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Finding Resolutions of Monomial Ideals

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FINDING RESOLUTIONS OF MONONOMIAL IDEALS

by

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ABSTRACT

In this paper we present two different combinatorial approaches to finding resolutions of polynomial ideals. Their goal is to get resolutions that are as small as possible while still preserving the structure of the zeroth syzygy module. Then we present the idea of a differential graded algebra and discuss when the minimal resolutions of a polynomial ideals admits such a structure.

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CHAPTER 1

INTRODUCTION

Let $S = k[x_1, \dots, x_n]$ be a polynomial ring over a field k and $I \subset S$ be a monomial ideal. In many situations we are interested in studying the resolutions of S/I over S . For many purposes it is often useful to be able to find a resolution which is a minimal free resolution. (For a definition of minimal free resolution one may look to [5], pages 472-473.) One way we might go about finding minimal free resolutions is through studying the combinatorial interpretations of the ideal.

As [7] and [9] show given the case of monomial ideals in three variables the combinatorial object we get is a planar graph. Ideally when trying to describe these ideals combinatorially, we want to preserve their algebraic structure. In particular, we are interested in preserving the structure of their syzygy module. This allows us to translate the combinatorial properties into structural statements about the syzygy module. In the case where the syzygy module structure is preserved we are able to obtain a minimal free resolution of the ideal using this structure. In chapter 2 we discuss how to obtain the planar graph, known as the Buchberger Graph, and under what conditions the minimal free resolution is achieved.

For monomial ideals with more than three variables we often look at cellular resolutions. One cellular resolution discussed in [8] and [9] is the Taylor Resolution. However, this resolution is often not a minimal resolution. In trying to find a cellular resolution that is minimal one may look at the Scarf complex of a given ideal and how this relates to the hull resolution. Given a particular relation between these one can obtain a minimal free resolution. In chapter 3 we will discuss what the Scarf

complex is and when this leads to a minimal free resolution.

In chapters 2 and 3 we present ways of finding minimal free resolution, however, some times these resolutions don't admit all properties needed to be useful. In chapter 4 we discuss specific type of resolution called a differential graded algebra. This type of resolution admits a useful multiplicative structure that not all resolutions have. In this section we discuss when we get a differential graded algebra resolution, when these resolutions are minimal, and what are the consequences of having a minimal differential graded algebra resolution.

CHAPTER 2

PRELIMINARIES AND DEFINITIONS

Key definitions from homological algebra and graph theory will be presented. Much of this material is taken from [9].

2.1 GRAPH THEORY

First we will start with some preliminary graph theory definitions.

Definition 2.1. A **planar graph** is an abstract graph that can be drawn in such a way that no two edges meet in a point other than a common vertex.

We say that a graph is **3-connected** if it has at least three vertices and if deleting any pair of vertices along with all edges incident to them yields a connected graph.

Definition 2.2. Given a set \mathcal{V} of vertices in G , define the **suspension** of G over \mathcal{V} by adding a new vertex to G and connecting it by edges to all vertices in \mathcal{V} .

The graph G is **almost 3-connected** if it comes with a set \mathcal{V} of three distinguished vertices such that the suspension of G over \mathcal{V} is 3-connected.

2.2 HOMOLOGICAL ALGEBRA

Next we give some definitions related to monomial ideals which will be used for many of the theorems in Chapter 3.

Definition 2.3. We say that the **staircase surface** of a monomial ideal I in $k[x, y, z]$ is the topological boundary of the set of vertices $(v_x, v_y, v_z) \in \mathbb{R}^3$ for which there is some monomial $x^{u_x}y^{u_y}z^{u_z} \in I$ satisfying $u_i \leq v_i$ for all $i \in \{x, y, z\}$.

Definition 2.4. A monomial ideal over a polynomial ring in three variables is said to be an **artinian ideal** if its minimal generators include pure powers in each of the three variables.

Next we define the notions of complex and free resolution. Then we give the definitions of two common resolutions. These definitions relate to the information in Chapters 4 and 5.

Definition 2.5. A sequence $\mathcal{F}_\bullet : 0 \leftarrow F_0 \xleftarrow{\phi_1} F_1 \leftarrow \cdots \leftarrow F_{\ell-1} \xleftarrow{\phi_\ell} F_\ell \leftarrow 0$ of maps of free S -modules is a **complex** if $\phi_i \circ \phi_{i+1} = 0$ for all i . The complex is **exact** in homological degree i if $\ker(\phi_i) = \text{im}(\phi_{i+1})$.

Definition 2.6. A complex \mathcal{F}_\bullet is a **free resolution** of a module M over $S = k[x_1, \dots, x_n]$ if \mathcal{F}_\bullet is exact everywhere except in homological degree 0, where $M = F_0/\text{im}(\phi_1)$. The image in F_i of the homomorphism ϕ_{i+1} is the i^{th} **syzygy module** of M .

Definition 2.7. An (abstract) **simplicial complex** Δ on a vertex set $\{1, \dots, n\}$ is a collection of sunsets called simplices or faces that are closed under taking subsets, that is: if $\sigma \in \Delta$ is a face and $\tau \subseteq \sigma$, then $\tau \in \Delta$.

Definition 2.8. We take X to be the full $(r-1)$ -dimensional simplex whose r vertices are labeled by given monomials $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r}$. For any vector $\mathbf{b} \in \mathbb{N}^n$, the subcomplex $X_{\leq \mathbf{b}}$ is a face of X ; namely, it is the full simplex on all monomials $\mathbf{x}^{\mathbf{a}_i}$ dividing $\mathbf{x}^{\mathbf{b}}$. Then we have that \mathcal{F}_X is the **Taylor resolution** of S/I , where $I = \langle \mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_r} \rangle$ is the ideal generated by all vertex labels of X .

We see that the Betti numbers of S/I in the Taylor resolution are given by the homology of the simplicial complexes $\mathbf{X}_{\leq \mathbf{b}}$. Therefore, since the faces of X are labeled by least common multiples of the generators of I , the Betti numbers can occur only in such degrees.

Next we define the hull resolution but first we must give some background information which we take from [4].

Let M be a monomial module in $T = k[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. For $\mathbf{a} \in \mathbb{Z}^n$ and $t \in \mathbb{R}$ we abbreviate $t^{\mathbf{a}} = (t^{a_1}, \dots, t^{a_n})$. Fix any real number t larger than $(n + 1)!$. We define $\mathcal{P}_t = \text{conv}\{\mathbf{t}^{\mathbf{a}} | \mathbf{x}^{\mathbf{a}} \in I\}$; that is \mathcal{P}_t is the convex hull of the point set $\{\mathbf{t}^{\mathbf{a}} | \mathbf{a} \text{ is the exponent of a monomial } \mathbf{x}^{\mathbf{a}} \in M\} \subseteq \mathbb{R}^n$. The set \mathcal{P}_t is an unbounded n -dimensional convex polyhedron.

From this we get the following definition.

Definition 2.9. The **hull complex** $\text{hull}(I)$ of a monomial ideal I is the polyhedral cell complex of all bounded faces of \mathcal{P}_t for $t \gg 0$. This complex is naturally labeled, with each vertex corresponding to a minimal generator of I . The cellular free complex $\mathcal{F}_{\text{hull}(I)}$ is called the **hull resolution** of I .

We are now ready to define the hull resolution and state our main result. The hull complex of a monomial module M , denoted $\text{hull}(M)$, is the complex of bounded faces of the polyhedron \mathcal{P}_t for large t .

CHAPTER 3

BUCHBERGER GRAPHS AND RESOLUTIONS

In this chapter we primarily discuss the combinatorial structure of monomial ideals over $k[x, y, z]$. This structure turns out to be a planar graph we call the Buchberger Graph. We will discuss how to obtain the Buchberger graph for a given ideal and then discuss some properties that can be found from this graph.

First we discuss some notation that is used throughout this chapter. We take $S = k[x_1, \dots, x_n]$ and define $\langle m_1, m_2, \dots, m_r \rangle$ to be the minimal generating set for an ideal I .

3.1 BASICS OF BUCHBERGER GRAPHS

In order to understand how to build the Buchberger graph and how it preserves the structure of the syzygy module we must first understand Buchberger's Criterion; theorem as given in [9].

Theorem 3.1 (Buchberger's Criterion). *Let $\{f_i\}_{i=1}^r$ be a set of polynomials and $m_i = \text{lm}(f_i)$, the leading monomial of f_i . The set $\{f_i\}_{i=1}^r$ is a Gröbner basis under the term order $<$ if each s -pair*

$$s(f_i, f_j) := \frac{\text{lcm}(m_i, m_j)}{m_i} f_i - \frac{\text{lcm}(m_i, m_j)}{m_j} f_j$$

can be reduced to zero by $\{f_1, \dots, f_r\}$ using the division algorithm.

Each s -pair $s(f_i, f_j)$ yields an element σ_{ij} of the free module S^r , namely

$$\sigma_{ij} = \frac{\text{lcm}(m_i, m_j)}{m_i} \mathbf{e}_i - \frac{\text{lcm}(m_i, m_j)}{m_j} \mathbf{e}_j.$$

The elements σ_{ij} generate the module of the first syzygies, $\text{syz}(I) = \ker_s[m_1 \ m_2 \ \cdots \ m_r]$ of the monomial ideal $I = \langle m_1, m_2, \dots, m_r \rangle$, however, they do not always do so minimally.

