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An Implementation of the Kapustin-Li Formula

Jessica Otis

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by

Jessica Otis

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Accepted by:

Matthew Ballard, Director of Thesis

Jesse Kass, Reader

Cheryl L. Addy, Vice Provost and Dean of the Graduate School

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Abstract

Let R be a regular local ring and take ω to be an isolated singularity on R. Taking $\mathbb{Z}/2$-graded R-modules, X and Y, a matrix factorization of ω is a pair of morphisms (φ, ψ) such that $\varphi \circ \psi = \omega$ and $\psi \circ \varphi = \omega$ are satisfied in the diagram $X \xrightarrow{\varphi} Y \xrightarrow{\psi} X$. We will discuss the category of matrix factorizations of ω in R and lead into the homotopy category of matrix factorizations as well as its historical development. Finally, we will conclude with the statement of the Kapustin-Li formula for the duality pairing on the morphisms in the matrix factorization category of (R, ω) and discuss its implementation in SageMath.

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Chapter 1

Introduction

Matrix factorizations were introduced by Eisenbud. He showed in [6] that by taking a finitely generated maximal Cohen-Macaulay module over the ring $R/(f)$, where R is regular and local and (f) is a principal ideal in R, its minimal free resolution is obtained from a matrix factorization of f in R.

The homotopy category of matrix factorizations of f in R was established by Auslander in [2] as a Calabi-Yau category which gave rise to the following interpretation by Kapustin and Li: considering the homotopy category, which is triangulated, as the category of boundary conditions in the Landau-Ginzburg B-model that corresponds to (R, f) allowed the derivation of a formula for the duality pairing on the morphism complexes in the matrix factorization category of (R, f), discussed in [11].

The focus of this thesis is the implementation of the Kapustin-Li Formula in an affine setting. In order to understand the statement and the code related to it, we start by introducing basic algebraic and complex analytic definitions and theorems in Chapter 2. We also look at introductory category theory and begin the intuition for the homotopy category. Then, in Chapter 3, we discuss matrix factorizations, the category comprised of them as objects, and the homotopy category of matrix factorizations. We end the chapter with the statement and a proof of Eisenbud's matrix factorization theorem from [13].

We then move to the statement of the formula and its interpretation in Chapter 4. Finally, discussion of the implementation of the Kapustin-Li formula in SageMath through the writing of Python scripts is our concluding chapter of content. The computation in the $n = 1$ variable of the form x^d case can be seen in full in Appendix A and is discussed in Chapter 5. For $n \neq 1$, a complete calculation of the pairing could not be demonstrated, though discussed and attempted. The final chapter expresses the desire for future work concerning the formula to include the full implementation for any n-variable singularity.

Chapter 2

Definitions and Preliminary Knowledge

First to be introduced will be key definitions from algebra, category theory, and complex analysis. Formulations of these definitions can also be found in the standard references ([3], [5], [10], [1]). As they arise, important and relevant theorems and propositions will be presented.

Notation 2.1. The identity element of a ring R or R-module M will be denoted id_R or id_M, respectively.

Notation 2.2. The Krull dimension of a ring R we denote as $\dim R$.

2.1 Algebra

Remark 2.1.1. A common statement on the Krull dimension is the following of $\dim R = 0$ if and only if every prime ideal P in R is a maximal ideal. A notion which follows is: if R is Noetherian, then R has finite length if and only if $\dim R = 0$.

Definition 2.1.2. Let R be a Noetherian local ring with maximal ideal $\underline{m} = (a_1, ..., a_n)$ where n is minimal. Then R is *regular* if $\dim R = n$. Here we refer to $a_1, ..., a_n$ as a *regular system of parameters*.

Definition 2.1.3. Given additive subgroups $R_i \subseteq R$ such that $R = \bigoplus_i R_i$ and $R_i R_j \subseteq R_{i+j}$, then we say R is $\mathbb{Z}$*-graded*, or $\mathbb{Z}/2\mathbb{Z}$*-graded*.

This property of gradedness can also extend to the modules of the ring under certain conditions.

Definition 2.1.4. Let R be a graded ring. Given additive subgroups $M_i \subseteq M$ such that $M = \bigoplus_i M_i$ and $R_i M_j \subseteq M_{i+j}$, then we say M is $\mathbb{Z}$-*graded* over R.

Definition 2.1.5. For a R-module M, a sequence of elements $a_1, ..., a_n$ is a *regular sequence* on M if $(a_1, ..., a_n)M \neq M$ and, for $i = 1, ..., n$, a_i is a nonzero divisor on $M/(a_1, ..., a_{n-1})$.

We have seen the geometric measure of a module in the idea of Krull dimension; now we may look at the homological measure of the size of a module M.

Definition 2.1.6. If R is local Noetherian with maximal ideal $\underline{m}$ and M is a nonzero finitely generated R-module, then grade$(\underline{m}, M)$, or the *depth* of M, is the length of the longest regular sequence in $\underline{m}$ on M.

Definition 2.1.7. A R-module M is a *(maximal) Cohen-Macaulay module* if the depth of M is equal to the Krull dimension of R.

We denote a Cohen-Macaulay module by the abbreviation CM. The ring R is a *Cohen-Macaulay ring* if R is a CM over R.

Definition 2.1.8. An *exact sequence* is a sequence of objects F_i and morphisms between these objects f_i, written as

$$F_0 \xrightarrow{f_1} F_1 \xrightarrow{f_2} ... \xrightarrow{f_n} F_n,$$

such that the image of a morphism is the kernel of the following morphism, im $f_i =$ ker f_{i+1}.

Definition 2.1.9. A *resolution* is an exact sequence of modules. A *free resolution* is one where each module F_i is free.

Definition 2.1.10. Given an exact sequence of R-modules,

$$0 \longrightarrow N \longrightarrow F_{n-1} \longrightarrow ... \longrightarrow F_1 \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

for F_i free R-modules, then N is the n-th *syzygy* of M.

Definition 2.1.11. An *injective resolution* of the module M is an exact sequence of the form

$$0 \longrightarrow M \longrightarrow I^0 \longrightarrow I^1 \longrightarrow ... \tag{2.1}$$

where I^j are injective modules.

An important conclusion is that every module M has an injective resolution. The dual notion of these resolutions is the *projective resolution*. Every module N also has a projective resolution, an exact sequence of the form

$$... \longrightarrow P_1 \longrightarrow P_0 \longrightarrow N \longrightarrow 0 \tag{2.2}$$

where P_i are projective modules. Definitions of injective and projective modules can be found in [3] also containing propositions, with proof, relaying their properties.

Definition 2.1.12. For a module M which admits a finite injective resolution, the minimal length among all finite injective resolutions of M as seen in (2.1) is its *injective dimension*, denoted $\text{inj}_R(M)$.

