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Moving Off Collections and Their Applications, in Particular to Function Spaces

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by

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Abstract

The main focus of this paper is the concept of a moving off collection of sets. We will be looking at how this relatively lesser known idea connects and interacts with other more widely used topological properties. In particular we will examine how moving off collections act with the function spaces $C_p(X)$, $C_0(X)$, and $C_K(X)$. We conclude with a chapter on the Cantor tree and its moving off connections.

Many of the discussions of important theorems in the literature are expressed in terms that do not suggest the concept of moving off but can be rephrased using it. The main goal of this paper is to bring these scattered pieces of information together into a single organized work. As a secondary goal we will endeavor to make a number of important theorems in the literature easier for non-specialists to understand by giving expanded versions of their existing proofs.

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Chapter 1

Preliminaries

It is difficult, and perhaps impossible to find a mathematical statement that cannot be stated in a different way with alternative approaches to the same idea. Often times an alternative definition can make the statement of a concept appear entirely alien and yet be equivalent to its original form. This is a theme that will take hold in this paper. The concept of moving off and the related moving off property are lesser known and rarely studied topological definitions through which many commonly studied ideas can be restated and analyzed in a different light. Such an exercise is sometimes of worth and sometimes of less use, indeed there is typically good reason a definition or theorem is stated the way it is. However it is worth exploring as doing so results in new connections and insights that otherwise may not be seen.

We will begin with a discussion of introductory topics and definitions.

Definition 1.1. A topological space X is called *Tychonoff* if it is a completely regular T_1 space. Otherwise put, each pair of distinct points in X have neighborhoods not containing the other and for any closed set $A \subset X$, $x \notin A$, there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$.

Note that the choice of $[0, 1]$ could easily be replaced with the real number line in this definition.

Henceforth the word "space" will mean a Tychonoff space. Most of our results involve sets of real valued functions so this is a prudent decision. We will now introduce the definition of a moving off collection of sets. Later on we will introduce one use of the concept which has come to be known as the moving off property (MOP).

Definition 1.2. Let $\mathcal{K}$ be a collection of nonempty subsets of X. Let $\mathcal{C} \subseteq \mathscr{P}(X)$. Then $\mathcal{K}$ is said to *move off* $\mathcal{C}$ if for all $C \in \mathcal{C}$ there exists $K \in \mathcal{K}$ such that $C \cap K = \emptyset$. We also define the statement "$\mathcal{K}$ moves off $A \subset X$" to mean $\mathcal{K}$ moves off $\{A\}$. *[Nyikos 2003]*

The last bit of this definition is added to make our discussions more convenient. At times we will consider moving off with regards to a set, in which case we simply mean the collection of sets consisting of that set alone.

From here on, all collections of sets will be understood not to include the empty set. Again this is a stipulation made for the sake of convenience.

We will begin the illustration of moving off by presenting some examples as well as a trio of lemmas.

Example 1: $X = \mathbb{R}$, $\mathcal{K} = \mathcal{C} =$\{compact sets\}. In other words, the collection of compact sets on the real number line moves off itself. This is because the set of compact singletons moves off every compact set.

Example 2: In any compact space, no collection will move off the compact sets. And more generally it is impossible to ever move off a family of sets that includes the space itself.

Example 3: If $\mathcal{K}$ has two disjoint members, then it moves off the singletons.

Lemma 1.3. *Let X be a set and $\mathcal{K}$ be the collection of all singleton sets in X. Let $\mathcal{C}$ be the collection of finite sets in X. Then $\mathcal{K}$ moves off $\mathcal{C}$ if and only if $\mathcal{K}$ is infinite.*

Proof. $\Longrightarrow$ Suppose $\mathcal{K}$ moves off $\mathcal{C}$. By contradiction suppose $\mathcal{K}$ is finite. Then X is finite, so $X \in \mathcal{C}$ and therefore $\mathcal{K}$ does not move off $\mathcal{C}$. So $\mathcal{K}$ must be infinite.

$\Longleftarrow$ Suppose $\mathcal{K}$ is infinite. Then X is infinite. Therefore we have that $C \subset X$ and $C \neq X$. So for each $C \in \mathcal{C}$ there exists a point $p \in X$ such that $p \notin C$ for all $C \in \mathcal{C}$ and by definition $\{p\} \in \mathcal{K}$. Therefore $\mathcal{K}$ moves off $\mathcal{C}$. $\square$

In as much as every collection of singletons is disjoint, the following is a generalization of Lemma 1.3.

Lemma 1.4. *Let X be a set and let $\mathcal{K}$ be a collection of disjoint finite sets in X. Let $\mathcal{C}$ be the collection of finite sets in X. Then $\mathcal{K}$ moves off $\mathcal{C}$ iff $\mathcal{K}$ is infinite.*

Proof. $\Longrightarrow$ Suppose $\mathcal{K}$ moves off $\mathcal{C}$. By contradiction suppose $\mathcal{K}$ is finite. If that is the case then we have $\mathcal{K} = \{K_1, K_2, ..., K_n\}$. Let $J = \{p_1, p_2, ..., p_n\}$ such that $p_1 \in K_1, p_2 \in K_2$, etc. Then J is finite but $\mathcal{K}$ does not move off $\{J\}$ which is a contradiction. Therefore $\mathcal{K}$ must be infinite.

$\Longleftarrow$ Let $\mathcal{K}$ be infinite and suppose $\mathcal{K}$ does not move off $\mathcal{C}$. Then there exists $F = \{p_1, p_2, ..., p_n\}$ such that for every $K \in \mathcal{K}$, $K \cap F \neq \emptyset$. But since $\mathcal{K}$ is disjoint we have that for each $p_i \in F$ there is at most one $K \in \mathcal{K}$ containing it. But this would imply $\mathcal{K}$ is finite, which is a contradiction. So we have that $\mathcal{K}$ moves off $\mathcal{C}$. $\square$

Our third lemma expands on the previous two by discussing a collection of point-finite sets and how it relates to the finite sets. We first define what it means to be point-finite.