We can use the structure of the syzygy module $\text{syz}(I)$ to strengthen Buchberger's Criterion.

Theorem 3.2 (Buchberger's Second Criterion). *If G is any subset of the pairs (i, j) with $1 \leq i < j \leq r$ such that the set $\{\sigma_{ij} \mid (i, j) \in G\}$ generates $\text{syz}(I)$, then it suffices that only the s -pairs $s(f_i, f_j)$ with $(i, j) \in G$ reduce to zero in order to imply the Gröbner basis property for $\{f_1, f_2, \dots, f_r\}$.*

This leads to the following definition ([9], Def. 3.4).

Definition 3.3. The **Buchberger graph**, $\text{Buch}(I)$, of a monomial ideal

$I = \langle m_1, m_2, \dots, m_r \rangle$ has vertices $1, \dots, r$ and an edge (i, j) whenever there is no monomial m_k such that m_k divides $\text{lcm}(m_i, m_j)$ and the degree of m_k is different from $\text{lcm}(m_i, m_j)$ in every variable that occurs in $\text{lcm}(m_i, m_j)$.

First, we discuss a result about Buchberger graphs which deals with ideals over $k[x, y]$ and can be found in [6].

Proposition 3.4. *Let I be a monomial ideal over $k[x, y]$ with r generators. The Buchberger graph of I , $\text{Buch}(I)$, has r vertices and $r - 1$ consecutive edges.*

Proof. First we must note that for $I = \langle m_1, m_2, \dots, m_r \rangle$ a monomial ideal over $k[x, y]$ then for generators $m_i = x^{a_i}y^{b_i}$ and $m_j = x^{a_j}y^{b_j}$ if $a_i > a_j$ then $b_i < b_j$. We write $I = \langle m_1, m_2, \dots, m_r \rangle$ where $a_1 > a_2 > \cdots > a_r$ and $b_1 < b_2 < \cdots < b_r$.

First, consider m_i and m_k where $k > i + 1$. We see that $\text{lcm}(m_i, m_k) = x^{a_i}y^{b_k}$. But then $m_{i+1} = x^{a_{i+1}}y^{b_{i+1}} \mid \text{lcm}(m_i, m_k)$ since $a_{i+1} < a_i$ and $b_{i+1} < b_k$.

Next, consider m_i and m_j where $j < i - 1$. We see that $\text{lcm}(m_i, m_j) = x^{a_j}y^{b_i}$. But then $m_{i-1} = x^{a_{i-1}}y^{b_{i-1}} \mid \text{lcm}(m_i, m_j)$ since $a_{i-1} < a_j$ and $b_{i-1} < b_i$.

So the only edges we get with m_i as a vertex are between (m_{i-1}, m_i) and (m_i, m_{i+1}) . thus we have that the Buchberger graph must look like in Figure 3.1.

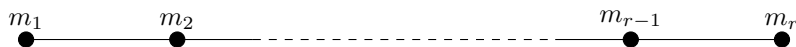


Figure 3.1: Buchberger graph of an ideal over $k[x,y]$.

□

From here on we give results about the Buchberger graphs of ideals over any number of variables. However, all examples will deal with ideals over $k[x, y, z]$ as these give graphs that are easy to draw and work with.

Next, we discuss how the Buchberger graph relates to and preserves the structure of the syzygy module, $\text{syz}(I)$. We see in the following proposition how the generators of the syzygy module correspond to the edges of the Buchberger graph. ([9] pg. 48)

Proposition 3.5. *The syzygy module $\text{syz}(I)$ is generated by syzygies σ_{ij} corresponding to edges (i, j) in the Buchberger graph $\text{Buch}(I)$.*

Proof. In order to prove this we must first note an identity about the generator of the syzygy module. The following holds for all $i, j, k \in 1, \dots, r$:

$$\frac{\text{lcm}(m_i, m_j, m_k)}{\text{lcm}(m_i, m_j)}\sigma_{ij} + \frac{\text{lcm}(m_i, m_j, m_k)}{\text{lcm}(m_j, m_k)}\sigma_{jk} + \frac{\text{lcm}(m_i, m_j, m_k)}{\text{lcm}(m_k, m_i)}\sigma_{ki} = 0$$

If (i, j) is not an edge of $\text{Buch}(I)$, then for some k , the coefficient of σ_{ij} is 1 while the coefficients of σ_{jk} and σ_{ki} are non-constant monomials. Hence σ_{ij} lies in the S -module generated by other first syzygies of strictly smaller degree. This means that we can remove σ_{ij} from the generators of $\text{syz}(I)$ without running into a cycle. □

Next we give an example of the Buchberger graph for a monomial ideal. When labeling or referencing a vertex of the Buchberger graph or staircase surface we use the notation $(\alpha\beta\gamma)$ to represent the vertex given by the monomial $x^\alpha y^\beta z^\gamma$.

Example 3.6. Given the ideal $J = \langle x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3 \rangle \in k[x, y, z]$ we find that the Buchberger graph and staircase surface of J are given as in Figure 3.3 and Figure 3.2 respectively.

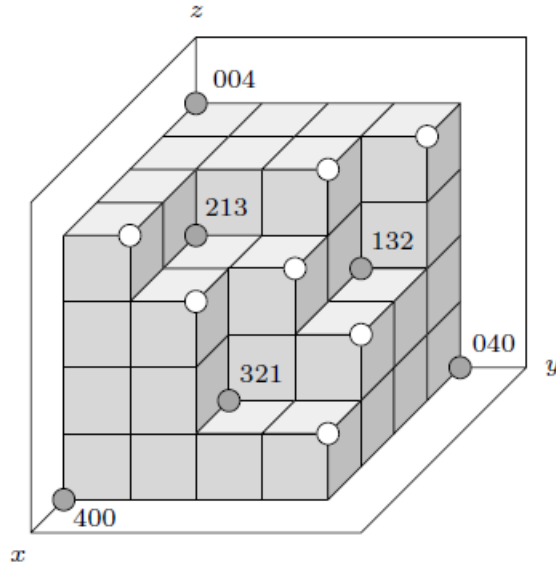


Figure 3.2: The staircase diagram for J .

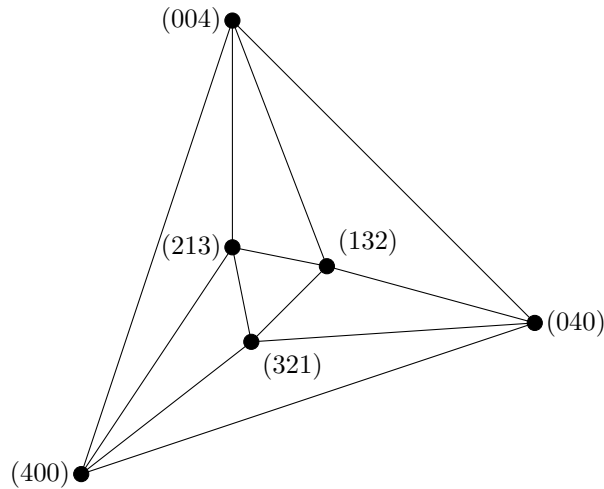


Figure 3.3: The Buchberger graph for J .

We see in Example 3.6 that the Buchberger graph for J can be nicely embedded in the staircase surface. We often use this embedding to help us prove certain theorems

about the Buchberger graph. Actually we can precisely describe how the Buchberger graph of an ideal is embedded into its staircase surface. We do this in a way so that the exponent vector of x^a corresponds to the point $(a, 0, 0)$, and so that an edge on $\text{Buch}(I)$ is the union of two line segments: one connecting the exponent vectors of m and $\text{lcm}(m, m')$ and the other connecting m' and $\text{lcm}(m, m')$.

Now we can discuss how the Buchberger graph of a monomial ideal encodes a free resolution. The free resolution given by the graph G with v vertices, e edges, and f faces, all labeled by monomials, has the form:

$$0 \leftarrow S \longleftarrow S^v \xleftarrow{\sigma_E} S^e \xleftarrow{\sigma_F} S^f \leftarrow 0 \quad (3.1)$$

Writing $m_{ij} = \text{lcm}(m_i, m_j)$ for each edge $\{i, j\}$ of G , and m_R for the least common multiple of the monomial labels on the edges in each region R . Then

$$\sigma_E(\mathbf{e}_{ij}) = \frac{m_{ij}}{m_j} \cdot \mathbf{e}_j - \frac{m_{ij}}{m_i} \cdot \mathbf{e}_i$$

if an edge oriented toward m_j joins the vertices labeled m_i and m_j , whereas

$$\sigma_F(\mathbf{e}_R) = \sum_{\substack{\text{edges} \\ \{i,j\} \subset R}} \pm \frac{m_R}{m_{ij}} \cdot \mathbf{e}_{ij}$$

for each region R , where the sign is positive precisely when the edge $\{i, j\}$ is oriented counterclockwise around R .

From the graph of $J = \langle x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3 \rangle$ in Example 3.6 we find that one resolution is given by:

$$0 \leftarrow S \longleftarrow S^6 \longleftarrow S^{12} \longleftarrow S^7 \leftarrow 0$$

This resolution is actually the minimal resolution for J .

We will see from the following example that the Buchberger graph does not always result in a minimal resolution.

Example 3.7. Given the ideal $I = \langle x^2z, xyz, y^2z, x^3y^5, x^4y^4, x^5y^3 \rangle$ we find the staircase diagram and Buchberger graph of I is given as in Figure 3.5 and Figure 3.4 respectively.