Definition 2.1.13. Similarly, if a module N admits a finite projective resolution, the minimal length of all finite projective resolutions of N as seen in (2.2) is its *projective dimension*, denoted $\text{pd}_R(N)$.

Considering these homological measures, we come to a formulation by Auslander-Buchsbaum. The equation below can be seen in [5], [6], and [13] and shown in [12] where first two preliminary lemmata are proven and then used in the proof of the statement.

Theorem 2.1.14. (Auslander-Buchsbaum) For R a commutative Noetherian local ring and M a nonzero finitely generated R-module of finite projective dimension, then

$$\text{pd}_R M = \text{depth } R - \text{depth } M \tag{2.3}$$

Definition 2.1.15. A *Gorenstein ring* is a commutative Noetherian ring such that each localization at a prime ideal is a *Gorenstein local ring* — a commutative Noetherian local ring R with finite injective dimension as a R-module.

The following, which can be found in [13], are propositions detailing some useful properties of CM modules over a ring R.

Proposition 2.1.16. Let R be a commutative Noetherian local ring and let $0 \longrightarrow L \longrightarrow M \longrightarrow N \longrightarrow 0$ be an exact sequence of R-modules. Then the following are true:

1. If L and N are CM, then M is CM.

2. If M and N are CM, then so is L.

Proposition 2.1.17. If R is a regular local ring, then any CM module over R is a free module.

Finally, an important algebraic structure required for all proceeding material is that of the chain complex. There is a dual notion, cochain complex, however we will use the convention of chain complex for this paper.

Definition 2.1.18. A *chain complex* $(A_\bullet, d_\bullet)$ is a sequence of modules $..., A_0, A_1, ...$ connected by homomorphisms, called *differentials*, $d_n : A_n \to A_{n-1}$ such that $d_n \circ d_{n+1} = 0$. This can be represented by

$$... \xrightarrow{d_3} A_2 \xrightarrow{d_2} A_1 \xrightarrow{d_1} A_0 \xrightarrow{d_0} ... \tag{2.4}$$

An important complex used in homological algebra is the *Koszul complex.*

Definition 2.1.19. For R a commutative ring, E a free R-module with finite rank r, and R-linear map $\varphi : E \to R$, the *Koszul complex* associated to φ is the chain complex of R-modules

$$0 \longrightarrow \bigwedge^r E \xrightarrow{d_r} \bigwedge^{r-1} E \xrightarrow{d_{r-1}} \dots \xrightarrow{d_2} \bigwedge^1 E \xrightarrow{d_1} \bigwedge^0 E \longrightarrow 0 \tag{2.5}$$

where d_k is the differential between the exterior powers of the free module; note, $\bigwedge^0 E = R$. For any $e_i \in E$, d_k is defined as

$$d_k(e_1 \wedge \dots \wedge e_k) := \sum_{i=1}^{k} (-1)^{i+1} e_1 \wedge \dots \wedge \varphi(e_i) \wedge \dots \wedge e_k \tag{2.6}$$

2.2 Category Theory

In this section, we introduce basic category theory, including additive categories and triangulated categories. These concepts are important to and applied throughout the contents of Chapters 3 and 4, and each reference therein.

Definition 2.2.1. A *category* $\mathscr{C}$ consists of:

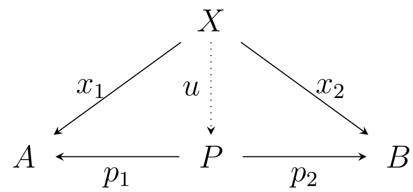
1. A class of objects $\mathrm{Obj}(\mathscr{C})$,

2. A class $\mathrm{Hom}_{\mathscr{C}}$ of morphisms between two objects, where each morphism has a source and a target in $\mathrm{Obj}(\mathscr{C})$, and

3. For three objects $A, B,$ and C, a binary operation called the *composition of morphisms* where $\mathrm{Hom}_{\mathscr{C}}(A,B) \times \mathrm{Hom}_{\mathscr{C}}(B,C) \to \mathrm{Hom}_{\mathscr{C}}(A,C)$, $f \times g \mapsto g \circ f$.

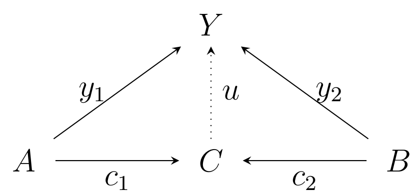
such that there is exactly one identity morphism for every object and that associativity of morphisms holds.

Definition 2.2.2. Let A and B be two objects in a category $\mathscr{C}$. A *product* of A and B is an object P along with morphisms $A \xleftarrow{p_1} P \xrightarrow{p_2} B$ such that, given any diagram $A \xleftarrow{x_1} X \xrightarrow{x_2} B$, there exists a unique morphism $u : X \longrightarrow P$ so that the following diagram commutes.



(2.7)

Definition 2.2.3. Let A and B be two objects in a category $\mathscr{C}$. A *coproduct* of A and B is an object C along with morphisms $A \xrightarrow{c_1} C \xleftarrow{c_2} B$ such that, given any diagram $A \xrightarrow{y_1} Y \xleftarrow{y_2} B$, there exists a unique morphism $u : C \longrightarrow Y$ so that the following diagram commutes.



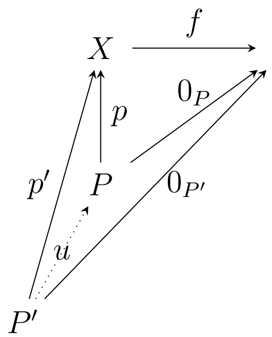
(2.8)

A coproduct is the dual notion of a product, and they are often denoted $A \prod B$ and $A \coprod B$ for products and coproducts, respectively.

Just as in ring theory, kernels and cokernels also have applications in category theory, though the added context of an object is necessary to understand more than just how the morphism acts. There is also an assumption that the category contains zero morphisms.

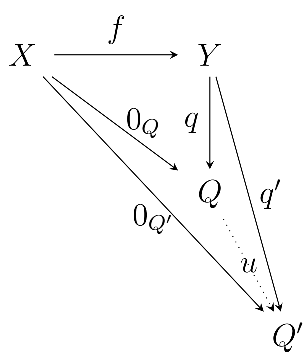
Definition 2.2.4. For a category $\mathscr{C}$, let $a : X \to Y$ be some morphism between objects X, Y in $\mathscr{C}$. Then for any morphisms $g, h : A \to X$ for some object A in $\mathscr{C}$, if $ag = ah$ we call a the *zero morphism.*

Definition 2.2.5. For a category $\mathscr{C}$ which contains zero morphisms and for some morphism $f : X \to Y$, the *kernel* of f is an object P in $\mathscr{C}$ with the morphism $p : P \to X$, written $\ker f = (P, p)$, such that the composition $f \circ p$ is the zero morphism from P to Y, which we denote 0_P in the following diagram. Explicitly, given any $p' : P' \to X$ such that $f \circ p'$ is the zero morphism, there is a unique morphism $u : P' \to P$ such that $p \circ u = p'$.



