An Equivariant Count of Nodal Orbits in an Invariant Pencil of Conics

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by

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Dedication

For Bernard, Cerita, Christopher, Catherine, Grandpa and, of course, Essie Mae Jones.

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Abstract

Enumerative geometry studies the number of geometric objects in a given class satisfying specific geometric conditions. For example, we can ask how many conics in the plane pass through four points. One such example is a specific case of Göttsche's conjecture, stated in (Göttsche 1998): given a pencil of conics in $\mathbb{P}^2_{\mathbb{C}}$, how many curves in the pencil have nodal singularities? The answer is three as long as the defining conics of the pencil are general. The conjecture was proved in full generality by Y. Tzeng in 2010 in (Tzeng 2012), and another proof exists due to Kool, Shende, and Thomas in 2011 in (Kool, Shende, and Thomas 2011).

Recent developments in motivic homotopy theory have led to enrichments of many enumerative results over non-algebraically closed fields, and examples can be found in (Hoyois 2014), (Kass and Wickelgren 2019), and (Levine 2017). However, the question of replicating enumerative results in the presence of a group action is unstudied.

This work takes a classical enumerative problem in the presence of a group action and enriches it using equivariant topology. Specifically, let X be a pencil of general conics in $\mathbb{P}^2_{\mathbb{C}}$ which is invariant under the linear action of a finite group. The main result is that for finite groups not isomorphic to $\mathbb{Z}/2\times\mathbb{Z}/2$ or D_8, there is a weighted sum valued in the Burnside ring of G-sets of the orbits of nodal conics in X in terms of the base locus of the defining equations. Counterexamples for $\mathbb{Z}/2\times\mathbb{Z}/2$ and D_8 are also given. This is a direct generalization of the specific aforementioned case of Göttsche's conjecture as well as its real analogue, as the classical case is obtained by taking a trivial group action and the case over $\mathbb{R}$ is obtained by taking $G \cong \mathbb{Z}/2$ and the action on $\mathbb{P}^2_{\mathbb{C}}$ to be coordinate-wise conjugation.

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Chapter 1

Introduction and Notation

1.1 Motivation

We will say that two equations for conics, f and g, in $\mathbb{P}^2_{\mathbb{C}}$ are *general*, or in *general position*, if $\Sigma := \{p \in \mathbb{P}^2_{\mathbb{C}} \colon f(p) = g(p) = 0\}$ contains four points with no three co-linear. Given a pair of conics defined by f and g in general position in $\mathbb{P}^2_{\mathbb{C}}$, we can form a family of curves parameterized by $\mathbb{P}^1_{\mathbb{C}}$, $X := \{\mu f + \lambda g = 0 \colon \mu, \lambda \in \mathbb{C}\} \subseteq \mathbb{P}^2_{\mathbb{C}}$, where μ and λ are not both equal to 0. This is called a *pencil* of conics, and X is referred to as the *pencil of conics spanned by f and g*. The set Σ is called the *base locus* of X. Specializing μ and λ gives a different element in X.

A natural question to ask is the following: how many curves in X have nodal singularities? In other words, for how many values of $[\mu : \lambda] \in \mathbb{P}^1_{\mathbb{C}}$ is $\mu f + \lambda g$ nodal? A conic is said to be *nodal* if there exists a point at which it can be parameterized as the product of two lines. As long as f and g are general pair of conics, there are always exactly three nodal conics in X independent of the defining equations of f and g. This is a very special case of Göttsche's conjecture, first stated in (Göttsche 1998), and proven by Y. Tzeng in 2010 in (Tzeng 2012). An elementary proof can be found in (Eisenbud and Harris 2016). One can ask if there's a similar result in the presence of a finite group action. This question is answered by Theorem 2.2, stated and proved in Chapter 2.

1.2 Definitions and Notation

In this section, we will introduce all definitions and notation from group theiry needed to read this text. We will always assume that G is a finite group unless otherwise stated. A *finite G-set* is a finite set S together with an action of G on this set. All group actions are assumed to be left actions in this paper. Given G-sets S and T, a set map $f\colon S \to T$ is *G-equivariant* if $g \cdot f(s) = f(g \cdot s)$ for all g in G. A *G-isomorphism* is a G-equivariant map of G-sets which is as isomorphism as a set function. Given a G-set S and a subset S' of S, we will say S' is *G-invariant*, or that G *acts invariantly on S'*, if $g \cdot s' \in S'$ for all $s' \in S'$.

Given any two G-sets S and T, there are natural set operations on S and T by which other G-sets can be obtained. We can take the disjoint union of S and T, $S \amalg T$, or the cartesian product, $S \times T$, and obtain G-sets by letting G act diagonally in either case. Let $A(G)^+$ denote the monoid of G-isomorphism classes of G-sets with addition given by disjoint union.

Definition 1.1. Given a group G, the *Burnside ring* of G is the Grothendieck group completion of $A(G)^+$. The Burnside ring of a group G will always be denoted by $A(G)$. Given any G-set S, we will denote its isomorphism class in $A(G)$ by $[S]$. The set $\{*\}$ will always denote the one-point set, which can only be given the trivial action. $A(G)$ also has a ring structure with multiplication given by Cartesian product.

Any G-set can be written as a disjoint union of its orbits, each of which is G-isomorphic to G/H for some $H \leq G$ by the Orbit-Stabilizer theorem. Thus given a finite G-set S, S can be written as $\amalg G/H_i$ where $\{H_i\}$ is some finite collection of subgroups of G. Recall from basic group theory that two subgroups H and K of a finite group G are conjugate in G if and only if G/H is isomorphic to G/K. This implies that the isomorphism class $[G/H]$ in $A(G)$ is determined by the conjugacy class of H in G, which we will denote by (H). Combining this with the fact that every

G-set can be written as the disjoint union of its orbits, each of which is G-isomorphic to a quotient of G by a subgroup, implies that every genuine G-set $[S]$ in $A(G)$ can be written as $[S] = \sum_{H_i \leq G} n_i[G/H_i]$ uniquely up to (H_i) for each H_i. Any G-set which comes from a genuine set with a group action will be called a *genuine* G-set. This is in contrast to a *virtual* G-set, which can refer to any element of $A(G)$ even if it's expression as a sum of orbits contains differences. For example, $-\{*\}$ is the virtual G-set that is the formal additive inverse of the genuine G-set $\{*\}$. More information on the structure of the Burnside Ring of a finite group and its properties can be found in (Dieck 1979).

Keep in mind that two G-sets may have the same cardinality and be represented differently in A(G), for example $[S_1] = [G/\langle(14)(23)\rangle]$ and $[S_2] = [G/\langle(13)(24)]$ for $G = \{(), (12)(34), (14)(23), (13)(24)\}$. In the example above we could determine if $\langle(14)(23)\rangle$ and $\langle(13)(24)\rangle$ are conjugate in G to determine if $[S_1] = [S_2]$, but it can be inefficient to use this method of determining when two G-sets are equal in $A(G)$ for larger groups. A classical result in equivariant topology, see (Dieck 1979) Proposition 1.2.2, says that two G-sets S_1 and S_2 are equal in $A(G)$ if $|S_1|^H = |S_2|^H$ for all subgroups H of G. Said more succinctly, $S \mapsto (|S|^H) \subseteq \Pi_{H \leq G}\mathbb{Z}$ is injective. We will use this result multiple times in the proof of Theorem 2.2. In the example given above, the number of $\langle(14)(23)\rangle$ fixed-point subsets of $[S_1]$ is 2, but the number of $\langle(14)(23)\rangle$ fixed-point subsets of $[S_2]$ is 0. Thus $[S_1]$ and $[S_2]$ are not equal in $A(G)$ by (Dieck 1979) Proposition 1.2.2.

Given two G-sets S and T, we have already described two ways to produce new G-sets. These two ways are precisely the ring operations in $A(G)$, cartesian product and disjoint union. Given a finite group G and a subgroup H of G, we will now describe a way to obtain a G-set from any H-set.

Definition 1.2. Let G be a finite group and let $H \leq G$. Let X be an H-set. The *inflation of X from H to G*, denoted $\mathrm{inf}_H^G(X)$, is the G-set whose underlying set structure is $(G \times X)/\sim$ where $(gh, x) \sim (g, hx)$ for all $g \in G$, $h \in H$, and $x \in X$. The G-action on $\inf_H^G(X)$ is given by $g' \cdot (g, x) = (g'g, x)$ for all $g' \in G$ and $(g, x) \in \inf_H^G(X)$.

It is worth noting that the inflation is a natural construction to consider because it satisfies the universal property of being adjoint to restriction, i.e. $\mathrm{Hom}_H(X, \mathrm{Res}_H^G(Y)) = \mathrm{Hom}_G(\mathrm{Inf}_H^G(X), Y)$. Here, given a G set X and a subgroup $H \leq G$, $\mathrm{Res}_H^G(X)$ is the H-set with set X and H-action coming from the action of G.

Every G-set we will encounter in this paper will already be represented as a formal sum of orbits, each equal to $[G/K]$ for some $K \leq G$. Therefore for any $H \leq G$ it will be useful to have a description of the inflation of an H-set to G when represented by a sum of orbits of this form. The following lemma gives such a description.

Lemma 1.3. Let H be a subgroup of a finite group G and let $[X] = \sum_{i=1}^m n_i[H/K_i]$ for some $m \in \mathbb{N}$, some $n_1, \ldots, n_m \in \mathbb{Z}$, and $K_i \leq H$ for $1 \leq i \leq m$ be an H-set in $A(H)$. Then $\inf_H^G(X) = \sum_{i=1}^m n_i[G/K_i]$ in $A(G)$.

Proof. First note that $\inf_H^G(\sum_{i=1}^m n_i[H/K_i]) = \sum_{i=1}^n n_i \inf_H^G([H/K_i])$ because cartesian products commute with disjoint unions and the action on a disjoint union is assumed to be diagonal, that is by linearity we only need to show that $\inf_H^G(H/K) = [G/K]$ in $A(G)$ for any $K \leq H$. We will explicitly construct a set isomorphism between the two sets and show that it is G-equivariant, which by definition will imply that the two G-sets are equal in $A(G)$.

Define $f\colon \inf_H^G(H/K) \to G/K$ by $f((g, hK)) = ghK$. Observe that $(gh_1, h_2K) \mapsto gh_1h_2K$ and $(g, h_1h_2K) \mapsto gh_1h_2K$, so f is well-defined. Furthermore, given $gK \in G/K$, $f((g, K)) = gK$. Thus f is surjective. Now suppose that $f((g_1, h_1K)) = f((g_2, h_2K))$. This means that $g_1h_1K = g_2h_2K$, so there exists some $k \in K$ such that

$g_1h_1k = g_2h_2$. Therefore

$$\begin{aligned}(g_2, h_2K) &= (g_2h_2K)\\ &= (g_1h_1k, K)\\ &= (g_1, h_1kK)\\ &= (g_1, h_1K)\end{aligned}$$

in $\inf_H^G(H/K)$. Hence f is injective, and we conclude that f is a set isomorphism.

The last step to show that $\inf_H^G(H/K) = [G/K]$ in $A(G)$ is to show that f is G-equivariant. Note that the G-action on G/K is left-multiplication. Observe that $g' \cdot f((g, hK)) = g' \cdot ghK = g'ghK$, and $f(g' \cdot (g, hK)) = f((g'g, hK)) = g'ghK$. Therefore $g' \cdot f((g, hK)) = f(g'(g, hK))$ for all $g' \in G$ and $(g, hK) \in \inf_H^G(H/K)$, so f is G-equivariant as desired. □

We have now introduced all terminology and notation needed to understand Theorem 2.2, and will introduce the main question and the proof of the classical theorem that the answer generalizes before proving Theorem 2.2 in Chapter 2.

1.3 Main Question

Let G be a finite group acting linearly on $\mathbb{P}^2_{\mathbb{C}}$, and let f and g be equations defining a general pair of conics in $\mathbb{P}^2_{\mathbb{C}}$ such that $X := \{\mu f + \lambda g = 0 \colon \mu, \lambda \in \mathbb{C}\}$ is G-invariant. By a linear action on $\mathbb{P}^2_{\mathbb{C}}$ we mean a linear action on $(\mathbb{C}^3)^\vee = \mathrm{Span}\{x, y, z\}$, which in turn gives us an action on $\mathbb{P}^2_{\mathbb{C}} = \{[x : y : z] \colon x, y, z \text{ not all } 0\}$. Note that from this linear action we can obtain an action on $\mathrm{Sym}^2((\mathbb{C}^3)^\vee) = \mathrm{Span}\{x^2, y^2, z^2, xy, xz, yz\}$. Recall that X being G-invariant means for all conics $C_t \in X$, writing $t = [\mu : \lambda] \in \mathbb{P}^1_{\mathbb{C}}$ for simplicity and remembering that any $(\mu, \lambda) \in \mathbb{C}^2$ can be scaled by any non-zero scalar, and for all $g' \in G$, we have $g' \cdot C_t \in X$. Note that this is equivalent to asking that $\langle f, g\rangle$ is a G-invariant subspace of $\mathrm{Sym}^2((\mathbb{C}^3)^\vee)$. Instead of asking how many nodal conics are in X, one can ask how many orbits of nodal conics are in X and the cardinality of each orbit. Since an orbit is a set, instead of asking for the number of nodal conics as an integer we can ask for the count of nodal orbits of X as a G-set in the Burnside ring $A(G)$ of G. This brings us to the main question: Given a finite group G and a general pair of conics defined by f and g in $\mathbb{P}^2_{\mathbb{C}}$ such that X is G-invariant, is there an $A(G)$-valued formula for the number of nodal orbits of conics in X? The question is answered in the main theorem of this work:

Theorem 1.4. Let G be a finite group not isomorphic to either $\mathbb{Z}/2 \times \mathbb{Z}/2$ or D_8, and assume that G acts linearly on $\mathbb{P}^2_{\mathbb{C}}$. Let X be a G-invariant pencil spanned by a pair of general conics and let $[\Sigma] \in A(G)$ be the base locus of X. Then

$$\sum_{\{t \in \mathbb{P}^1_{\mathbb{C}} \colon C_t \in X \text{ is nodal}\}} \mathrm{wt}^G(C_t) = [\Sigma] - \{*\} \tag{1.1}$$

in $A(G)$. That is, there is a weighted sum of the number of nodal conics in X, valued in the Burnside ring of G.