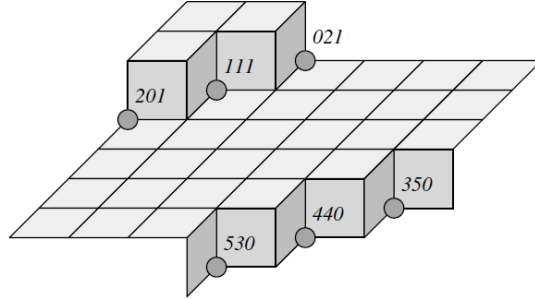


Figure 3.4: The staircase diagram for I .

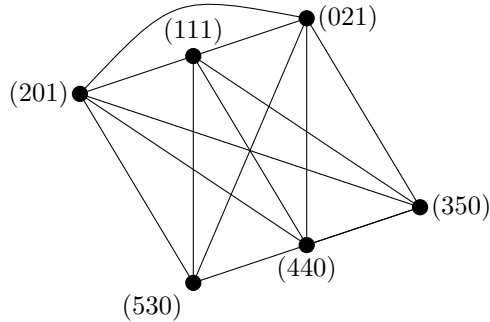


Figure 3.5: The Buchberger graph for I .

We see that we don't get a line connecting (350) to (530) as $\text{lcm}(x^3y^5, x^5y^3) = x^5y^5$ and x^4y^4 divides this and the powers on both variables differ.

Now we see that the resolution of I given by the graph in Figure 3.5 is:

$$0 \leftarrow S \leftarrow S^6 \leftarrow S^{14} \leftarrow S^{16} \leftarrow 0$$

However, we know this is not the minimal free resolution of I . In the following chapter we will see why we are guaranteed to get the minimal resolution for J but not for I .

3.2 GENERICITY

In this chapter we will give some parameters on our ideal that will guarantee we can get a minimal resolution from the Buchberger Graph of this ideal. First, we must define some algebraic notions and then we can give the theorem which states when we can get a minimal resolution.

Definition 3.8. A monomial ideal I in $k[x, y, z]$ is **strongly generic** if every pair of minimal generators $x^i y^j z^k$ and $x^{i_0} y^{j_0} z^{k_0}$ of I satisfies:

- 1) $i \neq i_0$ or $i = i_0 = 0$ and
- 2) $j \neq j_0$ or $j = j_0 = 0$ and
- 3) $k \neq k_0$ or $k = k_0 = 0$.

In other words, no two generators agree in the exponent on any variable that appears in both of them.

This definition lead us the following result about how certain algebraic properties of the polynomial ideal relate to graphical properties of it's Buchberger graph. The following proposition is from [9] page 50 and the proof is an expanded version of the proofs given there and in [7].

Proposition 3.9. *If I is a strongly generic monomial ideal in $k[x, y, z]$, then the Buchberger graph $\text{Buch}(I)$ is planar and connected. If, in addition, I is artinian, then $\text{Buch}(I)$ consist of the edges in a triangulated triangle.*

Proof. First, observe that it suffices to consider artinian monomial ideals I , meaning that the minimal generators of I include pure powers in each of the three variables, say x^a, y^b , and z^c . Indeed, erasing all edges and regions incident to one or more of $\{x^a, y^b, z^c\}$ yields the Buchberger graph for the ideal without the corresponding generator, and what results is connected because planar triangulations are 3-connected (Definition 2.1).

The idea now is that the bounded faces in the staircase surface of the monomial ideal I form a topological disk bounded by a piecewise linear triangle with vertices $(a, 0, 0)$, $(b, 0, 0)$, $(c, 0, 0)$ corresponding to x^a , y^b , and z^c , the pure power generators of I . Each edge $\{m, m'\}$ of $\text{Buch}(I)$ is drawn in the staircase surface as the union of the two line segments from m to $\text{lcm}(m, m')$ and from m' to $\text{lcm}(m, m')$. We want to show that $\text{lcm}(m, m')$ lies on the staircase.

Take $m, m' \in I$. Denote their exponent vectors as (x, y, z) , (x', y', z') . Since I is strongly generic, we have that $x, x' < a$, $y, y' < b$, and $z, z' < c$. Moreover the exponent vector of $\text{lcm}(m, m') = (\max(x, x'), \max(y, y'), \max(z, z'))$. Hence $\text{lcm}(m, m')$ lies on the staircase surface. Finally since I is generic, there can be no other edges passing through this point. We thus obtain an embedding of $\text{Buch}(I)$ in the staircase surface.

What remains to be shown is that $\text{Buch}(I)$ consists of the edges of a triangulated triangle i.e., each region is bounded by exactly three edges. This is proved by showing that each of the two regions containing any interior Buchberger edge $\{m, m'\}$ is a triangle. This triangle is produced by finding a uniquely determined third generator m'' such that the least common multiple of $\{m, m', m''\}$ lies in the staircase surface; the region is then bounded by the Buchberger edges $\{m, m'\}$, $\{m, m''\}$, and $\{m', m''\}$.

□

This brings us to our main theorem from [9] about when the resolution given by the Buchberger graph of a given ideal is guaranteed to be a minimal resolution. Theorem 3.10 says that planar maps encode minimal free resolutions since they organize into single diagrams the syzygies and their interrelations.

Theorem 3.10. *Given a strongly generic monomial ideal I in $k[x, y, z]$, the planar map $\text{Buch}(I)$ provides a minimal free resolution of I .*

Proof. (Sketch) Begin by throwing high powers x^a, y^b , and z^c into I . What results is still strongly generic, but now artinian. If we are given a minimal free resolution of this new ideal by a planar map, then deleting all edges and regions incident to one or more of $\{x^a, y^b, z^c\}$ leaves a minimal free resolution of I . Indeed, these deletions have no effect on the \mathbb{N}^3 -graded components of degree $\preceq (a-1, b-1, c-1)$, which remain exact, and I has no syzygies in any other degree. Therefore we assume that I is artinian.

Each triangle in $\text{Buch}(I)$ contains a unique "mountain peak" in the surface of the staircase, located at the outside corner $\text{lcm}(m, m', m'')$. That peak is surrounded by three "mountain passes" $\text{lcm}(m, m')$, $\text{lcm}(m, m'')$, and $\text{lcm}(m', m'')$, each of which represents a minimal first syzygy of I by Theorem 1.34 in [9] (check that the simplicial complex $K^b(I)$ from [9] Definition 1.33 is disconnected precisely when a mountain pass sits in degree \mathbf{b}). The mountain peak represents a second syzygy relating these three first syzygies by the identity in the proof of Proposition 3.5, and all minimal second syzygies arise this way by Theorem 1.34 in [9]. \square

We can see that the ideal $J = \langle x^4, y^4, z^4, x^3y^2z, xy^3z^2, x^2yz^3 \rangle$ given in Example 3.6 is strongly generic and so Theorem 3.10 tells us that the resolutions given by the graph $\text{Buch}(J)$ is minimal.

However, the ideal $I = \langle x^2z, xyz, y^2z, x^3y^5, x^4y^4, x^5y^3 \rangle$ is not strongly generic since the generators x^2z and xyz have the same exponent on z . This is why Theorem 3.10 does not guarantee that a minimal resolution is obtained from $\text{Buch}(J)$.

Ideally we would like to be able to get a minimal resolution for all monomial ideals. We will discuss in the next section a method for finding such minimal resolutions.

3.3 DEFORMATIONS

Next we discuss how we can take arbitrary monomial ideals and approximate them by strongly generic ones. The idea is we add small rational numbers to the exponents

of the generators of I without reversing any existing strict inequalities between the degrees in x , y , or z of any two generators. This occurs inside the polynomial ring $S_\epsilon = k[x_\epsilon, y_\epsilon, z_\epsilon]$, where $\epsilon = 1/N$ for some large positive integer N ; we see $S = k[x, y, z]$ is a subring of S_ϵ . This causes equalities among x -, y -, and z -degrees to turn into strict inequalities potentially going either way.

Definition 3.11. Let $I = \langle m_1, \dots, m_r \rangle$ and $I_\epsilon = \langle m_{\epsilon,1}, \dots, m_{\epsilon,r} \rangle$ be monomial ideals in S and S_ϵ , respectively. Call I_ϵ a **strong deformation** of I if the partial order on $1, \dots, r$ by x -degree of the $m_{\epsilon,i}$ refines the partial order by x -degree of the m_i , and the same holds for y and z . We also say that I is a **specialization** of I_ϵ .

Example 3.12. The ideal in S_ϵ given by

$$\langle x^3, x^{2+\epsilon}y^{1+\epsilon}, x^2z^1, x^{1+2\epsilon}y^2, x^{1+\epsilon}y^1z^{1+\epsilon}, x^1z^{2+\epsilon}, y^3, y^{2-\epsilon}z^{1+2\epsilon}, y^{1+2\epsilon}z^2, z^3 \rangle$$

is one possible strong deformation of the ideal $\langle x, y, z \rangle^3$ in S .

Our goal is to get a strong deformation I_ϵ that is now a generic monomial ideal. Then we would be able to use the results of Theorem 3.10 along with the specialization to obtain similar results for monomial ideals I which are not strongly generic.