(2.9)

Definition 2.2.6. Similarly, for $\mathscr{C}$ which again contains zero morphisms and some morphism $f : X \to Y$, the *cokernel* of f is an object Q in $\mathscr{C}$ with the morphism $q : Y \to Q$, written $\operatorname{coker} f = (Q, q)$, such that the composition $q \circ f$ is the zero morphism from X to Q, denoted 0_Q in the following diagram. Explicitly, given any $q' : Y \to Q'$ such that $q' \circ f$ is the zero morphism, there is a unique morphism $u : Q \to Q'$ such that $u \circ q = q'$.



(2.10)

Definition 2.2.7. A category $\mathscr{C}$ is an *additive category* if the following hold:

1. For every $X, Y \in \mathrm{Obj}(\mathscr{C}), \mathrm{Hom}(X, Y)$ is an abelian group and the composition of morphisms is bilinear,

2. $\mathscr{C}$ contains a zero object (an object that is both initial and terminal), and

3. For any $X, Y \in \mathrm{Obj}(\mathscr{C})$, there exists a coproduct $X \coprod Y$ in $\mathscr{C}$.

A functor $F : \mathscr{C} \to \mathscr{D}$ between two such categories is *additive* if for all $X, Y \in \mathrm{Obj}(\mathscr{C})$ F induces a homomorphism of groups $\mathrm{Hom}_{\mathscr{C}}(X, Y) \to \mathrm{Hom}_{\mathscr{D}}(X, Y)$.

Definition 2.2.8. Let $\mathscr{T}$ be an additive category and let $\Sigma : \mathscr{T} \to \mathscr{T}$ be an additive automorphism. A *triangle* in $\mathscr{T}$ is a sequence $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ of objects and morphisms in $\mathscr{T}$.

Let $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ and $B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \Sigma B_1$ be two triangles in $\mathscr{T}$. A *morphism of triangles* is a commutative diagram.

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \Sigma A_1 \\ \downarrow{\scriptstyle \varphi_1} & & \downarrow{\scriptstyle \varphi_2} & & \downarrow{\scriptstyle \varphi_3} & & \downarrow{\scriptstyle \Sigma\varphi_1} \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \Sigma B_1 \end{array} \tag{2.11}$$

If $\varphi_1, \varphi_2, \text{and} \varphi_3$ are isomorphisms in $\mathscr{T}$, then $(\varphi_1, \varphi_2, \varphi_3)$ is called an *isomorphism of triangles.*

Definition 2.2.9. Let $\mathscr{T}$ be an additive category. Then $\mathscr{T}$, an additive automorphism Σ, and a collection Δ of distinguished triangles is a *triangulated category* if all of the following are satisfied:

1. If a triangle is isomorphic to a triangle in Δ, then it is in Δ,

2. For every $A \in \mathrm{Obj}(\mathscr{T})$ the triangle $A \xrightarrow{1} A \longrightarrow 0 \longrightarrow \Sigma A$ is in Δ,

3. For every $A_1, A_2 \in \mathrm{Obj}(\mathscr{T})$ and $\alpha \in \mathrm{Hom}_{\mathscr{T}}(A_1, A_2)$ there is a triangle in Δ of the form $A_1 \xrightarrow{\alpha} A_2 \longrightarrow A_3 \longrightarrow \Sigma A_1$,

4. For $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ in Δ, then $A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1 \xrightarrow{-\Sigma\alpha_1} \Sigma A_2$ is in Δ,

5. Given two triangles $A_1 \xrightarrow{\alpha_1} A_2 \xrightarrow{\alpha_2} A_3 \xrightarrow{\alpha_3} \Sigma A_1$ and $B_1 \xrightarrow{\beta_1} B_2 \xrightarrow{\beta_2} B_3 \xrightarrow{\beta_3} \Sigma B_1$ in Δ, each commutative diagram can be completed to morphisms of triangles.

$$\begin{array}{ccccccc} A_1 & \xrightarrow{\alpha_1} & A_2 & \xrightarrow{\alpha_2} & A_3 & \xrightarrow{\alpha_3} & \Sigma A_1 \\ \varphi_1 \downarrow & & \varphi_2 \downarrow & & \varphi_3 \downarrow & & \downarrow \Sigma\varphi_1 \\ B_1 & \xrightarrow{\beta_1} & B_2 & \xrightarrow{\beta_2} & B_3 & \xrightarrow{\beta_3} & \Sigma B_1 \end{array} \tag{2.12}$$

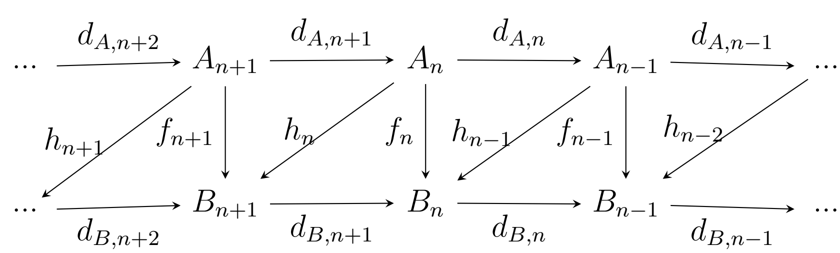
Definition 2.2.10. Let $\mathscr{C}$ be an additive category. Let $C_\bullet$ be a chain complex with boundary maps $d_{C,n} : C_n \longrightarrow C_{n-1}$. For any $k \in \mathbb{Z}$, the *k-shifted chain complex $C[k]_\bullet$* is defined by $d_{C[k],n} = (-1)^k d_{C,n+k}$ as $C[k]_n = C_{n+k}$, written

$$d_{C[k],n} : C[k]_n \longrightarrow C[k]_{n-1} \tag{2.13}$$

Definition 2.2.11. Given two chain complexes A and B, and two chain maps, $f, g : A \to B$, a *chain homotopy* is a sequence of homomorphisms $h_n : A_n \to B_{n+1}$ so that

$$f - g = hd_A + d_B h \tag{2.14}$$

This can be represented by the following diagram.



(2.15)

The map $hd_A + d_b h$ induces the zero map on homology for any h, thus f and g induce the same map on homology. The maps f and g are said to be *(chain) homotopic*.