The weighting convention for nodal conics in X appearing in the left-hand side of equation 1.1 is defined in full in Section 2.2 before the main theorem is restated, and is a direct generalization of the weighting convention needed to extend the classical formula from $\mathbb{C}$ to $\mathbb{R}$. An example of the weighting convention used in the real analogue of the classical result is also given in Chapter 2. It is also worth noting that a linear action on $\mathbb{P}^2_{\mathbb{C}}$ is meant to refer to a linear action on $(\mathbb{C}^3)^\vee$ from which a linear action on $\mathrm{Sym}^2((\mathbb{C}^3)^\vee)$, and therefore $\langle f, g \rangle$, can be obtained.

1.4 Proof of the Classical Result

Before answering the main question stated above in Chapter 2, for completeness we will first sketch a topological proof and an elementary proof that the number of nodal conics in a general pencil of conics over $\mathbb{C}$ is three. Everything is assumed to be defined over $\mathbb{C}$. Let f and g be a general pair of conics in $\mathbb{P}^2$. Let $X := \{\mu f + \lambda g = 0 \colon \mu, \lambda \in$

$\mathbb{C}\} \subseteq \mathbb{P}^2$. Write $t = [\mu : \lambda]$ and let $C := \{(t,p) \in \mathbb{P}^1 \times \mathbb{P}^2 \colon \mu f(p) + \lambda g(p) = 0\}$. We have two projections of C to projective space: $\pi_1 \colon C \to \mathbb{P}^1$ by projecting onto the first coordinate, and $\pi_2 \colon C \to \mathbb{P}^2$ by projecting onto the second coordinate. We will compute $\chi(C)$ in two ways, and then set the two computations equal to each other to obtain the main result.

First, we will compute $\chi(C)$ using the projection $\pi_1 \colon C \to \mathbb{P}^1$. Let $D \subseteq \mathbb{P}^1$ be the set of $[\mu : \lambda] \in \mathbb{P}^1$ such that $\mu f + \lambda g = 0 \in X$ is nodal. Note that $\#D$ is equal to the number of singular conics in X, which is what we want to find. The fibers of π_1 over D are singular conics, and the fibers over $\mathbb{P}^1 - D$ are smooth conics. Using the fact that C is the disjoint union of fibers over D and fibers over $\mathbb{P}^1 - D$, we have

$$\begin{aligned}\chi(C) &= \chi(C|D) + \chi(C|\mathbb{P}^1 - D)\\ &= \chi(C_{sing}) \cdot \chi(D) + \chi(C_{sm}) \cdot \chi(\mathbb{P}^1 - D)\end{aligned}$$

where C_{sm} denotes any smooth conic in a fiber over $\mathbb{P}^1 - D$ and C_{sing} denotes any singular conic in a fiber over D. Here we have used additivity of the compactly supported Euler characteristic as well as the fact that if $E \to B$ is a fiber bundle with fiber F where B is path connected, then the compactly supported Euler characteristic satisfies the formula $\chi(F) \cdot \chi(B) = \chi(E)$, described in detail in Theorem 17, page 481, in (Spanier 1981). Recall that the Euler characteristic of a smooth projective curve is $2 - 2g$ where g is the genus, and the Euler characteristic of a singular curve is $2 - 2g + \mu(C_{sing})$ where $\mu(C)$ is the Milnor number of a curve C. Since conics have genus 0 and the Milnor number of a nodal conic is 1, we have

$$\begin{aligned}\chi(C) &= \chi(C_{sing}) \cdot \chi(D) + \chi(C_{sm}) \cdot \chi(\mathbb{P}^1 - D)\\ &= \#D(2 - 2g + \mu(C_{sing})) + (2 - 2g)(2 - \#D)\\ &= \#D(2 + \mu(C_{sing})) + 2(2 - \#D)\\ &= \#D + 4.\end{aligned}$$

Another way to see that $\chi(C_{sing}) = 3$ is to recall that a nodal conic is isomorphic to $\mathbb{P}^1 \vee \mathbb{P}^1$, which has Euler characteristic $2 + 2 - 1 = 3$.

The second way to compute $\chi(C)$ is to use the fact that $\pi_2 \colon C \to \mathbb{P}^2$ is the blow-up of $\mathbb{P}^2$ at $d^2 = 4$ points where $d = 2$ is the degree of f and g as homogenous polynomials in three variables. In general, again using additivity for the compactly supported Euler characteristic, we have

$$\begin{aligned}
\chi(C) &= \chi(\mathbb{P}^2) - 4\chi(\{*\}) + 4\chi(\mathbb{P}^1) \\
&= 3 - 4 + 2 \cdot 4 \\
&= 7.
\end{aligned}$$

Combining the two computations of $\chi(C)$ we get that $\#D + 4 = 7$, and we can conclude that the number of singular conics in X is $\#D = 3$. A more general version of this proof with even more detail shown can be found in (Eisenbud and Harris 2016).

It is worth noting that another way to write the formula for $\#D$ is $\#D = \chi(\Sigma) - 1 = \#\Sigma - 1$. This formula for the number of nodal conics in X in terms of $\#\Sigma$ motivates the formula in Theorem 2.2.

An elementary proof of the same result is as follows. If f and g are a general pair of conics, then they intersect in exactly the four points of Σ. A nodal conic geometrically looks like a pair of lines, and the general pair hypothesis rules out the possibility that the two curves share a common line. Thus, asking how many conics in X are nodal is equivalent to asking how many ways there are to draw a pair of lines through four points in $\mathbb{P}^2$, which is three. We need to check that $h \in X$ if and only if h vanishes on Σ, done in the next paragraph. Labeling the points of Σ as b_1, b_2, b_3, and b_4, the three pairs of lines are $\overline{b_1 b_2} \cup \overline{b_3 b_4}$, $\overline{b_1 b_3} \cup \overline{b_2 b_4}$, and $\overline{b_1 b_4} \cup \overline{b_2 b_3}$. This way of thinking of the number of nodal conics in X will be useful to us going forward. When describing the G-orbits of nodal conics, we can instead look at orbits of configurations of lines through Σ.

As described above, the fact that f and g are a general pair of conics implies that for $X = \{\mu f + \lambda g = 0 \colon \mu, \lambda \in \mathbb{C}\}$ we have a uniquely determined base locus Σ. It is also worth noting that any set of four points in $\mathbb{P}^2$ with no three co-linear uniquely determines a pair of general conics. Indeed, that the vector space of conics in $\mathbb{P}^2_{\mathbb{C}}$ is 5-dimensional, $\mathrm{Sym}^2((\mathbb{C}^3)^\vee) = \mathrm{Span}_{\mathbb{C}}\{x^2, y^2, z^2, xy, xz, yz\}$. Requiring that a conic passes through a point imposes a 1-dimensional condition on the space of conics, so requiring that a conic passes through four points results in a 1-dimensional linear span of conics, or a 2-dimensional projective span of conics, i.e. a pencil of conics. Therefore, any Σ which is a set of four points in $\mathbb{P}^2_{\mathbb{C}}$ with no three co-linear uniquely determines a pencil of conics. In particular, both a pencil of conics and the number of nodal conics in it are dependent only on Σ. When we introduce a G-action, every $[\Sigma] \in A(G)$ does still uniquely determine an invariant pencil of conics, but using the same weights to count the number of nodal orbits does not work for all groups.

Chapter 2

An Equivariant Counting Formula for Nodal Elements in a Pencil of Conics

2.1 Example over $\mathbb{R}$

Before proving the theorem in general, it is worth looking at an example over $\mathbb{R}$ to illustrate counting with a weight assigned to each node. Let $G = \mathbb{Z}/2$ act on $\mathbb{P}^2_{\mathbb{C}}$ by conjugation, so $g \cdot [x : y : z] \mapsto [\overline{x} : \overline{y} : \overline{z}]$ for $g \in \mathbb{Z}/2$ nontrivial. Given a G-invariant pencil of conics, X, we can ask how many of the three nodal conics in X are defined over $\mathbb{R}$. We know that there are exactly three nodal conics, call them C_1, C_2, and C_3, in X defined over $\mathbb{C}$. If the determinant of the Hessian matrix of C_i is negative, C_i is a *split* node. If the determinant of the Hessian matrix of C_i is positive, C_i is a *non-split* node. Weighting a split node with -1 and a non-split node with $+1$, the number of nodal conics in X defined over $\mathbb{R}$ is the weighted sum of the nodal conics in X over $\mathbb{C}$. Algebraically, a split node has the form $ax^2 - by^2$ with $a, b > 0$, so splits over $\mathbb{R}$ as $(\sqrt{a}x + \sqrt{b}y)(\sqrt{a}x - \sqrt{b}y)$, hence the name split node. Likewise, a non-split node has the form $ax^2 + by^2$, so factors as $(\sqrt{a}x + i\sqrt{b}y)(\sqrt{a}x - i\sqrt{b}y)$ over $\mathbb{C}$ but does not split over $\mathbb{R}$.

Another way of thinking about the weight of a node is as the topological degree of the gradient of its defining equation, remembering that the equation of the node either has the form $ax^2 - by^2$ or $ax^2 + by^2$. If C_i is a split node, the topological degree of the gradient is -1. If C_i is a non-split node, the topological degree of the gradient is $+1$.

Example 2.1. Let $f = X^2 + Y^2 - Z^2$ and let $g = X^2 - Y^2 + 2Z^2$. Then the pencil spanned by f and g is

$$\begin{aligned} X &= \{\mu f + \lambda g = 0 \colon \mu, \lambda \in \mathbb{C}\} \\ &= \{(\mu + \lambda)X^2 + (\mu - \lambda)Y^2 + (-\mu + 2\lambda)Z^2 = 0 \colon [\mu : \lambda] \in \mathbb{P}^1_{\mathbb{C}}\}, \end{aligned}$$

and X is invariant under conjugation. The values of $[\mu : \lambda] \in \mathbb{P}^1$ that define nodal conics in X are $[1 : -1], [1 : 1]$, and $[1 : \frac{1}{2}]$. Note that all of the nodal conics must be defined over $\mathbb{C}$ but not necessarily over $\mathbb{R}$. We will write $t = [\mu : \lambda]$ for simplicity of notation so that C_t is an element of X obtained by specializing μ and λ. When $t = [1 : -1]$, $C_t = Y^2 - 3Z^2$ and thus is split with weight -1. When $t = [1 : 1]$, $C_t = 2X^2 + 4Z^2$ and thus is non-split with weight $+1$. When $t = [1 : \frac{1}{2}]$, $C_t = \frac{3}{2}X^2 + \frac{1}{2}Y^2$ and is non-split with weight $+1$. The weighted sum of nodal conics in X is $-1 + 1 + 1 = 1$, and therefore one of the nodal conics is defined over $\mathbb{R}$. In fact, we can see that the nodal conic defined over $\mathbb{R}$ is C_t with $t = [1 : -1]$.

In this example, the base locus is

$$\Sigma = \left\{ \left[i\sqrt{\frac{1}{3}} : 1 : \sqrt{\frac{2}{3}}\right], \left[-i\sqrt{\frac{1}{3}} : 1 : \sqrt{\frac{2}{3}}\right], \left[i\sqrt{\frac{1}{3}} : 1 : -\sqrt{\frac{2}{3}}\right], \left[-i\sqrt{\frac{1}{3}} : 1 : -\sqrt{\frac{2}{3}}\right] \right\}.$$

Writing $\Sigma(\mathbb{R}) = \{[X : Y : Z] \in \Sigma \colon X, Y, Z \in \mathbb{R}\}$, we can see that $\Sigma(\mathbb{R}) = 0$ for the above example. In fact, the number of nodal conics defined over $\mathbb{R}$ is $-(\Sigma(\mathbb{R}) - 1)$ and the number of nodal conics defined over $\mathbb{C}$ is $\#\Sigma - 1$. This observation about the number of nodal conics in a G-invariant pencil of conics that are defined over $\mathbb{C}^G$ being a formula in $\Sigma(\mathbb{C}^G)$ is what is generalized in the main theorem.

2.2 Main Theorem

We are now ready to state the main theorem. Let f and g be a general pair of conics and let $X := \{\mu f + \lambda g = 0 \colon \mu, \lambda \in \mathbb{C}\}$. Henceforth for simplicity of notation we will write $t = [\mu : \lambda] \in \mathbb{P}^1_{\mathbb{C}}$, so that $C_t \in X$ denotes the element of X obtained by specializing to specific values of μ and λ, which can be scaled since $X \subseteq \mathbb{P}^2_{\mathbb{C}}$. Given C_t in X, we will write X_t to denote the irreducible components of C_t, so that $X_t = \{L_1, L_2\}$ is a set of size two corresponding to the branches of C_t if $C_t = L_1 \cdot L_2$ is a nodal conic parameterized as the product of lines L_1 and L_2. Define the G-weight of the orbit of C_t to be $\mathrm{wt}^G(C_t) = \inf^G_H(\mathrm{wt}^H(C_t))$ where $H = \mathrm{stab}(X_t)$ is the stabilizer of X_t, $\inf^G_H(X_t)$ is the inflation of X_t from H to G, and $\mathrm{wt}^H(C_t) = [X_t] - \{*\}$ in $A(H)$.