One thing we may notice about the definition of strong deformation is that it does not specify a specific way to deform the ideal. This can lead to more than one strong deformation. For example, we see that both

$$I_\epsilon = \langle x^3y^{2+\epsilon}z, xy^2z^3, x^2y^3z^2 \rangle \text{ and } I_\epsilon = \langle x^3y^2z, xy^{2+\epsilon}z^3, x^2y^3z^2 \rangle$$

are specializations of the ideal $I = \langle x^3y^2z, xy^2z^3, x^2y^3z^2 \rangle$. We will later discuss how we fix this issue that we one get one strong deformation. This brings us to the following result ([9] page 52).

Proposition 3.13. *Suppose I is a monomial ideal in $k[x, y, z]$ and I_ϵ is a strong deformation resolved by a planar map G_ϵ . Specializing the vertices (hence also the edges and regions) of G_ϵ yields a planar map resolution of I .*

Proof. Consider the minimal free resolution \mathcal{F}_{G_ϵ} determined by the triangulation G_ϵ as in (3.1). The specialization G of the labeled planar map G_ϵ still gives a complex \mathcal{F}_G of free modules over $k[x, y, z]$, and we need to demonstrate its exactness. Considering any fixed \mathbb{N}^3 -degree $\omega = (a, b, c)$, we must demonstrate exactness of the complex of vector spaces over k in the degree ω part of \mathcal{F}_G . Define ω_ϵ as the exponent vector on

$$\text{lcm}(m_{\epsilon,i} \mid m_i \text{ divides } x^a y^b z^c).$$

The summands contributing to the degree ω part of \mathcal{F}_G are exactly those summands of \mathcal{F}_{G_ϵ} contributing to its degree ω_ϵ part, which is exact. \square

Soon we will be able to demonstrate a method to make any planar map resolution a minimal one by successively removing edges and joining adjacent regions. However, first we go over some properties of our graphs which are used in the next theorem.

For our graphs we take the vertex sets to be the sets of monomials that minimally generating some ideal I inside $k[x, y, z]$. Note that when I is artinian, such a vertex set contains a distinguished set \mathcal{V} consisting of the three vertices relating to the pure-power generators x^a, y^b, z^c . Now we come to the main result in this chapter.

Theorem 3.14. *Every monomial ideal I in $k[x, y, z]$ has a minimal free resolution by some planar map. If I is artinian then the graph G underlying any such planar map is almost 3-connected.*

Proof. See *Proof of Theorem 3.17* in [9] on page 56-57. \square

Corollary 3.15. *The converse to Theorem 3.14 holds as well: every planar graph G that is almost 3-connected appears as the minimal free resolution of some monomial ideal.*

The vertices, edges, and bounded regions of this planar map are labeled by their associated "staircase corners" as in the examples above. This determines a complex of free modules over $S = k[x, y, z]$ as in (3.1). Next we present an algorithm for finding

a planar map resolution as in Theorem 3.14 for artinian ideals. We will do this by using the process of strong deformations.

As we have seen before we can deform some ideal in more than one way, however, we want to be able to give generic deformations in the same manner every time. Specifically, Algorithm 3.16 requires a generic deformation satisfying the conditions:

$$\begin{aligned}
& \text{if } a_i = a_j \text{ and } c_i < c_j \text{ then } a_{\epsilon,i} < a_{\epsilon,j} \\
& \text{if } b_i = b_j \text{ and } a_i < a_j \text{ then } b_{\epsilon,i} < b_{\epsilon,j} \\
& \text{if } c_i = c_j \text{ and } b_i < b_j \text{ then } c_{\epsilon,i} < c_{\epsilon,j}
\end{aligned} \tag{3.2}$$

Observe that $c_i < c_j$ is equivalent to $b_i > b_j$ when the condition $a_i = a_j$ is assumed; in other words, if two generators lie at the same distance in front of the yz -plane, then the lower one lies farther to the right (as seen from far out on the x -axis). The first condition of (3.2) says that among generators that start at the same distance from the yz -plane, the deformation pulls increasingly farther from the yz -plane as the generators move up and to the left.

Given a deformation I_ϵ of a monomial ideal $I = \langle m_1, \dots, m_r \rangle$ with $m_i = x^{a_i} y^{b_i} z^{c_i}$, we write the i^{th} deformed generator as $m_{\epsilon,i} = x^{a_{\epsilon,i}} y^{b_{\epsilon,i}} z^{c_{\epsilon,i}}$. Now we explain the process for finding a planar graph for non-generic ideals.

Algorithm 3.16. Fix an artinian monomial ideal I inside $k[x, y, z]$.

- initialize $I_\epsilon =$ the strongly generic deformation of I in specification reference, and $G = \text{Buch}(I_\epsilon)$.
- while $I_\epsilon \neq I$ do
 - choose $u \in \{a, b, c\}$ and an index i such that $u_{\epsilon,i}$ is minimal among the deformed u -coordinates satisfying $u_{\epsilon,i} \neq u_i$. Assume for the sake of notation that $u = a$, by cyclic symmetry of (a, b, c) .
 - find the region of G whose monomial label $x^\alpha y^\beta z^\gamma$ has $\alpha = a_{\epsilon,i}$ and γ minimal.

- find the generator $m_{\epsilon,j}$ with the least x -degree among those with y -degree β and z -degree strictly less than γ .
 - redefine I_ϵ and G by setting $a_{\epsilon,i} = a_i$ and leaving all other generators alone.
 - if $a_j = a_i$ then delete from G the edge labeled $x^{a_i}y^\beta z^\gamma$, else leave G unchanged
- output G

Note: In the case where $u = b$ we change γ to α and β to γ . In the case where $u = c$ we change γ to β and β to α .

Proof of Correctness. See [9] pages 55-56. □

Now that we have an Algorithm for getting a planar graph for any monomial ideal in $k[x, y, z]$ that preserves key factors of the syzygy module we can use this to find the minimal resolution of non-generic monomial ideals. We now go back to the ideal given in Example 3.7 and use Algorithm 3.16 to find the minimal resolution.

Example 3.17. Consider the ideal I given in Example 3.7. Figure 3.6 shows us each step of what happens as we use Algorithm 3.16 to find a planar graph for the ideal I .

In order to use Algorithm 3.16 we must first add in some x^a, y^b, z^c in order to make I artinian. We use $I = \langle x^2z, xyz, y^2z, x^3y^5, x^4y^4, x^5y^3, x^6, y^6, z^6 \rangle$. Then we see that $I_\epsilon = \langle x^2z, xyz^{1.1}, y^2z^{1.2}, x^3y^5, x^4y^4, x^5y^3, x^6, y^6, z^6 \rangle$ is the strongly generic deformation satisfying the conditions above. We get $G = \text{Buch}(I_\epsilon)$ is the graph in the top left of Figure 3.6. In using the Algorithm we must remove the edges in blue in the top right of Figure 3.6 to get the graph in the bottom right. Then, the final step in our process is the remove the vertices corresponding to x^6, y^6, z^6 , indicated in red, and all adjacent edges. We are left with the planar map in the bottom right of Figure 3.6.

This final planar graph we get through using the Algorithm is the graph that gives us the minimal resolution for I . From this we find that the minimal resolution is:

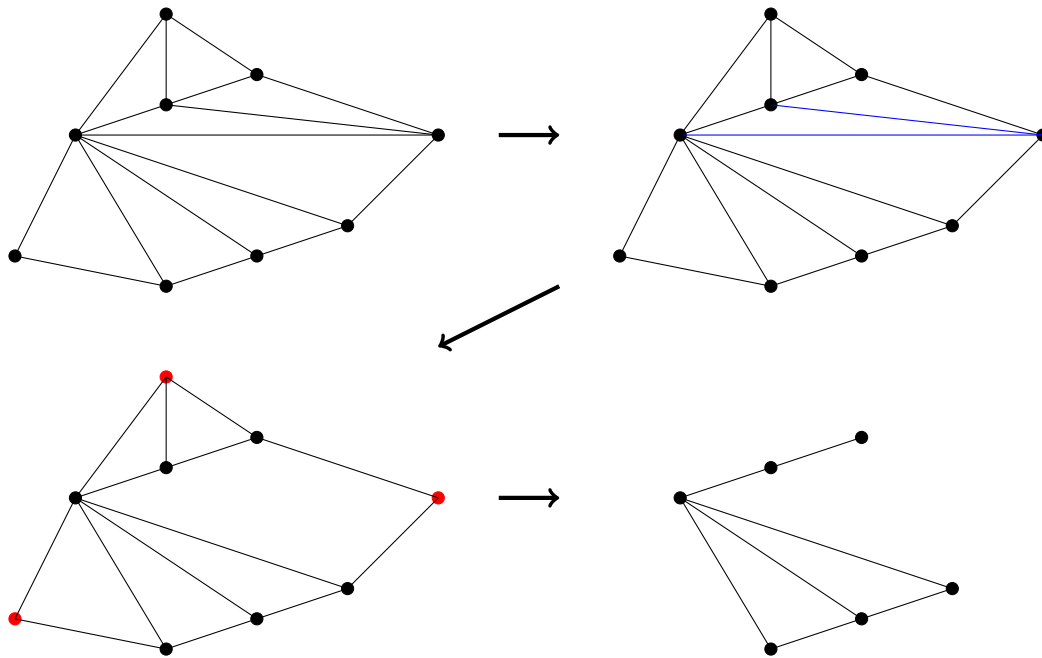


Figure 3.6: Algorithmic specialization from Example 3.7

$$0 \leftarrow S \leftarrow S^6 \leftarrow S^7 \leftarrow S^2 \leftarrow 0$$

We also see that this is much smaller than the resolution we originally got for I .