Definition 1.5. For a set S, a collection $\mathcal{K} \subseteq \mathscr{P}(S)$ is *point-finite* if for all $s \in S$ there exist only finitely many $K \in \mathcal{K}$ such that $s \in K$.

Lemma 1.6. *Let X be an infinite set. Let $\mathcal{K}$ be a point-finite collection of finite subsets. Let $\mathcal{C}$ be the collection of finite sets. Then $\mathcal{K}$ is infinite if and only if $\mathcal{K}$ moves off $\mathcal{C}$.*

Proof. $\Longrightarrow$ Let $\mathcal{K}$ be infinite and point-finite. So every point in X is in only finitely many sets in $\mathcal{K}$. Let $C \in \mathcal{C}$, then each point in C is in only finitely many sets in $\mathcal{K}$. But $\mathcal{K}$ is infinite so there exists $K \in \mathcal{K}$ such that $K \cap C = \emptyset$. So $\mathcal{K}$ moves off $\mathcal{C}$.

$\Longleftarrow$ Suppose $\mathcal{K}$ moves off the finite sets and suppose by contradiction that $\mathcal{K}$ is finite. Then $C = \cup\{K : K \in \mathcal{K}\}$ is a finite set which has nonempty intersection with every set in $\mathcal{K}$. Therefore $\mathcal{K}$ does not move off it, a contradiction. $\square$

These lemmas are meant to get the reader's feet wet, showing the idea of moving off and how it relates to other concepts of topology.

Chapter 2

Moving Off and Function Spaces

Here we develop further some various topological ideas centered around moving off and function spaces. No theorems will be proven in this chapter but some will be restated and proven in the next. Our goal is to present a sequence of mathematical facts that display the usefulness of the notion of moving off. A deeper dive into some of these statements will be reserved for Chapter 3. To begin, we introduce the notion of four different kinds of spaces.

Definition 2.1. A space X is *first countable* if every point has a countable neighborhood base.

Definition 2.2. A space X is *Fréchet-Urysohn* if for every set $A \subset X$ and for every $x \in \bar{A}$ there is a sequence of points $\{x_n\}$ in A that converges to x.

Definition 2.3. A space X is *countably tight* if for each $A \subset X$ and for each $x \in \bar{A}$ there exists a countable $B \subset A$ such that $x \in \bar{B}$.

Definition 2.4. A space X is a *sequential* space if a subset is closed if and only if it contains the limit of every convergent sequence contained within it.

It is worth noting that First Countable $\Longrightarrow$ Fréchet-Urysohn $\Longrightarrow$ Sequential $\Longrightarrow$ Countably Tight. First countable, countably tight, and Fréchet-Urysohn are all hereditary properties, meaning that any subspace of a space having these properties will also have these properties. Sequential is not a hereditary property. It is fairly easy to see the hereditary properties of the first countable, Fréchet-Urysohn, and countably tight spaces. Regarding sequential spaces, the argument goes that given a sequential space X which is not Fréchet-Urysohn, let p be in the closure of a set A but without a sequence from A converging to p. Let $\tilde{A}$ be the set of all $x \in X$ for which there is a sequence from A converging to x. Now $Y = (X \setminus \tilde{A}) \cup A$ is not sequential, because A is not closed, but it is sequentially closed because no sequence in A converges to a point of Y outside of A.

We will introduce one more topological space before this chapter's first pair of theorems which connect these concepts to the idea of moving off.

Definition 2.5. The space of continuous, real valued functions on X with the product topology is denoted as $C_p(X)$.

The following pair of theorems are not stated explicitly in Gerlits and Nagy's paper, however as will be shown in Chapter 3 they are equivalent to theorems proven there and so are cited to them here.

Theorem 2.6. *$C_p(X)$ is countably tight if and only if every collection of closed subsets of X which moves off the finite sets has a countable subcollection which also moves off the finite sets. [Gerlits and Nagy 1982]*

Theorem 2.7. *$C_p(X)$ is Fréchet-Urysohn if and only if every collection of closed subsets of X which moves off the finite sets has an infinite point-finite subcollection. [Gerlits and Nagy 1982]*

We will now have a brief discussion on the compact-open topology as it becomes relevant presently. If X and Y are topological spaces and $C(X, Y)$ is the set of continuous functions from X to Y then the *compact-open topology*, denoted $C_K(X, Y)$, on $C(X, Y)$ has a subbase given by the sets $V(K, U) = \{f \in C(X, Y) | f(K) \subseteq U\}$ where K ranges over all compact subsets of X, and U ranges over all open subsets of Y. So $\{V(K, U)\}$ generates the compact-open topology, in other words the compact-open topology is the minimal topology such that $V(K, U)$ are open sets. Therefore a subset $W \subseteq C(X,Y)$ is open in the compact-open topology if and only if it is an arbitrary union of finite intersections of elements of $\{V(K,U)\}$. In this paper when discussing $C_K(X)$ it will be implied we mean $C_K(X,\mathbb{R})$.

Our main focus in this paper will be on real valued functions, and therefore $C(X,\mathbb{R})$ will be simply denoted as $C(X)$. To extend this to the one-point compactification of a space we introduce the notation $C_0(X)$. For the following definitions as well as anywhere we are discussing $C_0(X)$ we will assume X to be locally compact.

