$ pull back to $L_\alpha \mathcal{S}$ and these are the locally free sheaves considered by Kapranov. By abuse of notation, we will consider $\mathcal{K}_{d,m}$ as a collection of locally free sheaves on $\operatorname{Hom}(V, W) \oplus \operatorname{Hom}(W', V)$ or $\operatorname{Hom}(W', V)$ (again, by pulling back along the structure morphism). Note that when $\Bbbk = \mathbb{C}$, this is exactly the dual of the zero$^{\text{th}}$ window $W_0$ from Donovan and Segal 2014, Section 3.1.

It is the objective of this section to show that the thick triangulated subcategory generated by elements of $\mathcal{K}_{d,m}$ is equivalent to $\operatorname{Im}\left(\Phi_{\hat{Q}_Z^+}\right)$. We show one containment in Proposition 5, which relies on the work of Section 3.4.

### 3.5.1 Windows from a resolution

Consider the projection $\pi_1 : Z_s^{\mathrm{ss}} \to \mathrm{Hom}(V,W)_s^{\mathrm{ss}}$. To demonstrate that the image of $\Phi_{\hat{Q}_Z^+}$ is contained in $\langle \mathbb{K}_{d,m} \rangle$, we exhibit a particular $\mathrm{GL}(V)^L \times \mathrm{GL}(V)^R$-equivariant resolution $\mathcal{K}_\bullet$ of $(\mathrm{Id}_Z \times \pi_1)_* \widehat{Q}_Z^+$ over $\mathrm{Hom}(V,W)_s^{\mathrm{ss}} \times Z$. Equivalently, this is a $\mathrm{GL}(V)^R$-equivariant resolution of $(\mathrm{Id}_Z \times \pi_1)_* \widehat{Q}_Z^+$ over $\mathrm{Gr}(d,W) \times Z$. The resolution obtained in equation (3.22) in Section 3.4.2 is the one we are looking for and resolves the functor $\Phi_{\hat{Q}_Z^+} \circ \pi_1^*$.

In this subsection, we will show that the components $\mathcal{K}^i$ of the resolution have a filtration whose associated graded pieces are of the form $J \boxtimes K$ with $K \in \mathbb{K}_{d,m}$. This decomposition of the Fourier-Mukai transform $\Phi_{\hat{Q}_Z^+} \circ \pi_1^*$ yields a functorial way to describe $\Phi_{\hat{Q}_Z^+} \circ \pi_1^*(M)$ using objects of $\mathbb{K}_{d,m}$ for all objects $\pi_1^*(M) \in \mathsf{D}^{\mathsf{b}}([Z_s^{\mathrm{ss}} / \mathrm{GL}(V)])$. As such objects generate $\mathsf{D}^{\mathsf{b}}([Z_s^{\mathrm{ss}} / \mathrm{GL}(V)])$ this is enough to conclude the goal of this section, $\mathrm{Im}\left(\Phi_{\hat{Q}_Z^+}\right) \subseteq \mathbb{K}_{d,m}$.

**Proposition 5.** *With notation as above, we have*

$$\mathrm{Im}\left(\Phi_{\hat{Q}_Z^+}\right) \subseteq \left\langle \mathbb{K}_{d,m} \right\rangle,$$

*where* $\left\langle \mathbb{K}_{d,m} \right\rangle$ *is the thick triangulated subcategory generated by elements in* $\mathbb{K}_{d,m}$.

*Proof.* By Corollary 3.24, we have a quasi-isomorphism with the Koszul complex

$$\mathcal{K}_\bullet \cong (\mathrm{Id}_Z \times \pi_1)_* \widehat{Q}_Z^+$$

The components of the Koszul complex are $\bigwedge^l \pi^* \mathscr{H}om(t^*V, q^*\mathcal{Q})^\vee$ for $0 \leqslant l \leqslant d$. We can appeal to the Cauchy Formula, e.g. Weyman 2003, Theorem 2.3.2(a), to get a filtration on $\bigwedge^i t^*\pi^* \mathscr{H}om(t^*V, q^*\mathcal{Q})^\vee$ whose associated graded pieces are

$$\pi^* \left( \bigoplus_{|\lambda|=i} L_\lambda V \boxtimes L_{\lambda'} \mathcal{Q}^\vee \right).$$