Theorem 3.14 grants that every ideal I over $k[x, y, z]$ has a planar graph that results in its minimal resolution. We see that Algorithm 3.16 gives a precise way to find this such planar graph for all monomial ideals over $k[x, y, z]$. However, this does not tell us how to find minimal resolutions for ideals I over $k[x_1, \dots, x_r]$ for $r > 3$. In the next chapter we will discuss some methods for finding minimal resolutions of ideals in any number of variables.

CHAPTER 4

SCARF COMPLEX AND RESOLUTIONS

In this chapter we discuss a type of combinatorial structure we can find for monomial ideals over $k[x_1, \dots, m_r]$, namely where r can be greater than 3. This structure we will work with is called the Scarf Complex. We will also discuss how this relates to the Buchberger graph and then when this gives us useful resolutions.

4.1 BASICS OF THE SCARF COMPLEX

In this section we will define what the Scarf complex is and then we will go over some basic lemmas about the structure of the Scarf complex. First we start with the definition of the scarf complex.

Definition 4.1. Let I be a monomial ideal with minimal generating set $\{m_1, \dots, m_r\}$. The **Scarf complex** Δ_I is the collection of all subsets of $\{m_1, \dots, m_r\}$ whose least common multiple is unique:

$$\Delta_I = \{\sigma \subset \{1, \dots, r\} \mid m_\sigma = m_\tau \implies \sigma = \tau\}$$

Now we give a lemma found in [9] that shows that a subset of a set in Δ_I is again a set in Δ_I ; meaning the Scarf complex is indeed a simplicial complex. Also we give a bound on the dimension of the Scarf complex.

Lemma 4.2. *The Scarf complex Δ_I is a simplicial complex. Its dimension is at most $n - 1$.*

Proof. If σ is a face of the Scarf complex and i is an element of σ , let $\tau = \sigma \setminus i$. Suppose that $m_\tau = m_\rho$ for some index set ρ . Then $m_\sigma = m_{\rho \cup i}$ and consequently $\rho \cup i = \sigma$, because σ lies in the Scarf complex. It follows that either $\rho = \tau$ or $\rho = \sigma$. However, the latter is impossible, since that would mean $m_\tau = m_\sigma$. Hence $\tau = \rho$ and we conclude that τ is a face of Δ_I .

For the dimension count, a facet σ of Δ_I has cardinality at most n because for each index $i \in \sigma$, the generator m_i contributes at least one coordinate to m_σ —that is, there is some variable x_k such that m_i is the only generator dividing m_σ and having the same degree in x_k as m_σ . \square

If we are working over $k[x, y]$ then the Scarf complex is one-dimensional, and its facets are the adjacent pairs of generators in the staircase. For a simple example working over $k[x, y, z]$ we see that the Scarf Complex of the ideal $I = \langle x^2, xy, y^2z, z^2 \rangle$ consisted of the triangles $\{x^2, xy, z^2\}$ and $\{xy, y^2z, z^2\}$ connected by the edge $\{xy, z^2\}$. Note that in some cases the Scarf complex may be disconnected.

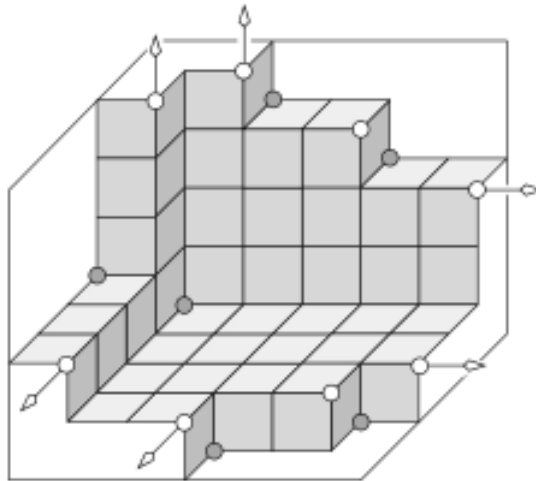


Figure 4.1: The staircase diagram for Example 3.11.

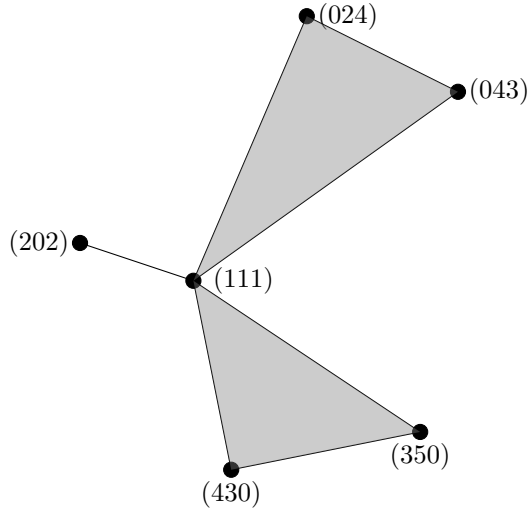


Figure 4.2: The Scarf Complex for Example 3.11.

Next we give another example of finding the Scarf complex for an ideal over $k[x, y, z]$.

Example 4.3. The generic ideal $I = \langle x^2z^2, xyz, y^2z^4, y^4z^3, x^3y^5, x^4y^3 \rangle$ has staircase diagram and Scarf complex as in Figures 4.1 and 4.2.

We see that the Scarf Complex given in Figure 4.2 does not include either the triangle $\{xyz, y^4z^3, x^3y^5\}$ or the edge $\{y^4z^3, x^3y^5\}$. This is because $\text{lcm}(xyz, y^4z^3, x^3y^5) = x^3y^5z^3$ and $\text{lcm}(y^4z^3, x^3y^5) = x^3y^5z^3$ and so the least common multiples are not unique.

4.2 RELATIONS BETWEEN THE SCARF COMPLEX AND OTHER STRUCTURES

In this section we will discuss how the Scarf Complex is related to other methods used for finding the resolutions of monomial ideals. Primarily we will discuss the relation between the Scarf complex and structures such as the Taylor Complex and the Buchberger Graph.

Example 4.4. When $I = \langle xy, xz, yz \rangle$, the Scarf complex Δ_I consists of three isolated points and $\text{Buch}(I)$ is the triangle. We see here that the edges of Scarf complex are not the same as the Buchberger graph but they are a subset.

In all dimensions, every edge of the Scarf complex of a monomial ideal is an edge of the Buchberger graph:

$$\text{edges}(\Delta_I) \subseteq \text{Buch}(I).$$

However, the converse is usually not true. We will give a lemma that states exactly when the converse holds but first we must give some definitions of criteria on our monomial ideals.

Definition 4.5. A monomial m' **strictly divides** another monomial m if m' divides m/x_i for all variables x_i dividing m . A monomial ideal $\langle m_1, \dots, m_r \rangle$ is **generic** if whenever two distinct minimal generators m_i and m_j have the same positive (nonzero) degree in some variable, a third generator m_k strictly divides their least common multiple $\text{lcm}(m_i, m_j)$.

This definition is more inclusive than the definition of strongly generic given in Chapter 2. This is given by the fact that any ideal that is strongly generic is also generic. However, there are examples of ideals which are generic but not strongly generic.

Example 4.6. For example, the ideal $\langle x^2, xy, y^2z, z^2 \rangle$ is strongly generic and therefore also generic.

However, the ideal $\langle x^2z, xy, y^2z, z^2 \rangle$ is generic but not strongly generic.

Finally, the ideal $\langle x^2, xy, yz, z^2 \rangle$ is neither strongly generic nor even generic.

Now we give the conditions needed for the edges of the scarf complex to be exactly the same as the Buchberger graph [9].

Lemma 4.7. *For I a generic monomial ideal, $\text{edges}(\Delta_I) = \text{Buch}(I)$.*

Proof. (\subseteq) Assume we have an edge in Δ_I that connects the generators m_i and m_j . Then, we know that $\text{lcm}(m_i, m_j)$ is unique. Now we see that if $m_k | \text{lcm}(m_i, m_j)$ for some monomial m_k , then $\text{lcm}(m_i, m_j, m_k) = \text{lcm}(m_i, m_j)$. This contradicts the uniqueness. Thus we have that $\text{Buch}(I)$ contains an edge connecting m_i and m_j .

(\supseteq) We prove this by contrapositive. Assume that there is no edge in Δ_I connecting m_i and m_j . Then we must have that $\text{lcm}(m_i, m_j) = \text{lcm}(P)$ for some other set P of generators. Let $m_k \in P$, then since $m_k | \text{lcm}(P)$ we must have that $m_k | \text{lcm}(m_i, m_j)$. If m_k has no exponent the same as $\text{lcm}(m_i, m_j)$ then we know that $\text{Buch}(I)$ does not contain an edge connecting m_i and m_j . Now we assume that m_k has some exponent the same as $\text{lcm}(m_i, m_j)$. Then since i is generic we know that there exist a generator m_l so that m_l strictly divides $\text{lcm}(m_i, m_j, m_k) = \text{lcm}(m_i, m_j)$. Therefore we have that $\text{Buch}(I)$ does not contain an edge connecting m_i and m_j . \square

Now consider the ideal $I = \langle x^4, y^4, z^4, xy^2z^3, x^3yz^2, x^2y^3z \rangle$ we first saw in Example 3.7. We see that the Scarf complex of I consist the edges in the Buchberger graph along with each of the triangle faces between these edges. Since I is generic here we know that $\text{edges}(\Delta_I) = \text{Buch}(I)$.

