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Classification of Non-Singular Cubic Surfaces up to e-invariants

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Dedication

To my late mother who gave me everything she could...and my wife and beloved daughters...I love you so much!

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Abstract

In this thesis, we use the Clebsch map to construct cubic surfaces with twenty-seven lines in $PG(3, q)$ from 6 points in general position in $PG(2, q)$ for $q = 17, 19, 23, 29, 31$. We classify the cubic surfaces with twenty-seven lines in three dimensions (up to e-invariants) by introducing computational and geometrical procedures for the classification. All elliptic and hyperbolic lines on a non-singular cubic surface in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$ are calculated. We define an operation on triples of lines on a non-singular cubic surface with 27 lines which help us to determine the exact value of the number of Eckardt point on a cubic surface. Moreover, we discuss the irreducibility of classes of smooth cubic surfaces in $PG(19, \mathbb{C})$, and we give the corresponding codimension of each class as a subvariety of $PG(19, \mathbb{C})$.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

The intensive study of a cubic surface started in 1849, when the British mathematicians Salmon and Cayley published the results of their correspondence on the number of lines on a non-singular cubic surface (see [6], Pages 118-132 and [26], Pages 252-260). Moreover, Cayley and Salmon show that a non-singular cubic surface over the complex field contains exactly twenty-seven lines. In 1858, Schläfli ([27]) found the helpful notation for the complete figure formed by these 27 lines. Clebsch constructed the famous Diagonal surface in ([7], Pages 284-345) and showed that it contained 27 real lines. In 21^{st} century, mathematicians can do even more in addition to making static models, they can use computers to manipulate them interactively. For this purpose, we have the following important theorem of Clebsch ([7], Pages 359-380).

Theorem: Every non-singular cubic surface can be represented in the plane using 4 plane cubic curves through six points in general position and vice versa.

In 1849, Cayley and Salmon showed that a general cubic surface over the complex field contains exactly 27 lines [6]. However Cayley observed that through each line of a non-singular cubic surface, there are 5 planes meeting it in two other lines and these planes are called tritangent planes. Further he showed that the equation of a non-singular cubic surface can be written as $L_1L_2L_3 + L_1'L_2'L_3' = 0$ where $L_1, L_2, L_3, L_1', L_2', L_3'$ are certain homogeneous linear polynomials in 4 variables. These homogeneous linear polynomials are associated to objects called trihedral pairs (see Section 3.2). In 1858 Schläfli introduced the double-six theorem, namely

Theorem: (Schläfli [27]) Given five skew lines a_1, a_2, a_3, a_4, a_5 with a single transversal b_6 such that each set of four a_i omitting a_j has a unique further transversal b_j, then the five lines b_1, b_2, b_3, b_4, b_5 also have a transversal a_6. These twelve lines form a double-six.

A double-six in $PG(3, k)$ is a set of 12 lines, namely

$$D: \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{array}$$

such that each line only meets the five lines which are not in the same row or column.

The main property of a double-six is that it determines a unique cubic surface with 27 lines.

There are three main related problems stated in [17], namely

1. Characterize when a double-six exists over $GF(q)$.
2. Determine the particular properties of cubic surfaces over $GF(q)$. For example, the number of Eckardt points on a line of the cubic surface, arithmetical properties of cubic surface and configuration of Eckardt points.
3. Classify the cubic surfaces with twenty-seven lines over $GF(q)$.

Problem (1) has been solved, Problem (2) has seen much progress in [18]. In this thesis, we will discuss the Problem (2) in more detail when $q > 6$ and q is prime by defining an operation on the triples of lines on a non-singular cubic surface with 27 lines. Furthermore, we will discuss the Problem (3) by classifying the non-singular cubic surfaces with 27 lines in $PG(3, q)$ up to e-invariants. In fact, every non-singular cubic surface with 27 lines in $PG(3, q)$ is a blow-up of $PG(2, q)$ at six points in general position (such a configuration of points is a 6-arc not on a conic). In general, an r-arc ($r > 3$) in $PG(2, q)$ is a set of r points no three of them on the same line. The e-invariants of a non-singular cubic surface $\mathscr{S}$ with 27 lines are e_0, e_1, e_2, e_3, where e_r denotes the number of points of $\mathscr{S}$ that lie on exactly r lines of $\mathscr{S}$. So e_3 is the number of Eckardt points of $\mathscr{S}$.

A non-singular cubic surface $\mathscr{S}$ with 27 lines is said to be of type $[e_0, e_1, e_2, e_3]$ if e_0, e_1, e_2, e_3 are the e-invariants of $\mathscr{S}$. Two non-singular cubic surfaces with 27 lines, namely $\mathscr{S}$ and $\mathscr{S}'$ are said to be equivalent if they have the same type.

For an r-arc $\mathcal{K}$ in $PG(2, q)$, we define the isotropy subgroup of $\mathcal{K}$ as follows:

$$G(\mathcal{K}) := \mathrm{PGL}_3(q)_{\mathcal{K}} = \Big\{\gamma \in \mathrm{PGL}_3(q) : \gamma(\mathcal{K}) = \mathcal{K}\Big\},$$

where $\mathrm{PGL}_3(q)$ is the projective general linear group over $GF(q)$. Moreover, two r-arcs $\mathcal{K}$ and $\mathcal{K}'$ in $PG(2, q)$ are said to be projectively equivalent if

$$\mathrm{PGL}_3(q)_{\mathcal{K}} \cong \mathrm{PGL}_3(q)_{\mathcal{K}'}.$$

In this case, two 5-arcs, namely $\mathcal{F}$ and $\mathcal{F}'$, are projectively distinct if they have different isotropy groups. Similarly, two 6-arcs, namely $\mathcal{S}$ and $\mathcal{S}'$, are projectively distinct if they have different isotropy groups.

To find the equation of a non-singular cubic surface $\mathscr{S}$ with 27 lines, we consider a 6-arcs not on a conic $\mathcal{S}$, where

$$\mathcal{S} = \{P_1, P_2, P_3, P_4, P_5, P_6\}.$$

There exists one half of a double-six on the corresponding non-singular cubic surface $\mathscr{S}$ with 27 lines, namely

$$a_1, a_2, a_3, a_4, a_5, a_6,$$

such that if $\mathcal{S}^*$ is the set of points on the lines a_i then the restriction of the Clebsch map (see Section 3.1, Section 3.6) is a bijection

$$s : \mathscr{S} \backslash \mathcal{S}^* \to PG(2, q) \backslash \mathcal{S}.$$

If we find all 15 bisecants of $\mathcal{S}$ and the six conics $\mathscr{C}_j$, where $\mathscr{C}_j$ is a conic through the 5 points of $\mathcal{S}$ except P_j, then we get 30 plane cubic curves of the form $\mathbb{V}(\mathscr{C}_j \cdot \overline{P_i P_j})$, and 15 plane cubic curves of the form $\mathbb{V}(\overline{P_i P_j} \cdot \overline{P_k P_l} \cdot \overline{P_m P_n})$.

Among the 45 plane cubics above we choose four base cubic curves through $\mathcal{S}$

$$\omega_1 = \mathbb{V}(W_1),$$

$$\omega_2 = \mathbb{V}(W_2),$$

$$\omega_3 = \mathbb{V}(W_3),$$

$$\omega_4 = \mathbb{V}(W_4).$$

The corresponding tritangent planes on $\mathscr{S}$ are chosen as

$$\pi_{\omega_1} = \mathbb{V}(\Pi_{W_1}),$$

$$\pi_{\omega_2} = \mathbb{V}(\Pi_{W_2}),$$

$$\pi_{\omega_3} = \mathbb{V}(\Pi_{W_3}),$$

$$\pi_{\omega_4} = \mathbb{V}(\Pi_{W_4}),$$

where Π_{W_j} is a linear form defining π_{ω_j} and corresponds to the cubic form W_j defining ω_j. Every tritangent plane on $\mathscr{S}$ can be written as a linear combination of $\Pi_{W_1}, \Pi_{W_2}, \Pi_{W_3}$ and Π_{W_4}.

We choose one trihedral pair among the 120 trihedral pairs, namely

$$\begin{array}{lcccl}
T_{123}: & c_{23} & b_3 & a_2 & \rightsquigarrow \pi_{\omega_1} = \mathbb{V}(\Pi_{W_1}) \\
 & a_3 & c_{13} & b_1 & \rightsquigarrow \pi_{\omega_2} = \mathbb{V}(\Pi_{W_2}) \\
 & b_2 & a_1 & c_{12} & \rightsquigarrow \pi_{\omega_i} = \mathbb{V}(\Pi_{W_i}) \\
 & \wr\downarrow & \wr\downarrow & \wr\downarrow & \\
 & \pi_{\omega_3} & \pi_{\omega_4} & \pi_{\omega_j} & \\
 & \shortparallel & \shortparallel & \shortparallel & \\
 & \mathbb{V}(\Pi_{W_3}) & \mathbb{V}(\Pi_{W_3}) & \mathbb{V}(\Pi_{W_j}) &
\end{array}$$

where $\pi_{\omega_1}, \pi_{\omega_2}, \pi_{\omega_3}, \pi_{\omega_4}$ are the 4 tritangent planes on $\mathscr{S}$ corresponding to the six plane cubics $\omega_1, \omega_2, \omega_3, \omega_4$ which pass through $\mathcal{S}$. The tritangent planes on $\mathscr{S}$, which correspond to the third row and third column, are respectively $\pi_{\omega_i} = \mathbb{V}(\Pi_{W_i})$ and $\pi_{\omega_j} = \mathbb{V}(\Pi_{W_j})$. Consequently, the equation of the non-singular cubic surface $\mathscr{S}$ is

$$\mathscr{S} = \mathbb{V}\Big(\Pi_{W_1}\Pi_{W_2}\Pi_{W_i} + \lambda\Pi_{W_2}\Pi_{W_4}\Pi_{W_j}\Big),$$

where the plane cubics W_i, W_j can be written as a linear combination of the four base cubics W_1, W_2, W_3, W_4 and λ is some non-zero element in $GF(q)$.

The classification of 5-arcs and 6-arcs (up to the group of projectivities), and the classification of the non-singular cubic surfaces with twenty-seven lines (up to e-invariants) over the finite fields of seventeen, nineteen, twenty-three, twenty-nine, and thirty-one elements are the main theme of this work. In fact, the later theme helps me to discuss the non-singular cubic surface with 27 lines and m Eckardt points.

Among the k-arcs with interesting properties are the 5-arcs and 6-arcs. In the beginning of our work, we prove that there are $2, 4, 2, 4$ and 4 projectively distinct 5-arcs in the projective planes $PG(2, q)$ for $q = 17, 19, 23, 29, 31$ respectively. Furthermore, we prove that there are $9, 10, 8, 10$ and 11 projectively distinct 6-arcs in the projective planes $PG(2, q)$ for $q = 17, 19, 23, 29, 31$ respectively. Amongst the $9, 10, 8, 10$ and 11 projectively distinct 6-arcs, in $PG(2, q)$ for $q = 17, 19, 23, 29, 31$, there are 6,8,6,7 and 9 of which do not lie on a conic respectively.

The projectively distinct 6-arcs not on a conic in the projective planes $PG(2, q)$ for $q = 17, 19, 23, 29, 31$ all correspond to non-singular cubic surfaces with twenty-seven lines in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$. These surfaces fall into equivalence classes up to e-invariants which will be defined in Section 3.3.

In Hirschfeld [17], the existence of a cubic surface which arises from a double-six over the finite field of order four was considered. In Hirschfeld [18], the existence and the properties of the cubic surfaces over the finite field of odd and even order was discussed and classified over the fields of order seven, eight, nine. Cubic surfaces with twenty-seven lines over the finite field of thirteen elements are classified in [2]. In this thesis, a non-singular cubic surfaces with twenty-seven lines over the finite field of seventeen, nineteen, twenty-three, twenty-nine, and thirty-one elements are classified up to e-invariants, and hence they are classified up to the number of Eckardt points.

For $q \leq 16$, an upper and lower bound for the number of Eckardt points on a non-singular cubic surface over $GF(q)$ are given in [16]. The exact value of maximum number of Eckardt points over $GF(13)$ is discussed in [2]. However, in our work, we count the exact value of maximum and minimum number of Eckardt points on a non-singular cubic surface in $PG(3,q)$ for $q = 17, 19, 23, 29, 31$.

When all 27 lines on a non-singular cubic surface are defined over $GF(q)$, the total number of elliptic lines on cubic surface is always even [21]. In our work, we determined which even number of elliptic lines can occur depending on some conditions related to the number of Eckardt points on cubic surface. The arithmetic of all elliptic and hyperbolic lines on a non-singular cubic surface in $PG(3,q)$ for $q = 17, 19, 23, 29, 31$ are indicated in Section 3.6.

In Chapter 4 of the thesis, we classify classes of smooth cubic surfaces with 27 lines in $PG(19,k)$ up to Eckardt points where $k = \mathbb{C}$ or $k = GF(q)$ for $q > 7$ and q is prime. By considering configurations of 6 points in general position in the projective plane $PG(2,k)$, we can describe subsets of projective space $PG(19,k)$ that correspond to non-singular cubic surfaces with m Eckardt points. Recall that a non-singular cubic surface, namely X, can be viewed as the blow up of $PG(2,k)$ at 6 points in general position. Furthermore, there are 45 tritangent planes on X. Classification of cubic surfaces with m Eckardt points, have been studied by Segre 1946 [28]. However, we give another way to classify cubic surfaces and give the possible number of Eckardt points on them. Moreover, we discuss the irreducibility of classes of smooth cubic surfaces in $PG(19,\mathbb{C})$, and we give the codimension of each class as a subvariety of $PG(19,\mathbb{C})$.

The main results in our work are

Theorem 2.3. There are respectively 2,4,2,4 and 4 projectively distinct 5-arcs in $PG(2,q)$ for $q = 17, 19, 23, 29, 31$.

Theorem 2.4. There are respectively 9,10,8,10 and 11 projectively distinct 6-arcs in $PG(2,q)$ for $q = 17, 19, 23, 29, 31$.

Theorem 3.5. Let $\mathscr{S}$ be a non-singular cubic surface with 27 lines. Then for $q = 17, 19, 23, 29, 31$, the minimal and maximum value for e_3 are given in Table 3.3.

Theorem 3.6. For $q = 17, 19, 23, 29, 31$, the possible number of elliptic lines on a non-singular cubic surface with 27 lines over $GF(q)$ are represented by the entries of Table 3.4.

Let $\mathscr{S}^{(j)}(q)$ denotes the smooth cubic surface with j Eckardt points over $GF(q)$ that corresponds to the 6-arcs $\mathcal{S}$ not on a conic in $PG(2, q)$. Then we get the following facts.

Theorem 3.7. There are $4, 7, 5, 7, 9$ distinct non-singular cubic surfaces with 27 lines (up to e-invariants) in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$ respectively, namely,

$$\mathscr{S}^{(m)}(17), m = 1, 3, 4, 6.$$

$$\mathscr{S}^{(m)}(19), m = 2, 3, 4, 6, 9, 10, 18.$$

$$\mathscr{S}^{(m)}(23), m = 1, 2, 3, 4, 6.$$

$$\mathscr{S}^{(m)}(29), m = 0, 1, 2, 3, 4, 6, 10.$$

$$\mathscr{S}^{(m)}(31), m = 0, 1, 2, 3, 4, 6, 9, 10, 18.$$

Corollary 3.1. The maximal number of Eckardt points on a non-singular cubic surfaces with 27 lines in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$, are $6, 18, 6, 10, 18$ respectively. Moreover, the minimal number of Eckardt points on a non-singular cubic surfaces with 27 lines in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$, are $1, 2, 1, 0, 0$ respectively.

Corollary 3.2. The number of elliptic lines on a non-singular cubic surfaces with 27 lines in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$ is either 0 or 12 or 16.

Corollary 3.3. For q odd prime, the number of elliptic lines on a non-singular cubic surfaces $\mathscr{S}^{(3)}(q)$ with 27 lines in $PG(3, q)$ is 12.

Corollary 3.4. For q odd prime, the number of elliptic lines on a non-singular cubic surfaces $\mathscr{S}^{(4)}(q)$ with 27 lines in $PG(3, q)$ is 12.

Corollary 3.5. For q odd prime, all the 27 lines on a non-singular cubic surfaces $\mathscr{S}^{(18)}(q)$ with 27 lines in $PG(3, q); q = 1 (\mathrm{mod}\, 3)$ are hyperbolic.

Corollary 3.6. For q odd prime, the number of elliptic lines on a non-singular cubic surfaces with 27 lines, $\mathscr{S}^{(0)}(q)$ in $PG(3,q)$, is 16.

Theorem 4.1. For $q > 7$ and q prime, any non-singular cubic surface with 27 lines $\mathscr{S}^{(0)}(q)$ is of type $[(q-10)^2+9, 27(q-9), 135, 0]$.

Theorem 4.2. For $q > 7$ and q prime, any non-singular cubic surface with 27 lines $\mathscr{S}^{(1)}(q)$ is of type $[(q-10)^2+8, 27(q-9)+3, 132, 1]$.

Theorem 4.3. For $q \geq 7$ and q prime, the only non-singular cubic surfaces with 27 lines and all points lying on those lines, i.e, surfaces of type $[0, e_1, e_2, e_3]$, are $\mathscr{S}^{(18)}(7), \mathscr{S}^{(10)}(11)$ and $\mathscr{S}^{(18)}(13)$.

Let $\mathbb{T}$ be the set of all triples of lines on a non-singular cubic surface $\mathscr{S}$. Define

$$
\begin{aligned}
c(\mathscr{S}) &:= (c_1 : \ldots : c_{20}) \in \mathbb{P}_k^{19} \\
&= \text{class of coefficients of } g \text{ as a point in } \mathbb{P}_k^{19} \\
&= \{\lambda(c_1, \ldots, c_{20}) : \lambda \in k^*\}, \\
\mathbb{S}_{sm} &:= \{c(\mathscr{S}) \in \mathbb{P}_k^{19} : \mathscr{S} \text{ is a smooth cubic surface in } \mathbb{P}_k^3\}, \\
\mathbb{T}^{(3)} &:= \{t \in \mathbb{T} : \text{lines of } t \text{ form an Eckardt point}\}, \\
\mathbb{S}^{(m)} &:= \{c(\mathscr{S}) \in \mathbb{S}_{sm} : \mathscr{S} \text{ has at least } m \text{ Eckardt points}\}, \\
\mathbb{E}^{(m,k)} &:= \{c(\mathscr{S}) \in \mathbb{S}^{(k)} : \mathscr{S} \text{ has } m \text{ Eckardt points}\} \\
\mathbb{E}^{(2)} &:= \{c(\mathscr{S}) \in \mathbb{E}^{(2,2)} : \mathscr{S} \text{ has } t_1, t_2 \in \mathbb{T}^{(3)} \text{ with one common line}\}, \\
\mathbb{E}^{(3)} &:= \{c(\mathscr{S}) \in \mathbb{E}^{(2,2)} : \mathscr{S} \text{ has } t_1, t_2 \in \mathbb{T}^{(3)} \text{ with no common line}\}.
\end{aligned}
$$

Then we get

Lemma 4.2.2. Let $T = \{t_1, t_2, t_3\}, T' = \{t'_1, t'_2, t'_3\}$ be two triads constructed by some lines on $\mathscr{S}$ where $c(\mathscr{S}) \in \mathbb{S}_{sm}$. Then T can transformed to T' via some permutations and quadratic transformations.

Proposition 4.1. Let $t_1 = (l_1 l_2 l_3), t_2 = (l'_1 l'_2 l'_3)$ and $t_3 = (l''_1 l''_2 l''_3)$ be three triples in $\mathbb{T}$. Then

1. if $t_1 \cap t_2 = l$, $t_1 \cap t_3 = l'$ and $t_2 \cap t_3 = l''$. Then l, l', l'' have common point. Furthermore, if $(l_1 l_1' l_1'')$ forms an Eckardt point, namely E, then $E \in l \cap l' \cap l''$.

2. if t_1 and t_2 form 2 Eckardt points, namely E_1, E_2 respectively, then t_3 forms another Eckardt points, namely E_3 so that E_1, E_2, E_3 are collinear.

Proposition 4.2. Let $\mathscr{S}$ be the non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(2)}$. Then

1. If $c(\mathscr{S}) \in \mathbb{E}^{(2)}$ then $\mathscr{S}$ has two Eckardt points of one of the following kinds:

   (a) $(a_\alpha b_\beta c_{\alpha\beta}), (a_{\alpha^*} b_\beta c_{\alpha^*\beta})$,

   (b) $(a_r b_s c_{rs}), (a_r b_{s^*} c_{rs^*})$,

   (c) $(a_i b_j c_{ij}), (c_{kh} c_{mn} c_{ij})$,

   (d) $(c_{xy} c_{zw} c_{pq}), (c_{xy} c_{wq} c_{pz})$,

   where $\alpha, \beta, \alpha^*, \beta^*, r, s, r^*, s^*, i, j, m, n, k, h, x, y, z, w, p, q \in \{1, \ldots, 6\}$. Furthermore, $\{(a)\} \sim \{(b)\} \sim \{(c)\} \sim \{(d)\}$.

2. If $c(\mathscr{S}) \in \mathbb{E}^{(3)}$ then $\mathscr{S}$ has two Eckardt points of one of the following kinds:

   (a) $(c_{ik} c_{jm} c_{nh}), (a_i b_j c_{ij})$,

   (b) $(c_{xy} c_{zw} c_{pq}), (c_{wp} c_{yq} c_{xz})$,

   where $i, j, m, n, k, h, x, y, z, w, p, q \in \{1, \ldots, 6\}$. Furthermore, $\{(a)\} \sim \{(b)\}$.

Proposition 4.3.

1. Any non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(3)}$ has exactly three Eckardt points.

2. Any non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(2)}$ has exactly two Eckardt points.

Corollary 4.2. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(2)}(q)$ that corresponds to $c(\mathscr{S}^{(2)}) \in \mathbb{E}^{(2)}$ is of type $[(q-10)^2+7, 27(q-9)+6, 129, 2]$.

Corollary 4.3. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(3)}(q)$ that corresponds to $c(\mathscr{S}^{(3)}) \in \mathbb{E}^{(3)}$ is of type $[(q-10)^2+6, 27(q-9)+9, 126, 3]$.

Define the following classes of $\mathbb{S}_{sm}$

$$\mathbb{E}^{(4)} := \left\{ \begin{array}{c} c(\mathscr{S}) \in \mathbb{E}^{(4,4)} : \mathscr{S} \text{ has } T \vee t \subset \mathbb{T}^{(3)} \text{ such that} \\ t \text{ has three common line with } T \end{array} \right\},$$

$$\mathbb{E}^{(6)} := \left\{ \begin{array}{c} c(\mathscr{S}) \in \mathbb{E}^{(6,4)} : \mathscr{S} \text{ has } T \vee t \subset \mathbb{T}^{(3)} \text{ such that} \\ t \text{ has one common line with } T \end{array} \right\},$$

$$\mathbb{E}^{(9)} := \left\{ \begin{array}{c} c(\mathscr{S}) \in \mathbb{E}^{(4,4)} : \mathscr{S} \text{ has } T \vee t \subset \mathbb{T}^{(3)} \text{ such that} \\ t \text{ has no common line with } T \end{array} \right\}.$$

Then we get

Proposition 4.4. Let $\mathscr{S}$ be the non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(4)}$. There are two possible kinds for the set $T \vee t$ (as in Definition 3.6).

Corollary 4.4. Any non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(4)}$ has exactly 4 Eckardt points and one triad.

Corollary 4.5. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(4)}(q)$ that corresponds to $c(\mathscr{S}^{(4)}) \in \mathbb{E}^{(4)}$ is of type $[(q-10)^2+5, 27(q-9)+12, 123, 4]$.

Proposition 4.5. Let $\mathscr{S}$ be the non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(6)}$. There are 3 possible kinds for the set $T \vee t$ (as in Definition 4.6).

Corollary 4.6. Let $\mathscr{S}$ be a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(6)}$. Then $\mathscr{S}$ has exactly 6 Eckardt points and 4 triads which contain 15 lines among the 27 lines on cubic surface.

Corollary 4.7. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(6)}(q)$ that corresponds to $c(\mathscr{S}^{(6)}) \in \mathbb{E}^{(6)}$ is of type $[(q-10)^2+3, 27(q-9)+18, 117, 6]$.

Proposition 4.6. Let $\mathscr{S}$ be a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(9)}$. There is one possible kind for the set $T \vee t$ (as in Definition 4.6).

Corollary 4.8. Let $\mathscr{S}$ be a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(9)}$. Then $\mathscr{S}$ has exactly nine Eckardt points and 12 triads.

Corollary 4.9. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(9)}(q)$ that corresponds to $c(\mathscr{S}^{(9)}) \in \mathbb{E}^{(9)}$ is of type $[(q-10)^2, 27(q-8), 108, 9]$.

Theorem 4.4. Let $\mathscr{S}$ be a non-singular cubic surface with the six triples $\{t_1, t_2, t_3, t_4, t_5, t_6\}$ mentioned in the proof of Corollary 4.6, and let t_7 be another triple on $\mathscr{S}$, that is $t_7 \in \mathbb{T}^{(3)} \backslash \{t_i : i \in \{1, \ldots, 6\}\}$. Then

I. $\mathscr{S}$ has at least 10 Eckardt points and at least 10 triads if all lines of t_7 are in common with one of the 4 triads generated by $t_1, \ldots, t_6$.

II. Otherwise, $\mathscr{S}$ has at least 18 Eckardt points and at least 42 triads.

Let $\mathbb{E}^{(10)}$ and $\mathbb{E}^{(18)}$ denote the subsets of $\mathbb{E}^{(10,10)}$ and $\mathbb{E}^{(18,10)}$ respectively that correspond to the non-singular cubic surfaces of kind I and II of Theorem 4.4 respectively. Note that according to Theorem 4.4, the two sets $\mathbb{E}^{(10,10)}$ and $\mathbb{E}^{(18,10)}$ are subsets of $\mathbb{E}^{(6,4)}$. Then we get

Corollary 4.10. The non-singular cubic surfaces that corresponds to members in $\mathbb{E}^{(10)}$ and $\mathbb{E}^{(18)}$ have exactly 10 and 18 Eckardt points respectively.

Corollary 4.11. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(10)}(q)$ that corresponds to $c(\mathscr{S}^{(10)}) \in \mathbb{E}^{(10)}$ is of type $[(q-10)^2 - 1, 27(q-8) + 3, 105, 10]$.

Corollary 4.12. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(18)}(q)$ that corresponds to $c(\mathscr{S}^{(18)}) \in \mathbb{E}^{(18)}$ is of type $[(q-10)^2 - 9, 27(q-7), 81, 18]$.

Proposition 4.7. Let $\mathscr{S}$ be a cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(4)}$ with 4 triples, namely the set $T \vee t$ mentioned in Corollary 4.4. Let $t' \in \mathbb{T}^{(3)}$ be another triple on $\mathscr{S}$ whose all lines in common with T. Then $c(\mathscr{S}) \in \mathbb{E}^{(9)}$.

Proposition 4.8. $\mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)} = \mathbb{S}^{(6)} = \mathbb{S}^{(5)}$.

Proposition 4.9. $\mathbb{E}^{(9,4)} \cup \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)} = \mathbb{S}^{(7)} = \mathbb{S}^{(8)} = \mathbb{S}^{(9)}$.

Proposition 4.10. $\mathbb{S}^{(10)} = \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}$.

Proposition 4.11.

$$\mathbb{S}^{(11)} = \mathbb{S}^{(12)} = \mathbb{S}^{(13)} = \mathbb{S}^{(14)} = \mathbb{S}^{(15)} = \mathbb{S}^{(16)} = \mathbb{S}^{(17)} = \mathbb{S}^{(18)} = \mathbb{E}^{(18,10)}.$$

Corollary 4.13. For every $k > 18$, we have $\mathbb{S}^{(k)} = \emptyset$.

Proposition 4.12. Let $s = \kappa_{123456} \in \mathbb{S}_6$ and

$$t = \{Q_1, Q_2, Q_3, \varphi_{123}(P_4), \varphi_{123}(P_5), \varphi_{123}(P_6)\}.$$

Then

$$\operatorname{blw}_s \mathbb{P}^2_X \cong \operatorname{blw}_t \mathbb{P}^2_Y.$$

In particular, if $s' \in \mathbb{S}_6$ is obtained from s via quadratic transformation, then

$$\operatorname{blw}_s \mathbb{P}^2_X \cong \operatorname{blw}_{s'} \mathbb{P}^2_X.$$

Lemma 4.3.1. Let $\mathbb{K}^{(1)} := \left\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(12, 34, 56) \neq \emptyset \right\}$. Then $\mathbb{K}^{(1)}$ is an irreducible subset of $\mathbb{S}_6$ and $\operatorname{codim} \mathbb{K}^{(1)} = 1$.

Theorem 4.8. $\mathbb{S}^{(1)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 1.

Lemma 4.3.2. Let

$$\mathbb{K}^{(2)} := \left\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(12, 34, 56) = \{P_7\} \text{ } \textit{and} \text{ } \wedge (12, 35, 46) = \{P_8\} \right\}.$$

Then $\mathbb{K}^{(2)}$ is an irreducible subset of $\mathbb{S}_6$ and $\operatorname{codim} \mathbb{K}^{(2)} = 2$.

Theorem 4.9. $\mathbb{E}^{(2)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 2.

Lemma 4.3.3. Let

$$\mathbb{K}^{(3)} := \left\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(12, 34, 56) = \{P_7\} \text{ } \textit{and} \text{ } \wedge (13, 45, 26) = \{P_8\} \right\}.$$

Then $\mathbb{K}^{(3)}$ is an irreducible subset of $\mathbb{S}_6$ and $\operatorname{codim} \mathbb{K}^{(3)} = 2$.

Theorem 4.10. $\mathbb{E}^{(3)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 2.

Corollary 4.14. $\mathbb{S}^{(1)}$ and $\mathbb{S}^{(2)}$ are closed subset of $\mathbb{S}_{sm}$. Moreover, $\mathbb{S}^{(2)}$ has two irreducible components $\mathbb{E}^{(2)}$ and $\mathbb{E}^{(3)}$ in $\mathbb{S}_{sm}$ with codimension 2.

Lemma 4.3.4. Let

$$\mathbb{K}^{(4)} := \left\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(13, 24, 56) = \{P_7\} \ \textit{and} \ l_{12}, l_{13} \ \textit{tangents to} \ \mathscr{C}_1 \right\}.$$

Then $\mathbb{K}^{(4)}$ is an irreducible subset of $\mathbb{S}_6$ and $\operatorname{codim} \mathbb{K}^{(4)} = 3$.

Theorem 4.11. $\mathbb{E}^{(4)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 3.

Lemma 4.3.5. Let

$$\mathbb{K}^{(6)} := \left\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(14, 23, 56) = \{P_7\} \ \textit{and} \ l_{12}, l_{13} \ \textit{tangents to} \ \mathscr{C}_1 \right\}.$$

Then $\mathbb{K}^{(6)}$ is an irreducible subset of $\mathbb{S}_6$ with $\operatorname{codim} \mathbb{K}^{(6)} = 3$.

Theorem 4.12. $\mathbb{E}^{(6)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 3.

Lemma 4.3.6. Let $s = \kappa_{123456} \in \mathbb{S}_6$ and define

$$\mathbb{K}^{(9)} := \left\{ \begin{array}{c} s \in \mathbb{S}_6 : \wedge(12, 34, 56) = \{P_8\}, \wedge(15, 24, 36) = \{P_7\} \ \textit{and} \\ l_{14} \ \textit{tangent to} \ \mathscr{C}_1 \ \textit{at} \ P_4 \end{array} \right\}.$$

Then $\mathbb{K}^{(9)}$ is an irreducible subset of $\mathbb{S}_6$ with codimension equal 3.

Theorem 4.13. $\mathbb{E}^{(9)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 3.

Corollary 4.15. $\mathbb{E}^{(4)}, \mathbb{E}^{(6)}$ and $\mathbb{E}^{(9)}$ are closed subset of $\mathbb{S}_{sm}$.

Lemma 4.3.7. Let $s = \kappa_{123456} \in \mathbb{S}_6$ and define

$$\mathbb{K}^{(10)} := \left\{ \begin{array}{c} s \in \mathbb{S}_6 : \wedge(12, 34, 56) = \{P_7\}; \wedge(14, 23, 56) = \{P_8\} \ \textit{and} \\ l_{12}, l_{13} \ \textit{tangents to} \ \mathscr{C}_1 \end{array} \right\}.$$

Then $\mathbb{K}^{(10)}$ is an irreducible subset of $\mathbb{S}_6$ with codimension equal 4.

Theorem 4.14. $\mathbb{E}^{(10)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 4.

Lemma 4.3.8. Let $s = \kappa_{123456} \in \mathbb{S}_6$ and define

$$\mathbb{K}^{(18)} := \left\{ \begin{array}{c} s \in \mathbb{S}_6 : \wedge(15, 24, 36) = \{P_7\}; \wedge(14, 23, 56) = \{P_8\} \ \textit{and} \\ l_{12}, l_{13} \ \textit{tangents to} \ \mathscr{C}_1 \end{array} \right\}.$$

Then $\mathbb{K}^{(18)}$ is an irreducible subset of $\mathbb{S}_6$ with codimension equal 4.

Theorem 4.15. $\mathbb{E}^{(18)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 4.

Chapter 2

Classification of 6-arcs over some finite fields

The principal aim of this chapter is to classify all the projectively distinct 6-arcs in the projective plane, namely $PG(2,q)$ for $q = 17, 19, 23, 29, 31$. A great deal of work has been done on classifying the 6-arcs in the projective plane over the Galois field of order q for $q = 2, \ldots, 9$. A detailed account of these results can be found in ([15], Pages 389-413).

This chapter is subdivided into 7 sections as follows: Section 2.1 consists of some preliminary results on Galois fields, projective plane and projective space, and projectivities. Section 2.2 and Section 2.3 deal with some aspects of k-arcs and conics in $PG(2,q)$ for later reference.

Section 2.4, Section 2.5 and Section 2.6 proceeds from the projectively distinct 5-arcs to the construction of the projectively distinct 6-arcs and their group of projectivities with the help of a computer program. In Section 2.7 we will give some facts about the blow up of the plane at six points in general position.

2.1 The fundamental theorem of projective geometry

A Galois field is a finite field with $q = p^h$ elements, where p is a prime number and h is a natural number. This field is denoted by $GF(q)$ or $\mathbb{F}_q$. The prime number p is called the characteristic of this field and it is also the smallest integer such that $px = 0$ for all x in this field.

Let $f(x)$ be an irreducible polynomial of degree h over $\mathbb{F}_p$, then

$$GF(p^h) = \mathbb{F}_{p^h} = \mathbb{F}_p[x]/\langle f(x)\rangle.$$

So

$$\mathbb{F}_{p^h} = \{\alpha_0 + \alpha_0 t + \ldots + \alpha_{h-1} t^{h-1} : \alpha_i \in \mathbb{F}_p\}.$$

If q is a prime then the elements of $\mathbb{F}_q$ can be represented as the residue classes $\pmod q$. Therefore,

$$\mathbb{F}_q = \{0, 1, \ldots, q-1\}.$$

As an example,

$$\begin{aligned}\mathbb{F}_{17} &= \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16\} \\ &= \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5, \pm 6, \pm 7, \pm 8\}.\end{aligned}$$

Here the operations addition and multiplication are performed $\pmod{17}$.

The elements of $\mathbb{F}_q$, $q = p^h$, satisfy,

$$x^q - x = 0,$$

and there exists β in $\mathbb{F}_q$ such that

$$\mathbb{F}_q = \{0, 1, \beta, \ldots, \beta^{q-2}\}$$

where β is called a primitive element or primitive root of $\mathbb{F}_q$.

The $(n+1)$-dimensional vector space $\mathbb{F}^{\oplus n+1}$ over an arbitrary field $\mathbb{F}$ and with origin $\mathbf{0}$ is denoted by $V = V(n+1, \mathbb{F})$. We define an equivalent relation on $V \backslash \{\mathbf{0}\}$ as follows: Let $X, Y \in V \backslash \{\mathbf{0}\}$ and for some basis $X = (x_0, \ldots, x_n), Y = (y_0, \ldots, y_n)$. Then we say that X is equivalent to Y and we write $X \sim Y$ if there is $t \in \mathbb{F} \backslash \{0\}$ such that $Y = tX$. The equivalence classes of previous relation are just the one-dimensional subspaces of V with the origin deleted. The set of equivalence classes is the n-dimensional projective space over $\mathbb{F}$ which is denoted by $PG(n, \mathbb{F})$ or, if $\mathbb{F} = GF(q)$, by $PG(n, q) = \mathbb{P}_q^n$. That is

$$PG(n, q) = \frac{V(n+1, q) \backslash \{\mathbf{0}\}}{\sim}.$$

A subspace of dimension m, or m-space, of $PG(n, \mathbb{F})$ is a set of points all of whose representing vectors form (together with the origin) a subspace of dimension $m+1$ of $V(n+1, \mathbb{F})$. Subspaces of dimension zero, one, two and three are respectively called a point, a line, a plane, and a solid. A subspace of dimension $n-1$ is called a prime or a hyperplane. So a prime is the set of points $P(X)$ whose vectors $X = (x_0, \ldots, x_n)$ satisfy an equation $\alpha_0 x_0 + \alpha_1 x_1 + \ldots + \alpha_n x_n = 0$ where $\alpha_i \in \mathbb{F}$. Let Π_r denote an r-space and let $\mathbb{F} = GF(q)$. If S and S' are two n-dimensional projective spaces, then a collineation $\Sigma : S \to S'$ is a bijection which preserves incidence; that is, if $\Pi_r \subseteq \Pi_s \subseteq S$ then $\Sigma(\Pi_r) \subseteq \Sigma(\Pi_s) \subseteq S'$.

For example, consider the projective plane $PG(2, 7)$, and let

$$\Sigma : PG(2,7) \to PG(2,7) \text{ defined by } (x_0, x_1, x_2) \mapsto (3x_2, x_0 + 3x_2, x_1 + 3x_2).$$

Then Σ is a collineation which preserves incidence. In fact, Σ is a bijection. Furthermore, Σ maps the line $L := \mathbb{V}(x_0 - x_2)$ to the line $\mathbb{V}(x_0 + x_1)$. Note that $P := (0 : 1 : 0) \in \mathbb{V}(x_0 - x_2)$ and $\Sigma(P) = (0 : 0 : 1) \in \Sigma(L) = \mathbb{V}(x_0 + x_1)$.

If the point $P(X)$ is the equivalence class of the vector X, then we will say that X is a vector representing $P(X)$. Then, a projectivity $\Sigma : S \to S'$ is a bijection given by a non-singular matrix T; if $P(X') = \Sigma(P(X))$, then $tX' = XT$ where X' and X are coordinate vectors for $P(X')$ and $P(X)$ respectively, and $t \in \mathbb{F}\backslash\{0\}$. Write $\Sigma = M(T)$, then $\Sigma = M(\lambda T)$ for any $\lambda \in \mathbb{F}\backslash\{0\}$.

For instance, the collineation in previous example, namely Σ, is a projectivity and $\Sigma = M(\lambda T)$ where

$$T = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & 3 & 3 \end{pmatrix}.$$

Recall that the projective linear group $PGL(n, q)$ is the group of projectivities of $PG(n-1, q)$. The general linear group $GL(n, q)$ is the group of all non-singular matrices of $V(n, q) = V(n, \mathbb{F})$ where $\mathbb{F} = GF(q)$. Write $Z(G)$ for the center of the group G. Let $Z = Z(GL(n,q))$. Then $Z = \left\{tI_n : t \in \mathbb{F}\backslash\{0\}\right\}$, I_n is the identity of $GL(n,q)$. Then $PGL(n,q)$ is isomorphic to $GL(n,q)/Z$.

Theorem 2.1. ([15], Pages 30-31)[The fundamental theorem of projective geometry] Let S be a projective subspace of $PG(n,q)$, then

1. If $\Sigma' : S \to S$ is a collineation, then $\Sigma' = \sigma\Sigma$ where σ is an automorphism of the field and Σ is a projectivity. In particular if $\mathbb{F} = GF(p^h)$ and $P(X') = \Sigma'(P(X))$ then there exists m in $\{1, 2, \ldots, h\}$, t_{ij} in $\mathbb{F}$ such that for i, j in $\{0, 1, 2, \ldots, n\}$, and t in $\mathbb{F}_0$ we have

$$tX' = X^{p^m}T$$
$$\text{where } X^{p^m} = (x_0^{p^m}, \ldots, x_n^{p^m})$$
$$\text{and } T = (t_{ij}), i, j \in \{0, 1, 2, \ldots, n\};$$
$$\text{that is, } tx'_i = x_0^{p^m}t_{0i} + \ldots + x_n^{p^m}t_{ni}.$$

2. If $\{P_1, \ldots, P_{n+2}\}$, $\{P'_1, \ldots, P'_{n+2}\}$ are sets of $n+2$ points of $PG(n, \mathbb{F})$ such that no $n+1$ points chosen from the same set lie in a prime (or, in the language of k-arcs, the two sets form an $(n+2)$-arc), then there exists a unique projectivity Σ such that $P'_i = \Sigma(P_i)$, for all i in $\{1, 2, \ldots, n+2\}$.

For example, consider the points

$$P_1 := (1:0:0),\ P_2 := (0:1:0),\ P_3 := (0:0:1) \text{ and } P_4 := (1:1:1),$$

in $PG(2,7)$. The unique projectivity Σ_i which respectively maps the points P_1, P_2, P_3 and P_4 into the points $P'_1 := \Sigma_i(P_1)$, $P'_2 := \Sigma_i(P_2)$, $P'_3 := \Sigma_i(P_3)$ and $P'_4 := \Sigma_i(P_4)$ is shown in Table 2.1.

2.2 Projective plane and k-arcs

Recall from Section 2.1, $PG(n,q)$ is defined as the n-dimensional projective space over $GF(q)$. In $PG(n,q)$ there is principle of duality, that is, there is a dual space

Table 2.1: The unique projectivity Σ_i

i	$\Sigma_i((x_0 : x_1 : x_2)) =$	$P_1,\ P_2,\ P_3,\ P_4 \mapsto$
1	$(x_0 : x_1 : x_2)$	$P_1,\ P_2,\ P_3,\ P_4$
2	$(x_2 : x_2 - x_0 : x_2 - x_1)$	$P_2,\ P_3,\ P_4,\ P_1$
3	$(x_1 - x_2 : x_1 : x_1 - x_0)$	$P_3,\ P_4,\ P_1,\ P_2$
4	$(x_0 - x_1 : x_0 - x_2 : x_0)$	$P_4,\ P_1,\ P_2,\ P_3$
5	$(x_1 : x_0 : x_2)$	$P_2,\ P_1,\ P_3,\ P_4$
6	$(x_2 - x_0 : x_2 : x_2 - x_1)$	$P_1,\ P_3,\ P_4,\ P_2$
7	$(x_1 : x_1 - x_2 : x_2 - x_0)$	$P_3,\ P_4,\ P_2,\ P_1$
8	$(x_0 - x_2 : x_0 - x_1 : x_0)$	$P_4,\ P_2,\ P_1,\ P_3$
9	$(x_2 : x_1 : x_0)$	$P_3,\ P_2,\ P_1,\ P_4$
10	$(x_2 - x_1 : x_2 - x_0 : x_2)$	$P_2,\ P_1,\ P_4,\ P_3$
11	$(x_1 - x_0 : x_1 : x_1 - x_2)$	$P_1,\ P_4,\ P_3,\ P_2$
12	$(x_0 : x_0 - x_2 : x_0 - x_1)$	$P_4,\ P_3,\ P_2,\ P_1$
13	$(x_0 : x_0 - x_1 : x_0 - x_2)$	$P_4,\ P_2,\ P_3,\ P_1$
14	$(x_2 : x_0 : x_1)$	$P_2,\ P_3,\ P_1,\ P_4$
15	$(x_2 - x_1 : x_2 : x_2 - x_0)$	$P_3,\ P_1,\ P_4,\ P_2$
16	$(x_1 - x_0 : x_1 - x_2 : x_1)$	$P_1,\ P_4,\ P_2,\ P_3$
17	$(x_0 : x_2 : x_1)$	$P_1,\ P_3,\ P_2,\ P_4$
18	$(x_2 : x_2 - x_1 : x_2 - x_0)$	$P_3,\ P_2,\ P_4,\ P_1$
19	$(x_1 - x_2 : x_1 - x_0 : x_1)$	$P_2,\ P_4,\ P_1,\ P_3$
20	$(x_0 - x_1 : x_0 : x_0 - x_2)$	$P_4,\ P_1,\ P_3,\ P_2$
21	$(x_2 - x_0 : x_2 - x_1 : x_2)$	$P_1,\ P_2,\ P_4,\ P_3$
22	$(x_1 : x_1 - x_0 : x_1 - x_2)$	$P_2,\ P_4,\ P_3,\ P_1$
23	$(x_0 - x_2 : x_0 : x_0 - x_1)$	$P_4,\ P_3,\ P_1,\ P_2$
24	$(x_1 : x_2 : x_0)$	$P_3,\ P_1,\ P_2,\ P_4$

$PG(n,q)^*$ whose points and primes are respectively the primes and points of $PG(n,q)$. For any theorem true in $PG(n,q)$, there is an equivalent theorem true in $PG(n,q)^*$. In particular, if **T** is a theorem in $PG(n,q)$ stated in terms of points, primes, and incidence, the same theorem is true in $PG(n,q)^*$ and gives a dual theorem $\mathbf{T}^*$ in $PG(n,q)$ by interchanging (point) and (prime) whenever they occur. Thus (join) and (meet) are dual. Hence the dual of an r-space in $PG(n,q)$ is an $(n-r-1)$-space.

In particular, in $PG(2,q)$, point and line are dual; in the projective space $PG(3,q)$ point and plane are dual, whereas the dual of a line is a line in 3-dimensional projective space ([15], Page 31).

In the projective plane $PG(2,q)$, each point P is joined to the remaining points by a pencil which consists of $q+1$ lines; each of these lines contains P and q other points. Hence the plane contains

$$q(q+1)+1 = q^2+q+1$$

points, and by duality a plane contains q^2+q+1 lines. The integer q is called the order of the plane.

A set $\mathcal{K}$ of $r > 3$ points in a finite projective plane such that no three of them are collinear is called an r-arc. A line containing exactly one point of $\mathcal{K}$ is called a tangent or a unisecant of $\mathcal{K}$. A line containing two points of $\mathcal{K}$ is called a bisecant of $\mathcal{K}$, and a line that does not contain any points of $\mathcal{K}$ will be called an exterior line of $\mathcal{K}$.

For example, in $PG(2,7)$, the bisecants, the tangents and the exterior lines of a 4-arc, namely

$$\mathcal{K} := \{(1:0:0),(0:1:0),(0:0:1),(1:1:1)\},$$

are shown in Table2.2.

An r-arc containing $q+1$ points for q odd and $q+2$ points for q even, is called an oval.

A subset $\mathscr{V}$ of $PG(n,q)$ is a variety over $GF(q)$ if there exist homogenous polynomials $F_1,\ldots,F_m$ in $GF(q)[x_0,\ldots,x_n]$ such that

$$\begin{aligned}\mathscr{V} &= \{P(A) \in PG(n,q) : F_1(A) = \ldots = F_m(A) = 0\}\\ &:= \mathbb{V}(F_1,\ldots,F_m),\end{aligned}$$

where $GF(q)[x_0,\ldots,x_n]$ is the polynomial ring in $x_0,\ldots,x_n$ over $GF(q)$ and the points $P(A)$ are the points of $\mathscr{V}$.

A variety $\mathbb{V}(F)$ determined by one homogenous polynomial is called a hypersurface. A hypersurface in $PG(2,q)$ is called a curve, and hypersurface in $PG(3,q)$ is called a surface. The degree of a hypersurface is the degree of F.

Table 2.2: The bisecants, tangents and exterior lines of a 4-arc $\mathcal{K}$

lines of $\mathcal{K}$		tangents of $\mathcal{K}$	bisecants of $\mathcal{K}$
$\mathbb{V}(x_0+x_1+x_2)$	$\mathbb{V}(x_0-x_1-3x_2)$	$\mathbb{V}(x_0+3x_1)$	$\mathbb{V}(x_0)$
$\mathbb{V}(x_0-x_1-x_2)$	$\mathbb{V}(x_0-3x_1-2x_2)$	$\mathbb{V}(x_1+3x_2)$	$\mathbb{V}(x_1)$
$\mathbb{V}(x_0+3x_1-x_2)$	$\mathbb{V}(x_0+2x_1-2x_2)$	$\mathbb{V}(x_0+x_1-2x_2)$	$\mathbb{V}(x_2)$
$\mathbb{V}(x_0+2x_1+2x_2)$	$\mathbb{V}(x_0+2x_1+3x_2)$	$\mathbb{V}(x_0+2x_2)$	$\mathbb{V}(x_0-x_2)$
$\mathbb{V}(x_0+3x_1+x_2)$	$\mathbb{V}(x_0-2x_1+2x_2)$	$\mathbb{V}(x_0-3x_1+2x_2)$	$\mathbb{V}(x_0-x_1)$
$\mathbb{V}(x_1-x_2)$	$\mathbb{V}(x_0-x_1-2x_2)$	$\mathbb{V}(x_0-3x_2)$	$\mathbb{V}(x_0+x_1+x_2)$
$\mathbb{V}(x_0-x_1+x_2)$	$\mathbb{V}(x_0+x_1-x_2)$	$\mathbb{V}(x_0+2x_1-3x_2)$	
$\mathbb{V}(x_0-x_1+3x_2)$	$\mathbb{V}(x_0-2x_1-x_2)$	$\mathbb{V}(x_0-3x_1)$	
$\mathbb{V}(x_0-2x_1-3x_2)$	$\mathbb{V}(x_0+3x_1-3x_2)$	$\mathbb{V}(x_1-3x_2)$	
$\mathbb{V}(x_0-3x_1+x_2)$	$\mathbb{V}(x_0-3x_1+3x_2)$	$\mathbb{V}(x_0-2x_2)$	
$\mathbb{V}(x_0+x_1+3x_2)$	$\mathbb{V}(x_0-2x_1+3x_2)$	$\mathbb{V}(x_0+3x_2)$	
$\mathbb{V}(x_0-2x_1-2x_2)$	$\mathbb{V}(x_0+x_1+2x_2)$	$\mathbb{V}(x_0-2x_1+x_2)$	
$\mathbb{V}(x_0+2x_1-x_2)$		$\mathbb{V}(x_0+2x_1)$	
$\mathbb{V}(x_0+3x_1-2x_2)$		$\mathbb{V}(x_1+2x_2)$	
$\mathbb{V}(x_0+x_1-3x_2)$		$\mathbb{V}(x_0+x_1)$	
$\mathbb{V}(x_0-3x_1-3x_2)$		$\mathbb{V}(x_1+x_2)$	
$\mathbb{V}(x_0-3x_1-x_2)$		$\mathbb{V}(x_0+3x_1+3x_2)$	
$\mathbb{V}(x_0+3x_1+2x_2)$		$\mathbb{V}(x_0-2x_1)$	
$\mathbb{V}(x_0+2x_1+x_2)$		$\mathbb{V}(x_1-2x_2)$	

2.3 Quadrics and conics

A hypersurface of degree two in $PG(n,q)$ is called a quadric. Let $\mathscr{Q}$ be a quadric. Then $\mathscr{Q}=\mathbb{V}(Q)$ where Q is a quadratic form; that is,

$$\begin{aligned} Q &= \sum_{i,j=0}^{n} a_{ij}x_ix_j \\ &= a_{00}x_0^2 + a_{01}x_0x_1 + \dots \qquad . \end{aligned}$$

When $p \neq 2$, write $t_{ij}=t_{ji}=(a_{ij}+a_{ji})/2$. Therefore

$$\begin{aligned} Q &= \sum_{i,j=0}^{n} t_{ij}x_ix_j \\ &= XTX^* \text{ where } T=(t_{ij}). \end{aligned}$$

If there is a change of coordinate system which reduces the form into a form of fewer variables, then the form is called degenerate. So a quadric is degenerate if and only if it is singular.

Theorem 2.2. ([15], Pages 106-107) In $PG(n,q)$, the number of projectively distinct non-singular quadrics is one or two as n is even or odd. They have the following canonical forms.

1. $n = 2m, m \geq 0$:

$$\mathscr{P}_{2m} = \mathbb{V}(x_0^2 + x_1x_2 + x_3x_4 + \ldots + x_{2m-1}x_{2m});$$

2. $n = 2m-1, m \geq 1$:

$$\mathscr{H}_{2m-1} = \mathbb{V}(x_0x_1 + x_2x_3 + \ldots + x_{2m-2}x_{2m-1}),$$
$$\mathscr{E}_{2m-1} = \mathbb{V}\Big(f(x_0,x_1) + x_2x_3 + x_4x_5 + \ldots + x_{2m-2}x_{2m-1}\Big),$$

where f is any irreducible binary quadratic form. The quadrics $\mathscr{P}_{2m}$, $\mathscr{H}_{2m-1}$ and $\mathscr{E}_{2m-1}$ are called parabolic, hyperbolic and elliptic respectively.

When $m = 1$ in the above theorem, $\mathscr{P}_2$ is a conic.

Let $Q := Q(2,q)$ be the set of quadrics in $PG(2,q)$. Then

$$Q = \Big\{\mathbb{V}(F) : F = a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{12}x_1x_2; a_{ij} \in \mathbb{F}_q\Big\}.$$

So the number of quadrics in $PG(2,q)$ is

$$|Q(2,q)| := \theta(5,q) = \frac{q^6-1}{q-1} = q^5 + q^4 + q^3 + q^2 + q + 1.$$

The quadric $\mathbb{V}(F)$ is a conic when it is non-singular. The $\theta(5,q)$ quadrics of $Q(2,q)$ fall into four orbits under the projective general linear group, $PGL(3,q)$ as follows:

1. Singular F:

   a) $\mathbb{V}(F) = \mathbb{V}(x_0^2)$,

   b) $\mathbb{V}(F) = \mathbb{V}(x_0x_1)$,

   c) $\mathbb{V}(F) = \mathbb{V}(x_0^2 + \alpha x_0x_1 + \beta x_1^2)$;

2. Non-singular, $\mathbb{V}(F) = \mathbb{V}(x_0^2 + x_1x_2)$.

Case (a) is a double line, the number of which is q^2+q+1. Case (b) is a pair of distinct lines defined over $GF(q)$, the number of which is $q(q+1)(q^2+q+1)/2$. Case (c) is a pair of lines defined over $GF(q^2)$ whose point of intersection is defined over $GF(q)$, the number of which is $q(q-1)(q^2+q+1)/2$. So the total number of reducible conics is $(q^2+q+1)(q^2+1)$. Hence the number of irreducible conics is

$$\Big(q^5+q^4+q^3+q^2+q+1\Big)-\Big((q^2+q+1)(q^2+1)\Big)=q^5-q^2,$$

which is the number of conics in orbit (2). Each conic contains $q+1$ points. Note that no 3 points of these $q+1$ points are collinear.

A conic is determined by the ratios of the coefficients $(a_{00}, a_{11}, a_{22}, a_{01}, a_{02}, a_{12})$. Here is some important results on conics which can be found in ([15], Pages 140-143):

1. Denote the number of n-arcs by $L(n,q)$, and let N be the number of conics which are $(q+1)$-arcs, then

   $$N = L(5,q)/\binom{q+1}{5} = q^5 - q^2.$$

   So all irreducible conics comprise $q+1$ points.

2. The number of 6-arcs in $PG(2,q)$ which do not lie on a conic is equal to

   $$L(5,q)\Big(\ell^*(5,q)-(q-4)\Big)/6,$$

   where $\ell^*(n,q)$ denotes the number points which are not on any bisecants of n-arcs in $PG(2,q)$.

3. If $\mathscr{F}=\mathbb{V}(F)=\mathbb{V}(a_{00}x_0^2+a_{11}x_1^2+a_{22}x_2^2+a_{01}x_0x_1+a_{02}x_0x_2+a_{12}x_1x_2)$, then $\mathscr{F}$ is singular if and only if $\Delta=0$, where

   $$\Delta = 4a_{00}a_{11}a_{22}+a_{01}a_{02}a_{12}-a_{00}a_{12}^2-a_{11}a_{02}^2-a_{22}a_{01}^2.$$

   $\mathscr{F}$ is singular at $Q=P(y_0,y_1,y_2)$ if

   $$\frac{\partial F}{\partial x_0}=\frac{\partial F}{\partial x_1}=\frac{\partial F}{\partial x_2}=0 \text{ at } Q.$$

4. In $PG(2,q)$ with $q \geq 4$ there is a unique conic through a 5-arc.

5. If a conic contains at least one point, it contains exactly $q+1$.

6. Every conic in $PG(2,q)$ is a $(q+1)$-arc.

2.4 Construction of 6-arcs

By the definition of a r-arc, given a 5-arc $\mathcal{F}$, a 6-arc is constructed by adding a point to $\mathcal{F}$ which is not on any bisecant of $\mathcal{F}$. The 6-arcs constructed in this way are not necessarily all projectively distinct. However, in this chapter we give all the projectively distinct 6-arcs and their groups of projectivities. Moreover, we will distinguish between the ones that lie on a conic and the ones that do not lie on a conic.

Let $\mathcal{S} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ be a 6-arc in $PG(2,q)$. If the three lines $\overline{P_iP_j}, \overline{P_kP_l}$ and $\overline{P_mP_n}, i \neq j \neq k \neq l \neq m \neq n$, meet at a point B, then B is called a Brianchon point, or a B-point for short. We write

$$(ij, kl, mn) = \overline{P_iP_j} \cap \overline{P_kP_l} \cap \overline{P_mP_n}$$

for a B-point. Thus there are the following 15 possibilities for the Brianchon point B, namely:

1. $(12, 34, 56)$	6. $(13, 26, 45)$	11. $(15, 24, 36)$
2. $(12, 35, 46)$	7. $(14, 23, 56)$	12. $(15, 26, 34)$
3. $(12, 36, 45)$	8. $(14, 25, 36)$	13. $(16, 23, 45)$
4. $(13, 24, 56)$	9. $(14, 26, 35)$	14. $(16, 24, 35)$
5. $(13, 25, 46)$	10. $(15, 23, 46)$	15. $(16, 25, 34)$.

Definition 2.1. For an k-arc $\mathcal{K}$ in $PG(2,q)$ we define the isotropy subgroup of $\mathcal{K}$ as follows:

$$G(\mathcal{K}) := \mathrm{PGL}_3(q)_{\mathcal{K}} = \Big\{\gamma \in \mathrm{PGL}_3(q) : \gamma(\mathcal{K}) = \mathcal{K}\Big\}.$$

Moreover, two n-arcs $\mathcal{K}$ and $\mathcal{K}'$ in $PG(2,q)$ are said to be projectively equivalent if $\mathrm{PGL}_3(q)_{\mathcal{K}} \cong \mathrm{PGL}_3(q)_{\mathcal{K}'}$.