Definition 2.8. A continuous function f *vanishes at infinity* if and only if for all $\epsilon > 0$ there exists a compact subset C of X such that $f(C^c) \subset (-\epsilon, \epsilon)$.

Definition 2.9. The set of real valued continuous functions that vanish at infinity with the compact-open topology is denoted $C_0(X)$.

Lemma 2.10. *Define* $f^* : X \cup \{\infty\} \to \mathbb{R}$*:*

$$f^*(x) = \begin{cases} f(x) & \text{if } x \in X \\ 0 & \text{if } x = \infty \end{cases}$$

Then $C_0(X)$ *is the subcollection of functions* $f \in C(X)$ *for which* f^* *is continuous.*

When talking about $C_0(X)$ in this paper it will be implied that we are discussing the compact-open topology on this space.

Theorem 2.11. *The relative topology on* $C_0(X)$ *is the same as the compact-open topology.*

We will at times use the phrase *canonically moving off*, or *moves off canonically* to mean that our subject of conversation is a compact collection that moves off the compact sets.

The next pair of statements display that $C_K(X)$ and $C_0(X)$ are related to certain topological properties which can be rewritten using the notion of moving off. Lemma

2.12 as well as the forward direction of Theorem 2.13 will be restated and proven in Chapter 3.

Lemma 2.12. *If $C_K(X)$ is countably tight, every canonically moving off family has a countable moving off subcollection. [Nyikos 2003]*

Theorem 2.13. *$C_0(X)$ is countably tight if and only if every canonically moving off collection in X has a countable moving off subcollection. [Nyikos 2003]*

Having not forgotten about the compact-open topology we must introduce two more definitions before we can see it in use for the first time.

Definition 2.14. On a space X, a collection of subsets $\mathcal{D}$ is *discrete* if for every $x \in X$ there exists an open neighborhood of x that meets at most one member of $\mathcal{D}$.

Definition 2.15. A space X is *Baire* if given any countable collection $\{A_n\}$ of closed sets in X each of which with empty interior, $\bigcup A_n$ also has empty interior in X.

Theorem 2.16. *If $C_K(X)$ is Baire then every collection of canonically moving off sets has an infinite discrete subcollection. If X is locally compact then the converse is true. [Gruenhage and Ma 1997]*

Towards introducing the definition of the moving off property (MOP) we must first state what it means to expand a collection of sets.

Definition 2.17. An *expansion* of a collection $\mathcal{C}$ is a collection of sets, $\mathcal{D}$, with the property that for all $C \in \mathcal{C}$ there exists $D(C) \in \mathcal{D}$ such that $C \subset D(C)$ and $D(C) = D(C')$ if and only if $C = C'$.

Definition 2.18. A space X has the *moving off property*, or *MOP*, if every canonically moving off collection has an infinite subcollection $\mathcal{L}$ that has a discrete open expansion in X.

We will start our discussion of the MOP with a lemma connecting it to a concept more commonly used. Both of the following lemmas will be proven in Chapter 3.

Lemma 2.19. *Let X be a normal space. Then the MOP is equivalent to the simpler condition that the subcollection $\mathcal{L}$ is discrete. [Nyikos 2003]*

Lemma 2.20. *Suppose in a locally compact space X every canonically moving off collection has an infinite discrete moving off subcollection. Then X has the MOP.*

Theorem 2.21. *$C_0(X)$ is Fréchet-Urysohn if and only if X has the MOP. [Nyikos 2003]*

We will now introduce the notion of the γ-property and a γ-set. Usually these ideas are formulated using open sets instead of closed sets as we do here. We will also introduce the notion of an ω-cover before connecting these ideas with Theorem 2.24.

Definition 2.22. A space X has the *γ-property* if every collection of closed sets which moves off the finite sets has an infinite point-finite subcollection. A *γ-set* is a subset of the real line with the γ-property.

Definition 2.23. An open cover of a space is an *ω-cover* if every finite subset of the space is contained in some member of the cover.

Theorem 2.24. *A space X has the γ-property if and only if every open ω-cover has a subcover $\mathcal{V}$ such that every element of X is contained in all but finitely many members of $\mathcal{V}$. [Nyikos 2003]*

The following pair of lemmas as well as Theorem 2.27 will be proven in the next chapter. Part (2) of Theorem 2.28 will also be shown.

Lemma 2.25. *For any compact set K and open set G containing K there exists a continuous $h : X \to [0, 1]$ taking K to 1 and G^c to 0.*

Lemma 2.26. *For locally compact X, if $\mathcal{K}$ is a canonically moving off family then for each $K \in \mathcal{K}$ there is a family $\mathcal{V}_K$ which is a base of neighborhoods of K with compact closure, and which is also a canonically moving off family.*

Theorem 2.27. *$C_K(X)$ being Baire implies X has the moving off property. [Gruenhage and Ma 1997]*

Theorem 2.28. *Let $\sigma = \langle d_n : n \in \mathbb{N} \rangle$ be a one-to-one sequence in a locally compact space X. Then:*

(1) ∞ is a cluster point of σ if and only if $\mathcal{K} = \{\{d_n\} : n \in \mathbb{N}\}$ moves off the compact sets.

(2) σ converges to $\infty \iff D = \{d_n : n \in \mathbb{N}\}$ is a closed discrete subspace of X if and only if the collection $\mathcal{K}$ of all cofinite subsets of D moves off the compact sets.

Chapter 3

More Results and Proofs

In this chapter we will restate and prove many of the theorems and lemmas seen previously, as well as introduce others that have not yet been discussed. We start by introducing a new theorem for which a definition and lemma are required.

Definition 3.1. A space X is *feebly compact* if every discrete collection of open sets is finite.