We have seen that the edges of the Scarf complex coincided with the Buchberger graph for generic ideals. Next we will discuss how the Scarf complex relates to the Taylor complex.

Definition 4.8. The Taylor complex \mathcal{F}_{Δ_I} supported on the Scarf complex Δ_I is called the **algebraic Scarf complex** of the monomial ideal I .

This leads us to the following proposition from [9] about how the Scarf complex relates to the other free resolutions of S/I .

Proposition 4.9. *If I is a monomial ideal in S , then every free resolution of S/I contains the algebraic Scarf complex \mathcal{F}_{Δ_I} as a subcomplex.*

Proof. Every free resolution contains a minimal free resolution, so it is enough to show that \mathcal{F}_{Δ_I} is contained in some minimal free resolution \mathcal{F} of S/I . In particular, we may choose \mathcal{F} to be a subcomplex of the full Taylor resolution, which is supported on the entire simplex whose vertices are the minimal generators of I . Every basis vector \mathbf{e}_σ for $\sigma \in \Delta_I$ must lie in \mathcal{F} by Theorem 4.7 and the uniqueness of a \mathbf{a}_σ as a face label. \square

The next two theorems which can be found as one theorem, Theorem 6.13 on page 111 of [9], relate the Scarf complex and the hull resolution and state a consequence of this relation.

Theorem 4.10. *If I is a monomial ideal, then its Scarf complex Δ_I is a subcomplex of the hull complex $\text{hull}(I)$.*

Proof. Let $F = \{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_p}\}$ be a face of the Scarf complex Δ_I with $m_F = \mathbf{x}^{\mathbf{u}}$. For any index $i \in \{1, \dots, p\}$, the least common multiple $m_{F \setminus \{\mathbf{x}^{\mathbf{a}_i}\}}$ strictly divides m_F in at least one variable. After relabeling, we may assume that this variable is x_i . Hence the x_i -degree of $\{\mathbf{x}^{\mathbf{a}_i}\}$ is strictly larger than the x_i -degree of $m_{F \setminus \{\mathbf{x}^{\mathbf{a}_i}\}}$. We conclude that $a_{ki} < a_{ii}$ for any two distinct indices i and k in $\{1, \dots, p\}$. This condition ensures that the determinant of the $p \times p$ matrix (t^{k_i}) is nonzero, so the points $t^{\mathbf{a}_1}, \dots, t^{\mathbf{a}_p}$ are affinely independent in \mathbb{R}^n , and their convex hull is a simplex.

The points $t^{\mathbf{a}_1}, \dots, t^{\mathbf{a}_p}$ constitute the vertex set of the restricted hull complex $\text{hull}(I)_{\preceq \mathbf{u}}$. It follows that every face of $\text{hull}(I)$ labeled by $\mathbf{x}^{\mathbf{u}}$ has vertices with labels from among $\{\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_p}\}$. There can be at most one such face of $\text{hull}(I)$, since F is a Scarf face, and there must be at least one by Proposition 4.9. We conclude that the simplex F is a face of the polyhedral cell complex $\text{hull}(I)_{\preceq \mathbf{u}}$. \square

We have seen that the Scarf complex is always a subcomplex of the hull complex. However, what we really want to know is when do these two complexes coincide. In the next theorem, from [9], we impose conditions on our ideal I that guarantee that

the scarf complex equals the hull complex. We also show that in this case we have that the scarf complex gives a minimal resolution.

Theorem 4.11. *If I is generic then $\Delta_I = \text{hull}(I)$, so its algebraic Scarf complex \mathcal{F}_{Δ_I} minimally resolves the quotient S/I .*

In order to prove this theorem we need the following lemma.

Lemma 4.12. *Let I be a monomial ideal and F a face of $\text{hull}(I)$. For each monomial $m \in I$ there is a variable x_j such that $\deg_{x_j}(m) \geq \deg_{x_j}(m_F)$.*

Proof. Suppose that $m = \mathbf{x}^{\mathbf{u}}$ strictly divides m_F in each coordinate. Let $t^{\mathbf{a}_1}, \dots, t^{\mathbf{a}_p}$ be the vertices of the face F and consider their barycenter

$$v(t) = \frac{1}{p} \cdot (t^{\mathbf{a}_1} + \dots + t^{\mathbf{a}_p}) \in F.$$

The j^{th} coordinate of $\mathbf{v}(t)$ is a polynomial in t of degree equal to $\deg_{x_j}(m_F)$. The j^{th} coordinate of $t^{\mathbf{u}}$ is a monomial of strictly lower degree. Hence $t^{\mathbf{u}} < \mathbf{v}(t)$ coordinatewise for $t \gg 0$. Let \mathbf{w} be a nonzero linear functional that is nonnegative on \mathbb{R}_+^n and whose minimum over \mathcal{P}_t is attained at the face F . Then $\mathbf{w} \cdot \mathbf{v}(t) = \mathbf{w} \cdot \mathbf{a}_1 = \dots = \mathbf{w} \cdot \mathbf{a}_p$, but our discussion implies $\mathbf{w} \cdot t^{\mathbf{u}} < \mathbf{w} \cdot \mathbf{v}(t)$, a contradiction. \square

Now we continue on to the proof of Theorem 4.11.

Proof of Theorem 4.11. Let F be any face of $\text{hull}(I)$ and let $\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_p}$ be the monomial generators of I corresponding to the vertices of F . We may assume that all n variables x_j appear in the monomial $m_F = \text{lcm}(\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_p})$. Suppose that F is not a face of the Scarf complex Δ_I . Then either

- (i) $\text{lcm}(\mathbf{x}^{\mathbf{a}_1}, \dots, \mathbf{x}^{\mathbf{a}_{i-1}}, \mathbf{x}^{\mathbf{a}_{i+1}}, \dots, \mathbf{x}^{\mathbf{a}_p}) = m_F$ for some $i \in \{1, \dots, p\}$, or
- (ii) there exists another generator $\mathbf{x}^{\mathbf{u}}$ of I such that $t^{\mathbf{u}} \notin F$ and $\mathbf{x}^{\mathbf{u}}$ divides m_F .

Consider first case (i). By Lemma 4.12 applied to $m = \mathbf{x}^{\mathbf{a}_i}$, there exists a variable x_j such that $\deg_{x_j}(\mathbf{x}^{\mathbf{a}_i}) = \deg_{x_j}(m_F)$, and hence $\deg_{x_j}(\mathbf{x}^{\mathbf{a}_i}) = \deg_{x_j}(\mathbf{x}^{\mathbf{a}_k})$ for some $k \neq i$. Since I is generic, there exists another generator m of I strictly dividing $\text{lcm}(\mathbf{x}^{\mathbf{a}_i}, \mathbf{x}^{\mathbf{a}_k})$ in all of its positive coordinates. Since $\text{lcm}(\mathbf{x}^{\mathbf{a}_i}, \mathbf{x}^{\mathbf{a}_k})$ divides m_F , it follows that m divides m_F in all n coordinates. This is a contradiction to Lemma 4.12.

Consider now case (ii), and suppose that we are not in case (i). For any variable x_j there exists $i \in \{1, \dots, p\}$ such that $\deg_{x_j}(\mathbf{x}^{\mathbf{a}_i}) = \deg_{x_j}(m_F) \geq \deg_{x_j}(xu)$. If the inequality " \geq " is an equality " $=$ ", then there exists a new monomial generator m strictly dividing m_F in all of its positive coordinates, a contradiction to Lemma 4.12, as before. Therefore " \geq " is a strict inequality " $>$ " for all variables x_j . This means that $\mathbf{x}^{\mathbf{u}}$ strictly divides m_F in all coordinates, again a contradiction to Lemma 4.12.

Hence both cases (i) and (ii) lead to a contradiction, and we conclude that every face of the hull complex $\text{hull}(I)$ is a face of the Scarf complex Δ_I . This implies that $\text{hull}(I) = \Delta_I$, by Theorem 4.10. The algebraic Scarf complex \mathcal{F}_{Δ_I} is minimal because no two faces in Δ_I have the same degree. \square

Now we are able to draw some algebraic conclusions about the Scarf complex from theorem 4.11. We present these conclusions in the following two corollaries.

Corollary 4.13. *The minimal free resolution of a generic monomial ideal I is independent of the characteristic of the field k . The total Betti number $\beta_i(I) = \sum_{\mathbf{a} \in \mathbb{N}^n} \beta_{i,\mathbf{a}}(I)$ equals the number $f_i(\Delta_I)$ of i -dimensional faces of its Scarf complex Δ_I .*

Corollary 4.14. *If $I = \langle m_1, \dots, m_r \rangle$ is generic and S/I is artinian, with $m_i = x_i^{d_i}$ for $i = 1, \dots, n$, then the Scarf complex Δ_I is a regular triangulation (usually with additional vertices, some of which may lie on the boundary) of the $(n - 1)$ -simplex with vertex set $\{1, \dots, n\}$.*

We have seen that for generic ideals the scarf complex gives us a minimal resolution. However, this is generally not true for ideals that are not generic; often times the scarf complex does not even result in a resolution. In the next section we will discuss a procedure for making these ideals generic in order to find resolutions.