Theorem 2.2. Let G be a finite group not isomorphic to either $\mathbb{Z}/2 \times \mathbb{Z}/2$ or D_8, and assume that G acts linearly on $\mathbb{P}^2_{\mathbb{C}}$. Let X be a G-invariant pencil spanned by a pair of general conics and let $[\Sigma] \in A(G)$ be the base locus of X. Then

$$\sum_{\{t \in \mathbb{P}^1_{\mathbb{C}} \colon C_t \in X \text{ is nodal}\}} \mathrm{wt}^G(C_t) = [\Sigma] - \{*\} \tag{2.1}$$

in $A(G)$. That is, there is a weighted sum of the number of nodal conics in X, valued in the Burnside ring of G.

Proof. We will prove the theorem is true for each finite group G that can act linearly on $\mathbb{P}^2_{\mathbb{C}}$ and invariantly on a pencil of conics. Any such group must be a finite subgroup of $PGL(3, \mathbb{C})$, which have been classified and can be found in (Hambleton and Lee 1988). If G is a finite group that acts linearly on $\mathbb{P}^2_{\mathbb{C}}$ and G can act invariantly on a pencil of conics, then G must fix the base locus of the pencil, i.e. must act bijectively on a set of four distinct points, but the only elements of $PGL(3, \mathbb{C})$ that act trivially on four points in general position are necessarily the identity. Therefore the only group actions that we need to consider must be actions of subgroups of S_4, which indeed is a finite subgroup of $PGL(3, \mathbb{C})$.

It is known that if $H_1, H_2 \leq G$ are conjugate subgroups of a finite group G, then $A(H_1) \cong A(H_2)$. Bouc (2000) gives a proof of this fact. Thus we will only check that the theorem is true for each conjugacy class of subgroups of S_4. The isomorphism classes of conjugacy classes of subgroups of S_4 are: $\langle () \rangle$, $\mathbb{Z}/2 \cong \langle (12) \rangle$,

$S_3 = \langle(123),(12)\rangle$, $A_3 = \{(),(123),(132)\}$, $\mathbb{Z}/4 \cong \langle(1234)\rangle$, $A_4 = \langle(123),(12)(34)\rangle$, $\mathbb{Z}/2\times\mathbb{Z}/2 \cong \langle(12)(34),(13)(24)\rangle$, $D_8 \cong \langle(1234),(13)\rangle$, and S_4. For each one of these groups except $\mathbb{Z}/2\times\mathbb{Z}/2$ and D_8, we will show that the theorem is true. In the next chapter, we will provide counterexamples for $\mathbb{Z}/2\times\mathbb{Z}/2$ and D_8.

We will use the following notation throughout. We will write $[\Sigma] = \{b_1, b_2, b_3, b_4\} \in A(G)$ for the base locus of a pencil, and the line through b_i and b_j, $1 \le i, j \le 4$, will be denoted by L_{ij}. Since any nodal conic through $[\Sigma]$ is geometrically a pair of lines, the G-isomorphism class of a nodal conic will be denoted by $[L_{ij} \cup L_{kl}]$ in $A(G)$.

Recall that the set of pencils of general conics in $\mathbb{P}^2_{\mathbb{C}}$ is in bijection with the set of collections of four points in $\mathbb{P}^2_{\mathbb{C}}$ with no three co-linear by a vector space argument given at the end of Section 1.4. Every G-invariant pencil of general conics in $\mathbb{P}^2_{\mathbb{C}}$ uniquely determines a G-set of four points satisfying the linearity condition, and every G-set of four points in $\mathbb{P}^2_{\mathbb{C}}$ satisfying the linearity condition that no three points in $[\Sigma]$ are collinear uniquely determines *at most one* G-invariant pencil of general conics depending on whether or not the unique 1-dimensional subspace of $\mathbb{P}\,\mathrm{Sym}^2((\mathbb{C}^3)^\vee)$ corresponding to Σ not considered as a G-set is G-invariant. Showing for each G listed above that equation (2.1) holds for any possible configuration of $[\Sigma] \in A(G)$ will prove the theorem, keeping in mind that not every configuration of $[\Sigma]$ may not actually correspond to a pencil of conics which is also G-invariant.

2.2.1 G=$\langle()\rangle$

If $G = \langle()\rangle$ is the trivial group, then any group action on $\mathbb{P}^2_{\mathbb{C}}$ is trivial. Therefore, this is simply the classical result over $\mathbb{C}$.

2.2.2 $G = \mathbb{Z}/2 \cong \langle(12)\rangle$

The only genuine size four G-sets, and therefore the only combinatorially possible choices for $[\Sigma]$ in $A(G)$, are the following:

1. $[\Sigma] = 4\{*\}$

2. $[\Sigma] = 2[G]$

3. $[\Sigma] = 2\{*\} + [G]$

The fact that any $[\Sigma]$ that is the base locus of a G-invariant pencil of conics must be one of these options relies on the fact that any genuine G-set $[S] \in A(G)$ has the form $S = \sum_{(H_i)\colon H_i \leq G} n_i[G/H_i] = n_0[G/G] + n_1[G/\langle()\rangle]$ with $n_0, n_1\mathbb{Z}_{\geq 0}$ being the number of orbits with stabilizer equal to G or $()$ respectively. Since $[\Sigma]$ is an actual set with a group action, i.e. a genuine G-set, it must have one of the three configurations listed above and cannot have n_0 or n_1 negative.

Given a configuration of $[\Sigma]$, if there is a G-invariant pencil of conics X determined by $[\Sigma]$, then any nodal conic in X geometrically looks like one of the three configurations of a pair of lines through $[\Sigma]$. Therefore, to see that the theorem is true for every configuration of $[\Sigma]$, and therefore true for G, we will calculate the weight of each pair of lines through $[\Sigma]$ and show that the sum of the weights is equal to $[\Sigma] - \{*\}$, the right-hand side of equation (2.1).

First consider the case where $[\Sigma] = 4\{*\}$. All four points of $[\Sigma]$ are fixed, so

$$\mathrm{stab}([L_{12} \cup L_{34}]) = \mathrm{stab}([L_{13} \cup L_{24}]) = \mathrm{stab}([L_{14} \cup L_{23}]) = G.$$

Note then that $\mathrm{wt}^G([L_{ij} \cup L_{kl}]) = \{[L_{ij} \cup L_{kl}]\} - \{*\} = 2\{*\} - \{*\} = \{*\}$ for any distinct $i, j, k, l \in \{1, 2, 3, 4\}$. Hence the left-hand side of equation (2.1) is $\sum \mathrm{wt}^G(X_t) = \{*\} + \{*\} + \{*\} = 3\{*\}$, and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 4\{*\} - \{*\} = 3\{*\}$.

Consider the second case where $[\Sigma] = 2[G]$, and say that $\{b_1, b_2\}$ and $\{b_3, b_4\}$ are the orbits of $[\Sigma]$. In this case, $()$ is the element that acts trivially and (12) is the element that acts nontrivially on each orbit i.e. swaps b_1 and b_2 and swaps b_3 and b_4.

Then for $g \in G$,

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{21} \cup L_{43} & , g = (12) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{24} \cup L_{13} & , g = (12) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{23} \cup L_{14} & , g = (12) \end{cases}.$$

This notation means, for example, that $() \cdot L_{12} \cup L_{34} = L_{12} \cup L_{34}$ and that $(12) \cdot L_{14} \cup L_{23} = L_{23} \cup L_{14}$. As a G-set, $[L_{12} \cup L_{34}]$ records the information of how $G = \{(), (12)\}$ acts on $L_{12} \cup L_{34}$.

For each node, the stabilizer is G, and so $\mathrm{wt}^G([L_{ij} \cup L_{kl}]) = [L_{ij} \cup L_{kl}] - \{*\}$. Note that $[L_{12} \cup L_{34}] = 2\{*\} \in A(G)$, but $[L_{13} \cup L_{24}]$ and $[L_{14} \cup L_{23}]$ are both isomorphic to $[G]$ in $A(G)$. Hence $\mathrm{wt}^G([L_{12} \cup L_{34}]) = 2\{*\} - \{*\} = \{*\}$ and $\mathrm{wt}^G([L_{13} \cup L_{24}]) = \mathrm{wt}^G([L_{14} \cup L_{23}]) = [G] - \{*\}$. Thus the left-hand side of equation (2.1) is $\{*\} + 2[G] - 2\{*\} = 2[G] - \{*\}$, and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 2[G] - \{*\}$, as desired.

The last configuration of $[\Sigma]$ to prove the theorem for is $[\Sigma] = 2\{*\} + [G]$. Say that b_1 and b_2 are the fixed points and $\{b_3, b_4\}$ are an orbit. Hence $()$ acts trivially on all points in $[\Sigma]$, and (12) acts nontrivially on $\{b_3, b_4\}$ and trivially on b_1 and b_2. Thus

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{12} \cup L_{43} & , g = (12) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{14} \cup L_{23} & , g = (12) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{13} \cup L_{24} & , g = (12) \end{cases}.$$

In this case, $\mathrm{stab}([L_{12} \cup L_{34}]) = G$ and both L_{12} and L_{34} are fixed, so $\mathrm{wt}^G([L_{12} \cup L_{34}]) = [L_{12} \cup L_{34}] - \{*\} = 2\{*\} - \{*\} = \{*\}$. Note that $\mathrm{stab}([L_{13} \cup L_{24}]) =$

$\mathrm{stab}([L_{14} \cup L_{23}]) = \langle () \rangle$. Furthermore, $(12) \cdot [L_{13} \cup L_{24}] = [L_{14} \cup L_{23}]$ and $(12) \cdot [L_{14} \cup L_{23}] = [L_{13} \cup L_{24}]$, i.e. they are both in the same (12)-orbit of the G-set $\{[L_{12} \cup L_{34}], [L_{13} \cup L_{24}], [L_{14} \cup L_{23}]\}$. Therefore we only need to count one of $[L_{13} \cup L_{24}]$ or $[L_{14} \cup L_{23}]$ in the weighted sum of nodal curves in the G-invariant pencil determined by $[\Sigma]$. Making an arbitrary choice and using Lemma 1.3 to compute the inflation, $\mathrm{wt}^G([L_{13} \cup L_{24}]) = \inf_{\langle () \rangle}^G(\mathrm{wt}^{\langle () \rangle}([L_{13} \cup L_{24}])) = \inf_{\langle () \rangle}^G(2\{*\} - \{*\}) = \inf_{\langle () \rangle}^G(\{*\}) = [G/\langle () \rangle] = [G]$.

Finally, the left-hand side of equation (2.1) is $\mathrm{wt}^G([L_{12} \cup L_{34}]) + \mathrm{wt}^G([L_{13} \cup L_{24}]) = \{*\} + [G]$ and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = [G] + 2\{*\} - \{*\} = [G] + \{*\}$, as desired. Therefore the theorem is true for $\mathbb{Z}/2$.

2.2.3 $G = S_3 \cong \langle (123), (12) \rangle$

The only possibilities for $[\Sigma]$ in $A(G)$ are:

1. $[\Sigma] = 4\{*\}$
2. $[\Sigma] = \{*\} + [G/\langle (12) \rangle]$
3. $[\Sigma] = 2\{*\} + [G/\langle (123) \rangle]$

First consider the case where $[\Sigma] = 4\{*\}$. All four points of $[\Sigma]$ are fixed, so

$$\mathrm{stab}([L_{12} \cup L_{34}]) = \mathrm{stab}([L_{13} \cup L_{24}]) = \mathrm{stab}([L_{14} \cup L_{23}]) = G.$$

Note then that $\mathrm{wt}^G([L_{ij} \cup L_{kl}]) = \{[L_{ij} \cup L_{kl}]\} - \{*\} = 2\{*\} - \{*\} = \{*\}$ for any distinct $i, j, k, l \in \{1, 2, 3, 4\}$. Hence the left-hand side of equation (2.1) is $3\{*\}$, and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 4\{*\} - \{*\} = 3\{*\}$.

Next consider the second case where $[\Sigma] = \{*\} + [G/\langle (12) \rangle]$. Say b_4 is fixed and $\{b_1, b_2, b_3\}$ are an orbit. We can actually use the coset structure of $G/\langle (12) \rangle = \{[()], [(123)], [(132)]\}$ to say that $[\Sigma] = \{b_1 = [()], b_2 = [(123)], b_3 = [(132)], b_4 = \{*\}\}$.

Then

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{12} \cup L_{34} & , g = (12) \\ L_{21} \cup L_{34} & , g = (13) \\ L_{32} \cup L_{14} & , g = (23) \\ L_{23} \cup L_{14} & , g = (123) \\ L_{31} \cup L_{24} & , g = (132) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{13} \cup L_{24} & , g = (12) \\ L_{23} \cup L_{14} & , g = (13) \\ L_{31} \cup L_{24} & , g = (23) \\ L_{12} \cup L_{34} & , g = (123) \\ L_{32} \cup L_{14} & , g = (132) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{14} \cup L_{23} & , g = (12) \\ L_{24} \cup L_{12} & , g = (13) \\ L_{34} \cup L_{21} & , g = (23) \\ L_{24} \cup L_{31} & , g = (123) \\ L_{34} \cup L_{12} & , g = (132) \end{cases}.$$

Observe that $\mathrm{stab}([L_{14} \cup L_{23}]) = \langle (12) \rangle$. There are two subgroups of S_3 that stabilize $[L_{12} \cup L_{34}]$, $\langle (12) \rangle$ and $\langle (13) \rangle$, which are conjugate in S_3 because $(132)\langle (12) \rangle (132)^{-1} = \langle (13) \rangle$. The branches of $[L_{12} \cup L_{34}]$ are equal to $2\{*\}$ in both $A(\langle (12) \rangle)$ and $A(\langle (13) \rangle)$, and $\inf^G_{\langle (12) \rangle}(2\{*\} - \{*\}) = [G/\langle (12) \rangle]$ and $\inf^G_{\langle (13) \rangle}(2\{*\} - \{*\}) = [G/\langle (13) \rangle]$. Since $\langle (12) \rangle$ and $\langle (13) \rangle$ are conjugate, $G/\langle (12) \rangle \cong G/\langle (13) \rangle$. Hence

$$\inf^G_{\langle (12) \rangle}(2\{*\} - \{*\}) = [G/\langle (12) \rangle] = [G/\langle (13) \rangle] = \inf^G_{\langle (13) \rangle}(2\{*\} - \{*\}),$$

and therefore $\mathrm{wt}^G([L_{12} \cup L_{34}])$ can be computed with either $\langle (12) \rangle$ or $\langle (13) \rangle$ as the stabilizer. A similar argument holds for $[L_{13} \cup L_{24}]$ having both $\langle (12) \rangle$ and $\langle (23) \rangle$ as stabilizers. Since $[L_{14} \cup L_{23}]$ has stabilizer $\langle (12) \rangle$, we will use $\langle (12) \rangle$ as the stabilizer for all three nodal orbits.