Often times the definition given for feebly compact is every locally finite family of open sets is finite. This is equivalent to Definition 3.1 in a regular space, so for this paper they are the same.

Lemma 3.2. *Every feebly compact space satisfying the moving off property is compact. [Gruenhage and Ma 1997]*

Proof. Let X be feebly compact but not compact. Then the collection of all singletons of X is a canonically moving off collection. If X had the moving off property then X would contain an infinite subset with a discrete open expansion, contradicting feebly compact. □

As promised, we will now prove Lemma 2.19 from the previous chapter.

Lemma 3.3. *Let X be a normal space. Then the MOP is equivalent to the simpler condition that the subcollection $\mathcal{L}$ is discrete.*

Proof. We may assume $\mathcal{L}$ is countable in the definition of MOP. Then $\mathcal{L} = \{C_n : n \in \mathbb{N}\}$ is a discrete collection of closed sets. We first expand to a collection of disjoint open sets, $\{U_n : n \in \mathbb{N}\}$, and then shrink each U_n down to V_n so that $C_n \subset V_n \subset U_n$ and $\{V_n : n \in \mathbb{N}\}$ is discrete. Let $K_n = \bigcup_{i=n+1}^{\infty} C_i$. Then K_n is closed. Let U_1 and V_1 be disjoint open sets containing C_1 and K_1 respectively. Let U_2 and V_2 be disjoint open subsets of V_1 containing C_2 and K_2 respectively and so on, in general let U_n and V_n be disjoint open subsets of V_{n-1} containing C_n and K_n respectively. Let $K_0 = \bigcup\{C_n : n \in \mathbb{N}\}$ and let $W_0 = \bigcup\{U_n : n \in \mathbb{N}\}$. Let V_0 be an open set such that $K_0 \subset V_0$ and $\overline{V_0} \subset W_0$ by the normality of X. Let $H_n = U_n \cap V_0$.

Claim: $\{H_n : n \in \mathbb{N}\}$ is a discrete open expansion of $\{C_n : n \in \mathbb{N}\}$.

Proof of claim: Let $\mathcal{H} = \{H_n : n \in \mathbb{N}\}$. Clearly every element in $\mathcal{H}$ is open. Also for every $n \in \mathbb{N}$, $C_n \subset U_n$, so $C_n \subset H_n$ and therefore $\mathcal{H}$ is an expansion. If $p \in X$, and $p \in U_n$ for some n, then U_n is a neighborhood of p that misses all of the sets H_k for which $k \neq n$. Otherwise $p \notin W_0$, and $\overline{V}_0^c$ is a neighborhood of p missing $\cup\mathcal{H}$. □

The following theorem restate the equivalences of Theorem 2.28 part (2).

Theorem 3.4. *Let $\sigma = \langle d_n : n \in \mathbb{N}\rangle$ be a one-to-one sequence in a locally compact space X. Then the following are equivalent:*

1. *σ converges to ∞.*

2. *The range of σ is a closed discrete subset of X.*

3. *The collection $\mathcal{K}$ of all cofinite subsets of D moves off the compact sets.*

Proof. *1 $\Longrightarrow$ 2* Suppose σ converges to ∞. Let p be any point in X. Then there are disjoint open neighborhoods U of p and V of ∞ in $X \cup \{\infty\}$ since $X \cup \{\infty\}$ is Hausdorff and V must contain $\{d_n : n \geq k\}$ for some k. Thus U contains only finitely many terms of σ, and there is a neighborhood $W \subset U$ of p which contains at most one point in the range of σ.

2 $\Longrightarrow$ 1 Suppose the range R of σ is a closed discrete subset of X. If V is an arbitrary open neighborhood of ∞, then $(X \cup \{\infty\}) \setminus V$ is a compact subset of X and thus it contains only finitely many points of R. Therefore, V contains $\{d_n : n \geq k\}$ for some k, which shows that σ converges to ∞.

2 $\implies$ *3* Suppose $D = \{d_n : n \in \mathbb{N}\}$ is a closed discrete subspace of X. Then for every point $p \in X$ there exists an open neighborhood of p that meets at most one member of D. Let $C \subset X$ be compact. Then for every point $p \in C$ there exists an open neighborhood of p containing at most one member of D. Let $\mathcal{U}$ be a collection of neighborhoods of this form. Then $\mathcal{U}$ is an open cover of C and hence there exists a finite subcover, call this $\mathcal{U}'$. For each $U \in \mathcal{U}'$ there exists at most one d_n in U. Hence there are only finitely many d_n's in the sets of $\mathcal{U}'$. Let $A = \{d_n : d_n \in U$ where $U \in \mathcal{U}'\}$. Then $D \backslash A$ is a cofinite subset of D that has empty intersection with C. Therefore the collection of cofinite subsets of D moves off the compact sets.

3 $\implies$ *2* Let $D = \{d_n : n \in \mathbb{N}\}$. Suppose the collection $\mathcal{K}$ of all cofinite subsets of D moves off the compact sets.

To first show that D is closed suppose by contradiction that D is not closed. Then there exists a limit point of D that is not contained in D. Call this point q. But then by local compactness there exists a compact neighborhood V of q. Since the cofinite sets of D move off V, we have that $D \cap V$ is finite, meaning q is not a limit point of D, a contradiction.

Let p be a point in X. Then by local compactness there exists a compact neighborhood C of p. Therefore there exists a cofinite subset of $\mathcal{K}$ which has empty intersection with C. Hence there are at most finitely many points in $A = C \cap D$. But since X is Hausdorff we have that there exists an open neighborhood U of p not containing any point in A that is not p itself. So U contains at most one point in D, so D is discrete.