4.3 DEFORMATIONS OF IDEALS

Consider $I = \langle m_1, \dots, m_r \rangle$ an arbitrary monomial ideal; assume I is not generic. In this section we construct a free resolution of S/I by deforming the exponent vectors of the generators of I . While this approach does not usually result in a minimal resolution it has an advantage. Namely, the resolution by deformation of exponents has length at most the number of variables; thus in general it is much smaller and shorter than Taylor's resolution.

Definition 4.15. A **deformation** ϵ of a monomial ideal $I = \langle m_1, \dots, m_r \rangle$ is a choice of vectors $\epsilon_i = (\epsilon_{i1}, \dots, \epsilon_{in}) \in \mathbb{R}^n$ for $i \in \{1, \dots, r\}$ satisfying

$$a_{is} < a_{js} \implies a_{is} + \epsilon_{is} < a_{js} + \epsilon_{js} \quad \text{and} \quad a_{is} = 0 \implies \epsilon_{is} = 0,$$

where $\mathbf{a}_i = (a_{i1}, \dots, a_{in})$ is the exponent vector of m_i . Formally introduce the monomial ideal (in a polynomial ring with real exponents):

$$I_\epsilon = \langle m_1 \cdot \mathbf{x}^{\epsilon_1}, m_2 \cdot \mathbf{x}^{\epsilon_2}, \dots, m_r \cdot \mathbf{x}^{\epsilon_r} \rangle = \langle \mathbf{x}^{\mathbf{a}_1 + \epsilon_1}, \mathbf{x}^{\mathbf{a}_2 + \epsilon_2}, \dots, \mathbf{x}^{\mathbf{a}_r + \epsilon_r} \rangle.$$

A deformation ϵ is called **generic** if I_ϵ is a generic monomial ideal.

The Scarf complex Δ_{I_ϵ} of the deformed ideal I_ϵ still makes sense, as a combinatorial object, and has the same vertex set $\{1, \dots, r\}$ as Δ_I . Indeed, the combinatorics of a deformation depends only on the coordinatewise order that results on generating exponents, and there is always a choice of deformation that results in integer exponents inducing the same coordinatewise order.

Taking Δ_{I_ϵ} , the Scarf complex of I_ϵ , we label the vertex of Δ_{I_ϵ} corresponding to $m_i \cdot \mathbf{x}^{\epsilon_i}$ with the original monomial m_i . Let $\mathcal{F}_{\Delta_{I_\epsilon}}$ be the complex of S -modules defined by this labeling of Δ_{I_ϵ} as in Construction 2.1 in [3]. For generic deformations ϵ , the Scarf complex Δ_{I_ϵ} of the deformed ideal gives an easy simplicial (but typically nonminimal) free resolution of I .

Theorem 4.16. *The complex $\mathcal{F}_{\Delta_{I_\epsilon}}$ is a free resolution of S/I over S . [3]*

Proof. Fix a monomial m . Let J be the largest subset of $\{1, \dots, r\}$ such that m_J divides m . The following conditions are equivalent for a subset I of $\{1, \dots, r\}$:

$$m_I \text{ divides } m \iff I \subseteq J \iff m_I \text{ divides } m_J \iff m_I(\epsilon) \text{ divides } m_J(\epsilon).$$

Here $m_I(\epsilon) := \text{lcm}(m_i \mathbf{x}^{\epsilon_i} : i \in I)$. The last equivalence follows from our choice of the ϵ_{ij} . The set of all faces of Δ_{I_ϵ} which satisfy the four equivalent conditions above is an acyclic simplicial complex, by Theorem 4.11 and (Lemma 2.2, [3]) applied to $M_\epsilon[m_J(\epsilon)]$. Now apply (Lemma 2.2, [3]) to M and m with $\Delta = \Delta_{I_\epsilon}$. \square

The resolution $\mathcal{F}_{\Delta_{I_\epsilon}}$ in Theorem 4.16 has length less than or equal to the bound n provided by the Hilbert Syzygy Theorem, by Lemma 4.2, but it is generally not minimal. Note that this reduction to the generic situation actually produces a free resolution of S/I for any I .

Example 4.17. The square m^2 of the maximal ideal $m = \langle x, y, z \rangle$ is not generic, and indeed, its Scarf complex is 1-dimensional and not contractible. However, we can find a generic deformation as depicted in Figure 4.3. The resolution of m^2 afforded by the right-hand diagram but with labels as in the left-hand diagram is not minimal.

Corollary 4.18. *The Betti numbers of I are less than or equal to those of any deformation I_ϵ , that is, less than or equal to the face numbers of the Scarf complex Δ_{M_ϵ} . [3]*

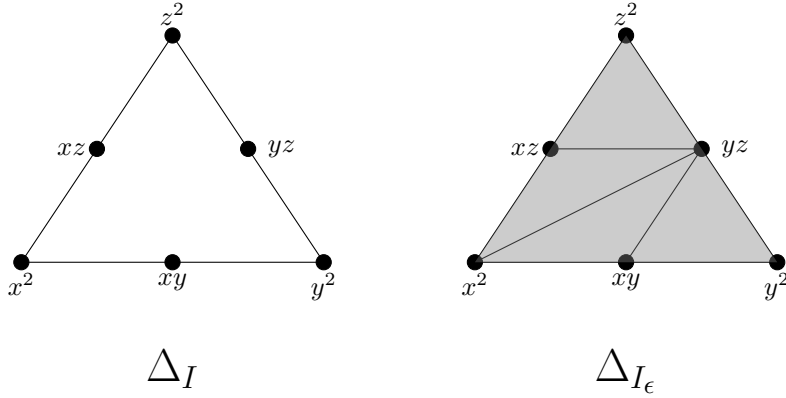


Figure 4.3: Generic Deformation of $\langle x, y, z \rangle^2$

We emphasize that the Betti numbers of I_ϵ depend on the choice of the generic deformation.

Next we discuss the example first proposed in [1]. Here we will give a deformation and find the resolution generated by the Scarf complex of this deformation. Then we will show how this compares to the minimal resolution of the ideal.

Example 4.19. Consider the ideal $I = \langle x^2, xy^2z, y^2z^2, yz^2w, w^2 \rangle$ in Example 3.5. A generic deformation is $I_\epsilon = \langle x^2, xy^2z, y^3z^3, yz^2w, w^2 \rangle$. Label the generators as 1, 2, 3, 4, 5 in the given order. The Scarf complex of I_ϵ consists of the tetrahedron $\{1, 2, 4, 5\}$ and the triangle $\{2, 3, 4\}$.

We can obtain a nonminimal free resolution $\mathcal{F}_{\Delta_\epsilon}$ of S/I by using the process above. This resolution for S/I_ϵ is

$$\mathcal{F}_{\Delta_\epsilon} : 0 \rightarrow S^1 \rightarrow S^5 \rightarrow S^8 \rightarrow S^5 \rightarrow S \rightarrow S/I_\epsilon \rightarrow 0.$$

While the minimal resolution for S/I is

$$\mathcal{F}_{\Delta_\epsilon} : 0 \rightarrow S^1 \rightarrow S^5 \rightarrow S^7 \rightarrow S^4 \rightarrow S \rightarrow S/I_\epsilon \rightarrow 0.$$

Thus $\mathcal{F}_{\Delta_\epsilon}$ differs from the minimal resolution by a single summand $0 \rightarrow S \rightarrow S \rightarrow 0$, placed in homological degrees 2 and 3. However, this makes a big difference struc-

turally: we have that the resolution $\mathcal{F}_{\Delta_f^\epsilon}$ is a DG-algebra (with a simple multiplication rule) while the minimal free resolution admits no DG-algebra structure at all.

Note that Taylor's resolution is one step longer than $\mathcal{F}_{\Delta_f^\epsilon}$ and it has Betti numbers 1, 5, 10, 10, 5, 1. So we see that the Scarf complex is usually shorter than the Taylor complex.

CHAPTER 5

DIFFERENTIAL GRADED ALGEBRAS

In this chapter we look at differential grade algebra (or DGA) structure on monomial ideals. We will discuss the notions of when the minimal resolutions admits a DGA structure, we call this a minimal DGA resolution. Then we will discuss what other constructions (generally not minimal) of resolutions give DGA resolutions.

In this section we take S be a polynomial ring over a field, $I \subseteq S$ be a monomial ideal, and let \mathbb{F} denote the minimal free resolution of S/I over S . Now we start off by giving the definition of a differential graded algebra structure, which we often abbreviate DGA.

Definition 5.1. A differential graded algebra (DGA) structure on \mathbb{F} is an S -linear map $*$: $\mathbb{F} \otimes_S \mathbb{F} \rightarrow \mathbb{F}$ satisfying the following axioms for $a, b, c \in \mathbb{F}$:

- 1) $*$ extends the usual multiplication on $\mathbb{F}_0 = S$,
- 2) $\partial(a * b) = \partial(a) * b + (-1)^{|a|} a * \partial b$ (Leibniz rule),
- 3) $|a * b| = |a| + |b|$ (homogeneity with respect to the homological grading),
- 4) $a * b = (-1)^{|a| \cdot |b|} b * a$ (graded commutativity), and
- 5) $(a * b) * c = a * (b * c)$ (associativity).