Definition 2.2.12. The homotopic maps define an equivalence relation on the abelian groups of morphisms in the category $\mathscr{C}$ which we call the *equivalence class* of the morphisms.

Definition 2.2.13. Given an additive category of chain complexes, $\mathscr{C}$, the *homotopy category*, denoted $\mathrm{H}(\mathscr{C})$, retains the same objects as the category $\mathscr{C}$, however the morphisms are the equivalence classes of chain maps.

2.3 Complex Analysis

In this final section of definitions, we state concepts in complex analysis that will be used in Chapters 4 and 5.

Definition 2.3.1. Let f be holomorphic everywhere except at a point z_0. We say z_0 is an *isolated singularity*.

Definition 2.3.2. The *residue* of f at z_0 is defined as

$$\mathrm{Res}_{z_0} f = \frac{1}{2\pi i} \oint f(z)dz \tag{2.16}$$

In calculating the residue, we can consider the series expansion of f about z_0, written:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \text{ where } c_n = \frac{1}{2\pi i} \oint_{\Gamma} \frac{f(\zeta)d\zeta}{(\zeta - z_0)^{n+1}} \tag{2.17}$$

Thus $\mathrm{Res}_{z_0} f$ is the coefficient of $(z - z_0)^{-1}$ in the expansion, so

$$\mathrm{Res}_{z_0} f = c_{-1} = \frac{1}{2\pi i} \oint_{\Gamma} d\zeta f(\zeta) \tag{2.18}$$

This alternate residue calculation will make the implementation of the Kapustin-Li formula described in Chapter 4 easier and more explicit to calculate in SageMath.

Chapter 3

Matrix Factorizations

In this chapter, we will define matrix factorizations as well as the category which contains them as objects. Next, understanding of the homotopy category of matrix factorizations will be developed in order to understand our final portion of the chapter on the first contribution of matrix factorizations which can be referenced in [4], [7], [9], [13].

Definition 3.1.1. Let R be a commutative ring, with $\omega \in R$. A *matrix factorization* (A, B, φ, ψ), or shortened (φ, ψ), of ω in R is a diagram

$$A \underset{\psi}{\overset{\varphi}{\rightleftarrows}} B \tag{3.1}$$

for A, B finitely generated free R-modules and φ, ψ R-homomorphisms such that the following are satisfied:

$$\varphi \circ \psi = \omega \cdot \mathrm{id}_B \quad \text{and} \quad \psi \circ \varphi = \omega \cdot \mathrm{id}_A \tag{3.2}$$

Example 3.1.2. Let $R = \mathbb{C}[\![x]\!]$ and $\omega = x^n$. Considering R as the R-modules, we have the factorizations

$$R \underset{\psi}{\overset{\varphi}{\rightleftarrows}} R \tag{3.3}$$

where φ is just multiplication by x^d and ψ by x^{n-d} so that $\psi \circ \varphi = x^n$.

Remark 3.1.3. We can see for matrix factorization (φ, ψ) of ω that ω annihilates coker φ as defined in (2.10). So $\omega(\text{coker } \varphi) = 0$. This will be used in (3.1.4) to clarify a condition.

The following propositions are shown in [6]

Proposition 3.1.4. Let R be a Noetherian ring and ... $\xrightarrow{\varphi} A \xrightarrow{\psi} B \xrightarrow{\varphi} A \xrightarrow{\psi} B$ be a free resolution of finitely generated R-modules which is periodic of period 2. Then rank $A =$ rank B.

Proposition 3.1.5. Let $\omega \in R$ be a nonzero divisor, and let $\varphi : A \to B$ be a map between free modules. There exists a matrix factorization of the form (φ, ψ) if and only if

1. rank $A =$ rank B
2. det $\varphi =$ rank B, and
3. $\omega \cdot \text{Fit }_1(\varphi) \subset (\det \varphi)$

Remark 3.1.6. The third condition above refers to the fitting invariant of coker φ, denoted $\text{Fit}_1(\varphi)$. However, Eisenbud discusses that the assumption of the proposition along with the first condition of equal rank implies the annihilator of coker φ, written $\text{ann}_R(\text{coker } \varphi)$, is equivalent to $(\text{Fit}_1(\varphi) : \det \varphi)$. So for $\omega \cdot \text{Fit }_1(\varphi) \subset (\det \varphi)$ we may instead write $\omega(\text{coker } \varphi) = 0$. The proof found in [6] uses this idea of condition three.

Definition 3.1.7. A *morphism* θ between two matrix factorizations $(A_1, B_1, \varphi_1, \psi_1)$ and $(A_2, B_2, \varphi_2, \psi_2)$ of ω is a pair of maps $\alpha : A_1 \to A_2$ and $\beta : B_1 \to B_2$ such that the following diagram commutes.

$$\begin{array}{ccccc} A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\psi_1} & A_1 \\ \alpha \downarrow & & \beta \downarrow & & \alpha \downarrow \\ A_2 & \xrightarrow{\varphi_2} & B_2 & \xrightarrow{\psi_2} & A_2 \end{array} \tag{3.4}$$

The commutativity of the left side implies the commutativity of the right side of (3.4) so we can also state, though redundant, that $\alpha \circ \psi_1 = \psi_2 \circ \beta$ must also be satisfied by α and β to be a morphism between matrix factorizations.

Definition 3.1.8. Two matrix factorizations of ω are *equivalent* if α and β are isomorphisms.

Definition 3.1.9. Equivalent matrix factorizations which have non-unit maps are referred to as *reduced matrix factorizations*.

For R a regular local ring, any matrix factorization can be written, using the differential (φ, ψ), as the following direct sum with a reduced matrix factorization (φ_r, ψ_r) and $a, b \in \mathbb{Z}^{\geq 0}$

$$(\varphi, \psi) = (\varphi_r, \psi_r) \oplus (\mathrm{id}_A, \omega)^a \oplus (\omega, \mathrm{id}_B)^b \tag{3.5}$$

Specifically, if the matrix φ contains a unit, then we can write it as the sum of a reduced matrix factorization and (id_A, ω) to some power a. Similarly, if ψ contains a unit, we can write it as the sum of a reduced matrix factorization and (ω, id_B) to some power b.

Example 3.1.10. Again considering the example in one variable of the matrix factorizations of x^n, written (x^d, x^{n-d}), we see for $d \neq 0$ the matrix factorization of ω is reduced.

Now we can build the category of matrix factorizations to have the collection of matrix factorizations as objects and to let the morphisms between the matrix factorizations as defined in (3.1.7) be the morphisms of the category.

Definition 3.1.11. The *category of matrix factorizations* $MF(R, \omega)$ is the collection of matrix factorizations of ω in R and the morphisms between them.

$MF(R, \omega)$ can be observed to be an additive category with the expected zero object and direct sums as the coproduct.