□

Lemmas 3.5 and 3.6 are the restatements of Lemmas 2.25 and 2.26 respectively.

Lemma 3.5. *For any compact set K and open set G containing K there exists a continuous $h : X \to [0,1]$ taking K to 1 and G^c to 0.*

Proof. Let $x \in K$ and let $f_x : X \to [0,1]$ be a continuous function taking x to 1 and G^c to 0. Let $W = (\frac{1}{2}, 1]$. Then $\{f_x^{\leftarrow} W : x \in K\}$ is a family of open sets covering K. Take the finite subcover $\{f_{x_1}^{\leftarrow} W, f_{x_2}^{\leftarrow} W, ..., f_{x_n}^{\leftarrow} W\}$. Now construct the continuous $g : X \to [0,1]$ by letting $g(x) = (f_{x_1} \vee f_{x_2}, ..., \vee f_{x_n})(x) = max\{f_{x_1}(x), f_{x_2}(x), ..., f_{x_n}(x)\}$. Then we have that $g : K \to [\frac{1}{2}, 1]$, and $g : G^c \to \{0\}$. Let g^* be a continuous, onto function such that $g^* : [0,1] \to [0,1]$, $g^*([\frac{1}{2}, 1] = \{1\}$, and $g^*(G^c) = 0$ (for example $g^*(x) = 2x$ on $[0, \frac{1}{2})$, $g^*(x) = 1$ on $[\frac{1}{2}, 1]$). Then the function $h = g^* \circ g$ is as desired. □

As a notational note, if h is as in the preceding lemma we let $V(h, K) = h^{\leftarrow}(\frac{1}{2}, 1]$.

Lemma 3.6. *Let X be locally compact X and let $\mathcal{K}$ be a canonically moving off family. Then for each $K \in \mathcal{K}$ there is a family $\mathcal{V}_K$ which is a base of neighborhoods of K with compact closure, and $\cup\{\mathcal{V}_K : K \in \mathcal{K}\}$ is a canonical moving off family.*

Proof. Since X is locally compact then we can restrict our attention in the preceding lemma to the G's that have compact closure. Then $V(h, K) \subset G$ is a neighborhood of K with compact closure. This gives K a base $\mathcal{V}_K$ of neighborhoods with compact closure, namely the set of all such $V(h, K) = h^{-1}(\frac{1}{2}, 1]$. If C is any compact set, then by hypothesis there is a set $K \in \mathcal{K}$ such that $C \cap K = \emptyset$. Hence we may choose a $V(h, K) \subset C^c$ and it follows that $V(h, K) \cap C = \emptyset$. □

Lemma 3.7. *Suppose every canonical moving off collection has an infinite discrete moving off subcollection. If X is locally compact then X has the moving off property.*

Proof. Replace $\mathcal{K}$ with $\mathcal{V}(\mathcal{K}) = \bigcup\{\mathcal{V}_K : K \in \mathcal{K}\}$ where the sets $\mathcal{V}_K$ are as in Lemma 3.6. Now the hypothesis gives us an infinite discrete moving off subcollection of the sets $\mathcal{V}(\mathcal{K})$. This gives a discrete open expansion of the K's involved. □

Lemma 3.7 was the restatement of Lemma 2.20 from Chapter 2. We will now prove Lemma 2.12, followed by the forward direction of Theorem 2.13.

Lemma 3.8. *If $C_K(X)$ is countably tight, then every canonical moving off family has a countable moving off subcollection. [Nyikos 2003]*

Proof. Let $F_K = \{f \in C(X) : f^{\rightarrow}K = \{1\}\}$, and let $H = \bigcup\{F_K : K \in \mathcal{K}\}$ where $\mathcal{K}$ is moving off and canonical. Then $\overrightarrow{0} \in \overline{H}$ because $\mathcal{K}$ is moving off. Indeed, given a basic neighborhood $V(C, U)$ of $\overrightarrow{0}$ we have $0 \in U$. Assume without loss of generality that $U \subset (-\frac{1}{2}, \frac{1}{2})$ and take $K \in \mathcal{K}$ such that $K \cap C = \emptyset$. Let $f : X \to [0, 1]$ take K to 1 and C to 0. Then $f \in V(C, U)$ and $f \in H$ as desired.

Since $C_K(X)$ is countably tight, there exist $\{f_n : n \in \mathbb{N}\} \subset H$ and sets $\{K_n : n \in \mathbb{N}\} \subset \mathcal{K}$ such that $\overrightarrow{0} \in \overline{\{f_n : n \in \mathbb{N}\}}$ and $f_n \in F_{K_n}$ for each $n \in \mathbb{N}$. To show $\{K_n : n \in \mathbb{N}\}$ is moving off, suppose $C \cap K_n \neq \emptyset$ for some compact C and all K_n. Then $1 \in f_n(C)$ for each n, and this implies that $f_n \notin V(C, U)$ whenever $U \subset (-\frac{1}{2}, \frac{1}{2})$, which is a contradiction. □

Theorem 3.9. *If $C_0(X)$ is countably tight then every moving off collection in X has a countable moving off subcollection. [Nyikos 2003]*

Proof. In the proof of the preceding lemma all that needs to be changed is to have $f \in C_0(X)$ in the definition of F_K. □

We now turn our attention to a curious and fascinating theorem which ties together many seemingly disassociated properties.

Definition 3.10. A space X has *property* ε if any open ω-cover of X contains a countable ω-subcover.

Definition 3.11. A space X is *Lindelöf* if every open cover of X has a countable subcover.

Note that in the following theorem the equivalence of properties *1* and *4* is the restatement of Theorem 2.6.