Thus, each term in the Koszul complex can be generated using iterated exact sequences from the locally free sheaves

$$\pi^* \left( L_\lambda V \boxtimes L_{\lambda'} \mathcal{Q}^\vee \right).$$

These components, in turn, generate $\widehat{Q}^+_Z$. Hence, for all $M$, $\Phi_{\widehat{Q}^+_Z}(\pi_1^* M)$ is generated by objects of the form

$$\Phi_{\pi^*(L_\lambda V \boxtimes L_\lambda \mathcal{Q}^\vee)}(\pi_1^* M) = \mathbf{R}\Gamma(M \otimes L_\lambda \mathcal{Q}^\vee) \otimes_{\Bbbk} L_\lambda V$$

all of which lie in $\mathsf{K}_{d,m}$. Now, since $\pi_1$ is an affine map, $\mathsf{D}^{\mathsf{b}}([Z^{\mathrm{ss}}/\operatorname{GL}(V)])$ is generated by the essential image of $\pi_1^*$. The result follows. □

### 3.5.2 Truncation operator

In this section we will see that $\Phi_{\hat{Q}^+_Z}$ has a useful description on $\operatorname{GL}(V)$-representations. Yet before we go deeper into the representation theory we define a truncation operator over our field $\Bbbk$ of characteristic zero.

**Definition 3.25.** Let $M \in \operatorname{Mod}^{\operatorname{GL}(V)}(\Bbbk[\operatorname{Hom}(V, W)])$, we define the *truncation operator* as follows

$$M_{\geqslant 0} := \Big(M \otimes \Bbbk[\operatorname{End}(V)]\Big)^{\operatorname{GL}(V)}$$

Recall, further that there is a $\operatorname{GL}(V) \times_{\Bbbk} \operatorname{GL}(V)$-module decomposition

$$\Bbbk[\operatorname{End}(V)] \cong \bigoplus N_i^\vee \otimes_{\Bbbk} N_i, \tag{3.12}$$

where we sum over all irreducible representations of $\operatorname{GL}(V)$ with all positive weights **lie**, these representations are also referred to as *polynomial representations*. Since $\operatorname{GL}(V)$ is linearly reductive over a field of characteristic zero, we may decompose any $\operatorname{GL}(V)$-module $M$ as $M \cong \bigoplus M_i$, where $M_i$ is irreducible and we have the following description of the truncation operator 3.25.

**Lemma 3.26.** *Let* $M \in \operatorname{Mod}^{\mathrm{GL}(V)}(\Bbbk[\operatorname{Hom}(V,W)])$*; then decompose* $M$ *over* $\Bbbk$ *into irreducibles as*

$$M = \bigoplus_{M_i \text{ irreducible}} M_i. \tag{3.13}$$

*Then the truncation operator may be described as follows*

$$M_{\geqslant 0} = \bigoplus_{\substack{M_i \text{ irreducible} \\ \text{and polynomial}}} M_i$$

□

**Lemma 3.27.** *For any* $M \in \operatorname{Mod}^{\mathrm{GL}(V)}(\Bbbk[\operatorname{Hom}(V,W)])$*,* $M_{\geqslant 0}$ *is a* $\Bbbk[\operatorname{Hom}(V,W)]$*-submodule of* $M$ *and* $(\_)_{\geqslant 0}$ *is exact.*

*Proof.* The exactness of the functor follows since $\mathrm{GL}(V)$ is linearly reductive and thus our operator is just a projection. That $M_{\geqslant 0}$ is a $\Bbbk[\operatorname{Hom}(V,W)]$-submodule follows since $\Bbbk[\operatorname{Hom}(V,W)]_{\geqslant 0} = \Bbbk[\operatorname{Hom}(V,W)]$ since $\Bbbk[\operatorname{Hom}(V,W)]$ is a polynomial representation. □

To deliver a cleaner picture we define some more notation $Y' := \operatorname{Hom}(V,W)$. For the remainder of this subsection we will exploit the commutativity of the following diagram.