Since all three nodes are in the $\langle(123)\rangle$-orbit of each other, as in the third case for $\mathbb{Z}/2$ we only need to count one of them to obtain the left-hand side of equation (2.1). Arbitrarily choosing $[L_{12} \cup L_{34}]$, recall that $\mathrm{wt}^G([L_{12} \cup L_{34}]) = [G/\langle(12)\rangle]$. Therefore, the left-hand side of equation (2.1) is $[G/\langle(12)\rangle]$ and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = \{*\} + [G/\langle(12)\rangle] - \{*\} = [G/\langle(12)\rangle]$, as desired.

The last case to consider for S_3 is when $[\Sigma] = 2\{*\} + [G/\langle(123)\rangle]$. Say that b_3 and b_4 are fixed. Using the coset structure of $G/\langle(123)\rangle = \{[()], [(12)]\}$, we can say $b_1 = [()]$ and $b_2 = [(12)]$. Then

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{21} \cup L_{34} & , g = (12) \\ L_{21} \cup L_{34} & , g = (13) \\ L_{21} \cup L_{34} & , g = (23) \\ L_{12} \cup L_{34} & , g = (123) \\ L_{12} \cup L_{34} & , g = (132) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{23} \cup L_{14} & , g = (12) \\ L_{23} \cup L_{14} & , g = (13) \\ L_{23} \cup L_{14} & , g = (23) \\ L_{13} \cup L_{24} & , g = (123) \\ L_{13} \cup L_{24} & , g = (132) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{24} \cup L_{13} & , g = (12) \\ L_{24} \cup L_{13} & , g = (13) \\ L_{24} \cup L_{13} & , g = (23) \\ L_{14} \cup L_{23} & , g = (123) \\ L_{14} \cup L_{23} & , g = (132) \end{cases}.$$

In this case only one of $[L_{13} \cup L_{24}]$ or $[L_{14} \cup L_{23}]$ needs to be counted in the left-hand side of equation (2.1) because $(12) \cdot L_{13} \cup L_{24} = L_{14} \cup L_{23}$ and $(12) \cdot [L_{14} \cup L_{23}] = [L_{13} \cup L_{24}]$. That is, they are both in the (12)-orbit of the G-set $\{[L_{12} \cup L_{34}], [L_{13} \cup L_{24}], [L_{14} \cup L_{23}]\}$, so we only need to count one of $[L_{13} \cup L_{24}]$ or $[L_{14} \cup L_{23}]$.

Both have stabilizer $\langle(123)\rangle$. Arbitrarily choosing $[L_{13} \cup L_{24}]$ to count, we see that $\mathrm{wt}^G([L_{13} \cup L_{24}]) = \mathrm{inf}^G_{\langle(123)\rangle}(\{*\}) = [G/\langle(123)\rangle]$. Additionally, observe that $\mathrm{stab}([L_{12} \cup L_{34}]) = G$, and $\mathrm{wt}^G([L_{12} \cup L_{34}]) = 2\{*\} - \{*\} = \{*\}$. Therefore, the left-hand side of equation (2.1) is $[G/\langle(123)\rangle] + \{*\}$ and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 2\{*\} + [G/\langle(123)\rangle] - \{*\} = [G/\langle(123)\rangle] + \{*\}$, as desired. Therefore the theorem is true for S_3.

2.2.4 $G = A_3 = \{(), (123), (132)\}$

The only possibilities for $[\Sigma]$ in $A(G)$ are:

1. $[\Sigma] = 4\{*\}$
2. $[\Sigma] = \{*\} + [G]$

Consider the case where $[\Sigma] = 4\{*\}$. All four points of $[\Sigma]$ are fixed, so

$$\mathrm{stab}([L_{12} \cup L_{34}]) = \mathrm{stab}([L_{13} \cup L_{24}]) = \mathrm{stab}([L_{14} \cup L_{23}]) = G.$$

Note then that $\mathrm{wt}^G([L_{ij} \cup L_{kl}]) = \{[L_{ij} \cup L_{kl}]\} - \{*\} = 2\{*\} - \{*\} = \{*\}$ for any distinct $i, j, k, l \in \{1, 2, 3, 4\}$. Hence the left-hand side of equation (2.1) is $3\{*\}$, and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 4\{*\} - \{*\} = 3\{*\}$.

The only other case to consider is when $[\Sigma] = \{*\} + [G]$. Say b_4 is the fixed point and $\{b_1, b_2, b_3\}$ are an orbit with (123) and (132) permuting the indices of the b_i. Then

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{23} \cup L_{14} & , g = (123) \\ L_{31} \cup L_{24} & , g = (132) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{21} \cup L_{34} & , g = (123) \\ L_{32} \cup L_{14} & , g = (132) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{24} \cup L_{31} & , g = (123) \\ L_{34} \cup L_{12} & , g = (132) \end{cases} .$$

Since $\langle(123)\rangle \cdot [L_{12} \cup L_{34}] = [L_{13} \cup L_{24}]$, $\langle(123)\rangle \cdot [L_{13} \cup L_{24}] = [L_{14} \cup L_{23}]$, and $\langle(123)\rangle \cdot [L_{14} \cup L_{23}] = [L_{12} \cup L_{34}]$, counting any one of the nodes will give the weighted sum of orbits of nodal conics in the pencil. Arbitrarily choosing $[L_{12} \cup L_{34}]$, $\mathrm{stab}([L_{12} \cup L_{34}]) = \langle()\rangle$. Therefore $\mathrm{wt}^G([L_{12} \cup L_{34}]) = \mathrm{inf}^G_{\langle()\rangle}(2\{*\} - \{*\}) = \mathrm{inf}^G_{\langle()\rangle}(\{*\}) = [G]$, and so the left-hand side of equation (2.1) is $[G]$. The right-hand side of equation (2.1) is $[\Sigma] - \{*\} = [G] + \{*\} - \{*\} = [G]$, as desired.

2.2.5 $G = \mathbb{Z}/4 \cong \langle(1234)\rangle = \{(), (1234), (13)(24), (1432)\}$

The only possibilities for $[\Sigma]$ in $A(G)$ are:

1. $[\Sigma] = 4\{*\}$
2. $[\Sigma] = 2\{*\} + [G/\langle(13)(24)\rangle]$
3. $[\Sigma] = 2[G/\langle(13)(24)\rangle]$
4. $[\Sigma] = [G]$

Consider the case where $[\Sigma] = 4\{*\}$. All four points of $[\Sigma]$ are fixed, so

$$\mathrm{stab}([L_{12} \cup L_{34}]) = \mathrm{stab}([L_{13} \cup L_{24}]) = \mathrm{stab}([L_{14} \cup L_{23}]) = G.$$

Note then that $\mathrm{wt}^G([L_{ij} \cup L_{kl}]) = \{[L_{ij} \cup L_{kl}]\} - \{*\} = 2\{*\} - \{*\} = \{*\}$ for any distinct $i, j, k, l \in \{1, 2, 3, 4\}$. Hence the left-hand side of equation (2.1) is $3\{*\}$, and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 4\{*\} - \{*\} = 3\{*\}$.

Now consider the second case where $[\Sigma] = 2\{*\} + [G/\langle(13)(24)\rangle] = 2\{*\} + \{[()], [(1234)]\}$. Note that $(), (13)(24) \in [()]$ and $(1234), (1432) \in [(1234)]$. Say b_1

and b_2 are the fixed points, $b_3 = [()]$, and $b_4 = [(1234)]$. Then

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{12} \cup L_{43} & , g = (1234) \\ L_{12} \cup L_{34} & , g = (13)(24) \\ L_{12} \cup L_{43} & , g = (1432) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{14} \cup L_{23} & , g = (1234) \\ L_{13} \cup L_{24} & , g = (13)(24) \\ L_{14} \cup L_{23} & , g = (1432) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{13} \cup L_{24} & , g = (1234) \\ L_{14} \cup L_{23} & , g = (13)(24) \\ L_{13} \cup L_{24} & , g = (1432) \end{cases}.$$

Observe that stab $= [L_{12} \cup L_{34}] = G$ and each of L_{12} and L_{34} are fixed by every element of G. Thus the branches of $[L_{12} \cup L_{34}]$ are equal to $2\{*\}$ in $A(G)$. Therefore $\mathrm{wt}^G([L_{12} \cup L_{34}]) = 2\{*\} - \{*\} = \{*\}$. Now observe that $G/\langle(13)(24)\rangle \cdot L_{13} \cup L_{24} = L_{14} \cup L_{23}$ and $G/\langle(13)(24)\rangle \cdot L_{14} \cup L_{23} = L_{13} \cup L_{24}$, so we only need to count one of $[L_{13} \cup L_{24}]$ or $[L_{14} \cup L_{23}]$. Arbitrarily choosing $[L_{13} \cup L_{24}]$, $\mathrm{stab}([L_{13} \cup L_{24}]) = \langle(13)(24)\rangle$ and the branches are fixed, equal to $2\{*\}$ as a $\langle(13)(24)\rangle$-set. Therefore $\mathrm{wt}^G([L_{13} \cup L_{24}]) = \inf^G_{\langle(13)(24)\rangle}(2\{*\} - \{*\}) = \inf^G_{\langle(13)(23)\rangle}(\{*\}) = [G/\langle(13)(24)]$. Adding the weights of $[L_{12} \cup L_{34}]$ and $[L_{13} \cup L_{24}]$, the left-hand side of equation (2.1) is $\{*\} + [G/\langle(13)(24)\rangle]$. The right hand side of equation (2.1) is $[\Sigma] - \{*\} = 2\{*\} + [G/\langle(13)(24)\rangle] - \{*\} = [G/\langle(13)(24)\rangle] - \{*\}$, as desired.

Consider the third case where $[\Sigma] = 2[G/\langle(13)(24)\rangle] = \{[()], [(1234)]\} + \{[()], [(1234)]\}$. Note that $(), (13)(24) \in [()]$ and $(1234), (1432) \in [(1234)]$. We will say that $b_1 = b_3 = [()]$ and $b_2 = b_4 = [(1234)]$ with $\{b_1, b_2\}$ one $[G/\langle(13)(24)\rangle]$-orbit and $\{b_3, b_4\}$ the other. Then

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{21} \cup L_{43} & , g = (1234) \\ L_{12} \cup L_{34} & , g = (13)(24) \\ L_{21} \cup L_{43} & , g = (1432) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{24} \cup L_{13} & , g = (1234) \\ L_{13} \cup L_{24} & , g = (13)(24) \\ L_{24} \cup L_{13} & , g = (1432) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{23} \cup L_{14} & , g = (1234) \\ L_{14} \cup L_{23} & , g = (13)(24) \\ L_{23} \cup L_{14} & , g = (1432) \end{cases}.$$

Each of $[L_{12} \cup L_{34}]$, $[L_{13} \cup L_{24}]$, and $[L_{14} \cup L_{23}]$ has stabilizer G. The branches of $[L_{12} \cup L_{34}]$ are fixed, equal to $2\{*\}$. Thus $\mathrm{wt}^G([L_{12} \cup L_{34}]) = 2\{*\} - \{*\} = \{*\}$. Both of $[L_{13} \cup L_{24}]$ and $[L_{14} \cup L_{23}]$ have branches equal to $[G/\langle(13)(24)\rangle]$. Therefore $\mathrm{wt}^G([L_{13} \cup L_{24}]) = \mathrm{wt}^G([L_{14} \cup L_{23}]) = [G/\langle(13)(24)] - \{*\}$. Adding the weights of all three nodal orbits together, the left-hand side of equation (2.1) is equal to $\{*\} + [G/\langle(13)(24)] - \{*\} + [G/\langle(13)(24)] - \{*\} = 2[G/\langle(13)(24)\rangle] - \{*\}$. The right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 2[G/\langle(13)(24)\rangle] - \{*\}$, as desired.