Theorem 3.12. *The following are equivalent:*

1. *Every collection of closed sets in X that moves off the finite sets has a countable subcollection that moves off the finite sets.*

2. *X has property ε.*

3. *X^n is Lindelöf for all $n \in \omega$.*

4. *$C_p(X)$ has countable tightness.*

Proof. *3* $\implies$ *4* was shown by Arhangel'skiĭ. *[Arhangel'skiĭ 1978]*

4 $\implies$ *3* was shown by Pytkeev. *[Pytkeev 1982]*

1 $\implies$ *2* Suppose every open ω-cover contains a countable subcover. Let $\mathcal{C}$ be a collection of closed sets that moves off the finite sets. Then $\mathcal{C}^* = \{C^c : C \in \mathcal{C}\}$ is an open ω-cover, hence it contains $\mathcal{A}^* \subset \mathcal{C}^*$, a countable ω-subcover. Then $\mathcal{A} = \{A^c : A \in \mathcal{A}^*\}$ is a countable subcollection of $\mathcal{C}$ which also moves off the finite sets.

2 $\implies$ *1* Let $\mathcal{A}$ be an open ω-cover. Consequently, $\mathcal{C} = \{A^c : A \in \mathcal{A}\}$ is a collection of closed sets that moves off the finite sets of X. Therefore we can reduce to a countable subcollection $\mathcal{C}^* \subset \mathcal{C}$ that moves off the finite sets of X. Then $\mathcal{A}^* = \{C^c : C \in \mathcal{C}^*\}$ is a countable ω-subcover of $\mathcal{A}$.

The equivalence of *2* and *3* was shown by Gerlits and Nagy. Here we will display a considerably expanded version of their proof.

3 $\implies$ *2* Suppose X^n is Lindelöf for all n. Let $\mathcal{G}$ be an open ω-cover of X, and let $\mathcal{G}^n = \{G^n : G \in \mathcal{G}\}$.

Claim: $\mathcal{G}^n$ is an open cover of X^n.

Proof of Claim: Let $\langle x_1, ..., x_n\rangle \in X^n$. Since $\mathcal{G}$ is an ω-cover, $\{x_1, ..., x_n\} \subset G$ for some $G \in \mathcal{G}$, hence $\langle x_1, ..., x_n\rangle \in G^n$.

Now let $\mathcal{G}_n$ be a countable subset of $\mathcal{G}$ such that $\mathcal{G}_n^n = \{G^n : G \in \mathcal{G}_n\}$ covers X^n using the Lindelöf property. Then $\mathcal{G}_\omega = \cup\{\mathcal{G}_n : n \in \omega\}$ is a countable ω-subcover. Indeed, let $\{x_1, ..., x_n\} \in X$, so $\langle x_1, ..., x_n\rangle \in X^n$. Let $G \in \mathcal{G}_n$ satisfy $\langle x_1, ..., x_n\rangle \in G^n$. Then $\{x_1, ..., x_n\} \subset G$. Therefore X has property ε.

2 $\Longrightarrow$ *3* Let $\mathcal{U}$ be a basic open cover of X^n. Let $\mathcal{G} = \{G \subset X : G$ is open, G^n can be covered with finitely many sets in $\mathcal{U}\}$.

Claim: $\mathcal{G}$ is an open ω-cover of X. It is clear that the cover is open, however checking that it is an ω-cover requires work.

Assuming this claim, we can use the ε property to get a countable ω-subcover, we will call this $\mathcal{G}_0$. Let $\mathcal{G}_0^n = \{G^n : G \in \mathcal{G}_0\}$. This is a countable collection and its union can be covered by a countable family of sets in $\mathcal{U}$. It is also a cover of X^n: if $\langle x_1, ..., x_n\rangle \in X^n$ let $G \in \mathcal{G}_0$ satisfy $\{x_1, ..., x_n\} \subset G$. Then $\langle x_1, ..., x_n\rangle \in G^n \in \mathcal{G}_0^n$. Therefore we have that X^n is Lindelöf.

Proof of Claim: Let $A = \{x_1, ..., x_n\}$ be a finite subset of X. So we have that $A^n = \{(y_1, ..., y_n) : y_i \in A$ for all $i\}$. For each $\overline{y} = (y_1, ..., y_n) \in A^n$ let $H(\overline{y}) \in \mathcal{U}$ be a basic open neighborhood of $\overline{y}$ where $H(\overline{y}) = G_1(\overline{y}) \times ... \times G_n(\overline{y})$ and $G_i(\overline{y})$ is an open neighborhood of y_i. Now for each $a \in A$ and each $\overline{y}$ in which a appears as any coordinate let $G(a, \overline{y}) = \cap\{G_i(\overline{y}) : y_i = a\}$. If a is not one of the coordinates of $\overline{y}$, let $G(a, \overline{y}) = X$. Let $G(a) = \cap\{G(a, \overline{y}) : \overline{y} \in A^n\}$. Each $G(a)$ is open because it is the intersection of finitely many open sets. Let $G(A) = \cup\{G(a) : a \in A\}$, then $A \subset G(A) \in \mathcal{G}$. In other words $[G(A)]^n$ is covered by finitely many $G_1 \times ... \times G_n \in \mathcal{U}$, in other words by finitely many $H(\overline{y}) \in \mathcal{U}$ for some $\overline{y}$. Therefore we have that $\mathcal{G}$ is an ω-cover. □

Theorem 2.27 will now be restated and proven, this is the final proof we will show of a statement from Chapter 2.

Theorem 3.13. *If $C_K(X)$ is Baire, then X has the moving off property. [Gruenhage* *and Ma 1997]*

Proof. Let $\mathcal{K}$ be a moving off collection of compact subsets of X. For each $n \in \mathbb{N}$, let $U_n = \{f \in C_K(X) :$ there exists $K \in \mathcal{K}$ with $f(K) > n\}$. Then U_n is open in $C_K(X)$, and using the fact that $\mathcal{K}$ moves off the compact sets, we claim that U_n is dense.