Next we introduce the notion that the morphisms between matrix factorizations can be chain homotopic and the conditions required to be such. This will be used in each remaining chapter and is important to the idea of the Kapustin-Li formula as well as the major theorem in this chapter. Each component of the morphisms between matrix factorizations will have to satisfy applications of (2.14) to be homotopic.

Definition 3.1.12. Let $\theta, \theta' : (A_1, B_1, \varphi_1, \psi_1) \to (A_2, B_2, \varphi_2, \psi_2)$ be two morphisms in $MF(R, \omega)$ where $\theta = (\alpha, \beta)$ and $\theta' = (\alpha', \beta')$. Then θ, θ' are *homotopic* if there exist maps s, t which satisfy

$$\alpha - \alpha' = s \circ \varphi_1 + \psi_2 \circ t \tag{3.6}$$

$$\beta - \beta' = t \circ \psi_1 + \varphi_2 \circ s \tag{3.7}$$

seen in the following diagram.

$$\begin{array}{ccccc} A_1 & \xrightarrow{\varphi_1} & B_1 & \xrightarrow{\psi_1} & A_1 \\ {\scriptstyle \alpha', \alpha}\downarrow & \overset{s}{\swarrow} & {\scriptstyle \beta'}\downarrow{\scriptstyle \beta} & \overset{t}{\swarrow} & \downarrow{\scriptstyle \alpha', \alpha} \\ A_2 & \xrightarrow{\varphi_2} & B_2 & \xrightarrow{\psi_2} & A_2 \end{array} \tag{3.8}$$

These homotopic maps define an equivalence relation on the abelian groups of morphisms in the category $MF(R, \omega)$ and we denote the equivalence class of a morphism θ by $\left[\theta\right]$.

Next we introduce the homotopy category of matrix factorizations which has the collection of matrix factorizations as objects and those homotopic equivalence classes described above as morphisms.

Definition 3.1.13. The *homotopy category* $HMF(R, \omega)$ is the category which retains the same objects as $MF(R, x)$, but the morphisms are the homotopy equivalence classes of morphisms.

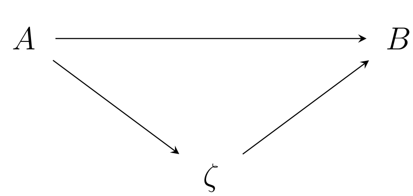
It can be observed that $HMF(R, \omega)$ is an additive category. The morphisms, which are the homotopy equivalence classes, form an abelian group and the composition is bilinear; these maps act as matrix multiplication which is bilinear. Finally, the zero object and coproduct are retained from the matrix factorization category.

Next, we provide the concluding material necessary to understand Eisenbud's theorem on matrix factorizations.

We first define a quotient which will lead to the categories used in Eisenbud's matrix factorization theorem (3.1.15).

Definition 3.1.14. Let $\mathscr{C}$ be a category with the homomorphism sets being abelian groups, and $\mathscr{A}$ be a set of objects in $\mathscr{C}$. We can now define the category $\mathscr{C}/\mathscr{A}$ as the category which retains the same objects but whose morphisms between objects A and B are the elements of the quotient $\mathrm{Hom}_{\mathscr{C}}(A, B)/\mathscr{A}(A, B)$, where $\mathscr{A}(A, B)$ are all morphisms from A to B which factor through direct sums of the objects of $\mathscr{A}$.

Letting ζ represent a direct sum of objects in $\mathscr{A}$, we have the following diagram describing the behavior of the morphisms $\mathscr{A}(A, B)$.



(3.9)

The zero objects in $\mathscr{A}$ are retained in the category $\mathscr{C}/\mathscr{A}$ which can be shown to be an additive category.

For R a regular local ring and (f) a principal ideal in R, we let $S = R/(f)$. Then we can define $\mathscr{C}(S)$ to be the category of all CM modules over S and the quotient as $\underline{\mathscr{C}}(S) = \mathscr{C}(S)/\{S^{(n)}\}$ for all n. $\underline{\mathscr{C}}(S)$ has morphisms in the quotient Hom_S over a S-submodule of Hom_S. Explicitly, let $\mathscr{B}(M, N)$ be the set of S-homomorphisms of M to N which factor through a free module, F, written as $M \to F \to N$. This $\mathscr{B}(M, N)$ is a submodule of S so the quotient is written $\underline{\mathrm{Hom}}_S(M, N) = \mathrm{Hom}_S(M, N)/\mathscr{B}(M, N)$.

We also note that the quotient $MF(R, f)/\{(\mathrm{id}_{R^{(n)}}, f)\}$ for all n is denoted $\underline{MF}(R, f)$. Considering $RMF(R, f)$, the reduced matrix factorization category, the quotient $RMF(R, f)/\{(\mathrm{id}_{R^{(n)}}, f), (f, \mathrm{id}_{R^{(n)}})\}$ is denoted $\underline{RMF}(R, f)$, where (id_R, f) and (f, id_R) are obviously not reduced matrix factorizations as discussed and seen in (3.5).

Finally, we define the additive functor coker : $\mathrm{MF}(R, f) \to \mathscr{C}(S)$, where coker (α, β) is a homomorphism of CM modules coker $(\varphi_1, \psi_1) \to$ coker (φ_2, ψ_2) and the object associated to coker (id_R, f) is the zero object and the object associated to coker (f, id_R) is S.

Now we are ready to state the following theorem. There is a proof outlined in [13] which follows.

Theorem 3.1.15. (Eisenbud's Matrix Factorization Theorem) Suppose R is a regular local ring and (f) is a principal ideal. If $S = R/(f)$ is a hypersurface, then coker induces an equivalence:

$$\underline{MF}(R, f) \cong \mathscr{C}(S) \tag{3.10}$$

Furthermore, $\underline{RMF}(R, f) \cong \underline{\mathscr{C}}(S)$.