Now, let $\mathcal{F}$ be a 5-arc in $PG(2,q)$. According to our computer program (see Algorithm 2 in appendix), we have the set of c_0 points $Q_i, i = 1, \ldots, c_0$ which are on no bisecant of the 5-arc $\mathcal{F}$. Note that the set $S = \Big\{\mathcal{S}_i := \mathcal{F} \cup \{Q_i\} : i = 1, \ldots, c_0\Big\}$ forms a collection of 6-arcs in $PG(2,q)$. Let $\mathcal{S}_i, \mathcal{S}_j \in S$, and let $G(\mathcal{S}_i) := \mathrm{PGL}_3(q)_{\mathcal{S}_i}$, $G(\mathcal{S}_j) := \mathrm{PGL}_3(q)_{\mathcal{S}_j}$. Then $\mathcal{S}_i, \mathcal{S}_j$ are projectively equivalent if $G(\mathcal{S}_i) \cong G(\mathcal{S}_i)$, otherwise we say $\mathcal{S}_i, \mathcal{S}_j$ are projectively distinct.

Henceforth, we write $\mathbf{1}, \mathbf{2}, \mathbf{3}$, and $\mathbf{4}$ to denote the points, namely $(1:0:0), (0:1:0), (0:0:1)$, and $(1:1:1)$ respectively. By using our computer program (see Algorithm 1 in appendix), we get all the projectively distinct 5-arcs in $PG(2,q)$ for $q = 17, 19, 23, 29, 31$, as shown in tables, namely Table 2.1, Table 2.2, Table 2.3, Table 2.4, and Table 2.5.

For example, let us consider the 4-arc $\mathcal{A} = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$ in $PG(2,17)$. The number of points which do not lie on any bisecant of $\mathcal{A}$ is $\ell^*(4,q) = (q-2)(q-3) = 210$. These points partitioned into two sets, namely $\mathcal{B}$ and $\mathcal{B}'$ such that $G(\mathcal{A} \cup \{Q\}) \cong \mathbb{Z}/(2)$ for any $Q \in \mathcal{B}$ and $G(\mathcal{A} \cup \{Q'\}) \cong \mathbb{Z}/(4)$ for any $Q' \in \mathcal{B}'$ where the points of $\mathcal{B}'$ are shown Table2.3.

Table 2.3: Points of $\mathcal{B}'$

(1:7:9)	(1:11:4)	(1:16:13)	(1:14:2)	(1:14:13)	(1:4:16)	(1:13:7)	(1:2:14)
(1:4:13)	(1:13:4)	(1:7:14)	(1:9:7)	(1:5:4)	(1:11:9)	(1:14:5)	(1:13:16)
(1:9:11)	(1:16:4)	(1:7:11)	(1:5:11)	(1:2:5)	(1:11:5)	(1:13:14)	(1:7:13)
(1:11:7)	(1:14:7)	(1:5:2)	(1:5:14)	(1:4:11)	(1:4:5)		

and the points the of $\mathcal{B}$ are shown Table2.4.

If we denote to the points $(1:7:10)$ and $(1:7:9)$ by $\mathbf{6}$ and $\mathbf{7}$ respectively, we

Table 2.4: Points of $\mathcal{B}$

(1:7:10)	(1:2:8)	(1:10:14)	(1:15:5)	(1:13:11)	(1:8:7)	(1:12:4)	(1:16:11)	(1:8:11)
(1:11:13)	(1:3:6)	(1:11:14)	(1:15:2)	(1:14:9)	(1:2:15)	(1:5:9)	(1:2:6)	(1:3:16)
(1:10:2)	(1:14:12)	(1:6:3)	(1:4:2)	(1:3:12)	(1:6:16)	(1:9:15)	(1:9:16)	(1:9:5)
(1:10:15)	(1:5:7)	(1:12:5)	(1:13:9)	(1:15:12)	(1:6:8)	(1:10:4)	(1:16:15)	(1:15:3)
(1:6:4)	(1:16:6)	(1:11:8)	(1:10:7)	(1:12:9)	(1:2:13)	(1:3:5)	(1:13:3)	(1:4:6)
(1:4:3)	(1:3:9)	(1:2:4)	(1:16:14)	(1:15:4)	(1:16:5)	(1:13:6)	(1:11:12)	(1:6:5)
(1:16:9)	(1:5:10)	(1:15:14)	(1:15:7)	(1:12:13)	(1:3:8)	(1:10:12)	(1:10:13)	(1:3:4)
(1:6:13)	(1:3:13)	(1:3:7)	(1:8:10)	(1:7:15)	(1:5:12)	(1:6:9)	(1:2:7)	(1:12:6)
(1:14:8)	(1:10:9)	(1:2:11)	(1:8:15)	(1:5:16)	(1:12:15)	(1:5:15)	(1:16:8)	(1:10:8)
(1:11:16)	(1:9:4)	(1:14:10)	(1:11:10)	(1:7:16)	(1:9:6)	(1:11:6)	(1:2:12)	(1:6:11)
(1:2:9)	(1:2:3)	(1:4:7)	(1:12:11)	(1:12:10)	(1:7:5)	(1:16:10)	(1:7:12)	(1:6:2)
(1:8:14)	(1:15:11)	(1:8:4)	(1:16:2)	(1:9:3)	(1:8:12)	(1:6:7)	(1:12:16)	(1:9:12)
(1:6:14)	(1:3:10)	(1:7:2)	(1:12:2)	(1:14:4)	(1:16:7)	(1:12:7)	(1:12:14)	(1:15:16)
(1:14:16)	(1:8:13)	(1:9:10)	(1:7:4)	(1:16:3)	(1:4:15)	(1:4:10)	(1:7:8)	(1:10:5)
(1:13:10)	(1:8:16)	(1:9:14)	(1:15:8)	(1:10:3)	(1:4:14)	(1:15:6)	(1:11:15)	(1:8:2)
(1:4:9)	(1:13:8)	(1:10:16)	(1:9:13)	(1:3:2)	(1:14:6)	(1:15:13)	(1:3:14)	(1:15:9)
(1:15:10)	(1:7:6)	(1:11:3)	(1:2:10)	(1:3:15)	(1:5:13)	(1:3:11)	(1:8:5)	(1:13:12)
(1:5:8)	(1:10:11)	(1:8:3)	(1:4:8)	(1:6:10)	(1:7:3)	(1:13:15)	(1:14:15)	(1:5:6)
(1:8:6)	(1:9:8)	(1:12:8)	(1:4:12)	(1:12:3)	(1:13:5)	(1:16:12)	(1:11:2)	(1:10:6)
(1:8:9)	(1:14:11)	(1:6:12)	(1:9:2)	(1:13:2)	(1:14:3)	(1:2:16)	(1:6:15)	(1:5:3)

get the projectively distinct 5-arcs, namely

$$\mathcal{F}_1 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}\}$$

$$\mathcal{F}_2 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{7}\}.$$

The group of projectivities of $\mathcal{F}_1$ consists of two elements, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -3 & -8 & -6 \\ 0 & -8 & -8 \\ 0 & -8 & 8 \end{pmatrix}.$$

The group of projectivities of $\mathcal{F}_2$ consists of 4 elements, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 2 & 0 & -2 \\ -3 & 3 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 2 \\ -1 & 4 & -3 \\ -1 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & 0 & 1 \\ 0 & 6 & 1 \\ 8 & 0 & 1 \end{pmatrix}.$$

Thus we have two projectively distinct 5-arcs (see Table 2.5).

A 6-arc is constructed by adding a point to a 5-arc $\mathcal{F}_i$ which is not on any bisecant of $\mathcal{F}_i$. In fact, the 6-arcs constructed in this manner are not necessarily all projectively distinct. However, we will give all projectively distinct 6-arcs in the next section.

Table 2.5: Projectively distinct 5-arcs in $PG(2,17)$

$\mathcal{F}_i$ in $PG(2,17)$	$\lvert G(\mathcal{F}_i)\rvert$	$G(\mathcal{F}_i)$
$\mathcal{F}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{F}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{7}\}$	4	$\mathbb{Z}/(4)$

where $\mathbf{6} = (1:7:-7)$, and $\mathbf{7} = (1:7:-8)$.

Table 2.6: Projectively distinct 5-arcs in $PG(2,19)$

$\mathcal{F}_i$ in $PG(2,19)$	$\lvert G(\mathcal{F}_i)\rvert$	$G(\mathcal{F}_i)$
$\mathcal{F}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{F}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{7}\}$	1	Trivial
$\mathcal{F}_3 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{13}\}$	10	D_5
$\mathcal{F}_4 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{48}\}$	6	$\mathfrak{S}_3$

where $\mathbf{6} = (1:-5:-8), \mathbf{7} = (1:-5:-7), \mathbf{13} = (1:-5:6)$, and $\mathbf{48} = (1:8:7)$.

Table 2.7: Projectively distinct 5-arcs in $PG(2,23)$

$\mathcal{F}_i$ in $PG(2,23)$	$\lvert G(\mathcal{F}_i)\rvert$	$G(\mathcal{F}_i)$
$\mathcal{F}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{F}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{11}\}$	1	Trivial

where $\mathbf{6} = (1:9:-10), \mathbf{11} = (1:-1:-5)$.

Table 2.8: Projectively distinct 5-arcs in $PG(2,29)$

$\mathcal{F}_i$ in $PG(2,29)$	$\lvert G(\mathcal{F}_i)\rvert$	$G(\mathcal{F}_i)$
$\mathcal{F}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{F}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{7}\}$	1	Trivial
$\mathcal{F}_3 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{25}\}$	4	$\mathbb{Z}/(4)$
$\mathcal{F}_4 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{185}\}$	10	D_5

where $\mathbf{6} = (1:11:-13), \mathbf{7} = (1:11:-5), \mathbf{25} = (1:-1:-12), \mathbf{185} = (1:5:-4)$.

Table 2.9: Projectively distinct 5-arcs in $PG(2,31)$

$\mathcal{F}_i$ in $PG(2,31)$	$\lvert G(\mathcal{F}_i)\rvert$	$G(\mathcal{F}_i)$
$\mathcal{F}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{F}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{7}\}$	1	Trivial
$\mathcal{F}_3 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{88}\}$	10	D_{10}
$\mathcal{F}_4 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{121}\}$	6	$\mathfrak{S}_3$

where $\mathbf{6} = (1:9:-9), \mathbf{7} = (1:9:11), \mathbf{88} = (1:-12:-11), \mathbf{121} = (1:5:-6)$.

Theorem 2.3. There are respectively 2,4,2,4 and 4 projectively distinct 5-arcs in $PG(2,q)$ for $q = 17, 19, 23, 29, 31$.

Proof. Our program used in this section have been used to prove the theorem. The projectively distinct 5-arcs in $PG(2,q)$ with their automorphism groups, where $q = 17, 19, 23, 29$ and 31, are shown in Table 2.5, Table 2.6, Table 2.7, Table 2.8 and Table 2.9 respectively. □

2.5 The projectively distinct 6-arcs

In this section, we will give the arithmetic method used in the computer program for determining the projectively distinct 6-arcs in $PG(2,q)$ for $q = 17, 19, 23, 29, 31$, and their groups of projectivities (see Algorithm 2 in appendix).

Let $PG(2,q)$ be the 2-dimensional projective space over a field $GF(q)$ where q odd. We consider 6-arcs with the same number of B-points since any two 6-arcs with different numbers of B-points are projectively distinct.

Let $\mathcal{S} = \{X_1, X_2, X_3, X_4, X_5, X_6\}$ and $\mathcal{S}' = \{Y_1, Y_2, Y_3, Y_4, Y_5, Y_6\}$ be two 6-arcs, where the coordinates of the points X_i and Y_i are

$$X_i = \big(x_i(0), x_i(1), x_i(2)\big)$$
$$\text{and } Y_i = \big(y_i(0), y_i(1), y_i(2)\big).$$

Let $\mathcal{S}$ and $\mathcal{S}'$ have the same number of B-points. Then according to the fundamental theorem of projective geometry, there exists a unique projectivity which takes any set of four points of the 6-arc $\mathcal{S}$ to a set of four points of $\mathcal{S}'$. Let the 3×3 matrix be $A = (a_{ij}), i, j = 1, 2, 3,$ take a fixed set of four points of the 6-arcs $\mathcal{S}$, namely $\{X_1, X_2, X_3, X_4\}$ to any set of four points of $\mathcal{S}'$, namely $\{Y_1, Y_2, Y_3, Y_4\}$. The 6-arcs $\mathcal{S}$ and $\mathcal{S}'$ are said to be projectively equivalent if $AX_i = Y_j, i, j = 5, 6$; that is if the 3×3 projectivity matrix A takes the fifth point X_5 and sixth point X_6 of $\mathcal{S}$ to the corresponding points of $\mathcal{S}'$, namely the points Y_5, and Y_6. The following conditions must be satisfied for $\mathcal{S}$ and $\mathcal{S}'$ to be projectively equivalent:

1. if $A(X_5) = Y_5$, then we must have $A(X_6) = Y_6$;

2. if $A(X_5) = Y_6$, then we must have $A(X_6) = Y_5$.

To determine A, we fix a set of four points of $\mathcal{S}$. Then we work out the projectivity matrix A' that takes the fixed set of four points of $\mathcal{S}$ to one of $\binom{6}{4} \cdot (4!)$ set of four points of $\mathcal{S}'$. Note that there are $\binom{6}{4} = 15$ sets of four points, each of which has 4! unordered sets of four points. Therefore there are 360 matrices A' to be checked. Now A' is the projectivity matrix A if the conditions (1) and (2) above are satisfied for the two remaining points of $\mathcal{S}$ and $\mathcal{S}'$.

At this stage of our research the help of a computer was needed. For this a program was written in Fortran to classify all 6-arcs in $PG(2, q)$ for $q = 17, 19, 23, 29, 31$.

The following is the matrix arithmetic to determine the matrix A'.

Let

$$\mathcal{X} = \begin{pmatrix} x_1(0) & x_2(0) & x_3(0) \\ x_1(1) & x_2(1) & x_3(1) \\ x_1(2) & x_2(2) & x_3(2) \end{pmatrix} \quad \text{and} \quad \mathcal{Y} = \begin{pmatrix} y_1(0) & y_2(0) & y_3(0) \\ y_1(1) & y_2(1) & y_3(1) \\ y_1(2) & y_2(2) & y_3(2) \end{pmatrix}$$

Now the (3×3) matrix A' is such that it takes three points X_i of $\mathcal{S}$ to three points Y_i of $\mathcal{S}'$, when $i = 1, 2, 3$. So

$$A'\mathcal{X} = \mathcal{Y}(\lambda_1 \;\; \lambda_2 \;\; \lambda_3)^t.$$

Thus

$$A' = \mathcal{Y}(\lambda_1 \;\; \lambda_2 \;\; \lambda_3)^t \mathcal{X}^{-1} \tag{2.5.1}$$

where $\lambda_1, \lambda_2, \lambda_3 \in GF(q)\backslash\{0\}$. The matrix A' also has to take the fourth points $A'(X_4) = Y_4$. So Equation (2.5.1) gives

$$\mathcal{Y}(\lambda_1 \;\; \lambda_2 \;\; \lambda_3)^t \mathcal{X}^{-1} X_4 = Y_4 \tag{2.5.2}$$

Let

$$\mathcal{X}^{-1}X_4 = (\gamma_1 \;\; \gamma_2 \;\; \gamma_3)^t,$$

where $\gamma_1, \gamma_2, \gamma_3 \in GF(q)\backslash\{0\}$, and let

$$\Gamma = \begin{pmatrix} \gamma_1 & 0 & 0 \\ 0 & \gamma_2 & 0 \\ 0 & 0 & \gamma_3 \end{pmatrix}.$$

Thus Equation (2.5.2) can be written as

$$\mathcal{Y}\Gamma(\lambda_1 \;\; \lambda_2 \;\; \lambda_3)^t = Y_4.$$

So

$$(\lambda_1 \;\; \lambda_2 \;\; \lambda_3)^t = \Gamma^{-1}\mathcal{Y}^{-1}Y_4.$$

Substituting the values of λ_i in Equation (2.5.1), we will have the matrix A'.

There is also a program that checks whether or not the two remaining points of each 6-arc are mapped to each other under the matrix A'.

Let $L(6, q)$ denotes the number of 6-arcs in $PG(2, q)$, and let $G(\mathcal{S}_i)$ denotes the group of projectivities of the corresponding 6-arc $\mathcal{S}_i$.

In the tables, namely Table 2.17, Table 2.18, Table 2.19, Table 2.20, and Table 2.21, all the 6-arcs $\mathcal{S}$ in $PG(2, q)$ for $q = 17, 19, 23, 29, 31$ are classified up to their groups of projectivities, and the number of 6-arcs, $L(6, q)$ for $q = 17, 19, 23, 29, 31$, are indicated.

For example, let us consider the 5-arc $\mathcal{F}_1 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}\}$ in $PG(2, 17)$. The number of points which do not lie on any bisecant of $\mathcal{F}_1$ is $\ell^*(5, q) = (q-4)(q-5)+1 = 157$. These points partitioned into seven sets, namely $\mathcal{B}_1, \mathcal{B}_2, \mathcal{B}_3, \mathcal{B}_4, \mathcal{B}_5, \mathcal{B}_6$ and $\mathcal{B}_7$ such that

$$G(\mathcal{F}_1 \cup \{Q\}) \cong I \text{ for every } Q \in \mathcal{B}_1,$$

$$G(\mathcal{F}_1 \cup \{Q\}) \cong \mathbb{Z}/(2) \text{ for every } Q \in \mathcal{B}_2,$$

$$G(\mathcal{F}_1 \cup \{Q\}) \cong \mathbb{Z}/(2) \times \mathbb{Z}/(2) \text{ for every } Q \in \mathcal{B}_3,$$

$$G(\mathcal{F}_1 \cup \{Q\}) \cong \mathbb{Z}/(3) \text{ for every } Q \in \mathcal{B}_4,$$

$$G(\mathcal{F}_1 \cup \{Q\}) \cong \mathbb{Z}/(4) \text{ for every } Q \in \mathcal{B}_5,$$

$$G(\mathcal{F}_1 \cup \{Q\}) \cong \mathfrak{S}_3 \text{ for every } Q \in \mathcal{B}_6,$$

$$G(\mathcal{F}_1 \cup \{Q\}) \cong \mathfrak{A}_3 \text{ for every } Q \in \mathcal{B}_7,$$

where $\mathfrak{S}_3$ denotes the symmetric group on 3 letters, and $\mathfrak{A}_3$ denotes the alternating group on 3 letters. In fact, the points of $\mathcal{B}_2$ are shown in Table 2.10.

Table 2.10: Points of $\mathcal{B}_2$

(1:13:11)	(1:12:4)	(1:16:13)	(1:3:16)	(1:14:2)	(1:9:15)	(1:9:5)	(1:13:7)	(1:4:6)
(1:15:14)	(1:15:7)	(1:10:12)	(1:10:13)	(1:16:12)	(1:8:12)	(1:6:14)	(1:14:4)	(1:15:16)
(1:9:11)	(1:16:3)	(1:10:5)	(1:13:8)	(1:11:5)	(1:4:8)	(1:3:9)	(1:14:16).	

The points of $\mathcal{B}_3$ are shown in Table 2.11.

Table 2.11: Points of $\mathcal{B}_3$

(1:8:11)	(1:6:8)	(1:2:3).

The points of $\mathcal{B}_4$ are shown in Table 2.12.

Table 2.12: Points of $\mathcal{B}_4$

(1:8:6)	(1:6:16)	(1:6:13)	(1:10:9)	(1:16:8)	(1:10:8)	(1:16:2)	(1:13:16)	(1:6:7)
(1:10:16)	(1:5:8)	(1:12:7).						

The points of $\mathcal{B}_5$ are shown in Table 2.13.

Table 2.13: Points of $\mathcal{B}_5$

(1:6:3)	(1:3:13)	(1:4:7).

The points of $\mathcal{B}_6$ are shown in Table 2.14.

Table 2.14: Points of $\mathcal{B}_6$

(1:10:15)	(1:11:8)	(1:6:9)	(1:5:16).

Table 2.15: Points of $\mathcal{B}_7$

(1:16:7).

The points of $\mathcal{B}_7$ are shown in Table 2.15.

The points of $\mathcal{B}_1$ are shown in Table 2.16.

Table 2.16: Points of $\mathcal{B}_1$

(1:2:8)	(1:10:14)	(1:8:7)	(1:16:11)	(1:11:13)	(1:3:6)	(1:11:14)	(1:14:9)	(1:5:9)
(1:11:4)	(1:10:2)	(1:4:2)	(1:14:13)	(1:3:12)	(1:5:3)	(1:4:16)	(1:9:16)	(1:12:8)
(1:2:14)	(1:15:12)	(1:10:4)	(1:5:14)	(1:15:3)	(1:4:12)	(1:6:4)	(1:16:6)	(1:2:13)
(1:13:3)	(1:11:7)	(1:12:3)	(1:4:3)	(1:2:4)	(1:16:14)	(1:15:4)	(1:16:5)	(1:13:6)
(1:6:5)	(1:13:5)	(1:16:9)	(1:12:13)	(1:3:8)	(1:3:7)	(1:2:7)	(1:12:6)	(1:11:2)
(1:8:15)	(1:9:7)	(1:12:15)	(1:5:15)	(1:5:4)	(1:9:4)	(1:9:6)	(1:11:9)	(1:2:12)
(1:12:11)	(1:6:2)	(1:14:11)	(1:8:14)	(1:15:11)	(1:8:4)	(1:14:5)	(1:9:3)	(1:4:11)
(1:9:12)	(1:6:12)	(1:14:7)	(1:12:2)	(1:12:14)	(1:9:2)	(1:8:13)	(1:16:4)	(1:4:15)
(1:9:14)	(1:15:8)	(1:10:3)	(1:15:6)	(1:11:15)	(1:5:11)	(1:8:2)	(1:4:9)	(1:2:5)
(1:14:6)	(1:13:14)	(1:15:13)	(1:15:9)	(1:2:16)	(1:11:3)	(1:3:15)	(1:5:13)	(1:3:11)
(1:13:12)	(1:6:15)	(1:10:11)	(1:4:5)	(1:13:15)	(1:5:2)	(1:14:15)	(1:5:6)	(1:2:6)
(1:13:9)	(1:3:5)	(1:11:12)	(1:14:8)	(1:2:9)	(1:12:16)	(1:8:16)	(1:3:2)	(1:8:5).

If we denoted to the points

$$(1:2:8),(1:13:11),(1:8:11),(1:8:6),(1:6:3),(1:10:15) \text{ and } (1:16:7)$$

by $\mathbf{8},\mathbf{13},\mathbf{17},\mathbf{18},\mathbf{35},\mathbf{50}$ and $\mathbf{207}$ respectively, we get the projectively distinct 6-arcs which are construct from the 5-arc $\mathcal{F}_1$, namely

$$\mathcal{S}_1=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{8}\},$$

$$\mathcal{S}_2=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{13}\},$$

$$\mathcal{S}_3=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{17}\},$$

$$\mathcal{S}_4=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{18}\},$$

$$\mathcal{S}_5=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{35}\},$$

$$\mathcal{S}_6=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{50}\},$$

$$\mathcal{S}_7=\{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{207}\}.$$

The group of projectivities of $\mathcal{S}_1$ consists of one element, while the group of projectivities of $\mathcal{S}_2$ consists of two elements, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \text{ and } \begin{pmatrix} -3 & -8 & -6 \\ 0 & -8 & -8 \\ 0 & -8 & 8 \end{pmatrix}.$$

The group of projectivities of $\mathcal{S}_3$ consists of 4 elements, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 1 & 0 & -1 \\ 1 & -1 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 8 & -1 & 0 \\ -6 & 0 & -1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 7 & 0 & 1 \\ -7 & 1 & 0 \end{pmatrix}.$$

The group of projectivities of $\mathcal{S}_4$ consists of 3 elements, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 8 & 0 & -7 \\ 5 & 0 & -5 \\ -5 & -4 & -8 \end{pmatrix} \text{ and } \begin{pmatrix} 6 & -5 & 0 \\ 6 & -6 & 7 \\ 6 & 4 & 0 \end{pmatrix}.$$

The group of projectivities of $\mathcal{S}_5$ consists of 4 elements, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} -1 & 0 & 1 \\ 0 & -1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & -1 & 2 \\ -8 & 0 & -3 \\ 0 & 0 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 0 & -8 & -8 \\ 4 & 0 & 3 \\ 0 & 0 & -7 \end{pmatrix}.$$

The group of projectivities of $\mathcal{S}_6$ consists of 6 elements, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 0 & 1 \\ 0 & -7 & 0 \\ -2 & 0 & 0 \end{pmatrix}, \begin{pmatrix} -3 & -8 & -6 \\ 0 & -8 & -8 \\ 0 & -8 & 8 \end{pmatrix},$$

$$\begin{pmatrix} -4 & 4 & 1 \\ 6 & 4 & 0 \\ -6 & 4 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 6 & -6 \\ 0 & -8 & -8 \\ 4 & 5 & 8 \end{pmatrix} \text{ and } \begin{pmatrix} 3 & -2 & 0 \\ 4 & -3 & 0 \\ -4 & 4 & 1 \end{pmatrix}.$$

The group of projectivities of $\mathcal{S}_7$ consists of 12 elements, namely

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 & 1 \\ 7 & 0 & 0 \\ 0 & -7 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ 7 & 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} -3 & -8 & -6 \\ 0 & -8 & -8 \\ 0 & -8 & 8 \end{pmatrix},$$

$$\begin{pmatrix} -8 & -8 & 0 \\ -5 & 8 & -3 \\ 5 & -5 & 0 \end{pmatrix}, \quad \begin{pmatrix} -8 & 6 & 2 \\ -8 & 8 & 0 \\ -8 & -8 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & -8 & -8 \\ 0 & 8 & -8 \\ -4 & -5 & -8 \end{pmatrix}, \quad \begin{pmatrix} -3 & 1 & 3 \\ -4 & 0 & 3 \\ 4 & 0 & 3 \end{pmatrix},$$

$$\begin{pmatrix} 8 & -8 & 0 \\ -8 & -8 & 0 \\ 5 & -8 & 3 \end{pmatrix}, \quad \begin{pmatrix} 0 & 6 & -6 \\ 3 & 8 & 6 \\ 0 & -8 & -8 \end{pmatrix}, \quad \begin{pmatrix} 3 & 0 & -2 \\ -3 & 1 & 3 \\ 4 & 0 & -3 \end{pmatrix}, \quad \begin{pmatrix} 4 & 0 & -3 \\ 4 & 0 & 3 \\ 4 & -7 & -4 \end{pmatrix}.$$

See Table 2.17. for the projectively distinct 6-arcs in $PG(2,17)$.

Table 2.17: Projectively distinct 6-arcs in $PG(2,17)$

$L(6,17) = 318261467328$		
$\mathcal{S}_i$	$\|G(\mathcal{S}_i)\|$	$G(\mathcal{S}_i)$
$\mathcal{S}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{8}\}$	1	Trivial
$\mathcal{S}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{13}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{S}_3 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{17}\}$	4	$\mathbb{Z}/(2)\times\mathbb{Z}/(2)$
$\mathcal{S}_4 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{18}\}$	3	$\mathbb{Z}/(3)$
$\mathcal{S}_5 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{35}\}$	4	$\mathbb{Z}/(4)$
$\mathcal{S}_6 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{50}\}$	6	$\mathfrak{S}_3$
$\mathcal{S}_7 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{207}\}$	12	$\mathfrak{A}_4$
$\mathcal{S}_8 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{7},\mathbf{213}\}$	24	$\mathfrak{S}_4$
$\mathcal{S}_9 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{24},\mathbf{29}\}$	12	D_6

where

$$\begin{array}{lll} \mathbf{8} = (1:2:8), & \mathbf{13} = (1:-4:-6), & \mathbf{17} = (1:8:-6), \\ \mathbf{35} = (1:6:3), & \mathbf{50} = (1:-7:-2), & \mathbf{207} = (1:-1:7), \\ \mathbf{18} = (1:8:6), & \mathbf{213} = (1:9:-6), & \\ \mathbf{24} = (1:2:-2), & \text{and} & \mathbf{29} = (1:3:-1). \end{array}$$

Table 2.18: Projectively distinct 6-arcs in $PG(2,19)$

$L(6,19) = 1349831775312$		
$\mathcal{S}_i$	$\lvert G(\mathcal{S}_i)\rvert$	$G(\mathcal{S}_i)$
$\mathcal{S}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{14}\}$	1	Trivial
$\mathcal{S}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{32}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{S}_3 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{33}\}$	6	$\mathfrak{S}_3$
$\mathcal{S}_4 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{50}\}$	3	$\mathbb{Z}/(3)$
$\mathcal{S}_5 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{84}\}$	4	$\mathbb{Z}/(2)\times\mathbb{Z}/(2)$
$\mathcal{S}_6 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{137}\}$	4	$\mathbb{Z}/(4)$
$\mathcal{S}_7 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{285}\}$	12	$\mathfrak{A}_4$
$\mathcal{S}_8 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{13},\mathbf{137}\}$	60	$\mathfrak{A}_5$
$\mathcal{S}_9 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{48},\mathbf{260}\}$	36	9 elements of order 2 8 elements of order 3
$\mathcal{S}_{10} = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{16},\mathbf{38}\}$	12	$D_6 = \mathbb{Z}_2\times\mathfrak{S}_3$

where

$\mathbf{14} = (1:9:-1)$, $\mathbf{32} = (1:-6:2)$, $\mathbf{33} = (1:6:9)$,
$\mathbf{84} = (1:2:3)$, $\mathbf{137} = (1:6:-5)$, $\mathbf{285} = (1:3:-5)$,
$\mathbf{50} = (1:-3:-6)$, $\mathbf{260} = (1:9:8)$, $\mathbf{38} = (1:5:9)$,
$\mathbf{48} = (1:8:7)$ and $\mathbf{16} = (1:9:5)$.

Table 2.19: Projectively distinct 6-arcs in $PG(2,23)$

$L(6,23) = 15637818086968$		
$\mathcal{S}_i$	$\lvert G(\mathcal{S}_i)\rvert$	$G(\mathcal{S}_i)$
$\mathcal{S}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{8}\}$	1	Trivial
$\mathcal{S}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{9}\}$	3	$\mathbb{Z}/(3)$
$\mathcal{S}_3 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{30}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{S}_4 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{33}\}$	6	$\mathfrak{S}_3$
$\mathcal{S}_5 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{102}\}$	4	$\mathbb{Z}/(4)$
$\mathcal{S}_6 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{136}\}$	12	$\mathfrak{A}_4$
$\mathcal{S}_7 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{386}\}$	4	$\mathbb{Z}/(2)\times\mathbb{Z}/(2)$
$\mathcal{S}_8 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{80},\mathbf{114}\}$	12	$D_6 = \mathbb{Z}/(2)\times\mathfrak{A}_3$

where

$\mathbf{8} = (1:4:5)$, $\mathbf{9} = (1:8:6)$, $\mathbf{30} = (1:4:11)$,
$\mathbf{102} = (1:-4:-5)$, $\mathbf{136} = (1:6:9)$, $\mathbf{386} = (1:-8:-11)$,
$\mathbf{33} = (1:-10:11)$, $\mathbf{80} = (1:-1:-8)$, $\mathbf{114} = (1:-11:8)$.

Table 2.20: Projectively distinct 6-arcs in $PG(2,29)$

$L(6,29) = 292771335510108$		
$\mathcal{S}_i$	$\lvert G(\mathcal{S}_i)\rvert$	$G(\mathcal{S}_i)$
$\mathcal{S}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{8}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{S}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{9}\}$	1	Trivial
$\mathcal{S}_3 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{117}\}$	3	$\mathbb{Z}/(3)$
$\mathcal{S}_4 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{183}\}$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathcal{S}_5 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{342}\}$	6	$\mathfrak{S}_3$
$\mathcal{S}_6 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{562}\}$	4	$\mathbb{Z}/(4)$
$\mathcal{S}_7 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{589}\}$	12	$\mathfrak{A}_4$
$\mathcal{S}_8 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{25},\mathbf{700}\}$	24	$\mathfrak{S}_4$
$\mathcal{S}_9 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{185},\mathbf{758}\}$	60	$\mathfrak{A}_5$
$\mathcal{S}_{10} = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{57},\mathbf{100}\}$	12	$D_6 = \mathbb{Z}_2 \times \mathfrak{S}_3$

where

$$\begin{array}{ccc} \mathbf{57} = (1:-1:-3), & \mathbf{8} = (1:-2:-3), & \mathbf{9} = (1:-4:11), \\ \mathbf{183} = (1:2:3), & \mathbf{342} = (1:-13:-5), & \mathbf{562} = (1:14:-8), \\ \mathbf{700} = (1:-12:9) & \mathbf{758} = (1:-4:5) & \mathbf{117} = (1:-12:-7), \\ \mathbf{589} = (1:-12:11), & \text{and} & \mathbf{100} = (1:14:-1). \end{array}$$

Theorem 2.4. There are respectively 9,10,8,10 and 11 projectively distinct 6-arcs in $PG(2,q)$ for $q = 17, 19, 23, 29, 31$.

Proof. Our program used in this section has been used to prove the theorem. The projectively distinct 6-arcs in $PG(2,q)$ with their automorphism groups, where $q = 17, 19, 23, 29$ and 31, are respectively shown in Table 2.17, Table 2.18, Table 2.19, Table 2.20 and Table 2.21. □

2.6 The 6-arcs not on a conic in $PG(2,q)$

In this section, we give an arithmetic method for determining the equation of the conic through a set of five points of a 6-arc not on a conic. Our program is based on the following general algebraic method of solving $n-1$ equations with n unknowns. Recall if we have $n-1$ equations (see Equation(2.6.1))

Table 2.21: Projectively distinct 6-arcs in $PG(2,31)$

$L(6,31) = 675469393962720$		
$\mathcal{S}_i$	$\|G(\mathcal{S}_i)\|$	$G(\mathcal{S}_i)$
$\mathcal{S}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{8}\}$	1	Trivial
$\mathcal{S}_2 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{21}\}$	2	$\mathbb{Z}/(2)$
$\mathcal{S}_3 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{259}\}$	4	$\mathbb{Z}/(4)$
$\mathcal{S}_4 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{315}\}$	6	$\mathfrak{S}_3$
$\mathcal{S}_5 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{383}\}$	3	$\mathbb{Z}/(3)$
$\mathcal{S}_6 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{605}\}$	4	$\mathbb{Z}_2 \times \mathbb{Z}_2$
$\mathcal{S}_7 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{954}\}$	12	$\mathfrak{A}_4$
$\mathcal{S}_8 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{7},\mathbf{290}\}$	5	$\mathbb{Z}/(5)$
$\mathcal{S}_9 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{88},\mathbf{576}\}$	60	$\mathfrak{A}_5$
$\mathcal{S}_{10} = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{121},\mathbf{553}\}$	36	9 elements of order 2 8 elements of order 3
$\mathcal{S}_{11} = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{202},\mathbf{271}\}$	12	$D_6 = \mathbb{Z}_2 \times \mathfrak{S}_3$

where

$$\begin{array}{ccc}
\mathbf{202} = (1:15:-15), & \mathbf{121} = (1:5:-6), & \mathbf{8} = (1:-14:-10), \\
\mathbf{259} = (1:3:4), & \mathbf{315} = (1:8:-10), & \mathbf{383} = (1:-1:10), \\
\mathbf{954} = (1:-1:9), & \mathbf{290} = (1:13:14), & \mathbf{576} = (1:-13:-12), \\
\mathbf{271} = (1:-15:-14), & \mathbf{21} = (1:14:-2), & \mathbf{88} = (1:-12:-11) \\
\mathbf{605} = (1:2:13), & \text{and} & \mathbf{553} = (1:-6:5).
\end{array}$$

$$\begin{aligned}
a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= 0 \\
a_{21}x_1 + a_{22}x_2 + \ldots + a_{1n}x_n &= 0 \\
\vdots \qquad \vdots \qquad \qquad \vdots & \\
a_{n-1,1}x_1 + a_{n-1,2}x_1 + \ldots + a_{n-1,n}x_n &= 0,
\end{aligned} \tag{2.6.1}$$

and $A = (a_{ij})$ is the $(n-1) \times n$ matrix representing the system (2.6.1), then we have

$$x_i = (-1)^{i-1} \det A_i,$$

where A_k is an $(n-1 \times n-1)$ matrix obtained from A by omitting the k-th column.

Now consider a conic $\mathscr{C}$ in $PG(2,q)$ of the form

$$\mathscr{C} = \mathbb{V}(a_{00}x_0^2 + a_{11}x_1^2 + a_{22}x_2^2 + 2a_{01}x_0x_1 + 2a_{02}x_0x_2 + 2a_{12}x_1x_2).$$

If $\mathscr{C}$ passes through a five points $P_i = P(x_{i1}, x_{i2}, x_{i3})$ for $i \in \{1,2,3,4,5\}$, then we have the following five equations:

$$\begin{array}{ccccc} a_{00}x_{11}^2 + a_{11}x_{12}^2 + a_{22}x_{13}^2 + 2a_{01}x_{11}x_{12} + 2a_{02}x_{11}x_{13} + 2a_{12}x_{12}x_{13} = 0, \\ \vdots \qquad\qquad \vdots \qquad\qquad\qquad \vdots \qquad\qquad \vdots \\ a_{00}x_{51}^2 + a_{11}x_{52}^2 + a_{22}x_{53}^2 + 2a_{01}x_{51}x_{52} + 2a_{02}x_{51}x_{53} + 2a_{12}x_{52}x_{53} = 0. \end{array}$$

Consider the 5×6 matrix $\mathcal{X}$ which is the same as matrix A above with $n = 6$, namely

$$\mathcal{X} = \begin{pmatrix} x_{11}^2 & x_{12}^2 & x_{13}^2 & 2x_{11}x_{12} & 2x_{11}x_{13} & 2x_{12}x_{13} \\ x_{21}^2 & x_{22}^2 & x_{23}^2 & 2x_{21}x_{22} & 2x_{21}x_{23} & 2x_{22}x_{23} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{51}^2 & x_{52}^2 & x_{53}^2 & 2x_{51}x_{52} & 2x_{51}x_{53} & 2x_{52}x_{53} \end{pmatrix}.$$

Let $\mathcal{X}_k$ be a 5×5 matrix obtained from $\mathcal{X}$ by omitting the k-th column. So we have:

$$\alpha_i = (-1)^{i-1} \det \mathcal{X}_i,$$

where $\alpha_1 = a_{00}, \alpha_2 = a_{11}, \alpha_3 = a_{22}, \alpha_4 = 2a_{01}, \alpha_5 = 2a_{02}$ and $\alpha_6 = 2a_{12}$.

All the conics $\mathscr{C}_j$ passing through set of five points of the projectively distinct 6-arcs $\mathcal{S}_k$ in $PG(2,q)$ for $q = 17, 19, 23, 29, 31$, are given in tables, namely Table 2.11, Table 2.12, Table 2.13, Table 2.14, and Table 2.15. The conics $\mathscr{C}_j$ in $PG(2,q)$ passing through five points of a 6-arc $\mathcal{S}_k$, namely $\mathcal{S}_k \backslash \{P_j\}(q)$, are given in the form

$$\mathbb{V}\Big(\sum_{i,j=0}^{2} a_{ij}x_ix_j\Big) = \mathbb{V}(XTX^*).$$

Let us consider the 6-arc $\mathcal{S}_1 = \{\mathbf{1},\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{8}\}$ in $PG(2,17)$. Let $\mathscr{C}_1$ be a conic passing through the five points $\{\mathbf{2},\mathbf{3},\mathbf{4},\mathbf{6},\mathbf{8}\}$ of $\mathcal{S}_1$. Then we have

$$\mathcal{X} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & -2 & -2 & -3 & 3 & 4 \\ 1 & 4 & -4 & 4 & -1 & -2 \end{pmatrix} \quad \text{and} \quad \begin{array}{l} \det \mathcal{X}_1 = -2 \\ \det \mathcal{X}_2 = \mathcal{X}_3 = 0 \\ \det \mathcal{X}_4 = 2 \\ \det \mathcal{X}_5 = -1 \\ \det \mathcal{X}_6 = -4. \end{array}$$

Hence $\mathscr{C}_1 = \mathbb{V}(x_0^2 + 2x_0x_1 + x_0x_2 - 4x_1x_2)$.

Let $\mathscr{C}_2$ be a conic passing through the five points $\{\mathbf{1}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{8}\}$ of $\mathcal{S}_1$. Then we have

$$\mathcal{X} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 \\ 1 & -2 & -2 & -3 & 3 & 4 \\ 1 & 4 & -4 & 4 & -1 & -2 \end{pmatrix} \quad \text{and} \quad \begin{array}{l} \det \mathcal{X}_1 = \mathcal{X}_3 = 0 \\ \det \mathcal{X}_2 = -2 \\ \det \mathcal{X}_4 = -2 \\ \det \mathcal{X}_5 = -5 \\ \det \mathcal{X}_6 = -2. \end{array}$$

Hence $\mathscr{C}_2 = \mathbb{V}(x_1^2 + 2x_0x_1 - 5x_0x_2 + 2x_1x_2)$.

By the similar argument used previously, we can find the equations of the conics $\mathscr{C}_3, \mathscr{C}_4, \mathscr{C}_5$ and $\mathscr{C}_6$ (see Table 2.22).

2.7 Blowing-up the plane in six points

First let us construct the blowing-up of the affine plane $\mathbb{A}^2_{\mathbb{R}}$ at the origin $\mathbf{O} = (0, 0)$ (see [19],[23] and [30], Pages 47-50). Let (x_0, x_1) be the affine coordinates of $\mathbb{A}^2_{\mathbb{R}}$ and let $(y_0 : y_1)$ be the projective coordinates of $\mathbb{P}^1_{\mathbb{R}}$. The blowing-up of $\mathbb{A}^2_{\mathbb{R}}$ at the origin $\mathbf{O}$ is the closed subset $\widetilde{\mathbb{A}^2_{\mathbb{R}}}$ of $\mathbb{A}^2_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ which is defined by the equation $x_0y_1 = x_1y_0$. In fact, we have a natural morphism $\beta : \widetilde{\mathbb{A}^2_{\mathbb{R}}} \to \mathbb{A}^2_{\mathbb{R}}$ which is obtained by restricting

Table 2.22: Conics through $\mathcal{S}_k \backslash \{P_j\}(17)$

$\mathcal{S}_j$	$\mathscr{C}_m :=$ conic through $\mathcal{S}_j \backslash \{P_m\}$
$\mathcal{S}_1$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 2x_0x_1 + x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 2x_0x_1 - 5x_0x_2 + 2x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 3x_0x_1 + 5x_0x_2 + 8x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 5x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 6x_0x_2 + 5x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 7x_0x_2 + 6x_1x_2)$
$\mathcal{S}_2$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 4x_0x_1 + 4x_0x_2 + 8x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 8x_0x_1 - 3x_0x_2 - 7x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 4x_0x_1 + 3x_0x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 2x_0x_2 - 5x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 7x_0x_2 - 8x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 7x_0x_2 + 6x_1x_2)$
$\mathcal{S}_4$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 3x_0x_1 - 6x_0x_2 + 2x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 5x_0x_1 - 7x_0x_2 - 6x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + x_0x_1 + 2x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 6x_0x_2 - 8x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 5x_0x_2 + 4x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 7x_0x_2 + 6x_1x_2)$

$\mathcal{S}_j$	$\mathscr{C}_m :=$ conic through $\mathcal{S}_j \backslash \{P_m\}$
$\mathcal{S}_5$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 8x_0x_1 + 3x_0x_2 + 4x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 4x_0x_1 - 2x_0x_2 - 3x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + x_0x_1 + 2x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 3x_0x_2 + 3x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 6x_0x_2 - 7x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 7x_0x_2 + 6x_1x_2)$
$\mathcal{S}_6$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 6x_0x_1 + 6x_0x_2 - x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 4x_0x_1 - 2x_0x_2 - 3x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 3x_0x_1 + 5x_0x_2 + 8x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 3x_0x_2 + 7x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 4x_0x_2 - 5x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 7x_0x_2 + 6x_1x_2)$
$\mathcal{S}_7$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 2x_0x_1 + x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 7x_0x_1 - 6x_0x_2 - 2x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 8x_0x_1 - 3x_0x_2 - 7x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 4x_0x_2 + 8x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 2x_0x_2 - 3x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 7x_0x_2 + 6x_1x_2)$

Table 2.23: Conics through $\mathcal{S}_k \backslash \{P_j\}(19)$

$\mathcal{S}_j$	$\mathscr{C}_m :=$ conic through $\mathcal{S}_j \backslash \{P_m\}$
$\mathcal{S}_1$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 6x_0x_1 + 9x_0x_2 + 3x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 2x_0x_1 - 2x_0x_2 + 3x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 5x_0x_1 - 3x_0x_2 - 3x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 4x_0x_2 + 9x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 7x_0x_2 + 6x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 + 5x_0x_2 - 6x_1x_2)$
$\mathcal{S}_3$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 8x_0x_1 - 9x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 3x_0x_1 + 4x_0x_2 - 8x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 2x_0x_1 + x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 9x_0x_2 - 9x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 4x_0x_2 - 5x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 + 5x_0x_2 - 6x_1x_2)$
$\mathcal{S}_4$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 8x_0x_1 - 4x_0x_2 - 8x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + x_0x_1 - 6x_0x_2 + 4x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 4x_0x_1 + 9x_0x_2 - 6x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 8x_0x_2 + 6x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 8x_0x_2 + 7x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 + 5x_0x_2 - 6x_1x_2)$
$\mathcal{S}_6$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 2x_0x_2 + x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 8x_0x_2 - 9x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 2x_0x_1 + x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 8x_0x_2 - x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 6x_0x_2 + 5x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 + 5x_0x_2 - 6x_1x_2)$

$\mathcal{S}_j$	$\mathscr{C}_m :=$ conic through $\mathcal{S}_j \backslash \{P_m\}$
$\mathcal{S}_7$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 9x_0x_1 + 5x_0x_2 + 4x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 6x_0x_1 - 3x_0x_2 + 8x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 8x_0x_1 - 7x_0x_2 - 2x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 4x_0x_2 + 9x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 2x_0x_2 - 3x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 + 5x_0x_2 - 6x_1x_2)$
$\mathcal{S}_8$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 3x_0x_1 - 3x_0x_2 + 5x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + x_0x_1 + 2x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 2x_0x_1 + x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + x_0x_2 + 7x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 6x_0x_2 + 5x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 + 3x_0x_2 - 4x_1x_2)$
$\mathcal{S}_9$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 4x_0x_1 - 5x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 6x_0x_1 + x_0x_2 + 4x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 5x_0x_1 - 6x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 4x_0x_2 - x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 + 6x_0x_2 - 7x_1x_2)$

Table 2.24: Conics through $\mathcal{S}_k\backslash\{P_j\}(23)$

$\mathcal{S}_j$	$\mathscr{C}_m :=$ conic through $\mathcal{S}_j\backslash\{P_m\}$
$\mathcal{S}_1$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 3x_0x_1 - 6x_0x_2 + 2x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 7x_0x_1 + 3x_0x_2 - 11x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 8x_0x_1 - 8x_0x_2 - 8x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 4x_0x_2 + 10x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 2x_0x_2 - 3x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$
$\mathcal{S}_2$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 2x_0x_1 + 3x_0x_2 - 6x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 6x_0x_1 + 5x_0x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 8x_0x_1 - 8x_0x_2 - 8x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 10x_0x_2 + 3x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 10x_0x_2 - 11x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$
$\mathcal{S}_3$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 2x_0x_2 + x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 11x_0x_1 - 10x_0x_2 - 2x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 5x_0x_1 - 10x_0x_2 + 11x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 10x_0x_2 + 11x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 4x_0x_2 + 3x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$
$\mathcal{S}_5$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 5x_0x_1 - 3x_0x_2 + x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 3x_0x_1 + x_0x_2 + x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 6x_0x_1 + 4x_0x_2 - 11x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 3x_0x_2 - x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 4x_0x_2 + 3x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$
$\mathcal{S}_6$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 5x_0x_1 - 3x_0x_2 + 7x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 3x_0x_1 + x_0x_2 + x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 6x_0x_1 + 4x_0x_2 - 11x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 3x_0x_2 - x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 4x_0x_2 + 3x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$
$\mathcal{S}_7$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 5x_0x_1 - 3x_0x_2 + 7x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 2x_0x_1 + 2x_0x_2 - 5x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 11x_0x_1 + 5x_0x_2 + 6x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 7x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 6x_0x_2 - 7x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$

Table 2.25: Conics through $\mathcal{S}_k\backslash\{P_j\}(29)$

$\mathcal{S}_j$	$\mathscr{C}_m :=$ conic through $\mathcal{S}_j\backslash\{P_m\}$
$\mathcal{S}_1$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 4x_0x_1 + 12x_0x_2 + 12x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 14x_0x_1 - 3x_0x_2 - 12x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 4x_0x_1 - 5x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 5x_0x_2 - 2x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 12x_0x_2 - 13x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 11x_0x_2 + 10x_1x_2)$
$\mathcal{S}_2$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 14x_0x_1 - 11x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 3x_0x_1 + 2x_0x_2 - 6x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 8x_0x_1 - 13x_0x_2 - 9x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 12x_0x_2 - 11x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 6x_0x_2 + 5x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 11x_0x_2 + 10x_1x_2)$
$\mathcal{S}_3$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 2x_0x_1 - 9x_0x_2 + 10x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 6x_0x_1 + 14x_0x_2 - 9x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 11x_0x_1 + 10x_0x_2 + 7x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 4x_0x_2 + 6x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 5x_0x_2 + 23x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 11x_0x_2 + 10x_1x_2)$
$\mathcal{S}_5$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 10x_0x_1 - 8x_0x_2 - 12x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 8x_0x_1 + 5x_0x_2 - 14x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 3x_0x_1 + 11x_0x_2 + 14x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 7x_0x_2 + 11x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 8x_0x_2 - 9x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 11x_0x_2 + 10x_1x_2)$
$\mathcal{S}_6$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 7x_0x_1 + 8x_0x_2 + 13x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 11x_0x_1 + x_0x_2 - 13x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 11x_0x_1 - 9x_0x_2 - 10x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 6x_0x_2 - x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 4x_0x_2 + 3x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 11x_0x_2 + 10x_1x_2)$
$\mathcal{S}_7$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 7x_0x_1 + 8x_0x_2 + 13x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + x_0x_1 - 5x_0x_2 + 3x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 3x_0x_1 - 10x_0x_2 + 12x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 3x_0x_2 + 14x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 2x_0x_2 - 3x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 11x_0x_2 + 10x_1x_2)$
$\mathcal{S}_9$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 9x_0x_1 - 9x_0x_2 - 12x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 7x_0x_1 + 8x_0x_2 + 13x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 8x_0x_1 + 7x_0x_2 + 13x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + x_0x_2 - 13x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 4x_0x_2 - 5x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 7x_0x_2 + 6x_1x_2)$

Table 2.26: Conics through $\mathcal{S}_k\backslash\{P_j\}(31)$

$\mathcal{S}_j$	$\mathscr{C}_m :=$ conic through $\mathcal{S}_j\backslash\{P_m\}$	$\mathcal{S}_j$	$\mathscr{C}_m :=$ conic through $\mathcal{S}_j\backslash\{P_m\}$
$\mathcal{S}_1$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 10x_0x_1 + x_0x_2 - 12x_1x_2)$	$\mathcal{S}_5$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 4x_0x_1 - 7x_0x_2 + 2x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 10x_0x_1 - 7x_0x_2 - 4x_1x_2)$		$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 6x_0x_1 + 13x_0x_2 - 8x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 5x_0x_1 + 5x_0x_2 - 13x_1x_2)$		$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 6x_0x_1 - 2x_0x_2 - 5x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 13x_0x_2 + 9x_1x_2)$		$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 15x_0x_2 - 12x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 6x_0x_2 - 7x_1x_2)$		$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 2x_0x_2 + x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$		$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$
$\mathcal{S}_2$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 9x_0x_1 - 14x_0x_2 - 9x_1x_2)$	$\mathcal{S}_7$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 5x_0x_1 + 12x_0x_2 - 8x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 6x_0x_1 + 13x_0x_2 - 8x_1x_2)$		$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 15x_0x_1 + 10x_0x_2 + 5x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 7x_0x_1 - 9x_0x_2 + 15x_1x_2)$		$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 3x_0x_1 - 14x_0x_2 - 15x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 15x_0x_2 - 5x_1x_2)$		$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 6x_0x_2 - 13x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 4x_0x_2 + 3x_1x_2)$		$\mathscr{C}_5 = \mathbb{V}(x_0x_1 + 3x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$		$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$
$\mathcal{S}_3$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 5x_0x_1 + 12x_0x_2 - 8x_1x_2)$	$\mathcal{S}_9$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 10x_0x_1 - 6x_0x_2 - 5x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 15x_0x_1 + x_0x_2 + 13x_1x_2)$		$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 10x_0x_1 - 5x_0x_2 - 6x_1x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 9x_0x_1 + 2x_0x_2 - 12x_1x_2)$		$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 9x_0x_1 + 4x_0x_2 + 4x_1x_2)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + 10x_0x_2 - x_1x_2)$		$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 4x_0x_2 - 4x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 5x_0x_2 + 4x_1x_2)$		$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 13x_0x_2 + 12x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$		$\mathscr{C}_6 = \mathbb{V}(x_0x_1 + 12x_0x_2 - 13x_1x_2)$
$\mathcal{S}_4$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - 6x_0x_1 - 10x_0x_2 + 15x_1x_2)$	$\mathcal{S}_{10}$	$\mathscr{C}_1 = \mathbb{V}(x_0^2 - x_1x_2)$
	$\mathscr{C}_2 = \mathbb{V}(x_1^2 - 4x_0x_1 - 5x_0x_2 + 8x_1x_2)$		$\mathscr{C}_2 = \mathbb{V}(x_1^2 - x_0x_2)$
	$\mathscr{C}_3 = \mathbb{V}(x_2^2 - 14x_0x_1 - 8x_0x_2 - 10x_1x_2)$		$\mathscr{C}_3 = \mathbb{V}(x_2^2 - x_0x_1)$
	$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 7x_0x_2 - 6x_1x_2)$		$\mathscr{C}_4 = \mathbb{V}(x_0x_1 + x_0x_2 + x_1x_2)$
	$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 11x_0x_2 + 10x_1x_2)$		$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 6x_0x_2 + 5x_1x_2)$
	$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 9x_0x_2 + 8x_1x_2)$		$\mathscr{C}_6 = \mathbb{V}(x_0x_1 + 5x_0x_2 - 6x_1x_2)$

the projection $\mathbb{A}^2_{\mathbb{R}} \times \mathbb{P}^1_{\mathbb{R}}$ to the first factor as we shown in the following diagram (see Figure 2.1).

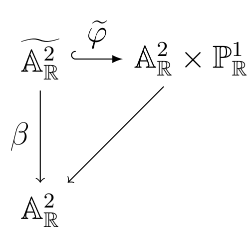

Figure 2.1: Blowing-up the plane.

In ([13], Pages 28,400,401), there are many facts demonstrate properties of the blowing-up:

Fact 2.1. With the previous notations, we have

- The restriction of β to the set $\widetilde{\mathbb{A}^2_{\mathbb{R}}}\backslash\beta^{-1}(\mathbf{O})$ is bijection.

- $\beta^{-1}(\mathbf{O}) \cong \mathbb{P}^1_{\mathbb{R}}$.
- The points of $\beta^{-1}(\mathbf{O})$ are in $1-1$ correspondence with set of lines through $\mathbf{O}$ in $\mathbb{A}^2_{\mathbb{R}}$.

If $C \subseteq \mathbb{A}^2_{\mathbb{R}}$ is a curve in $\mathbb{A}^2_{\mathbb{R}}$, we will call $\widetilde{C} := \overline{\beta^{-1}\big|_{\mathbb{A}^2_{\mathbb{R}}\setminus\mathbf{O}}(C\setminus\{\mathbf{O}\})}$ the strict transform of C (see [31]). The generalization of the above notions can be found in ([13], Pages 14,136-171 or [11].

Fact 2.2. Let $\mathcal{S} := \{P_1, \ldots, P_6\} \subseteq \mathbb{P}^2$ be a six points in the plane, such that no three are collinear and not all the six points are on a common conic. Then the blowing-up $\widetilde{\mathbb{P}^2}$ of the projective plane $\mathbb{P}^2$ in S can be embedded as a smooth cubic surface in projective three-space $\mathbb{P}^3$.

According to the embedding $\varphi : \mathbb{P}^2 \hookrightarrow \mathbb{P}^3$ and the projection as in the figure above, we have:

Fact 2.3. Let $\mathcal{S} := \{P_1, \ldots, P_6\} \subseteq \mathbb{P}^2$ be a six points in the plane, such that no three are collinear and not all the six points are on a common conic. Let $\mathscr{C}_i \subseteq \mathbb{P}^2$ denote the unique conic through the five points $S\setminus\{P_i\}$ for $i = 1, 2, \ldots, 6$, and let $l_{ij} \subseteq \mathbb{P}^2$ denote the line through the points P_i and P_j for $i, j = 1, 2, \ldots, 6$, $i \neq j$. Then the 27 lines lying on the cubic surface are as follows:

- The 6 exceptional lines over the 6 base points P_i:

$$a_i := \widetilde{\varphi}(\beta^{-1}(P_i)) \subseteq \mathbb{P}^3,\ i = 1, 2, \ldots, 6.$$

- The 6 strict transforms of the 6 plane conics $\mathscr{C}_i$:

$$b_i := \widetilde{\varphi}(\widetilde{\mathscr{C}_i}) \subseteq \mathbb{P}^3,\ i = 1, 2, \ldots, 6.$$

- The 15 strict transforms of the $\binom{6}{2} = 15$ lines l_{ij} joining the P_i:

$$c_{ij} := \widetilde{\varphi}(\widetilde{l_{ij}}) \subseteq \mathbb{P}^3,\ i, j = 1, 2, \ldots, 6,\ i \neq j.$$

Moreover, the lines a_i and b_i, $i = 1, 2, \ldots, 6$ form a double six, and c_{ij}, $i \neq j$ are the remaining 15 lines on the smooth cubic surface.

An Eckardt point on a non-singular cubic surface is a smooth point, where three of its lines meet. Next we will discuss the problem of number of Eckardt points on a non-singular cubic surface with 27 lines in more detail.

Fact 2.4. Let $\mathcal{S} := \{P_1, \ldots, P_6\} \subseteq \mathbb{P}^2$ be a six points in the plane, such that no three are collinear and not all the six points are on a common conic. Denote by l_{ij} the 15 lines through the points P_i and P_j, $i \neq j$ and by $\mathscr{C}_i \subseteq \mathbb{P}^2$ the unique conic through the five points $S \backslash \{P_i\}$ for $i = 1, 2, \ldots, 6$. Then

- If three of l_{ij} meet in a point $E \in \mathbb{P}^2 \backslash S$, then the corresponding lines $\widetilde{l_{ij}}$ on the cubic surface meet in an Eckardt point.
- If l_{ij} touches the conic $\mathscr{C}_i$ in P_j for some $i, j = 1, 2, \ldots, 6$, $i \neq j$, then the corresponding lines $a_j := (\beta^{-1} \circ \widetilde{\varphi})((P_j)$, $b_i := \widetilde{\mathscr{C}_i}$ and $c_{ij} := \widetilde{l_{ij}}$ on the cubic surface meet in an Eckardt point.
- The application $\beta^{-1} \circ \widetilde{\varphi}$ is bijection on $\mathbb{P}^2 \backslash S$.
- Both $\mathscr{C}_i$ and l_{ij} have the same tangent direction in P_j, so the corresponding lines $b_i := \widetilde{\mathscr{C}_i}$ and $c_{ij} := \widetilde{l_{ij}}$ meet the line $a_j := \widetilde{P_j}$ in the same point.