To see this claim, let C be any compact subset of X and I any open interval in $\mathbb{R}$. We will show $U_n \cap V(C, I) \neq \emptyset$.

Since $\mathcal{K}$ is moving off, there exists $K \in \mathcal{K}$ such that $C \cap K = \emptyset$. Choose a number $a \in I$ and a number $b \in \mathbb{R}$ such that $a \neq b$ and $b > n$. Then since C and K are compact we have, by routine modification of Lemma 3.5, that there exists $f \in C(X)$ such that $f(C) = a$ and $f(K) = b$. Hence, $f \in U_n \cap V(C, I)$.

So there must be some $g \in \bigcap_{n \in \omega} U_n$. For each $n \in \omega$ there is $K_n \in \mathcal{K}$ such that $g(K_n) > n$. Since each $g(K_n)$ is compact, by passing to a subsequence if necessary we may assume $\{g(K_n) : n \in \omega\}$ is a discrete collection on $\mathbb{R}$. So in $\mathbb{R}$ there is a discrete open expansion $\{O_n : n \in \omega\}$ of $\{g(K_n) : n \in \omega\}$, and so $\{g^{-1}(O_n) : n \in \omega\}$ is a discrete open expansion of $\{K_n : n \in \omega\}$ in X. □

The following is an application of Theorem 3.13 whose proof has ideas in common with that of Theorem 3.13 itself.

Theorem 3.14. *Let X have the moving off property. Then given a sequence $\mathcal{K}_0, \mathcal{K}_1, ...$ of moving off collections in X there are $K_i \in \mathcal{K}_i$ such that $\{K_i : i \in \omega\}$ has a discrete open expansion.*

Proof. Suppose X has the moving off property. If X is compact then there are no moving off collections so the theorem holds vacuously. Suppose X is not compact. Then by the Lemma 3.2 we have that X is additionally not feebly compact. Hence there is an infinite closed discrete subset of distinct points $\{x_n : n \in \omega\}$ of X.

Let $\mathcal{K}_0, \mathcal{K}_1, ...$ be a sequence of moving off collections in X and let

$$\mathcal{L}_n = \{\{x_n\} \cup K_0 \cup K_1 \cup ... \cup K_n : K_i \in \mathcal{K}_i\}.$$

Now let $\mathcal{L} = \bigcup_{n\in\omega} \mathcal{L}_n$. We wish to claim that $\mathcal{L}$ is a moving off collection.

Let C be a compact subset of X. Since C is countably compact, $\{x_i : i \in \omega\} \cap C$ must be finite. Thus there is some x_n not in C. For each i, $0 \leq i \leq n$, choose some $K_i \in \mathcal{K}_i$ such that $K_i \cap C = \emptyset$. Then $(\{x_n\} \cup K_0 \cup K_1 \cup ... \cup K_n) \cap C = \emptyset$, which shows that $\mathcal{L}$ is a moving off collection.

Therefore there is a strictly increasing sequence $\{m_n : n \in \omega\}$ of natural numbers and sets $\{L_n \in \mathcal{L}_{m_n} : n \in \omega\}$ such that each $m_n \geq n$ and $\{L_n : n \in \omega\}$ has a discrete open expansion. Hence $m_n \geq n$ for each n implies each L_n contains a set $K_n \in \mathcal{K}_n$ so $\{K_n : n \in \omega\}$ has a discrete open expansion. $\square$

Definition 3.15. A space X is *strictly Fréchet-Urysohn* at $x \in X$ if for any sequence $(A_n)_{n\in\omega}$ such that $x \in \bigcap_{n\in\omega} \overline{A_n}$ there exists a sequence $(x_n)_{n\in\omega} \in \prod_{n\in\omega} A_n$ such that $x_n \to x$. A space with this property at all its points is strictly Fréchet-Urysohn.

Definition 3.16. If $\langle A_n : n \in \omega\rangle$ is a sequence of subsets of a set X, then we define $\underline{Lim} A_n = \{x \in X : \text{there exists } n_0 \in \omega \text{ such that for all } n \geq n_0,\ x \in A_n\}$

Definition 3.17. For a space X, the following properties will be given the following labels:

- *(a)*: $C_p(X)$ strictly Fréchet-Urysohn.
- *(b)*: $C_p(X)$ is Fréchet-Urysohn.
- *(c)*: $C_p(X)$ is sequential.
- (γ): if $\mathcal{G}$ is an open ω-cover of X, then there is a sequence $G_n \in \mathcal{G}$ with $\underline{Lim}\, G_n = X$.
- (δ): if $\mathcal{G}$ is an open ω-cover of X, then $X \in L(\mathcal{G})$ where $L(\mathcal{G})$ is the smallest family of subsets of X containing $\mathcal{G}$ and closed under $\underline{\text{Lim}}$.
- (ε): any open ω-cover of X contains a countable ω-subcover.

It is of note that property γ in this definition is equivalent to the γ-property presented earlier in this paper, a fact we will now prove.