In the context of monomial ideals there are a few cases of ideals for which it is known that they admit a DGA structure. Some of these cases are stable ideals, matroidal ideals and edge ideals of cointerval graphs.

It was also previously thought that the Scarf complex of strongly generic monomial ideals admitted a minimal DGA resolution. However, we will state a theorem from [8] which presents a strongly generic ideal whose minimal resolution does not admit a DGA structure.

Theorem 5.2. *The ideal $\langle x^2, xy, y^2z^2, zw, w^2 \rangle \in k[x, y, z, w]$ is strongly generic, but its minimal free resolution does not admit the structure of a DGA, whose multiplication respects the standard \mathbb{Z} -grading.*

Proof. The proof of this theorem can be found on page 14 of [8]. □

This also gives rise to the following corollary. [8]

Corollary 5.3. *1. There exists a monomial ideal whose hull resolution does not admit a DGA structure.*

2. There exists a monomial ideal whose Lyubeznik resolution does not admit a DGA structure.

3. There exists a monomial ideal whose minimal free resolution is supported on a simplicial complex (and in particular cellular), but it does not admit a DGA structure.

Next we present a theorem given by Khattan in [8] which states that we can find an element s for which $S/(sI)$ gives a minimal DGA resolution for any ideal I . This theorem is important as what it really implies that all that is need in order to ensure we get a DGA structure is to modify the last map of the minimal free resolution

Theorem 5.4. *Let S be a regular local ring and $I \in S$ an ideal. There exists an element $s \in S, s \neq 0$ such that the minimal free resolution of $S/(sI)$ admits a DGA structure.*

The same conclusion holds if S is a polynomial ring and I is a homogeneous ideal. In this case, s can be chosen homogeneous. If I is even a monomial ideal (and S a polynomial ring), then s can be chosen to be the least common multiple of the generators of I .

In order to prove this theorem we need the following lemma.

Lemma 5.5. *Let $Q = k[t_1^\pm, \dots, t_n^\pm]$ be a \mathbb{Z}^n -graded Laurent polynomial ring for $n \geq 0$.*

Let

$$\mathbb{F} : 0 \rightarrow \mathbb{F}_p \rightarrow \dots \rightarrow \mathbb{F}_1 \rightarrow \mathbb{F}_0 \rightarrow 0$$

be an exact complex of graded Q -modules and assume that $\mathbb{F}_0 = Q$. Then the multiplication on \mathbb{F}_0 can be extended to a graded-commutative DGA structure on \mathbb{F} .

Proof. Let ∂ denote the differential of \mathbb{F} . We claim that there exists a map $\sigma : \mathbb{F} \rightarrow \mathbb{F}$ of homological degree 1 such that $\partial \circ \sigma + \sigma \circ \partial = \text{id}_{\mathbb{F}}$ and $\sigma \circ \sigma = 0$. Indeed, every graded module over Q is free, so we can choose splittings $\mathbb{F}_i \cong V_i \oplus \partial(\mathbb{F}_{i+1})$. Note that $\partial(\mathbb{F}_{i+1}) = \partial(V_{i+1})$ and the restriction of ∂ to V_{i+1} is injective. We define $\sigma_i|_{V_i} = 0$ and $\sigma_i(\partial(f)) = f$ for $f \in V_{i+1}$. It is not difficult to see that this gives indeed a map σ with the claimed properties.

We define the DGA structure on \mathbb{F} inductively by the formula

$$a * b := \begin{cases} ab & \text{if } |a| = 0 \text{ or } |b| = 0 \\ \sigma(\partial(a) * b + (-1)^{|a|} a * \partial(b)) & \text{otherwise,} \end{cases}$$

where the multiplication in the first case is the one from $F_0 = Q$.

This multiplication clearly satisfies the Leibniz rule and it extends the multiplication on \mathbb{F}_0 . It remains to show that $*$ is graded-commutative and associative. Note that $\sigma = \sigma \circ \partial \circ \sigma$. We proceed by induction on the homological degree, the base case being clear. It holds for $a, b, c \in F$:

$$\begin{aligned}
a * (b * c) &= \sigma\left(\partial(a * (b * c))\right) \\
&= \sigma\left(\partial a * (b * c) + (-1)^{|a|} a * (\partial b * c) + (-1)^{|a|+|b|} a * (b * \partial c)\right) \\
&\stackrel{(\#)}{=} \sigma\left((\partial a * b) * c + (-1)^{|a|} (a * \partial b) * c + (-1)^{|a|+|b|} (a * b) * \partial c\right) \\
&= \sigma\left(\partial((a * b) * c)\right) \\
&= (a * b) * c
\end{aligned}$$

where in $(\#)$ we use the induction hypothesis. The commutativity is verified analogously. \square

Proof of Theorem 5.4. Let \mathbb{F} be the minimal free resolution of S/I . In the local case, let \mathcal{Q} be the field of fractions of I , while in the polynomial case, let \mathcal{Q} be the subring of the field of fractions of S where we adjoin inverses for all homogeneous elements of S . Set $\mathbb{F}_{\mathcal{Q}} := F \otimes S_{\mathcal{Q}}$. In both cases, \mathcal{Q} is a multivariate Laurent polynomial ring over some field k , though without variables in the local case. Moreover, \mathcal{Q} is flat over S and hence $\mathbb{F}_{\mathcal{Q}}$ is exact. So by Lemma 5.5, it can be endowed with a DGA structure $*$. Choose a basis for each \mathbb{F}_i . Then the multiplication on $F_{\mathcal{Q}}$ can be represented as a matrix with entries in \mathcal{Q} . Hence we can choose an element $s \in S$ such that $sq \in S$ for each entry q of this matrix. Let \mathbb{F}' be the subcomplex of \mathbb{F} defined by $\mathbb{F}'_0 := \mathbb{F}_0$ and $\mathbb{F}'_i := s\mathbb{F}_i$ for $i \geq 1$. We claim that \mathbb{F}' is closed under multiplication. Indeed, for $sa, sb \in \mathbb{F}'$ the choice of s implies that $s(a * b) \in \mathbb{F}$ and thus $(sa * sb) \in \mathbb{F}'$. Note that \mathbb{F}' is isomorphic to \mathbb{F} except in degree 0, so in particular it is exact in every other degree. In degree 0, it holds that $H_0(\mathbb{F}') = \mathbb{F}'_0 / \partial(s\mathbb{F}'_1) = S / (sI)$. Thus \mathbb{F}' is the minimal free resolution of $S/(sI)$. Finally, consider the case that I is a monomial ideal in a polynomial ring $S = k[x_1, \dots, x_n]$. Let $a, b \in \mathbb{F}$ be two homogeneous elements. The product $a * b \in \mathbb{F}_{\mathcal{Q}}$ can be written as a sum

$$\sum_{\mathbf{g} \in \mathcal{B}} \lambda_{\mathbf{g}} \mathbf{x}^{deg(a)+deg(b)-deg(\mathbf{g})} \mathbf{g}$$

where \mathcal{B} is an S -basis of \mathbb{F} and $\lambda_{\mathbf{g}} \in k$. Now let s be the lcm of all generators of I . Then the multidegrees of all elements of \mathcal{B} are less or equal to $\deg(s)$, and hence $s(a * b) \in \mathbb{F}$. Now one can argue as above. \square

Finally, we conclude with some consequences that arise when the minimal free resolution of an ideal admits a DGA structure. These consequence are all given in [8].

First we start with a theorem about the characterization of the possible Betti vectors of squarefree monomial ideals admitting minimal DGA resolutions.

Theorem 5.6. *Let $f = (1, f_1, f_2, \dots) \in \mathbb{N}^{\nu}$ be a finite sequence of natural numbers. Then the following conditions are equivalent:*

1. *There exists a squarefree monomial ideal I in some polynomial ring S whose minimal free resolution is a DGA, such that $\beta_i^S(S/I) = f_i$ for all i .*
2. *f is the f -vector of a simplicial complex Δ which is a cone.*

Proof. A full proof and related proposition can be found on pages 1236-1237 of [8]. \square

For a monomial ideal $I \subseteq S$ and $0 \leq i \leq \text{pdim}S/I$ we define

$$t_i := \max\{j : \beta_{i,j}^S(S/I) \neq 0\}.$$

We say that the syzygies of I are subadditive when $t_b \leq t_a + t_{b-a}$ for all a, b such that $1 \leq a < b \leq \text{pdim}S/I$. We see that not every ideal has this property [2], however there is no counterexample known among monomial ideals. Next we present a proposition which shows that for squarefree monomial ideals, the existence of a DGA structure on the minimal free resolution implies the subadditivity of syzygies.

Proposition 5.7. *If $I \subseteq S$ is a squarefree monomial ideal which admits a minimal DGA resolution \mathbb{F} , then its syzygies are subadditive.*

Proof. A proof of the proposition can be found on page 13 of [8]. □

If it is not assumed that \mathbb{F} admits a DGA structure, then the methods used in the proof still suffice to show the case $a = 1$.

Now we end this chapter by giving the corollary which states this more precisely.

Corollary 5.8. *For a monomial ideal $I \subseteq S$, it holds that $t_i \leq t_1 + t_{i-1}$ for all $2 \leq i \leq \text{pdim}S/I$.*

Proof. We may replace I by its polarization and so assume it is squarefree. Further, choose any multiplication on \mathbb{F} . Now we just note that the proof of Proposition 5.7 does not require the multiplication to be associative if $a = 1$. □

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