Chapter 3

Classification of non-singular cubic surfaces up to e-invariants

The main result of this chapter is the classification of non-singular cubic surfaces with twenty-seven lines up to e-invariants over the finite fields $\mathbb{F}_q$ where $q = 17, 19, 23, 29, 31$. Some structures on cubic surfaces with 27 lines, in $PG(2, q)$ with $q = 17, 19, 23, 29, 31$, are discussed. Furthermore, the classification of these cubic surfaces is done by classifying the 6-arcs not lying on a conic (6 points in general position) in the projective planes $PG(2, q)$ for $q = 17, 19, 23, 29, 31$.

3.1 Double-sixes and Eckardt points on a non-singular cubic surface

Let $PG(3, q)$ be the 3-dimensional projective space over the Galois field $GF(q)$. The space $PG(3, q)$ contains $q^3 + q^2 + q + 1$ points and planes, as well as $(q^2 + q + 1)(q^2 + 1)$ lines. There are $q^2 + q + 1$ lines through every point, and $q + 1$ planes through a line.

In $PG(3, q)$, planes and lines are characterized as follows: a subset Π_2 is a plane if and only if it has $q^2 + q + 1$ points and meets every line; a subset Π_1 is a line if and only if it has $q + 1$ points and meets every plane ([16], Pages 3,4).

A point $P(X) = P(x_0, x_1, x_2, x_3)$ in $PG(3, q)$, where $x_0, x_1, x_2, x_3 \in GF(q)$ and not all zero, is denoted by $(x_0 : x_1 : x_2 : x_3)$.

In $PG(3,q)$, a cubic surface $\mathscr{S}$, is the zero set of a homogeneous cubic polynomial in four variables over $GF(q)$, that is

$$\mathscr{S} = \mathbb{V}\Big(\sum a_{ijkl}x_0^i x_1^j x_2^k x_3^l\Big),$$

where $i,j,k,l \in \{0,1,2,3\}, i+j+k+l = 3$, and $a_{ijkl} \in GF(q)$. Therefore, to determine a cubic surface $\mathscr{S}$, 19 conditions are required since there are 20 monomials of degree 3 in four variables.

A double-six D in $PG(3,q)$ is a set of 12 lines

$$D: \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix} \tag{3.1.1}$$

such that each line only meets the five lines which are not in the same row or column.

The intensive study of the cubic surfaces started in 1849, when the British mathematicians Salmon and Cayley published the results of their correspondence on the number of lines on a non-singular cubic surface (see [6], pages 118-132 and [26], Pages 252-60). Moreover, Cayley and Salmon showed that a non-singular cubic surface over the complex field contains exactly twenty-seven lines. In 1858, Schläfli ([27]) found the required notation for the complete figure formed by these 27 lines. In fact, Clebsch constructed the famous Diagonal surface in ([7], Pages 284-345) and showed that it contained 27 real lines. The computer programs allow mathematicians of the 21st century not only to make static models of surfaces and curves, but also to manipulate them interactively. For this purpose, we have the following important theorem of Clebsch ([7], Pages 359-380).

Theorem 3.1. Every non-singular cubic surface can be represented in the plane using 4 plane cubic curves through six points in general position and vice versa.

The construction and existences of a double-six are described by the following main facts in ([16], Pages 182,187,188).

1. Given five skew lines a_1, a_2, a_3, a_4, a_5 with a single transversal b_6 such that each set of four a_i omitting a_j has a unique further transversal b_j, then the five lines b_1, b_2, b_3, b_4, b_5 also have a transversal a_6. These twelve lines form a double-six.

2. A double-six lies on a unique cubic surface $\mathscr{S}$ with 15 further lines c_{ij} given by the intersection of $[a_i, b_j]$ (sometimes denoted by $a_i \vee b_j$) and $[a_j, b_i]$, where $[a_i, b_j]$ is the plane containing a_i and b_j.

3. A necessary and sufficient condition for the existence of a double-six, and so of a cubic surface with 27 lines in $PG(3, q)$, is the existence over the same field of a plane 6-arc not on a conic. This occurs when $q \neq 2, 3, 5$.

Given a **double-six** D

$$D: \begin{matrix} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{matrix}$$

then the incidence diagram of a double-six D is shown in Table 3.1.

Table 3.1: Incidence diagram

	a_1	a_2	a_3	a_4	a_5	a_6
b_1		✓	✓	✓	✓	✓
b_2	✓		✓	✓	✓	✓
b_3	✓	✓		✓	✓	✓
b_4	✓	✓	✓		✓	✓
b_5	✓	✓	✓	✓		✓
b_6	✓	✓	✓	✓	✓	

where ✓ indicates that the two lines intersect.

The following are some results on skew lines in $PG(3, q)$:

(a) If five skew lines a_1, a_2, a_3, a_4, a_5 have a transversal b, then each set of four a_i has a unique, distinct second transversal if and only if each set of five of the six lines is linearly independent.

(b) The configuration in part (a) exists if and only if a plane 6-arc not on a conic exists in $PG(2,q)$. This occurs for $q \neq 2,3$ or 5.

(c) Given five skew lines a_1, a_2, a_3, a_4, a_5 with a transversal b_6 such that each five of the six lines are linearly independent, the second transversals b_1, b_2, b_3, b_4, b_5 of sets of four of the a_i have themselves a transversal a_6.

(d) A double-six lies on a unique cubic surface $\mathscr{S}$, which contains a further 15 lines.

A cubic surface is determined by 19 conditions. According to Bézout's Theorem, if four points of a line l lie on a cubic surface, then the whole line lies on it. Let D be the double-six above. Then, to put $a_1, b_2, b_3, b_4, b_5, b_6$ on a cubic surface requires $4 + 5 \cdot 3 = 19$ conditions. So there exists a cubic surface $\mathscr{S}$ containing these lines. Then, each of a_2, a_3, a_4, a_5, a_6 meets four of these lines and therefore lies on $\mathscr{S}$. Now $\mathscr{S}$ is unique, since if there was another cubic surface $\mathscr{S}'$ containing these lines, then $\mathscr{S}$ and $\mathscr{S}'$ would intersect in a curve of degree at least 12 (number of points on that curve), which is impossible unless $\mathscr{S}$ and $\mathscr{S}'$ have a common component of lower order, but the definition of double-six does not allow this. So $\mathscr{S} = \mathscr{S}'$. Note that $\mathscr{S}$ also contains the 15 lines

$$c_{ij} = [a_i, b_j] \cap [a_j, b_i],\ i, j = 1, \ldots, 6 \text{ and } i \neq j.$$

In $PG(2,q)$, consider a 6-arc $\mathcal{S} = \{P_i : i = 1, \ldots, 6\}$ not on a conic (sometimes these points are called six points in general position). Then a set of plane cubic curves through all six P_i is called the **web** W of cubic curves. The lines $\overline{P_1P_2}, \overline{P_3P_4}, \overline{P_5P_6}$ compose a curve of W.

Given a line ℓ in $PG(3,q)$, two of whose points are $P(Y)$ and $P(Y')$, where $Y = (y_0 : y_1 : y_2 : y_3)$ and $Y' = (y'_0 : y'_1 : y'_2 : y'_3)$. The Plücker coordinates of ℓ is defined as

$$L := (\ell_{01}, \ell_{02}, \ell_{03}, \ell_{12}, \ell_{13}, \ell_{23})$$

where $\ell_{ij} = y_i y'_j - y'_i y_j$. Note that $L \in PG(5,q)$.

Define a map from the set of all of lines $\mathscr{L}$ (as points of $PG(5,q)$) to the set of points of the hyperbolic quadric $\mathscr{H}_5 = \mathbb{V}(z_0 z_1 + z_2 z_3 + z_4 z_5)$ in $PG(5,q)$ as

$$\mathscr{K} : \mathscr{L} \to \mathscr{H}_5,\ L = (\ell_{01}, \ell_{02}, \ell_{03}, \ell_{12}, \ell_{13}, \ell_{23}) \mapsto (\ell_{01}, \ell_{23}, -\ell_{02}, \ell_{13}, \ell_{03}, \ell_{12}).$$

The map $\mathscr{K}$ is bijective and its image is the simplest, non-trivial example of a Grassmannian. In fact, $\mathscr{H}_5 = \mathbb{G}_{2,4}$

Now, Let ℓ_1 and ℓ'_1 be two lines in $PG(3,q)$ with Plücker coordinates

$$L = (\ell_{01}, \ell_{02}, \ell_{03}, \ell_{12}, \ell_{13}, \ell_{23}),$$
$$L' = (\ell'_{01}, \ell'_{02}, \ell'_{03}, \ell'_{12}, \ell'_{13}, \ell'_{23}),$$

then ℓ_1 and ℓ'_1 intersect if and only if $\varpi(\mathscr{K}(\ell_1), \mathscr{K}(\ell_2)) = 0 \pmod q$, where

$$\varpi(\mathscr{K}(\ell_1), \mathscr{K}(\ell'_1)) = \ell_{01}\ell'_{23} + \ell_{23}\ell'_{01} - \ell_{02}\ell'_{13} - \ell_{13}\ell'_{02} + \ell_{03}\ell'_{12} + \ell_{12}\ell'_{03}.$$

It follows that ℓ_1 and ℓ'_1 are skew if $\varpi(\mathscr{K}(\ell_1), \mathscr{K}(\ell'_1)) \neq 0 \pmod q$ (see [8], Pages 4,28).

Let us give an example. Consider the non-singular cubic surface with 27 lines in $PG(3,13)$, namely

$$\mathscr{S} = \mathbb{V}(y_3^3 - y_0^2 y_3 - y_1^2 y_3 - y_2^2 y_3 - 4 y_0 y_1 y_2).$$

Let us consider the following twelve lines on $\mathscr{S}$:

$$a_1 = \{(\lambda : 2\lambda : \mu : \mu) : (\lambda : \mu) \in PG(1,13)\},$$
$$a_2 = \{(\lambda : -2\lambda : \mu : -\mu) : (\lambda : \mu) \in PG(1,13)\},$$
$$a_3 = \{(\lambda : \mu : 6\lambda : -\mu) : (\lambda : \mu) \in PG(1,13)\},$$
$$a_4 = \{(\lambda : \mu : -6\lambda : \mu) : (\lambda : \mu) \in PG(1,13)\},$$

$$a_5 = \{(\lambda : \mu : -2\mu : -\lambda) : (\lambda : \mu) \in PG(1, 13)\},$$

$$a_6 = \{(\lambda : \mu : 2\mu : \lambda) : (\lambda : \mu) \in PG(1, 13)\},$$

$$b_1 = \{(\lambda : 6\lambda : \mu : -\mu) : (\lambda : \mu) \in PG(1, 13)\},$$

$$b_2 = \{(\lambda : -6\lambda : \mu : \mu) : (\lambda : \mu) \in PG(1, 13)\},$$

$$b_3 = \{(\lambda : \mu : 2\lambda : \mu) : (\lambda : \mu) \in PG(1, 13)\},$$

$$b_4 = \{(\lambda : \mu : -2\lambda : -\mu) : (\lambda : \mu) \in PG(1, 13)\},$$

$$b_5 = \{(\lambda : \mu : -6\mu : \lambda) : (\lambda : \mu) \in PG(1, 13)\},$$

$$b_6 = \{(\lambda : \mu : 6\mu : -\lambda) : (\lambda : \mu) \in PG(1, 13)\}.$$

The values of $\mathscr{K}(a_i), \mathscr{K}(b_i)$ for $i, j = 1, \ldots, 6$ are:

$$\mathscr{K}(a_1) = (0, 0, -1, 2, 1, 2),$$

$$\mathscr{K}(a_2) = (0, 0, -1, 2, -1, -2),$$

$$\mathscr{K}(a_3) = (1, -6, 0, 0, -1, -6),$$

$$\mathscr{K}(a_4) = (1, -6, 0, 0, 1, 6),$$

$$\mathscr{K}(a_5) = (1, -2, 2, 1, 0, 0),$$

$$\mathscr{K}(a_6) = (1, -2, -2, -1, 0, 0),$$

$$\mathscr{K}(b_1) = (0, 0, -1, -6, -1, 6),$$

$$\mathscr{K}(b_2) = (0, 0, -1, -6, 1, -6),$$

$$\mathscr{K}(b_3) = (1, 2, 0, 0, 1, -2),$$

$$\mathscr{K}(b_4) = (1, 2, 0, 0, -1, 2),$$

$$\mathscr{K}(b_5) = (1, 6, 6, -1, 0, 0),$$

$$\mathscr{K}(b_6) = (1, 6, -6, 1, 0, 0).$$

Consequently, we have the symmetric table (see Table 3.2)

Table 3.2: The values of $\varpi(\mathscr{K}(\ell_1), \mathscr{K}(\ell_1'))\ (\mathrm{mod}\ 13)$

	a_1	a_2	a_3	a_4	a_5	a_6	b_1	b_2	b_3	b_4	b_5	b_6
a_1	0	5	5	8	3	10	8	0	0	0	0	0
a_2	5	0	8	5	3	10	0	8	0	0	0	0
a_3	5	8	0	2	5	5	0	0	5	0	0	0
a_4	8	5	2	0	5	5	0	0	0	5	0	0
a_5	3	3	5	5	0	5	0	0	0	0	8	0
a_6	10	10	5	5	5	0	0	0	0	0	0	8
b_1	8	0	0	0	0	0	0	11	8	5	4	9
b_2	0	8	0	0	0	0	11	0	5	8	4	9
b_3	0	0	5	0	0	0	8	5	0	8	8	8
b_4	0	0	0	0	0	0	5	8	8	0	8	8
b_5	0	0	0	8	8	0	4	4	8	8	0	11
b_6	0	0	0	0	0	8	9	9	8	8	11	0

where the entries of table represent the values of $\varpi(\mathscr{K}(\ell_1), \mathscr{K}(\ell_1'))\ (\mathrm{mod}\ 13)$ for

$$\ell_1, \ell_1' \in \{a_1, a_2, a_3, a_4, a_5, a_6, b_1, b_2, b_3, b_4, b_5, b_6\}.$$

It follows that the above six lines form a double-six (see Figure 3.1).

The 15 further lines c_{ij} on $\mathscr{S}$ are given by the intersection of the plane $[a_i, b_j]$ and the plane $[a_j, b_i]$ (see Table 3.3). For instant,

$$c_{12} = [a_1, b_2] \cap [a_2, b_1] = \mathbb{V}(y_2 - y_3) \cap \mathbb{V}(y_2 + y_3) = \{(\lambda : \mu : 0 : 0) : (\lambda : \mu) \in PG(1, 13)\}.$$

A **Clebsch mapping** $s : \mathscr{S} \dashrightarrow PG(2, q)$ is a special birational map that we will describe in more detail later. Such a map s induces a bijection

$$\mathscr{S} \backslash \bigcup_{i=1}^{6} a_i \to PG(2, q) \backslash \mathcal{S}.$$

The lines a_i form one half of the double-six D; any five of them have a unique transversal and these six transversals b_i form the other half of D. Let $\mathcal{S} = \{P_i : i = 1, \ldots, 6\}$ be a 6-arc not on any conic in $PG(2, q)$. Every point of $PG(2, q)$ other than the $P_i \in \mathcal{S}$ is the image of a single point of $\mathscr{S}$. Every curve in $PG(2, q)$ is the image of a curve on $\mathscr{S}$. An intersection of two curves in $PG(2, q)$ that is not equal to P_i is also the image of an intersection of the corresponding curves on $\mathscr{S}$. Two curves in

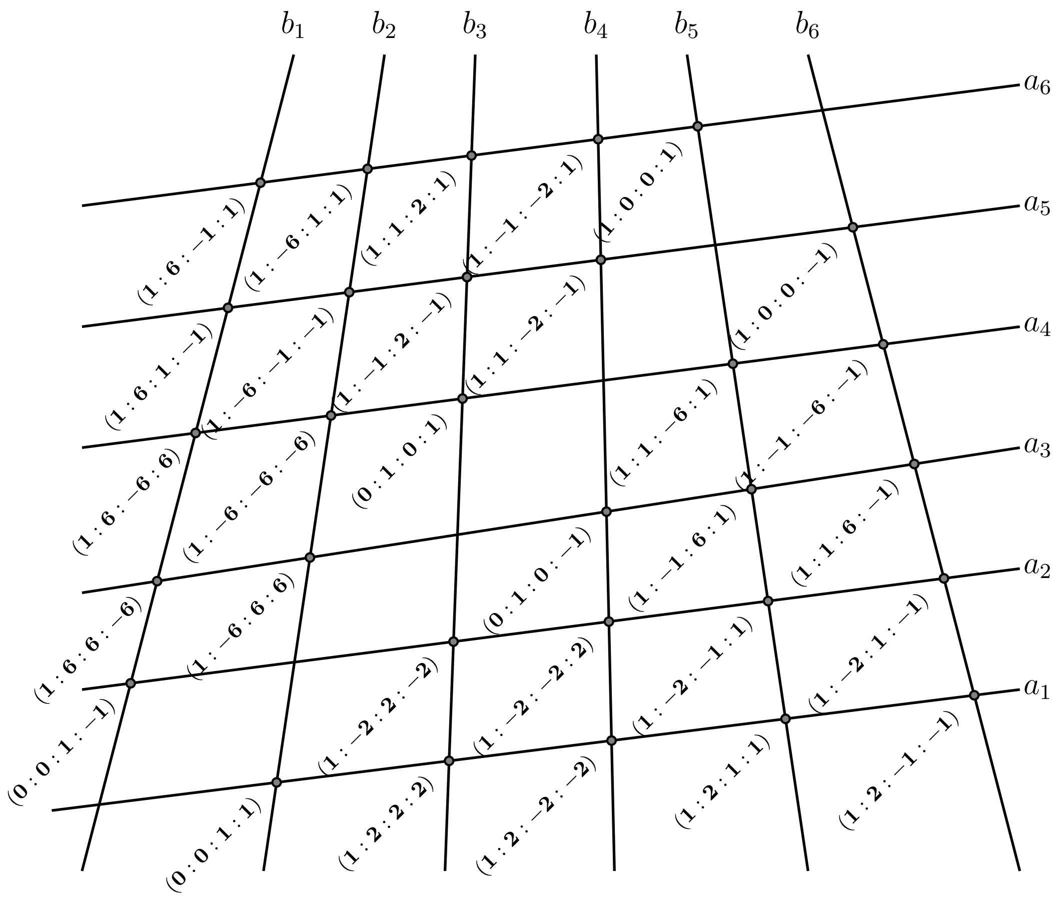

Figure 3.1: Configuration of a double-six on $\mathscr{S}$

$PG(2,q)$ through $P_i \in \mathcal{S}$ correspond to two curves on $\mathscr{S}$ meeting a_i. Two curves in $PG(2,q)$ touching at $P_i \in \mathcal{S}$ also correspond to two curves on $\mathscr{S}$ meeting at a_i. The plane sections of $\mathscr{S}$ (sometimes called plane cubic curves) are mapped to the web W of cubic curves through all six P_i. Every plane section is mapped to such a cubic curve and every such plane cubic curve is the image a plane section. If the cubic of W is composite, then so is the plane section of $\mathscr{S}$. The number of intersections of a curve on $\mathscr{S}$ with a plane is called the order of that curve. This is the number of intersections of the image curve in $PG(2,q)$ with a cubic of W such that the points P_i are excluded from this number of intersections. If the order of a curve on $\mathscr{S}$ is one then it is a line on $\mathscr{S}$. For example, a line $\overline{P_iP_j}$ meets a cubic curve in $PG(2,q)$ in three points and if the cubic belongs to W, there is a single point of intersection

Table 3.3: The 15 lines c_{ij} on $\mathscr{S}$

$[i,j]$	$c_{ij} = [a_i, b_j] \cap [a_j, b_i]$
$[1,2]$	$c_{12} = \mathbb{V}(y_2 - y_3) \cap \mathbb{V}(y_2 + y_3)$
$[1,3]$	$c_{13} = \mathbb{V}(y_0 + 6y_1 + 6y_2 - 6y_3) \cap \mathbb{V}(y_0 + 2y_1 + 2y_2 + 2y_3)$
$[1,4]$	$c_{14} = \mathbb{V}(y_0 + 6y_1 - 6y_2 + 6y_3) \cap \mathbb{V}(y_0 + 2y_1 - 2y_2 - 2y_3)$
$[1,5]$	$c_{15} = \mathbb{V}(y_0 + 6y_1 + y_2 - 12y_3) \cap \mathbb{V}(y_0 + 2y_1 + y_2 + y_3)$
$[1,6]$	$c_{16} = \mathbb{V}(y_0 + 6y_1 - y_2 + y_3) \cap \mathbb{V}(y_0 + 2y_1 - y_3)$
$[2,3]$	$c_{23} = \mathbb{V}(y_0 - 6y_1 + 6y_2 + 6y_3) \cap \mathbb{V}(y_0 - 2y_1 + 2y_2 - 2y_3)$
$[2,4]$	$c_{24} = \mathbb{V}(y_0 - 6y_1 - 6y_2 - 6y_3) \cap \mathbb{V}(y_0 - 2y_1 - 2y_2 + 2y_3)$
$[2,5]$	$c_{25} = \mathbb{V}(y_0 - 6y_1 - y_2 - y_3) \cap \mathbb{V}(y_0 - 2y_1 - y_2 + y_3)$
$[2,6]$	$c_{26} = \mathbb{V}(y_0 - 6y_1 + y_2 + y_3) \cap \mathbb{V}(y_0 - 2y_1 + y_2 - y_3)$
$[3,4]$	$c_{34} = \mathbb{V}(y_1 + y_3) \cap \mathbb{V}(y_1 - y_3)$
$[3,5]$	$c_{35} = \mathbb{V}(y_0 - y_1 + 2y_2 - y_3) \cap \mathbb{V}(y_0 - y_1 + 6y_2 + y_3)$
$[3,6]$	$c_{36} = \mathbb{V}(y_0 + y_1 + 2y_2 + y_3) \cap \mathbb{V}(y_0 + y_1 + 6y_2 - y_3)$
$[4,5]$	$c_{45} = \mathbb{V}(y_0 + y_1 - 2y_2 - y_3) \cap \mathbb{V}(y_0 + y_1 - 6y_2 + y_3)$
$[4,6]$	$c_{46} = \mathbb{V}(y_0 - y_1 - 2y_2 + y_3) \cap \mathbb{V}(y_0 - y_1 - 6y_2 - y_3)$
$[5,6]$	$c_{56} = \mathbb{V}(y_0 + y_3) \cap \mathbb{V}(y_0 - y_3)$

with the cubic apart from the points P_i and P_j. The line $\overline{P_iP_j}$ is the image of a line c_{ij} , a transversal to a_i and a_j, and there are 15 bisecants $\overline{P_iP_j}$ of a 6-arc $\mathcal{K}$ as there are 15 c_{ij}. Two of c_{ij} intersect when they do not share a suffix, and they are skew if they have a suffix in common. That is, the lines c_{12}, c_{34}, c_{56} intersect each other and the lines c_{12}, c_{23} are skew.

Let $\mathscr{C}_i$ be the conic through the five points of $\mathcal{S}\backslash\{P_i\}$ for $i = 1, \ldots, 6$. A conic and a cubic in $PG(2,q)$ have six points of intersection; so the conic $\mathscr{C}_i$ has one free intersection with a cubic of W. Therefore, a $\mathscr{C}_i$ maps a line b_i on $\mathscr{S}$, where b_i is the transversal to five $a_i, i \neq j$. The line c_{ij} of $\mathscr{S}$ is the line of intersection of two planes, one containing the two intersecting lines a_i and b_j and the other containing the two intersecting lines a_j and b_i.

The cubic surface $\mathscr{S}$ has a tangent plane $T_P\mathscr{S}$ at every point $P \in \mathscr{S}$. Note that $\mathscr{S}$ has to be non-singular. If the point P lies on the line l of $\mathscr{S}$, then the tangent plane $T_P\mathscr{S}$ contains l.

We know that the lines $\overline{P_1P_2}, \overline{P_3P_4}, \overline{P_5P_6}$ compose a cubic curve of the web W, and we also know that a curve of W maps a plane section of the cubic surface $\mathscr{S}$. So the curve composed of the bisecants $\overline{P_1P_2}, \overline{P_3P_4}, \overline{P_5P_6}$, of the 6-arc $\mathcal{S}$ maps a plane section of $\mathscr{S}$ composed of c_{12}, c_{34}, c_{56}. This plane section meeting $\mathscr{S}$ in three lines is called a **tritangent plane**. Note that a tritangent plane can meet $\mathscr{S}$ in three lines a_i, b_j and c_{ij}, $i \neq j = 1, \dots, 6$.

Theorem 3.2. ([16], Page 191) Let $\mathscr{S}$ be a non-singular cubic surface and let P be a point of $\mathscr{S}$. Then

1. If a point P is on no line of $\mathscr{S}$, then $T_P\mathscr{S} \cap \mathscr{S}$ is an irreducible cubic with a double point at P.

2. If a point P is on exactly one line l of $\mathscr{S}$, then $T_P\mathscr{S} \cap \mathscr{S}$ consists of l and a conic through P.

3. If a point P is on exactly two lines l_1 and l_2 of $\mathscr{S}$, then $T_P\mathscr{S} \cap \mathscr{S}$ consists of l_1, l_2 and a third line forming a triangle.

4. If a point P is on exactly three lines l_1, l_2 and l_3 of $\mathscr{S}$, then $T_P\mathscr{S} \cap \mathscr{S}$ consists of these three lines.

We note here that, in cases (3) and (4) above, $T_P\mathscr{S}$ is a tritangent plane. In Case (4), the point P at which three lines of a cubic surface $\mathscr{S}$ are meet is called an **Eckardt point** or an **E-point**.

Let these three concurrent lines of $\mathscr{S}$ be c_{12}, c_{34} and c_{56}. We denote this E-point as $E_{12,34,56}$. The image of $E_{12,34,56}$ in the plane $PG(2,q)$ is the Brianchon point $(12, 34, 56)$ of the 6-arc $\mathcal{S} = \{P_i : i = 1, \dots, 6\}$ not on a conic. Thus, a possible type of an E-point is $E_{ij,kl,mn} = c_{ij} \cap c_{kl} \cap c_{mn}$. Also another possible type of an E-point is $E_{ij} = a_i \cap b_j \cap c_{ij}$. The image of E_{ij} in the projective plane $PG(2,q)$ is a conic $\mathscr{C}_j$ having a tangent $\overline{P_iP_j}$ where $P_i \in \mathscr{C}_j$ and $P_j \notin \mathscr{C}_j$.

Theorem 3.3. ([27], Pages 192,193) Let $\mathscr{S}$ be a non-singular cubic surface in $PG(3,q)$ containing at least one line. Then the following are equivalent:

1. The $q+1$ residual intersections with $\mathscr{S}$ of the planes through any line of $\mathscr{S}$ contain exactly five line pairs;

2. $\mathscr{S}$ has 27 lines;

3. $\mathscr{S}$ has q^2+7q+1 points.

3.2 Some classical structures on cubic surfaces

Consider a set of six tritangent planes divided into two triads, such that the three planes of each triad contain the same set of nine distinct lines of the cubic surface $\mathscr{S}$. This set is called a **trihedral pair** (See [14], Page 11).

Let $\mathscr{S}$ be a cubic surface with 27 lines $a_i, b_j, c_{ij}, i, j \in \{1,2,3,4,5,6\}$, where $c_{ij} = c_{ji}$. Then they are as follows:

$$\begin{array}{cccccc}
a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\
b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \\
c_{12} & c_{13} & c_{14} & c_{15} & c_{16} & \\
c_{23} & c_{24} & c_{25} & c_{26} & & \\
c_{34} & c_{35} & c_{36} & & & \\
c_{45} & c_{46} & & & & \\
c_{56} & & & & &
\end{array}$$

Each line meets 10 others, namely

$$\begin{array}{llll}
a_i & \text{meets} & b_j, c_{ij} & \text{with } i \neq j \\
b_j & \text{meets} & a_i, c_{ij} & \text{with } i \neq j \\
c_{ij} & \text{meets} & a_i, a_j, b_i, b_j, c_{rs} & \text{with } r, s \neq i, j.
\end{array}$$

For example,

$$\begin{array}{lll} a_1 & \text{meets} & b_2, b_3, b_4, b_5, b_6, c_{12}, c_{13}, c_{14}, c_{15}, c_{16}; \\ b_2 & \text{meets} & a_1, a_3, a_4, a_5, a_6, c_{12}, c_{23}, c_{24}, c_{25}, c_{26}; \\ c_{12} & \text{meets} & a_1, a_2, b_1, b_2, c_{34}, c_{35}, c_{36}, c_{45}, c_{46}, c_{56}. \end{array}$$

A tritangent plane meets $\mathscr{S}$ in three lines of the form $c_{ij}c_{kl}c_{mn}$ or $a_ib_jc_{ij}$ with $i, j, k, l, m, n \in \{1, 2, 3, 4, 5, 6\}$. Thus, there are 45 tritangent planes, namely,

$$\begin{array}{lll} 30 & \text{of the kind} & a_ib_jc_{ij}; \\ 15 & \text{of the kind} & c_{ij}c_{kl}c_{mn}. \end{array}$$

The 27 lines form 36 double-sixes: D, D_{ij}, D_{ijk}, namely,

$$D: \begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & a_5 & a_6 \\ b_1 & b_2 & b_3 & b_4 & b_5 & b_6 \end{array}$$

$$D_{12}: \begin{array}{cccccc} a_1 & b_1 & c_{23} & c_{24} & c_{25} & c_{26} \\ a_2 & b_2 & c_{13} & c_{14} & c_{15} & c_{16} \end{array}$$

$$D_{123}: \begin{array}{cccccc} a_1 & a_2 & a_3 & c_{56} & c_{46} & c_{45} \\ c_{23} & c_{13} & c_{12} & b_4 & b_5 & b_6 \end{array}$$

There is one double-six of type D, 15 of type D_{ij}, and 20 of type D_{ijk}. From the definitions of the tritangent planes and trihedral pair, the 45 tritangent planes form 120 trihedral pairs. Let the six planes of a trihedral pair be given the rows and columns of a 3×3 array. Then the 120 trihedral pairs are the following arrays:

$$T_{123}: \begin{array}{ccc} c_{23} & a_3 & b_2 \\ b_3 & c_{13} & a_1 \\ a_2 & b_1 & c_{12} \end{array} \qquad T_{12,43}: \begin{array}{ccc} a_1 & b_4 & c_{14} \\ b_3 & a_2 & c_{23} \\ c_{13} & c_{24} & c_{56} \end{array} \qquad T_{123,456}: \begin{array}{ccc} c_{14} & c_{25} & c_{36} \\ c_{26} & c_{34} & c_{15} \\ c_{35} & c_{16} & c_{24} \end{array}$$

There are $20, 90$ and 10 trihedral pairs of kind $T_{123}, T_{12,34}$ and $T_{123,456}$ respectively. Now consider a trihedral pair, namely T_{123}

$$
\begin{array}{rcccl}
 & c_{23} & a_3 & b_2 & \to \mathbb{V}(L_1) \\
T_{123}: & b_3 & c_{13} & a_1 & \to \mathbb{V}(L_2) \\
 & a_2 & b_1 & c_{12} & \to \mathbb{V}(L_3) \\
 & \downarrow & \downarrow & \downarrow & \\
 & \mathbb{V}(L'_1) & \mathbb{V}(L'_2) & \mathbb{V}(L'_3) &
\end{array}
$$

Let the three planes sections $c_{23}a_3b_2$, $b_3c_{13}a_1$, $a_2b_1c_{12}$ be given by $\mathbb{V}(L_1)$, $\mathbb{V}(L_2)$, $\mathbb{V}(L_3)$ for L_1, L_2, L_3 linear polynomials, and let the three planes sections $c_{23}b_3a_2$, $a_3c_{13}b_1$, $b_2a_1c_{12}$ be given by $\mathbb{V}(L'_1)$, $\mathbb{V}(L'_2)$, $\mathbb{V}(L'_3)$ for L'_1, L'_2,L'_3 linear polynomials. Then the equation of a cubic surface $\mathscr{S}$ is given by

$$
\mathscr{S} = \mathbb{V}(L_1L_2L_3 + \lambda L'_1L'_2L'_3), \tag{3.2.1}
$$

for some $\lambda \in GF(q)\backslash\{0\}$ (See [27], Page 196 and [4], pages 4,7). Note that the nine lines each being the intersection of two cubic plane sections of a trihedral pair, are: $\mathbb{V}(L_i, L'_j), i, j \in \{1, 2, 3\}$. More precisely, for a trihedral pair T_{123} we have:

$$
\begin{aligned}
\mathbb{V}(L_1, L'_1) &= \mathbb{V}(L_1) \cap \mathbb{V}(L'_1) = c_{23}, \\
\mathbb{V}(L_1, L'_2) &= \mathbb{V}(L_1) \cap \mathbb{V}(L'_2) = a_3, \\
\mathbb{V}(L_1, L'_3) &= \mathbb{V}(L_1) \cap \mathbb{V}(L'_3) = b_2, \\
\mathbb{V}(L_2, L'_1) &= \mathbb{V}(L_2) \cap \mathbb{V}(L'_1) = b_3, \\
\mathbb{V}(L_2, L'_2) &= \mathbb{V}(L_2) \cap \mathbb{V}(L'_2) = c_{13}, \\
\mathbb{V}(L_2, L'_3) &= \mathbb{V}(L_2) \cap \mathbb{V}(L'_3) = a_1, \\
\mathbb{V}(L_3, L'_1) &= \mathbb{V}(L_3) \cap \mathbb{V}(L'_1) = a_2, \\
\mathbb{V}(L_3, L'_2) &= \mathbb{V}(L_3) \cap \mathbb{V}(L'_2) = b_1, \\
\mathbb{V}(L_3, L'_3) &= \mathbb{V}(L_3) \cap \mathbb{V}(L'_3) = c_{12}.
\end{aligned}
$$

Let $\mathscr{S}$ be a non-singular cubic surface with 27 lines. Then we have the following facts in ([27], Pages 198,199):

1. If $\mathscr{S}$ contains exactly 6 E-points, they form 6 vertices of a plane quadrilateral.

2. If $\mathscr{S}$ contains exactly 10 E-points, they form the 10 vertices of a pentahedron $\mathcal{P}$, lying 2 on each of the 15 lines residual to a double-six and whose collinear triples lie on the 10 edges of $\mathcal{P}$; that is, a non-planar Desargues configuration.

A cubic surface $\mathscr{S}$ with 27 lines in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$, exist and have $q^2 + 7q + 1$ points. The possible two types of E-point, namely, $E_{ij,kl,mn}$ and E_{st} which correspond to $(ij, kl, mn) := \overline{P_iP_j} \cap \overline{P_kP_l} \cap \overline{P_mP_n}$ and $\mathscr{C}_t \cap \overline{P_sP_t}$ respectively are found, as we will show in the following example:

Let us consider the 6-arc $\mathcal{S}_1 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{8}\}$ in $PG(2, 17)$. From Table 2.11, we have

$$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 2x_0x_1 + x_0x_2 - 4x_1x_2),$$

$$\mathscr{C}_2 = \mathbb{V}(x_1^2 + 2x_0x_1 - 5x_0x_2 + 2x_1x_2),$$

$$\mathscr{C}_3 = \mathbb{V}(x_2^2 + 3x_0x_1 + 5x_0x_2 + 8x_1x_2),$$

$$\mathscr{C}_4 = \mathbb{V}(x_0x_1 - 5x_0x_2 - 4x_1x_2),$$

$$\mathscr{C}_5 = \mathbb{V}(x_0x_1 - 6x_0x_2 + 5x_1x_2),$$

$$\mathscr{C}_6 = \mathbb{V}(x_0x_1 - 7x_0x_2 + 6x_1x_2),$$

where $\mathscr{C}_j$ is a conic passing through the points of $\mathcal{S}_1 \backslash \{P_j\}$. The 15 bisecants of $\mathcal{S}_1$ are:

$$\overline{P_1P_2} = \mathbb{V}(x_2),$$

$$\overline{P_1P_3} = \mathbb{V}(x_1),$$

$$\overline{P_1P_4} = \mathbb{V}(x_1 - x_2),$$

$$\overline{P_1P_5} = \mathbb{V}(x_1 + x_2),$$

$$\overline{P_1P_6} = \mathbb{V}(x_1 + 4x_2),$$

$$\overline{P_2P_3} = \mathbb{V}(x_0),$$

$$\overline{P_2P_4} = \mathbb{V}(x_0 - x_2),$$

$$\overline{P_2P_5} = \mathbb{V}(x_0 + 5x_2),$$

$$\overline{P_2P_6} = \mathbb{V}(x_0 + 2x_2),$$

$$\overline{P_3P_4} = \mathbb{V}(x_0 - x_1),$$

$$\overline{P_3P_5} = \mathbb{V}(x_0 - 5x_1),$$

$$\overline{P_3P_6} = \mathbb{V}(x_0 + 8x_1),$$

$$\overline{P_4P_5} = \mathbb{V}(x_0 - 3x_1 + 2x_2),$$

$$\overline{P_4P_6} = \mathbb{V}(x_0 - 4x_1 + 3x_2),$$

$$\overline{P_5P_6} = \mathbb{V}(x_0 + x_1 + 6x_2).$$

Consequently, there is only E-point of type $E_{ij,kl,mn}$ which corresponds to Brianchon point, namely

$$(13, 26, 45) = \overline{P_1P_3} \cap \overline{P_2P_6} \cap \overline{P_4P_5} = (1 : 0 : 8).$$

However, there are two E-points of type E_{ij}, namely E_{25} and E_{46} which correspond to

$$\mathscr{C}_5 \cap \overline{P_2P_5} = (0 : 1 : 0),$$

$$\mathscr{C}_6 \cap \overline{P_4P_6} = (1 : 1 : 1).$$

Over the Galois field $GF(q)$ with $q = 17, 19$, we have the Table 3.4 and Table 3.5 which illustrate the Brianchon points of 6-arcs over $GF(17)$, $GF(19)$ respectively.

Let $\mathscr{S}^{(j)}(q)$ denotes the smooth cubic surface corresponding to the 6-arcs $\mathcal{S}$ not on a conic, with j Eckardt points. Then from Theorem 2.3, such non-singular cubic

Table 3.4: The Briancohn points of $\mathcal{S}_j$ over $GF(17)$

$PG(2,17)$				$PG(2,17)$			
$\mathcal{S}_j$	$\mathscr{C}_t$	E_{st}	$E_{ij,kl,mn}$	$\mathcal{S}_j$	$\mathscr{C}_t$	E_{st}	$E_{ij,kl,mn}$
$\mathcal{S}_1$	$\mathscr{C}_1$	-	$E_{13,26,45}$	$\mathcal{S}_5$	$\mathscr{C}_1$	-	$E_{12,35,46}$
	$\mathscr{C}_2$	-			$\mathscr{C}_2$	-	
	$\mathscr{C}_3$	-			$\mathscr{C}_3$	E_{43}	$E_{12,36,45}$
	$\mathscr{C}_4$	-			$\mathscr{C}_4$	E_{34}	
	$\mathscr{C}_5$	E_{25}			$\mathscr{C}_5$	E_{65}	
	$\mathscr{C}_6$	E_{46}			$\mathscr{C}_6$	E_{56}	
$\mathcal{S}_2$	$\mathscr{C}_1$	-	$E_{16,23,45}$	$\mathcal{S}_6$	$\mathscr{C}_1$	-	$E_{12,34,56}$
	$\mathscr{C}_2$	-	$E_{16,25,34}$		$\mathscr{C}_2$	-	$E_{14,25,36}$
	$\mathscr{C}_3$	E_{23}			$\mathscr{C}_3$	-	$E_{16,23,45}$
	$\mathscr{C}_4$	-			$\mathscr{C}_4$	-	$E_{16,25,34}$
	$\mathscr{C}_5$	E_{45}			$\mathscr{C}_5$	-	
	$\mathscr{C}_6$	-			$\mathscr{C}_6$	-	
$\mathcal{S}_4$	$\mathscr{C}_1$	-	$E_{12,34,56}$	$\mathcal{S}_7$	$\mathscr{C}_1$	-	$E_{12,35,46}$
	$\mathscr{C}_2$	E_{52}	$E_{13,25,46}$		$\mathscr{C}_2$	-	$E_{13,24,56}$
	$\mathscr{C}_3$	E_{43}	$E_{16,24,35}$		$\mathscr{C}_3$	-	$E_{14,26,35}$
	$\mathscr{C}_4$	-			$\mathscr{C}_4$	-	$E_{15,24,36}$
	$\mathscr{C}_5$	-			$\mathscr{C}_5$	-	$E_{16,23,45}$
	$\mathscr{C}_6$	E_{16}			$\mathscr{C}_6$	-	$E_{16,25,34}$

surfaces $\mathscr{S}^{(j)}(q)$ exist and have $409, 495, 691, 1045$ and 1179 points in $PG(3,q)$ for $q = 17, 19, 23, 29, 31$ respectively.

3.3 e-INVARIANTS AND A NON-SINGULAR CUBIC SURFACES WITH 27 LINES

A cubic surface $\mathscr{S}$ always has 27 lines in this section. Recall a point E is said to be an Eckardt point if it lies on exactly three lines of the non-singular cubic surface $\mathscr{S}$ in $PG(3,q)$. Let us define $e_r = e_r(\mathscr{S})$ to be the number of points of $\mathscr{S}$ on exactly r lines of $\mathscr{S}$. So e_3 is the number of Eckardt points of $\mathscr{S}$. The numbers e_r have been computed for any non-singular cubic surface over $GF(q)$ for $q = 17, 19, 23, 29, 31$, as we will show later in Section 3.6.

Let n_q be the total number of points on the lines of $\mathscr{S}$ over $GF(q)$. Then

$$n_q = e_3 + e_2 + e_1.$$

Table 3.5: The Briancho points of $\mathcal{S}_j$ over $GF(19)$

$PG(2,19)$			
$\mathcal{S}_j$	$\mathscr{C}_t$	E_{st}	$E_{ij,kl,mn}$
$\mathcal{S}_1$	$\mathscr{C}_1$	-	-
	$\mathscr{C}_2$	-	
	$\mathscr{C}_3$	-	
	$\mathscr{C}_4$	-	
	$\mathscr{C}_5$	-	
	$\mathscr{C}_6$	E_{46}, E_{56}	
$\mathcal{S}_3$	$\mathscr{C}_1$	E_{31}, E_{51}	$E_{13,26,45}$
	$\mathscr{C}_2$	E_{52}, E_{62}	$E_{15,23,46}$
	$\mathscr{C}_3$	-	$E_{16,23,45}$
	$\mathscr{C}_4$	E_{34}, E_{64}	$E_{16,25,34}$
	$\mathscr{C}_5$	-	
	$\mathscr{C}_6$	-	
$\mathcal{S}_4$	$\mathscr{C}_1$	-	$E_{12,35,46}$
	$\mathscr{C}_2$	-	$E_{14,23,56}$
	$\mathscr{C}_3$	-	$E_{15,26,34}$
	$\mathscr{C}_4$	-	
	$\mathscr{C}_5$	-	
	$\mathscr{C}_6$	-	
$\mathcal{S}_6$	$\mathscr{C}_1$	E_{21}, E_{41}	$E_{14,26,35}$
	$\mathscr{C}_2$	E_{12}, E_{62}	$E_{16,24,35}$
	$\mathscr{C}_3$	-	
	$\mathscr{C}_4$	E_{24}, E_{64}	
	$\mathscr{C}_5$	-	
	$\mathscr{C}_6$	E_{16}, E_{46}	

$PG(2,19)$			
$\mathcal{S}_j$	$\mathscr{C}_t$	E_{st}	$E_{ij,kl,mn}$
$\mathcal{S}_7$	$\mathscr{C}_1$	-	$E_{12,35,46}$
	$\mathscr{C}_2$	-	$E_{13,24,56}$
	$\mathscr{C}_3$	-	$E_{14,26,35}$
	$\mathscr{C}_4$	-	$E_{15,24,36}$
	$\mathscr{C}_5$	-	$E_{16,23,45}$
	$\mathscr{C}_6$	-	$E_{16,25,34}$
$\mathcal{S}_8$	$\mathscr{C}_1$	-	$E_{12,34,56}, E_{12,36,45}$
	$\mathscr{C}_2$	-	$E_{13,24,56}, E_{13,25,46}$
	$\mathscr{C}_3$	-	$E_{14,25,36}, E_{14,26,35}$
	$\mathscr{C}_4$	-	$E_{15,23,46}, E_{15,26,34}$
	$\mathscr{C}_5$	-	$E_{16,23,45}, E_{16,24,35}$
	$\mathscr{C}_6$	-	
$\mathcal{S}_9$	$\mathscr{C}_1$	E_{31}, E_{61}	$E_{12,34,56}$
	$\mathscr{C}_2$	E_{42}, E_{52}	$E_{12,35,46}$
	$\mathscr{C}_3$	E_{13}, E_{63}	$E_{14,23,56}$
	$\mathscr{C}_4$	E_{24}, E_{54}	$E_{14,26,35}$
	$\mathscr{C}_5$	E_{25}, E_{45}	$E_{15,23,46}$
	$\mathscr{C}_6$	E_{16}, E_{36}	$E_{15,26,34}$

Now let $l_i, i = 1, \ldots, 27$, be the 27 lines of $\mathscr{S}$ and let $e_r^{(i)}$ be the number of points of l_i on exactly r lines of $\mathscr{S}$. then

$$\sum_{i=1}^{27} e_3^{(i)} = 3e_3,$$
$$\sum_{i=1}^{27} e_2^{(i)} = 2e_2,$$
$$\sum_{i=1}^{27} e_1^{(i)} = e_1.$$

Also we know that each line meets ten others. So we have

$$2e_3^{(i)} + e_2^{(i)} = 10,$$

$$e_3^{(i)} + e_2^{(i)} + e_1^{(i)} = q + 1.$$

So when we take sum over all $i = 1, \dots, 27$, we get

$$6e_3 + 2e_2 = 270,$$

$$3e_3 + 2e_2 + e_1 = 27(q+1).$$

More precisely, we obtain

$$e_2 + e_1 = 27(q-4),$$

$$n_q = 27(q-4) + e_3.$$

From Section 2.1, we have $\#(\mathscr{S}) = q^2 + 7q + 1$. Hence

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1,$$

or

$$n_q = 27(q-4) + e_3,$$

$$e_3 + e_0 = q^2 + 7q + 1 - 27(q-4) = (q-10)^2 + 9.$$

So

$$\#(\mathscr{S}) = n_q + e_0.$$

We define the e-invariants that correspond to a non-singular cubic surface with 27 lines, $\mathscr{S}$, as the set $\{e_0, e_1, e_2, e_3\}$. Our classifications over $GF(q)$ for $q = 17$, 19, 23, 29, 31 of non-singular cubic surfaces with 27 lines (up to e-invariants), are given in Section 3.6.

Theorem 3.4. ([27], Page 194) Let $\mathscr{S}$ be a non-singular cubic surface with 27 lines. Then for $q \leq 16$, upper and lower bounds for e_3 are given by the Table 3.6.

Table 3.6: Lower and upper bound for e_3

q	4	5	7	8	9	11	13	16
Upper bound for e_3	45	34	18	13	10	10	18	16
Lower bound for e_3	45	36	18	9	0	0	0	45

Table 3.6 does not give exactly the minimal and maximal value of e_3 over $GF(q)$ for $q \leq 16$. Furthermore, in Table 3.6 we can see that for the case $q = 5$ the upper

bound for e_3 is 34 while the lower bound is 36. This is impossible. Thus there is no cubic surface with 27 lines for $q = 5$. In fact, if there is such cubic surface, we get

$$e_3 \leq e_3 + e_0 = (q - 10)^2 + 9 = 34$$

which is contradiction.

The exact value of minimal and maximal value of e_3 over $GF(13)$, is given by [2]. In our work, we completed the above table and give the exact value of minimal and maximal value of e_3 over $GF(q)$ for $q = 17, 19, 23, 29, 31$, and we have the following theorem.

Theorem 3.5. Let $\mathscr{S}$ be a non-singular cubic surface with 27 lines. Then for $q = 17, 19, 23, 29, 31$, the minimal and maximum value for e_3 are given in Table 3.7

Table 3.7: Minimal and maximal value of e_3

q	17	19	23	29	31
Maximal value of e_3	6	18	6	10	18
Minimal value of e_3	1	2	1	0	0

Proof. All the detail of the above table are given in our work in Section 3.6. □

In the next three sections of this chapter, we will discuss the geometrical configuration formed by six E-points in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$. Furthermore, we will determine the equations of all the correspond non-singular cubic surfaces with 27 lines. The maximal number of Eckardt points on a non-singular cubic surface will be indicated. Additionally, we will find all elliptic and hyperbolic lines on a non-singular cubic surface with 27 lines in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$.

3.4 The geometrical configuration formed by six and ten E-points in $PG(3, q)$

In this section, we will give some examples to explain the geometrical configuration formed by six, ten and eighteen E-points in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$.

From previous chapter, and by using our program, we know that a six E-points of a non-singular cubic surface are the 6 vertices of a plane quadrilateral. The non-singular cubic surface correspond to the 6-arc $\mathcal{S}_7$ in $PG(2,19)$, has 6 E-points, which correspond to $E_{12,35,46}, E_{13,24,56}, E_{14,26,35}, E_{15,24,36}, E_{16,23,45}$ and $E_{16,25,34}$, is denoted by $\mathscr{S}^{(6)}(19)$. Also, the 10 E-points of a non-singular cubic surface $\mathscr{S}^{(10)}(19)$, which correspond to

$$E_{31}, E_{51}, E_{52}, E_{62}, E_{34}, E_{64}, E_{13,26,45}, E_{15,23,46}, E_{16,23,45} \text{ and } E_{16,25,34}$$

of $\mathcal{S}_3$ in $PG(2,19)$, are the 10 vertices of a pentahedron $\mathcal{P}$ lying 2 on each of the 15 lines residual to a double-six and whose collinear triples lie on the 10 edges of $\mathcal{P}$. In fact, such configuration of the E-points exists since $x^2 - x - 1 = 0$ has two roots in $GF(19)$, namely 5 and -4 ([16], Page 201). A cubic surface whose 10 E-points form such a configuration is called a diagonal surface, and we have that, in a suitable coordinate system, the diagonal surface can be written in the form

$$\mathscr{S} = \mathbb{V}(x_0^3 + x_1^3 + x_2^3 + x_3^3 + x_4^3) \text{ for } p \neq 3$$

where $x_0 + x_1 + x_2 + x_3 + x_4 = 0$.

Furthermore, in $PG(2,19)$, we have the configuration of 18 E-points, two on each of the 27 lines lying by threes on the six axes of a triad of trihedral pairs $T_{123}, T_{456}, T_{123,456}$. In fact, such configuration exists in $PG(2,19)$ since $19 \equiv 1 \bmod 3$, that is, the equation $x^2 + x + 1 = 0$ has two roots in $GF(19)$, namely 7 and -8 ([27], Page 200). For the later case and in a suitable coordinate system, the cubic surfaces can be written in the form

$$\mathscr{S} = \mathbb{V}(x_0^3 + x_1^3 + x_2^3 + x_3^3).$$

The following configurations, namely $\mathcal{Q}^{(6)}(19)$ which is the plane quadrilateral formed by the 6 E-points of $\mathscr{S}^{(6)}(19)$, and $\mathcal{Q}^{(10)}(19)$ which is pentahedron formed by the 10 E-points of $\mathscr{S}^{(10)}(19)$, are shown in Figure 3.2 and Figure 3.3 respectively.

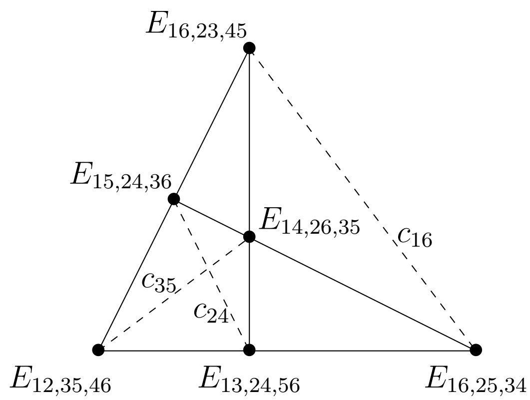

Figure 3.2: The configuration $\mathcal{Q}^{(6)}(19)$.

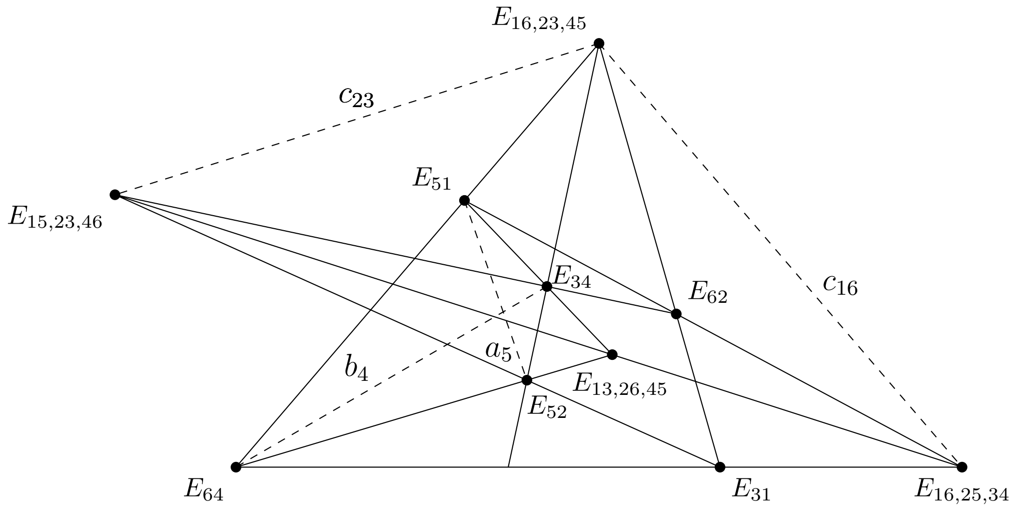

Figure 3.3: The configuration $\mathcal{Q}^{(10)}(19)$.

3.5 Elliptic and hyperbolic lines on a non-singular cubic surfaces

In this section, we introduce the concepts of resultant and involution. Furthermore, we define the concepts of elliptic and hyperbolic lines on a non-singular cubic surface, and then we demonstrate these concepts by giving some examples.

Let $f(x) = a_n x^n + \ldots + a_0$ and $g(x) = b_m x^m + \ldots + b_0$ be two polynomials of degrees n and m respectively, with coefficients in an arbitrary field $\mathbb{F}$. Their resultant $R(f, g) = R_{n,m}(f, g)$ is the element of $\mathbb{F}$ given by the determinant of the

$(m+n)\times(m+n)$ Sylvester matrix $Syl(f,g) = Syl_{n,m}(f,g)$ defined as

$$Syl_{n,m}(f,g) = \begin{pmatrix} a_n & a_{n-1} & a_{n-2} & \ldots & 0 & 0 & 0 \\ 0 & a_n & a_{n-1} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & a_1 & a_0 & 0 \\ 0 & 0 & 0 & \ldots & a_2 & a_1 & a_0 \\ b_m & b_{m-1} & b_{m-2} & \ldots & 0 & 0 & 0 \\ 0 & b_m & b_{m-1} & \ldots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \ldots & b_1 & b_0 & 0 \\ 0 & 0 & 0 & \ldots & b_2 & b_1 & b_0 \end{pmatrix}$$

where the m first rows contain the coefficients $a_n, a_{n-1}, \ldots, a_0$ of f shifted $0, 1, \ldots, m-1$ steps and padded with zeros, and the n last rows contain the coefficients b_m, b_{m-1}, …, b_0 of g shifted $0, 1, \ldots, n-1$ steps and padded with zeros. The following facts are taken from ([8], Pages 162,163 and [20]):

1. Let $f(x) = a_n x^n + \ldots + a_0$ and $g(x) = b_m x^m + \ldots + b_0$ be two polynomials of degrees n and m respectively, with coefficients in an arbitrary field $\mathbb{F}$. Suppose that, in some extension field of $\mathbb{F}$ (for example in an algebraically closed extension), f has n roots $\xi_1, \ldots, \xi_n$ and g has m roots $\eta_1, \ldots, \eta_m$ (not necessarily distinct). Then

$$\begin{aligned} R(f,g) &= a_n^m b_m^n \prod_{i=1}^{n} \prod_{j=1}^{m} (\xi_i - \eta_j) \\ &= a_n^m \prod_{i=1}^{n} g(\xi_i) \\ &= (-1)^{nm} b_m^n \prod_{j=1}^{m} f(\eta_j). \end{aligned}$$

2. Let f and g be two non-zero polynomials of degrees n and m respectively with coefficients in an arbitrary field $\mathbb{F}$. Then f and g have a common root in some extension of $\mathbb{F}$ if and only if $R(f,g)=0$.

3. If f and g are polynomials of degrees n and $m \geq 1$, then

$$\Delta(fg) = \Delta(f)\Delta(g)R(f,g)^2$$

where $\Delta(f)$ is the discriminant of f, which is given by

$$\Delta(f) = {a_n}^{2n-2} \prod_{1\leq i<j\leq n} (\xi_i - \xi_j).$$

For the special case, say $n=m=2$, we have

$$\begin{aligned} R(f,g) &= \begin{vmatrix} a_2 & a_1 & a_0 & 0 \\ 0 & a_2 & a_1 & a_0 \\ b_2 & b_1 & b_0 & 0 \\ 0 & b_2 & b_1 & b_0 \end{vmatrix} \\ &= (a_2b_0 - b_2a_0)^2 - (a_2b_1 - b_2a_1)(a_1b_0 - b_1a_0). \end{aligned}$$

Segre divided the lines on cubic surfaces into two species called hyperbolic and elliptic. Consider a real line on the cubic surface. The tangent plane to any point on this line will intersect the surface in the line itself and a further residual conic (perhaps another pair of lines). This residual conic will intersect the line in two points, one of which being the point where we took the tangent plane from. We define an involution on the line by exchanging these two points of intersection. The fixed points of this involution are called parabolic points [28]. It is possible that the parabolic points only exist in the complexification. The real line is called a hyperbolic line if the involution has two real parabolic points. The real line is called an elliptic line if it has a pair of complex conjugate parabolic points.