Lemma 3.18. *On a space X, for any open ω-cover $\mathcal{G}$ of X there is a sequence $G_n \in \mathcal{G}$ with $\underline{Lim}\, G_n = X$ if and only if every collection of closed subsets of X which moves off the finite sets has an infinite point-finite subcollection.*

Proof. $\Longrightarrow$ Suppose on a space X, for any open ω-cover $\mathcal{G}$ of X there is a sequence $G_n \in \mathcal{G}$ with $\underline{Lim}\, G_n = X$. Let $\mathcal{A}$ be a collection of closed subsets of X which move off the finite sets. Then $\mathcal{B} = \{A^c : A \in \mathcal{A}\}$ is an open ω-cover of X. Therefore there exists a sequence $\langle B_n : n \in \mathbb{N}\rangle$ in $\mathcal{B}$ such that $\underline{Lim} B_n = X$. Let $\mathcal{B}' = \{B_n : n \in \mathbb{N}\}$, and let $\mathcal{A}' = \{B_n^c : n \in \mathbb{N}\}$. Then for any point p in X we have that there exists n_0 such that $p \in B_n$ for all $n \geq n_0$. So there are only finitely many elements in $\mathcal{A}'$ that contain p, therefore $\mathcal{A}'$ is point-finite. Hence we have that $\mathcal{A}'$ is an infinite point-finite subcollection of $\mathcal{A}$.

$\Longleftarrow$ Suppose for a space X every collection of closed subsets of X which moves off the finite sets has an infinite point-finite subcollection. Let $\mathcal{A}$ be an open ω-cover of X. So every finite subset of X is contained in some member of $\mathcal{A}$. Therefore $\mathcal{B} = \{A^c : A \in \mathcal{A}\}$ is a collection of closed sets which moves off the finite sets of X. So $\mathcal{B}$ has an infinite point-finite subcollection, call this $\mathcal{B}'$. Let $\mathcal{A}' = \{B^c : B \in \mathcal{B}'\} = \{A'_n : n \in \mathcal{N}\}$. Let p be an arbitrary point of X. Then since $\mathcal{B}'$ is point-finite, p is in only finitely many elements of $\mathcal{B}'$, hence p is in all but finitely many elements of $\mathcal{A}'$. So there exists n_0 such that $p \in A'_n$ for all $n \geq n_0$. So $p \in \underline{Lim} A'_n$ and therefore, since p was an arbitrary point, we have that $\underline{Lim}\mathcal{A}' = X$.

□

Theorem 3.19. *If X is Tychonoff the following holds:*

$$(a) \Longleftrightarrow (b) \Longleftrightarrow (\gamma) \Longrightarrow (c) \Longrightarrow (\delta) \Longrightarrow (\varepsilon)$$

[Gerlits and Nagy 1982]

Part of this theorem comes from the well known fact that $(a) \implies (b) \implies (c)$ for all spaces. Additionally, $(b) \implies (a)$ for any topological group *[Nyikos 1981]*. Hence the two are equivalent for any of the function spaces discussed in this paper.

Taking into account Lemma 3.18 we have that Theorem 3.19 encapsulates within it Theorem 2.7 from the previous chapter.

Chapter 4

Moving off and The Cantor Tree

Here we will discuss the Cantor tree, some of its properties, and how it relates to moving off and the moving off property. There are multiple ways to define the Cantor tree. The set theoretic way is the full binary tree of high $\omega + 1$ with the tree topology. The Cantor tree is separable, it has a dense set of isolated points, and a closed discrete subspace of cardinality 2^ω formed by its set of nonisolated points. It is also locally compact and locally countable, and therefore first countable as well. To be more explicit, the Cantor tree has the following construction: *[Nyikos 1989]*

The full binary tree of height α, where α is an ordinal number, is the set of all sequences of 0's and 1's whose domain is an ordinal number less than α with the following extension. Given two sequences σ and τ we have $\sigma \leq \tau$ if the domain of σ is a subset of the domain of τ and if τ agrees with σ on its domain. Therefore the full binary tree of height ω is the set of all finite sequences of 0's and 1's whose domain is a finite ordinal with the extension order. Call this set T. Let C be the set of all sequences of 0's and 1's whose domain is ω. The Cantor tree is formed by taking $T \cup C$. *[Nyikos 1989]*

The interval topology on the tree is the one whose base sets are of the form $(s, t] = \{t' : s < t' \leq t\}$ and $\{t\}$ where t is a minimal number. So in the Cantor tree all points of T are isolated and a neighborhood of a point x of the set C is a set containing x and a cofinite subset of the branch of T where it lies. *[Nyikos 1989]*

The main result in this chapter is a series of equivalencies that connect the Cantor tree to a host of concepts that have appeared in this paper.

Consider the subsets of the Cantor tree, written as $T \cup X$ where $X \subseteq C$. Note that $T \cup X$ is a locally compact space.

Recall the following from Theorem 3.17, translating Theorem 2.7.

Theorem 4.1. *For an arbitrary space X, $C_p(X)$ is a Fréchet-Urysohn space if and only if X has the γ-property. [Gerlits and Nagy 1982]*

Theorem 4.2. *Let X be a subset of the Cantor set. Then the following are equivalent.*

1. *$C_K(T \cup X)$ is a Baire space.*
2. *The space $T \cup X$ has the MOP.*
3. *X is a γ-set in the relative Euclidean topology on $\mathbb{R}$.*
4. *$C_K(T \cup X)$ is of second category.*

[Ma 1993]

It is noteworthy that the Cantor set itself is not a γ-set, nor is $\mathbb{R}$. In fact, the existence of an uncountable γ-set is independent of the usual axioms of set theory (ZFC). In models of ZFC where every γ-set is countable, the space $T \cup X$ is metrizable by Urysohn's metrization theorem for all γ-sets X. In general, as demonstrated in Theorem 3.4.16 of Ryszard Engelking's *General Topology*, if $T \cup X$ is countable then it is metrizable and so is $C_K(T \cup X)$ *[Engelking 1975]*. On the other hand, if X is uncountable, then $T \cup X$ is not metrizable, because it is separable but not second countable.

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