Finally we will consider the case where $[\Sigma] = [G] = \{(), (1234), (13)(24), (1432)\}$, so there are no fixed points. Say $b_1 = ()$, $b_2 = (1234)$, $b_3 = (13)(24)$, and $b_4 = (1432)$. Then

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{23} \cup L_{41} & , g = (1234) \\ L_{34} \cup L_{12} & , g = (13)(24) \\ L_{41} \cup L_{23} & , g = (1432) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{24} \cup L_{31} & , g = (1234) \\ L_{31} \cup L_{42} & , g = (13)(24) \\ L_{42} \cup L_{13} & , g = (1432) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{21} \cup L_{34} & , g = (1234) \\ L_{32} \cup L_{41} & , g = (13)(24) \\ L_{43} \cup L_{12} & , g = (1432) \end{cases}.$$

Observe that $\mathrm{stab}([L_{13} \cup L_{24}]) = G$ with the branches equal to $G/\langle(13)(24)\rangle$. Therefore $\mathrm{wt}^G([L_{13} \cup L_{24}]) = [G/\langle(13)(24)\rangle] - \{*\}$. Also observe that $G/\langle(13)(24)\rangle \cdot L_{12} \cup L_{34} = L_{14} \cup L_{23}$ and $G/\langle(13)(24)\rangle \cdot L_{14} \cup L_{23} = L_{12} \cup L_{34}$, so we only need to count one of $[L_{12} \cup L_{34}]$ or $[L_{14} \cup L_{23}]$. Arbitrarily choosing $[L_{12} \cup L_{34}]$, $\mathrm{stab}([L_{12} \cup L_{34}]) = \langle(13)(24)\rangle$ and the branches are equal to $[\langle(13)(24)\rangle]$ in $A(\langle(12)(34)\rangle)$. Thus $\mathrm{wt}^G([L_{12} \cup L_{34}]) = \mathrm{inf}^G_{\langle(13)(24)\rangle}(\langle(13)(24)\rangle - \{*\}) = [G] - [G/\langle(13)(24)\rangle]$. Therefore the left-hand side of equation (2.1) is

$$\begin{aligned} \mathrm{wt}^G([L_{12} \cup L_{34}]) + \mathrm{wt}^G([L_{13} \cup L_{24}]) &= [G] - [G/\langle(13)(24)\rangle] + [G/\langle(13)(24)\rangle] - \{*\} \\ &= [G] - \{*\}. \end{aligned}$$

The right-hand side of equation (2.1) is $[\Sigma] - \{*\} = [G] - \{*\}$, as desired.

2.2.6 $G = A_4$

The possible options for $[\Sigma]$ are:

1. $[\Sigma] = 4\{*\}$
2. $[\Sigma] = \{*\} + [G/(\mathbb{Z}/2)^2]$, with $\mathbb{Z}/2 \times \mathbb{Z}/2 = \{(), (12)(34), (13)(24), (14)(23)\}$
3. $[\Sigma] = [G/A_3]$

Consider the case where $[\Sigma] = 4\{*\}$. All four points of $[\Sigma]$ are fixed, so

$$\mathrm{stab}([L_{12} \cup L_{34}]) = \mathrm{stab}([L_{13} \cup L_{24}]) = \mathrm{stab}([L_{14} \cup L_{23}]) = G.$$

Note then that $\mathrm{wt}^G([L_{ij} \cup L_{kl}]) = \{[L_{ij} \cup L_{kl}]\} - \{*\} = 2\{*\} - \{*\} = \{*\}$ for any distinct $i, j, k, l \in \{1, 2, 3, 4\}$. Hence the left-hand side of equation (2.1) is $3\{*\}$, and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 4\{*\} - \{*\} = 3\{*\}$.

Consider the second case, $[\Sigma] = \{*\} + [G/(\mathbb{Z}/2)^2] = \{*\} + \{[()], [(123)], [(132)]\}$. For ease of referencing, note that

$$
\begin{gathered}
(), (12)(34), (13)(24), (14)(23) \in [()] \\
(123), (123), (243), (142) \in [(123)] \\
(132), (234), (124), (143) \in [(132)]
\end{gathered}
$$

Observe

$$
g \cdot L_{12} \cup L_{34} = \begin{cases}
L_{12} \cup L_{34} & , g = () \\
L_{13} \cup L_{42} & , g = (123) \\
L_{14} \cup L_{23} & , g = (124) \\
L_{14} \cup L_{23} & , g = (132) \\
L_{13} \cup L_{42} & , g = (134) \\
L_{13} \cup L_{42} & , g = (142) \\
L_{14} \cup L_{23} & , g = (143) \\
L_{14} \cup L_{23} & , g = (234) \\
L_{13} \cup L_{42} & , g = (243) \\
L_{12} \cup L_{34} & , g = (12)(34) \\
L_{12} \cup L_{34} & , g = (13)(24) \\
L_{12} \cup L_{34} & , g = (14)(23)
\end{cases}
\qquad
g \cdot L_{13} \cup L_{24} = \begin{cases}
L_{13} \cup L_{24} & , g = () \\
L_{14} \cup L_{32} & , g = (123) \\
L_{12} \cup L_{43} & , g = (124) \\
L_{12} \cup L_{43} & , g = (132) \\
L_{14} \cup L_{32} & , g = (134) \\
L_{14} \cup L_{32} & , g = (142) \\
L_{12} \cup L_{43} & , g = (143) \\
L_{12} \cup L_{43} & , g = (234) \\
L_{14} \cup L_{32} & , g = (243) \\
L_{13} \cup L_{24} & , g = (12)(34) \\
L_{13} \cup L_{24} & , g = (13)(24) \\
L_{13} \cup L_{24} & , g = (14)(23)
\end{cases}
$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{12} \cup L_{34} & , g = (123) \\ L_{13} \cup L_{42} & , g = (124) \\ L_{13} \cup L_{42} & , g = (132) \\ L_{12} \cup L_{34} & , g = (134) \\ L_{12} \cup L_{34} & , g = (142) \\ L_{13} \cup L_{42} & , g = (143) \\ L_{13} \cup L_{42} & , g = (234) \\ L_{12} \cup L_{34} & , g = (243) \\ L_{14} \cup L_{23} & , g = (12)(34) \\ L_{14} \cup L_{23} & , g = (13)(24) \\ L_{14} \cup L_{23} & , g = (14)(23) \end{cases}.$$

Observe that all of $L_{12} \cup L_{34}$, $L_{13} \cup L_{24}$, and $L_{14} \cup L_{23}$ can be obtained from each other by the action of A_3 for any one of the three conjugate copies of A_3 in A_4. Therefore, we only need to count one of $[L_{12} \cup L_{34}]$, $[L_{13} \cup L_{24}]$, or $[L_{14} \cup L_{23}]$ in the left-hand side of equation (2.1). Arbitrarily choosing $[L_{12} \cup L_{34}]$, observe that $\mathrm{stab}([L_{12} \cup L_{34}]) = \{(), (12)(34), (13)(24), (14)(23)\} \cong (\mathbb{Z}/2)^2 =: H$. As an H-set, the branches of $L_{12} \cup L_{34}$ are fixed, equal to $2\{*\}$, in $A(G)$. Therefore $\mathrm{wt}^G([L_{12} \cup L_{34}]) = \inf_H^G(2\{*\} - \{*\}) = \inf_H^G(\{*\}) = [G/H]$. Therefore the left-hand side of equation (2.1) is $[G/(\mathbb{Z}/2)^2]$ and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = \{*\} + [G/(\mathbb{Z}/2)^2] - \{*\} = [G/(\mathbb{Z}/2)^2]$, as desired.

Now consider the final case, $[\Sigma] = [G/A_3]$. We can write $[\Sigma] = \{b_1 = [()], b_2 = [(124)], b_3 = [(142)], b_4 = [(243)]\}$. For ease of referencing, note that

$$(),(123),(132) \in [()] \qquad (124),(14)(23),(134) \in [(124)]$$

$$(142),(234),(13)(24) \in [(142)] \quad (243),(143),(12)(34) \in [(243)]$$

Observe

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{13} \cup L_{42} & , g = (123) \\ L_{23} \cup L_{14} & , g = (124) \\ L_{14} \cup L_{23} & , g = (132) \\ L_{24} \cup L_{12} & , g = (134) \\ L_{31} \cup L_{24} & , g = (142) \\ L_{41} \cup L_{23} & , g = (143) \\ L_{32} \cup L_{14} & , g = (234) \\ L_{42} \cup L_{12} & , g = (243) \\ L_{43} \cup L_{12} & , g = (12)(34) \\ L_{34} \cup L_{12} & , g = (13)(24) \\ L_{21} \cup L_{43} & , g = (14)(23) \end{cases} \quad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{14} \cup L_{23} & , g = (123) \\ L_{21} \cup L_{34} & , g = (124) \\ L_{12} \cup L_{43} & , g = (132) \\ L_{23} \cup L_{41} & , g = (134) \\ L_{32} \cup L_{14} & , g = (142) \\ L_{43} \cup L_{12} & , g = (143) \\ L_{34} \cup L_{21} & , g = (234) \\ L_{41} \cup L_{23} & , g = (243) \\ L_{42} \cup L_{12} & , g = (12)(34) \\ L_{31} \cup L_{42} & , g = (13)(24) \\ L_{24} \cup L_{12} & , g = (14)(23) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{12} \cup L_{34} & , g = (123) \\ L_{24} \cup L_{31} & , g = (124) \\ L_{13} \cup L_{42} & , g = (132) \\ L_{21} \cup L_{43} & , g = (134) \\ L_{34} \cup L_{12} & , g = (142) \\ L_{42} \cup L_{12} & , g = (143) \\ L_{31} \cup L_{24} & , g = (234) \\ L_{43} \cup L_{21} & , g = (243) \\ L_{41} \cup L_{32} & , g = (12)(34) \\ L_{32} \cup L_{41} & , g = (13)(24) \\ L_{23} \cup L_{14} & , g = (14)(23) \end{cases}.$$

Again, all of $L_{12} \cup L_{34}$, $L_{13} \cup L_{24}$, and $L_{14} \cup L_{23}$ can be obtained from each other by the action of A_3 for any one of the three conjugate copies of A_3 in A_4. Therefore, we only need to count one of $[L_{12} \cup L_{34}]$, $[L_{13} \cup L_{24}]$, or $[L_{14} \cup L_{23}]$ in the left-hand side of equation (2.1). Making an arbitrary choice, we will count $[L_{12} \cup L_{34}]$. Note that the stabilizer of $[L_{12} \cup L_{34}]$ is $H := \{(), (12)(34), (13)(24), (14)(23)\}$ and is isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$. The branches of $[L_{12} \cup L_{34}]$ as an H-set are $[H/\langle(14)(23)\rangle]$. Therefore the weight of $[L_{12} \cup L_{34}]$, and therefore the left-hand side of equation (2.1), is $\mathrm{wt}^G([L_{12} \cup L_{34}]) = \mathrm{inf}_H^G([\frac{H}{\langle(14)(23)\rangle}] - \{*\}) = [G/\langle(14)(23)\rangle] - [G/H]$.

Since the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = [G/A_3] - \{*\}$, we need to show that $[G/\langle(14)(23)\rangle] - [G/H] = [G/A_3] - \{*\}$. In order to show both sides are equal, we will use (Dieck 1979) Proposition 1.2.2. In particular, we need to show that for each $K \leq G$, $\left|(G/\langle(14)(23)\rangle] - [G/H])^K\right| = \left|([G/A_3] - \{*\})^K\right|$.

Before doing so, we will describe the coset structure of $[G/\langle(14)(23)\rangle] - [G/H]$ to make following the fixed-point calculations easier for the reader. Observe that $G/\langle(14)(23) = \{[()], [(123)], [(132)], [(124)], [(134)], [(12)(34)]\}$ with

$$\begin{array}{cc} (), (14)(23) \in [()] & (124), (234) \in [(124)] \\ (123), (142) \in [(123)] & (134), (243) \in [(134)] \\ (132), (143) \in [(132)] & (12)(34), (13)(24) \in [(12)(34)] \end{array}$$

Additionally, $G/H = \{[()], [(123)], [(132)]\}$ with

$$(), (12)(34), (13)(24), (14)(23) \in [()]$$

$$(123), (134), (243), (142) \in [(123)]$$

$$(132), (234), (124), (143) \in [(132)]$$

With this coset structure of both $[G/\langle(14)(23)\rangle] - [G/H]$ and $[G/A_3] - \{*\}$ in mind, the K-fixed points of either G-set refers to cosets that are fixed under left-multiplication by any element of K for $K \leq G$. Creating a table to record fixed points and writing $|(-)|$ for cardinality,

Table 2.1 $G = A_4$, $[\Sigma] = [G/A_3]$ fixed points.

$K \leq G$	$\lvert([G/\langle(14)(23)\rangle] - [G/H])^K\rvert$	$\lvert([G/A_3] - \{*\})^K\rvert$
$\langle()\rangle$	3	3
$\mathbb{Z}/2 = \{(), (12)(34)\}$	-1	-1
$H = \{(), (12)(34), (13)(23), (14)(23)\}$	-1	-1
$A_3 = \{(), (123), (132)\}$	0	0
G	-1	-1

Since for each $K \leq G$, the number of K-fixed points of $[G/\langle(14)(23)\rangle] - [G/H]$ and $[G/A_3] - \{*\}$ are equal, the two G-sets are equal in $A(G)$ by (Dieck 1979) Proposition 1.2.2. Therefore, equation (2.1) is true for $G = A_3$ and $[\Sigma] = [G/A_3]$. Since this was the last case to check for $G = A_4$, the main theorem is true for A_4.

2.2.7 $G = S_4$

The only possibilities for $[\Sigma]$ in $A(G)$ are

1. $[\Sigma] = 4\{*\}$
2. $[\Sigma] = 2\{*\} + [G/A_4]$
3. $[\Sigma] = 2[G/A_4]$
4. $[\Sigma] = \{*\} + [G/D_8]$

Consider the case where $[\Sigma] = 4\{*\}$. All four points of $[\Sigma]$ are fixed, so

$$\mathrm{stab}([L_{12} \cup L_{34}]) = \mathrm{stab}([L_{13} \cup L_{24}]) = \mathrm{stab}([L_{14} \cup L_{23}]) = G.$$

Note then that $\mathrm{wt}^G([L_{ij} \cup L_{kl}]) = \{[L_{ij} \cup L_{kl}]\} - \{*\} = 2\{*\} - \{*\} = \{*\}$ for any distinct $i, j, k, l \in \{1, 2, 3, 4\}$. Hence the left-hand side of equation (2.1) is $3\{*\}$, and the right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 4\{*\} - \{*\} = 3\{*\}$.