More concretely, choose projective coordinates x_0, x_1, x_2, x_3 on $\mathbb{P}^3_{\mathbb{R}}$ so that the line l is given by $x_0 = x_1 = 0$. Then the defining polynomial of the surface $\mathscr{S}$ has the form

$$f = x_2^2\mathfrak{l}_{11} + 2x_2x_3\mathfrak{l}_{12} + x_3^2\mathfrak{l}_{22} + x_2\mathfrak{q}_1 + x_3\mathfrak{q}_2 + \mathfrak{c},$$

where $\mathfrak{l}_{ij}$,$\mathfrak{q}_i$ and $\mathfrak{c}$ are of degree one, two, and three homogeneous polynomial in x_0, x_1. Any plane containing l is given by the equation $bx_0 - ax_1 = 0$ for projective coordinates a, b. The pairs of conjugate points (the intersection of l and residual conic) are given by the roots of the projective quadratic in x_2, x_3

$$x_2^2\mathfrak{l}_{11} + 2x_2x_3\mathfrak{l}_{12} + x_3^2\mathfrak{l}_{22}, \tag{3.5.1}$$

with $\mathfrak{l}_{ij}$ being evaluated at $x_0 = a, x_1 = b$. A parabolic point is given by the unique root of this quadratic when its discriminant $\mathfrak{l}_{12}^2 - \mathfrak{l}_{11}\mathfrak{l}_{22}$ is zero. This discriminant is a quadratic form in $x_0 = a$ and $x_1 = b$. Let us call it $\mathcal{Q}$. If $\mathcal{Q}$ is indefinite, there are two real values of $[a : b]$ which make $\mathcal{Q}$ zero, and each of these values gives a real parabolic point by plugging it into Equation 3.3 and finding the root. If $\mathcal{Q}$ is definite, there are two complex conjugate values of $[a : b]$ making $\mathcal{Q}$ zero, which give the complex conjugate parabolic points. If we let $\mathfrak{l}_{ij} = l_{ij}x_0 + m_{ij}x_1$, then the $\mathcal{Q}$ is explicitly

$$(l_{12}^2 - l_{11}l_{22})x_0^2 + (2l_{12}m_{12} - l_{11}m_{22} - l_{22}m_{11})x_0x_1 + (m_{12}^2 - m_{11}l_{22})x_1^2.$$

Write this as $\mathcal{Q} = Ax_0^2 + Bx_0x_1 + Cx_1^2$. Then $\mathcal{Q}$ is definite (l is elliptic) if $B^2 - 4AC < 0$ and indefinite (l hyperbolic) if $B^2 - 4AC > 0$.

In [12], Finashin–Kharlamov and Okonek–Teleman observed that the equality

$$\#\,\text{real hyperbolic lines on } \mathscr{S} - \#\,\text{real elliptic lines on } \mathscr{S} = 3$$

can be deduced from Segre's work.

We discussed these results over some finite fields, namely $GF(q)$ for $q = 17$, 19, 23, 29, 31. Just as in the real case, a line $l \subset \mathscr{S}$ admits a distinguished involution, and we classify l as either hyperbolic or elliptic using Proposition 14 in [21].

When all 27 lines on $\mathscr{S}$ are defined over $GF(q)$, we have

$$\#\,\text{elliptic lines on } \mathscr{S} = 0 \bmod 2.$$

In [21], we see that if $\mathscr{S}$ is any smooth cubic surface over $GF(q)$ and $\mathfrak{e}$ denoted to the total number of elliptic lines on $\mathscr{S}$ with field of definition $GF(q^{\alpha})$ for α odd, and $\mathfrak{h}$ denoted to the total number of hyperbolic lines on $\mathscr{S}$ with field of definition $GF(q^{\alpha})$ for α even, then we have

$$\mathfrak{e} + \mathfrak{h} = 0 \bmod 2.$$

In our work, we used an arithmetic method to determine the hyperbolic and elliptic lines on a non-singular cubic surface by using Proposition 14 in [21]. The residual intersections of a non-singular cubic surface $\mathscr{S}$ with the hyperplanes containing l are conic curves that determine an involution of l, defined so that two points are exchanged if they lie on a common conic. Lines are classified as either hyperbolic or elliptic according to whether the involution is hyperbolic or elliptic as an element of $PGL(2, q)$ (i.e. whether the fixed points are defined over $GF(q)$ or not).

Let l be any line among the 27 lines on a non-singular cubic surface $\mathscr{S} = \mathbb{V}(f)$ over $GF(q)$. Let $(a_0 : a_1 : a_2 : a_3)$, $(b_0 : b_1 : b_2 : b_3)$ be any points on l, then we can parameterize l by

$$l = \Big\{(a_0\lambda + b_0\mu : a_1\lambda + b_1\mu : a_2\lambda + b_2\mu : a_3\lambda + a_3\mu) : (\lambda : \mu) \in PG(1, q)\Big\}.$$

On the other hand, we can pick a basis $\beta_3, \beta_2, \beta_1, \beta_0$ for $GF(q)^{\oplus 4}$ so that $\beta_3 = (a_0, a_1, a_2, a_3)$ and $\beta_2 = (b_0, b_1, b_2, b_3)$.

According to Proposition 14 ([21], Page 9) if we compute the resultant of the partial derivative of homogeneous polynomial defining $\mathscr{S}$ with respect to β_0 and β_1 and then restrict the result to the line l, then we get

$$\left.\frac{\partial f}{\partial \beta_0}\right|_l = a\lambda^2 + b\lambda\mu + c\mu^2,$$
$$\left.\frac{\partial f}{\partial \beta_1}\right|_l = a'\lambda^2 + b'\lambda\mu + c'\mu^2,$$

where $a, b, c, a', b', c' \in GF(q)$. Hence

$$R\left(\left.\frac{\partial f}{\partial \beta_0}\right|_l, \left.\frac{\partial f}{\partial \beta_1}\right|_l\right) = \begin{vmatrix} a & b & c & 0 \\ 0 & a & b & c \\ a' & b' & c' & 0 \\ 0 & a' & b' & c' \end{vmatrix}.$$

The line l is elliptic (hyperbolic) if $R\left(\left.\frac{\partial f}{\partial \beta_0}\right|_l, \left.\frac{\partial f}{\partial \beta_1}\right|_l\right)$ is non-square (square) as an element of $GF(q)$.

Let us first give an example for the real case: The Clebsch diagonal surface, namely

$$\mathscr{S} = \mathbb{V}\Big(x_0^3 + x_1^3 + x_2^3 + x_3^3 - (x_0 + x_1 + x_2 + x_3)^3\Big),$$

has 27 lines on it. The following are some lines on $\mathscr{S}$ which are given parametrically in $PG(3, \mathbb{R})$ by:

1. $(\mu : -\mu : \lambda : 0)$, **2**. $(\mu : \lambda : -\mu : 0)$,

3. $(\mu : \lambda : -\lambda : 0)$, **4**. $(\lambda : \mu : -\mu : 0)$,

5. $(-\mu : \lambda : \mu : 0)$, **6**. $(\lambda : -\lambda : \mu : \mu)$,

7. $(\lambda : -\lambda : 0 : \mu)$, **8**. $(\lambda : -\mu : -\lambda : \mu)$,

9. $(\lambda : 0 : -\lambda : \mu)$, **10**. $(-\mu : \lambda : -\lambda : \mu)$,

11. $(\lambda : -\lambda + a\mu : a\lambda - \mu : \mu)$, **12**. $(\lambda : -\lambda + b\mu : b\lambda - \mu : \mu)$,

13. $(\lambda : -\lambda + a\mu : -a\lambda - a\mu : \mu)$, **14**. $(\lambda : -\lambda + b\mu : -b\lambda - b\mu : \mu)$,

15. $(\lambda : a\lambda - \mu : -\lambda + a\mu : \mu)$, **16**. $(\lambda : b\lambda - \mu : -\lambda + b\mu : \mu)$,

17. $(\lambda : -a\lambda - a\mu : -\lambda + a\mu : \mu)$, **18**. $(\lambda : -b\lambda - b\mu : -\lambda + b\mu : \mu)$,

19. $(\lambda : -a\lambda - \mu : a\lambda + a\mu : \mu)$, **20**. $(\lambda : -b\lambda - \mu : b\lambda + b\mu : \mu)$,

21. $(\lambda : a\lambda + a\mu : -a\lambda - \mu : \mu)$, **22**. $(\lambda : b\lambda + b\mu : -b\lambda - \mu : \mu)$,

where $a = \frac{1+\sqrt{5}}{2}$ and $b = \frac{1-\sqrt{5}}{2}$ ([[3], Page 25). Let $l := (\mu : -\mu : \lambda : 0)$. For l, picking the basis $\beta_3 = (0,0,1,0)$, $\beta_2 = (1,-1,0,0)$, $\beta_1 = (0,0,0,1)$, and $\beta_0 = (0,0,0,1)$ for $\mathbb{R}^{\oplus 4}$, we get

$$R\left(\frac{\partial f}{\partial \beta_0}\bigg|_l, \frac{\partial f}{\partial \beta_1}\bigg|_l\right) = \begin{vmatrix} -3 & 0 & 3 & 0 \\ 0 & -3 & 0 & 3 \\ -3 & 0 & 0 & 0 \\ 0 & -3 & 0 & 0 \end{vmatrix} = 81 \ \text{square}\,.$$

Hence l is hyperbolic line. In fact, all the first 10 lines are hyperbolic.

Now let us work over the Galois field $GF(q)$: consider the non-singular cubic surface $\mathscr{S} = \mathbb{V}(f)$ over $GF(17)$, where

$$f(y_0, y_1, y_2, y_3) = y_0^2 y_1 + 4y_1^2 y_0 + y_2^2 y_3 + 2y_3^2 y_2 - 8y_0 y_1 y_2 - 4y_0 y_2 y_3 - 5y_1 y_2 y_3.$$

Let us determine the kind of the following two lines l and m on $\mathscr{S}$, namely

$$l = \{(\lambda : 0 : \mu : 0) : (\lambda : \mu) \in PG(1,17)\},$$

$$m = \{(0 : \lambda : \mu : 0) : (\lambda : \mu) \in PG(1,17)\}.$$

For l, picking the basis $\beta_3 = (0,0,1,0)$, $\beta_2 = (1,0,0,0)$, $\beta_1 = (0,0,0,1)$, and $\beta_0 = (0,1,0,0)$, we get

$$R\left(\frac{\partial f}{\partial \beta_0}\bigg|_l, \frac{\partial f}{\partial \beta_1}\bigg|_l\right) = \begin{vmatrix} 1 & -8 & 0 & 0 \\ 0 & 1 & -8 & 0 \\ 0 & -4 & 1 & 0 \\ 0 & 0 & -4 & 1 \end{vmatrix} = 3 \ \text{non-square} \pmod{17}.$$

Hence l is an elliptic line.

For m, picking the basis $\beta_3 = (0,0,1,0)$, $\beta_2 = (0,1,0,0)$, $\beta_1 = (0,0,0,1)$, and $\beta_0 = (1,0,0,0)$, we get

$$R\left(\frac{\partial f}{\partial \beta_0}\bigg|_m, \frac{\partial f}{\partial \beta_1}\bigg|_m\right) = \begin{vmatrix} 8 & -8 & 0 & 0 \\ 0 & 8 & -8 & 0 \\ 0 & -5 & 1 & 0 \\ 0 & 0 & -5 & 1 \end{vmatrix} = 16 \text{ square} \pmod{17}.$$

Hence m is hyperbolic line.

All the hyperbolic and elliptic lines on a non-singular cubic surface with 27 lines over $GF(q)$ for $q = 17, 19, 23, 29, 31$ are indicated in tables for the next section.

3.6 The equation of a non-singular cubic surface

In this section, we will give the equation of a non-singular cubic surface with 27 lines in $PG(3,q)$ for $q = 17, 19, 23, 29, 31$. Furthermore, we will find the lines and Eckardt points on each of them, and we will determine whether the line is elliptic or hyperbolic.

Recall that a cubic surface $\mathscr{S}$ with 27 lines in $PG(3,q)$ can be mapped onto the plane in the following way. Let $\mathscr{S} = V(F)$ be given, as in Equation 3.2, by

$$F = L_1L_2L_3 + L_1'L_2'L_3' = \begin{vmatrix} 0 & L_1 & L_3' \\ L_1' & 0 & L_2 \\ L_3 & L_2' & 0 \end{vmatrix} = 0,$$

where $L_1, L_2, L_3, L_1', L_2'$ and L_3' are homogeneous linear polynomials in 4 variables, namely y_0, y_1, y_2 and y_3. The above homogeneous linear polynomials can be written as

$$L_1 = \alpha_{00}y_0 + \alpha_{01}y_1 + \alpha_{02}y_2 + \alpha_{03}y_3,$$

$$L_2 = \alpha_{10}y_0 + \alpha_{11}y_1 + \alpha_{12}y_2 + \alpha_{13}y_3,$$

$$L_3 = \alpha_{30}y_0 + \alpha_{31}y_1 + \alpha_{32}y_2 + \alpha_{33}y_3,$$

$$L'_1 = \beta_{00}y_0 + \beta_{01}y_1 + \beta_{02}y_2 + \beta_{03}y_3,$$

$$L'_2 = \beta_{10}y_0 + \beta_{11}y_1 + \beta_{12}y_2 + \beta_{13}y_3,$$

$$L'_3 = \beta_{30}y_0 + \beta_{31}y_1 + \beta_{32}y_2 + \beta_{33}y_3,$$

where α_{ij} and β_{ij} are elements in $GF(q)$. Let $P(Y) := P(y_0, y_1, y_2, y_3)$ be a point in the projective space $PG(3, q)$ such that $P(Y) \notin \mathbb{V}(L_1) \cup \mathbb{V}(L_2) \cup \mathbb{V}(L_3) \cup \mathbb{V}(L'_1) \cup \mathbb{V}(L'_2) \cup \mathbb{V}(L'_3)$. A point $P(Y)$ is on $\mathscr{S}$ if and only if

$$\begin{vmatrix} 0 & L_1 & L'_3 \\ L'_1 & 0 & L_2 \\ L_3 & L'_2 & 0 \end{vmatrix} = 0.$$

Consequently, there exists a point $P(X) = (x_0, x_1, x_2) \in PG(2, q)$ such that the following system has a non-trivial unique solution:

$$\begin{aligned} L'_1(Y)x_1 + L_3(Y)x_2 &= 0, \\ L_1(Y)x_0 + L'_2(Y)x_2 &= 0, \\ L'_3(Y)x_0 + L_2(Y)x_1 &= 0. \end{aligned} \tag{3.6.1}$$

Let us illustrate the Clebsch map, namely $s : \mathscr{S} \dashrightarrow PG(2, q)$, in more detail: This map takes $P(Y) := P(y_0, y_1, y_2, y_3)$ to $P(X) = (x_0, x_1, x_2)$, where

$$\frac{x_0}{x_1} = -\frac{L_2(Y)}{L'_3(Y)},$$
$$\frac{x_0}{x_2} = -\frac{L'_2(Y)}{L_1(Y)},$$
$$\frac{x_1}{x_2} = -\frac{L_3(Y)}{L'_1(Y)} = -\frac{L'_2(Y)L'_3(Y)}{L_1(Y)L_2(Y)}.$$

It follows that

$$x_0 = L_2(Y)L_3(Y)L_2'(Y),$$

$$x_1 = -L_3(Y)L_2'(Y)L_3'(Y),$$

$$x_2 = -L_1(Y)L_2(Y)L_3(Y).$$

Moreover, if we assume that $x_2 = 1$ then

$$\begin{aligned} x_0 &= \frac{x_0}{x_1}\frac{x_1}{x_2} = \frac{L_2(Y)L_3(Y)}{L_1'(Y)L_3'(Y)}, \\ x_1 &= \frac{x_1}{x_2} = -\frac{L_3(Y)}{L_1'(Y)}, \\ x_2 &= \frac{x_1}{x_2}\cdot\frac{x_2}{x_0}\cdot\frac{x_0}{x_1} = -\frac{L_1(Y)L_2(Y)L_3(Y)}{L_1'(Y)L_2'(Y)L_3'(Y)} = 1. \end{aligned}$$

At this stage, we find rational functions, namely

$$\begin{aligned} x_0 &= \rho_0(y_0, y_1, y_2, y_3) = \frac{L_2(Y)L_3(Y)}{L_1'(Y)L_3'(Y)}, \\ x_1 &= \rho_1(y_0, y_1, y_2, y_3) = -\frac{L_3(Y)}{L_1'(Y)}, \\ x_2 &= \rho_1(y_0, y_1, y_2, y_3) = 1. \end{aligned}$$

Hence the Clebsch map, namely s, is a rational map. In order to prove that s is a birational map, it is enough to show there ia a map $s^{-1} : PG(2, q) \dashrightarrow \mathscr{S}$ such that $ss^{-1} = I$. Rewrite the system in Equation 3.6.1 as

$$L_{11}(x_0, x_1, x_2)y_0 + L_{12}(x_0, x_1, x_2)y_1 + L_{13}(x_0, x_1, x_2)y_2 + L_{14}(x_0, x_1, x_2)y_3 = 0,$$

$$L_{21}(x_0, x_1, x_2)y_0 + L_{22}(x_0, x_1, x_2)y_1 + L_{23}(x_0, x_1, x_2)y_2 + L_{24}(x_0, x_1, x_2)y_3 = 0,$$

$$L_{31}(x_0, x_1, x_2)y_0 + L_{32}(x_0, x_1, x_2)y_1 + L_{33}(x_0, x_1, x_2)y_2 + L_{34}(x_0, x_1, x_2)y_3 = 0,$$

where $L_{ij}; i, j \in \{1, 2, 3, 4\}$ are linear forms in the variables x_0, x_1 and x_2. Hence, we have the system $LY^t = 0$ where

$$L = \begin{pmatrix} L_{11} & L_{12} & L_{13} & L_{14} \\ L_{21} & L_{22} & L_{23} & L_{24} \\ L_{31} & L_{32} & L_{33} & L_{34} \end{pmatrix}.$$

Let

$$L' = \begin{pmatrix} L_{11} & L_{12} & L_{13} \\ L_{21} & L_{22} & L_{23} \\ L_{31} & L_{32} & L_{33} \end{pmatrix}.$$

Then

$$\begin{pmatrix} y_0 \\ y_1 \\ y_2 \end{pmatrix} = \mathrm{Adj}(L') \begin{pmatrix} L_{14} \\ L_{24} \\ L_{34} \end{pmatrix} \text{ and } y_3 = -\det(L'),$$

where $\mathrm{Adj}(L')$ is the adjoint of L'. In other words, we get

$$\begin{aligned}
y_0 &= \gamma_{01}x_0^2x_1 + \gamma_{02}x_0^2x_2 + \gamma_{03}x_1^2x_0 + \gamma_{04}x_1^2x_2 + \gamma_{05}x_2^2x_0 + \gamma_{06}x_2^2x_1 + \gamma_{07}x_0x_1x_2 \\
&:= W_0(X), \\
y_1 &= \gamma_{11}x_0^2x_1 + \gamma_{12}x_0^2x_2 + \gamma_{13}x_1^2x_0 + \gamma_{14}x_1^2x_2 + \gamma_{15}x_2^2x_0 + \gamma_{16}x_2^2x_1 + \gamma_{17}x_0x_1x_2 \\
&:= W_1(X), \\
y_2 &= \gamma_{21}x_0^2x_1 + \gamma_{22}x_0^2x_2 + \gamma_{23}x_1^2x_0 + \gamma_{24}x_1^2x_2 + \gamma_{25}x_2^2x_0 + \gamma_{26}x_2^2x_1 + \gamma_{27}x_0x_1x_2 \\
&:= W_2(X), \\
y_3 &= \gamma_{31}x_0^2x_1 + \gamma_{32}x_0^2x_2 + \gamma_{33}x_1^2x_0 + \gamma_{34}x_1^2x_2 + \gamma_{35}x_2^2x_0 + \gamma_{36}x_2^2x_1 + \gamma_{37}x_0x_1x_2 \\
&:= W_3(X),
\end{aligned}$$

where γ_{ij} is in $GF(q)$ and each W_i is plane cubic curve. So There is a map s^{-1} such that $s^{-1} : PG(2,q) \dashrightarrow \mathscr{S}$, and s^{-1} maps $P(X) = P(x_0, x_1, x_2)$ to $P(Y) = P(y_0, y_1, y_2, y_3)$ where $y_i = W_i(X), i = 0, 1, 2, 3$. Note that s^{-1} is again a rational map and $ss^{-1} = I$. It follows that s is a birational map.

Since the L_i and L_i' are linear, solving for the y_i gives

$$(y_0 : y_1 : y_2 : y_3) = \Big(W_0(X) : W_1(X) : W_2(X) : W_3(X)\Big),$$

where each W_i is plane cubic curve. So we have a birational map

$$s : \mathscr{S} \dashrightarrow PG(2,q)$$

given by $s(P(Y)) = P(X)$ which is the Clebsch mapping mentioned in Section 3.1. Recall that s maps plane sections of $\mathscr{S}$ to cubic curves through the set $\mathscr{B}$ of base points, where

$$\mathscr{B} = \mathbb{V}(W_0, W_1, W_2, W_3).$$

In fact, $\mathscr{B} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ is a 6-arc not on a conic. Then there exists one half of a double-six on $\mathscr{S}$, namely $a_1\, a_2\, a_3\, a_4\, a_5\, a_6$, such that if $\mathscr{A}$ is the set of points on the lines a_i then the restriction of s is a bijection

$$s : \mathscr{S} \backslash \mathscr{A} \to PG(2, q) \backslash \mathscr{B}.$$

Let $\mathscr{C}_j$ be the conic through the 5 points of $\mathscr{B}$ other than P_j. Then we have

$$s(a_i) = P_i,$$

$$s(b_j) = \mathscr{C}_j,$$

$$s(c_{ij}) = \overline{P_i P_j}.$$

Let us give some examples: consider the 6-arc not on a conic in $PG(2, 13)$, namely

$$\mathcal{S} := \{P_1, P_2, P_3, P_4, P_5, P_6\},$$

where

$$P_1 = (1 : 7 : 0), \quad P_4 = (1 : 0 : 7),$$

$$P_2 = (1 : 6 : 0), \quad P_5 = (0 : 1 : 11),$$

$$P_3 = (1 : 0 : 6), \quad P_6 = (0 : 1 : 2).$$

In fact, the six points form a 6-arc, and $\mathscr{C}_1 := \mathbb{V}(x_0^2 + 3x_1^2 - 4x_2^2 - 3x_0x_1)$ is a conic through $\mathcal{S} \backslash \{P_1\}$. The corresponding non-singular cubic surface with 27 lines, namely $\mathscr{S}$ in $PG(3, 13)$ can be determined in 5 stages as follows:

Stage(1): We find all bisecants $\overline{P_i P_j}$ of the 6-arc $\mathcal{S}$. The 15 bisecants of $\mathcal{S}$ are

$$\overline{P_1P_2} = \mathbb{V}(x_2), \qquad \overline{P_2P_6} = \mathbb{V}(x_0 + 2x_1 - x_2),$$

$$\overline{P_1P_3} = \mathbb{V}(x_0 - 2x_1 + 2x_2), \qquad \overline{P_3P_4} = \mathbb{V}(x_1),$$

$$\overline{P_1P_4} = \mathbb{V}(x_0 - 2x_1 - 2x_2), \qquad \overline{P_3P_5} = \mathbb{V}(x_0 + 4x_1 + 2x_2),$$

$$\overline{P_1P_5} = \mathbb{V}(x_0 - 2x_1 - x_2), \qquad \overline{P_3P_6} = \mathbb{V}(x_0 - 3x_1 + 2x_2),$$

$$\overline{P_1P_6} = \mathbb{V}(x_0 - 2x_1 + x_2), \qquad \overline{P_4P_5} = \mathbb{V}(x_0 - 3x_1 - 2x_2),$$

$$\overline{P_2P_3} = \mathbb{V}(x_0 + 2x_1 + 2x_2), \qquad \overline{P_4P_6} = \mathbb{V}(x_0 + 4x_1 - 2x_2),$$

$$\overline{P_2P_4} = \mathbb{V}(x_0 + 2x_1 - 2x_2), \qquad \overline{P_5P_6} = \mathbb{V}(x_0).$$

$$\overline{P_2P_5} = \mathbb{V}(x_0 + 2x_1 + x_2),$$

Stage(2): We find all the conics $\mathscr{C}_i$ through $S\backslash\{P_i\}; i = 1, 2, 3, 4, 5, 6$. The conics $\mathscr{C}_i$ are

$$\mathscr{C}_1 = \mathbb{V}(x_0^2 + 3x_1^2 - 4x_2^2 - 3x_0x_1),$$

$$\mathscr{C}_2 = \mathbb{V}(x_0^2 + 3x_1^2 - 4x_2^2 + 3x_0x_1),$$

$$\mathscr{C}_3 = \mathbb{V}(x_0^2 - 4x_1^2 + x_2^2 + 4x_0x_1),$$

$$\mathscr{C}_4 = \mathbb{V}(x_0^2 - 4x_1^2 + x_2^2 - 4x_0x_1),$$

$$\mathscr{C}_5 = \mathbb{V}(x_0^2 - 4x_1^2 - 4x_2^2 - 3x_0x_1),$$

$$\mathscr{C}_6 = \mathbb{V}(x_0^2 - 4x_1^2 - 4x_2^2 + 3x_0x_1).$$

Stage(3): We find all the plane cubics $\omega_i := \mathbb{V}(W_i)$ through $\mathcal{S}$. The 30 cubic curves of the form $\mathbb{V}(\mathscr{C}_j \cdot \overline{P_iP_j})$ are

$$\omega_1 = \mathbb{V}(x_0^3 + 5x_1^3 + 2x_2^3 + 2x_0^2x_1 + 6x_0^2x_2 - 4x_1^2x_0 + 5x_1^2x_2 - 4x_2^2x_0 + 2x_2^2x_1 - 5x_0x_1x_2),$$

$$\omega_2 = \mathbb{V}(x_0^3 - 6x_1^3 + 5x_2^3 - 5x_0^2x_1 + 2x_0^2x_2 - 4x_1^2x_0 + 6x_1^2x_2 - 4x_2^2x_0 - 5x_2^2x_1 - 6x_0x_1x_2),$$

$$\omega_3 = \mathbb{V}(x_0^3 + 6x_1^3 + 5x_2^3 + 5x_0^2x_1 + 2x_0^2x_2 - 4x_1^2x_0 + 6x_1^2x_2 - 4x_2^2x_0 + 5x_2^2x_1 + 6x_0x_1x_2),$$

$$\omega_4 = \mathbb{V}(x_0^3 - 5x_1^3 + 2x_2^3 - 2x_0^2x_1 + 6x_0^2x_2 - 4x_1^2x_0 + 5x_1^2x_2 - 4x_2^2x_0 - 2x_2^2x_1 + 5x_0x_1x_2),$$

$$\omega_5 = \mathbb{V}(x_0^3 - 5x_1^3 - 2x_2^3 - 2x_0^2x_1 - 6x_0^2x_2 - 4x_1^2x_0 - 5x_1^2x_2 - 4x_2^2x_0 - 2x_2^2x_1 - 5x_0x_1x_2),$$

$$\omega_6 = \mathbb{V}(x_0^3 - 6x_1^3 - 5x_2^3 - 5x_0^2x_1 - 2x_0^2x_2 - 4x_1^2x_0 - 6x_1^2x_2 - 4x_2^2x_0 - 5x_2^2x_1 + 6x_0x_1x_2),$$

$$\omega_7 = \mathbb{V}(x_0^3 - 5x_1^3 + 4x_2^3 - 2x_0^2x_1 - x_0^2x_2 - 4x_1^2x_0 - 3x_1^2x_2 - 4x_2^2x_0 - 2x_2^2x_1 - 3x_0x_1x_2),$$

$$\omega_8 = \mathbb{V}(x_0^3 - 6x_1^3 + 4x_2^3 - 5x_0^2x_1 - x_0^2x_2 - 4x_1^2x_0 - 3x_1^2x_2 - 4x_2^2x_0 - 5x_2^2x_1 + 3x_0x_1x_2),$$

$$\omega_9 = \mathbb{V}(x_0^3 - 5x_1^3 - 4x_2^3 - 2x_0^2x_1 + x_0^2x_2 - 4x_1^2x_0 + 3x_1^2x_2 - 4x_2^2x_0 - 2x_2^2x_1 + 3x_0x_1x_2),$$

$$\omega_{10} = \mathbb{V}(x_0^3 - 6x_1^3 - 4x_2^3 - 5x_0^2x_1 + x_0^2x_2 - 4x_1^2x_0 + 3x_1^2x_2 - 4x_2^2x_0 - 5x_2^2x_1 - 3x_0x_1x_2),$$

$$\omega_{11} = \mathbb{V}(x_0^3 + 5x_1^3 - 2x_2^3 + 2x_0^2x_1 - 6x_0^2x_2 - 4x_1^2x_0 - 5x_1^2x_2 - 4x_2^2x_0 + 2x_2^2x_1 + 5x_0x_1x_2),$$

$$\omega_{12} = \mathbb{V}(x_0^3 + 6x_1^3 - 5x_2^3 + 5x_0^2x_1 - 2x_0^2x_2 - 4x_1^2x_0 - 6x_1^2x_2 - 4x_2^2x_0 + 5x_2^2x_1 - 6x_0x_1x_2),$$

$$\omega_{13} = \mathbb{V}(x_0^3 + 5x_1^3 - 4x_2^3 + 2x_0^2x_1 + x_0^2x_2 - 4x_1^2x_0 + 3x_1^2x_2 - 4x_2^2x_0 + 2x_2^2x_1 - 3x_0x_1x_2),$$

$$\omega_{14} = \mathbb{V}(x_0^3 + 6x_1^3 - 4x_2^3 + 5x_0^2x_1 + x_0^2x_2 - 4x_1^2x_0 + 3x_1^2x_2 - 4x_2^2x_0 + 5x_2^2x_1 + 3x_0x_1x_2),$$

$$\omega_{15} = \mathbb{V}(x_0^3 + 5x_1^3 + 4x_2^3 + 2x_0^2x_1 - x_0^2x_2 - 4x_1^2x_0 - 3x_1^2x_2 - 4x_2^2x_0 + 2x_2^2x_1 + 3x_0x_1x_2),$$

$$\omega_{16} = \mathbb{V}(x_0^3 + 6x_1^3 + 4x_2^3 + 5x_0^2x_1 - x_0^2x_2 - 4x_1^2x_0 - 3x_1^2x_2 - 4x_2^2x_0 + 5x_2^2x_1 - 3x_0x_1x_2),$$

$$\omega_{17} = \mathbb{V}(x_0^3 - 3x_1^3 + 5x_2^3 + 4x_0^2x_1 + 2x_0^2x_2 - 4x_1^2x_0 + 6x_1^2x_2 - 4x_2^2x_0 + 4x_2^2x_1 - 3x_0x_1x_2),$$

$$\omega_{18} = \mathbb{V}(x_0^3 - 3x_1^3 + 2x_2^3 + 4x_0^2x_1 + 6x_0^2x_2 - 4x_1^2x_0 + 5x_1^2x_2 - 4x_2^2x_0 + 4x_2^2x_1 + 3x_0x_1x_2),$$

$$\omega_{19} = \mathbb{V}(x_0^3 + 3x_1^3 + 5x_2^3 - 4x_0^2x_1 + 2x_0^2x_2 - 4x_1^2x_0 + 6x_1^2x_2 - 4x_2^2x_0 - 4x_2^2x_1 + 3x_0x_1x_2),$$

$$\omega_{20} = \mathbb{V}(x_0^3 + 3x_1^3 + 2x_2^3 - 4x_0^2x_1 + 6x_0^2x_2 - 4x_1^2x_0 + 5x_1^2x_2 - 4x_2^2x_0 - 4x_2^2x_1 - 3x_0x_1x_2),$$

$$\omega_{21} = \mathbb{V}(x_0^3 + 3x_1^3 - 5x_2^3 - 4x_0^2x_1 - 2x_0^2x_2 - 4x_1^2x_0 - 6x_1^2x_2 - 4x_2^2x_0 - 4x_2^2x_1 - 3x_0x_1x_2),$$

$$\omega_{22} = \mathbb{V}(x_0^3 + 3x_1^3 - 2x_2^3 - 4x_0^2x_1 - 6x_0^2x_2 - 4x_1^2x_0 - 5x_1^2x_2 - 4x_2^2x_0 - 4x_2^2x_1 + 3x_0x_1x_2),$$

$$\omega_{23} = \mathbb{V}(x_0^3 - 3x_1^3 - 5x_2^3 + 4x_0^2x_1 - 2x_0^2x_2 - 4x_1^2x_0 - 6x_1^2x_2 - 4x_2^2x_0 + 4x_2^2x_1 + 3x_0x_1x_2),$$

$$\omega_{24} = \mathbb{V}(x_0^3 - 3x_1^3 - 2x_2^3 + 4x_0^2x_1 - 6x_0^2x_2 - 4x_1^2x_0 - 5x_1^2x_2 - 4x_2^2x_0 + 4x_2^2x_1 - 3x_0x_1x_2),$$

$$\omega_{25} = \mathbb{V}(x_0^3 - 4x_1^2x_0 - 4x_2^2x_0 + 3x_0x_1x_2),$$

$$\omega_{26} = \mathbb{V}(x_1^3 + 3x_0^2x_1 + 3x_2^2x_1 + x_0x_1x_2),$$

$$\omega_{27} = \mathbb{V}(x_2^3 + 3x_0^2x_2 - 4x_1^2x_2 - 4x_0x_1x_2),$$

$$\omega_{28} = \mathbb{V}(x_0^3 - 4x_1^2x_0 - 4x_2^2x_0 - 3x_0x_1x_2),$$

$$\omega_{29} = \mathbb{V}(x_1^3 + 3x_0^2x_1 + 3x_2^2x_1 - x_0x_1x_2),$$

$$\omega_{30} = \mathbb{V}(x_2^3 + 3x_0^2x_2 - 4x_1^2x_2 + 4x_0x_1x_2).$$

The 15 cubic curves of the form $\mathbb{V}(\overline{P_iP_j} \cdot \overline{P_kP_l} \cdot \overline{P_mP_n})$ are

$$\omega_{31} = \mathbb{V}(x_0^3 - 3x_1^3 - 4x_2^3 + 4x_0^2x_1 + x_0^2x_2 - 4x_1^2x_0 + 3x_1^2x_2 - 4x_2^2x_0 + 4x_2^2x_1 + x_0x_1x_2),$$

$$\omega_{32} = \mathbb{V}(x_0^3 + 3x_1^3 + 4x_2^3 - 4x_0^2x_1 - x_0^2x_2 - 4x_1^2x_0 - 3x_1^2x_2 - 4x_2^2x_0 - 4x_2^2x_1 + 2x_0x_1x_2),$$

$$\omega_{33} = \mathbb{V}(x_0^3 + 3x_1^3 - 4x_2^3 - 4x_0^2x_1 + x_0^2x_2 - 4x_1^2x_0 + 3x_1^2x_2 - 4x_2^2x_0 - 4x_2^2x_1 - x_0x_1x_2),$$

$$\omega_{34} = \mathbb{V}(x_0^3 - 3x_1^3 - 4x_2^3 + 4x_0^2x_1 + x_0^2x_2 - 4x_1^2x_0 + 3x_1^2x_2 - 4x_2^2x_0 + 4x_2^2x_1 + 2x_0x_1x_2),$$

$$\omega_{35} = \mathbb{V}(x_0^3 + 3x_1^3 - 4x_2^3 - 4x_0^2x_1 + x_0^2x_2 - 4x_1^2x_0 + 3x_1^2x_2 - 4x_2^2x_0 - 4x_2^2x_1 - 2x_0x_1x_2),$$

$$\omega_{36} = \mathbb{V}(x_0^3 - 3x_1^3 + 4x_2^3 + 4x_0^2x_1 - x_0^2x_2 - 4x_1^2x_0 - 3x_1^2x_2 - 4x_2^2x_0 + 4x_2^2x_1 - 2x_0x_1x_2),$$

$$\omega_{37} = \mathbb{V}(x_0^3 - 3x_1^3 + 4x_2^3 + 4x_0^2x_1 - x_0^2x_2 - 4x_1^2x_0 - 3x_1^2x_2 - 4x_2^2x_0 + 4x_2^2x_1 - x_0x_1x_2),$$

$$\omega_{38} = \mathbb{V}(x_0^3 + 3x_1^3 + 4x_2^3 - 4x_0^2x_1 - x_0^2x_2 - 4x_1^2x_0 - 3x_1^2x_2 - 4x_2^2x_0 - 4x_2^2x_1 + x_0x_1x_2),$$

$$\omega_{39} = \mathbb{V}(x_1^3 + 3x_0^2x_1 + 3x_2^2x_1 + 6x_0x_1x_2),$$

$$\omega_{40} = \mathbb{V}(x_1^3 + 3x_0^2x_1 + 3x_2^2x_1 - 6x_0x_1x_2),$$

$$\omega_{41} = \mathbb{V}(x_0^3 - 4x_1^2x_0 - 4x_2^2x_0 + 5x_0x_1x_2),$$

$$\omega_{42} = \mathbb{V}(x_0^3 - 4x_1^2x_0 - 4x_2^2x_0 - 5x_0x_1x_2),$$

$$\omega_{43} = \mathbb{V}(x_2^3 + 3x_0^2x_1 - 4x_1^2x_2 - 2x_0x_1x_2),$$

$$\omega_{44} = \mathbb{V}(x_2^3 + 3x_0^2x_1 - 4x_1^2x_2 + 2x_0x_1x_2),$$

$$\omega_{45} = \mathbb{V}(x_0x_1x_2).$$

Stage(4): We choose four base cubic curves through $\mathcal{S}$, namely $\omega_1, \omega_2, \omega_3, \omega_4$. The corresponding tritangent planes on $\mathscr{S}$ (called base tritangent planes) are chosen as

$$\pi_{\omega_1} = \mathbb{V}(\Pi_{W_1}) := \mathbb{V}(y_0),$$

$$\pi_{\omega_2} = \mathbb{V}(\Pi_{W_2}) := \mathbb{V}(y_1),$$

$$\pi_{\omega_3} = \mathbb{V}(\Pi_{W_3}) := \mathbb{V}(y_2),$$

$$\pi_{\omega_4} = \mathbb{V}(\Pi_{W_4}) := \mathbb{V}(y_3),$$

where Π_{W_j} is a linear form defining π_{ω_j} and correspond to the cubic form W_j defining ω_j. Every tritangent plane on $\mathscr{S}$ can be written as a linear combination of y_0, y_1, y_2, y_3. For instance, the plane cubic curve ω_{27} is a linear combination of the 4 base cubic curves:

$$\begin{aligned}
\omega_{27} = \mathbb{V}(W_{27}) &= \mathbb{V}(x_2^3 + 3x_0^2x_2 - 4x_1^2x_2 - 4x_0x_1x_2) \\
W_{27} := s(\Pi_{W_{27}}) &= \lambda_1W_1 + \lambda_2W_2 + \lambda_3W_3 + \lambda_4W_4;\ \lambda_i \in GF(13) \\
&= (\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4)x_0^3 + (5\lambda_1 - 6\lambda_2 + 6\lambda_3 - 5\lambda_4)x_1^3 \\
&+ (2\lambda_1 + 5\lambda_2 + 5\lambda_3 + 2\lambda_4)x_2^3 + (2\lambda_1 - 5\lambda_2 + 5\lambda_3 - 2\lambda_4)x_0^2x_1
\end{aligned}$$

$$+ (6\lambda_1 + 2\lambda_2 + 2\lambda_3 + 6\lambda_4)x_0^2x_2 + (-4\lambda_1 - 4\lambda_2 - 4\lambda_3 - 4\lambda_4)x_1^2x_0$$

$$+ (5\lambda_1 + 6\lambda_2 + 6\lambda_3 + 5\lambda_4)x_1^2x_2 + (-4\lambda_1 - 4\lambda_2 - 4\lambda_3 - 4\lambda_4)x_2^2x_0$$

$$+ (2\lambda_1 - 5\lambda_2 + 5\lambda_3 - 2\lambda_4)x_2^2x_1 + (-5\lambda_1 - 6\lambda_2 + 6\lambda_3 + 5\lambda_4)x_0x_1x_2.$$

By some calculations, we get $\lambda_1 = -6, \lambda_2 = 5, \lambda_3 = 0$ and $\lambda_4 = 1$. It follows that

$$\begin{aligned} s(\Pi_{W_{27}}) &= -6W_1 + 5W_2 + W_4 \\ &= -6s(\Pi_{W_1}) + 5s(\Pi_{W_2}) + s(\Pi_{W_4}) \\ &= s(-6\Pi_{W_1} + 5\Pi_{W_2} + \Pi_{W_4}) \\ &= s(-6y_0 + 5y_1 + y_3). \end{aligned}$$

So

$$\Pi_{W_{27}} = -6y_0 + 5y_1 + y_3 \text{ and } \pi_{\omega_{27}} = \mathbb{V}(-6y_0 + 5y_1 + y_3).$$

By using similar augment as above, we get all the 45 tritangent planes on $\mathscr{S}$, namely

$$\begin{aligned} \pi_{\omega_1} &= \mathbb{V}(y_0), \\ \pi_{\omega_2} &= \mathbb{V}(y_1), \\ \pi_{\omega_3} &= \mathbb{V}(y_2), \\ \pi_{\omega_4} &= \mathbb{V}(y_3), \\ \pi_{\omega_5} &= \mathbb{V}(y_0 - 3y_1 + y_3), \\ \pi_{\omega_6} &= \mathbb{V}(y_0 + 5y_1 + 5y_2 - 6y_3), \\ \pi_{\omega_7} &= \mathbb{V}(y_0 - 3y_1 + 4y_3), \\ \pi_{\omega_8} &= \mathbb{V}(y_0 - 2y_1 + 5y_2 - 6y_3), \\ \pi_{\omega_9} &= \mathbb{V}(y_0 - 3y_1 - 3y_3), \\ \pi_{\omega_{10}} &= \mathbb{V}(y_0 - 6y_1 + 5y_2 - 6y_3), \\ \pi_{\omega_{11}} &= \mathbb{V}(y_0 - 3y_2 + y_3), \end{aligned}$$

$$\pi_{\omega_{12}} = \mathbb{V}(y_0 - 3y_1 - 3y_2 + 2y_3),$$

$$\pi_{\omega_{13}} = \mathbb{V}(y_0 + y_2 + 4y_3),$$

$$\pi_{\omega_{14}} = \mathbb{V}(y_0 - 3y_1 + y_2 + 2y_3),$$

$$\pi_{\omega_{15}} = \mathbb{V}(y_0 - 4y_2 - 3y_3),$$

$$\pi_{\omega_{16}} = \mathbb{V}(y_0 - 3y_1 - 4y_2 + 2y_3),$$

$$\pi_{\omega_{17}} = \mathbb{V}(y_1 - 4y_2),$$

$$\pi_{\omega_{18}} = \mathbb{V}(y_0 + 4y_3),$$

$$\pi_{\omega_{19}} = \mathbb{V}(y_1 + 3y_2),$$

$$\pi_{\omega_{20}} = \mathbb{V}(y_0 - 3y_3),$$

$$\pi_{\omega_{21}} = \mathbb{V}(y_0 - 5y_1 - 5y_2 + 4y_3),$$

$$\pi_{\omega_{22}} = \mathbb{V}(y_0 + 2y_1 - 5y_2 + y_3),$$

$$\pi_{\omega_{23}} = \mathbb{V}(y_0 - 5y_1 + 2y_2 + y_3),$$

$$\pi_{\omega_{24}} = \mathbb{V}(y_0 + 2y_1 + 2y_2 - 3y_3),$$

$$\pi_{\omega_{25}} = \mathbb{V}(y_0 - 4y_1 - 3y_2 - 3y_3),$$

$$\pi_{\omega_{26}} = \mathbb{V}(y_1 - y_2),$$

$$\pi_{\omega_{27}} = \mathbb{V}(y_0 - 3y_1 + 2y_3),$$

$$\pi_{\omega_{28}} = \mathbb{V}(y_0 + y_1 - 3y_2 + 4y_3),$$

$$\pi_{\omega_{29}} = \mathbb{V}(y_0 - y_3),$$

$$\pi_{\omega_{30}} = \mathbb{V}(y_0 + 5y_2 - 6y_3).$$

$$\pi_{\omega_{31}} = \mathbb{V}(y_1 + 5y_3),$$

$$\pi_{\omega_{32}} = \mathbb{V}(y_1 - 6y_3),$$

$$\pi_{\omega_{33}} = \mathbb{V}(y_0 - 4y_1 + y_2 - 3y_3),$$

$$
\begin{aligned}
\pi_{\omega_{34}} &= \mathbb{V}(y_0 + 4y_1 - 3y_2 + 4y_3),\\
\pi_{\omega_{35}} &= \mathbb{V}(y_0 - 5y_2),\\
\pi_{\omega_{36}} &= \mathbb{V}(y_0 + y_1 - 4y_2 + 4y_3),\\
\pi_{\omega_{37}} &= \mathbb{V}(y_0 - y_1 - 3y_2 - 3y_3),\\
\pi_{\omega_{38}} &= \mathbb{V}(y_1 - 6y_3),\\
\pi_{\omega_{39}} &= \mathbb{V}(y_0 - y_1 + y_2 - y_3),\\
\pi_{\omega_{40}} &= \mathbb{V}(y_0 + 4y_1 - 4y_2 - y_3),\\
\pi_{\omega_{41}} &= \mathbb{V}(y_0 - 3y_2),\\
\pi_{\omega_{42}} &= \mathbb{V}(y_1 + 4y_3),\\
\pi_{\omega_{43}} &= \mathbb{V}(y_0 + y_1 + 2y_2 + 4y_3),\\
\pi_{\omega_{44}} &= \mathbb{V}(y_0 - 6y_1 - 5y_2 - 3y_3),\\
\pi_{\omega_{45}} &= \mathbb{V}(y_0 + 3y_1 - 3y_2 - y_3).
\end{aligned}
$$

Stage(5): We choose one trihedral pairs among the 120 trihedral pairs, namely

$$
\begin{array}{lccll}
T_{123}: & c_{23} & b_3 & a_2 & \rightsquigarrow \pi_{\omega_1} = V(y_0)\\
& a_3 & c_{13} & b_1 & \rightsquigarrow \pi_{\omega_2} = \mathbb{V}(y_1)\\
& b_2 & a_1 & c_{12} & \\
& \wr\!\downarrow & \wr\!\downarrow & & \\
& \pi_{\omega_3} & \pi_{\omega_4} & & \\
& \shortparallel & \shortparallel & & \\
& \mathbb{V}(y_2) & \mathbb{V}(y_3) & &
\end{array}
$$

where $\pi_{\omega_1}, \pi_{\omega_2}, \pi_{\omega_3}, \pi_{\omega_4}$ are the 4 tritangent planes on $\mathscr{S}$ corresponding to the six plane cubics $\omega_1, \omega_2, \omega_3, \omega_4$ which passing through S. From Stage 4, the tritangent planes on $\mathscr{S}$, which correspond to the third row and third column, are respectively

$\pi_{\omega_{27}}$ and $\pi_{\omega_{30}}$. Consequently, the equation of the non-singular cubic surface $\mathscr{S}$ is

$$\mathscr{S} = \mathbb{V}\Big(y_0y_1(y_0 - 3y_1 + 2y_3) + \lambda y_2y_3(y_0 + 5y_2 - 6y_3)\Big).$$

To find the non-zero value $\lambda \in GF(13)$, we pick a point $P(x_0 : x_1 : x_2) \in PG(2, 13)$ which is not on any of the basic plane cubics $\omega_1, \omega_2, \omega_3$ and ω_4. Let us choose the point $P(1 : 1 : 1)$. Then according to Clebsch map, we have

$$P \mapsto (W_1(P) : W_2(P) : W_3(P) : W_4(P)) = (1 : 1 : -5 - 5) \in \mathscr{S}.$$

Therefore, we find $\lambda = -2$. It follows that

$$\mathscr{S} = \mathbb{V}(y_0^2y_1 - 3y_1^2y_0 + 3y_2^2y_3 - y_3^2y_2 + 2y_0y_1y_3 - 2y_0y_2y_3).$$

The 27 lines on $\mathscr{S}$ are

$$\ell_1 = \{(\lambda : 0 : 0 : \mu)\},$$

$$\ell_2 = \{(\mu : -4\mu : \lambda : 0)\},$$

$$\ell_3 = \{(\mu : 0 : \lambda + \mu : 3\lambda + \mu)\},$$

$$\ell_4 = \{(\mu : \lambda : \lambda : 3\lambda - \mu)\},$$

$$\ell_5 = \{(\lambda : \mu : -3\mu : 3\lambda)\},$$

$$\ell_6 = \{(\lambda : \mu : 4\mu : -4\lambda)\},$$

$$\ell_7 = \{(\lambda + \mu : -4\lambda : 0 : 6\mu)\},$$

$$\ell_8 = \{(\mu : \lambda : \lambda - 4\mu : 3\lambda)\},$$

$$\ell_9 = \{(\mu : \lambda : 3\lambda + 6\mu : 3\mu)\},$$

$$\ell_{10} = \{(0 : \lambda : 0 : \mu)\},$$

$$\ell_{11} = \{(\mu : \lambda : 6\mu : 4\lambda + 3\mu)\},$$

$$\ell_{12} = \{(\mu : \lambda : 6\lambda - \mu : 5\lambda)\},$$

$$\ell_{13} = \{(\mu : \lambda : -3\lambda : 4\lambda + 3\mu)\},$$

$$\ell_{14} = \{(\mu : \lambda + 2\mu : -4\lambda : -4\mu)\},$$

$$\ell_{15} = \{(\mu : \lambda - 4\mu : -4\mu : 3\lambda)\},$$

$$\ell_{16} = \{(\lambda : \mu : 6\lambda : 5\mu)\},$$

$$\ell_{17} = \{(\lambda : 0\mu : 0)\},$$

$$\ell_{18} = \{(\lambda : \mu : -5\lambda : -2\mu)\},$$

$$\ell_{19} = \{(\lambda + \mu : 2\lambda : -3\mu : -4\lambda)\},$$

$$\ell_{20} = \{(\mu : \lambda : 4\lambda : -\lambda - 4\mu)\},$$

$$\ell_{21} = \{(0 : \lambda : \mu : 3\mu)\},$$

$$\ell_{22} = \{(\mu : \lambda : -5\mu : -\lambda - 4\mu)\},$$

$$\ell_{23} = \{(0 : \lambda : \mu : 0)\},$$

$$\ell_{24} = \{(\lambda + \mu : \lambda : \mu : \lambda + \mu)\},$$

$$\ell_{25} = \{(\lambda + \mu : -4\mu : -6\lambda + 5\mu : 6\lambda)\},$$

$$\ell_{26} = \{(\mu : \lambda : \lambda : \mu)\},$$

$$\ell_{27} = \{(\mu : \lambda : -4\mu : 3\lambda)\}.$$

where $(\lambda : \mu) \in PG(1, 13)$. Furthermore, $\mathscr{S}$ has 18 Eckardt points which are

$$Q_1 = (0 : 1 : 0 : 0),$$

$$Q_2 = (0 : 0 : 1 : 0),$$

$$Q_3 = (0 : 1 : 1 : 3),$$

$$Q_4 = (0 : 1 : 0 : 3),$$

$$Q_5 = (1 : 0 : 0 : 3),$$

$$Q_6 = (1 : 2 : 0 : -4),$$

$$Q_7 = (1 : -4 : -4 : 0),$$
$$Q_8 = (1 : 0 : 6 : 3),$$
$$Q_9 = (1 : -5 : 6 : 1),$$
$$Q_{10} = (1 : 6 : 6 : 1),$$
$$Q_{11} = (1 : 0 : 0 : -4),$$
$$Q_{12} = (1 : 0 : -5 : -4),$$
$$Q_{13} = (1 : -5 : -5 : 1),$$
$$Q_{14} = (1 : 0 : -4 : 0),$$
$$Q_{15} = (1 : -2 : 0 : 3),$$
$$Q_{16} = (1 : 6 : -5 : 1),$$
$$Q_{17} = (1 : 2 : -5 : -4),$$
$$Q_{18} = (1 : -2 : 6 : 3).$$

As another example: consider the plane quadrilaterals $\mathcal{Q}^{(6)}(19)$ and the configuration $\mathcal{Q}^{(10)}(19)$ in Section 3.4.

For the plane quadrilaterals $\mathcal{Q}^{(6)}(19)$, we are able to determine the equation of the correspond non-singular cubic surface, namely $\mathscr{S}^{(6)}(19)$. From Section 3.4, the lines c_{16}, c_{24}, c_{35} lie on the cubic surface $\mathscr{S}^{(6)}(19)$. The trihedral pairs containing the plane section consisting the c_{16}, c_{24}, and c_{35} are $T_{123,654}, T_{124,635}, T_{123,645}$ and $T_{124,653}$.

To determine the equation of $\mathscr{S}^{(6)}(19)$, we consider only one of the four trihedral pairs, namely

$$\begin{array}{llll} T_{123,654}: & c_{16} & c_{25} & c_{34} \\ & c_{24} & c_{13} & c_{56} \\ & c_{35} & c_{46} & c_{12} \end{array}$$

Let the three faces of one triad, namely the row plane sections $c_{16}c_{25}c_{34}$, $c_{24}c_{13}c_{56}$ and $c_{35}c_{46}c_{12}$, be $\mathfrak{p}_1, \mathfrak{p}_2$ and $\mathfrak{p}_3$ respectively. The three faces of the second triad (the conjugate triad), namely the column planes sections $c_{16}c_{24}c_{35}, c_{25}c_{13}c_{46}$ and $c_{34}c_{56}c_{12}$, are denoted by $\mathfrak{p}'_1, \mathfrak{p}'_2$ and $\mathfrak{p}'_3$ respectively. From Section 3.2, we know that the six plane sections of $T_{123,654}$ in $PG(3, 19)$ corresponds to 6 cubic curves of the web W, through the 6 points in general position $P_i \in \mathcal{S}_7$, where $\mathcal{S}_7$ is a 6-arc not on a conic in $PG(2, 19)$, namely

$$\mathcal{S}_7 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{285}\}.$$

Let the planes $\mathfrak{p}_1 = \mathbb{V}(L_1), \mathfrak{p}_2 = \mathbb{V}(L_2), \mathfrak{p}'_2 = \mathbb{V}(L'_2)$ and $\mathfrak{p}'_3 = \mathbb{V}(L'_3)$ correspond to the cubic curves $\omega_1 = \mathbb{V}(W_1)$, $\omega_2 = \mathbb{V}(W_2)$, $\omega'_2 = \mathbb{V}(W'_2)$ and $\omega'_3 = \mathbb{V}(W'_3)$ respectively, where

$$\begin{aligned}
\omega_1 &= \mathbb{V}(\overline{P_1P_6}) \cup \mathbb{V}(\overline{P_2P_5}) \cup \mathbb{V}(\overline{P_3P_4}) \\
&= \mathbb{V}(x_0^2x_1 - 7x_0^2x_2 - x_1^2x_0 + 7x_1^2x_2 - 8x_2^2x_0 + 8x_2^2x_1); \\
\omega_2 &= \mathbb{V}(\overline{P_1P_6}) \cup \mathbb{V}(\overline{P_2P_5}) \cup \mathbb{V}(\overline{P_3P_4}) \\
&= \mathbb{V}(x_0^2x_1 - 2x_1^2x_0 + 2x_1^2x_2 + x_2^2x_1 - 2x_0x_1x_2); \\
\omega'_2 &= \mathbb{V}(\overline{P_1P_6}) \cup \mathbb{V}(\overline{P_2P_5}) \cup \mathbb{V}(\overline{P_3P_4}) \\
&= \mathbb{V}(x_0^2x_1 + 4x_1^2x_0 - 9x_1^2x_2 - 3x_2^2x_1 + 7x_0x_1x_2); \\
\omega'_3 &= \mathbb{V}(\overline{P_1P_6}) \cup \mathbb{V}(\overline{P_2P_5}) \cup \mathbb{V}(\overline{P_3P_4}) \\
&= \mathbb{V}(x_0^2x_2 + 2x_1^2x_2 - x_2^2x_0 + x_2^2x_1 - 3x_0x_1x_2).
\end{aligned}$$

From above argument, we have the following configuration

$$\begin{array}{lllllll}
T_{123,654}: & c_{16} & c_{25} & c_{34} & \rightarrow \mathfrak{p}_1 = \mathbb{V}(y_0) & \leftrightsquigarrow \omega_1 = \mathbb{V}(W_1) \\
& c_{24} & c_{13} & c_{56} & \rightarrow \mathfrak{p}_2 = \mathbb{V}(y_1) & \leftrightsquigarrow \omega_2 = \mathbb{V}(W_2) \\
& c_{35} & c_{46} & c_{12} & \rightarrow \mathfrak{p}_3 = \mathbb{V}(L_3) & \leftrightsquigarrow \omega_3 = \mathbb{V}(W_3) \\
& \downarrow & \downarrow & \downarrow & & \\
& \mathfrak{p}'_1 & \mathfrak{p}'_2 & \mathfrak{p}'_3 & &
\end{array}$$

where

$$\mathfrak{p}_1' = \mathbb{V}(L_1') \leftrightsquigarrow \omega_1' = \mathbb{V}(W_1'),$$

$$\mathfrak{p}_2' = \mathbb{V}(y_2) \leftrightsquigarrow \omega_2' = \mathbb{V}(W_2'),$$

$$\mathfrak{p}_3' = \mathbb{V}(y_3) \leftrightsquigarrow \omega_3' = \mathbb{V}(W_3'),$$

and L_3, L_1' are homogeneous linear forms in y_0, y_1, y_2, y_3, and W_1 , W_2, W_3, W_1', W_2', W_3' are homogeneous cubic forms in x_0, x_1, x_2. By some algebraic calculations, we get W_1, W_2, W_2' and W_3', are linearly independent. The plane cubics W_3 and W_1', where

$$\omega_3 = \mathbb{V}(W_3) = \mathbb{V}(\overline{P_3P_5}) \cup \mathbb{V}(\overline{P_4P_6}) \cup \mathbb{V}(\overline{P_1P_1}),$$

$$\omega_1' = \mathbb{V}(W_1') = \mathbb{V}(\overline{P_1P_6}) \cup \mathbb{V}(\overline{P_2P_4}) \cup \mathbb{V}(\overline{P_3P_5}),$$

can be written in terms of W_1, W_2, W_2' and W_3' as

$$W_3 = -W_1 + 4W_2 + 2W_2' - 4W_3',$$

$$W_1' = W_2' - 7W_3'.$$

It follows that

$$L_3 = -y_0 + 4y_1 + 2y_2 - 4y_3,$$

$$L_1' = y_2 - 7y_3.$$

Consequently, the the equation of the non-singular cubic surface $\mathscr{S}^{(6)}(19)$ with $\lambda = -1$ is of the form:

$$\begin{aligned}\mathscr{S}^{(6)}(19) &= \mathbb{V}(L_1L_2L_3 - \lambda L_1'L_2'L_3') \\ &= \mathbb{V}(y_0y_1(-y_0 + 4y_1 - 3y_2 - 6y_3) - y_2y_3(y_2 - 7y_3) \\ &= \mathbb{V}(y_0^2y_1 - 4y_1^2y_0 + y_2^2y_3 - 7y_3^2y_2 + 3y_0y_1y_2 + 6y_0y_1y_3).\end{aligned}$$

Again from Section 3.4, the lines c_{16}, c_{23}, c_{45} lie on the cubic surface $\mathscr{S}^{(10)}(19)$. The trihedral pairs containing the plane section consisting the c_{16}, c_{23}, and c_{45} are $T_{12,36}$, $T_{13,26}$, $T_{34,52}$ and $T_{14,56}$.