Proof. As coker $(\mathrm{id}_R, f) = 0$, coker induces the functor $\underline{MF}(R, f) \to \mathscr{C}(S)$, which will be denoted Coker. For a nontrivial CM module M we have a free resolution of

M

$$0 \longrightarrow R^{(n)} \xrightarrow{\varphi} R^{(n)} \longrightarrow M \longrightarrow 0$$

and we can obtain (φ, ψ) which satisfies conditions in (3.2) where $\varphi \circ \psi = f \cdot \mathrm{id}_{R^{(n)}}$ and $\psi \circ \varphi = \mathrm{id}_{R^{(n)}} \cdot f$. In order to find (φ, ψ), we first note that since M is a S-module it follows that $fM = 0$, so for any $x \in R^{(n)}$ there is a unique element $y \in R^{(n)}$ where $f \cdot x = \varphi(y)$. Letting $y = \psi(y)$, then we see ψ is a linear mapping from $R^{(n)}$ to itself and therefore satisfies $\varphi \cdot \psi = f \cdot \mathrm{id}_{R^{(n)}}$. This defines a functor $F : \mathscr{C}(S) \to \underline{MF}(R, f)$. So we set $F(M) = (\varphi, \psi)$, which is determined uniquely as an object in $\underline{MF}(R, f)$ since we may neglect (id_R, f), and we note that if we make the choice for φ to be minimal and if (φ_1, ψ_1) is another matrix factorization obtained from M, then there are invertible matrices α and β such that the following is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^{n_1+1} & \xrightarrow{\gamma} & R^{n_1+1} & \xrightarrow{\delta} & M & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow & & \\ 0 & \longrightarrow & R^{n_1} & \xrightarrow{\varphi_1} & R^{n_1} & \xrightarrow{\psi_1} & M & \longrightarrow & 0 \end{array} \tag{3.11}$$

with $\gamma = \begin{pmatrix} \varphi & 0 \\ 0 & \mathrm{id}_R \end{pmatrix}$ and $\delta = \begin{pmatrix} \psi & 0 \\ 0 & f \cdot \mathrm{id}_R \end{pmatrix}$. Therefore (α, β) is a morphism from (γ, δ) to (φ_1, ψ_1). Now given a morphism $g : M_1 \to M_2$ in $\mathscr{C}(S)$, there is a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & R^{n_1} & \xrightarrow{\varphi_1} & R^{n_1} & \xrightarrow{\psi_1} & M_1 & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle g} & & \\ 0 & \longrightarrow & R^{n_2} & \xrightarrow{\varphi_2} & R^{n_2} & \xrightarrow{\psi_2} & M_2 & \longrightarrow & 0 \end{array} \tag{3.12}$$

Therefore (α, β) gives a morphism of functors $F(M_1) \to F(M_2)$, denoted $F(g)$. If we take (α', β') to be another morphism which makes (3.13) commute, then we can define a homotopy $\mu : R^{n_1} \to R^{n_2}$ such that (3.7) and (3.8) are satisfied. So the morphism $(\alpha, \beta) \to (\alpha', \beta')$ is a composition $(\mu, \mu \cdot \varphi_1) \circ (\varphi_2, \mathrm{id}_R)$, where

$(\mu, \mu \cdot \varphi_1) : (\varphi_1, \psi_1) \to (\mathrm{id}_R, f \cdot \mathrm{id}_R)$ and $(\varphi_2, \mathrm{id}_R) : (\mathrm{id}_R, f \cdot \mathrm{id}_R) \to (\varphi_2, \psi_2)$. Therefore the morphism is in $\underline{MF}(R, f)$. Thus $F(g)$ is uniquely determined. Then we can check that $F \cdot \mathrm{Coker} = \mathrm{id}_R$ and $\mathrm{Coker} \cdot F = \mathrm{id}_R$ which shows the desired equivalence $\underline{MF}(R, f) \cong \mathscr{C}(S)$. Similarly, since $\mathrm{coker}(f, \mathrm{id}_R) = S$, the second equivalence $\underline{RMF}(R, f) \cong \underline{\mathscr{C}}(S)$ follows. □

Chapter 4

Kapustin-Li Formula

Here we will first introduce the statement of the formula discovered by Kapustin and Li for the duality pairing as stated and discussed in [11]. We then follow with an accessible description of the formula.

4.1 Statement of Formula

According to Kapustin and Li, the formula provides the duality pairing on the morphism complexes in the matrix factorization category of an isolated hypersurface singularity.

We consider a regular local ring R with isolated singularity ω found by taking a maximal ideal $\underline{m}$ in R with $\omega \in \underline{m}$. A matrix factorization (A, B, φ, ψ) of ω in R in this context consists of $\mathbb{Z}/2$-graded finite free R-modules A and B equipped with an odd endomorphism d which satisfies $d^2 = \omega$. This object then corresponds to the pair of square matrices φ and ψ which can be combined into a supermatrix

$$Q = \begin{pmatrix} 0 & \varphi \\ \psi & 0 \end{pmatrix}$$

where $Q^2 = \omega \cdot \mathrm{id}$. So we see the odd endomorphism $d = \varphi \oplus \psi = (\varphi, \psi)$.

Furthermore, we take the homotopy category of matrix factorizations, denoted $HMF(R, \omega)$. For X, Y matrix factorizations, $HMF(R, \omega)(X, Y)$ denotes the morphisms, more explicitly the homotopy equivalence classes, in the homotopy category between X and Y which we have called α and β for $\left[\theta\right]$ in (3.8).

Similarly, $HMF(R,\omega)(Y,X[n])$ denotes the morphisms in the homotopy category between Y and $X[n]$. $X[n]$ denotes the shifted complex of the matrix factorization X as described in (2.13) and to be shown in the following diagram. For $X = (X^0, X^1, \varphi_X, \psi_X)$ and $Y = (Y^0, Y^1, \varphi_Y, \psi_Y)$, and for the homotopy equivalence class $\left[\theta_X\right]$, say (α_X, β_X), from $HMF(R,\omega)(X,Y)$ and for $\left[\theta_Y\right]$, (α_Y, β_Y), from $HMF(R,\omega)(Y,X[n])$ we have

$$\begin{array}{ccccc}
X^0 & \xrightarrow{\varphi_X} & X^1 & \xrightarrow{\psi_X} & X^0 \\
\alpha_X \downarrow & & \beta_X \downarrow & & \alpha_X \downarrow \\
Y^0 & \xrightarrow{\varphi_Y} & Y^1 & \xrightarrow{\psi_Y} & Y^0 \\
\alpha_Y \downarrow & & \beta_Y \downarrow & & \alpha_Y \downarrow \\
X^n & \xrightarrow[\varphi_{X[n]}]{} & X^{n+1} & \xrightarrow[\psi_{X[n]}]{} & X^n
\end{array} \tag{4.1}$$

Then we have the following formula.

Definition 4.1.1. For some F in $HMF(R,\omega)(X,Y)$ and for some G in $HMF(R,\omega)(Y,X[n])$,

$$(F,G) \longmapsto \frac{1}{(2\pi i)^n n!} \oint_{\{|\partial_i \omega| = \epsilon\}} \frac{\mathrm{tr}(FG(dQ)^{\wedge n})}{\partial_1\omega\partial_2\omega \ldots \partial_n\omega}$$

The following theorem discusses the property of the non-degeneracy. The discussion and proof can be seen in [8].