Consider the second case, $[\Sigma] = 2\{*\} + [G/A_4]$. Say b_1 and b_2 are fixed, $b_3 = [()]$, and $b_4 = [(12)]$. For the sake of brevity, we will not say where each element of S_4 maps each of $[L_{12} \cup L_{34}]$, $[L_{13} \cup L_{24}]$, and $[L_{13} \cup L_{24}]$. One can check that

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g \in S_4 \end{cases}$$

$$g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g \in A_4 \\ L_{14} \cup L_{23} & , g \in G \setminus A_4 \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{13} \cup L_{24} & , g \in G \setminus A_4 \end{cases}.$$

Observe $\mathrm{stab}([L_{12} \cup L_{34}]) = G$, and both branches are fixed by every element of G. Therefore $\mathrm{wt}^G([L_{12} \cup L_{34}]) = 2\{*\} - \{*\} = \{*\}$. Also observe that $[L_{13} \cup L_{24}]$ and $[L_{14} \cup L_{23}]$ have the same stabilizer, A_4, and that they're both in the same orbit of any $g \in G \setminus A_4$ in $\{[L_{12} \cup L_{34}], [L_{13} \cup L_{24}], [L_{14} \cup L_{23}]\}$. Therefore we only need to count one of $[L_{13} \cup L_{24}]$ or $[L_{14} \cup L_{23}]$. Arbitrarily choosing $[L_{13} \cup L_{24}]$, observe $\mathrm{wt}^G([L_{13} \cup L_{24}]) = \mathrm{inf}_{A_4}^G(2\{*\} - \{*\}) = [G/A_4]$. Therefore the left-hand side of equation (2.1) is $\{*\} + [G/A_4]$. The right-hand side of equation (2.1) is $[\Sigma] - \{*\} = 2\{*\} + [G/A_4] - \{*\} = [G/A_4] + \{*\}$, as desired.

Consider the third case, $[\Sigma] = 2[G/A_4]$. Say $[\Sigma] = \{b_1 = [()], b_2 = [(12)]\} + \{b_3 = [()], b_4 = [(12)]\}$. Then

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g \in A_4 \\ L_{21} \cup L_{43} & , g \in G \setminus A_4 \end{cases}$$

$$g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g \in A_4 \\ L_{24} \cup L_{13} & , g \in G \setminus A_4 \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{23} \cup L_{14} & , g \in G \setminus A_4 \end{cases}.$$

Each node has stabilizer equal to G. The branches of $[L_{12} \cup L_{34}]$ are equal to $2\{*\}$, so $\mathrm{wt}^G([L_{12} \cup L_{34}]) = \{*\}$. For both $[L_{13} \cup L_{24}]$ and $[L_{14} \cup L_{23}]$, the branches are equal to $[G/A_4]$. Thus $\mathrm{wt}^G([L_{13} \cup L_{24}]) = \mathrm{wt}^G([L_{14} \cup L_{23}]) = [G/A_4] - \{*\}$. Therefore, the left-hand side of equation (2.1) is equal to $2[G/A_4] - 2\{*\} + \{*\} = 2[G/A_4] - \{*\}$. The right-hand side of equation (2.1) is equal to $[\Sigma] - \{*\} = 2[G/A_4] - \{*\}$, as desired.

Now consider the last case, $[\Sigma] = \{*\} + [G/D_8] = \{*\} + \{[()], [(1324)], [(1342)]\}$. Note first that there are three copies of D_8 in S_4, all conjugate. They are $H_1 := \langle (13), (1234) \rangle$, $H_2 := \langle (12), (1324) \rangle$, and $H_3 := \langle (14), (1243) \rangle$. We will not write where every element of S_4 maps each of $[L_{12} \cup L_{34}], [L_{13} \cup L_{24}]$, and $[L_{14} \cup L_{23}]$. However, $\mathrm{stab}([L_{12} \cup L_{34}]) = H_1$, $\mathrm{stab}([L_{13} \cup L_{24}]) = H_2$, and $\mathrm{stab}([L_{14} \cup L_{23}]) = H_3$. Furthermore, $(123) \cdot L_{12} \cup L_{34} = L_{14} \cup L_{23}$, $(123) \cdot L_{14} \cup L_{23} = L_{13} \cup L_{24}$, and $(123) \cdot L_{13} \cup L_{24} = L_{12} \cup L_{34}$. Since all of $[L_{12} \cup L_{34}], [L_{13} \cup L_{24}]$, and $[L_{14} \cup L_{23}]$ are all in the same orbit of $\langle(123)\rangle$, we can count any one of them to obtain the left-hand side of equation (2.1). Arbitrarily choosing to count $[L_{12} \cup L_{34}]$, $\mathrm{wt}^G([L_{12} \cup L_{34}]) = \inf_{D_8}^G(2\{*\}) = [G/D_8]$. Thus the left-hand side of equation (2.1) is equal to $[G/D_8]$. The right-hand side of equation (2.1) is equal to $[\Sigma] - \{*\} = [G/D_8]$, as desired. $\square$

Chapter 3

Counterexamples

This chapter will give counterexamples where equation (2.1) does not hold, which is for groups isomorphic to $\mathbb{Z}/2 \times \mathbb{Z}/2$ or D_8.

3.1 $G = \mathbb{Z}/2 \times \mathbb{Z}/2$

Let $G = \{(), (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. We have an action of S_4, and therefore of G, on $\mathbb{P}^2_{\mathbb{C}}$ using the standard $PGL(3, \mathbb{C})$-representation of S_4. The G standard $PGL(3, \mathbb{C})$ representation is $G \to PGL(3, \mathbb{C})$ given by

$$g_1 := () \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad , \quad g_2 := (12)(34) \mapsto \begin{bmatrix} -1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 1 & -1 \end{bmatrix},$$

$$g_3 := (13)(24) \mapsto \begin{bmatrix} 0 & -1 & 1 \\ 0 & -1 & 0 \\ 1 & -1 & 0 \end{bmatrix} \quad , \text{ and } \quad g_4 := (14)(23) \mapsto \begin{bmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{bmatrix}.$$

Consider the point $[1 : 2 : 3] \in \mathbb{P}^2_{\mathbb{C}}$. Using the matrices above to represent the action of G on $\mathbb{P}^2_{\mathbb{C}}$, $g_1 \cdot [1 : 2 : 3] = [1 : 2 : 3]$, $g_2 \cdot [1 : 2 : 3] = [1 : 2 : -1]$, $g_3 \cdot [1 : 2 : 3] = [1 : -2 : -1]$, and $g_4 \cdot [1 : 2 : 3] = [-3 : -2 : -1]$. Define the size 4 G-set $[\Sigma] := \{b_1 := [1 : 2 : 3], b_2 := [1 : 2 : -1], b_3 := [1 : -2 : -1], b_4 := [-3 : -2 : -1]\}$. We will show that $[\Sigma]$ is a G-invariant base locus of a general pencil, i.e. no three points in $[\Sigma]$ are colinear, but that (2.1) does not hold for the G-invariant pencil of conics associated to $[\Sigma]$.

Note first that by definition, $[\Sigma]$ is G-invariant with $g \cdot b_i = b_{i+1}$ for $1 \leq i \leq 3$ and $g \cdot b_4 = b_1$ for all $g \neq ()$ in G. Furthermore, $[\Sigma]$ was defined to be isomorphic to G as a G-set, with the isomorphism being given by $b_i \mapsto g_i$ for $1 \leq i \leq 4$. Therefore $[\Sigma] = [G]$ in $A(G)$.

Now we will check that no three points in $[\Sigma]$ lie on a line. The possible lines through any two points in $[\Sigma]$ are:

1. $L_{12} = \overline{b_1 b_2} = \{-8x + 4y = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$
2. $L_{13} = \overline{b_1 b_3} = \{4x + 4y - 4z = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$
3. $L_{14} = \overline{b_1 b_4} = \{4x - 8y + 4z = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$
4. $L_{23} = \overline{b_2 b_3} = \{-4z = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$
5. $L_{24} = \overline{b_2 b_4} = \{-4x + 4y + 4z = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$
6. $L_{34} = \overline{b_3 b_4} = \{4y - 8z = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$

For each of the lines L_{ij}, $1 \leq i, j \leq 4$, listed above, it is easy to show that b_k, b_l do not lie on L_{ij} for $k, l \in \{1, 2, 3, 4\} - \{i, j\}$. Therefore $[\Sigma]$ satisfies the conditions necessary to define a G-equivariant pencil of general conics in $\mathbb{P}^2_{\mathbb{C}}$.

Now we will show that equation (2.1) does not hold for $[\Sigma]$. Observe

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{12} \cup L_{43} & , g = (12)(34) \\ L_{34} \cup L_{12} & , g = (13)(24) \\ L_{43} \cup L_{21} & , g = (14)(23) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{24} \cup L_{13} & , g = (12)(34) \\ L_{31} \cup L_{42} & , g = (13)(24) \\ L_{42} \cup L_{31} & , g = (14)(23) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{23} \cup L_{14} & , g = (12)(34) \\ L_{32} \cup L_{41} & , g = (13)(24) \\ L_{41} \cup L_{32} & , g = (14)(23) \end{cases}.$$

Each of $[L_{12} \cup L_{34}]$, $[L_{13} \cup L_{24}]$, and $[L_{14} \cup L_{23}]$ has stabilizer equal to G and branches equal to $[G/H]$ for $H \leq G$ the subgroup of G that fixes both branches in addition to the union. Thus $\mathrm{wt}^G([L_{12} \cup L_{34}]) = [G/\langle(12)(34)\rangle] - \{*\}$, $\mathrm{wt}^G([L_{13} \cup L_{24}]) = [G/\langle(13)(24)\rangle] - \{*\}$, and $\mathrm{wt}^G([L_{14} \cup L_{23}]) = [G/\langle(14)(23)\rangle] - \{*\}$. Therefore the left-hand side of (2.1) is $[G/\langle(12)(34)\rangle] + [G/\langle(13)(24)\rangle] + [G/\langle(14)(23)] - 3\{*\}$. The right-hand side of (2.1) is $[\Sigma] - \{*\} = [G] - \{*\}$.

As with the group A_4 and the base locus equal to $[A_4/A_3]$, we will use (Dieck 1979) Proposition 1.2.2 to determine whether the left-hand and right-hand sides of equation (2.1) are equal in $A(G)$. In particular, we need to compute for each $K \leq G$ the number of K-fixed points of the left-hand side and the right-hand side of (2.1). The number K-fixed points of either G-set is the number of cosets that are fixed under left-multiplication by any element of K. Creating a table to record fixed points:

Table 3.1 $G = \mathbb{Z}/2 \times \mathbb{Z}/2$, $[\Sigma] = [G]$ fixed points.

$K \leq G$	$\mid ([G/\langle(12)(34)\rangle]+[G/\langle(13)(24)\rangle]+ [G/\langle(14)(23)] - 3\{*\})^K \mid$	$\mid ([G] - \{*\})^K \mid$
$\langle()\rangle$	3	3
$\langle(12)(34)\rangle$	-2	-1
$\langle(13)(24)\rangle$	-2	-1
$\langle(14)(23)\rangle$	-2	-1
G	-3	-1

The fact that there are subgroups of G for which the number of fixed points of $[G/\langle(12)(34)\rangle] + [G/\langle(13)(24)\rangle] + [G/\langle(14)(23)] - 3\{*\}$ and $[G] - \{*\}$ are not equal implies that the two sets are not equal in $A(G)$. Therefore Theorem 2.2 is not true for $G = \mathbb{Z}/2 \times \mathbb{Z}/2$ and $[\Sigma] = [G] = \{[1:2:3], [1:2:-1], [1:-2:-1], [-3:-2:-1]\}$. Furthermore, we can say what the equations of the nodal conics in the pencil associated to $[\Sigma]$ are. They are $(L_{12} \cdot L_{34}) = y^2 - 2xy + 4xz - 2yz$, $(L_{13} \cdot L_{24}) = -x^2 + y^2 - z^2 + 2xz$, and $(L_{14} \cdot L_{13}) = -z^2 - xz + 2yz$.

3.2 $G = D_8$

For $D_8 = \{(), (1234), (13)(24), (1432), (12)(34), (14)(23), (13), (24)\}$ we will use a different approach to provide counterexamples to Theorem 2.2. We will start with a 3-dimensional representation of D_8 on $W := (\mathbb{C}^3)^\vee$ to obtain a 6-dimensional representation of D_8 on $V := \mathrm{Sym}^2((\mathbb{C}^3)^\vee)$. The G-invariant vector space V has a decomposition into irreducible sub-representations using the common eigenspaces of the generators of D_8, and from these irreducible sub-representations the pencils of conics can be read off as the spans of irreducible 1-dimensional sub-representations.

We will say $r := (13)$ and $s := (1234)$ so that $D_8 = \langle r, s \colon r^2 = s^4 = 1, rxr^{-1} = s^{-1}\rangle$. For reference, the character table of D_8 is:

Table 3.2 Character table of D_8.

	e	r^2	r	s	sr
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	1	-1	1	-1
χ_4	1	1	-1	-1	1
σ	2	-2	0	0	0

where χ_1, χ_2, χ_3, and χ_4 are the four 1-dimensional representations of D_8 and σ is the unique 2-dimensional representation of D_8. The character of any 3-dimensional representation of D_8 is then given by $\chi = \sigma + \chi_i$ or $\chi = \chi_i + \chi_j + \chi_k$, $i, j, k \in \{1, 2, 3, 4\}$. We will produce two counterexamples to Theorem 2.2 using a 3-dimensional representation of W with character $\chi = \sigma + \chi_i$.