To determine the equation of $\mathscr{S}^{(10)}(19)$, we consider only one of the four trihedral pairs, namely

$$
\begin{array}{lcccll}
T_{12,36}: & a_1 & b_3 & c_{13} & \rightarrow \mathfrak{p}_1 = \mathbb{V}(y_0) & \leftrightsquigarrow \omega_1 = \mathbb{V}(W_1) \\
 & b_6 & a_2 & c_{26} & \rightarrow \mathfrak{p}_2 = \mathbb{V}(y_1) & \leftrightsquigarrow \omega_2 = \mathbb{V}(W_2) \\
 & c_{16} & c_{23} & c_{45} & \rightarrow \mathfrak{p}_3 = \mathbb{V}(L_3) & \leftrightsquigarrow \omega_3 = \mathbb{V}(W_3) \\
 & \downarrow & \downarrow & \downarrow & & \\
 & \mathfrak{p}'_1 & \mathfrak{p}'_2 & \mathfrak{p}'_3 & & \\
 & \shortparallel & \shortparallel & \shortparallel & & \\
 & \mathbb{V}(L'_1) & \mathbb{V}(y_2) & \mathbb{V}(y_3) & & \\
 & \updownarrow & \updownarrow & \updownarrow & & \\
 & \omega'_1 & \omega'_2 & \omega'_3 & & \\
 & \shortparallel & \shortparallel & \shortparallel & & \\
 & \mathbb{V}(W'_1) & \mathbb{V}(W'_2) & \mathbb{V}(W'_3) & &
\end{array}
$$

where L_3, L'_1 are homogeneous linear forms in y_0, y_1, y_2, y_3, and W_1, W_2, W_3, W'_1, W'_2, W'_3 are homogeneous cubic forms in x_0, x_1, x_2.

From Section 3.2, we know that the six plane sections of $T_{12,36}$ in $PG(3,19)$ corresponds to 6 cubic curves of the web W, through the 6 points in general position $P_i \in \mathcal{S}_3$, where $\mathcal{S}_3$ is a 6-arc not on a conic in $PG(2,19)$, namely

$$\mathcal{S}_3 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{33}\}.$$

Let the planes $\mathfrak{p}_1 = \mathbb{V}(L_1) = \mathbb{V}(y_0), \mathfrak{p}_2 = \mathbb{V}(L_2) = \mathbb{V}(y_1), \mathfrak{p}'_2 = \mathbb{V}(L'_2) = \mathbb{V}(y_2)$ and $\mathfrak{p}'_3 = \mathbb{V}(L'_3) = \mathbb{V}(y_3)$ correspond to the plane cubic curves $\omega_1 = \mathbb{V}(W_1), \omega_2 = \mathbb{V}(W_2), \omega'_2 = \mathbb{V}(W'_2)$ and $\omega'_3 = \mathbb{V}(W'_3)$ respectively, where

$$
\begin{aligned}
\omega_1 &= \mathbb{V}(\overline{P_1P_3}) \cup \mathbb{V}(\mathscr{C}_3)\\
&= \mathbb{V}(\overline{P_1P_3}) \cup \mathbb{V}(x_2^2 + 2x_0x_1 + x_0x_2 - 4x_1x_2)\\
&= \mathbb{V}(x_1^2x_0 - 2x_1^2x_2 - 9x_2^2x_1 - 9x_0x_1x_2);
\end{aligned}
$$

$$
\begin{aligned}
\omega_2 &= \mathbb{V}(\overline{P_2P_6}) \cup \mathbb{V}(\mathscr{C}_6)\\
&= \mathbb{V}(\overline{P_2P_6}) \cup \mathbb{V}(x_0x_1 + 5x_0x_2 - 6x_1x_2)\\
&= \mathbb{V}(x_0^2x_1 + 5x_0^2x_2 - 9x_2^2x_0 + 7x_2^2x_1 - 4x_0x_1x_2);\\
\omega_2' &= \mathbb{V}(\overline{P_2P_3}) \cup \mathbb{V}(\mathscr{C}_3)\\
&= \mathbb{V}(\overline{P_2P_3}) \cup \mathbb{V}(x_2^2 + 2x_0x_1 + x_0x_2 - 4x_1x_2)\\
&= \mathbb{V}(x_0^2x_1 - 9x_0^2x_2 - 9x_2^2x_0 - 2x_0x_1x_2);\\
\omega_3' &= \mathbb{V}(\overline{P_1P_3}) \cup \mathbb{V}(\overline{P_2P_6}) \cup \mathbb{V}(\overline{P_4P_5})\\
&= \mathbb{V}(x_0^2x_1 - 3x_1^2x_0 - 6x_1^2x_2 + 4x_2^2x_1 + 4x_0x_1x_2).
\end{aligned}
$$

These four plane cubics, namely W_1, W_2, W_2' and W_3', are linearly independent. Furthermore, the plane cubics W_3 and W_1' where

$$
\begin{aligned}
\omega_3 &= \mathbb{V}(W_3) = \mathbb{V}(\overline{P_1P_6}) \cup \mathbb{V}(\overline{P_2P_3}) \cup \mathbb{V}(\overline{P_4P_5}),\\
\omega_1' &= \mathbb{V}(W_1') = \mathbb{V}(\overline{P_1P_6}) \cup \mathbb{V}(\mathscr{C}_6),
\end{aligned}
$$

can be written in terms of W_1, W_2, W_2' and W_3' as

$$
\begin{aligned}
W_3 &= 8W_1 - 9W_2 - 9W_3',\\
W_1' &= 2W_1 - 7W_2 - 6W_2' - 6W_3'.
\end{aligned}
$$

It follows that

$$
\begin{aligned}
L_3 &= 8y_0 - 9y_1 - 9y_2,\\
L_1' &= 2y_0 - 7y_1 - 6y_2 - 6y_3.
\end{aligned}
$$

Consequently, the the equation of the non-singular cubic surface $\mathscr{S}^{(6)}(19)$ with $\lambda = -1$ has the form:

$$\begin{aligned}\mathscr{S}^{(10)}(19) &= \mathbb{V}(L_1L_2L_3 - \lambda L_1'L_2'L_3') \\ &= \mathbb{V}(y_0y_1(8y_0 - 9y_1 - 9y_3) - y_2y_3(2y_0 - 7y_1 - 6y_2 - 6y_3) \\ &= \mathbb{V}(y_0^2y_1 + 6y_1^2y_0 - 4y_2^2y_3 - 4y_3^2y_2 + 6y_0y_1y_3 - 5y_0y_2y_3 + 8y_1y_2y_3).\end{aligned}$$

Let $\left\{(a_0\lambda + b_0\mu : a_1\lambda + b_1\mu : a_2\lambda + b_2\mu : a_3\lambda + b_3\mu)\right\}_{\mathcal{H}}$ denote the hyperbolic line on cubic surface passing through the points $(a_0 : a_1 : a_2 : a_3)$ and $(b_0 : b_1 : b_2 : b_3)$, and let $\left\{(c_0\lambda + d_0\mu : c_1\lambda + d_1\mu : c_2\lambda + d_2\mu : c_3\lambda + d_3\mu)\right\}_{\mathcal{E}}$ denotes the elliptic line on cubic surface passing through the points $(c_0 : c_1 : c_2 : c_3)$ and $(d_0 : d_1 : d_2 : d_3)$. Then all distinct non-singular cubic surfaces (up to e-invariants) in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$ with their 27 lines are shown in the tables: Table 3.8, Table 3.9, Table 3.10, Table 3.11, Table 3.12, Table 3.13, Table 3.14, Table 3.15, Table 3.16, Table 3.18, Table 3.19, Table 3.20, Table 3.21, Table 3.22, Table 3.23, Table 3.24, Table 3.25, Table 3.26, Table 3.27, Table 3.28, Table 3.29, Table 3.30, Table 3.31, Table 3.32, Table 3.33, Table 3.34, Table 3.35, Table 3.36, Table 3.37, Table 3.38, Table 3.39 and Table 3.40.

Theorem 3.6. For $q = 17, 19, 23, 29, 31$, the possible number of elliptic lines on a non-singular cubic surface with 27 lines over $GF(q)$ are represented by the entries of Table 3.41.

Proof. All the detail are shown in the tables of Section 3.6. □

Let l be a line on a non-singular cubic surface $\mathscr{S}$ with 27 lines over $GF(q)$ where q is odd number, namely

$$\mathscr{S} = \mathbb{V}(f) = \mathbb{V}(L_1L_2L_3 + \lambda L_1'L_2'L_3'),$$

Table 3.8: The non-singular cubic surface $\mathscr{S}^{(1)}(17)$

$\mathscr{S}^{(1)}(17) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 4y_2^2y_3 - 3y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + 2y_1y_2y_3)$		
$e_3 = 1, e_2 = 132, e_1 = 219, e_0 = 57$ *and* $\lvert\mathscr{S}^{(1)}(17)\rvert = 409$		
27 Lines on $\mathscr{S}^{(1)}(17) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{17}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : 5\lambda + \mu : -5\mu : -6\lambda)\}_{\mathcal{E}}$	$(0 : 1 : 6 : 3)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -7\lambda + 3\mu : 8\lambda)\}_{\mathcal{E}}$	
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : -8\mu : -5\lambda : 7\mu)\}_{\mathcal{H}}$	
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(0 : \mu : \lambda + 8\mu : 7\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -3\mu : 8\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -3\lambda - 8\mu : 2\lambda + 7\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 2\lambda : -3\mu)\}_{\mathcal{E}}$	$\{(\lambda : 7\lambda + \mu : -7\mu : -8\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : \lambda : 3\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : \lambda : -4\lambda - 2\mu)\}_{\mathcal{E}}$	
$\{(\tau : -\lambda : 0 : -\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 6\mu : -2\lambda + 3\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 6\mu : -8\lambda)\}_{\mathcal{E}}$	$\{(\tau : 4\mu : -7\lambda : -5\mu)\}_{\mathcal{H}}$	
$\{(\tau : 8\lambda : 8\lambda : \mu)\}_{\mathcal{H}}$	$\{(\lambda : 3\lambda + \mu : -4\lambda : -4\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 2\lambda : 3\lambda + \mu)\}_{\mathcal{E}}$	$\{(\tau : -8\mu : 7\mu : -4\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : 0 : \mu : 6\lambda + 7\mu)\}_{\mathcal{H}}$	$\{(\lambda : -\lambda + \mu : -\mu : 0)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : \mu : -6\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 3\lambda + 4\mu : 6\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -4\lambda : 6\mu)\}_{\mathcal{E}}$		
# Elliptic lines= 16 #Hyperbolic lines= 11		

Table 3.9: The non-singular cubic surface $\mathscr{S}^{(3)}(17)$

$\mathscr{S}^{(3)}(17) = \mathbb{V}(y_0^2y_1 + 4y_1^2y_0 + y_2^2y_3 + 2y_3^2y_2 - 8y_0y_1y_2 - 4y_0y_2y_3 - 5y_1y_2y_3)$		
$e_3 = 3, e_2 = 126, e_1 = 225, e_0 = 55$ *and* $\lvert\mathscr{S}^{(3)}(17)\rvert = 409$		
27 Lines on $\mathscr{S}^{(3)}(17) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{17}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -6\lambda : 5\lambda + 7\mu)\}_{\mathcal{H}}$	$(0 : 0 : 0 : 1)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : 4\lambda + \mu : -3\mu : -4\lambda)\}_{\mathcal{E}}$	$(0 : 1 : 3 : 1)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 2\lambda : \lambda + 2\mu)\}_{\mathcal{E}}$	$(0 : 1 : 3 : 0)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : \lambda + 4\mu : -7\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 7\lambda : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -5\mu : -7\lambda)\}_{\mathcal{H}}$	
$\{(0 : \tau : 5\mu : -6\lambda)\}_{\mathcal{H}}$	$\{(\tau : 4\mu : -5\lambda : -\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 3\mu : 8\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 6\mu : 6\lambda + 8\mu)\}_{\mathcal{E}}$	
$\{(\tau : 6\lambda : \lambda : -6\mu)\}_{\mathcal{H}}$	$\{(\lambda : 0 : \mu : 2\lambda + 8\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -6\lambda : 4\mu)\}_{\mathcal{H}}$	$\{(\lambda : 4\lambda + \mu : 3\mu : 8\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : 4\lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 2\lambda : -8\mu)\}_{\mathcal{E}}$	
$\{(\tau : 4\lambda : 5\mu : 4\lambda)\}_{\mathcal{H}}$	$\{(\tau : 4\mu : -2\lambda : 3\lambda - 5\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 6\mu : -4\lambda)\}_{\mathcal{E}}$	$\{(\tau : 4\mu : -2\lambda : 0)\}_{\mathcal{E}}$	
$\{(\lambda : \tau : 7\lambda : -7\mu)\}_{\mathcal{E}}$	$\{(\tau : 5\mu : -8\mu : -8\lambda)\}_{\mathcal{H}}$	
$\{(\tau : 4\mu : \lambda : 2\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 12 #Hyperbolic lines= 15		

and let $\mathscr{S}_\lambda$ denote the residual plane section, namely

$$\mathscr{S}_\lambda = f\Big|_l.$$

Then we have the following facts ([27], Pages 196,197):

1. The residual plane sections $\mathscr{S}_\lambda$ define an involution on l.

Table 3.10: The non-singular cubic surface $\mathscr{S}^{(4)}(17)$

$\mathscr{S}^{(4)}(17) = \mathbb{V}(y_0^2y_1 - y_1^2y_0 + 5y_2^2y_3 + y_3^2y_2 + 4y_0y_1y_2 - 5y_0y_1y_3 - y_0y_2y_3 - y_1y_2y_3)$		
$e_3 = 4, e_2 = 123, e_1 = 228, e_0 = 54$ *and* $\lvert\mathscr{S}^{(4)}(17)\rvert = 409$		
27 Lines on $\mathscr{S}^{(4)}(17) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{17}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 8\lambda + \mu : -5\lambda)\}_{\mathcal{E}}$	$(0 : -8 : 5 : -4)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : 4\lambda + \mu : 5\lambda : 6\mu)\}_{\mathcal{H}}$	$(1 : 6 : 5 : -5)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 3\lambda : 3\lambda + 5\mu)\}_{\mathcal{E}}$	$(1 : 0 : 0 : -5)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : \lambda + 4\mu : -4\lambda)\}_{\mathcal{H}}$	$(0 : 1 : -7 : 2)$
$\{(\lambda : \mu : 3\lambda : -8\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -3\lambda : -\lambda + 3\mu)\}_{\mathcal{E}}$	
$\{(\tau : \mu : 0 : 7\lambda)\}_{\mathcal{H}}$	$\{(\tau : -2\mu : -5\mu : -6\lambda)\}_{\mathcal{E}}$	
$\{(\tau : 0 : 7\mu : \lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -7\mu : -3\lambda + 2\mu)\}_{\mathcal{H}}$	
$\{(\tau : \mu : 4\mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -7\mu : -4\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 5\lambda : -4\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -4\lambda - 7\mu : 2\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -6\mu : -8\lambda)\}_{\mathcal{E}}$	$\{(\tau : 5\lambda : 6\mu : 6\lambda + 5\mu)\}_{\mathcal{H}}$	
$\{(\tau : 2\lambda : -4\lambda : -5\mu)\}_{\mathcal{H}}$	$\{(\tau : 8\lambda : -8\mu : 2\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -3\lambda : 2\mu)\}_{\mathcal{H}}$	$\{(\lambda : 7\lambda + \mu : 8\mu : -8\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -2\mu : -5\lambda)\}_{\mathcal{H}}$	$\{(\tau : -7\mu : -\lambda : 5\mu)\}_{\mathcal{H}}$	
$\{(0 : \tau : 7\mu : \lambda)\}_{\mathcal{H}}$		
# Elliptic lines= 12 #Hyperbolic lines= 15		

Table 3.11: The non-singular cubic surface $\mathscr{S}^{(6)}(17)$

$\mathscr{S}^{(6)}(17) = \mathbb{V}(y_0^2y_1 - 6y_1^2y_0 - 8y_2^2y_3 + 7y_3^2y_2 + 5y_0y_1y_2 + 6y_0y_1y_3 + 5y_0y_2y_3 + 5y_1y_2y_3)$		
$e_3 = 6, e_2 = 117, e_1 = 234, e_0 = 52$ *and* $\lvert\mathscr{S}^{(6)}(17)\rvert = 409$		
27 Lines on $\mathscr{S}^{(6)}(17) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{17}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 5\lambda - 7\mu : 5\lambda)\}_{\mathcal{H}}$	$(0 : 1 : -4 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 5\lambda : 5\lambda + 8\mu)\}_{\mathcal{H}}$	$(1 : -2 : 0 : -5)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : 4\lambda + \mu : 0 : \tau)\}_{\mathcal{H}}$	$(1 : 0 : 6 : -6)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -5\lambda + 6\mu : -6\mu)\}_{\mathcal{H}}$	$(1 : 0 : 5 : 5)$
$\{(\lambda : \mu : 5\lambda : -5\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -5\mu : 5\lambda)\}_{\mathcal{H}}$	$(1 : -5 : 8 : 5)$
$\{(\tau : 3\mu : -7\lambda : 0)\}_{\mathcal{H}}$	$\{(\tau : 2\mu : -8\mu : -4\lambda)\}_{\mathcal{H}}$	$(1 : 8 : 8 : -6)$
$\{(\tau : 0 : 7\mu : -8\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 6\lambda + 2\mu : -6\lambda)\}_{\mathcal{H}}$	
$\{(\tau : \mu : \mu : 2\lambda)\}_{\mathcal{H}}$	$\{(\lambda : -2\lambda + \mu : -4\mu : -5\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : \mu : -6\lambda)\}_{\mathcal{H}}$	$\{(\lambda : 8\lambda + \mu : 6\lambda : 5\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -4\mu : -5\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 8\lambda : 6\lambda + 7\mu)\}_{\mathcal{H}}$	
$\{(\tau : -7\mu : -6\lambda : 7\mu)\}_{\mathcal{H}}$	$\{(\tau : 7\mu : -3\lambda : 8\lambda + 4\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 8\lambda : -6\mu)\}_{\mathcal{H}}$	$\{(0 : \mu : \lambda + 7\mu : 6\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 6\lambda : -\mu)\}_{\mathcal{H}}$	$\{(\tau : -6\lambda : -4\lambda : -7\mu)\}_{\mathcal{H}}$	
$\{(\tau : -8\lambda : 4\mu : 6\lambda)\}_{\mathcal{H}}$		
# Elliptic lines= 0 #Hyperbolic lines= 27		

2. Since $\mathscr{S}_\lambda$ is either a line pair or a conic, there are five possible configurations for $(l, \mathscr{S}_\lambda)$ as in Figure 3.4.

3. The line l contains 0, 1, or 2 Eckardt points and $\mathscr{S}$ has at most 18 Eckardt points.

4. The number of the type I and type II of the configurations $(l, \mathscr{S}_\lambda)$ for the

Table 3.12: The non-singular cubic surface $\mathscr{S}^{(2)}(19)$

$\mathscr{S}^{(2)}(19) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 2y_2^2y_3 - 4y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 2, e_2 = 129, e_1 = 276, e_0 = 88$ *and* $\lvert\mathscr{S}^{(2)}(19)\rvert = 495$		
27 Lines on $\mathscr{S}^{(2)}(19) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{19}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -2\lambda : 4\lambda - 4\mu)\}_{\mathcal{H}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : 6\lambda + \mu : 4\mu : -7\lambda)\}_{\mathcal{E}}$	$(1 : 1 : -2 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -2\mu : -4\lambda + 4\mu)\}_{\mathcal{H}}$	
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -3\lambda + 8\mu : 9\mu)\}_{\mathcal{H}}$	
$\{(\tau : 5\lambda : 9\lambda : -3\tau)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -6\lambda : 2\lambda - 8\mu)\}_{\mathcal{E}}$	
$\{(\tau : 0 : 9\lambda : 5\mu)\}_{\mathcal{E}}$	$\{(\tau : 9\mu : 9\mu : -2\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : -\lambda + \mu : -\mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -6\mu : -8\lambda + 2\mu)\}_{\mathcal{E}}$	
$\{(0 : \lambda : \mu : 5\lambda - 9\mu)\}_{\mathcal{E}}$	$\{(\lambda : 2\lambda + \mu : 8\mu : -3\lambda)\}_{\mathcal{E}}$	
$\{(\tau : 8\lambda : 2\lambda + 4\mu : \lambda)\}_{\mathcal{E}}$	$\{(\lambda : -\lambda + \mu : -4\mu : 3\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -6\lambda : 9\mu)\}_{\mathcal{E}}$	$\{(\tau : -9\lambda : 8\mu : 8\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -2\lambda : -3\mu)\}_{\mathcal{H}}$	$\{(\lambda : -2\lambda + \mu : \lambda : -2\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 0 : -\tau)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 8\lambda - 3\mu : 9\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : \mu : -7\lambda)\}_{\mathcal{H}}$	$\{(\tau : 5\lambda : 8\lambda : 9\tau)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : \lambda : -7\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

Table 3.13: The non-singular cubic surface $\mathscr{S}^{(3)}(19)$

$\mathscr{S}^{(3)}(19) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 4y_2^2y_3 - 7y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 3, e_2 = 126, e_1 = 279, e_0 = 87$ *and* $\lvert\mathscr{S}^{(3)}(19)\rvert = 495$		
27 Lines on $\mathscr{S}^{(3)}(19) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{19}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(0 : \lambda : \mu : -8\lambda + 6\mu)\}_{\mathcal{H}}$	$(1 : -5 : -6 : -5)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 5\lambda : 3\lambda - 9\mu)\}_{\mathcal{H}}$	$(1 : -1 : 0 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : 0 : \mu : -8\lambda + 6\mu)\}_{\mathcal{H}}$	$(1 : -4 : 5 : 1)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : 5\lambda + \mu : -6\lambda : 5\mu)\}_{\mathcal{E}}$	
$\{(\tau : -\lambda : -\mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : -8\lambda + \mu : -8\mu : 7\lambda)\}_{\mathcal{H}}$	
$\{(\tau : 9\mu : -4\lambda : 9\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 7\lambda - 9\mu : -5\mu)\}_{\mathcal{H}}$	
$\{(\tau : 4\mu : -5\mu : 5\lambda)\}_{\mathcal{E}}$	$\{(\tau : 2\mu : -3\mu : -4\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -6\lambda : \mu)\}_{\mathcal{H}}$	$\{(\lambda : -2\lambda + \mu : -4\mu : \lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 0 : -\tau)\}_{\mathcal{H}}$	$\{(\lambda : 4\lambda + \mu : 7\mu : -5\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 5\lambda : 7\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 8\mu : -5\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 8\lambda : -5\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 8\lambda : 2\lambda - 4\mu)\}_{\mathcal{H}}$	
$\{(\tau : -\mu : \lambda : -2\lambda)\}_{\mathcal{H}}$	$\{(\tau : 7\mu : -8\lambda : -8\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -6\mu : \lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 5\mu : -9\lambda + 3\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 5\mu : 7\lambda)\}_{\mathcal{E}}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

different involutions on l are given by Table 3.42.

Theorem 3.7. There are $4, 7, 5, 7, 9$ distinct non-singular cubic surfaces with 27 lines

Table 3.14: The non-singular cubic surface $\mathscr{S}^{(4)}(19)$

$\mathscr{S}^{(4)}(19) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 8y_2^2y_3 - 9y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 4, e_2 = 123, e_1 = 282, e_0 = 86$ *and* $\lvert\mathscr{S}^{(4)}(19)\rvert = 495$		
27 Lines on $\mathscr{S}^{(4)}(19) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{19}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -5\mu : 9\lambda + 2\mu)\}_{\mathcal{H}}$	$(1 : 1 : -9 : 7)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\tau : 7\lambda - \mu : 2\lambda : 9\lambda)\}_{\mathcal{H}}$	$(1 : 9 : -5 : 7)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : 0 : 7\lambda + \mu : 3\mu)\}_{\mathcal{H}}$	$(1 : -2 : -9 : 5)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 3\lambda - 2\mu : 7\lambda)\}_{\mathcal{E}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : \mu : 0 : -\tau)\}_{\mathcal{H}}$	$\{(\lambda : -6\lambda + \mu : -7\mu : 5\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : -\lambda + \mu : -\mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -5\lambda : 2\lambda + 9\mu)\}_{\mathcal{H}}$	
$\{(\tau : 7\lambda : -2\mu : -8\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -7\lambda - 4\mu : 5\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -5\mu : -3\lambda)\}_{\mathcal{E}}$	$\{(0 : \mu : \lambda : 3\lambda - 2\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -9\mu : 7\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 4\lambda : -9\lambda + 2\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -9\lambda : 7\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -3\lambda + 6\mu : -3\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -5\lambda : -3\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -9\lambda : 9\lambda + 6\mu)\}_{\mathcal{E}}$	
$\{(\tau : -4\lambda : 3\lambda : 2\mu)\}_{\mathcal{H}}$	$\{(\tau : 2\mu : \mu : 6\lambda + 5\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 4\mu : 5\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 6\lambda - 3\mu : -3\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 4\lambda : 5\mu)\}_{\mathcal{E}}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

Table 3.15: The non-singular cubic surface $\mathscr{S}^{(6)}(19)$

$\mathscr{S}^{(6)}(19) = \mathbb{V}(y_0^2y_1 - 4y_1^2y_0 + y_2^2y_3 - 7y_3^2y_2 + 3y_0y_1y_2 + 6y_0y_1y_3)$		
$e_3 = 6, e_2 = 117, e_1 = 288, e_0 = 84$ *and* $\lvert\mathscr{S}^{(6)}(19)\rvert = 495$		
27 Lines on $\mathscr{S}^{(6)}(19) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{19}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : -8\lambda + \mu : -3\lambda - 9\mu : -6\lambda - 4\mu)\}_{\mathcal{H}}$	$(1 : 0 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : -6\lambda + 4\mu : -6\lambda + 4\mu : -\lambda - 5\mu)\}_{\mathcal{H}}$	$(0 : 1 : 0 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : 5\lambda - 8\mu : -3\lambda + \mu : -5\lambda + \mu)\}_{\mathcal{H}}$	$(1 : 0 : 3 : -5)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : 3\lambda + 2\mu : -7\lambda - \mu : -\lambda + 8\mu)\}_{\mathcal{H}}$	$(1 : 5 : 0 : 0)$
$\{(0 : \tau : -9\mu : -4\mu)\}_{\mathcal{H}}$	$\{(\lambda : 2\lambda + \mu : 3\lambda + 6\mu : -\lambda + 9\mu)\}_{\mathcal{H}}$	$(0 : 1 : 7 : 1)$
$\{(\tau : -2\lambda - 8\mu : -4\lambda + \mu : \tau)\}_{\mathcal{H}}$	$\{(\tau : 8\lambda - 6\mu : -5\tau : -7\lambda + 4\mu)\}_{\mathcal{H}}$	$(1 : -5 : 3 : -5)$
$\{(\lambda : -\lambda + \mu : 0 : -4\lambda + 7\mu)\}_{\mathcal{H}}$	$\{(\tau : -5\lambda + 6\mu : -3\lambda - 5\mu : -5\tau)\}_{\mathcal{H}}$	
$\{(\tau : 9\lambda + 7\mu : 3\tau : 9\lambda + 7\mu)\}_{\mathcal{H}}$	$\{(\tau : \lambda - 5\mu : \lambda - 3\mu : \lambda - 5\mu)\}_{\mathcal{H}}$	
$\{(\tau : 8\lambda - 5\mu : 3\tau : -8\lambda + 4\mu)\}_{\mathcal{H}}$	$\{(\tau : 5\lambda - 6\mu : -3\lambda - 4\mu : -5\tau)\}_{\mathcal{H}}$	
$\{(\tau : 0 : 9\mu : 4\mu)\}_{\mathcal{H}}$	$\{(\tau : 8\lambda - 6\mu : -5\lambda - \mu : 6\lambda - 7\mu)\}_{\mathcal{H}}$	
$\{(\tau : \lambda + 5\mu : \lambda + 5\mu : \tau)\}_{\mathcal{H}}$	$\{(\tau : 8\lambda + 5\mu : -5\lambda + 4\mu : -7\tau)\}_{\mathcal{H}}$	
$\{(\tau : -2\lambda - 8\mu : -4\tau : \lambda + 4\mu)\}_{\mathcal{H}}$	$\{(\tau : -9\lambda + 9\mu : -4\tau : 5\lambda + 2\mu)\}_{\mathcal{H}}$	
$\{(\lambda : -9\lambda + \mu : 7\lambda - 2\mu : -5\lambda)\}_{\mathcal{H}}$	$\{(\tau : 8\lambda - 5\mu : -5\tau : 6\lambda + \mu)\}_{\mathcal{H}}$	
$\{(\tau : 7\lambda + 6\mu : 9\lambda - 5\mu : 0)\}_{\mathcal{H}}$		
# Elliptic lines= 0 #Hyperbolic lines= 27		

(up to e-invariants) in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$ respectively, namely,

$$\mathscr{S}^{(m)}(17), m = 1, 3, 4, 6.$$

$$\mathscr{S}^{(m)}(19), m = 2, 3, 4, 6, 9, 10, 18.$$

$$\mathscr{S}^{(m)}(23), m = 1, 2, 3, 4, 6.$$

$$\mathscr{S}^{(m)}(29), m = 0, 1, 2, 3, 4, 6, 10.$$

$$\mathscr{S}^{(m)}(31), m = 0, 1, 2, 3, 4, 6, 9, 10, 18.$$

Table 3.16: The non-singular cubic surface $\mathscr{S}^{(9)}(19)$

$\mathscr{S}^{(9)}(19) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 3y_2^2y_3 - 6y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 9, e_2 = 108, e_1 = 297, e_0 = 81$ *and* $\lvert\mathscr{S}^{(9)}(19)\rvert = 495$		
27 Lines on $\mathscr{S}^{(9)}(19) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{19}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(0 : \tau : 6\lambda + \mu : 7\mu)\}_{\mathcal{H}}$	$(1 : 8 : 7 : 1)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : -\lambda + \mu : -5\mu : 4\mu)\}_{\mathcal{H}}$	$(1 : 0 : 7 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : 6\lambda + \mu : -5\mu : -7\lambda)\}_{\mathcal{H}}$	$(1 : 0 : 8 : 1)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 2\lambda : -2\lambda - 7\mu)\}_{\mathcal{H}}$	$(1 : 0 : 0 : -7)$
$\{(\tau : 7\mu : -8\mu : 6\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 4\lambda + 8\mu : \mu)\}_{\mathcal{H}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : \mu : 7\lambda : -4\mu)\}_{\mathcal{H}}$	$\{(\lambda : -2\lambda + \mu : 4\mu : \lambda)\}_{\mathcal{H}}$	$(0 : 1 : 8 : 1)$
$\{(\tau : -\mu : -\lambda : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 2\mu : -7\lambda - 2\mu)\}_{\mathcal{H}}$	$(1 : -7 : 8 : -7)$
$\{(\tau : 0 : 6\lambda : -3\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 7\lambda : -9\lambda + 6\mu)\}_{\mathcal{H}}$	$(0 : 1 : 0 : -7)$
$\{(\lambda : \mu : 2\mu : -7\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 7\lambda - 2\mu : -4\mu)\}_{\mathcal{H}}$	$(0 : 1 : 7 : 0)$
$\{(\lambda : \mu : 8\lambda : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 8\mu : -2\lambda + \mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 0 : -\tau)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -2\lambda + 7\mu : -4\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 8\mu : \lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 8\lambda : \lambda - 2\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 2\lambda : -7\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -5\lambda - 8\mu : -7\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 7\mu : -4\lambda)\}_{\mathcal{H}}$		
# Elliptic lines= 0 #Hyperbolic lines= 27		

Table 3.17: The non-singular cubic surface $\mathscr{S}^{(10)}(19)$

$\mathscr{S}^{(10)}(19) = \mathbb{V}(y_0^2y_1 + 6y_1^2y_0 - 4y_2^2y_3 - 4y_3^2y_2 + 6y_0y_1y_3 - 5y_0y_2y_3 + 8y_1y_2y_3)$		
$e_3 = 10, e_2 = 105, e_1 = 300, e_0 = 80$ *and* $\lvert\mathscr{S}^{(10)}(19)\rvert = 495$		
27 Lines on $\mathscr{S}^{(10)}(19) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{19}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : -6\lambda + 5\mu : -4\tau : -\lambda + 4\mu)\}_{\mathcal{E}}$	$(0 : 0 : 1 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\tau : 4\lambda + 7\mu : 5\lambda + 4\mu : -4\lambda + 3\mu)\}_{\mathcal{E}}$	$(1 : -7 : -4 : 6)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : 4\lambda + 7\mu : 9\lambda + 2\mu : -8\lambda + 5\mu)\}_{\mathcal{H}}$	$(1 : -4 : 1 : 8)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : 8\lambda + \mu : -9\lambda + 6\mu : 6\lambda)\}_{\mathcal{H}}$	$(1 : 3 : 0 : 0)$
$\{(\mu : 3\mu : \lambda - 7\mu : 0)\}_{\mathcal{H}}$	$\{(0 : \tau : -7\lambda + 8\mu : 9\lambda - 6\mu)\}_{\mathcal{E}}$	$(1 : -7 : -3 : 6)$
$\{(\tau : 7\lambda + 3\mu : 3\lambda + 4\mu : 9\lambda)\}_{\mathcal{H}}$	$\{(\tau : 4\lambda - 7\mu : \tau : -8\lambda - 5\mu)\}_{\mathcal{H}}$	$(1 : -3 : 0 : 6)$
$\{(\lambda : -8\lambda + \mu : -3\lambda : 8\lambda + \mu)\}_{\mathcal{E}}$	$\{(\tau : -8\lambda - 7\mu : -8\lambda - 7\mu : 8\lambda + 9\mu)\}_{\mathcal{E}}$	$(1 : -3 : 1 : 6)$
$\{(\tau : -8\lambda - \mu : -8\lambda - \mu : 8\tau)\}_{\mathcal{H}}$	$\{(\tau : -8\lambda + 6\mu : -3\lambda - 8\mu : 8\tau)\}_{\mathcal{H}}$	$(1 : -4 : -4 : 8)$
$\{(\tau : 4\lambda - 5\mu : \tau : -4\lambda - 6\mu)\}_{\mathcal{E}}$	$\{(\tau : -6\lambda - 5\mu : 7\lambda - \mu : -3\lambda + 7\mu)\}_{\mathcal{H}}$	$(1 : 0 : -3 : 0)$
$\{(\tau : 5\lambda - 6\mu : 6\lambda - 7\mu : 6\tau)\}_{\mathcal{H}}$	$\{(\tau : \lambda - 3\mu : -5\lambda - 4\mu : 9\tau)\}_{\mathcal{E}}$	$(0 : 1 : 1 : 0)$
$\{(\tau : -7\lambda - 3\mu : -7\lambda + 2\mu : 9\tau)\}_{\mathcal{E}}$	$\{(\tau : 7\lambda + \mu : -7\lambda - 6\mu : -4\lambda + 2\mu)\}_{\mathcal{H}}$	
$\{(\tau : 0 : 9\lambda - 6\mu : 4\lambda)\}_{\mathcal{E}}$	$\{(\tau : \lambda - 3\mu : 3\lambda + 2\mu : -3\lambda + 9\mu)\}_{\mathcal{E}}$	
$\{(\mu : \lambda - 5\mu : -4\mu : 7\lambda + \mu)\}_{\mathcal{H}}$	$\{(\tau : -6\lambda + 6\mu : -3\tau : -3\lambda + 3\mu)\}_{\mathcal{H}}$	
$\{(\tau : 7\lambda - 8\mu : 0 : -4\lambda - 8\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 12 #Hyperbolic lines= 15		

Proof. By the argument introduced in previous section, and our computer programs, the distinct non-singular cubic surfaces with 27 lines (up to e-invariants) that corresponding to 6-arcs not on a conic in $PG(2, q)$ for $q = 17, 19, 23, 29, 31$ are shown in Section 3.6. More precisely, if $T_i^{(q)}$ represents the type of a non-singular cubic surface with 27 lines over $GF(q)$, and $e_{0,i}, e_{1,i}, e_{2,i}, e_{3,i}$ denote the e-invariants correspond to $T_i^{(q)}$, then we have the Table 3.43 which illustrates the distinct non-singular cubic

Table 3.18: The non-singular cubic surface $\mathscr{S}^{(18)}(19)$

$\mathscr{S}^{(18)}(19) = \mathbb{V}(y_0^2y_1 + 8y_1^2y_0 + 9y_2^2y_3 - 9y_3^2y_2 + 3y_0y_1y_2 - 5y_0y_1y_3)$		
$e_3 = 18, e_2 = 81, e_1 = 324, e_0 = 72$ *and* $\|\mathscr{S}^{(18)}(19)\| = 495$		
27 Lines on $\mathscr{S}^{(18)}(19) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{19}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : 5\lambda - 4\mu : 0 : -7\lambda + 9\mu)\}_{\mathcal{H}}$	$(0 : 1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : 2\lambda - 6\mu : 6\lambda + \mu : -5\lambda + 9\mu)\}_{\mathcal{H}}$	$(1 : 0 : 0 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : -8\lambda - 2\mu : -3\lambda + 2\mu : -3\lambda + 2\mu)\}_{\mathcal{H}}$	$(1 : 0 : 0 : 8)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : -3\lambda - 2\mu : -9\lambda - 6\mu : 8\tau)\}_{\mathcal{H}}$	$(1 : -6 : 1 : 8)$
$\{(0 : \tau : -5\mu : -5\mu)\}_{\mathcal{H}}$	$\{(\tau : -6\lambda + 2\mu : 7\lambda + 6\mu : -4\lambda - 5\mu)\}_{\mathcal{H}}$	$(0 : 1 : 8 : 8)$
$\{(\tau : -5\lambda : -5\lambda : 2\lambda + 7\mu)\}_{\mathcal{H}}$	$\{(\tau : -7\lambda + 4\mu : -7\lambda + 4\mu : 3\tau)\}_{\mathcal{H}}$	$(1 : 7 : 0 : 0)$
$\{(\lambda : \tau : 8\tau : -6\lambda + 8\mu)\}_{\mathcal{H}}$	$\{(\tau : -3\lambda + 3\mu : -3\lambda + 8\mu : 4\lambda - 4\mu)\}_{\mathcal{H}}$	$(1 : -8 : -7 : 1)$
$\{(\tau : \lambda + 4\mu : -\lambda - 7\mu : \tau)\}_{\mathcal{H}}$	$\{(\tau : -3\lambda - 8\mu : -9\lambda + 7\mu : 8\tau)\}_{\mathcal{H}}$	$(0 : 1 : 0 : 7)$
$\{(\tau : 9\mu : -7\tau : -7\lambda + 3\mu)\}_{\mathcal{H}}$	$\{(\tau : 6\lambda + 7\mu : -2\tau : 4\lambda - 8\mu)\}_{\mathcal{H}}$	$(1 : 0 : 1 : 1)$
$\{(\tau : 6\lambda + 2\mu : 9\lambda + 7\mu : 0)\}_{\mathcal{H}}$	$\{(\tau : -5\lambda + 8\mu : -2\lambda + 7\mu : \tau)\}_{\mathcal{H}}$	$(1 : -5 : -2 : 3)$
$\{(\tau : 7\lambda - 2\mu : \tau : -4\lambda - 3\mu)\}_{\mathcal{H}}$	$\{(\tau : 2\lambda - 3\mu : -8\lambda + 9\mu : -3\lambda - 5\mu)\}_{\mathcal{H}}$	$(1 : 0 : -7 : 0)$
$\{(\tau : 2\lambda + 9\mu : 5\lambda - 7\mu : 3\tau)\}_{\mathcal{H}}$	$\{(\tau : 7\lambda + 3\mu : \tau : -\lambda + 5\mu)\}_{\mathcal{H}}$	$(1 : -7 : 1 : 1)$
$\{(\lambda : -7\lambda + \mu : -7\lambda : 3\lambda + 5\mu)\}_{\mathcal{H}}$	$\{(\lambda : -5\lambda + \mu : -2\lambda : \lambda + 7\mu)\}_{\mathcal{H}}$	$(1 : -7 : 0 : 8)$
$\{(\tau : 0 : -7\mu : -7\mu)\}_{\mathcal{H}}$		$(0 : 1 : 1 : 0)$
		$(1 : 4 : -7 : 1)$
		$(1 : -2 : -2 : 3)$
		$(1 : 1 : 1 : 8)$
		$(1 : -7 : -7 : 0)$
# Elliptic lines= 0 #Hyperbolic lines= 27		

Table 3.19: The non-singular cubic surface $\mathscr{S}^{(1)}(23)$

$\mathscr{S}^{(1)}(23) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 2y_2^2y_3 + 8y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 1, e_2 = 132, e_1 = 381, e_0 = 177$ *and* $\|\mathscr{S}^{(1)}(23)\| = 691$		
27 Lines on $\mathscr{S}^{(1)}(23) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{23}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : 5\lambda + \mu : -7\mu : -6\lambda)\}_{\mathcal{E}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\tau : 9\mu : -10\mu : -10\lambda)\}_{\mathcal{H}}$	
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : 2\lambda : -11\mu : -3\lambda)\}_{\mathcal{E}}$	
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 4\lambda : -4\lambda - 10\mu)\}_{\mathcal{H}}$	
$\{(\lambda : -\lambda + \mu : -\mu : 0)\}_{\mathcal{H}}$	$\{(\tau : -11\mu : 10\mu : 4\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 7\lambda : 10\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -6\lambda - 11\mu : 10\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 7\lambda : \lambda + 3\mu)\}_{\mathcal{H}}$	$\{(\lambda : 10\lambda + \mu : 6\mu : -11\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 4\mu : -11\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 7\mu : 3\lambda + \mu)\}_{\mathcal{H}}$	
$\{(\lambda : -\lambda + \mu : 0 : -\mu)\}_{\mathcal{H}}$	$\{(\tau : -\lambda + 5\mu : -7\mu : \mu)\}_{\mathcal{H}}$	
$\{(0 : \tau : 11\mu : -3\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -3\lambda : -6\mu)\}_{\mathcal{E}}$	
$\{(\tau : 0 : 11\mu : -3\lambda)\}_{\mathcal{E}}$	$\{(\tau : -9\mu : -7\lambda : 8\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 4\lambda : -11\mu)\}_{\mathcal{E}}$	$\{(\tau : 7\lambda : 6\mu : -8\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -3\mu : -6\lambda)\}_{\mathcal{E}}$	$\{(\lambda : 2\lambda + \mu : -3\lambda : 4\mu)\}_{\mathcal{H}}$	
$\{(\tau : 10\mu : \mu : 10\tau)\}_{\mathcal{E}}$		
# Elliptic lines= 16 #Hyperbolic lines= 11		

surfaces with 27 lines (up to e-invariants) over $GF(q)$ for $q = 17, 19, 23, 29, 31$:

□

Corollary 3.1. The maximal number of Eckardt points on a non-singular cubic surfaces with 27 lines in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$, are $6, 18, 6, 10, 18$ respec-

Table 3.20: The non-singular cubic surface $\mathscr{S}^{(2)}(23)$

$\mathscr{S}^{(2)}(23) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 3y_2^2y_3 - 6y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 2, e_2 = 129, e_1 = 384, e_0 = 176$ *and* $\lvert\mathscr{S}^{(2)}(23)\rvert = 691$		
27 Lines on $\mathscr{S}^{(2)}(23) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{23}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\tau : 3\mu : -4\mu : 8\lambda)\}_{\mathcal{E}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : 8\lambda + \mu : 8\mu : -9\lambda)\}_{\mathcal{E}}$	$(0 : 0 : 1 : -1)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\tau : -9\lambda : 8\lambda : -5\mu)\}_{\mathcal{E}}$	
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 2\lambda - 5\mu : -6\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 0 : -\tau)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 11\mu : -7\lambda + 8\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -\tau : 0)\}_{\mathcal{H}}$	$\{(\tau : 11\lambda : 11\tau : 8\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -6\mu : -6\lambda)\}_{\mathcal{H}}$	$\{(\lambda : 0 : \mu : -4\lambda - \mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -9\mu : -9\lambda)\}_{\mathcal{H}}$	$\{(0 : \lambda : \mu : -4\lambda - \mu)\}_{\mathcal{H}}$	
$\{(\lambda : -\lambda : \mu : -\mu)\}_{\mathcal{H}}$	$\{(\lambda : 8\lambda + \mu : -9\lambda : 8\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 11\lambda : 11\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -9\lambda : -9\mu)\}_{\mathcal{H}}$	
$\{(\tau : -2\lambda : -7\mu : \lambda)\}_{\mathcal{E}}$	$\{(\tau : 3\mu : 8\lambda : -4\mu)\}_{\mathcal{E}}$	
$\{(\tau : 11\mu : 8\lambda : 11\tau)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -6\lambda : 2\lambda - 5\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 11\mu : 11\lambda)\}_{\mathcal{H}}$	$\{(\tau : -9\mu : -5\lambda : 8\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -6\lambda : -6\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 16 #Hyperbolic lines= 11		

Table 3.21: The non-singular cubic surface $\mathscr{S}^{(3)}(23)$

$\mathscr{S}^{(3)}(23) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 7y_2^2y_3 + 8y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + 3y_1y_2y_3)$		
$e_3 = 3, e_2 = 126, e_1 = 387, e_0 = 175$ *and* $\lvert\mathscr{S}^{(3)}(23)\rvert = 691$		
27 Lines on $\mathscr{S}^{(3)}(23) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{23}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 5\lambda : 7\lambda + 5\mu)\}_{\mathcal{H}}$	$(0 : 1 : 1 : -7)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -4\lambda + \mu : -7\lambda)\}_{\mathcal{H}}$	$(1 : -9 : 2 : -2)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : 8\lambda + \mu : 9\mu : -9\lambda)\}_{\mathcal{H}}$	$(1 : -6 : 5 : 0)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\tau : 7\lambda : \lambda : 9\lambda + \mu)\}_{\mathcal{E}}$	
$\{(\tau : \lambda : \lambda : \lambda + 8\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 11\lambda + 2\mu : -5\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 0 : -\tau)\}_{\mathcal{H}}$	$\{(\tau : -10\lambda : 6\mu : 9\tau)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 5\lambda : -10\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 2\lambda : \lambda + 8\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 10\mu : -2\lambda)\}_{\mathcal{H}}$	$\{(\lambda : -6\lambda + \mu : -5\mu : 5\lambda + 4\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 4\lambda : -5\mu)\}_{\mathcal{H}}$	$\{(0 : \mu : \lambda : 2\lambda - 9\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -\tau : 0)\}_{\mathcal{H}}$	$\{(\tau : -5\mu : 9\lambda : 4\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 3\mu : -9\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \tau : 11\mu : -2\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : \mu : 9\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 4\lambda : 5\lambda + \mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 2\lambda : -7\mu)\}_{\mathcal{E}}$	$\{(\tau : 0 : 2\lambda : \lambda - 3\mu)\}_{\mathcal{H}}$	
$\{(\tau : -6\lambda : 5\lambda : 5\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

tively. Moreover, the minimal number of Eckardt points on a non-singular cubic surfaces with 27 lines in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$, are $1, 2, 1, 0, 0$ respectively.

Proof. See Table 3.41. □

Corollary 3.2. The number of elliptic lines on a non-singular cubic surfaces with 27 lines in $PG(3, q)$ for $q = 17, 19, 23, 29, 31$ is either 0 or 12 or 16.

Table 3.22: The non-singular cubic surface $\mathscr{S}^{(4)}(23)$

$\mathscr{S}^{(4)}(23) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 9y_2^2y_3 + 9y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 4, e_2 = 123, e_1 = 390, e_0 = 174$ *and* $\lvert\mathscr{S}^{(4)}(23)\rvert = 691$		
27 Lines on $\mathscr{S}^{(4)}(23) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{23}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : 10\lambda + \mu : -11\lambda : 3\mu)\}_{\mathcal{E}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -11\mu : -11\lambda)\}_{\mathcal{E}}$	$(0 : 0 : 1 : -1)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : -7\mu : 9\lambda + 8\mu : -11\mu)\}_{\mathcal{E}}$	$(1 : 1 : -2 : 0)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -5\mu : 9\lambda + 10\mu)\}_{\mathcal{E}}$	$(1 : 1 : 0 : -2)$
$\{(0 : \tau : 5\lambda : 5\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -2\lambda : 7\lambda - 7\mu)\}_{\mathcal{H}}$	
$\{(\tau : \mu : -7\lambda : -2\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -7\lambda + 3\mu : -11\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 0 : -\tau)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -2\mu : -7\lambda + 7\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -\tau : 0)\}_{\mathcal{H}}$	$\{(\lambda : 4\lambda + \mu : -5\lambda : 9\mu)\}_{\mathcal{E}}$	
$\{(\lambda : -\lambda : \mu : -\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -2\mu : -2\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -2\lambda : -2\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 3\lambda - 7\mu : -11\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -11\lambda : -11\mu)\}_{\mathcal{E}}$	$\{(\lambda : \tau : -7\mu : -2\lambda)\}_{\mathcal{H}}$	
$\{(\tau : 0 : 5\lambda : 5\mu)\}_{\mathcal{H}}$	$\{(\tau : 7\lambda : -8\lambda : 3\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -5\lambda : -5\mu)\}_{\mathcal{E}}$	$\{(\lambda : 4\lambda + \mu : 9\mu : -5\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -5\mu : -5\lambda)\}_{\mathcal{E}}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

Table 3.23: The non-singular cubic surface $\mathscr{S}^{(6)}(23)$

$\mathscr{S}^{(6)}(23) = \mathbb{V}(y_0^2y_1 - 10y_1^2y_0 - 10y_2^2y_3 + 5y_3^2y_2 - 3y_0y_1y_2 + 4y_0y_1y_3 - 10y_0y_2y_3)$		
$e_3 = 6, e_2 = 117, e_1 = 396, e_0 = 172$ *and* $\lvert\mathscr{S}^{(6)}(23)\rvert = 691$		
27 Lines on $\mathscr{S}^{(6)}(23) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{23}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -8\lambda : 9\lambda + \mu)\}_{\mathcal{H}}$	$(0 : 1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : \lambda : 4\lambda - 3\mu)\}_{\mathcal{H}}$	$(1 : 0 : -8 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 11\lambda + \mu : \lambda)\}_{\mathcal{H}}$	$(1 : -9 : -8 : 0)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : 0 : \mu : 2\tau)\}_{\mathcal{H}}$	$(0 : 1 : 1 : 0)$
$\{(\lambda : \mu : \mu : \lambda)\}_{\mathcal{H}}$	$\{(\lambda : 6\lambda + \mu : \lambda + 2\mu : 4\tau)\}_{\mathcal{H}}$	$(1 : 9 : 1 : 0)$
$\{(\tau : 9\lambda : \lambda : -2\mu)\}_{\mathcal{H}}$	$\{(\tau : 3\mu : \lambda + 2\mu : 2\mu)\}_{\mathcal{H}}$	$(1 : 0 : 1 : 0)$
$\{(\tau : 7\mu : 8\lambda : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : \mu : -9\lambda + 2\mu)\}_{\mathcal{H}}$	
$\{(\tau : 7\mu : 0 : -6\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 2\lambda - 10\mu : 6\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -8\lambda : 6\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -5\lambda + 11\mu : -8\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 6\mu : -8\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 6\mu : -7\lambda - 11\mu)\}_{\mathcal{H}}$	
$\{(\tau : 8\lambda : -7\mu : -9\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -8\lambda + 3\mu : 6\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : \lambda : -7\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -6\lambda : -4\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -5\mu : 6\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -6\lambda : -10\lambda - 2\mu)\}_{\mathcal{H}}$	
$\{(0 : \tau : 2\mu : 4\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 0 #Hyperbolic lines= 27		

Proof. See Table 3.41. □

Corollary 3.3. For q odd prime, the number of elliptic lines on a non-singular cubic surfaces $\mathscr{S}^{(3)}(q)$ with 27 lines in $PG(3, q)$ is 12.

Proof. Let l_1, l_2, and l_3 be any three lines on the non-singular cubic surface $\mathscr{S}^{(3)}(q)$.

Table 3.24: The non-singular cubic surface $\mathscr{S}^{(0)}(29)$

$\mathscr{S}^{(0)}(29) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 5y_2^2y_3 + 6y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + 3y_1y_2y_3)$		
$e_3 = 0, e_2 = 135, e_1 = 540, e_0 = 370$ *and* $\lvert\mathscr{S}^{(0)}(29)\rvert = 1045$		
27 Lines on $\mathscr{S}^{(0)}(29) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{29}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -11\lambda + 6\mu : 9\mu)\}_{\mathcal{E}}$	
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 10\lambda - 11\mu : 6\lambda)\}_{\mathcal{H}}$	
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 14\lambda : -7\lambda - 14\mu)\}_{\mathcal{H}}$	
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : 8\lambda + \mu : -3\mu : -9\lambda + 2\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -5\mu : 9\lambda)\}_{\mathcal{E}}$	$\{(\lambda : -10\lambda + \mu : -4\mu : 9\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -9\lambda : 5\mu)\}_{\mathcal{E}}$	$\{(\lambda : 8\lambda + \mu : -9\lambda : -13\mu)\}_{\mathcal{H}}$	
$\{(\tau : -\mu : 0 : -\lambda)\}_{\mathcal{E}}$	$\{(\tau : 3\lambda : -3\mu : -4\lambda)\}_{\mathcal{E}}$	
$\{(\tau : 0 : -6\lambda : -5\mu)\}_{\mathcal{H}}$	$\{(\tau : -4\lambda : 3\lambda : -14\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 14\lambda : 9\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -4\lambda + 5\mu : 5\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -\tau : 0)\}_{\mathcal{E}}$	$\{(\tau : -7\lambda : 6\lambda : -13\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -8\mu : 6\lambda)\}_{\mathcal{E}}$	$\{(\lambda : 7\lambda + \mu : -3\mu : -8\lambda)\}_{\mathcal{H}}$	
$\{(0 : \tau : 11\lambda : 14\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -13\lambda : -11\mu)\}_{\mathcal{E}}$	
$\{(\tau : 5\lambda : -6\lambda : 12\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -13\lambda : \lambda + 12\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -7\mu : -8\lambda)\}_{\mathcal{E}}$		
# Elliptic lines= 16 #Hyperbolic lines= 11		

Table 3.25: The non-singular cubic surface $\mathscr{S}^{(1)}(29)$

$\mathscr{S}^{(1)}(29) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 4y_2^2y_3 + 6y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 1, e_2 = 132, e_1 = 543, e_0 = 369$ *and* $\lvert\mathscr{S}^{(1)}(29)\rvert = 1045$		
27 Lines on $\mathscr{S}^{(1)}(29) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{29}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -4\lambda + 3\mu : -12\lambda)\}_{\mathcal{H}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : -9\lambda : -12\mu : 8\lambda)\}_{\mathcal{H}}$	
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -13\lambda - 3\mu : -3\mu)\}_{\mathcal{H}}$	
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(0 : \lambda : \mu : -5\lambda + 9\mu)\}_{\mathcal{E}}$	
$\{(\tau : 14\mu : 14\mu : 2\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 8\lambda : 9\lambda + \mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : \lambda : 4\lambda + 2\mu)\}_{\mathcal{E}}$	$\{(\lambda : 2\lambda + \mu : -13\mu : -3\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : \mu : -12\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -11\lambda - 12\mu : 12\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -7\lambda : 12\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 8\lambda : -3\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 0 : -\tau)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 8\mu : -3\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : \lambda : -12\mu)\}_{\mathcal{E}}$	$\{(\tau : -\mu : 12\lambda : -13\lambda)\}_{\mathcal{H}}$	
$\{(\tau : 8\lambda : 3\mu : -9\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -7\mu : 12\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 8\mu : \lambda + 9\mu)\}_{\mathcal{E}}$	$\{(\lambda : 6\lambda + \mu : -7\lambda : -8\mu)\}_{\mathcal{E}}$	
$\{(\tau : 0 : 7\mu : -5\lambda)\}_{\mathcal{E}}$	$\{(\tau : 5\lambda : -6\lambda : -8\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -\tau : 0)\}_{\mathcal{H}}$		
# Elliptic lines= 16 #Hyperbolic lines= 11		

Consider the following three configurations, namely

$$(l_1, \mathscr{S}_\lambda^{(3)}(q)), \quad (l_2, \mathscr{S}_\lambda^{(3)}(q)), \; and \quad (l_3, \mathscr{S}_\lambda^{(3)}(q)).$$

Then according to the previous facts mentioned in Table 3.42, each configuration has 1 hyperbolic involutions of type I, and 4 hyperbolic involutions of type II. So the total number of hyperbolic involutions corresponding to each configuration is 5. This

Table 3.26: The non-singular cubic surface $\mathscr{S}^{(2)}(29)$

$\mathscr{S}^{(2)}(29) = \mathbb{V}(y_0^2y_1 + 9y_1^2y_0 - 2y_2^2y_3 + 4y_3^2y_2 + 5y_0y_1y_2 + 12y_0y_1y_3 + 12y_0y_2y_3 - 12y_1y_2y_3)$		
$e_3 = 2, e_2 = 129, e_1 = 546, e_0 = 368$ *and* $\lvert\mathscr{S}^{(2)}(29)\rvert = 1045$		
27 Lines on $\mathscr{S}^{(2)}(29) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{29}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : 5\lambda + \mu : 10\mu : \lambda)\}_{\mathcal{E}}$	$(1 : 0 : 0 : 4)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : 6\mu : \mu : 4\lambda + 8\mu)\}_{\mathcal{H}}$	$(1 : 3 : -14 : 4)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\tau : \lambda : 3\lambda - 8\mu : -6\lambda)\}_{\mathcal{E}}$	
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : 4\lambda : 6\lambda + 14\mu : 8\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : \mu : \lambda)\}_{\mathcal{H}}$	$\{(\tau : 6\lambda : 5\lambda - 4\mu : 8\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : \lambda : 11\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -6\lambda - 7\mu : -6\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 3\mu : -6\lambda)\}_{\mathcal{E}}$	$\{(\tau : -13\lambda + 5\mu : 0 : \mu)\}_{\mathcal{H}}$	
$\{(\lambda : \tau : \lambda : \lambda - 11\mu)\}_{\mathcal{E}}$	$\{(\tau : 4\mu : -10\lambda : -8\lambda + 9\mu)\}_{\mathcal{H}}$	
$\{(\tau : 0 : 6\lambda : -3\mu)\}_{\mathcal{E}}$	$\{(\lambda : 2\lambda + \mu : 2\lambda + 4\mu : 0)\}_{\mathcal{H}}$	
$\{(0 : \tau : -6\mu : 3\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -14\lambda : -10\lambda - 5\mu)\}_{\mathcal{H}}$	
$\{(\tau : 2\mu : 2\mu : -7\lambda)\}_{\mathcal{H}}$	$\{(\lambda : 4\lambda + \mu : -13\lambda : 4\lambda + 7\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -14\lambda : 2\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -13\lambda : -6\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 5\mu : 4\lambda)\}_{\mathcal{H}}$	$\{(\tau : 3\lambda : 9\lambda : \lambda + 2\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \tau : -14\mu : 4\lambda)\}_{\mathcal{E}}$		
# Elliptic lines= 12 #Hyperbolic lines= 15		

Table 3.27: The non-singular cubic surface $\mathscr{S}^{(2)^*}(29)$

$\mathscr{S}^{(2)}(29) = \mathbb{V}(y_0^2y_1 - 9y_1^2y_0 - 8y_2^2y_3 + 14y_3^2y_2 + 11y_0y_1y_2 - 12y_0y_1y_3 - 3y_0y_2y_3 + 6y_1y_2y_3)$		
$e_3 = 2, e_2 = 129, e_1 = 546, e_0 = 368$ *and* $\lvert\mathscr{S}^{(2)}(29)\rvert = 1045$		
27 Lines on $\mathscr{S}^{(2)}(29) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{29}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\tau : -8\mu : \mu : -\lambda + 3\mu)\}_{\mathcal{E}}$	$(0 : 0 : 1 : 13)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : 0 : \mu : -6\lambda + 13\mu)\}_{\mathcal{H}}$	$(1 : -9 : -9 : 1)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : -14\lambda + \mu : -\lambda : 9\mu)\}_{\mathcal{E}}$	
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -9\mu : 11\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : \mu : \lambda)\}_{\mathcal{H}}$	$\{(\lambda : 8\lambda + \mu : -12\lambda : 13\mu)\}_{\mathcal{E}}$	
$\{(\tau : 14\mu : 14\mu : -2\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 8\lambda - \mu : 11\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : 13\lambda + \mu : 14\mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : 2\lambda + \mu : -9\lambda : -11\mu)\}_{\mathcal{E}}$	
$\{(\lambda : 2\lambda + \mu : 12\mu : \lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -9\mu : 5\lambda + 11\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -12\lambda : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 13\lambda - 12\mu : \mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -\lambda : -2\mu)\}_{\mathcal{H}}$	$\{(0 : \mu : \lambda : 13\lambda + 12\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -11\mu : 13\lambda)\}_{\mathcal{H}}$	$\{(\lambda : -9\lambda : \mu : 2\lambda + 13\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -9\lambda : -13\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -11\lambda + 7\mu : -13\mu)\}_{\mathcal{E}}$	
$\{(\lambda : 13\lambda + \mu : 0 : -8\mu)\}_{\mathcal{H}}$	$\{(\lambda : -14\lambda + \mu : 6\mu : 13\lambda)\}_{\mathcal{E}}$	
$\{(\tau : -2\lambda : 9\mu : 4\lambda)\}_{\mathcal{E}}$		
# Elliptic lines= 16 #Hyperbolic lines= 11		

means we have $15 = 5{\cdot}3$ hyperbolic involutions corresponding to the all configurations. Hence the number of hyperbolic lines on a non-singular cubic surfaces is 15 (see Figure 3.5).