Theorem 4.1.2. The formula defined above is a non-degenerate pairing, satisfying

1. For $F \in HMF(R,\omega)(X,Y)$, if $\Phi(F,G) = 0$ for all $G \in HMF(R,\omega)(Y,X[n])$ then $F = 0$, and

2. For $G \in HMF(R,\omega)(Y,X[n])$, if $\Phi(F,G) = 0$ for all $F \in HMF(R,\omega)(X,Y)$ then $G = 0$.

Finally, letting $k = \mathbb{C}$ allows the application of path integral methods to find the pairing.

4.2 Understanding the Formula

In this section we will discuss each component of the formula to be encoded and detailed in the following chapter.

Given the morphisms, $F = (\alpha_X, \beta_X)$ and $G = (\alpha_Y, \beta_Y)$, as shown in (4.1), we form matrices with the maps as entries. The following are for the singularity in odd n variables, where G is shifted by n, and thus has a matrix representation with entries within the off diagonal.

$$F = \begin{pmatrix} \alpha_X & 0 \\ 0 & \beta_X \end{pmatrix}, \quad G = \begin{pmatrix} 0 & \alpha_Y \\ \beta_Y & 0 \end{pmatrix}$$

Then we form the matrix FG

$$FG = \begin{pmatrix} 0 & \alpha_X \circ \alpha_Y \\ \beta_X \circ \beta_Y & 0 \end{pmatrix}$$

Then we take the differential (φ_Y, ψ_Y) and form the matrix

$$Q = \begin{pmatrix} 0 & \varphi_Y \\ \psi_Y & 0 \end{pmatrix}$$

We use the differential associated to Y as it is common to both morphisms, F and G. Then the matrix Q is differentiated by taking partial derivatives of the entries in each of the n variables of ω, denoted $\partial_i \omega$ for the i^{th} variable, and wedging the forms to construct $(dQ)^{\wedge n}$, shown explicitly as

$$(dQ)^{\wedge n} = \begin{pmatrix} 0 & \frac{\mathrm{d}\varphi_Y}{\partial_1 \omega} \wedge \ldots \wedge \frac{\mathrm{d}\varphi_Y}{\partial_n \omega} \\ \frac{\mathrm{d}\psi_Y}{\partial_1 \omega} \wedge \ldots \wedge \frac{\mathrm{d}\psi_Y}{\partial_n \omega} & 0 \end{pmatrix} \tag{4.2}$$

Then matrix representations of FG and $(dQ)^{\wedge n}$ as detailed above are multiplied. With this new matrix, we take its trace. This sum will produce a multivariate polynomial which then will be divided by the product of partial derivatives of the original singularity, seen in the formula as $\partial_1\omega...\partial_n\omega$.

Finally, we are left with the need to integrate. As shown in (2.17) and (2.18), we can take the series expansion of our quotient and find the coefficient of the term with degree -1. This will be the evaluation of the residue, therefore as described in (2.16) we can rewrite our formula as follows

$$\frac{1}{(2\pi i)^n n!}\oint_{\{|\partial_i\omega|=\epsilon\}}\frac{\mathrm{tr}(FG(dQ)^{\wedge n})}{\partial_1\omega\partial_2\omega...\partial_n\omega} = (-1)^{\binom{n+1}{2}}\frac{1}{n!}\mathrm{Res}\left[\begin{matrix}\mathrm{tr}(FG(dQ)^{\wedge n})\\ \partial_1\omega,\partial_2\omega,...,\partial_n\omega\end{matrix}\right] \tag{4.3}$$

Chapter 5

SageMath Calculations

In this chapter, we will discuss the encoding of the Kapustin-Li formula using SageMath. Included will be excerpts of SageMath input and output as well as contents of Python scripts utilized as functions or methods for parts of the formula, with full references found in the Appendix.

The conditions of the formula require R be a regular local ring. For the entirety this chapter we let R be the ring of formal power series in n variables with complex coefficients. We can declare this ring in SageMath via the following the PowerSeriesRing() command, with a two-variable example shown below.

```
sage:  R = PowerSeriesRing(CC,[x,y])
```

This command outputs a description of the ring declared including the adjoined variables, the type of ring, i.e. Power Series, Polynomial, etc., and the field.

```
Multivariate Power Series Ring in x, y over Complex Field with 53 bits
of precision
```

In general, the n-variable ring can be crafted using a while loop starting at index 1 as seen below. The loop creates notation of n-variables as indexed x's for simpler code and stores them in a string to be used in the declaration of the ring R. This is useful in that there is a simple input of the number of variables desired and an output of the ring with the corresponding number of variables. This string of variables is also convenient to have as it can be called on for later calculation.

```
while index <= n:
    str = str+"x_{},".format(index)
    index = index + 1
str = str[:-1]
R = PowerSeriesRing(CC,str)
```

Similar to the previous example, the output describes the ring with the variables and the field it is over. In the case of $n = 4$, we have the following output.

```
Multivariate Power Series Ring in x_1, x_2, x_3, x_4 over Complex
Field with 53 bits of precision
```

Since the formula begins with the input of morphisms F and G, we need the singularity ω and the matrix factorization, Y, in order to construct the matrix Q. We can establish F and G by constructing matrices from their provided maps as seen in (4.1). For the $n = 1$ variable case, the computation of $(dQ)^{\wedge n} = (dQ)^{\wedge 1} = dQ$ is easy and requires only differentiation of the entries in the matrix with respect to the single variable, here considered x. At the end of this chapter we have an example of a full implementation of the formula for $n = 1$ of the form x^d and specifically for the singularity x^4. For $n \neq 1$, we will skip the computation and construction of the differential matrix $(dQ)^{\wedge n}$ and return to discuss it in Chapter 6. Now we switch to discussion of the residue calculation, given that a polynomial is the output of the trace calculation. This will utilize the taylor() command, included in SageMath, to form the Taylor expansion and the residue computation. First, we must establish the denominator of our polynomial in order to expand the expression in a Taylor series.

As seen in (4.3), the trace evaluation is divided by the product of the partial derivatives of our singularity. With these declared n variables we can compute the partial derivatives. The general case Python script contains the while loop which takes the string of variables, parsed by variable and referred to below as "vari", and takes the partial derivative then the product of it and all preceding partials. The index again begins at 1 and the product of the partial derivatives denoted "der" is initially declared 1.

```
vari = str.split(",")
while index <= n:
   partial = sing.derivative(var(vari[index-1]))
   der = der*partial
   index = index+1
```

More simply, the one variable case script contains the command to differentiate the single variable singularity. Now we are ready to begin finding the Taylor series expansion. Given the polynomial output of the trace calculation, and the computed product of the partial derivatives of the singularity, taking the quotient we can find the Taylor series expansion. For the polynomial output, called "poly", and for the product of partial derivatives found above, still called "der", we can compute the quotient, referred "quot", by simple division: `quot = poly/der`.