The unique 2-dimensional representation of D_8 is given by

$$r \mapsto \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \qquad s \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Therefore, a 3-dimensional D_8 representation of W with basis $\{x, y, z\}$ and with character $\sigma + \chi_i$ is given by

$$r \mapsto \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & a \end{bmatrix} =: M_r, \qquad s \mapsto \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & b \end{bmatrix} =: M_s$$

where $a, b \in \{\pm 1\}$ are equal to the values of $\operatorname{tr} \chi_i(r)$ and $\operatorname{tr} \chi_i(s)$ respectively. Using the basis $\{x^2, y^2, z^2, yz, xz, xy\}$ for V, observe

$$\begin{aligned} M_r \cdot x^2 &= y^2 \\ M_r \cdot y^2 &= x^2 \\ M_r \cdot z^2 &= a^2 z^2 = z^2 \\ M_r \cdot yz &= -axz \\ M_r \cdot xz &= ayz \\ M_r \cdot xy &= -xy. \end{aligned}$$

Likewise,

$$\begin{aligned} M_s \cdot x^2 &= x^2 \\ M_s \cdot y^2 &= y^2 \\ M_s \cdot z^2 &= b^2 z^2 = z^2 \\ M_s \cdot yz &= -byz \\ M_s \cdot xz &= bxz \\ M_s \cdot xy &= -xy. \end{aligned}$$

Therefore, 6-dimensional representation of V obtained from the symmetric power of W is given by

$$r \mapsto \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & a & 0 \\ 0 & 0 & 0 & -a & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} = \mathrm{Sym}^2(M_r)$$

and

$$s \mapsto \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -b & 0 & 0 \\ 0 & 0 & 0 & 0 & b & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} = \mathrm{Sym}^2(M_s).$$

Table 3.3 1-dimensional sub-representations of any 6-dimensional representation of D_8.

	$\mathrm{Sym}^2(M_r)$	$\mathrm{Sym}^2(M_s)$
z^2	1	1
$x^2 - y^2$	-1	1
$x^2 + y^2$	1	1
xy	-1	-1

The 1-dimensional common G-invariant eigenspaces of $\mathrm{Sym}^2(M_r)$ and $\mathrm{Sym}^2(M_s)$ are z^2, x^2-y^2, x^2+y^2, and xy. The eigenvalues are recorded in Table 3.3. There is also a 2-dimensional common G-eigenspace with basis $\{yz, xz\}$. Therefore, the possible G-invariant pencils of conics in $\mathbb{P}^2$ with action coming from the representation of D_8 on V with character $\mathrm{Sym}^2(\sigma + \chi_i)$ are:

1. $\{\mu YZ + \lambda XZ = 0 \colon \mu, \lambda \in \mathbb{C}\}$

2. $\{\mu Z^2 + \lambda(X^2 - Y^2) = 0 \colon \mu, \lambda \in \mathbb{C}\}$

3. $\{\mu Z^2 + \lambda(X^2 + Y^2) = 0 \colon \mu, \lambda \in \mathbb{C}\}$

4. $\{\mu Z^2 + \lambda XY = 0 \colon \mu, \lambda \in \mathbb{C}\}$

5. $\{\mu(X^2 - Y^2) + \lambda(X^2 + Y^2) = 0 \colon \mu, \lambda \in \mathbb{C}\}$

6. $\{\mu(X^2 - Y^2) + \lambda XY = 0 \colon \mu, \lambda \in \mathbb{C}\}$

7. $\{\mu(X^2 + Y^2) + \lambda XY = 0 \colon \mu, \lambda \in \mathbb{C}\}$

8. $\{\mu(X^2 - Y^2) + \lambda(c(X^2 + Y^2) + dZ^2) = 0 \colon \mu, \lambda, c, d \in \mathbb{C}\}$

9. $\{\mu XY + \lambda(c(X^2 + Y^2) + dZ^2) = 0 \colon \mu, \lambda, c, d \in \mathbb{C}\}$

For each case, we will find the base locus of the associated pencil and either show that Theorem 2.2 doesn't hold if the conics defining the pencil are general or find that the conics defining the pencil are not in general position.

In the first case, $[\Sigma] = \{YZ = XZ = 0\}$. The conics defining $\{\mu YZ + \lambda XZ = 0 \colon \mu, \lambda \in \mathbb{C}\}$ are not in general position because both conics share the line $\{z = 0\} \subseteq \mathbb{P}^2_{\mathbb{C}}$. In particular, $[\Sigma]$ contains infinitely many points.

In the second case, $[\Sigma] = \{Z^2 = X^2 - Y^2 = 0\}$. The only points in $\mathbb{P}^2_{\mathbb{C}}$ that satisfy $Z^2 = 0$ have the form $[s : t : 0]$ with $s, t \in \mathbb{C}$ not both equal to 0. Notice that $[s : t : 0]$ is a solution to $X^2 - Y^2 = 0$ only if $s = \pm t$. Therefore $[\Sigma] = \{[1 : 1 : 0], [1 : -1 : 0]\}$, and the conics defining $\{\mu Z^2 + \lambda(X^2 - Y^2) = 0 \colon \mu, \lambda \in \mathbb{C}\}$ are not in general position.

In the third case, $[\Sigma] = \{Z^2 = X^2 + Y^2 = 0\}$. As in the previous case, the only points in $\mathbb{P}^2_{\mathbb{C}}$ that satisfy $Z^2 = 0$ have the form $[s : t : 0]$ with $s, t, \in \mathbb{C}$ not both equal to 0. The point $[s : t : 0]$ only satisfies $X^2 + Y^2 = 0$ if $s = \pm it$. Therefore $[\Sigma] = \{[1 : i : 0], [1 : -i : 0]\}$, and the conics defining $\{\mu Z^2 + \lambda(X^2 + Y^2) = 0 \colon \mu, \lambda \in \mathbb{C}\}$ are not in general position.

In the fourth case, $[\Sigma] = \{Z^2 = XY = 0\}$. As in the previous two cases, the only points in $\mathbb{P}^2_{\mathbb{C}}$ that satisfy $Z^2 = 0$ have the form $[s : t : 0]$ with $s, t \in \mathbb{C}$ not both equal to 0. A point $[s : t : 0]$ is a solution to $XY = 0$ only if at least one of s or t is equal to 0. Therefore $[\Sigma] = \{[0 : 1 : 0], [1 : 0 : 0]\}$, and so the conics defining $\{\mu Z^2 + \lambda XY = 0\colon \mu, \lambda \in \mathbb{C}\}$ are not in general position.

In the fifth case, $[\Sigma] = \{X^2 - Y^2 = X^2 + Y^2 = 0\}$. We can factor the two defining conics so that $[\Sigma] = \{(X + Y)(X - Y) = (X + iY)(X - iY) = 0\}$. The only two solutions to $(X+Y)(X-Y) = 0$ have the form $[s : s : t]$ or $[-s : s : t]$ with $s, t \in \mathbb{C}$ not both equal to 0. Points of both of these forms are solutions to $(X + iY)(X - iY) = 0$ if and only if $s = 0$. Similarly, the only solutions to $(X + iY)(X - iY) = 0$ have the form $[s : is : t]$ or $[s : -is : t]$ with $s, t \in \mathbb{C}$ not both equal to 0, and points of both of these forms are solutions to $(X + Y)(X - Y) = 0$ if and only if $s = 0$. Therefore, $[\Sigma] = \{[0 : 0 : 1]\}$, and the conics defining $\{\mu(X^2 - Y^2) + \lambda(X^2 + Y^2) = 0\colon \mu, \lambda \in \mathbb{C}\}$ are not in general position.

In the sixth case, $[\Sigma] = \{X^2 - Y^2 = XY = 0\}$. As in the previous case, the only solutions to $X^2 - Y^2 = 0$ either have the form $[s : s : t]$ or $[s : -s : t]$ with $s, t \in \mathbb{C}$ not both equal to 0. Points of either of these forms are solutions to $XY = 0$ if and only if $s = 0$. Therefore $[\Sigma] = \{[0 : 0 : 1]\}$, and the pencils defining $\{\mu(X^2 - Y^2) + \lambda XY = 0\colon \mu, \lambda \in \mathbb{C}\}$ are not in general position.

In the seventh case, $[\Sigma] = \{X^2 + Y^2 = XY = 0\}$. The only solutions to $X^2 + Y^2 = 0$ either have the form $[s : is : t]$ or $[s : -is : t]$ with $s, t \in \mathbb{C}$ not both equal to 0. Points of these forms are solutions to $XY = 0$ if and only if $s = 0$. Therefore $[\Sigma] = \{[0 : 0 : 1]\}$, and the conics defining $\{\mu(X^2 + Y^2) + \lambda XY = 0\colon \mu, \lambda \in \mathbb{C}\}$ are not general.

In the eighth case, $[\Sigma] = \{X^2 - Y^2 = c(X^2 + Y^2) + dZ^2 = 0\}$. In this case, $[\Sigma] = \{[1 : 1 : i\sqrt{\frac{2c}{d}}], [1 : -1 : i\sqrt{\frac{2c}{d}}], [1 : 1 : -i\sqrt{\frac{2c}{d}}], [1 : -1 : -i\sqrt{\frac{2c}{d}}]\}$ has four distinct points with no three co-linear. We will show equation (2.1) is not true in this case.

In the ninth case, $[\Sigma] = \{X^2 + Y^2 = c(X^2 + Y^2) + dZ^2 = 0\}$. In this case, $[\Sigma] = \{[1 : 0 : i\sqrt{\frac{c}{d}}], [0 : 1 : i\sqrt{\frac{c}{d}}], [1 : 0 : -i\sqrt{\frac{c}{d}}], [0 : -1 : i\sqrt{\frac{c}{d}}]\}$ has four distinct points with no three co-linear. We will show that equation (2.1) is not true for $\{\mu XY + \lambda(c(X^2 + Y^2) + dZ^2) = 0 \colon \mu, \lambda, c, d \in \mathbb{C}\}$.

3.2.1 $X := \{\mu(X^2 - Y^2) + \lambda(c(X^2 + Y^2) + dZ^2) = 0 \colon \mu, \lambda, c, d \in \mathbb{C}\}$

In this case, we will write $b_1 = [1 : 1 : i\sqrt{\frac{2c}{d}}], b_2 = [1 : -1 : i\sqrt{\frac{2c}{d}}], b_3 = [1 : 1 : -i\sqrt{\frac{2c}{d}}]$, and $b_4 = [1 : -1 : -i\sqrt{\frac{2c}{d}}]$ so that $[\Sigma] = \{b_1, b_2, b_3, b_4\}$. Recall the representation on V is the symmetric power of the representation on W given by $r \mapsto M_r$ and $s \mapsto M_s$, with M_r and M_s depending on the values of $a = \operatorname{tr} \chi_i(r)$ and $b = \operatorname{tr} \chi_i(s)$ respectively. The four cases we need to consider are $a = b = 1$, $a = 1$ and $b = -1$, $a = -1$ and $b = 1$, and $a = b = -1$.

$a = b = 1$

In this case,

$$\begin{array}{cc} () \cdot b_1 = b_1 & (14)(23) \cdot b_1 = b_1 \\ (13) \cdot b_1 = b_2 & (1432) \cdot b_1 = b_2 \\ (13)(24) \cdot b_1 = b_3 & (12)(34) \cdot b_1 = b_3 \\ (1234) \cdot b_1 = b_4 & (24) \cdot b_1 = b_4 \end{array}.$$

Therefore $[\Sigma] = [G/\langle(14)(23)\rangle] = \{b_1 = [()], b_2 = [(13)], b_3 = [(13)(24)], b_4 = [(24)]\}$. Observe

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{41} \cup L_{23} & , g = (1234) \\ L_{34} \cup L_{12} & , g = (13)(24) \\ L_{23} \cup L_{41} & , g = (1432) \\ L_{32} \cup L_{14} & , g = (12)(34) \\ L_{14} \cup L_{32} & , g = (14)(23) \\ L_{21} \cup L_{43} & , g = (13) \\ L_{43} \cup L_{21} & , g = (24) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{42} \cup L_{13} & , g = (1234) \\ L_{31} \cup L_{42} & , g = (13)(24) \\ L_{24} \cup L_{31} & , g = (1432) \\ L_{31} \cup L_{24} & , g = (12)(34) \\ L_{13} \cup L_{42} & , g = (14)(23) \\ L_{24} \cup L_{13} & , g = (13) \\ L_{42} \cup L_{31} & , g = (24) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{43} \cup L_{12} & , g = (1234) \\ L_{32} \cup L_{41} & , g = (13)(24) \\ L_{21} \cup L_{34} & , g = (1432) \\ L_{34} \cup L_{21} & , g = (12)(34) \\ L_{12} \cup L_{43} & , g = (14)(23) \\ L_{23} \cup L_{14} & , g = (13) \\ L_{41} \cup L_{32} & , g = (24) \end{cases}.$$

Observe $\mathrm{stab}([L_{12} \cup L_{34}]) = \mathrm{stab}([L_{14} \cup L_{23}]) = \{(), (13)(24), (13), (24)\} =: H_1$, and $H_1 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Furthermore, $[L_{12} \cup L_{34}] = [L_{14} \cup L_{23}]$ in $A(G)$ because $G/H_1 \cdot [L_{12} \cup L_{34}] = [L_{14} \cup L_{23}]$ and $G/H_1 \cdot [L_{14} \cup L_{24}] = [L_{12} \cup L_{34}]$. Therefore we only need to count one of $[L_{12} \cup L_{34}]$ or $[L_{14} \cup L_{23}]$ in the left-hand side of equation (2.1). Arbitrarily choosing $[L_{12} \cup L_{34}]$, observe that $[L_{12} \cup L_{34}] = [H_1/\langle(13)\rangle]$ in $A(H_1)$.