So

$$\#(\text{Elliptic lines}) = 27 - 15 = 12.$$

□

Table 3.28: The non-singular cubic surface $\mathscr{S}^{(3)}(29)$

$\mathscr{S}^{(3)}(29) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 3y_2^2y_3 + 5y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 3, e_2 = 126, e_1 = 549, e_0 = 367$ *and* $\lvert\mathscr{S}^{(3)}(29)\rvert = 1045$		
27 Lines on $\mathscr{S}^{(3)}(29) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{29}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : 3\lambda : 7\lambda : 4\lambda + 3\mu)\}_{\mathcal{E}}$	$(0 : 1 : -12 : 7)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : -6\lambda + \mu : 5\mu : 5\lambda)\}_{\mathcal{E}}$	$(1 : -1 : 0 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : -\lambda + \mu : -2\mu : \mu)\}_{\mathcal{H}}$	$(1 : 0 : -12 : 7)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 9\lambda : 14\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 5\lambda + \mu : 5\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 13\lambda - 12\mu : 7\mu)\}_{\mathcal{H}}$	
$\{(\tau : 0 : -10\mu : -6\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 9\lambda : 6\lambda - 11\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -12\mu : 7\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 12\lambda : 10\lambda + 3\mu)\}_{\mathcal{E}}$	
$\{(\tau : -\mu : 0 : -\lambda)\}_{\mathcal{H}}$	$\{(0 : \lambda : \mu : -6\lambda + 11\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -\tau : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -14\lambda + \mu : 14\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -12\lambda : 7\mu)\}_{\mathcal{H}}$	$\{(\tau : -3\lambda : 2\lambda : -11\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 9\mu : 14\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -12\lambda + 13\mu : 7\lambda)\}_{\mathcal{H}}$	
$\{(\tau : 8\lambda : -9\lambda : 2\mu)\}_{\mathcal{H}}$	$\{(\tau : -2\mu : \lambda : \mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 12\lambda : 5\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -12\lambda : 7\lambda + 2\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 12\mu : 5\lambda)\}_{\mathcal{E}}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

Table 3.29: The non-singular cubic surface $\mathscr{S}^{(4)}(29)$

$\mathscr{S}^{(4)}(29) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 4y_2^2y_3 + 8y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + 3y_0y_2y_3 + 3y_1y_2y_3)$		
$e_3 = 4, e_2 = 123, e_1 = 552, e_0 = 366$ *and* $\lvert\mathscr{S}^{(4)}(29)\rvert = 1045$		
27 Lines on $\mathscr{S}^{(4)}(29) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{29}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 5\mu : 11\lambda + 8\mu)\}_{\mathcal{H}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : 13\lambda + \mu : -6\mu : -14\lambda)\}_{\mathcal{H}}$	$(1 : 14 : 12 : 7)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -7\lambda + 9\mu : 6\mu)\}_{\mathcal{E}}$	$(1 : 1 : 12 : -14)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : -4\lambda + \mu : 3\lambda : -5\mu)\}_{\mathcal{E}}$	$(1 : -2 : 5 : -14)$
$\{(\lambda : \mu : 3\mu : 7\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 12\lambda : -10\lambda - 3\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -\tau : 0)\}_{\mathcal{H}}$	$\{(\lambda : 0 : -8\lambda + \mu : 14\mu)\}_{\mathcal{E}}$	
$\{(0 : \tau : -8\lambda : -4\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 5\lambda : 8\lambda + 11\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 12\mu : -14\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -6\lambda - 9\mu : -14\mu)\}_{\mathcal{H}}$	
$\{(\tau : -\mu : 0 : -\lambda)\}_{\mathcal{H}}$	$\{(\lambda : -8\lambda + \mu : -10\mu : 7\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 5\mu : 6\lambda)\}_{\mathcal{E}}$	$\{(\tau : -11\lambda : -10\mu : 10\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 5\lambda : 6\mu)\}_{\mathcal{E}}$	$\{(\lambda : -\lambda + \mu : 6\mu : -7\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 12\lambda : -14\mu)\}_{\mathcal{H}}$	$\{(\tau : -9\mu : 8\mu : -3\lambda)\}_{\mathcal{E}}$	
$\{(\tau : 7\mu : -8\mu : -5\lambda)\}_{\mathcal{E}}$	$\{(\tau : -7\lambda : 9\mu : 6\tau)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 3\lambda : 7\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

Corollary 3.4. For q odd prime, the number of elliptic lines on a non-singular cubic surfaces $\mathscr{S}^{(4)}(q)$ with 27 lines in $PG(3, q)$ is 12.

Proof. Let l_1, l_2, and l_3 be any three lines on the non-singular cubic surface $\mathscr{S}^{(3)}(q)$. Consider the following three configurations, namely

$$(l_1, \mathscr{S}_\lambda^{(4)}(q)), \quad (l_2, \mathscr{S}_\lambda^{(4)}(q)), \; and \quad (l_3, \mathscr{S}_\lambda^{(4)}(q)).$$

Table 3.30: The non-singular cubic surface $\mathscr{S}^{(6)}(29)$

$\mathscr{S}^{(6)}(29) = \mathbb{V}(y_0^2y_1 - y_1^2y_0 - y_2^2y_3 + 12y_3^2y_2 + 13y_0y_1y_3 - 11y_1y_2y_3)$		
$e_3 = 6, e_2 = 117, e_1 = 558, e_0 = 364$ *and* $\lvert\mathscr{S}^{(6)}(29)\rvert = 1045$		
27 Lines on $\mathscr{S}^{(6)}(29) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{29}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : 5\lambda + \mu : -11\mu : 7\lambda)\}_{\mathcal{H}}$	$(1 : 0 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : -6\mu : -6\lambda : -5\mu)\}_{\mathcal{H}}$	$(0 : 0 : 1 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(0 : \tau : \mu : 13\lambda + \mu)\}_{\mathcal{H}}$	$(1 : -1 : -12 : 1)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : \lambda : 12\lambda : -14\mu)\}_{\mathcal{H}}$	$(1 : 2 : -5 : -2)$
$\{(\lambda : \mu : \mu : 12\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 5\lambda + 6\mu : -2\lambda)\}_{\mathcal{H}}$	$(1 : -1 : 6 : 1)$
$\{(\lambda : \tau : 0 : 9\mu)\}_{\mathcal{H}}$	$\{(\lambda : \tau : 7\lambda - 5\mu : \lambda)\}_{\mathcal{H}}$	$(1 : 2 : -12 : -2)$
$\{(\tau : -2\lambda : 12\mu : 2\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -12\lambda : -\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \tau : 6\lambda : 14\mu)\}_{\mathcal{H}}$	$\{(\tau : -10\lambda : 5\mu : -12\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -5\lambda : 7\mu)\}_{\mathcal{H}}$	$\{(\tau : \lambda : 12\lambda : -2\tau)\}_{\mathcal{H}}$	
$\{(\lambda : \tau : -5\lambda : -2\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 6\lambda : -4\mu)\}_{\mathcal{H}}$	
$\{(\lambda : 0 : \mu : -12\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -\lambda - 12\mu : 12\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \tau : \tau : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -6\mu : \lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -6\mu : 2\lambda - 2\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \tau : -12\lambda : \mu)\}_{\mathcal{H}}$		
# Elliptic lines= 0 #Hyperbolic lines= 27		

Table 3.31: The non-singular cubic surface $\mathscr{S}^{(10)}(29)$

$\mathscr{S}^{(10)}(29) = \mathbb{V}(y_0^2y_1 - y_1^2y_0 + 6y_2^2y_3 - 5y_3^2y_2 - 10y_0y_1y_3)$		
$e_3 = 10, e_2 = 105, e_1 = 570, e_0 = 360$ *and* $\lvert\mathscr{S}^{(10)}(29)\rvert = 1045$		
27 Lines on $\mathscr{S}^{(10)}(29) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{29}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : 9\lambda + \mu : -\mu : 5\lambda)\}_{\mathcal{E}}$	$(1 : 0 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : -\mu : -5\lambda : 6\mu)\}_{\mathcal{H}}$	$(0 : 1 : 0 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : 4\mu : 4\lambda + 9\mu : 6\mu)\}_{\mathcal{E}}$	$(0 : 0 : 1 : 0)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -4\lambda : -13\mu)\}_{\mathcal{E}}$	$(1 : 0 : 5 : 6)$
$\{(\tau : \lambda : -12\mu : 3\mu)\}_{\mathcal{H}}$	$\{(\tau : 3\mu : \lambda + 3\mu : 14\mu)\}_{\mathcal{E}}$	$(1 : -1 : 5 : 6)$
$\{(\lambda : \mu : 4\mu : 13\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \tau : -4\lambda : -\mu)\}_{\mathcal{E}}$	$(0 : 1 : 0 : -6)$
$\{(\lambda : \tau : -\lambda : 7\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -5\mu : 6\lambda)\}_{\mathcal{H}}$	$(1 : 1 : 0 : 0)$
$\{(\tau : \mu : 0 : 3\lambda)\}_{\mathcal{H}}$	$\{(\tau : \mu : \mu : -7\lambda)\}_{\mathcal{E}}$	$(1 : 0 : 0 : 6)$
$\{(\tau : \mu : 4\mu : \lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 5\lambda : -6\mu)\}_{\mathcal{H}}$	$(1 : -1 : 0 : 6)$
$\{(\lambda : \mu : -\lambda : -5\mu)\}_{\mathcal{E}}$	$\{(\lambda : -13\lambda + \mu : -4\mu : 13\lambda)\}_{\mathcal{E}}$	$(0 : 1 : -5 : -6)$
$\{(\lambda : \mu : 5\tau : 6\lambda)\}_{\mathcal{H}}$	$\{(\lambda : 0 : \mu : 7\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : \mu : 5\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \tau : 5\lambda : -6\mu)\}_{\mathcal{H}}$	
$\{(0 : \lambda : \mu : 7\mu)\}_{\mathcal{H}}$	$\{(\tau : \lambda : -5\lambda : 6\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \lambda : \mu : 0)\}_{\mathcal{H}}$		
# Elliptic lines= 12 #Hyperbolic lines= 15		

Each configuration has 1 hyperbolic involutions of type I and 4 hyperbolic involutions of type II via the previous facts mentioned in Table 3.42. It follows that the number of hyperbolic involutions corresponding to each configuration, is 5. Consequently, we have $15 = 5 \cdot 3$ hyperbolic involutions corresponding to the all configurations. Hence the number of hyperbolic lines on a non-singular cubic surfaces is 15 (see Figure 3.6).

Table 3.32: The non-singular cubic surface $\mathscr{S}^{(0)}(31)$

$\mathscr{S}^{(0)}(31) = \mathbb{V}(y_0^2y_1 + 2y_1^2y_0 + 6y_2^2y_3 + 6y_3^2y_2 + 2y_0y_1y_2 + 5y_0y_1y_3 + 5y_0y_2y_3 + 5y_1y_2y_3)$		
$e_3 = 0, e_2 = 135, e_1 = 594, e_0 = 450$ *and* $\lvert\mathscr{S}^{(0)}(31)\rvert = 1179$		
27 Lines on $\mathscr{S}^{(0)}(31) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{31}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$ $\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$ $\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$ $\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$ $\{(\tau : 15\mu : 0 : 6\lambda)\}_{\mathcal{E}}$ $\{(\lambda : \tau : -7\mu : -13\lambda)\}_{\mathcal{H}}$ $\{(\lambda : \mu : 3\lambda : -\mu)\}_{\mathcal{E}}$ $\{(\lambda : \mu : 2\lambda : -12\mu)\}_{\mathcal{E}}$ $\{(\lambda : \mu : 4\mu : 9\lambda)\}_{\mathcal{E}}$ $\{(\lambda : \mu : 5\mu : -13\lambda)\}_{\mathcal{E}}$ $\{(\tau : 0 : -6\mu : -6\lambda)\}_{\mathcal{H}}$ $\{(\lambda : \mu : 5\lambda : \mu)\}_{\mathcal{E}}$ $\{(0 : \tau : -6\lambda : -6\mu)\}_{\mathcal{H}}$ $\{(\tau : -8\lambda : -14\mu : 3\lambda)\}_{\mathcal{E}}$	$\{(\lambda : 8\lambda + \mu : -2\mu : 9\lambda)\}_{\mathcal{H}}$ $\{(\tau : -10\mu : 12\lambda : 10\mu)\}_{\mathcal{E}}$ $\{(\lambda : \mu : 2\lambda : -8\lambda + 3\mu)\}_{\mathcal{H}}$ $\{(\lambda : 12\lambda + \mu : 3\lambda : -7\mu)\}_{\mathcal{H}}$ $\{(\tau : -13\lambda : -3\lambda : 12\mu)\}_{\mathcal{E}}$ $\{(\lambda : \mu : 4\lambda + 3\mu : -10\lambda)\}_{\mathcal{H}}$ $\{(\lambda : 2\lambda + \mu : 13\lambda + \mu : -7\mu)\}_{\mathcal{H}}$ $\{(\lambda : \mu : -10\mu : -10\lambda)\}_{\mathcal{E}}$ $\{(\tau : 8\lambda : 5\lambda - \mu : 8\lambda)\}_{\mathcal{E}}$ $\{(\tau : 3\mu : 12\mu : -\lambda)\}_{\mathcal{E}}$ $\{(\tau : 15\mu : 15\lambda : 0)\}_{\mathcal{E}}$ $\{(\lambda : 10\lambda + \mu : 5\lambda : -2\mu)\}_{\mathcal{H}}$ $\{(\tau : -12\lambda : -4\lambda : -14\mu)\}_{\mathcal{E}}$	
# Elliptic lines= 16 #Hyperbolic lines= 11		

Table 3.33: The non-singular cubic surface $\mathscr{S}^{(1)}(31)$

$\mathscr{S}^{(1)}(31) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 4y_2^2y_3 + 6y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 1, e_2 = 132, e_1 = 597, e_0 = 449$ *and* $\lvert\mathscr{S}^{(1)}(31)\rvert = 1179$		
27 Lines on $\mathscr{S}^{(1)}(31) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{31}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$ $\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$ $\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$ $\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$ $\{(\lambda : \mu : 12\lambda : 13\mu)\}_{\mathcal{E}}$ $\{(\lambda : -\lambda + \mu : -\mu : 0)\}_{\mathcal{H}}$ $\{(\tau : -12\mu : 11\mu : -5\lambda)\}_{\mathcal{H}}$ $\{(\tau : 0 : -8\mu : 5\lambda)\}_{\mathcal{H}}$ $\{(\lambda : \mu : 12\mu : 13\lambda)\}_{\mathcal{E}}$ $\{(\lambda : \mu : -9\lambda : 14\mu)\}_{\mathcal{E}}$ $\{(\lambda : \mu : -6\mu : 7\lambda)\}_{\mathcal{H}}$ $\{(\lambda : \mu : -9\mu : 14\lambda)\}_{\mathcal{E}}$ $\{(\tau : -\lambda : 0 : -\mu)\}_{\mathcal{H}}$ $\{(\lambda : \mu : -6\lambda : 7\mu)\}_{\mathcal{H}}$	$\{(\tau : -6\lambda : 5\lambda : -8\mu)\}_{\mathcal{E}}$ $\{(\lambda : \mu : 12\lambda : -3\lambda - 5\mu)\}_{\mathcal{H}}$ $\{(\tau : -\lambda : 13\mu : -14\mu)\}_{\mathcal{H}}$ $\{(\lambda : \mu : 2\lambda - 4\mu : 14\lambda)\}_{\mathcal{H}}$ $\{(\lambda : 8\lambda + \mu : -9\lambda : -13\mu)\}_{\mathcal{H}}$ $\{(\tau : -4\lambda : -12\mu : 3\lambda)\}_{\mathcal{E}}$ $\{(\lambda : -8\lambda + \mu : -13\mu : 7\lambda)\}_{\mathcal{E}}$ $\{(\lambda : \mu : 8\lambda - 12\mu : 13\mu)\}_{\mathcal{H}}$ $\{(0 : \lambda : \mu : 5\lambda - 11\mu)\}_{\mathcal{H}}$ $\{(\lambda : \mu : -4\lambda + 2\mu : 14\lambda)\}_{\mathcal{H}}$ $\{(\lambda : 5\lambda + \mu : -6\lambda : -8\mu)\}_{\mathcal{E}}$ $\{(\lambda : -13\lambda + \mu : 8\tau : 13\lambda)\}_{\mathcal{H}}$ $\{(\tau : 4\lambda : -5\lambda : -13\mu)\}_{\mathcal{H}}$	$(1 : -1 : 0 : 0)$
# Elliptic lines= 12 #Hyperbolic lines= 17		

Thus

$$\#(\text{Elliptic lines}) = 27 - 15 = 12.$$

□

Corollary 3.5. For q odd prime, all the 27 lines on a non-singular cubic surfaces $\mathscr{S}^{(18)}(q)$ with 27 lines in $PG(3, q); q = 1(\bmod 3)$ are hyperbolic.

Table 3.34: The non-singular cubic surface $\mathscr{S}^{(2)}(31)$

$\mathscr{S}^{(2)}(31) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 6y_2^2y_3 + 6y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 2, e_2 = 129, e_1 = 600, e_0 = 448$ *and* $\vert\mathscr{S}^{(2)}(31)\vert = 1179$		
27 Lines on $\mathscr{S}^{(2)}(31) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{31}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -10\lambda + 11\mu : 15\lambda)\}_{\mathcal{H}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : 2\lambda + \mu : -4\mu : -3\lambda)\}_{\mathcal{E}}$	$(0 : 0 : 1 : -1)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 15\mu : 15\lambda)\}_{\mathcal{E}}$	
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\tau : -5\mu : -2\lambda : 4\mu)\}_{\mathcal{H}}$	
$\{(\lambda : -\lambda : \mu : -\mu)\}_{\mathcal{H}}$	$\{(\lambda : 6\lambda + \mu : -2\mu : -7\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -3\mu : -3\lambda)\}_{\mathcal{H}}$	$\{(\lambda : 6\lambda + \mu : -7\lambda : -2\mu)\}_{\mathcal{H}}$	
$\{(\tau : -\lambda : 0 : -\mu)\}_{\mathcal{H}}$	$\{(\tau : -15\lambda : -4\mu : 14\lambda)\}_{\mathcal{E}}$	
$\{(\tau : -2\mu : 11\lambda : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 15\lambda : -10\lambda + 11\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : -7\lambda : -7\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : -7\mu : -7\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : -3\lambda : -3\mu)\}_{\mathcal{H}}$	$\{(\tau : -5\mu : 4\mu : -2\lambda)\}_{\mathcal{H}}$	
$\{(\tau : -2\lambda : \lambda : 11\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -3\lambda : 8\lambda - 4\mu)\}_{\mathcal{E}}$	
$\{(0 : \tau : 5\lambda : 5\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -3\mu : -4\lambda + 8\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 15\lambda : 15\mu)\}_{\mathcal{E}}$	$\{(\lambda : -\lambda + \mu : -\mu : 0)\}_{\mathcal{H}}$	
$\{(\tau : 0 : 5\lambda : 5\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

Table 3.35: The non-singular cubic surface $\mathscr{S}^{(3)}(31)$

$\mathscr{S}^{(3)}(31) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 2y_2^2y_3 + 3y_3^2y_2 + y_0y_1y_2 + y_0y_1y_3 + y_0y_2y_3 + y_1y_2y_3)$		
$e_3 = 3, e_2 = 126, e_1 = 603, e_0 = 447$ *and* $\vert\mathscr{S}^{(3)}(31)\vert = 1179$		
27 Lines on $\mathscr{S}^{(3)}(31) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{31}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : 6\lambda : 10\lambda : 5\lambda - 7\mu)\}_{\mathcal{E}}$	$(1 : -1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{E}}$	$\{(\tau : 4\lambda : 9\lambda : 8\lambda + 5\mu)\}_{\mathcal{E}}$	$(1 : 9 : 10 : -8)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 8\lambda : 15\lambda + 12\mu)\}_{\mathcal{H}}$	$(1 : 7 : 8 : 6)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{E}}$	$\{(\lambda : -\lambda + \mu : -2\mu : \mu)\}_{\mathcal{H}}$	
$\{(\lambda : -\lambda + \mu : -\mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : -8\lambda + \mu : -13\mu : 7\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 8\lambda : 7\mu)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 12\lambda : 2\lambda - 7\mu)\}_{\mathcal{E}}$	
$\{(\tau : -9\lambda : -8\mu : 8\lambda)\}_{\mathcal{H}}$	$\{(\lambda : 0 : 15\lambda + \mu : -11\mu)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 6\lambda - 8\mu : 6\lambda)\}_{\mathcal{H}}$	$\{(\lambda : 7\lambda + \mu : 5\mu : -8\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 10\mu : 6\lambda)\}_{\mathcal{H}}$	$\{(\tau : \mu : 5\lambda + \mu : -8\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 8\mu : 7\lambda)\}_{\mathcal{E}}$	$\{(\tau : -7\mu : 6\mu : 12\lambda)\}_{\mathcal{H}}$	
$\{(0 : \tau : 15\mu : 10\lambda)\}_{\mathcal{E}}$	$\{(\lambda : \mu : 10\lambda : -7\lambda + 5\mu)\}_{\mathcal{E}}$	
$\{(\tau : -\lambda : 0 : -\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 12\mu : -8\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \mu : 12\lambda : -8\mu)\}_{\mathcal{H}}$	$\{(\tau : -4\lambda : -13\mu : 3\lambda)\}_{\mathcal{E}}$	
$\{(\lambda : \mu : 10\lambda : 6\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

Proof. Let $\mathscr{S}^{(18)}(q) = \mathbb{V}(F)$ be a non-singular cubic surfaces with 18 Eckardt points then, in a suitable coordinate system, the cubic surfaces can be written in the form

$$\mathscr{S}^{(18)}(q) = \mathbb{V}(y_0^3 + y_1^3 + y_2^3 + y_3^3).$$

In fact, such cubic surface exists in $PG(3, q)$ if and only if $q = 1(\operatorname{mod} 3)$.

Table 3.36: The non-singular cubic surface $\mathscr{S}^{(4)}(31)$

$\mathscr{S}^{(4)}(31) = \mathbb{V}(y_0^2y_1 + y_1^2y_0 + 5y_2^2y_3 + 5y_3^2y_2 + 3y_0y_1y_2 + 3y_0y_1y_3 + 3y_0y_2y_3 + 3y_1y_2y_3)$		
$e_3 = 4, e_2 = 123, e_1 = 606, e_0 = 446$ *and* $\lvert\mathscr{S}^{(4)}(31)\rvert = 1179$		
27 Lines on $\mathscr{S}^{(4)}(31) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{31}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_\mathcal{E}$	$\{(\lambda : \mu : 13\mu : 4\lambda + 5\mu)\}_\mathcal{H}$	$(1 : 1 : -11 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_\mathcal{E}$	$\{(\lambda : \mu : -11\lambda : 2\lambda - 2\mu)\}_\mathcal{H}$	$(1 : -1 : 0 : 0)$
$\{(0 : \lambda : \mu : 0)\}_\mathcal{E}$	$\{(\lambda : 13\lambda + \mu : 7\mu : -15\lambda)\}_\mathcal{E}$	$(0 : 0 : 1 : -1)$
$\{(0 : \lambda : 0 : \mu)\}_\mathcal{E}$	$\{(\tau : -\lambda + 3\mu : 0 : 9\mu)\}_\mathcal{H}$	$(1 : 1 : 0 : -11)$
$\{(\tau : 0 : -13\lambda : -13\mu)\}_\mathcal{H}$	$\{(\lambda : \mu : 5\lambda + 4\mu : 13\lambda)\}_\mathcal{H}$	
$\{(0 : \tau : -13\lambda : -13\mu)\}_\mathcal{H}$	$\{(\lambda : \mu : -11\lambda : -11\mu)\}_\mathcal{E}$	
$\{(\lambda : \mu : 13\lambda : 13\mu)\}_\mathcal{E}$	$\{(\lambda : -9\lambda + \mu : 13\lambda : 4\mu)\}_\mathcal{H}$	
$\{(\lambda : \mu : -15\mu : -15\lambda)\}_\mathcal{H}$	$\{(\lambda : \mu : 7\lambda + 2\mu : -15\mu)\}_\mathcal{E}$	
$\{(\lambda : \mu : -15\lambda : -15\mu)\}_\mathcal{H}$	$\{(\lambda : \mu : 10\tau : 0)\}_\mathcal{H}$	
$\{(\lambda : \mu : 13\mu : 13\lambda)\}_\mathcal{E}$	$\{(\tau : 14\lambda : 7\lambda : 4\lambda + 7\mu)\}_\mathcal{E}$	
$\{(\lambda : -\lambda : \mu : -\mu)\}_\mathcal{H}$	$\{(\tau : -7\lambda : 4\mu : 2\lambda)\}_\mathcal{H}$	
$\{(\lambda : \tau : 2\mu : -11\lambda)\}_\mathcal{H}$	$\{(\lambda : \mu : -15\lambda : 2\lambda + 7\mu)\}_\mathcal{E}$	
$\{(\lambda : \tau : -11\lambda : 2\mu)\}_\mathcal{H}$	$\{(\lambda : \mu : -11\mu : -11\lambda)\}_\mathcal{E}$	
$\{(\tau : \lambda : 2\mu : -11\lambda)\}_\mathcal{H}$		
# Elliptic lines= 12 #Hyperbolic lines= 17		

Table 3.37: The non-singular cubic surface $\mathscr{S}^{(6)}(31)$

$\mathscr{S}^{(6)}(31) = \mathbb{V}(y_0^2y_1 - 13y_1^2y_0 - 8y_2^2y_3 + 11y_3^2y_2 - 10y_0y_1y_2 - 12y_0y_2y_3)$		
$e_3 = 6, e_2 = 117, e_1 = 612, e_0 = 444$ *and* $\lvert\mathscr{S}^{(6)}(31)\rvert = 1179$		
27 Lines on $\mathscr{S}^{(6)}(31) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{31}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_\mathcal{H}$	$\{(\lambda : \mu : 10\mu : -6\lambda - 4\mu)\}_\mathcal{H}$	$(0 : 1 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_\mathcal{H}$	$\{(\lambda : \mu : -15\lambda : -7\lambda - 15\mu)\}_\mathcal{H}$	$(0 : 0 : 0 : 1)$
$\{(0 : \lambda : \mu : 0)\}_\mathcal{H}$	$\{(\lambda : \mu : 4\lambda - 10\mu : 4\mu)\}_\mathcal{H}$	$(1 : 14 : -15 : 0)$
$\{(0 : \lambda : 0 : \mu)\}_\mathcal{H}$	$\{(\tau : \lambda : \lambda : -2\lambda - 14\mu)\}_\mathcal{H}$	$(1 : 0 : -15 : 0)$
$\{(\lambda : \mu : \mu : \lambda)\}_\mathcal{H}$	$\{(\lambda : \mu : -3\lambda + 8\mu : 0)\}_\mathcal{H}$	$(1 : 14 : -15 : -7)$
$\{(0 : \lambda : \mu : 12\mu)\}_\mathcal{H}$	$\{(\lambda : \mu : -15\lambda + 9\mu : -7\lambda)\}_\mathcal{H}$	$(1 : 0 : -15 : -7)$
$\{(\lambda : \mu : 8\lambda : 4\mu)\}_\mathcal{H}$	$\{(\lambda : \mu : -15\lambda + 9\mu : 15\mu)\}_\mathcal{H}$	
$\{(\lambda : 12\lambda : 0 : \mu)\}_\mathcal{H}$	$\{(\tau : -11\mu : 2\mu : 7\lambda)\}_\mathcal{H}$	
$\{(\lambda : \mu : 2\lambda + 5\mu : 11\lambda)\}_\mathcal{H}$	$\{(\lambda : \mu : 2\lambda + 5\mu : -2\mu)\}_\mathcal{H}$	
$\{(\lambda : \tau : \lambda : \lambda + 2\mu)\}_\mathcal{H}$	$\{(\tau : 0 : 14\lambda : -13\mu)\}_\mathcal{H}$	
$\{(\lambda : \mu : -15\lambda : 15\mu)\}_\mathcal{H}$	$\{(\lambda : \mu : -4\lambda - 10\mu : \lambda)\}_\mathcal{H}$	
$\{(\lambda : \mu : 10\mu : -7\lambda)\}_\mathcal{H}$	$\{(\lambda : \mu : 8\lambda : -10\lambda - 4\mu)\}_\mathcal{H}$	
$\{(\lambda : \mu : -3\mu : 11\lambda)\}_\mathcal{H}$	$\{(\lambda : \tau : 5\lambda + 8\mu : -15\lambda + 3\mu)\}_\mathcal{H}$	
$\{(\lambda : \mu : \lambda : -2\mu)\}_\mathcal{H}$		
# Elliptic lines= 0 #Hyperbolic lines= 27		

Assume that l is any line on $\mathscr{S}^{(18)}(q)$ and

$$l = \left\{(a_0\lambda + b_0\mu : a_1\lambda + b_1\mu : a_2\lambda + b_2\mu : a_3\lambda + b_3\mu) : (\lambda : \mu) \in PG(1, q)\right\}$$

Let $\beta_2 = (a_0, a_1, a_2, a_3)$, $\beta_3 = (b_0, b_1, b_2, b_3)$. Pick $\beta_0, \beta_1 \in GF(q)^{\oplus 4}$ so that

Table 3.38: The non-singular cubic surface $\mathscr{S}^{(9)}(31)$

$\mathscr{S}^{(9)}(31) = \mathbb{V}(y_0^2y_1 + 2y_1^2y_0 + 5y_2^2y_3 + 5y_3^2y_2 + 2y_0y_1y_2 + 2y_0y_1y_3 + 2y_0y_2y_3 + 3y_1y_2y_3)$		
$e_3 = 9, e_2 = 108, e_1 = 621, e_0 = 441$ *and* $\lvert\mathscr{S}^{(9)}(31))\rvert = 1179$		
27 Lines on $\mathscr{S}^{(9)}(31) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{31}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -3\lambda : 15\lambda - 6\mu)\}_{\mathcal{H}}$	$(0 : 0 : 1 : -1)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : -8\tau : -8\lambda : -8\mu)\}_{\mathcal{H}}$	$(1 : 2 : 0 : 13)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : -13\lambda + \mu : -6\mu : -3\lambda)\}_{\mathcal{H}}$	$(0 : 1 : 0 : 8)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -\lambda - 15\mu : 13\lambda)\}_{\mathcal{H}}$	$(1 : 0 : -3 : 0)$
$\{(\tau : 14\mu : \mu : -3\lambda)\}_{\mathcal{H}}$	$\{(\lambda : 2\lambda + \mu : 13\lambda : -15\mu)\}_{\mathcal{H}}$	$(1 : 2 : 13 : 0)$
$\{(\lambda : \mu : 8\mu : -6\lambda)\}_{\mathcal{H}}$	$\{(\tau : 14\mu : -3\lambda : \mu)\}_{\mathcal{H}}$	$(1 : -10 : 13 : -3)$
$\{(\lambda : \mu : -11\lambda : -3\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -3\lambda : -11\mu)\}_{\mathcal{H}}$	$(1 : -10 : -3 : 13)$
$\{(0 : \tau : -13\lambda : -13\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -9\mu : 13\lambda)\}_{\mathcal{H}}$	$(1 : 0 : 0 : -3)$
$\{(\lambda : \mu : 13\lambda : -9\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 4\lambda + 10\mu : 8\mu)\}_{\mathcal{H}}$	$(0 : 1 : 8 : 0)$
$\{(\tau : 15\mu : 0 : 15\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -6\lambda : 8\mu)\}_{\mathcal{H}}$	
$\{(\tau : 2\lambda : 8\mu : 13\lambda)\}_{\mathcal{H}}$	$\{(\lambda : -10\lambda + \mu : -6\lambda : 8\mu)\}_{\mathcal{H}}$	
$\{(\tau : 15\mu : 15\lambda : 0)\}_{\mathcal{H}}$	$\{(\tau : 0 : 12\mu : 12\lambda)\}_{\mathcal{H}}$	
$\{(\tau : 12\lambda : 3\lambda : 4\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -13\lambda + 8\mu : -6\lambda)\}_{\mathcal{H}}$	
$\{(\tau : 2\lambda : 13\lambda : 8\mu)\}_{\mathcal{H}}$		
# Elliptic lines= 0 #Hyperbolic lines= 27		

Table 3.39: The non-singular cubic surface $\mathscr{S}^{(10)}(31)$

$\mathscr{S}^{(10)}(31) = \mathbb{V}(y_0^2y_1 - y_1^2y_0 - 11y_2^2y_3 - 14y_3^2y_2 + 0y_0y_1y_2 + 0y_0y_1y_3 + 0y_0y_2y_3 - 2y_1y_2y_3)$		
$e_3 = 10, e_2 = 105, e_1 = 624, e_0 = 440$ *and* $\lvert\mathscr{S}^{(10)}(31)\rvert = 1179$		
27 Lines on $\mathscr{S}^{(10)}(31) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{31}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 10\mu : 8\lambda - 8\mu)\}_{\mathcal{H}}$	$(1 : 0 : 0 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : \tau : -4\lambda : -12\mu)\}_{\mathcal{H}}$	$(0 : 0 : 1 : 0)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \tau : -4\mu : -12\lambda)\}_{\mathcal{H}}$	$(0 : 0 : 0 : 1)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : \mu : 14\mu : -11\lambda)\}_{\mathcal{H}}$	$(1 : 1 : 14 : 11)$
$\{(\lambda : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(0 : \lambda : -3\lambda + \mu : -3\mu)\}_{\mathcal{H}}$	$(1 : 1 : 14 : 0)$
$\{(\lambda : 0 : \mu : -3\mu)\}_{\mathcal{H}}$	$\{(\lambda : \tau : -13\mu : -8\lambda)\}_{\mathcal{H}}$	$(0 : 1 : 14 : 11)$
$\{(\lambda : \mu : -13\lambda : -\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 10\mu : -8\lambda)\}_{\mathcal{H}}$	$(0 : 1 : 0 : 11)$
$\{(\tau : \mu : 13\lambda : -\mu)\}_{\mathcal{H}}$	$\{(\lambda : \tau : -13\lambda : -8\mu)\}_{\mathcal{H}}$	$(1 : 1 : 0 : 11)$
$\{(\lambda : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\tau : \lambda : -14\mu : 11\lambda)\}_{\mathcal{H}}$	$(0 : 0 : 1 : -3)$
$\{(\lambda : \mu : 14\lambda : 11\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : \mu : -12\lambda)\}_{\mathcal{H}}$	$(0 : 1 : 14 : 0)$
$\{(\tau : \lambda : \lambda : 12\mu)\}_{\mathcal{H}}$	$\{(\lambda : \lambda : -3\lambda + \mu : -3\mu)\}_{\mathcal{H}}$	
$\{(\tau : \lambda : 4\mu : 3\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -4\lambda : 3\mu)\}_{\mathcal{H}}$	
$\{(\lambda : \tau : 14\lambda : 11\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 14\mu : 11\lambda)\}_{\mathcal{H}}$	
$\{(\lambda : \tau : 14\mu : 11\lambda)\}_{\mathcal{H}}$		
# Elliptic lines= 0 #Hyperbolic lines= 27		

$\beta_0, \beta_1, \beta_2$ and β_3 form a basis of $GF(q)^{\oplus 4}$. Note that

$$\mathfrak{q}(\lambda, \mu) = \frac{\partial F}{\partial \beta_0}\bigg|_l = 3y_0^2\Big|_l = 3a_0^2\lambda^2 + 6a_0b_0\lambda\mu + 3b_0^2\mu^2$$

$$= \alpha_2\lambda^2 + \alpha_1\lambda + \alpha_0,$$

Table 3.40: The non-singular cubic surface $\mathscr{S}^{(18)}(31)$

$\mathscr{S}^{(18)}(31) = \mathbb{V}(y_0^2y_1 - y_1^2y_0 + 14y_2^2y_3 + y_3^2y_2 + 0y_0y_1y_2 - 13y_0y_1y_3 + 15y_0y_2y_3 - 15y_1y_2y_3)$		
$e_3 = 18, e_2 = 81, e_1 = 648, e_0 = 432$ *and* $\lvert\mathscr{S}^{(18)}(31)\rvert = 1179$		
27 Lines on $\mathscr{S}^{(18)}(31) : (\lambda : \mu) \in \mathbb{P}(\mathbb{F}_{31}); \tau = \lambda + \mu$		Eckardt points
$\{(\lambda : 0 : \mu : 0)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -6\mu : -6\lambda)\}_{\mathcal{H}}$	$(0 : 0 : 1 : 0)$
$\{(\lambda : 0 : 0 : \mu)\}_{\mathcal{H}}$	$\{(0 : \tau : -10\mu : -16\lambda)\}_{\mathcal{H}}$	$(1 : 0 : 0 : 25)$
$\{(0 : \lambda : \mu : 0)\}_{\mathcal{H}}$	$\{(\tau : \lambda : -5\lambda : 8\mu)\}_{\mathcal{H}}$	$(0 : 1 : 1 : 1)$
$\{(0 : \lambda : 0 : \mu)\}_{\mathcal{H}}$	$\{(\lambda : -\lambda + \mu : -5\mu : -7\lambda)\}_{\mathcal{H}}$	$(1 : -6 : 5 : -6)$
$\{(\lambda : \tau : 0 : -12\mu)\}_{\mathcal{H}}$	$\{(\tau : \mu : -6\mu : -6\lambda)\}_{\mathcal{H}}$	$(0 : 1 : 0 : 1)$
$\{(\tau : 11\lambda : 4\mu : 4\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : 6\lambda - 4\mu : -6\lambda)\}_{\mathcal{H}}$	$(1 : 0 : 0 : -1)$
$\{(\lambda : \mu : 5\tau : 7\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -\lambda - 11\mu : -\lambda)\}_{\mathcal{H}}$	$(1 : -1 : 5 : -7)$
$\{(\lambda : \tau : 6\lambda : 6\mu)\}_{\mathcal{H}}$	$\{(\lambda : 0 : \mu : -15\lambda - 14\mu)\}_{\mathcal{H}}$	$(1 : -1 : 0 : -7)$
$\{(\lambda : \tau : 6\lambda : 6\tau)\}_{\mathcal{H}}$	$\{(\lambda : \tau : -\lambda : \mu)\}_{\mathcal{H}}$	$(1 : 0 : 6 : -6)$
$\{(\lambda : \tau : \tau : \mu)\}_{\mathcal{H}}$	$\{(\tau : -11\lambda : 11\mu : -11\lambda)\}_{\mathcal{H}}$	$(1 : -6 : -1 : -6)$
$\{(\lambda : \mu : -5\mu : -7\lambda)\}_{\mathcal{H}}$	$\{(\lambda : \mu : \mu : -\lambda)\}_{\mathcal{H}}$	$(0 : 1 : -5 : 0)$
$\{(\lambda : \tau : 5\lambda : -8\mu)\}_{\mathcal{H}}$	$\{(\lambda : \mu : -\lambda : \mu)\}_{\mathcal{H}}$	$(1 : 5 : 6 : -1)$
$\{(\lambda : \mu : 5\lambda : 7\mu)\}_{\mathcal{H}}$	$\{(\tau : \mu : -13\lambda : 12\lambda)\}_{\mathcal{H}}$	$(1 : 5 : 5 : -1)$
$\{(\lambda : \lambda : \mu : 0)\}_{\mathcal{H}}$		$(0 : 1 : -6 : 6)$
		$(1 : 0 : 5 : 0)$
		$(1 : 0 : -1 : -1)$
		$(0 : 1 : 0 : 6)$
		$(1 : 1 : 0 : 0)$
# Elliptic lines= 0 #Hyperbolic lines= 27		

Table 3.41: Number of elliptic lines

q/e_3	0	1	2	3	4	5	6	7	8	9	10	11	17	18
17		16		12	12		0							
19			12	12	12		0			0	12			0
23		16	16	12	12		0							
29	16	16	16/12	12	12		0				12			
31	16	12	12	12	12		0			0	0			0

Table 3.42: Type of involutions

Type of involution	I	II
Elliptic	0	5
Hyperbolic	2	3
Hyperbolic	1	4
Hyperbolic	0	5

$$\mathfrak{q}'(\lambda, \mu) = \frac{\partial F}{\partial \beta_1}\bigg|_l = 3y_1^2\Big|_l = 3a_1^2\lambda^2 + 6a_1b_1\lambda\mu + 3b_1^2\mu^2$$
$$= \gamma_2\lambda^2 + \gamma_1\lambda + \gamma_0,$$

where $\alpha_2 = 3a_0^2,\ \alpha_1 = 6a_0b_0\mu,\ \alpha_0 = 3b_0^2\mu^2,\ \gamma_2 = 3a_1^2,\ \gamma_1 = 6a_1b_1\mu$ and $\gamma_0 =$

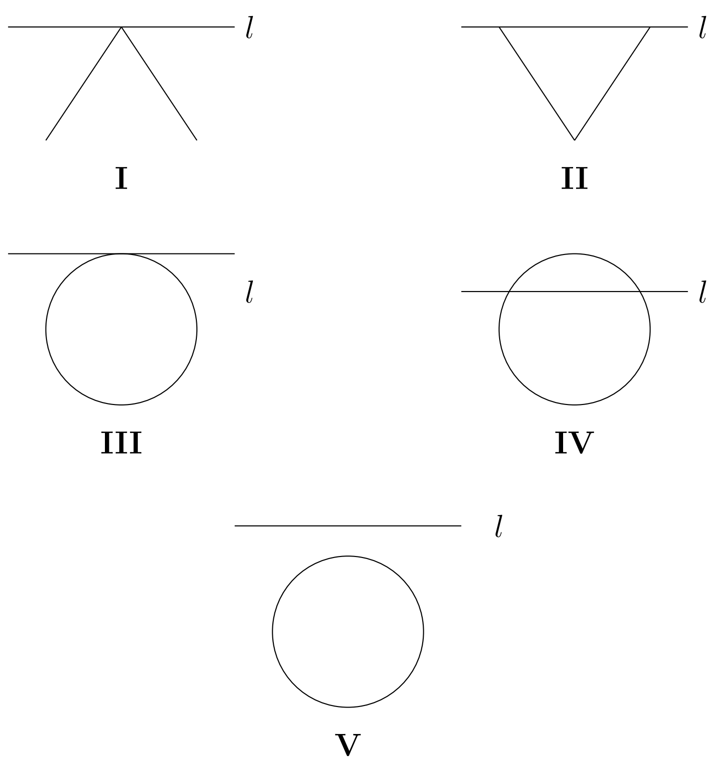

Figure 3.4: Configurations of $(l, \mathscr{S}_\lambda)$.

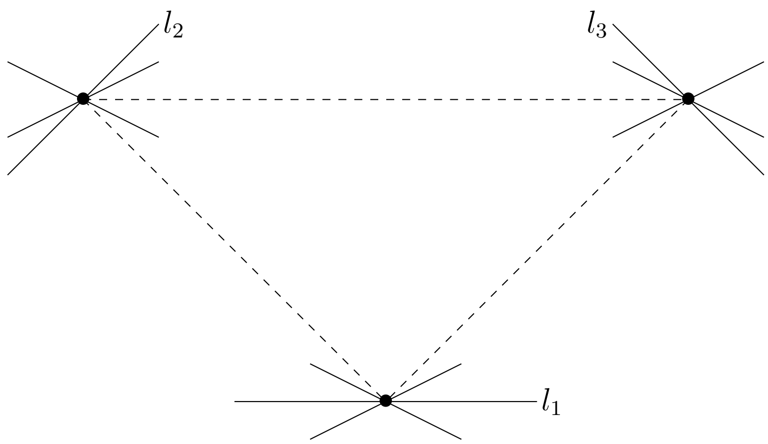

Figure 3.5: The configuration $(l, \mathscr{S}_\lambda^{(3)}(q))$.

$3b_1^2\mu^2$. Assume that ξ_1, ξ_1 roots of $\mathfrak{q}$ in some extension field of $GF(q)$. Then by using properties or resultant which was introduced in Section 3.5, we have

Table 3.43: The distinct non-singular cubic surfaces with 27 lines (up to e-invariants) over $GF(q)$ for $q = 17, 19, 23, 29, 31$

Over $GF(17)$				
$T_i^{(17)}$	$e_{0,i}$	$e_{1,i}$	$e_{2,i}$	$e_{3,i}$
$i=1$	57	219	132	1
$i=2$	55	255	126	3
$i=3$	54	228	123	4
$i=4$	52	234	117	6

Over $GF(19)$				
$T_i^{(19)}$	$e_{0,i}$	$e_{1,i}$	$e_{2,i}$	$e_{3,i}$
$i=1$	88	276	129	2
$i=2$	87	279	126	3
$i=3$	86	282	123	4
$i=4$	84	288	117	6
$i=5$	81	297	108	9
$i=6$	80	300	105	10
$i=7$	72	324	81	18

Over $GF(23)$				
$T_i^{(23)}$	$e_{0,i}$	$e_{1,i}$	$e_{2,i}$	$e_{3,i}$
$i=1$	177	381	132	1
$i=2$	176	384	129	2
$i=3$	175	387	126	3
$i=4$	174	390	123	4
$i=5$	172	396	117	6

Over $GF(29)$				
$T_i^{(29)}$	$e_{0,i}$	$e_{1,i}$	$e_{2,i}$	$e_{3,i}$
$i=1$	370	540	135	0
$i=2$	369	543	132	1
$i=3$	368	546	129	2
$i=4$	367	549	126	3
$i=5$	366	552	123	4
$i=6$	364	558	117	6
$i=7$	360	570	105	10

Over $GF(31)$				
$T_i^{(31)}$	$e_{0,i}$	$e_{1,i}$	$e_{2,i}$	$e_{3,i}$
$i=1$	450	594	135	0
$i=2$	449	597	132	1
$i=3$	448	600	129	2
$i=4$	447	603	126	3
$i=5$	446	606	123	4
$i=6$	444	612	117	6
$i=7$	441	621	108	9
$i=8$	440	624	105	10
$i=9$	432	648	81	18

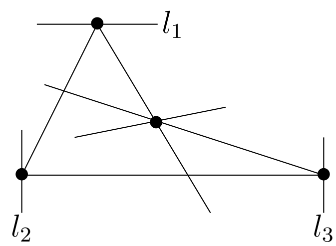

Figure 3.6: The configuration $(l, \mathscr{S}_\lambda^{(4)}(q))$.

$$R\left(\frac{\partial f}{\partial \beta_0}\bigg|_l, \frac{\partial f}{\partial \beta_1}\bigg|_l\right) = \alpha_2^2(\gamma_2\xi_1^2 + \gamma_1\xi_1 + \gamma_0)(\gamma_2\xi_2^2 + \gamma_1\xi_2 + \gamma_0)$$

$$= \alpha_2^2\gamma_2^2(\xi_1^2 + 2c\xi_1\mu + c^2\mu^2)(\xi_2^2 + 2c\xi_2\mu + c^2\mu^2);\ c = b_1/a_1$$

$$= \alpha_2^2\gamma_2^2(\xi_1 + c\mu)^2(\xi_2 + c\mu)^2.$$

If $a_1 = 0$, then $\mathfrak{q}'(\lambda, \mu) = 3b_1^2\mu^2$ and

$$R\left(\left.\frac{\partial f}{\partial \beta_0}\right|_l, \left.\frac{\partial f}{\partial \beta_1}\right|_l\right) = (9a_0^2 b_1^2 \mu)^2.$$

So l is hyperbolic line. Since l is an arbitrary line on the cubic surface $\mathscr{S}^{(18)}(q)$, all the 27 lines on $\mathscr{S}^{(18)}(q)$ are hyperbolic. □

Corollary 3.6. For q odd prime, the number of elliptic lines on a non-singular cubic surfaces with 27 lines, $\mathscr{S}^{(0)}(q)$ in $PG(3, q)$, is 16.

Proof. Let l be any line on the non-singular cubic surface $\mathscr{S}^{(0)}(q)$. Then we have the configuration, namely $(l, \mathscr{S}_\lambda^{(0)}(q))$.

The later configuration has 0 hyperbolic involutions of type I, and 5 hyperbolic involutions of type II depending on the previous facts (see Table 3.42). It follows that the number of hyperbolic involutions corresponding to this configuration, is $5 \cdot 2 + 1 = 11$ hyperbolic involutions corresponding to the configuration $(l, \mathscr{S}_\lambda^{(0)}(q))$. Hence the number of hyperbolic lines on a non-singular cubic surfaces $\mathscr{S}^{(0)}(q)$ is 11.

Thus

$$\#(\text{Elliptic lines}) = 27 - 11 = 16.$$

□

Chapter 4

Classification of classes of smooth cubic surfaces in $PG(19, k)$

In this chapter of the thesis, we classify classes of smooth cubic surfaces with 27 lines in $PG(19, k)$ (or equivalently $\mathbb{P}_k^{19}$) up to Eckardt points where $k = \mathbb{C}$ or $k = GF(q); q > 7$ and q prime. By considering the configurations of 6 points in general position in the projective plane $PG(2, k)$ (or equivalently $\mathbb{P}_k^2$), we can describe subsets of projective space $\mathbb{P}_k^{19}$ that correspond to non-singular cubic surfaces with m Eckardt points. Recall that a non-singular cubic surface, namely X, can be viewed as the blow up of $\mathbb{P}_k^2$ at 6 points in general position. Furthermore, there are 45 tritangent planes on X. Henceforth, we will denote the set of all triples of lines, which correspond to the 45 tritangent planes on X, by $\mathbb{T}$. The classification of such cubic surfaces with m Eckardt points has been studied by Segre in 1946. However, we give another way to classify cubic surfaces and give the possibilities for the number of Eckardt points on them. Moreover, we will discuss the irreducibility of classes of smooth cubic surfaces in $\mathbb{P}_{\mathbb{C}}^{19}$, and we will give the codimension of each class as a subvariety of $\mathbb{P}_{\mathbb{C}}^{19}$.

First of all, we give some notations and terminologies that we will be used later in this chapter. Recall the equation of a cubic surface $\mathscr{S}$ in $\mathbb{P}_k^3$ is $\mathscr{S} := \mathbb{V}(g) = \{g = 0\}$ where

$$\begin{aligned} g = c_1 y_0^3 + c_2 y_1^3 + c_3 y_2^3 + c_4 y_3^3 + c_5 y_0^2 y_1 + c_6 y_0^2 y_2 + c_7 y_0^2 y_3 \\ + c_8 y_1^2 y_0 + c_9 y_1^2 y_2 + c_{10} y_1^2 y_3 \end{aligned}$$

$$+ c_{11}y_2^2y_0 + c_{12}y_2^2y_1 + c_{13}y_2^2y_3 + c_{14}y_3^2y_0 + c_{15}y_3^2y_1 + c_{16}y_3^2y_2$$
$$+ c_{17}y_0y_1y_2 + c_{18}y_0y_1y_3 + c_{19}y_0y_2y_3 + c_{20}y_1y_2y_3 = 0. \tag{4.0.1}$$

The equation of a conic $\mathscr{C}$ in $\mathbb{P}_k^2$ is $\mathscr{C} := \mathbb{V}(h) = \{h = 0\}$ where

$$h = c_1y_0^2 + c_2y_1^2 + c_3y_2^2 + c_4y_0y_1 + c_5y_0y_2 + c_6y_1y_2. \tag{4.0.2}$$

Definition 4.1. Let $\mathscr{S}$ be a cubic surface in $\mathbb{P}_k^3$. Then $\mathscr{S} := \mathbb{V}(g) = \{g = 0\}$ where g is defined by Equation (4.0.1) above. Let $\mathscr{C}$ be a conic in $\mathbb{P}_k^2$. Then $\mathscr{C} := \mathbb{V}(h) = \{h = 0\}$ where h is defined by Equation (4.0.2) above. We define:

$$\begin{aligned}
c(\mathscr{S}) &:= (c_1 : \ldots : c_{20}) \in \mathbb{P}_k^{19} \\
&= \text{class of coefficients of } g \text{ as a point in } \mathbb{P}_k^{19} \\
&= \{\lambda(c_1, \ldots, c_{20}) : \lambda \in k^*\}, \\
c(\mathscr{C}) &:= (c_1 : \ldots : c_6) \in \mathbb{P}_k^5 \\
&:= \text{class of coefficients of } h \text{ as a point in } \mathbb{P}_k^5 \\
&= \{\lambda(c_1, \ldots, c_6) : \lambda \in k^*\}, \\
\mathbb{S}_{sm} &:= \{c(\mathscr{S}) \in \mathbb{P}_k^{19} : \mathscr{S} \text{ is a smooth cubic surface in } \mathbb{P}_k^3\}, \\
\mathbb{S}_{sn} &:= \{c(\mathscr{S}) \in \mathbb{P}_k^{19} : \mathscr{S} \text{ is a singular cubic surface in } \mathbb{P}_k^3\}, \\
\mathbb{C}_{sm} &:= \{c(\mathscr{C}) \in \mathbb{P}_k^5 : \mathscr{C} \text{ is a smooth conic in } \mathbb{P}_k^2\}, \\
\mathbb{C}_{sn} &:= \{c(\mathscr{C}) \in \mathbb{P}_k^5 : \mathscr{C} \text{ is a singular conic in } \mathbb{P}_k^2\}, \\
\mathbb{T}^{(3)} &:= \{t \in \mathbb{T} : \text{lines of } t \text{ form an Eckardt point}\}, \\
\mathbb{S}^{(m)} &:= \{c(\mathscr{S}) \in \mathbb{S}_{sm} : \mathscr{S} \text{ has at least } m \text{ Eckardt points}\}, \\
\mathbb{E}^{(m,k)} &:= \{c(\mathscr{S}) \in \mathbb{S}^{(k)} : \mathscr{S} \text{ has } m \text{ Eckardt points}\}.
\end{aligned}$$

Recall from ([22], Pages 5,6) that if V is a vector space of dimension $n \geq 2$ over the field k and d is any integer such that $1 \leq d \leq n$, then the Grassmannian $\mathbb{G}(d, n)$ or $\mathbb{G}_{d,n}$ is the set of all d-dimensional subspaces of V , i.e.

$$\mathbb{G}(d, n) = \{W : W \text{ subspace of } V \text{ of dimension } d\}.$$

Alternately, it is the set of all $(d-1)$-dimensional linear subspaces of the projective space $\mathbb{P}_k^{n-1}$. If we think of the Grassmannian this way, we denote it by $\mathbb{G}^{\mathbb{P}}(d-1, n-1)$. The simplest example of the Grassmannian could be $\mathbb{G}(1, n)$ which is the set of all 1-dimensional subspaces of the vector space V which is nothing but the projective space on V.

The Grassmannian $\mathbb{G}(d, n)$, as an algebraic variety, is a projective algebraic variety defined by quadratic polynomials called Plücker relations. The Grassmannian $\mathbb{G}(d, n)$ can be covered by open sets isomorphic to the affine space $\mathbb{A}^{d(n-d)}$ (see [22] and [1]), and so we have

$$\dim(\mathbb{G}(d, n)) = d(n-d).$$

The simplest Grassmannian that is not a projective space is $\mathbb{G}(2, 4)$, which may be parameterized via Plücker coordinates. Furthermore, $\dim \mathbb{G}(2, 4) = 2(4-2) = 4$.

Assume that $\kappa_{i_1 \dots i_m} := (P_{i_1}, \dots, P_{i_m})$ represents an order of m points in $(\mathbb{P}_k^2)^m$ such that no three collinear. We will write $\widehat{\kappa_{i_1 \dots i_m}}$ to denote to the set $\{P_{i_1}, \dots, P_{i_m}\}$. In this case, we say $\widehat{\kappa_{i_1 \dots i_m}}$ forms an m-arc in $\mathbb{P}_k^2$. Moreover, we define

$$\mathbb{S}_6 := \{s = \kappa_{i_1 \dots i_6} \in (\mathbb{P}_k^2)^6 : P_{i_1}, \dots, P_{i_6} \text{ form a } 6-arc \text{ not on a conic}\},$$

$$\mathbb{W}_s := \text{the space of all plane cubic passing through the six points of } s \in \mathbb{S}_6,$$

$$\operatorname{blw}_s \mathbb{P}_k^2 := \text{blow-up } \mathbb{P}_k^2 \text{ at the six points of } s = \kappa_{i_1 \dots i_6} \in \mathbb{S}_6.$$

Let $l_{i_1}, \dots, l_{i_m}$ be distinct lines in $\mathbb{P}_k^2$ (or in $\mathbb{P}_k^3$) and $w_{i_1}, \dots, w_{i_m} \in \mathbb{W}_s$. We define

$$\begin{aligned}
\lambda_{i_1 \dots i_m} &:= (l_{i_1}, \dots, l_{i_m}), \\
\widehat{\lambda_{i_1 \dots i_m}} &:= \{l_{i_1}, \dots, l_{i_m}\}, \\
\omega_{i_1 \dots i_m} &:= (w_{i_1}, \dots, w_{i_m}) \in \mathbb{W}_s^m, \\
\widehat{\omega_{i_1 \dots i_m}} &:= \{w_{i_1}, \dots, w_{i_m}\} \subset \mathbb{W}_s, \\
\wedge(\lambda_{i_1 \dots i_m}) &:= \wedge(i_1, \dots, i_m) := l_{i_1} \cap \dots \cap l_{i_m}.
\end{aligned}$$

Definition 4.2. A non-singular cubic surface with 27 lines, and e-invariants e_0, e_1, e_2, e_3 is called a cubic surface of type $[e_0, e_1, e_2, e_3]$.