$$\begin{tikzcd}
U_Z^+ \arrow[r, hook, "j"] \arrow[d, "q_1|_{U_Z^+}"'] & \operatorname{Hom}(V,W) \oplus \operatorname{Hom}(W',V) \arrow[d, "q_1"] \\
U_{Y'}^+ \arrow[r, hook, "i"'] & \operatorname{Hom}(V,W)
\end{tikzcd}$$

**Lemma 3.28.** *Let* $M \in \mathrm{Mod}^{\mathrm{GL}(V)}(\Bbbk[\mathrm{Hom}(V, W)])$ *then*

$$\Phi_{Q_{Y'}}(M) = M_{\geqslant 0}$$

*Proof.* The coaction map defines a morphism

$$M_{\geqslant 0} \to (\Bbbk[\mathrm{End}(V)] \otimes M_{\geqslant 0})^{\mathrm{GL}(V)} \hookrightarrow (\Bbbk[\mathrm{End}(V)] \otimes M)^{\mathrm{GL}(V)},$$

which we claim is an isomorphism. Notice that the coaction map lands in $\Bbbk[\mathrm{End}(V)] \subset \Bbbk[\mathrm{GL}(V)]$ as $M_{\geqslant 0}$ is a polynomial representation. To check that this map is an isomorphism, we may base change to $\overline{\Bbbk}$ (which is faithfully flat over $\Bbbk$). Hence, assume that $\Bbbk = \overline{\Bbbk}$.

Using equation (3.12) and Lemma 3.26, we get

$$\begin{aligned}(\Bbbk[\mathrm{End}(V)] \otimes M)^{\mathrm{GL}(V)} &\cong \bigoplus N_j \otimes (N_j^\vee \otimes M_i)^{\mathrm{GL}(V)} \\ &\cong M_{\geqslant 0}\end{aligned}$$

where we are considering the left $\mathrm{GL}(V)$ invariant submodule and the second line follows from Schur's Lemma.

Finally, by Lemma 3.8 we have $Q_{Y'} \cong \Bbbk[A] \otimes \Bbbk[\mathrm{End}(V)]$, and we get

$$\begin{aligned}(Q_{Y'} \otimes M)^{\mathrm{GL}(V)} &\cong \Bbbk[A] \otimes (\Bbbk[\mathrm{End}(V)] \otimes M)^{\mathrm{GL}(V)} \\ &\cong M_{\geqslant 0}.\end{aligned}$$

□

**Lemma 3.29.** *We have an isomorphism*

$$(q_1 \times \mathrm{Id})_{*\ s}(Q_Z)_p \cong (\mathrm{Id} \times q_1)^{*}{}_s(Q_{Y'})_p$$

*as objects of* $\mathrm{Mod}^{\mathrm{GL}(V) \times \mathrm{GL}(V)}(Y' \times Z)$.

*Proof.* This follows from the following calculation.

$$\begin{aligned} {}_sQ_Z &\cong \Bbbk[A^L, B^R, C] \\ &\cong \Bbbk[A^R, B^R] \otimes_{\Bbbk[A]} \Bbbk[A^L, C] \\ &\cong Z \otimes_{\Bbbk[Y']} Q_{Y'}, \end{aligned}$$

where the first isomorphism follows from Lemma 3.10 and in the second line, $k[A]$ acts on the left by going to $A^R$ and on the right by going $CA^L$. ☐

**Corollary 3.30.** *Let* $M \in \mathrm{Mod}^{\mathrm{GL}(V)}(k[Y'])$, *then*

$$\Phi_{Q_Z}(q_1^* M) \cong q_1^* \Phi_{Q_{Y'}}(M)$$

*Proof.* This follows from Lemma 3.29 which says that it is true at the level of the Fourier-Mukai kernels. ☐

**Lemma 3.31.** *For* $L_\alpha V \in \text{₭}_{d,m}$ *we have that*

$$(\boldsymbol{R}i_* \boldsymbol{L}i^* L_\alpha V)_{\geqslant 0} \cong L_\alpha V$$

*Proof.* To see this we will denote the irreducible components as $(\_)_\beta$ where $\beta$ is the highest weight corresponding to the isotypical piece, and by $\beta \geqslant 0$ we denote weights correspond to polynomial representations.