Then, we can use the `taylor()` command which takes 4 arguments: the function, the variables, the singularity, and the desired degree to which the terms be printed. The function to be used is our quotient, "quot". We have already established a list of variables so we implement that list "vari" for each a variable of the list in the second spot of the command. For the third argument, we want to find the solution set to each partial derivative, as seen in the formula statement in (4.1.1). Each solution set will yield values for that variable where the residue will be computed and summed for each. We can call the set an array and increment by 1 to evaluate the residue considering each entry. For the final argument, by the definition given in (2.18) we would like to find the coefficient of the degree -1 term of the series expansion so we place -1 in the final spot of our command. So to find our ith residue the command would be as follows.

```
taylor(quot,vari[a],soln[i],-1)
```

To retrieve the coefficient from this expression, we can convert the output to a string and parse the initial term. Converting each of these parsed terms back to variables allows us to take their sum and complete our calculation of the residues and thus the complex integral. Though not completed, these commands and points were the focus of the attempt to implement the formula from the trace calculation on.

We close the computation chapter with an example for the singularity $\omega = x^4$. The Python script used can be found in Appendix A. We let X be the factorization with maps (x^3, x) and Y be the factorization with maps (x^2, x^2). We can write the differentials in terms of the maps associated to X and Y, respectively.

$$X = \begin{pmatrix} 0 & x^3 \\ x & 0 \end{pmatrix}, Y = \begin{pmatrix} 0 & x^2 \\ x^2 & 0 \end{pmatrix}$$

Per the formula, the matrix Y will hold the role of Q in the computation. As the variable count is 1, the matrix factorization associated to the shifted complex of $X[1]$ is (x, x^3).

$$X[1] = \begin{pmatrix} 0 & x \\ x^3 & 0 \end{pmatrix}$$

Finally, we define F and G, consisting of maps of the respective homotopy equivalence classes, denoted (α_x, β_x) and (α_Y, β_Y) respectively as discussed in section 4.2. Here, we let F be defined by (x^2, x) and G be $(x, 1)$ so the composition of F and G, denoted FG in the formula, forms a matrix of the multiplication of the maps as entries on the off diagonal, as Q is defined.

$$FG = \begin{pmatrix} 0 & x^2 \times x \\ x \times 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & x^3 \\ x & 0 \end{pmatrix}$$

We then multiply the matrix FG by the differentiated matrix dQ, which is differentiated with respect to x by entry, associated to the factorization maps of Y

$$FG \times dQ = \begin{pmatrix} 0 & x^3 \\ x & 0 \end{pmatrix} \times \begin{pmatrix} 0 & 2x \\ 2x & 0 \end{pmatrix} = \begin{pmatrix} 2x^4 & 0 \\ 0 & 2x^2 \end{pmatrix}$$

Next, the trace of the matrix found above is computed and we form the quotient comprised of the trace output and the differentiated singularity, then we simplify the expression.

$$\frac{2x^4 + 2x^2}{4x^3} = \frac{x}{2} + \frac{1}{2x}$$

The Taylor expansion of this polynomial about $x = 0$ is the polynomial itself so we simply take the coefficient of the $\frac{1}{x}$ term as our residue. Therefore, for the singularity x^4 and for homotopy equivalence classes $F = (x^2, x)$ and $G = (x, 1)$, we have that the computed residue $\frac{1}{2}$.

Chapter 6

Conclusion

We have discussed matrix factorizations and their history, the category of matrix factorizations, and their homotopy category. We also focused on Eisenbud's Matrix Factorization Theorem as well as saw and discussed a proof. This intuition lead to a discussion of the Kapustin-Li formula as well as an effort to implement utilizing SageMath. A valuable next move would be to complete the affine application of the formula and develop a library of Python scripts for public use.

Neither the construction of the matrix representations called F, G, and the differential matrix $(dQ)^{\wedge n}$ nor the final component of residue calculation in the general $n \neq 1$ case were completed in this project, though discussed and attempted. In particular concerning the differential matrix, differential forms and the wedging of elements in SageMath requires background of their syntax concerning the context of manifolds which was not discussed in detail here. In the furthering of this project, this understanding would be crucial and greatly aid in the development and completion of the formula's calculation in this context and perhaps others.

After completion, it would also be interesting and helpful to compile a library of sample singularities and matrix factorizations for users to develop a closer understanding of the inner workings of the program. An ever-growing library could help to consider other statements and contexts leading to further study and potential research in this area. The completion of the implementation in SageMath of each portion of the formula in the context of the formal power series ring over the complex numbers would allow for quicker computation of the duality pairing and supply a referable material for other contexts.

Further research and implementation into the homological and topological understanding of the Kapustin-Li formula, and by extension matrix factorizations and their homotopy category, could be taken in many directions. Hopefully as questions are raised and solved, the impact and understanding of the formula and its applications across mathematics and other sciences will expand.

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Appendix A

One Variable x^n Case

The contents of the Python script which computes the duality pairing for any one variable case is below.

```
from sage.all import *


## Here we assume a single variable, x
## Ask for degree and the matrix factorizations X and Y, then the hom
## equiv classes, F and G
n = input("Enter the degree of the singularity:")
X1 = input("Enter the matrix factorization X, separated by a comma:")
X = matrix([[0,X1[0]],[X1[1],0]])
Y1 = input("Enter the matrix factorization Y, separated by a comma:")
Y = matrix([[0,Y1[0]],[Y1[1],0]])
Xn = matrix([[0,X1[1]],[X1[0],0]])
F1 = input("Enter the equivalence class morphisms alpha and beta for F,
separated by a comma:")
F = matrix([[F1[0],0],[0,F1[1]]])
G1 = input("Enter the equivalence class morphisms alpha and beta for G,
separated by a comma:")
G = matrix([[0,G1[0]],[G1[1],0]])
```

```
## Composes F and G into one matrix, called F, to then multiply
F = F*G

## Find the differential of Q associated to Y and multiply, then take
## trace of new matrix
dY = Y.derivative(x)
mult = F*dY
trace = mult[0,0] + mult[1,1]

## Compute the derivative of singularity to find the denominator of
## integral
sing = x**n
der = sing.derivative(x)
divide = trace/der

## Find the Taylor series expansion for the new polynomial and find
## the coefficient of -1 degree term for final residue
series = taylor(divide,x,0,-1)
set = series.operands()
l = len(set)
stset = str(set) k = len(stset)
star = stset.find('/x')
if star != -1:
   while star != -1:
      if l == 2:
         star = stset.find(' ')
         stset = stset[star+1:k-1]
```

```
        break
      stset = stset[1:star]
      star = stset.find(' ')
   fin = SR(stset)
   print(fin)
else:
   print("0")
```