Therefore,

$$\begin{aligned}
\mathrm{wt}^G([L_{12} \cup L_{34}]) &= \inf{}^G_{H_1}([H_1/\langle(13)\rangle] - \{*\}) \\
&= \left[\frac{H_1}{\langle(13)\rangle}\right] \cdot \left[\frac{G}{H_1}\right] - [G/H_1] \\
&= [G/\langle(13)\rangle] - [G/H_1].
\end{aligned}$$

If we had chosen to count with $[L_{14} \cup L_{23}]$, we would get $\mathrm{wt}^G([L_{14} \cup L_{23}]) = [G/\langle(24)\rangle] - [G/H_1]$. Since $\langle(24)\rangle$ and $\langle(13)\rangle$ are conjugate as subgroups of G, the quotients of G by each subgroup are equal and so $[G/\langle(13)\rangle] = [G/\langle(24)\rangle]$ in $A(G)$.

Observe that $\mathrm{stab}([L_{13} \cup L_{24}]) = G$, and $[L_{13} \cup L_{24}] = [G/H_2]$ in $A(G)$ where $H_2 := \{(), (12)(34), (13)(24), (14)(23)\}$. Thus $\mathrm{wt}^G([L_{13} \cup L_{24}]) = [G/H_2] - \{*\}$.

It is worth noting that $H_1 \cong H_2$ in S_4, but H_1 and H_2 are *not* conjugate in D_8. Therefore the two G-sets $[G/H_1]$ and $[G/H_2]$ are *not* equal in $A(G)$. The left-hand side of equation (2.1) is

$$\mathrm{wt}^G([L_{12} \cup L_{34}]) + \mathrm{wt}^G([L_{13} \cup L_{24}]) = [G/\langle(13)\rangle] - [G/H_1] + [G/H_2] - \{*\}.$$

Given that $[\Sigma] = [G/\langle(14)(23)\rangle]$, the right-hand side of equation (2.1) is $[G/\langle(14)(23)\rangle] - \{*\}$.

As with the the counter example for $\mathbb{Z}/2 \times \mathbb{Z}/2$, we will use (Dieck 1979) Proposition 1.2.2. In particular, to show that Theorem 2.2 is not true for the pencil $\{\mu(X^2 - Y^2) + \lambda(c(X^2 + Y^2) + dZ^2) = 0 \colon \mu, \lambda, c, d \in \mathbb{C}\}$ and $a = b = 1$ we need to show that for some $K \leq G$, the number of K-fixed points of the left-hand side of (2.1) is not equal to the number of K-fixed points of the right-hand side of (2.1). Creating a table to record fixed points:

The fact that the number of K-fixed points of the left-hand and right-hand sides of equation (2.1) are not equal for $K = H_1, H_2, \langle(13)\rangle$, and $\langle(14)(23)\rangle$ imply that the left-hand side and right-hand side are not equal in $A(G)$. Therefore Theorem 2.2 is not true in this case.

Table 3.4 $G = D_8$, $[\Sigma] = [G/\langle(14)(23)\rangle]$ fixed points.

$K \leq G$	$\mid ([G/\langle(13)\rangle] - [G/H_1] + [G/H_2] - \{*\})^K \mid$	$\mid ([G/\langle(14)(23)\rangle] - \{*\})^K \mid$
$\langle()\rangle$	3	3
G	-1	-1
H_1	-2	-1
H_2	-2	-1
$\langle(1234)\rangle$	-1	-1
$\langle(13)\rangle$	1	-1
$\langle(24)\rangle$	-1	-1
$\langle(13)(24)\rangle$	-1	-1
$\langle(12)(34)\rangle$	-1	-1
$\langle(14)(23)\rangle$	-1	3

It is worth noting that even if $[G/H_1] = [G/H_2]$ in $A(G)$, the left-hand side and right-hand side would still not be equal. In that case, the left-hand side of equation (2.1) would be $[G/\langle(13)\rangle] - \{*\}$ and the right-hand side would be $[\Sigma] - \{*\} = [G/\langle(14)(23)\rangle] - \{*\}$. The same issue arises, $[D_8/\langle(13)\rangle] = [D_8/\langle(14)(23)\rangle]$ in $A(S_4)$ because $\langle(13)\rangle$ and $\langle(14)(23)\rangle$ are conjugate in S_4, but $[D_8/\langle(13)\rangle] \neq [D_8/\langle(14)(23)\rangle]$ in $A(D_8)$. The issue with D_8 is that there exist subgroups of D_8 which are conjugate in S_4 but not in D_8, which is not true for all of the groups that the theorem is true for.

$a = 1, b = -1$ AND $a = -1, b = 1$

In both the case where $a = 1$ and $b = -1$ and the case where $a = -1$ and $b = 1$, $[\Sigma] = [G/\langle(12)(34)\rangle]$. We will write $b_1 = [()]$, $b_2 = [(24)]$, $b_3 = [(14)(23)]$, and $b_4 = [(13)]$. Doing a similar calculation as in the case where $a = b = 1$, the left hand side of equation (2.1) is $[G/\langle(24)\rangle] - [G/H_1] + [G/H_2] - \{*\}$, where H_1 and H_2 are as defined in the $a = b = 1$ case, and the right-hand side of equation (2.1) is $[G/\langle(12)(34)\rangle] - \{*\}$. The number of $\langle(24)\rangle$-fixed points of the left-hand side or equation (2.1) is 3 and the number of $\langle(24)\rangle$-fixed points of the right-hand side of equation (2.1) is -1. Therefore,

$[G/\langle(24)\rangle] - [G/H_1] + [G/H_2] - \{*\}$ and $[G/\langle(12)(34)\rangle] - \{*\}$ are not equal in $A(G)$. Again, this case fails because H_1 and H_2 are not conjugate in G and $\langle(24)\rangle$ and $\langle(12)(34)\rangle$ are not conjugate in G.

$a = b = -1$

In this case, $[\Sigma] = [G/\langle(14)(23)\rangle]$, and equation (2.1) does not hold using the exact same calculation as in the case where $a = b = 1$.

3.2.2 $\{\mu XY + \lambda(c(X^2 + Y^2) + dZ^2) = 0 \colon \mu, \lambda, c, d \in \mathbb{C}\}$

In this case, we will write $b_1 = [1 : 0 : i\sqrt{\frac{c}{d}}], b_2 = [0 : 1 : i\sqrt{\frac{c}{d}}], b_3 = [1 : 0 : -i\sqrt{\frac{c}{d}}]$ and $b_4 = [0 : 1 : -i\sqrt{\frac{c}{d}}]$ so that $[\Sigma] = \{b_1, b_2, b_3, b_4\}$. Again since M_r and M_s depend on $a = \operatorname{tr}\chi_i(r)$ and $b = \operatorname{tr}\chi_i(s)$, we must consider the cases $a = b = 1$, $a = 1$ and $b = -1$, $a = -1$ and $b = 1$, and $a = b = -1$. For brevity, we will only provide a counterexample to equation (2.1) with the $a = b = 1$ case, the other cases being similar.

$a = b = 1$

In this case,

$$\begin{array}{cc} () \cdot b_1 = b_1 & (14)(23) \cdot b_1 = b_2 \\ (13) \cdot b_1 = b_1 & (1432) \cdot b_1 = b_4 \\ (13)(24) \cdot b_1 = b_3 & (12)(34) \cdot b_1 = b_4 \\ (1234) \cdot b_1 = b_2 & (24) \cdot b_1 = b_3 \end{array}.$$

Therefore, $[\Sigma] = [G/\langle(13)\rangle]$. Therefore, the right-hand side of equation (2.1) is $[G/\langle(13)\rangle] - \{*\}$.

Observe

$$g \cdot L_{12} \cup L_{34} = \begin{cases} L_{12} \cup L_{34} & , g = () \\ L_{23} \cup L_{41} & , g = (1234) \\ L_{34} \cup L_{12} & , g = (13)(24) \\ L_{41} \cup L_{23} & , g = (1432) \\ L_{43} \cup L_{21} & , g = (12)(34) \\ L_{21} \cup L_{43} & , g = (14)(23) \\ L_{14} \cup L_{32} & , g = (13) \\ L_{32} \cup L_{14} & , g = (24) \end{cases} \qquad g \cdot L_{13} \cup L_{24} = \begin{cases} L_{13} \cup L_{24} & , g = () \\ L_{24} \cup L_{31} & , g = (1234) \\ L_{31} \cup L_{42} & , g = (13)(24) \\ L_{42} \cup L_{13} & , g = (1432) \\ L_{42} \cup L_{31} & , g = (12)(34) \\ L_{24} \cup L_{13} & , g = (14)(23) \\ L_{13} \cup L_{42} & , g = (13) \\ L_{31} \cup L_{24} & , g = (24) \end{cases}$$

$$g \cdot L_{14} \cup L_{23} = \begin{cases} L_{14} \cup L_{23} & , g = () \\ L_{21} \cup L_{34} & , g = (1234) \\ L_{32} \cup L_{41} & , g = (13)(24) \\ L_{43} \cup L_{12} & , g = (1432) \\ L_{41} \cup L_{32} & , g = (12)(34) \\ L_{23} \cup L_{14} & , g = (14)(23) \\ L_{12} \cup L_{43} & , g = (13) \\ L_{34} \cup L_{21} & , g = (24) \end{cases}.$$

Both $[L_{12} \cup L_{34}]$ and $[L_{13} \cup L_{24}]$ have stabilizer $H_1 := \{(), (12)(34), (13)(24), (14)(23)\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$. Furthermore, $G/H_1 \cdot [L_{12} \cup L_{34}] = [L_{14} \cup L_{23}]$ and $G/H_1 \cdot [L_{14} \cup L_{23}] = [L_{12} \cup L_{34}]$, so we only need to count one of $[L_{12} \cup L_{34}]$ or $[L_{14} \cup L_{23}]$ in the left-hand side of (2.1). Arbitrarily choosing $[L_{12} \cup L_{34}]$, $\mathrm{wt}^G([L_{12} \cup L_{34}]) = \mathrm{inf}_{H_1}^G([H/\langle (14)(23) \rangle] - \{*\}) = [G/\langle (14)(23) \rangle] - [G/H_1]$.

Now observe that $\mathrm{stab}([L_{13} \cup L_{24}]) = G$. Furthermore, the subgroup $H_2 := \{(), (13)(24), (13), (24)\} \cong \mathbb{Z}/2 \times \mathbb{Z}/2$ of G fixes the branches. Thus $\mathrm{wt}^G([L_{13} \cup L_{24}]) =$

$[G/H_2] - \{*\}$. Therefore the left-hand side of equation (2.1) is $[G/\langle(14)(23)\rangle] - [G/H_1] + [G/H_2] - \{*\}$. This is not equal to the right-hand side, $[G/\langle(13)\rangle] - \{*\}$, in $A(G)$ by (Dieck 1979) Proposition 1.2.2 because the number of $\langle(13)\rangle$-fixed points of both sides are not equal.

Chapter 4

Future Directions

This chapter will give a brief discussion of possible future questions to ask related to the topic of this dissertation.

Throughout this work, we have only considered pencils of conics where the defining equations are general, meaning that they intersect in four points with no three co-linear. However, it is certainly not true that all pencils are defined by conics in general position. In fact, the first seven cases of possible invariant pencils of conics for D_8, found in Section 3.2, are not in general position. A natural question to ask is the following:

Question 4.1. Given a finite group G and a G-invariant pencil of conics $X := \{\mu f + \lambda g = 0 \colon \mu, \lambda \in \mathbb{C}\} \subseteq \mathbb{P}^2_{\mathbb{C}}$ with f and g *not* in general position, is there a formula in terms of $[\Sigma] = \{p \in \mathbb{P}^2_{\mathbb{C}} \colon f(p) = g(p) = 0\}$ in $A(G)$ for the weighted sum of the number of nodal orbits in X?

Another topic of exploration alluded to but not explored is the relationship of each term in the left-hand side of equation (2.1) to the Milnor number of the defining equation of any nodal orbit. Again, while the global equivariant degree has been defined in (Roberts 1985), a local equivariant degree hasn't been defined or related to counting problems. This naturally leads to the following:

Question 4.2. Is there a way to define an equivariant Milnor number using local equivariant degrees, and if so how can it be used to reformulate equation (2.1) for pencils and groups for which Theorem 2.2 does hold?

Although counterexamples to Theorem 2.2 are given for groups isomorphic to D_8 and $\mathbb{Z}/2 \times \mathbb{Z}/2$, it is natural to ask if equation (2.1) can be modified to hold for all groups that can act invariantly on a pencil of general conics:

Question 4.3. Can equation(2.1) be modified so that a count of nodal orbits in a G-invariant pencil of general conics exists for G isomorphic to D_8 and $\mathbb{Z}/2 \times \mathbb{Z}/2$?

Finally, another area of consideration is whether or not a formula for the weighted sum of nodal conics can be found for pencils defined by higher degree curves. Göttsche's conjecture in full does not assume that f and g are conics. The proof given in this paper relies heavily on the fact that f and g are a general pair of conics, so intersect in exactly 4 points, and therefore all possible cases of $[\Sigma]$ can be exhausted for any finite G. As the degree of f and g increase, this becomes unrealistic. It is possible that the topological proof outlined in Section 1.4 can be modified to work equivariantly.

Question 4.4. Is there a count of nodal orbits in a G-invariant pencil of general conics, $X := \{\mu f + \lambda g = 0 \colon \mu, \lambda \in \mathbb{C}\}$ valued in $A(G)$ for f and g of arbitrary degree?

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