$$\begin{aligned}
(\mathbf{R}i_*\mathbf{L}i^*L_\alpha V)_{\geqslant 0} &= \bigoplus_{\beta \geqslant 0} (\mathbf{R}i_*\mathbf{L}i^*L_\alpha V)_\beta \\
&\cong \bigoplus_{\beta \geqslant 0} (\mathbf{R}i_*\mathbf{L}i^*L_\alpha V \otimes L_\beta V^\vee)^{\mathrm{GL}(V)} \\
&\cong \bigoplus_{\beta \geqslant 0} (\mathbf{R}i_*\mathbf{L}i^*(L_\alpha V \otimes L_\beta V^\vee))^{\mathrm{GL}(V)} \\
&\cong \bigoplus_{\beta \geqslant 0} \mathbf{R}\Gamma(\mathrm{Gr}(d, W), L_\alpha S \otimes L_\beta S^\vee) \\
&\cong \bigoplus_{\beta \geqslant 0} \Gamma(\mathrm{Gr}(d, W), L_\alpha S \otimes L_\beta S^\vee) && (3.14) \\
&\cong \bigoplus_{\beta \geqslant 0} \Gamma(\mathrm{Hom}(V, W), L_\alpha V \otimes L_\beta V^\vee)^{\mathrm{GL}(V)} && (3.15) \\
&\cong \bigoplus_{\beta \geqslant 0} (\mathrm{Sym}(\mathrm{Hom}(W, V)) \otimes L_\alpha V \otimes L_\beta V^\vee)^{\mathrm{GL}(V)} \\
&\cong \mathrm{Sym}(\mathrm{Hom}(W, V)) \otimes L_\alpha V && (3.16) \\
&\cong \mathcal{O}_{\mathrm{Hom}(V,W)} \otimes L_\alpha V
\end{aligned}$$

Equation (3.14) follows from M. M. Kapranov 1988, Lemma 3.2.a (this uses the assumption that $L_\alpha V \in \text{Ҡ}_{d,m}$ and the fact that the weights of the irreducible summands of $L_\alpha V \otimes L_\beta V^\vee$ are all strictly larger than $-(m-d)$.) Equation (3.15) follows as $\mathrm{Gr}(d, W)$ has co-dimension greater than 2 in the global quotient stack $[\mathrm{Hom}(V, W)/\,\mathrm{GL}(V)]$. Equation (3.16) follows from Schur's Lemma and the fact that all representations in $\mathrm{Sym}(\mathrm{Hom}(W, V)) \otimes L_\alpha V$ are polynomial (this uses the fact that $L_\alpha V$ is polynomial). $\square$

**Proposition 6.** *If* $L_\alpha V \in \mathscr{K}_{d,m}$ *then*

$$\Phi_{Q_Z^+}(L_\alpha V) \cong L_\alpha V$$

*Proof.* This result follows from another calculation,

$$\begin{aligned}\Phi_{Q_Z^+}(L_\alpha V) &\cong \Phi_Q(\mathbf{R}j_*\mathbf{L}j^*L_\alpha V)\\ &\cong \pi^*\Phi_{Q_{Y'}}(\mathbf{R}i_*\mathbf{L}i^*L_\alpha V)\\ &\cong \pi^*\Big(\mathbf{R}i_*\mathbf{L}i^*L_\alpha V\Big)_{\geqslant 0},\end{aligned}$$

where the second line follows from Corollary 3.30 and the last line by Lemma 3.28. Hence our result follows from Lemma 3.31. ☐

**Corollary 3.32.** $\operatorname{Im}\Phi_{\hat{Q}_Z^+} = \langle \mathscr{K}_{d,m}\rangle$.

*Proof.* This is an immediate consequence of Proposition 5 and Lemma 6. ☐

Note that we have a similar equality for $\Phi_{\hat{Q}^-}$.

**Corollary 3.33.** $\operatorname{Im}\Phi_{\hat{Q}_Z^-} = \langle (\mathscr{K}_{d,m'})^\vee\rangle = \langle \mathscr{K}_{d,m'} \otimes \det(V^*)^{m'-d}\rangle$.

*Proof.* We can switch the roles of $W$ and $W'$ by taking transposes. This is anti-equivariant, i.e., equivariant up to inversion in $\mathrm{GL}(V)$. Consequently, we replace all representations with their duals which gives the first equality. The second is a standard identity. ☐

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