As an example, the cubic surface $\mathscr{S}^{(4)}(17)$ is of type $[54, 228, 123, 4]$ (see Section 3.6). More precisely, the points on $\mathscr{S}^{(4)}(17)$ that do not lie on a line of the cubic surface are shown in Table 4.1.

Table 4.1: Points belong to $\mathscr{S}^{(4)}(17)\setminus\{\text{lines of } \mathscr{S}^{(4)}(17)\}$

(1:8:11:14)	(1:8:11:7)	(1:10:12:10)	(1:12:6:1)	(1:8:15:6)	(1:2:9:16)	(1:13:13:3)	(1:10:2:15)
(1:4:8:11)	(1:2:6:2)	(1:2:11:11)	(1:1:4:15)	(1:7:1:1)	(1:3:7:2)	(1:4:12:14)	(1:7:4:11)
(1:10:1:10)	(1:8:6:14)	(1:2:16:11)	(1:4:2:14)	(1:10:12:16)	(1:2:10:10)	(1:4:11:6)	(1:3:7:8)
(1:15:13:4)	(1:13:13:2)	(1:8:15:10)	(1:3:1:4)	(1:12:4:5)	(1:12:15:14)	(1:3:2:1)	(1:5:12:16)
(1:2:7:16)	(1:1:9:6)	(1:13:16:1)	(1:4:11:3)	(1:7:4:7)	(1:11:10:8)	(1:15:8:15)	(1:3:12:4)
(1:15:14:10)	(1:12:12:5)	(1:9:2:14)	(1:14:13:8)	(1:15:9:3)	(1:5:8:7)	(1:11:10:2)	(1:0:6:0)
(1:13:16:4)	(1:15:10:3)	(1:14:9:3)	(1:12:5:14)	(1:14:16:1)	(1:7:10:4)		

The points on $\mathscr{S}^{(4)}(17)$ that lie on exactly one line of the cubic surface are shown in Table 4.2.

Table 4.2: Points lie on exactly one line of $\mathscr{S}^{(4)}(17)$

(1:0:8:0)	(0:1:0:8)	(1:6:3:16)	(1:6:0:16)	(0:1:12:0)	(1:13:3:15)	(1:15:13:9)	(1:10:3:2)
(1:1:6:2)	(1:12:12:4)	(0:1:3:3)	(1:0: 0:3)	(1:11:5:8)	(1:0:1:0)	(1:15:12:9)	(1:3:1:10)
(1:0:10:0)	(1:0:0:8)	(1:10:5:2)	(1:5:7:12)	(1:11:14:15)	(1:12:1:13)	(1:4:6:13)	(0:1:16:0)
(1:13:8:12)	(1:4:3:6)	(1:16:6:14)	(1:13:4:12)	(1:7:1:3)	(1:11:8:2)	(1:7:14:3)	(1:4:5:1)
(1:8:10:10)	(1:5:16:7)	(1:4:13:2)	(1:7:9:7)	(1:3:14:8)	(1:12:0:8)	(0:1:3:0)	(0:1:6:0)
(1:0:0:15)	(1:10:7:13)	(1:14:13:1)	(1:12:10:12)	(1:8:14:6)	(1:13:0:1)	(1:2:14:4)	(1:11:4:5)
(1:0:2:8)	(1:0:12:0)	(0:1:0:12)	(1:3:16:2)	(1:0:11:14)	(1:12:6:9)	(1:13:11:6)	(1:16:8:0)
(1:11:14:5)	(1:8:16:13)	(1:0:15:11)	(1:3:11:11)	(1:14:1:9)	(1:2:7:2)	(1:3:5:11)	(1:0:11:0)
(1:16:7:13)	(1:12:4:3)	(0:1:13:4)	(1:13:4:11)	(1:16:8:4)	(1:11:8:13)	(1:7:15:14)	(1:10:8:15)
(1:5:13:12)	(1:15:5:15)	(1:10:15:13)	(1:13:2:15)	(1:10:8:9)	(1:4:14:8)	(1:10:1:12)	(1:3:9:0)
(1:15:6:12)	(1:10:11:7)	(1:4:10:16)	(1:15:0:4)	(1:7:16:6)	(0:1:2:8)	(1:9:0:12)	(1:7:10:0)
(1:7:5:1)	(1:1:11:8)	(1:2:0:10)	(1:15:3:16)	(1:0:0:10)	(1:8:14:16)	(1:7:2:11)	(1:5:16:13)
(1:15:4:7)	(0:1:5:10)	(1:11:11:0)	(1:3:14:6)	(1:0:0:2)	(1:15:9:16)	(1:12:7:6)	(1:7:2:13)
(1:12:5:14)	(1:0:6:0)	(1:8:6:0)	(1:5:1:9)	(1:0:10:2)	(0:1:2:0)	(0:1:16:6)	(1:12:14:1)
(1:2:9:13)	(1:0:0:6)	(0:1:15:0)	(1:3:9:6)	(1:14:4:8)	(1:4:6:5)	(1:8:10:4)	(1:5:14:10)
(1:4:14:11)	(1:16:4:8)	(1:3:3:1)	(1:12:10:8)	(1:0:0:5)	(1:9:1:1)	(1:4:9:5)	(0:1:0:4)
(0:1:0:7)	(1:0:0:4)	(1:2:10:12)	(1:0:13:4)	(0:1:6:5)	(1:8:5:2)	(1:4:8:16)	(1:11:2:9)
(1:6:8:13)	(0:1:14:0)	(1:0:0:16)	(1:0:16:6)	(1:8:8:16)	(1:15:7:8)	(1:4:2:8)	(1:0:2:0)
(1:11:0:15)	(1:8:1:12)	(1:12:15:13)	(1:13:7:6)	(1:3:5:5)	(0:1:8:12)	(1:0:4:15)	(1:4:3:2)
(1:4:9:12)	(1:0:1:13)	(1:14:16:0)	(0:1:14:16)	(1:0:15:0)	(0:1:0:15)	(1:9:3:14)	(1:1:3:8)
(1:10:2:11)	(1:12:1:4)	(1:16:6:9)	(1:13:2:13)	(1:12:5:3)	(1:13:11:13)	(1:5:8:11)	(1:2:13:12)
(1:9:16:12)	(1:1:5:16)	(1:1:10:16)	(1:16:0:14)	(1:3:12:5)	(1:13:14:4)	(1:10:11:3)	(1:7:5:6)
(0:1:9:0)	(1:15:4:12)	(1:6:15:9)	(1:8:5:7)	(1:0:9:7)	(0:1:0:14)	(1:16:2:1)	(1:7:12:13)
(1:2:6:1)	(1:14:14:11)	(1:13:8:2)	(1:5:12:10)	(1:2:16:4)	(1:16:4:9)	(1:11:11:13)	(1:2:14:5)
(1:11:12:12)	(1:12:7:0)	(1:5:1:0)	(1:8:8: 9)	(1:4 :13:1)	(0:1:0:11)	(1:7:9:9)	(1:3:13:3)
(1:10:3:5)	(1:3:3:10)	(1:14:1:3)	(1:4:12:12)	(1:5:5:6)	(1:14:3:5)	(1:1:9:12)	(1:16:3:8)
(1:11:15:9)	(0:1:4:15)	(1:1:15:6)	(1:3:2:9)	(1:16:2:12)	(1:7:14:14)	(1:10:5:11)	(1:9:1:15)
(1:8:16:12)	(1:5:5:14)	(1:5:7:16)	(0:1:5:0)	(1:14:3:7)	(1:14:4:13)	(1:3:0:3)	(1:8:1:15)
(1:13:5:3)	(1:11:5:7)	(1:10:7:9)	(0:1:0:16)				

The points on $\mathscr{S}^{(4)}(17)$ that lie on exactly two lines of the cubic surface are shown in Table 4.3.

Table 4.3: Points lie on exactly two lines of $\mathscr{S}^{(4)}(17)$

(1:0:0:0)	(0:1:0:0)	(0:0:1:0)	(0:0:0:1)	(1:16:14:13)	(1:0:0:9)	(1:11:2:12)	(0:0:1:12)
(1:5:0:6)	(1:0:16:0)	(1:2:11:9)	(1:7:15:12)	(1:2:3:13)	(1:2:5:9)	(1:12:13:9)	(0:1:0:1)
(0:1:0:10)	(1:5:4:13)	(1:1:3:9)	(1:6:5:10)	(1:0:0:13)	(1:1:11:9)	(1:14:0:11)	(1:4:10:9)
(1:10:14:12)	(1:0:0:1)	(1:1:10:13)	(1:0:3:0)	(0:1:0:3)	(1:5:3:11)	(0:1:12:9)	(1:6:3:3)
(1:7:0:9)	(1:14:5:12)	(1:3:16:9)	(1:15:14:13)	(1:10:0:5)	(1:0:13:0)	(0:1:0:13)	(1:1:14:2)
(1:4:0:13)	(1:13:7:9)	(1:8:3:9)	(0:1:9:7)	(1:9:14:9)	(1:8:0:2)	(1:16:3:15)	(1:8:3:4)
(1:0:14:16)	(1:6:15:10)	(1:10:15:0)	(1:14:5:9)	(1:7:3:12)	(1:16:14:15)	(0:1:7:0)	(1:13:5:16)
(1:8:12:13)	(1:6:14:12)	(1:5:14:14)	(1:14:6:13)	(0:1:11:0)	(1:14:14:7)	(1:6:14:0)	(1:0:0:11)
(1:15:3:10)	(1:0:12:9)	(0:1:15:11)	(1:1:15:12)	(1:0:4:0)	(0:1:11:14)	(1:0:7:0)	(1:6:9:13)
(1:16:5:4)	(1:7:3:4)	(1:11:3:14)	(1:6:9:9)	(0:1:0:2)	(1:9:3:13)	(1:9:16:9)	(1:15:12:0)
(1:2:5:5)	(1:12:14:7)	(1:0:8:12)	(1:0:9:0)	(0:1:0:9)	(1:15:14:10)	(1:1:0:0)	(0:1:1:0)
(1:9:2:0)	(1:0:0:14)	(0:1:1:13)	(1:0:5:0)	(0:1:0:5)	(1:13:3:0)	(1:10:14:3)	(1:1:5:13)
(1:3:13:13)	(1:0:14:0)	(0:1:4:0)	(1:9:14:1)	(1:13:14:9)	(0:1:13:0)	(1:3:11:12)	(1:0:3:3)
(1:2:13:0)	(0:1:10:0)	(1:0:0:7)	(1:11:3:7)	(1:0:5:10)	(1:11:12:14)	(1:14:6:12)	(0:1:0:6)
(1:9:5:15)	(1:5:4:9)	(1:12:3:6)	(0:1:8:0)	(1:0:6:5)	(1:2:3:1)	(1:15:5:8)	(1:4:5:0)
(1:15:10:13)	(1:12:3:12)	(1:16:7:12)					

The points on $\mathscr{S}^{(4)}(17)$ that lie on exactly three lines of the cubic surface are shown in Table 4.4.

Table 4.4: Points lie on exactly three lines of $\mathscr{S}^{(4)}(17)$

(1:9:5:13)	(1:6:5:12)	(1:0:0:12)	(0:1:10:2)

For other examples, see the proof of Theorem 3.7.

Theorem 4.1. For $q > 7$ and q prime, any non-singular cubic surface with 27 lines $\mathscr{S}^{(0)}(q)$ is of type $[(q-10)^2+9, 27(q-9), 135, 0]$.

Proof. Recall that $\mathscr{S}^{(0)}(q)$ denotes a non-singular cubic surface with 27 lines that has no Eckardt point over the Galois field $GF(q)$. If n_q be the total number of points on the lines of $\mathscr{S}^{(0)}(q)$, then

$$n_q = e_3 + e_2 + e_1.$$

Let $l_i, i = 1, \dots, 27$, be the 27 lines on $\mathscr{S}^{(0)}(q)$. Let $e_r^{(i)}$ be the number of points of l_i lying on exactly r lines of $\mathscr{S}^{(0)}(q)$. Then

$$\sum_{i=1}^{27} e_3^{(i)} = 3e_3,$$
$$\sum_{i=1}^{27} e_2^{(i)} = 2e_2,$$
$$\sum_{i=1}^{27} e_1^{(i)} = e_1.$$

Also we know that each line meets ten others. So we have

$$2e_3^{(i)} + e_2^{(i)} = 10.$$

Moreover, every line in $\mathbb{P}_q^2$ has exactly $q+1$ points. It follows that

$$e_3^{(i)} + e_2^{(i)} + e_1^{(i)} = q+1.$$

Taking the sum for both previous equations and over all $i = 1, \ldots, 27$ will give

$$6e_3 + 2e_2 = 270,$$

$$3e_3 + 2e_2 + e_1 = 27(q+1).$$

More precisely, we obtain

$$e_2 + e_1 = 27(q-4),$$

$$n_q = 27(q-4) + e_3.$$

From Section 3.1, we have $\#(\mathscr{S}^{(0)}(q)) = q^2 + 7q + 1$. Hence

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1,$$

or

$$e_3 + e_0 = q^2 + 7q + 1 - 27(q-4) = (q-10)^2 + 9.$$

Thus

$$\#(\mathscr{S}^{(0)}(q)) = n_q + e_0.$$

and hence

$$e_3 = 0,$$

$$e_2 = 270/2 = 135,$$

$$e_1 = 27(q-4) - 135 = 27(q-9),$$

$$e_0 = (q-10)^2 + 9.$$

□

Theorem 4.2. For $q > 7$ and q prime, any non-singular cubic surface with 27 lines $\mathscr{S}^{(1)}(q)$ is of type $[(q-10)^2+8, 27(q-9)+3, 132, 1]$.

Proof. Recall that $\mathscr{S}^{(1)}(q)$ denotes a non-singular cubic surface with 27 lines that has exactly one Eckardt point over the Galois field $GF(q)$. By the same argument used in the proof of Theorem 4.1, we have

$$n_q = e_3 + e_2 + e_1,$$

and

$$\sum_{i=1}^{27} e_3^{(i)} = 3e_3,$$

$$\sum_{i=1}^{27} e_2^{(i)} = 2e_2,$$
$$\sum_{i=1}^{27} e_1^{(i)} = e_1.$$

We know that

$$2e_3^{(i)} + e_2^{(i)} = 10,$$
$$e_3^{(i)} + e_2^{(i)} + e_1^{(i)} = q + 1.$$

So

$$6e_3 + 2e_2 = 270,$$

$$3e_3 + 2e_2 + e_1 = 27(q + 1).$$

Consequently, we get

$$e_2 + e_1 = 27(q - 4),$$

$$n_q = 27(q - 4) + e_3.$$

From Section 3.1, we have $\#(\mathscr{S}^{(1)}(q)) = q^2 + 7q + 1$. Hence

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1,$$

It follows that

$$e_3 = 1,$$
$$e_2 = 135 - 3 = 132,$$
$$e_1 = 27(q-4) - 132 = 27(q-9) + 3,$$
$$e_0 = (q-10)^2 + 8.$$

□

Theorem 4.3. For $q \geq 7$ and q prime, the only non-singular cubic surfaces with 27 lines and all points lying on those lines, i.e, surfaces of type $[0, e_1, e_2, e_3]$, are $\mathscr{S}^{(18)}(7), \mathscr{S}^{(10)}(11)$ and $\mathscr{S}^{(18)}(13)$.

Proof. Recall that $\mathscr{S}^{(m)}(q)$ denotes a non-singular cubic surface with 27 lines that has exactly m Eckardt points over the Galois field $GF(q)$. It is clear that for the case $q = 7$, a non-singular cubic surface with 27 lines exists. In fact, $\mathscr{S}^{(18)}(7)$ is such a surface because $7 = 1 \bmod 3$, and $x^2 + x + 1$ has two roots, namely $x = 2$ and $x = 4$ (see Section 3.4). In fact, the cubic surface $\mathscr{S}^{(18)}(7)$ is of type $[0, 0, 81, 18]$.

For the case $q = 11$, if

$$\mathscr{S}^{(10)}(11) := \mathbb{V}(x_0^3 + x_1^3 + x_2^3 + x_3^3 - (x_0 + x_1 + x_2 + x_3)^3),$$

then $\mathscr{S}^{(10)}(11)$ is a non-singular cubic surface with 27 lines, and it has exactly 10 Eckardt points (see Table 4.5).

Table 4.5: Eckardt points of $\mathscr{S}^{(10)}(11)$

(1:0:0:0)	(0:1:0:0)	(0:0:1:0)	(0:0:0:1)	(1:-1:0:0)
(0:1:-1:0)	(0:0:1:-1)	(1:0:-1:0)	(0:1:0:-1)	(1:0:0:-1)

Moreover, $\mathscr{S}^{(10)}(11)$ is of type $[0, 64, 105, 10]$.

For the case $q = 13$, if

$$\mathscr{S}^{(18)}(13) := \mathbb{V}(y_0^2 y_1 + 4y_1^2 y_0 + 6y_2^2 y_3 + 3y_3^2 y_2 + y_0 y_1 y_2 + 5y_0 y_1 y_3),$$

then $\mathscr{S}^{(18)}(13)$ a non-singular cubic surface with 27 lines, and it has exactly 18 Eckardt points (see Table 4.6)

Table 4.6: Eckardt points of $\mathscr{S}^{(18)}(13)$

(1:0:0:0)	(0:1:0:0)	(1:0:0:-3)	(1:3:0:0)	(1:-3:0:-3)	(1:6:6:-3)
(1:0:-2:0)	(1:-5:-3:8)	(1:-3:-2:0)	(1:0:6:1)	(1:-4:6:-3)	(1:-3:6:1)
(0:1:0:1)	(1:1:-2:1)	(0:1:-2:4)	(1:2:-3:-5)	(1:-2:-2:1)	(0:1:5:0)

In fact, the cubic surface $\mathscr{S}^{(18)}(13)$ is of type $[0, 162, 81, 18]$. It remain to show that, if $q > 13$, then there is no non-singular cubic surface with 27 lines such that all its points belong to the 27 lines, that is, every non-singular cubic surface with 27 lines, namely $\mathscr{S}^{(m)}(q), q > 13$ is of type $[e_0, e_1, e_2, e_3]$ such that $e_0 \neq 0$. By the way of contradiction, let us assume that there is such non-singular cubic surface, then from Section 3.3, we have

$$n_q = e_3 + e_2 + e_1$$

where n_q is the total number of points on the lines of $\mathscr{S}^{(m)}(q)$ and

$$e_2 + e_1 = 27(q - 4),$$

$$n_q = 27(q - 4) + e_3.$$

Moreover, from Section 3.1, we have $\#(\mathscr{S}^{(m)}(q)) = q^2 + 7q + 1$. Hence

$$e_3 + e_0 = e_3 = q^2 + 7q + 1 - 27(q - 4) = (q - 10)^2 + 9.$$

Thus for $q > 13$ and q is prime, we get $(q - 10)^2 > 9$ and hence $e_3 > 18$ which is impossible. □

4.1 Quadratic transformations

Recall a smooth cubic surface with 27 lines, namely X, is the blow-up of $\mathbb{P}^2_k$ at six points in general position, namely $s = \kappa_{123456} \in \mathbb{S}_6$. In this case, we write $X = \operatorname{blw}_s \mathbb{P}^2_k$, and we have

1. the exceptional curve $\widetilde{P_i} := a_i$ is defined to be the total transform of P_i in s,

2. the curve $\widetilde{l_{ij}} := c_{ij}$ is the strict transform of $l_{ij} = \overline{P_i P_j}$,

3. the curve $\widetilde{\mathscr{C}_j} := b_j$ is the strict transform of the conic $\mathscr{C}_j$ passing through all points of s except P_j.

Consider the rational map $\varphi_{123} : \mathbb{P}^2_k \dashrightarrow \mathbb{P}^2_k$ defined by:

$$\varphi_{123}(x_0 : x_1 : x_2) \mapsto (x_1 x_2 : x_0 x_2 : x_0 x_1)$$

which is called the quadratic elementary transformation. The points

$$P_1 = (1 : 0 : 0), P_2 = (0 : 1 : 0), P_3 = (0 : 0 : 1)$$

are called the fundamental points of φ_{123}, while the lines:

$$l_{23} = \overline{P_2 P_3} = \{x_0 = 0\},$$
$$l_{13} = \overline{P_1 P_3} = \{x_1 = 0\},$$
$$l_{12} = \overline{P_1 P_2} = \{x_2 = 0\},$$

are called the fundamental lines of φ_{123}. From the definition of φ_{123} it follows that φ_{123} is a morphism on $\mathbb{P}^2_k \backslash \{P_1, P_2, P_3\}$ and an isomorphism on $\mathbb{P}^2_k \backslash l_{12} \cup l_{l3} \cup l_{23}$, on which φ_{123} can be written:

$$\varphi_{123}(x_0 : x_1 : x_2) \mapsto \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right). \tag{4.1.1}$$

Note that $\varphi_{123}(P_1) = \varphi_{123}(P_2) = \varphi_{123}(P_3) = (0 : 0 : 0) \notin \mathbb{P}^2_k$. In this case, φ_{123} is birational and $\varphi_{123}^2 = id$. Moreover, on

$$\mathbb{P}^2_k \backslash l_{12} \cup l_{l3} \cup l_{23} = \mathbb{P}^2_k \backslash \mathbb{V}(x_0) \cup \mathbb{V}(x_1) \cup \mathbb{V}(x_2) = \mathbb{P}^2_k \backslash \mathbb{V}(x_0 x_1 x_2)$$

we have $x_0 x_1 x_2 \neq 0$ and hence

$$\varphi_{123}(x_0 : x_1 : x_2) \mapsto (x_1 x_2 : x_0 x_2 : x_0 x_1) = x_0 x_1 x_2 \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right) = \left(\frac{1}{x_0} : \frac{1}{x_1} : \frac{1}{x_2}\right).$$

The image under φ_{123} of a line not passing through the fundamental points is a conic passing through P_1, P_2 and P_3. More precisely, the net of lines of the domain plane is transformed into the net of conics (on the codomain, i.e $\mathbb{P}^2_k \backslash \mathbb{V}(x_0x_1x_2)$) passing through the P_1, P_2 and P_3 and vice versa. This explains why the term "quadratic" is used to indicate this Cremona transformation (see [10]). For example: Consider the irreducible conic

$$\mathscr{C} = \{x_1x_2 + x_0x_2 + x_0x_1 = 0\} = \mathbb{V}(x_1x_2 + x_0x_2 + x_0x_1)$$

which passes through the fundamental points P_1, P_2 and P_3. It follows that

$$\varphi_{123}(\mathscr{C}) = \frac{1}{x_1}\frac{1}{x_2} + \frac{1}{x_0}\frac{1}{x_2} + \frac{1}{x_0}\frac{1}{x_1} = x_0x_1x_2\left(\frac{1}{x_1}\frac{1}{x_2} + \frac{1}{x_0}\frac{1}{x_2} + \frac{1}{x_0}\frac{1}{x_1}\right) = x_0 + x_1 + x_2$$

which is a line not passing through the fundamental points P_1, P_2 and P_3. Conversely, consider the line

$$l = \mathbb{V}(x_0 + x_1 + x_2) = \{x_0 + x_1 + x_2 = 0\}$$

which is a line not passing through the fundamental points P_1, P_2 and P_3. Consequently, we have

$$\varphi_{123}(l) = \frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2} = x_0x_1x_2\left(\frac{1}{x_0} + \frac{1}{x_1} + \frac{1}{x_2}\right) = x_1x_2 + x_0x_2 + x_0x_1$$

which is an irreducible conic passing through the fundamental points P_1, P_2 and P_3. Now consider the line

$$l = \mathbb{V}(x_0 + x_1) = \{x_0 + x_1 = 0\}$$

which passes through only one point of the fundamental points P_1, P_2 and P_3, namely P_3. Then

$$\varphi_{123}(l) = \frac{1}{x_0} + \frac{1}{x_1} = x_0x_1\left(\frac{1}{x_0} + \frac{1}{x_1}\right) = x_1 + x_0$$

which is again a line passing through only the fundamental point P_3.

Consider the irreducible conic

$$\mathscr{C} = \{x_0^2 + x_1x_2 = 0\} = \mathbb{V}(x_0^2 + x_1x_2)$$

which passes through the 2 fundamental points of $\{P_1, P_2, P_3\}$, namely P_2 and P_3. Note that

$$\varphi_{123}(\mathscr{C}) = \frac{1}{x_0^2} + \frac{1}{x_1}\frac{1}{x_2} = x_0^2 x_1 x_2\left(\frac{1}{x_0^2} + \frac{1}{x_1}\frac{1}{x_2}\right) = x_1 x_2 + x_0^2$$

which is again an irreducible conic passing through 2 of the fundamental points of $\{P_1, P_2, P_3\}$, namely P_2 and P_3.

The birational map φ_{123} associated with $S(P) = \{P_1, P_2, P_3\}$ is called the elementary quadratic transformation with the fundamental points P_1, P_2 and P_3, and we write $\varphi_{123} = c(P_1, P_2, P_3)$ where c means "Cremonian".

We observe that each elementary quadratic transformation $c(P_1, P_2, P_3)$ is of the form (4.1.1), up to an automorphism of the plane. In fact, if $\varphi : \mathbb{P}^2_k \to \mathbb{P}^2_k$ is a projectivity such that:

$$\varphi(P_1) = (1 : 0 : 0), \varphi(P_2) = (0 : 1 : 0), \varphi(P_3) = (0 : 0 : 1),$$

then it immediately occurs that:

$$c(P_1, P_2, P_3) = \varphi^{-1} \circ \varphi_{123} \circ \varphi.$$

Let $\varphi_{123} = c(P_1, P_2, P_3)$ be an elementary quadratic transformation. Now we see how to interpret φ_{123} with blowing up. We consider the blowing up $\sigma : S \to \mathbb{P}^2_k$ of the points P_1, P_2 and P_3, and let $\sigma' : S' \to \mathbb{P}^2$ be a copy of $\sigma : S \to \mathbb{P}^2$. Then there exists an isomorphism ψ such that the diagram in Figure 4.1 commutes as we will explain. In particular, indicate by $\widetilde{l_{ij}} = c_{ij}$ the strict transforms of the line l_{ij} in S, and let $\widetilde{P_i} = a_i$. Then we have

$$\psi(a_i) = c'_{jk}, \psi(c_{jk}) = a'_i, i = 1, 2, 3; j, k \in \{1, 2, 3\}\backslash\{i\},$$

where a'_i and c'_{jk} are the copies in S' respectively of a_i and c_{jk} (see Figure 4.2).

In other words, $\varphi_{123} = c(P_1, P_2, P_3)$ is the blowing up of P_1, P_2 and P_3 and the contraction $\tau = \sigma' \circ \psi$ of c_{12}, c_{13} and c_{23} ([29], Page 260).

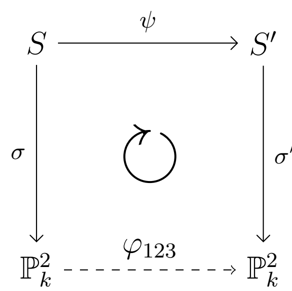

Figure 4.1: Elementary quadratic transformation φ_{123}.

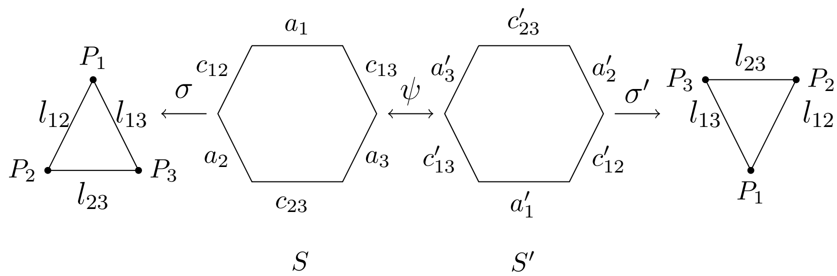

Figure 4.2: Elementary quadratic transformation

More precisely, if

$$P_1 = (1:0:0) = Q_1, P_2 = (0:1:0) = Q_2, P_3 = (0:0:1) = Q_3,$$

and

$$\pi_{S(P)} \text{ represents the blowing up of } \mathbb{P}^2_X \text{ at } S(P) := \{P_1, P_2, P_3\},$$

$$\pi_{S(Q)} \text{ represents the blowing up of } \mathbb{P}^2_Y \text{ at } S(Q) := \{Q_1, Q_2, Q_3\},$$

$$\mathcal{V} := \mathrm{blw}_{S(P)} \mathbb{P}^2_X,$$

then we have the following commutative diagram (see Figure 4.3).

Furthermore, we have the following correspondence via φ_{123}:

$$\begin{array}{ccc}
\text{irreducible conic containing } P_1, P_2, P_3 & \leftrightarrow & \text{line not containing any } Q_1, Q_2, Q_3, \\
\text{irreducible conic containing only} & \leftrightarrow & \text{irreducible conic containing only} \\
\text{the two points } P_i, P_j, & & \text{the two points } Q_i, Q_j, \\
i, j \in \{1,2,3\} & & i, j \in \{1,2,3\} \\
\text{line containing only one point} & \leftrightarrow & \text{line containing only one point.} \\
P_i,\ i \in \{1,2,3\} & & Q_i,\ i \in \{1,2,3\}.
\end{array}$$

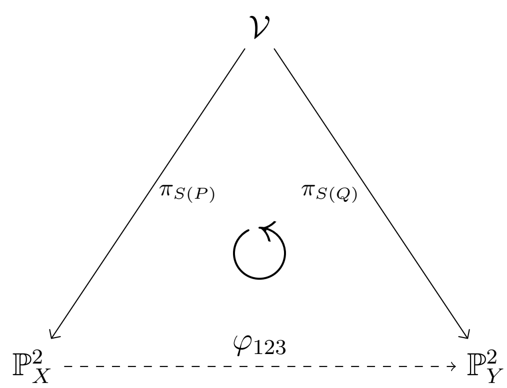

Figure 4.3: $\varphi_{123} = \pi_{S(Q)} \circ \pi^{-1}_{S(P)}$.

Remark 4.1. According to the above argument, if $s, s' \in \mathbb{S}_6$ are such that s' can be obtained from s by a quadratic transformation, then

$$\text{blw}_{s'} \mathbb{P}^2 = \text{blw}_s \mathbb{P}^2 \text{ (see Proposition 4.12).}$$

4.2 Operation on triples of lines on a smooth cubic surface

Recall from ([10], Page 477) that a tritangent trio (trihedron) is a set of three exceptional lines which span a tritangent plane. A pair of tritangent planes intersecting along a line not in the surface is classically known as a Cremona pair. A triad of tritangent trios is a set of three tritangent trios such that no two share a common exceptional line. Given a pair of tritangent trios with no common exceptional lines, there is a unique third tritangent trio forming a triad with the first two. Every triad of tritangent trios has a unique conjugate triad which contains the same 9 exceptional lines. We call these a conjugate pair of triads of tritangent trios. A trihedral pair is a triad with its conjugate triad. A representative example is the following set of 9 exceptional lines:

$$\begin{array}{lccccl} T_{123,456}: & c_{14} & c_{25} & c_{36} & \rightsquigarrow t'_1 \\ & c_{26} & c_{34} & c_{15} & \rightsquigarrow t'_2 \\ & c_{35} & c_{16} & c_{24} & \rightsquigarrow t'_3 \\ & \wr\downarrow & \wr\downarrow & \wr\downarrow & \\ & t_1 & t_2 & t_3 & \end{array}$$

where each row and column forms a tritangent trio, and the rows form one triad with the columns its conjugate triad.

Lemma 4.2.1. ([9], Page 43) Let X be a smooth cubic surface. Consider a trihedral pair $T = \{t_1, t_2, t_3\}, T' = \{t'_1, t'_2, t'_3\}$, where t_i, t'_i are tritangent trios (or trihedrons). The following are equivalent:

1. $t_1 \cap t_2 \cap t_3$ is a line,
2. t'_1, t'_2, t'_3 corresponds to Eckardt points on a line, and
3. two of t'_1, t'_2, t'_3 correspond to Eckardt points.

Proof. It suffices to assume the 9 exceptional lines are those from $T_{123,456}$ above. Assume (1). If $t_1 \cap t_2 \cap t_3$ contain a line then $t_1 \cap t_2 = t_1 \cap t_3$. Thus

$$c_{14} \cap c_{25} = t_1 \cap t_2 \cap t'_1 = t_1 \cap t_3 \cap t'_1 = c_{14} \cap c_{36}.$$

This means that $c_{14} \cap c_{25} \cap c_{36}$ is nonempty and so t'_1 has an Eckardt point contained in $t_1 \cap t_2 \cap t_3$. Similar arguments apply to t'_2 and t'_3 so (2) follows. Clearly, (2) implies (3). So it remains to show (3) implies (1). Suppose t'_1, t'_2 correspond to Eckardt points $E_1; E_2$. Then

$$E_1 = c_{14} \cap c_{25} \cap c_{36},$$

$$E_2 = c_{26} \cap c_{34} \cap c_{15}.$$

Now $t_1 \cap t_2$ contains both E_1 and E_2 and the same is true of $t_1 \cap t_3$. Thus $t_1 \cap t_2 \cap t_3$ contains both points and thus contains a line. □

Corollary 4.1. ([9], Page 44) Let X be a smooth cubic surface. If X contains two Eckardt points lying on a line ℓ not contained in X then there is a unique third Eckardt point such that ℓ is a trihedral line.

Recall a point P on a smooth cubic surface with 27 lines, namely X, is called an Eckardt point (sometimes called E-point) if it is an intersection of 3 lines among the 27 lines on X. Also recall that the set of all triples of lines on X is denoted by $\mathbb{T}$.

If $t_1, t_2, t_3 \in \mathbb{T}$, then the ordered triple $T = (t_1, t_2, t_3)$ is called a triad if t_1, t_2 have no line in common and every line of t_1 intersects exactly one line of t_2, and for every line $l_3 \in t_3$, we get $(l_1 l_2 l_3) \in \mathbb{T}$ for some $l_1 \in \widehat{t_1}$ and $l_2 \in \widehat{t_2}$ where $\widehat{t}$ denotes the set of lines that form t. In this case, we write $t_3 = t_{1,2}$.

Remark 4.2. Since the lines of $t \in \mathbb{T}^{(3)}$ form an Eckardt point, it follows that every $t \in \mathbb{T}^{(3)}$ is either of the form $(a_i b_j c_{ij})$ or $(c_{ij} c_{mn} c_{kh})$ where $i, j, m, n, k, h \in \{1, \ldots, 6\}$.

Let $\mathscr{S}$ be the blowing up $\mathbb{P}^2_k$ at six points, namely $s = \kappa_{123456} \in \mathbb{S}_6$, that is

$$\mathscr{S} = \operatorname{blw}_s \mathbb{P}^2_k,$$

then any triad of $\mathscr{S}$ has one of the following forms

$$(1) \qquad T_{ijk}: \qquad \{(a_i b_j c_{ij}), (a_j b_k c_{jk}), (a_k b_i c_{ik})\},$$

$$(2) \qquad T_{ij,km}: \qquad \{(a_i b_m c_{im}), (a_j b_k c_{jk}), (c_{ik} c_{jm} c_{nh})\},$$

$$(3) \qquad T_{ijk,mnh}: \qquad \{(c_{im} c_{jn} c_{kh}), (c_{jh} c_{km} c_{in}), (c_{kn} c_{ih} c_{jm}).$$

In fact, there are $20, 90$ and 10 trihedral pairs of type $T_{ijk}, T_{ij,km}$ and $T_{ijk,mnh}$ respectively ([14], Pages 28,29). Furthermore, the 9 lines of a triad can form a triad in exactly two ways as follow: Any triad, say $\{(ll'l''), (mm'm''), (nn'n'')\}$ can be reordered and written in a 3×3-array so that the 3 rows of the array form a triad, and the 3 columns of the array form another triad too. Such configurations are called trihedral pairs.

Clearly $c_{ij} = c_{ji} = \overline{P_iP_j}$. Thus we have the following forms for trihedral pairs:

$$\begin{array}{rcccl} T_{ijk}: & c_{jk} & a_k & b_j & \rightsquigarrow t_1 \\ & b_k & c_{ik} & a_i & \rightsquigarrow t_2 \\ & a_j & b_i & c_{ij} & \rightsquigarrow t_3 \\ & \wr\downarrow & \wr\downarrow & \wr\downarrow & \\ & t'_1 & t'_2 & t'_3 & \end{array}$$

$$\begin{array}{rcccl} T_{ij,km}: & a_i & b_m & c_{im} & \rightsquigarrow t_1 \\ & b_k & a_j & c_{jk} & \rightsquigarrow t_2 \\ & c_{ik} & c_{jm} & c_{nh} & \rightsquigarrow t_3 \\ & \wr\downarrow & \wr\downarrow & \wr\downarrow & \\ & t'_1 & t'_2 & t'_3 & \end{array}$$

$$\begin{array}{rcccl} T_{ijk,mnh}: & c_{im} & c_{jn} & c_{kh} & \rightsquigarrow t_1 \\ & c_{jh} & c_{km} & c_{in} & \rightsquigarrow t_2 \\ & c_{kn} & c_{ih} & c_{jm} & \rightsquigarrow t_3 \\ & \wr\downarrow & \wr\downarrow & \wr\downarrow & \\ & t'_1 & t'_2 & t'_3 & \end{array}$$

Remark 4.3. Recall that if $t_1, t_2 \in \mathbb{T}^{(3)}$ have no line in common, then they determine a third one, namely t_3, so that for any line l in t_3, we get $(ll_1l_2) \in \mathbb{T}^{(3)}$ for some $l_1 \in t_1$ and $l_2 \in t_2$. Note that $t_{1,2} = t_{2,1}$. Moreover, if $t_1, t_2, t_3 \in \mathbb{T}^{(3)}$ form a triad, then $t_3 = t_{1,2}$, $t_{1,3} = t_2$ and $t_{2,3} = t_1$. Consequently, if we assume that

$$\overline{\mathbb{T}^{(3)}} := \mathbb{T}^{(3)} \cup \{\emptyset\}$$

we can define an operation on $\overline{\mathbb{T}^{(3)}}$ as follow:

Definition 4.3. Let $t_1, t_2 \in \overline{\mathbb{T}^{(3)}}$. We define an operation on $\overline{\mathbb{T}^{(3)}}$ as follows:

$$\overline{\mathbb{T}^{(3)}} \times \overline{\mathbb{T}^{(3)}} \to \overline{\mathbb{T}^{(3)}} : (t_i, t_j) \mapsto t_it_j$$

where

$$t_i t_j := \begin{cases} \emptyset & \text{if} \quad t_i, t_j \text{ have at least one common line,} \\ t_i & \text{if} \quad t_j = \emptyset, \\ t_j & \text{if} \quad t_i = \emptyset, \\ t_{i,j} & \text{otherwise.} \end{cases}$$

For example: suppose that $\mathscr{S}$ is a non-singular cubic surface with 4 Eckardt points, and three of its Eckardt points correspond to the following 3 triples in $\mathbb{T}^{(3)}$:

$$t_1 := (c_{13}c_{24}c_{56}),$$

$$t_2 := (a_3b_4c_{34}),$$

$$t_3 := (a_2b_1c_{12}).$$

Note that the above 3 triples form a triad, namely $T = \{t_1, t_2, t_3\}$. Since $\mathscr{S}$ has 4 Eckardt points, there must be another triple $t \in \mathbb{T}^{(3)}$. Let t have 3 lines in common with the triad T. Then, up to permutations, $t \in \mathbb{T}^{(3)}$ has one of the following forms:

$$t := t_4 = (a_3b_1c_{13}),$$

$$t := t_5 = (c_{12}c_{34}c_{56}).$$

If $t = t_4$, then we have the symmetric Table 4.7).

Table 4.7: The symmetric table 1

	$\emptyset$	t_1	t_2	t_3	t_4
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$
t_2	t_2	t_3	$\emptyset$	t_1	$\emptyset$
t_3	t_3	t_2	t_1	$\emptyset$	$\emptyset$
t_4	t_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Similarly, if $t = t_5$, then we have the symmetric Table 4.8).

Lemma 4.2.2. Let $T = \{t_1, t_2, t_3\}, T' = \{t'_1, t'_2, t'_3\}$ be two triads constructed by some lines on $\mathscr{S}$ where $c(\mathscr{S}) \in \mathbb{S}_{sm}$. Then T can transformed to T' via some permutations and quadratic transformations.

Table 4.8: The symmetric table 2

	$\emptyset$	t_1	t_2	t_3	t_5
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_5
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$
t_2	t_2	t_3	$\emptyset$	t_1	$\emptyset$
t_3	t_3	t_2	t_1	$\emptyset$	$\emptyset$
t_5	t_5	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Proof. Let φ_{ijk} denote the quadratic transformation with respect to the fundamental points P_i, P_j, P_k. Then, up to a permutation of the 9 lines of triad, we can assume

$$T = \{(a_i b_m c_{im}), (a_j b_k c_{jk}), (c_{ik} c_{jm} c_{nh})\}, \text{ where } i,j,k,m,n,h \in \{1,2,3,4,5,6\}.$$

More precisely, we consider the following trihedral pair:

$$\begin{array}{lllll} T = T_{ij,km}: & a_i & b_m & c_{im} & \rightsquigarrow t_1 \\ & b_k & a_j & c_{jk} & \rightsquigarrow t_2 \\ & c_{ik} & c_{jm} & c_{nh} & \rightsquigarrow t_3 \end{array}$$

Now via the quadratic transformation φ_{imk}, the trihedral pair T is transformed to

$$\{(a_k b_m c_{mk}), (a_j b_k c_{jk}), (a_m b_j c_{mj})\}.$$

It follows that we have the trihedral pair:

$$\begin{array}{llll} T_{jmk}: & c_{mk} & a_k & b_m \\ & b_k & c_{jk} & a_j \\ & a_m & b_j & c_{mj} \end{array}$$

On the other hand and up to the quadratic transformation φ_{ijh}, where $h \notin \{i,j,k,m\}$, the trihedral pair T can be transformed to

$$\{(c_{jh} c_{kn} c_{im}), (c_{mn} c_{ih} c_{jk}), (c_{ik} c_{jm} c_{nh})\}.$$

More precisely, we have the following trihedral pair:

$$T_{jki,hnm}: \begin{array}{ccc} c_{jh} & c_{kn} & c_{im} \\ c_{mn} & c_{ih} & c_{jk} \\ c_{ik} & c_{jm} & c_{nh} \end{array}$$

where $n \notin \{i, j, k, m, h\}$. So we can transform from one triad to another via some sequence of permutations and quadratic transformations, namely

$$T_{jki,hnm} \xleftrightarrow{\text{via } \varphi_{ijh}} T = T_{ij,km} \xleftrightarrow{\text{via } \varphi_{imk}} T_{jmk}.$$

□

Definition 4.4. Let A, B are two subsets of triples in $\mathbb{T}$. Then A, B are said to be equivalent and we write $A \sim B$ if A can be obtained from B via some sequence of permutations and quadratic transformations.

Henceforth, for $A, B, T \subseteq \mathbb{T}$ and $t \in T$, we write $A \vee B$ to represent the union of A and B. In particular, $T \vee t$ represents $T \cup \{t\}$.

Proposition 4.1. Let $t_1 = (l_1 l_2 l_3), t_2 = (l'_1 l'_2 l'_3)$ and $t_3 = (l''_1 l''_2 l''_3)$ be three triples in $\mathbb{T}$. Then

1. if $t_1 \cap t_2 = l$, $t_1 \cap t_3 = l'$ and $t_2 \cap t_3 = l''$. Then l, l', l'' have common point. Furthermore, if $(l_1 l'_1 l''_1)$ forms an Eckardt point, namely E, then $E \in l \cap l' \cap l''$.

2. if t_1 and t_2 form 2 Eckardt points, namely E_1, E_2 respectively, then t_3 forms another Eckardt points, namely E_3 so that E_1, E_2, E_3 are collinear.

Proof. (1) Suppose that $\mathcal{P}_1$ is the tritangent plane formed by t_1, then $l, l' \subset \mathcal{P}_1$. So l, l' must intersect at some point. Similarly, if we assume that $\mathcal{P}_2$ is the tritangent plane formed by t_2, and $\mathcal{P}_3$ is the tritangent plane formed by t_3, then $l, l'' \subset \mathcal{P}_2$ and $l', l'' \subset \mathcal{P}_3$ and hence they must intersect. Now if $l = l'$ then either $l = l''$ and we are done. Otherwise, $l'' \not\subseteq \mathcal{P}_1$ and $l, l' \subset \mathcal{P}_1$, that is, l, l', l'' must have common point.

Let $(l_1l_1'l_1'')$ form an Eckardt point, namely E, i.e $l_1 \cap l_1' \cap l_1'' = \{E\}$. We can assume $l = l_1, l' = l_1'$ and $l'' = l_1''$. But according to previous argument, l, l', l'' have common point, that is $l \cap l' \cap l'' \neq \emptyset$. Thus $E \in l_1 \cap l_1' \cap l_1'' = l \cap l' \cap l''$.

(2) Replace t_i with t_i' and apply Lemma 4.2.1. □

Remark 4.4. Let $c(\mathscr{S}) \in \mathbb{S}^{(k)}$. Then from now on, $\mathscr{S}$ represents a non-singular cubic surface that is the blow up of $\mathbb{P}^2$ at six point in general position, and we write

$$\mathscr{S} := \mathrm{blw}_s \, \mathbb{P}^2 \ \text{ for some } \ s = \kappa_{123456} \in \mathbb{S}_6.$$

It is clear that $\mathbb{S}^{(2)} \subseteq \mathbb{S}^{(1)}$. Let $c(\mathscr{S}) \in \mathbb{S}^{(2)}$. Then $c(\mathscr{S}) \in \mathbb{S}^{(1)}$. So the corresponding non-singular cubic surface $\mathscr{S}$ has at least one Eckardt point. Recall that any Eckardt point on $\mathscr{S}$ is either of the form $(a_ib_jc_{ik})$ or the form $(c_{ij}c_{kh}c_{mn})$. However, $c(\mathscr{S}) \in \mathbb{S}^{(2)}$ implies that there is at least one other Eckardt point on $\mathscr{S}$. So assume that $\mathscr{S}$ has 2 Eckardt points corresponding to the two triples $t = (l_1l_2l_3)$ and $t' = (l_1'l_2'l_3')$ in $\mathbb{T}^{(3)}$. Then either t, t' have one line in common, or t, t' have no line in common since otherwise they coincide. Consequently, we have the following definition.

Definition 4.5. Let $\mathscr{S}$ be any non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(2)}$. We define

$$\mathbb{E}^{(2)} := \{c(\mathscr{S}) \in \mathbb{E}^{(2,2)} : \mathscr{S} \ \text{has} \ t_1, t_2 \in \mathbb{T}^{(3)} \ \text{with one common line}\},$$
$$\mathbb{E}^{(3)} := \{c(\mathscr{S}) \in \mathbb{E}^{(2,2)} : \mathscr{S} \ \text{has} \ t_1, t_2 \in \mathbb{T}^{(3)} \ \text{with no common line}\}.$$

Proposition 4.2. Let $\mathscr{S}$ be the non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(2)}$. Then

1. If $c(\mathscr{S}) \in \mathbb{E}^{(2)}$ then $\mathscr{S}$ has two Eckardt points of one of the following kinds:

   (a) $(a_\alpha b_\beta c_{\alpha\beta}), (a_{\alpha^*} b_\beta c_{\alpha^*\beta})$,

   (b) $(a_r b_s c_{rs}), (a_r b_{s^*} c_{rs^*})$,

   (c) $(a_i b_j c_{ij}), (c_{kh} c_{mn} c_{ij})$,

(d) $(c_{xy}c_{zw}c_{pq}), (c_{xy}c_{wq}c_{pz})$,

where $\alpha, \beta, \alpha^*, \beta^*, r, s, r^*, s^*, i, j, m, n, k, h, x, y, z, w, p, q \in \{1, \dots, 6\}$. Furthermore, $\{(a)\} \sim \{(b)\} \sim \{(c)\} \sim \{(d)\}$.

2. If $c(\mathscr{S}) \in \mathbb{E}^{(3)}$ then $\mathscr{S}$ has two Eckardt points of one of the following kinds:

(a) $(c_{ik}c_{jm}c_{nh}), (a_i b_j c_{ij})$,

(b) $(c_{xy}c_{zw}c_{pq}), (c_{wp}c_{yq}c_{xz})$,

where $i, j, m, n, k, h, x, y, z, w, p, q \in \{1, \dots, 6\}$. Furthermore, $\{(a)\} \sim \{(b)\}$.

Proof. Recall there are two kinds of Eckardt points on a non-singular cubic surface. Up to permutations (see Lemma 4.2.2), there is no ambiguity in assuming that $\mathscr{S}$ has two Eckardt points of one of the following kinds:

(a) $(a_\alpha b_\beta c_{\alpha\beta}), (a_{\alpha^*} b_\beta c_{\alpha^*\beta})$,

(b) $(a_r b_s c_{rs}), (a_r b_{s^*} c_{rs^*})$,

(c) $(a_i b_j c_{ij}), (c_{kh} c_{mn} c_{ij})$,

(d) $(c_{xy}c_{zw}c_{pq}), (c_{xy}c_{wq}c_{pz})$.

Consider the first kind (a):

$$(a_\alpha b_\beta c_{\alpha\beta}), (a_{\alpha^*} b_\beta c_{\alpha^*\beta}).$$

By renaming, say $a := b$, we have

$$(b_\alpha a_\beta c_{\alpha\beta}), (b_{\alpha^*} a_\beta c_{\alpha^*\beta})$$

Now via the permutation, namely $\tau := (\beta r)(\alpha s)(\alpha^* s^*)$, we get the kind (b), that is

$$(a_r b_s c_{rs}), (a_r b_{s^*} c_{rs^*}).$$

Consequently, via the quadratic transformation φ_{rs^*t}, we have

$$(c_{ts^*}c_{uv}c_{rs}), (c_{ts^*}b_{s^*}a_t).$$

Now, by using the permutation, namely $\sigma := (ti)(s^*j)(uk)(vh)(rm)(sn)$, we get the kind (c), that is

$$(a_i b_j c_{ij}), (c_{kh}c_{mn}c_{ij}).$$

Again via the quadratic transformation φ_{ikm}, we have

$$(c_{ij}c_{kh}c_{mn}), (c_{km}c_{nh}c_{ij}).$$

Applying the permutation, namely $\mu := (ix)(jy)(mz)(nw)(kp)(hq)$, gives us the kind (d), that is

$$(c_{xy}c_{zw}c_{pq}), (c_{xy}c_{wq}c_{pz}).$$

Thus we can transform from one kind to another. The diagram in Figure 4.4 illustrates the transformations between the kinds $(a), (b), (c)$ and (d):

$$\begin{array}{ccc} (a)\ (a_\alpha b_\beta c_{\alpha\beta}), (a_{\alpha^*} b_\beta c_{\alpha^*\beta}) & \xrightarrow{a := b \text{ and via } \tau} & (b)\ (a_r b_s c_{rs}), (a_r b_{s^*} c_{rs^*}) \\ \uparrow & & \downarrow \text{ via } \varphi_{rs^*t} \text{ and } \sigma \\ (d)\ (c_{xy}c_{zw}c_{pq}), (c_{xy}c_{wq}c_{pz}) & \xleftarrow{\text{via } \varphi_{ikm} \text{ and } \mu} & (c)\ (a_i b_j c_{ij}), (c_{kh}c_{mn}c_{ij}) \end{array}$$

Figure 4.4: Types of Eckardt points for $\mathbb{E}^{(2)}$.

(2) Let $c(\mathscr{S}) \in \mathbb{E}^{(3)}$. Similarly, as in the previous argument used in (1), and up to permutations, we can assume $\mathscr{S}$ has two Eckardt points of one of the following kinds (see Lemma 4.2.2):

(a) $(c_{ik}c_{jm}c_{nh}), (a_i b_j c_{ij})$,

(b) $(c_{xy}c_{zw}c_{pq}), (c_{wp}c_{yq}c_{xz})$.

Consider the first kind (a):

$$(c_{ik}c_{jm}c_{nh}), (a_i b_j c_{ij}).$$

Then the quadratic transformation φ_{imn} transforms the two triples above into

$$(c_{ik}c_{jm}c_{nh}), (c_{mn}c_{kh}c_{ij}),$$

because $j \notin \{m, n\}$. Again via the permutation $\sigma := (ix)(ky)(jz)(mw)(np)(hq)$, we get the kind (b), namely

$$(c_{xy}c_{zw}c_{pq}), (c_{wp}c_{yq}c_{xz}).$$

The following diagram in Figure 4.5 illustrates the transformations between the kinds (a) and (b):

$$\begin{array}{ccc} (a)\ (c_{ik}c_{jm}c_{nh}), (a_i b_j c_{ij}) & \xrightarrow{\text{via } \varphi_{imn}} & (c_{ik}c_{jm}c_{nh}), (c_{mn}c_{kh}c_{ij}) \\ & & \Big\downarrow \text{via } \sigma \\ & & (b)\ (c_{xy}c_{zw}c_{pq}), (c_{wp}c_{yq}c_{xz}) \end{array}$$

Figure 4.5: Types of Eckardt points for $\mathbb{E}^{(3)}$.

□

Remark 4.5. In $\mathbb{S}^{(2)}$, every non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(2)}$ has exactly 2 Eckardt points, and every non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(3)}$ has exactly 3 Eckardt points. All the results will be shown in the proof of the next proposition.

Proposition 4.3.

1. Any non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(3)}$ has exactly 3 Eckardt points.

2. Any non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(2)}$ has exactly 2 Eckardt points.

Proof. (1) Let $\mathscr{S}$ be a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(3)}$. Then according to Proposition 4.2 and the definition of $\mathbb{E}^{(3)}$, we can assume that $t_1 = (c_{12}c_{34}c_{56})$ and $t_2 = (c_{13}c_{26}c_{45})$ are two triples on $\mathscr{S}$. Applying our operation on t_1, t_2 implies

$$t_3 = t_1 t_2 = (c_{15}c_{24}c_{36}).$$

Let us construct the symmetric table for t_1, t_2 and t_3 (see Table 4.9).

Table 4.9: The symmetric table 3

	$\emptyset$	t_1	t_2	t_3
$\emptyset$	$\emptyset$	t_1	t_2	t_3
t_1	t_1	$\emptyset$	t_3	t_2
t_2	t_2	t_3	$\emptyset$	t_1
t_3	t_3	t_2	t_1	$\emptyset$

We see from the table above that there are no new possible triples in the table which give us new Eckardt points. Therefore, for any element $c(\mathscr{S}) \in \mathbb{E}^{(3)}$, the corresponding non-singular cubic surface $\mathscr{S}$ has exactly 3 Eckardt points which are associated to the triples t_1, t_2 and t_3.

(2) By arguing as in part (1), one can show that if $\mathscr{S}$ be a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(2)}$ then according to Proposition 4.2 and the definition of $\mathbb{E}^{(2)}$, we can assume that t_1 and t_2 are two triples on $\mathscr{S}$ with one common line. So by applying our operation on t_1 and t_2, we get $t_3 = t_1 t_2 = \emptyset$, and no new possible triples can give us new Eckardt points. Hence for any element $c(\mathscr{S}) \in \mathbb{E}^{(2)}$, the corresponding cubic surface $\mathscr{S}$ has exactly 2 Eckardt points which are associated to the triples t_1, t_2. □

Corollary 4.2. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(2)}(q)$ that corresponds to $c(\mathscr{S}^{(2)}) \in \mathbb{E}^{(2)}$ is of type $[(q-10)^2 + 7, 27(q-9) + 6, 129, 2]$.

Proof. Recall that $\mathscr{S}^{(2)}(q)$ denotes a non-singular cubic surface with 27 lines that has exactly two Eckardt points over the Galois field $GF(q)$. By the same argument used in the proof of Theorem 4.1, we have

$$n_q = e_3 + e_2 + e_1,$$

$$3e_3 + e_2 = 135,$$

$$3e_3 + 2e_2 + e_1 = 27(q+1),$$

and

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1.$$

Thus

$$e_3 = 2,$$

$$e_2 = 135 - 6 = 129,$$

$$e_1 = 27(q-4) - 129 = 27(q-9) + 6,$$

$$e_0 = (q-10)^2 + 7.$$

□

Corollary 4.3. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(3)}(q)$ that corresponds to $c(\mathscr{S}^{(3)}) \in \mathbb{E}^{(3)}$ is of type $[(q-10)^2 + 6, 27(q-9) + 9, 126, 3]$.

Proof. Recall that $\mathscr{S}^{(3)}(q)$ denotes a non-singular cubic surface with 27 lines that has exactly three Eckardt points over the Galois field $GF(q)$. By the same argument used in the proof of Corollary 4.2, we have

$$n_q = e_3 + e_2 + e_1,$$

$$3e_3 + e_2 = 135,$$

$$3e_3 + 2e_2 + e_1 = 27(q+1),$$

and

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1.$$

Thus

$$\begin{aligned}
e_3 &= 3,\\
e_2 &= 135 - 9 = 126,\\
e_1 &= 27(q-4) - 126 = 27(q-9) + 9,\\
e_0 &= (q-10)^2 + 6.
\end{aligned}$$

□

Let us assume that $c(\mathscr{S}) \in \mathbb{S}^{(4)}$. By the definition of $\mathbb{S}^{(4)}$, the corresponding non-singular cubic surface $\mathscr{S}$ has at least 4 trihedrons such that three of them form a triad, namely

$$T = \{t_1, t_2, t_3\} \subset \mathbb{T}^{(3)}.$$

The other triple, namely $t := t_4$ in $\mathbb{T}^{(3)}$ has the following possibilities:

1. t has all lines in common with T,

2. t has two lines in common with T,

3. t has one line in common with T,

4. t has zero line in common with T.

However, if t has 2 lines in common with T, then it has all lines in common with T (see the proof of Proposition 4.1). Therefore, t has either zero, or one, or three lines in common with T, and we can introduce the following definition.

Definition 4.6. Let $\mathscr{S}$ be the non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(4)}$. We define

$$\mathbb{E}^{(4)} := \left\{ \begin{array}{c} c(\mathscr{S}) \in \mathbb{E}^{(4,4)} : \mathscr{S} \text{ has } T \vee t \subset \mathbb{T}^{(3)} \text{ such that} \\ t \text{ has three common line with } T \end{array} \right\},$$

$$\mathbb{E}^{(6)} := \left\{ \begin{array}{c} c(\mathscr{S}) \in \mathbb{E}^{(6,4)} : \mathscr{S} \text{ has } T \vee t \subset \mathbb{T}^{(3)} \text{ such that} \\ t \text{ has one common line with } T \end{array} \right\},$$

$$\mathbb{E}^{(9)} := \left\{ \begin{array}{c} c(\mathscr{S}) \in \mathbb{E}^{(9,4)} : \mathscr{S} \text{ has } T \vee t \subset \mathbb{T}^{(3)} \text{ such that} \\ t \text{ has no common line with } T \end{array} \right\}.$$

In fact, in $\mathbb{S}^{(4)}$, every non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(4)}$ has exactly 4 Eckardt points. Every non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(6)}$ has exactly 6 Eckardt points, and every non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(9)}$ has exactly 9 Eckardt points. All the detail will be shown later.

Proposition 4.4. Let $\mathscr{S}$ be the non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(4)}$. There are two possible kinds for the set $T \vee t$ (as in the Definition 4.6).

Proof. Suppose that $\mathscr{S}$ is a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(4)}$. Then according to Lemma 4.2.2, we can assume $T = \{t_1, t_2, t_3\}$ where

$$t_1 = (a_i b_m c_{im}),$$

$$t_2 = (a_j b_k c_{jk}),$$

$$t_3 = t_{1,2} = (c_{ik} c_{jm} c_{nh}),$$

and $i, j, m, n, k, h \in \{1, \ldots, 6\}$. Up to permutations and quadratic transformations, we have the following two possibilities for triples $t \in \mathbb{T}^{(3)}$:

$$(a) \;\; t := (a_j b_m c_{jm}),$$

$$(b) \;\; t := t' = (c_{im} c_{jk} c_{nh}).$$

Now via the permutation (ij) and the quadratic transformation φ_{ijk}, we can transform between the two kinds, namely $(A): T \vee t$ and $(B): T \vee t'$. The diagram in Figure 4.6 illustrates the transformations between the two kinds.

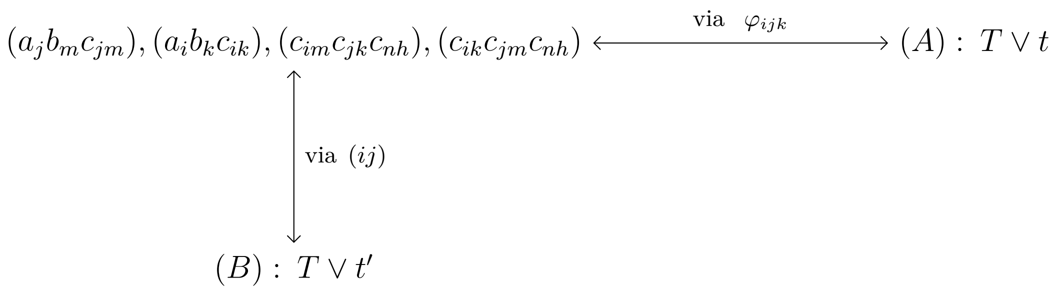

Figure 4.6: Types of Eckardt points for $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(4)}$.

□

Corollary 4.4. Any non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(4)}$ has exactly 4 Eckardt points and one triad.

Proof. Assume that $\mathscr{S}$ is a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(4)}$. Then according to the definition of $\mathbb{E}^{(4)}$, the non-singular cubic surface $\mathscr{S}$ has at least 4 Eckardt points. Furthermore, up to permutations and quadratic transformations, we can assume that $\mathscr{S}$ has $T \vee t$ where $T = \{t_1, t_2, t_3\}$ and

$$t_1 = (a_2 b_1 c_{12}),$$

$$t_2 = (a_3 b_4 c_{34}),$$

$$t_3 = (c_{13} c_{24} c_{56}),$$

$$t_4 := t = (a_3 b_1 c_{13}).$$

Consequently, we have the following symmetric table for t_1, t_2, t_3 and t_4 (see Table 4.10).

From the Table 4.10, it is obvious that there are no new possible triples which give us new Eckardt points. Hence for any element $c(\mathscr{S}) \in \mathbb{E}^{(4)}$, the corresponding cubic surface $\mathscr{S}$ has exactly 4 Eckardt points which are associated to the triad $T = \{t_1, t_2, t_3\}$ and the triple t_4. □

Table 4.10: The symmetric table 4

	$\emptyset$	t_1	t_2	t_3	t_4
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$
t_2	t_2	t_3	$\emptyset$	t_1	$\emptyset$
t_3	t_3	t_2	t_1	$\emptyset$	$\emptyset$
t_4	t_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$

Corollary 4.5. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(4)}(q)$ that corresponds to $c(\mathscr{S}^{(4)}) \in \mathbb{E}^{(4)}$ is of type $[(q-10)^2+5, 27(q-9)+12, 123, 4]$.

Proof. Recall that $\mathscr{S}^{(4)}(q)$ denotes a non-singular cubic surface with 27 lines that has exactly 4 Eckardt points over the Galois field $GF(q)$. By the same argument used in the proof of Corollary 4.2, we have

$$n_q = e_3 + e_2 + e_1,$$

$$3e_3 + e_2 = 135,$$

$$3e_3 + 2e_2 + e_1 = 27(q+1),$$

and

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1.$$

Thus

$$e_3 = 4,$$

$$e_2 = 135 - 12 = 123,$$

$$e_1 = 27(q-4) - 123 = 27(q-9) + 12,$$

$$e_0 = (q-10)^2 + 5.$$

□

Proposition 4.5. Let $\mathscr{S}$ be the non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(6)}$. There are 3 possible kinds for the set $T \vee t$ (as in the Definition 4.6).

Proof. Assume that $\mathscr{S}$ is a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(6)}$. Then $\mathscr{S}$ has 4 Eckardt points which correspond to the 4 triples in $T \vee t$, where the triple t has one common line with $T = \{t_1, t_2, t_3\}$. Then up to permutations and quadratic transformations, we can assume

$$t_1 = (a_2 b_1 c_{12}),$$
$$t = (a_3 b_1 c_{13}).$$

In this case, we have the following possibilities for t_2 and hence for t_3:

(a) $t_2 = (a_4 b_3 c_{34})$ and hence $t_3 = (c_{14} c_{23} c_{56})$,

(b) $t_2 := t_2' = (a_1 b_4 c_{14})$ and hence $t_3 := t_3' = (a_4 b_2 c_{24})$,

(c) $t_2 := t_2'' = (a_4 b_5 c_{45})$ and hence $t_3 := t_3'' = (c_{14} c_{25} c_{36})$.

First of all, assume t_2 have the form $(a_4 b_2 c_{24})$ or $(c_{ij} c_{km} c_{nh})$. Then up to permutations and quadratic transformations, t_3 has the form (b) or (c).

More precisely, we have the following three possibilities for the kinds of $T \vee t$:

(A) $T := \{t_1, t_2, t_3\}$ and $t_4 := t$,

(B) $T := T' = \{t_1', t_2', t_3'\}$, $t_1' := t_1$ and $t_4' := t$,

(C) $T := T'' = \{t_1'', t_2'', t_3''\}$, $t_1'' := t_1$ and $t_4'' := t$.

Let us consider the case where the kind is (B). Applying our operation on $T' \vee t_4'$ gives us the symmetric Table 4.11.

So $T' \vee t_4'$ is equivalent to $T \vee t_4$, where

$$T = \{t_1, t_2' t_4, t_3' t_4\} = \{t_1, t_2, t_3\}$$

Table 4.11: The symmetric table 5

	$\emptyset$	t_1	t'_2	t'_3	t_4
$\emptyset$	$\emptyset$	t_1	t'_2	t'_3	t_4
t_1	t_1	$\emptyset$	t'_3	t'_2	$\emptyset$
t'_2	t'_2	t'_3	$\emptyset$	t_1	t_2
t'_3	t'_3	t'_2	t_1	$\emptyset$	t_3
t_4	t_4	$\emptyset$	t_2	t_3	$\emptyset$

which forms one of kind (A).

On the other hand, let us consider the case where the kind is (C). Applying our operation on $T'' \vee t''_4$ gives the symmetric Table 4.12.

Table 4.12: The symmetric table 6

	$\emptyset$	t_1	t''_2	t''_3	t_4
$\emptyset$	$\emptyset$	t_1	t''_2	t''_3	t_4
t_1	t_1	$\emptyset$	t''_3	t''_2	$\emptyset$
t''_2	t''_2	t''_3	$\emptyset$	t_1	t_5
t''_3	t''_3	t''_2	t_1	$\emptyset$	t_6
t_4	t_4	$\emptyset$	t_5	t_6	$\emptyset$

where

$$t_5 := (c_{14}c_{35}c_{26}),$$

$$t_6 := (a_4b_6c_{46}).$$

Now via the quadratic transformation φ_{356}, we have

$$t_1 \longmapsto t'''_5 := t_1,$$

$$t''_2 \longmapsto t'''_2 := t''_2,$$

$$t''_3 \longmapsto t'''_4 := (a_5b_2c_{25}),$$

$$t_4 \longmapsto t'''_3 := (c_{13}c_{24}c_{56}),$$

$$t_5 \longmapsto t'''_1 := (a_6b_2c_{26}).$$

Thus applying the quadratic transformation φ_{356} gives us the new kind $T''' \vee t'''_4$ where $T''' := \{t'''_1, t'''_2, t'''_3\}$. Now via the permutation $\sigma := (62)(12)(53)$, the kind $T''' \vee t'''_4$ is transformed to one kind of (A).

Moreover, if the triples of kind (A) have the form $\widetilde{T} \vee \widetilde{t}$ where

$$\widetilde{T} := \{\widetilde{t_1}, \widetilde{t_2}, \widetilde{t_3}\} = \{(a_j b_i c_{ij}), (a_k b_h c_{kh}), (c_{ik} c_{jh} c_{mn}\} \text{ and } \widetilde{t} := (a_h b_i c_{ih}).$$

Then the permutation $\sigma = (i1)(j2)(h3)(k4)(m5)(n6)$ transforms $\widetilde{T} \vee \widetilde{t}$ to the triples of kind (A). In summary, we have the diagram (see Figure 4.7).

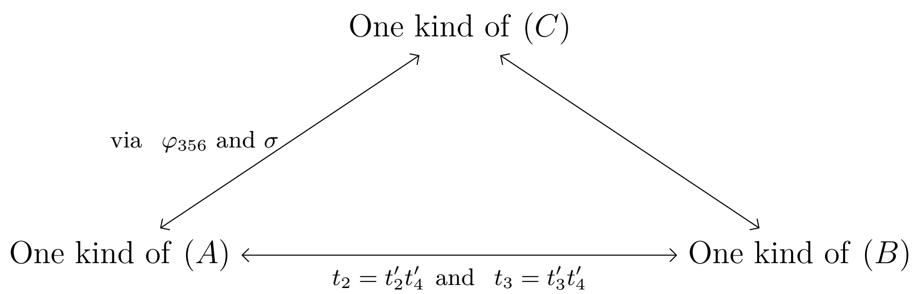

Figure 4.7: Types of Eckardt points for $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(6)}$.

$\square$

Corollary 4.6. Let $\mathscr{S}$ be a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(6)}$. Then $\mathscr{S}$ has exactly 6 Eckardt points and 4 triads which contain 15 lines among the 27 lines on cubic surface.

Proof. We know that $\mathscr{S}$ has a 4 triples of the form $T \vee t$ as we shown in Proposition 4.5. Up to permutations and quadratic transformations, we can assume that $T = \{t_1, t_2, t_3\}$, where

$$t_1 := (a_2 b_1 c_{12}),$$

$$t_2 := (a_4 b_3 c_{34}),$$

$$t_3 := (c_{14} c_{23} c_{56}),$$

$$t_4 := t = (a_3 b_1 c_{13}).$$

Let us construct the symmetric table determined by t_1, t_2, t_3, and t_4 (see Table 4.13).

Table 4.13: The symmetric table 7

	$\emptyset$	t_1	t_2	t_3	t_4
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$
t_2	t_2	t_3	$\emptyset$	t_1	t_5
t_3	t_3	t_2	t_1	$\emptyset$	t_6
t_4	t_4	$\emptyset$	t_5	t_6	$\emptyset$

where

$$t_5 := (a_1 b_4 c_{14}),$$

$$t_6 := (a_4 b_2 c_{24}).$$

Thus we have at least 6 Eckardt points generated by t_1, t_2, t_3, t_4, t_5, and t_6. Furthermore, we get the following 4 triads:

$$T_1 = \{t_1, t_2, t_3\},$$

$$T_2 = \{t_2, t_4, t_5\},$$

$$T_3 = \{t_3, t_4, t_6\},$$

$$T_4 = \{t_1, t_5, t_6\},$$

which contain 15 lines among the 27 lines on cubic surface, namely

$$a_1, a_2, a_3, a_4, b_1, b_2, b_3, b_4, c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34} \text{ and } c_{56}.$$

Again, applying our operation on the triples t_1, t_2, t_3, t_4, t_5 and t_6 will give us the symmetric Table 4.14.

It follows that there are no new possible triples in the table which give us new Eckardt points. Therefore, $\mathscr{S}$ has exactly 6 Eckardt points which correspond to the triples t_1, t_2, t_3, t_4, t_5 and t_6. $\square$

Corollary 4.7. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(6)}(q)$ that corresponds to $c(\mathscr{S}^{(6)}) \in \mathbb{E}^{(6)}$ is of type $[(q-10)^2 + 3, 27(q-9) + 18, 117, 6]$.

Table 4.14: The symmetric table 8

	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$	t_6	t_5
t_2	t_2	t_3	$\emptyset$	t_1	t_5	t_4	$\emptyset$
t_3	t_3	t_2	t_1	$\emptyset$	t_6	$\emptyset$	t_4
t_4	t_4	$\emptyset$	t_5	t_6	$\emptyset$	t_2	t_3
t_5	t_5	t_6	t_4	$\emptyset$	t_2	$\emptyset$	t_1
t_6	t_6	t_5	$\emptyset$	t_4	t_3	t_1	$\emptyset$

Proof. Recall that $\mathscr{S}^{(6)}(q)$ denotes a non-singular cubic surface with 27 lines that has exactly 6 Eckardt points over the Galois field $GF(q)$. By the same argument used in the proof of Corollary 4.2, we have

$$n_q = e_3 + e_2 + e_1,$$

$$3e_3 + e_2 = 135,$$

$$3e_3 + 2e_2 + e_1 = 27(q+1),$$

and

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1.$$

Thus

$$e_3 = 6,$$

$$e_2 = 135 - 18 = 117,$$

$$e_1 = 27(q-4) - 117 = 27(q-9) + 18,$$

$$e_0 = (q-10)^2 + 3.$$

□

Proposition 4.6. Let $\mathscr{S}$ be a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(9)}$. There is one possible kind for the set $T \vee t$ (as in Definition 4.6).

Proof. Let us assume that $\mathscr{S}$ is a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(9)}$. Then $\mathscr{S}$ has 4 Eckardt points which correspond to the 4 triples in $T \vee t$ where the triple t has no common line with triad $T = \{t_1, t_2, t_3\}$. Then up to permutations and quadratic transformations, we can assume

$$t_1 = (c_{12}c_{34}c_{56}),$$

$$t_2 = (c_{15}c_{24}c_{36}),$$

$$t_3 = t_1t_2 = (c_{13}c_{26}c_{45}).$$

Now if $t \in \mathbb{T}^{(3)} \backslash T$ has no common line with T, then t must be of the form $(a_ib_jc_{ij})$ where

$$(i,j) \in \{(1,4),(1,6),(2,3),(2,5),(3,2),(3,5),(4,1),(4,6),(5,2),(5,3),(6,1),(6,4)\}.$$

Again up to permutations and quadratic transformations, we have

$$\text{as } (i,j) = (1,4) : t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (1,6) : t = (a_1b_6c_{16}) \xrightarrow{\varphi_{146}} t = (a_4b_6c_{46}) \xrightarrow{(46)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (2,3) : t = (a_2b_3c_{23}) \xrightarrow{(12)(34)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (2,5) : t = (a_2b_5c_{25}) \xrightarrow{(12)(45)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (3,2) : t = (a_3b_2c_{23}) \xrightarrow{(13)(24)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (3,5) : t = (a_3b_5c_{35}) \xrightarrow{(13)(45)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (4,1) : t = (a_4b_1c_{14}) \xrightarrow{(14)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (4,6) : t = (a_4b_6c_{46}) \xrightarrow{\varphi_{146}} t = (a_1b_6c_{16}) \xrightarrow{(46)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (5,2) : t = (a_5b_2c_{25}) \xrightarrow{(15)(24)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (5,3) : t = (a_5b_3c_{35}) \xrightarrow{(15)(34)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (6,1) : t = (a_6b_1c_{16}) \xrightarrow{\varphi_{146}} t = (a_4b_1c_{14}) \xrightarrow{(14)} t = (a_1b_4c_{14}),$$

$$\text{as } (i,j) = (6,4) : t = (a_6b_4c_{46}) \xrightarrow{\varphi_{146}} t = (a_1b_4c_{14}).$$

Thus we can assume that $t = (a_4b_1c_{14})$ by applying the permutation (14). Therefore, up to equivalence of triples $T \vee t = \{(c_{12}c_{34}c_{56}), (c_{15}c_{24}c_{36}), (c_{13}c_{26}c_{45}), (a_4b_1c_{14})\}$. $\square$

Corollary 4.8. Let $\mathscr{S}$ be a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(9)}$. Then $\mathscr{S}$ has exactly nine Eckardt points and 12 triads.

Proof. Assume that $\mathscr{S}$ is a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(9)}$. Then we can assume $\mathscr{S}$ has the triples in $T \vee t$, where the triad $T = \{t_1, t_2, t_3\}$ and $t := t_4$ are as mentioned in Proposition 3.6, that is

$$t_1 = (c_{12}c_{34}c_{56}),$$

$$t_2 = (c_{15}c_{24}c_{36}),$$

$$t_3 = t_1t_2 = (c_{13}c_{26}c_{45}),$$

$$t_4 = (a_4b_1c_{14}).$$

By applying our operation on the triples t_1, t_2, t_3 and t_4, we have the symmetric Table 4.15.

Table 4.15: The symmetric table 9

	$\emptyset$	t_1	t_2	t_3	t_4
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4
t_1	t_1	$\emptyset$	t_3	t_2	t_5
t_2	t_2	t_3	$\emptyset$	t_1	t_6
t_3	t_3	t_2	t_1	$\emptyset$	t_7
t_4	t_4	t_5	t_6	t_7	$\emptyset$

where

$$t_5 := (a_2b_3c_{23}),$$

$$t_6 := (a_5b_2c_{25}),$$

$$t_7 := (a_3b_5c_{35}).$$

Again we can construct the following symmetric Table 4.16.

Table 4.16: The symmetric table 10

	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	t_1	$\emptyset$	t_3	t_2	t_5	t_4	t_8	t_9
t_2	t_2	t_3	$\emptyset$	t_1	t_6	t_9	t_4	t_8
t_3	t_3	t_2	t_1	$\emptyset$	t_7	t_8	t_9	t_4
t_4	t_4	t_5	t_6	t_7	$\emptyset$	t_1	t_2	t_3
t_5	t_5	t_4	t_9	t_8	t_1	$\emptyset$	t_7	t_6
t_6	t_6	t_8	t_4	t_9	t_2	t_7	$\emptyset$	t_5
t_7	t_7	t_9	t_8	t_4	t_3	t_6	t_5	$\emptyset$

where

$$t_8 := (a_1 b_6 c_{16}),$$

$$t_9 := (a_6 b_4 c_{46}).$$

Therefore, there are at least nine Eckardt points on $\mathscr{S}$ which correspond to the nine triples, namely

$$t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, \text{ and } t_9.$$

Moreover, we get the following 12 triads:

$$T_1 = \{t_1, t_4, t_5\},$$

$$T_2 = \{t_2, t_4, t_6\},$$

$$T_3 = \{t_3, t_4, t_7\},$$

$$T_4 = \{t_4, t_8, t_9\},$$

$$T_5 = \{t_1, t_2, t_3\},$$

$$T_6 = \{t_1, t_6, t_8\},$$

$$T_7 = \{t_1, t_7, t_9\},$$

$$T_8 = \{t_2, t_7, t_8\},$$

$$T_9 = \{t_2, t_5, t_9\},$$

$$T_{10} = \{t_3, t_5, t_8\},$$

$$T_{11} = \{t_3, t_6, t_9\},$$

$$T_{12} = \{t_5, t_6, t_7\}.$$

Now by applying our operation on $t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8$ and t_9, we have the symmetric Table 4.17.

Table 4.17: The symmetric table 11

	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9
t_1	t_1	$\emptyset$	t_3	t_2	t_5	t_4	t_8	t_9	t_6	t_7
t_2	t_2	t_3	$\emptyset$	t_1	t_6	t_9	t_4	t_8	t_7	t_5
t_3	t_3	t_2	t_1	$\emptyset$	t_7	t_8	t_9	t_4	t_5	t_6
t_4	t_4	t_5	t_6	t_7	$\emptyset$	t_1	t_2	t_3	t_9	t_8
t_5	t_5	t_4	t_9	t_8	t_1	$\emptyset$	t_7	t_6	t_3	t_2
t_6	t_6	t_8	t_4	t_9	t_2	t_7	$\emptyset$	t_5	t_1	t_3
t_7	t_7	t_9	t_8	t_4	t_3	t_6	t_5	$\emptyset$	t_2	t_1
t_8	t_8	t_6	t_7	t_5	t_9	t_3	t_1	t_2	$\emptyset$	t_4
t_9	t_9	t_7	t_3	t_6	t_8	t_2	t_3	t_1	t_4	$\emptyset$

It follows that there are no new possible triples in the table which give us new Eckardt points. Therefore $\mathscr{S}$ has exactly 9 Eckardt points which correspond to the triples t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8 and t_9. □

Remark 4.6. The nine triples $t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8,$ and t_9 in Corollary 4.8 contain all the 27 lines on a non-singular cubic surface $\mathscr{S}$. Furthermore, every triple in the previous list occurs 4 times among the 12 triads T_i; $i \in \{1, \ldots, 12\}$.

Corollary 4.9. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(9)}(q)$ that corresponds to $c(\mathscr{S}^{(9)}) \in \mathbb{E}^{(9)}$ is of type $[(q-10)^2, 27(q-8), 108, 9]$.

Proof. Recall that $\mathscr{S}^{(9)}(q)$ denotes a non-singular cubic surface with 27 lines that has exactly 9 Eckardt points over the Galois field $GF(q)$. By the same argument used in the proof of Corollary 4.7, we have

$$
\begin{aligned}
n_q &= e_3 + e_2 + e_1, \\
3e_3 + e_2 &= 135, \\
3e_3 + 2e_2 + e_1 &= 27(q+1),
\end{aligned}
$$

and

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1.$$

Thus

$$
\begin{aligned}
& e_3 = 9, \\
& e_2 = 135 - 27 = 108, \\
& e_1 = 27(q-4) - 108 = 27(q-8), \\
& e_0 = (q-10)^2.
\end{aligned}
$$

□

Theorem 4.4. Let $\mathscr{S}$ be a non-singular cubic surface with the six triples t_1, t_2, t_3, t_4, t_5, t_6 mentioned in the proof of Corollary 4.6, and let t_7 be another triple on $\mathscr{S}$, that is $t_7 \in \mathbb{T}^{(3)} \backslash \{t_i : i \in \{1, \dots, 6\}\}$. Then

I. $\mathscr{S}$ has at least 10 Eckardt points and at least 10 triads if all lines of t_7 are in common with one of the 4 triads generated by $t_1, \dots, t_6$.

II. Otherwise, $\mathscr{S}$ has at least 18 Eckardt points and at least 42 triads.

Proof. I. According to the argument introduced in the proof of Corollary 4.6, we have the following 4 triads:

$$T_1 = \{t_1, t_2, t_3\}, T_2 = \{t_2, t_4, t_5\}, T_3 = \{t_3, t_4, t_6\}, \text{ and } T_4 = \{t_1, t_5, t_6\},$$

where

$$t_1 := (a_2 b_1 c_{12}),$$

$$t_2 := (a_4 b_3 c_{34}),$$

$$t_3 := (c_{14} c_{23} c_{56}),$$

$$t_4 := (a_3 b_1 c_{13}),$$

$$t_5 := (a_1 b_4 c_{14}),$$

$$t_6 := (a_4 b_2 c_{24}).$$

Suppose that all lines of t_7 are in common with the triad T_1. Then there are three choices for the triple t_7:

$$(1)\ t_7 := (a_2 b_3 c_{23}),$$

$$(2)\ t_7 := (a_4 b_1 c_{14}),$$

$$(3)\ t_7 := (c_{12} c_{34} c_{56}).$$

However we can transform between the two choices (1) and (2) via the permutation (13)(24). Thus up to permutations and quadratic transformations, there are two cases for t_7, namely

$$\text{Case(1): } t_7 = (a_2 b_3 c_{23}),$$

$$\text{Case(2): } t_7 = (c_{12} c_{34} c_{56}).$$

For case (1), we construct the symmetric table which is determined by t_1, t_2, t_3, t_4, t_5, t_6 and t_7 (see Table 4.18).

where

$$t_8 := (a_1 b_2 c_{12}),$$

$$t_9 := (c_{13} c_{24} c_{56}),$$

$$t_{10} := (a_3 b_4 c_{34}).$$

Table 4.18: The symmetric table 12

	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$	t_6	t_5	$\emptyset$
t_2	t_2	t_3	$\emptyset$	t_1	t_5	t_4	$\emptyset$	$\emptyset$
t_3	t_3	t_2	t_1	$\emptyset$	t_6	$\emptyset$	t_4	$\emptyset$
t_4	t_4	$\emptyset$	t_5	t_6	$\emptyset$	t_2	t_3	t_8
t_5	t_5	t_6	t_4	$\emptyset$	t_2	$\emptyset$	t_1	t_9
t_6	t_6	t_5	$\emptyset$	t_4	t_3	t_1	$\emptyset$	t_{10}
t_7	t_7	$\emptyset$	$\emptyset$	$\emptyset$	t_8	t_9	t_{10}	$\emptyset$

Thus we have at least 10 Eckardt points determined by

$$t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, \text{ and } t_{10}.$$

Furthermore, we have the following 10 triads:

$$T_1 = \{t_1, t_2, t_3\},$$

$$T_2 = \{t_1, t_5, t_6\},$$

$$T_3 = \{t_1, t_9, t_{10}\},$$

$$T_4 = \{t_2, t_4, t_5\},$$

$$T_5 = \{t_2, t_8, t_9\},$$

$$T_6 = \{t_3, t_4, t_6\},$$

$$T_7 = \{t_3, t_8, t_{10}\},$$

$$T_8 = \{t_4, t_7, t_8\},$$

$$T_9 = \{t_5, t_7, t_9\},$$

$$T_{10} = \{t_6, t_7, t_{10}\}.$$

For the case (2), we have a table similar to the one in case (1) except some changes, namely

$$t_8 := (a_2 b_4 c_{24}),$$

$$t_9 := (a_3 b_2 c_{23}),$$

$$t_{10} := (a_1 b_3 c_{13}).$$

Consequently, we have at least 10 Eckardt points determined by

$$t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9, \text{ and } t_{10}.$$

In fact, we can transform from one case to another via some exchanges in triples so that the two cases represent the same kind:

$$\begin{array}{ccc} \text{Form for the case (2)} & \longleftrightarrow & \text{Form for the case (1)} \\ T := \{t_3, t_4, t_6\} & \longleftrightarrow & T := \{t_1, t_2, t_3\} \\ t := t_1 & \longleftrightarrow & t := t_4 \\ t_7 := t_9 = (c_{12}c_{34}c_{56}) & \longleftrightarrow & t_7 := (a_2b_3c_{23}) \end{array}$$

Thus we have the same 10 trihedral pairs for the case (1).

II. If t_7 has no line common in with one of the 4 triads, namely

$$T_1 = \{t_1, t_2, t_3\},$$

$$T_2 = \{t_2, t_4, t_5\},$$

$$T_3 = \{t_3, t_4, t_6\},$$

$$T_4 = \{t_1, t_5, t_6\},$$

then we can assume t_7 has no line common in with triad T_1. If t_7 and t_4 have a common line, we can replace T_1 by T_3, t_4 by t_5, and t_7 by t_9 so that we can avoid this case.

Now if t_7 has no line common in with triad T_1 and the triple t_4, then we have the following possibilities for t_7:

1. $(a_5b_2c_{25}), (a_5b_4c_{45}), (a_1b_5c_{15}),$ and $(c_{15}c_{24}c_{36})$,

2. $(a_6b_2c_{26}), (a_6b_4c_{46}), (a_1b_6c_{16}),$ and $(c_{16}c_{24}c_{35})$.

However we can transform between the two cases, namely (1) and (2), via the permutation (56) (see the diagram in Figure 4.8)

$$(a_5b_2c_{25}) \xleftrightarrow{(24)} (a_5b_4c_{45}), (a_1b_5c_{15}) \xleftrightarrow{\varphi_{124}} (c_{15}c_{24}c_{36}).$$

$$\updownarrow \text{ via } (56)$$

$$(a_6b_2c_{26}) \xleftrightarrow{(24)} (a_6b_4c_{46}), (a_1b_6c_{16}) \xleftrightarrow{\varphi_{124}} (c_{16}c_{24}c_{35}).$$

Figure 4.8: Cases of t_7 on $\mathscr{S}$.

Consequently, we get exactly the following two cases for t_7

$$\text{Case (i): } t_7 = (a_5b_2c_{25}) \text{ or } t_7 = (a_1b_5c_{15}),$$

$$\text{Case (ii): } t_7 = (a_5b_4c_{45}) \text{ or } t_7 = (c_{15}c_{24}c_{36}).$$

Let us consider the case (i): $t_7 = (a_5b_2c_{25})$ or $t_7 = (a_1b_5c_{15})$. If $t_7 = (a_5b_2c_{25})$, then we have the symmetric Table 4.19.

Table 4.19: The symmetric table 13

	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$	t_6	t_5	t_8
t_2	t_2	t_3	$\emptyset$	t_1	t_5	t_4	$\emptyset$	t_9
t_3	t_3	t_2	t_1	$\emptyset$	t_6	$\emptyset$	t_4	t_{10}
t_4	t_4	$\emptyset$	t_5	t_6	$\emptyset$	t_2	t_3	t_{11}
t_5	t_5	t_6	t_4	$\emptyset$	t_2	$\emptyset$	t_1	t_{12}
t_6	t_6	t_5	$\emptyset$	t_4	t_3	t_1	$\emptyset$	$\emptyset$
t_7	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	$\emptyset$	$\emptyset$

where

$$t_8 = (a_1b_5c_{15}),$$

$$t_9 = (c_{16}c_{24}c_{35}),$$

$$t_{10} = (a_3b_6c_{36}),$$

$$t_{11} = (c_{15}c_{23}c_{46}),$$

$$t_{12} = (c_{12}c_{36}c_{45}).$$

Thus we have at least 18 Eckardt points determined by the triples $t_j, j = 1, \ldots, 18$ where

$$t_{13} = t_2t_8 = (c_{13}c_{26}c_{45}), \qquad t_{14} = t_3t_8 = (a_6b_4c_{46}),$$

$$t_{15} = t_4t_8 = (a_5b_3c_{35}), \qquad t_{16} = t_4t_9 = (a_6b_5c_{56}),$$

$$t_{17} = t_5t_9 = (a_2b_6c_{26}), \qquad t_{18} = t_5t_{10} = (c_{16}c_{25}c_{34}).$$

Now if we assume that $t_7 = (a_1b_5c_{15})$, we get the symmetric Table 4.20.

Table 4.20: The symmetric table 14

	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$	t_6	t_5	t_8
t_2	t_2	t_3	$\emptyset$	t_1	t_5	t_4	$\emptyset$	t_9
t_3	t_3	t_2	t_1	$\emptyset$	t_6	$\emptyset$	t_4	t_{10}
t_4	t_4	$\emptyset$	t_5	t_6	$\emptyset$	t_2	t_3	t_{11}
t_5	t_5	t_6	t_4	$\emptyset$	t_2	$\emptyset$	t_1	$\emptyset$
t_6	t_6	t_5	$\emptyset$	t_4	t_3	t_1	$\emptyset$	t_{12}
t_7	t_7	t_8	t_9	t_{10}	t_{11}	$\emptyset$	t_{12}	$\emptyset$

where

$$t_8 = (a_5b_2c_{25}),$$

$$t_9 = (c_{13}c_{26}c_{45}),$$

$$t_{10} = (a_6b_4c_{46}),$$

$$t_{11} = (a_5b_3c_{35}),$$

$$t_{12} = (c_{12}c_{36}c_{45}).$$

Therefore we have at least 18 Eckardt points determined by the triples $t_j, j = 1, \ldots, 18$

where

$$t_{13} = t_2t_8 = (c_{16}c_{24}c_{35}), \qquad t_{14} = t_3t_8 = (a_3b_6c_{36}),$$

$$t_{15} = t_4t_8 = (c_{15}c_{23}c_{46}), \qquad t_{16} = t_6t_9 = (a_6b_5c_{56}),$$

$$t_{17} = t_5t_{13} = (a_2b_6c_{26}), \qquad t_{18} = t_5t_{14} = (c_{16}c_{25}c_{34}).$$

Case (ii): $t_7 = (a_5b_4c_{45})$ or $t_7 = (c_{15}c_{24}c_{36})$. In the same argument for the case (i), if we assume that $t_7 = (a_5b_4c_{45})$ then we get the symmetric Table 4.21.

Table 4.21: The symmetric table 15

	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$	t_6	t_5	t_8
t_2	t_2	t_3	$\emptyset$	t_1	t_5	t_4	$\emptyset$	t_9
t_3	t_3	t_2	t_1	$\emptyset$	t_6	$\emptyset$	t_4	t_{10}
t_4	t_4	$\emptyset$	t_5	t_6	$\emptyset$	t_2	t_3	t_{11}
t_5	t_5	t_6	t_4	$\emptyset$	t_2	$\emptyset$	t_1	t_{12}
t_6	t_6	t_5	$\emptyset$	t_4	t_3	t_1	$\emptyset$	$\emptyset$
t_7	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	$\emptyset$	$\emptyset$

where

$$t_8 = (c_{15}c_{24}c_{36}),$$

$$t_9 = (a_3b_5c_{35}),$$

$$t_{10} = (a_1b_6c_{16}),$$

$$t_{11} = (c_{15}c_{26}c_{34}),$$

$$t_{12} = (a_2b_5c_{25}).$$

Again we have at least 18 Eckardt points that correspond to the triples $t_j, j = 1, \ldots, 18$ where

$$t_{13} = t_2 t_8 = (a_6 b_2 c_{26}), \qquad t_{14} = t_3 t_8 = (c_{13} c_{25} c_{46}),$$

$$t_{15} = t_4 t_8 = (a_5 b_6 c_{56}), \qquad t_{16} = t_6 t_9 = (c_{16} c_{23} c_{45}),$$

$$t_{17} = t_4 t_{10} = (a_6 b_3 c_{36}), \qquad t_{18} = t_6 t_{10} = (c_{12} c_{35} c_{46}).$$

let us consider the case(ii) for $t_7 = (c_{15} c_{24} c_{36})$. In this case, we have the symmetric Table 4.22.

Table 4.22: The symmetric table 16

	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$	t_6	t_5	t_8
t_2	t_2	t_3	$\emptyset$	t_1	t_5	t_4	$\emptyset$	t_9
t_3	t_3	t_2	t_1	$\emptyset$	t_6	$\emptyset$	t_4	t_{10}
t_4	t_4	$\emptyset$	t_5	t_6	$\emptyset$	t_2	t_3	t_{11}
t_5	t_5	t_6	t_4	$\emptyset$	t_2	$\emptyset$	t_1	t_{12}
t_6	t_6	t_5	$\emptyset$	t_4	t_3	t_1	$\emptyset$	$\emptyset$
t_7	t_7	t_8	t_9	t_{10}	t_{11}	t_{12}	$\emptyset$	$\emptyset$

where

$$t_8 = (a_5 b_4 c_{45}),$$

$$t_9 = (a_6 b_2 c_{26}),$$

$$t_{10} = (c_{13} c_{25} c_{46}),$$

$$t_{11} = (a_5 b_6 c_{56}),$$

$$t_{12} = (a_2 b_5 c_{25}).$$

Again we have at least 18 Eckardt points determined by the triples $t_j, j = 1, \ldots, 18$ where

$$t_{13} = t_2 t_8 = (a_3 b_5 c_{35}), \qquad t_{14} = t_3 t_8 = (a_1 b_6 c_{16}),$$

$$t_{15} = t_4 t_8 = (c_{15} c_{26} c_{34}), \qquad t_{16} = t_6 t_{13} = (c_{16} c_{23} c_{45}),$$

$$t_{17} = t_4 t_{14} = (a_6 b_3 c_{36}), \qquad t_{18} = t_6 t_{14} = (c_{12} c_{35} c_{46}).$$

Note that we can transform from one case to another via some exchanges in triples so that the two cases represent the same kind:

$$\text{form for the case (i)} \xrightarrow{T:=T_1, t:=t_4, t_7:=t_{14}} \text{form for the case (ii)}$$
$$\text{form for the case (i)} \xleftarrow{T:=T_1, t:=t_4, t_7:=t_{13}} \text{form for the case (ii)}$$

Consequently, we also have 42 triads for case (ii). In fact, the 42 triads which correspond to case (i) are:

$$T_1 = \{t_1, t_2, t_3\}, \qquad T_2 = \{t_1, t_5, t_6\}, \qquad T_3 = \{t_1, t_7, t_8\},$$

$$T_4 = \{t_1, t_9, t_{14}\}, \qquad T_5 = \{t_1, t_{10}, t_{13}\}, \qquad T_6 = \{t_1, t_{11}, t_{15}\},$$

$$T_7 = \{t_1, t_{16}, t_{18}\}, \qquad T_8 = \{t_2, t_4, t_5\}, \qquad T_9 = \{t_2, t_7, t_9\},$$

$$T_{10} = \{t_2, t_8, t_{13}\}, \qquad T_{11} = \{t_2, t_{10}, t_{14}\}, \qquad T_{12} = \{t_2, t_{11}, t_{17}\},$$

$$T_{13} = \{t_2, t_{12}, t_{16}\}, \qquad T_{14} = \{t_3, t_4, t_6\}, \qquad T_{15} = \{t_3, t_7, t_{10}\},$$

$$T_{16} = \{t_3, t_8, t_{14}\}, \qquad T_{17} = \{t_3, t_9, t_{13}\}, \qquad T_{18} = \{t_3, t_{12}, t_{18}\},$$

$$T_{19} = \{t_3, t_{15}, t_{17}\}, \qquad T_{20} = \{t_4, t_7, t_{11}\}, \qquad T_{21} = \{t_4, t_8, t_{15}\},$$

$$T_{22} = \{t_4, t_9, t_{16}\}, \qquad T_{23} = \{t_4, t_{12}, t_{17}\}, \qquad T_{24} = \{t_4, t_{14}, t_{18}\},$$

$$T_{25} = \{t_5, t_7, t_{12}\}, \qquad T_{26} = \{t_5, t_9, t_{17}\}, \qquad T_{27} = \{t_5, t_{10}, t_{18}\},$$

$$T_{28} = \{t_5, t_{11}, t_{16}\}, \qquad T_{29} = \{t_5, t_{13}, t_{15}\}, \qquad T_{30} = \{t_6, t_8, t_{12}\},$$

$$T_{31} = \{t_6, t_{10}, t_{11}\}, \qquad T_{32} = \{t_6, t_{13}, t_{16}\}, \qquad T_{33} = \{t_6, t_{14}, t_{17}\},$$

$$T_{34} = \{t_6, t_{15}, t_{18}\}, \qquad T_{35} = \{t_7, t_{13}, t_{14}\}, \qquad T_{36} = \{t_7, t_{16}, t_{17}\},$$

$$T_{37} = \{t_8, t_9, t_{10}\}, \qquad T_{38} = \{t_8, t_{17}, t_{18}\}, \qquad T_{39} = \{t_9, t_{11}, t_{12}\},$$

$$T_{40} = \{t_{10}, t_{15}, t_{16}\}, \qquad T_{41} = \{t_{11}, t_{13}, t_{18}\}, \qquad T_{42} = \{t_{12}, t_{14}, t_{15}\}.$$

Furthermore, the triads $T_5, T_9, T_{16}, T_{23}, T_{28}$, and T_{34} contain all the 27 lines on $\mathscr{S}$.

□

Remark 4.7. Let $\mathbb{E}^{(10)}$ and $\mathbb{E}^{(18)}$ denote respectively the subsets of $\mathbb{E}^{(10,10)}$ and $\mathbb{E}^{(18,10)}$ that correspond to the non-singular cubic surfaces of kind I and II of Theorem 4.4 respectively. Note that according to Theorem 4.4, the two sets $\mathbb{E}^{(10,10)}$ and $\mathbb{E}^{(18,10)}$ are subsets of $\mathbb{E}^{(6,4)}$. A non-singular cubic surface $\mathscr{S}$, which corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(10)}$, has exactly 10 Eckardt points determined by the triples t_i; $i \in \{1, \dots, 10\}$ which contain 15 lines among the 27 lines on cubic surface, namely

$$a_1, a_2, a_3, a_4, b_1, b_2, b_2, b_4, c_{12}, c_{13}, c_{14}, c_{23}, c_{24}, c_{34}, \text{ and } c_{56},$$

and each t_i belongs to exactly three triads. On the other hand, a non-singular cubic surface $\mathscr{S}$, which corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(18)}$, has exactly 18 Eckardt points determined by the triples $t_i; i \in \{1, \dots, 18\}$. In fact, the triples $t_i; i \in \{1, \dots, 18\}$ contain all the 27 lines on cubic surface, and each t_i belongs to exactly seven triads. All the detail are shown in the following corollary.

Corollary 4.10. The non-singular cubic surfaces that corresponds to members in $\mathbb{E}^{(10)}$ and $\mathbb{E}^{(18)}$ have exactly 10 and 18 Eckardt points respectively.

Proof. According to Theorem 4.4 part I, a non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(10)}$ has at least 10 Eckardt points. Up to permutations and quadratic transformations, the Eckardt points on $\mathscr{S}$ are precisely the points that are associated to the triples t_i for $i \in \{1, \ldots, 10\}$, where

$$t_1 = (a_2 b_1 c_{12}),$$

$$t_2 = (a_4 b_3 c_{34}),$$

$$t_3 = (c_{14} c_{23} c_{56}),$$

$$t_4 = (a_3 b_1 c_{13}),$$

$$t_5 = (a_1 b_4 c_{14}),$$

$$t_6 = (a_4 b_2 c_{24}),$$

$$t_7 = (a_2 b_3 c_{23}),$$

$$t_8 = (a_1 b_2 c_{12}),$$

$$t_9 = (c_{13} c_{24} c_{56}),$$

$$t_{10} = (a_3 b_4 c_{34}).$$

Consequently, we have the symmetric table for the triples t_i; $i \in \{1, \ldots, 10\}$ (see Table 4.23).

Table 4.23: The symmetric table 17

	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5	t_6	t_7	t_8	t_9	t_{10}
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$	t_6	t_5	$\emptyset$	$\emptyset$	t_{10}	t_9
t_2	t_2	t_3	$\emptyset$	t_1	t_5	t_4	$\emptyset$	$\emptyset$	t_9	t_8	$\emptyset$
t_3	t_3	t_2	t_1	$\emptyset$	t_6	$\emptyset$	t_4	$\emptyset$	t_{10}	$\emptyset$	t_8
t_4	t_4	$\emptyset$	t_5	t_6	$\emptyset$	t_2	t_3	t_8	t_7	$\emptyset$	$\emptyset$
t_5	t_5	t_6	t_4	$\emptyset$	t_2	$\emptyset$	t_1	t_9	$\emptyset$	t_7	$\emptyset$
t_6	t_6	t_5	$\emptyset$	t_4	t_3	t_1	$\emptyset$	t_{10}	$\emptyset$	$\emptyset$	t_7
t_7	t_7	$\emptyset$	$\emptyset$	$\emptyset$	t_8	t_9	t_{10}	$\emptyset$	t_4	t_5	t_6
t_8	t_8	$\emptyset$	t_9	t_{10}	t_7	$\emptyset$	$\emptyset$	t_4	$\emptyset$	t_{10}	t_9
t_9	t_9	t_{10}	t_8	$\emptyset$	$\emptyset$	t_7	$\emptyset$	t_5	t_{10}	$\emptyset$	t_8
t_{10}	t_{10}	t_9	$\emptyset$	t_8	$\emptyset$	$\emptyset$	t_7	t_6	t_9	t_8	$\emptyset$

It follows that there are no new possible triples in the table which give us new Eckardt points. Therefore, $\mathscr{S}$ has exactly 10 Eckardt points which correspond to the triples $t_1, t_2, t_3, t_4, t_5, t_6, t_7, t_8, t_9$ and t_{10}.

Similarly, according to Theorem 4.4 part II, a non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(18)}$ has at least 18 Eckardt points. Up to permutations and quadratic transformations, the Eckardt points on $\mathscr{S}$ are precisely the points that are associated to the triples t_i; $i \in \{1, \ldots, 18\}$ where

$$
\begin{aligned}
t_1 &= (a_2b_1c_{12}), & t_2 &= (a_4b_3c_{34}),\\
t_3 &= (c_{14}c_{23}c_{56}), & t_4 &= (a_3b_1c_{13}),\\
t_5 &= (a_1b_4c_{14}), & t_6 &= (a_4b_2c_{24}),\\
t_7 &= (a_5b_4c_{45}), & t_8 &= (c_{15}c_{24}c_{36}),\\
t_9 &= (a_3b_5c_{35}), & t_{10} &= (a_1b_6c_{16}),\\
t_{11} &= (c_{15}c_{26}c_{34}), & t_{12} &= (a_2b_5c_{25}),\\
t_{13} &= t_2t_8 = (a_6b_2c_{26}), & t_{14} &= t_3t_8 = (c_{13}c_{25}c_{46}),\\
t_{15} &= t_4t_8 = (a_5b_6c_{56}), & t_{16} &= t_6t_9 = (c_{16}c_{23}c_{45}),\\
t_{17} &= t_4t_{10} = (a_6b_3c_{36}), & t_{18} &= t_6t_{10} = (c_{12}c_{35}c_{46}).
\end{aligned}
$$

When we construct the symmetric table for the triples t_i; $i \in \{1, \ldots, 18\}$, we are not have any new possible triples in the table. Consequently, there are no new Eckardt points on the cubic surface $\mathscr{S}$. Therefore, $\mathscr{S}$ has exactly 18 Eckardt points which correspond to the triples t_i; $i \in \{1, \ldots, 18\}$.

□

Corollary 4.11. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(10)}(q)$ corresponds to $c(\mathscr{S}^{(10)}) \in \mathbb{E}^{(10)}$ is of type $[(q-10)^2 - 1, 27(q-8) + 3, 105, 10]$.

Proof. Recall that $\mathscr{S}^{(10)}(q)$ denotes a non-singular cubic surface with 27 lines that has exactly ten Eckardt points over the Galois field $GF(q)$. By the same argument used in the proof of Corollary 4.2, we have

$$n_q = e_3 + e_2 + e_1,$$

$$3e_3 + e_2 = 135,$$

$$3e_3 + 2e_2 + e_1 = 27(q+1),$$

and

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1.$$

Thus

$$e_3 = 10,$$

$$e_2 = 135 - 30 = 105,$$

$$e_1 = 27(q-4) - 105 = 27(q-8) + 3,$$

$$e_0 = (q-10)^2 - 1.$$

□

Corollary 4.12. For $q > 7$ and q prime, every non-singular cubic surface $\mathscr{S}^{(18)}(q)$ corresponds to $c(\mathscr{S}^{(18)}) \in \mathbb{E}^{(18)}$ is of type $[(q-10)^2 - 9, 27(q-7), 81, 18]$.

Proof. Recall that $\mathscr{S}^{(18)}(q)$ denotes a non-singular cubic surface with 27 lines that has exactly eighteen Eckardt points over the Galois field $GF(q)$. By the same argument used in the proof of Corollary 4.2, we have

$$n_q = e_3 + e_2 + e_1,$$

$$3e_3 + e_2 = 135,$$

$$3e_3 + 2e_2 + e_1 = 27(q+1),$$

and

$$e_3 + e_2 + e_1 + e_0 = q^2 + 7q + 1.$$

Thus

$$\begin{aligned}
e_3 &= 18,\\
e_2 &= 135 - 54 = 81,\\
e_1 &= 27(q-4) - 81 = 27(q-7),\\
e_0 &= (q-10)^2 - 9.
\end{aligned}$$

□

Proposition 4.7. Let $\mathscr{S}$ be a cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(4)}$ with 4 triples, namely the set $T \vee t$ mentioned in Corollary 4.4. Let $t' \in \mathbb{T}^{(3)}$ be another triple on $\mathscr{S}$ all of whose lines are in common with T. Then $c(\mathscr{S}) \in \mathbb{E}^{(9,4)}$.

Proof. From Corollary 4.4, we have

$$\begin{aligned}
t_1 &:= (a_2 b_1 c_{12}),\\
t_2 &:= (a_3 b_4 c_{34}),\\
t_3 &:= (c_{13} c_{24} c_{56}),\\
t_4 &:= t = (a_3 b_1 c_{13}).
\end{aligned}$$

Up to the permutation $\sigma = (34)$, the triad $T = \{t_1, t_2, t_3\}$ transforms to

$$\begin{aligned}
t_1 &:= (a_2 b_1 c_{12}),\\
t_2 &:= (a_4 b_3 c_{34}),\\
t_3 &:= (c_{14} c_{23} c_{56}),
\end{aligned}$$

and the triple $t := t_4 = (a_3 b_1 c_{13})$ transforms to $t := t_4 = (a_4 b_1 c_{14})$. Again, up permutations and quadratic transformations, we can assume that $t' := (c_{12} c_{34} c_{56}) = t_5$. Let us construct the corresponding symmetric table for the triples t_1, t_2, t_3, t_4, t_5 (see Table 4.24).

Table 4.24: The symmetric table 18

	$\emptyset$	t_1	t_2	t_3	t_4	t_5
$\emptyset$	$\emptyset$	t_1	t_2	t_3	t_4	t_5
t_1	t_1	$\emptyset$	t_3	t_2	$\emptyset$	$\emptyset$
t_2	t_2	t_3	$\emptyset$	t_1	$\emptyset$	$\emptyset$
t_3	t_3	t_2	t_1	$\emptyset$	$\emptyset$	$\emptyset$
t_4	t_4	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	t_6
t_5	t_5	$\emptyset$	$\emptyset$	$\emptyset$	t_6	$\emptyset$

where $t_6 = (a_2b_3c_{23})$. Now if there is another triple $t' \in \mathbb{T}^{(3)}$ on $\mathscr{S}$ whose all lines in common with triad T then $\mathscr{S}$ has at least 6 Eckardt points that correspond to the triples t_i; $i \in \{1, \ldots, 6\}$. However, $c(\mathscr{S}) \in \mathbb{S}^{(4)}$ and has at least 6 Eckardt points. This implies that $c(\mathscr{S}) \in \mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)}$ (see the argument before Definition 4.6)

If $c(\mathscr{S}) \in \mathbb{E}^{(9,4)}$ then we are done. Otherwise, it is enough to show that there is another triple t_7 on $\mathscr{S}$ and hence $c(\mathscr{S}) \in \mathbb{E}^{(9,4)}$. By change of coordinates system over the complex field, we can assume that

$$P_1 = (1:0:0),\ P_2 = (0:1:0),\ P_3 = (1:1:1) \text{ and } P_4 = (0:0:1).$$

A little algebraic computations shows that the conic, namely $\mathscr{C}_1$, which pass through the points P_2, P_3, P_4, has the form:

$$\mathscr{C}_1 = \mathbb{V}(x_0^2 - x_1x_2).$$

The above conic has two tangents at the points P_2 and P_4, namely

$$l_{12} = \mathbb{V}(x_2) \text{ and } l_{14} = \mathbb{V}(x_1),$$

respectively. Now if the two points, namely $P_5 := (\alpha_5 : \beta_5 : \gamma_5)$ and $P_6 := (\alpha_6 : \beta_6 : \gamma_6)$, belong to the conic $\mathscr{C}_1$ then $\alpha_5^2 = \beta_5\gamma_5$ and $\alpha_6^2 = \beta_6\gamma_6$. It follows that

$$P_5 := (1 : \zeta : \overline{\zeta}) \text{ and } P_6 := (1 : \overline{\zeta} : \zeta),$$

or

$$P_5 := (1 : \overline{\zeta} : \zeta) \text{ and } P_6 := (1 : \zeta : \overline{\zeta}),$$

where ζ is a primitive cubic root of unity. Now consider the following lines:

$$l_{34} = \mathbb{V}(x_0 - x_1), \qquad l_{23} = \mathbb{V}(x_0 - x_2),$$
$$l_{16} = \mathbb{V}(x_1 - \bar{\zeta}x_2), \qquad l_{35} = \mathbb{V}((\zeta - \bar{\zeta})x_0 - (\zeta - 1)x_1 + (\bar{\zeta} - 1)x_2),$$
$$l_{24} = \mathbb{V}(x_0).$$

Note that

$$P_7 := (1 : 1 : 0) \in l_{12} \cap l_{34} \text{ and } P_8 := (1 : 0 : 1) \in l_{14} \cap l_{23}.$$

Moreover, $l_{78} = \mathbb{V}(x_0 - x_1 - x_2)$ and $l_{16} \cap l_{24} \cap l_{35} = \{P_9\}$ (see Figure 4.9).

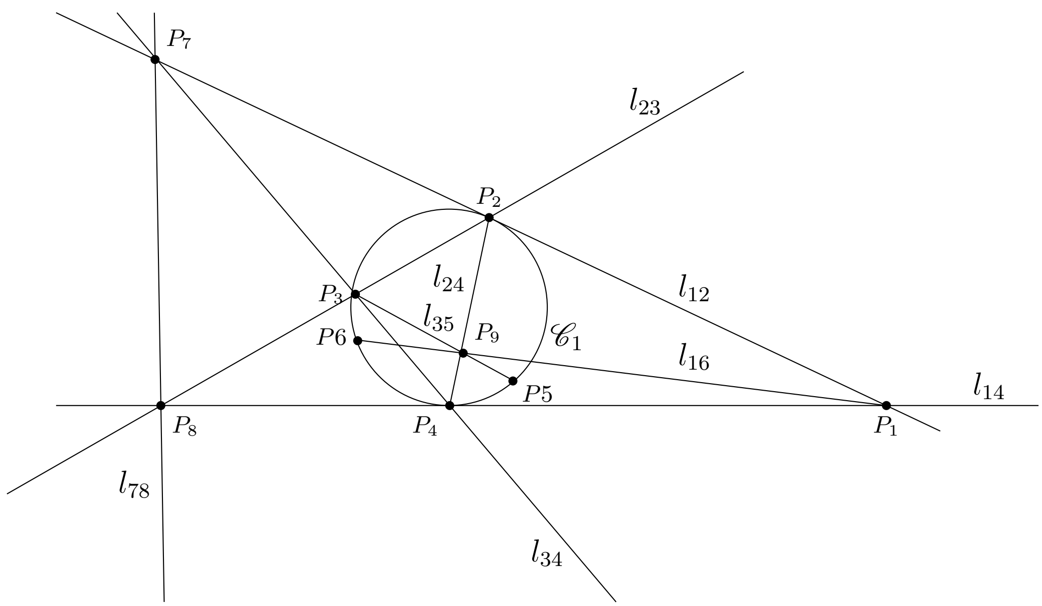

Figure 4.9: $l_{16} \cap l_{24} \cap l_{35} = \{P_9\}$.

So we can take $t_7 := (c_{16}c_{24}c_{35})$ where $c_{16} = \widetilde{l_{16}}, c_{24} = \widetilde{l_{24}}$ and $c_{35} = \widetilde{l_{35}}$. □

Proposition 4.8. $\mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)} = \mathbb{S}^{(6)} = \mathbb{S}^{(5)}$.

Proof. According to Corollary 4.6, every non-singular cubic surface $\mathscr{S}$ with 27 lines, which corresponds to $c(\mathscr{S}) \in \mathbb{E}^{(6,4)}$, has 6 Eckardt points. Also according to Proposition 4.6, every non-singular cubic surface $\mathscr{S}$ with 27 lines, which corresponds to

$c(\mathscr{S}) \in \mathbb{E}^{(9,4)}$, has 9 Eckardt points. Moreover, $\mathbb{S}^{(6)} \subset \mathbb{S}^{(5)} \subseteq \mathbb{S}^{(4)}$. Therefore, we have

$$\mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)} \subset \mathbb{S}^{(6)} \subset \mathbb{S}^{(5)} \subseteq \mathbb{S}^{(4)}.$$

If $\mathscr{S}$ is a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(5)} \subseteq \mathbb{S}^{(4)}$, then $\mathscr{S}$ has 4 triples $T \vee t$ on it (see Definition 4.6). On the other hand, $\mathscr{S}$ has another triple, namely t'. Now if t' has one or no common line with triad T then $c(\mathscr{S}) \in \mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)}$ and we are done. Otherwise, all lines of t' are in common with T and hence $c(\mathscr{S}) \in \mathbb{E}^{(9,4)} \subseteq \mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)}$ (see Proposition 4.7). Therefore

$$\mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)} \subseteq \mathbb{S}^{(6)} \subseteq \mathbb{S}^{(5)} \subseteq \mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)}.$$

□

Proposition 4.9. $\mathbb{E}^{(9,4)} \cup \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)} = \mathbb{S}^{(7)} = \mathbb{S}^{(8)} = \mathbb{S}^{(9)}$.

Proof. It is clear that

$$\mathbb{S}^{(9)} \subseteq \mathbb{S}^{(8)} \subseteq \mathbb{S}^{(7)} \subseteq \mathbb{S}^{(6)}.$$

Now according to Proposition 4.8, we have $\mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)} = \mathbb{S}^{(6)}$. It follows that

$$\mathbb{S}^{(7)} \subseteq \mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)}.$$

Consequently, we get

$$\begin{aligned}\mathbb{S}^{(7)} &= \mathbb{S}^{(7)} \cap (\mathbb{E}^{(6,4)} \cup \mathbb{E}^{(9,4)}) \\ &= (\mathbb{S}^{(7)} \cap \mathbb{E}^{(6,4)}) \cup (\mathbb{S}^{(7)} \cap \mathbb{E}^{(9,4)}) \\ &= (\mathbb{S}^{(7)} \cap \mathbb{E}^{(6,4)}) \cup \mathbb{E}^{(9,4)},\end{aligned}$$

since $\mathbb{E}^{(9,4)} \subseteq \mathbb{S}^{(7)}$. On the other hand, let $\mathscr{S}$ be any non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(7)} \cap \mathbb{E}^{(6,4)}$. Then $c(\mathscr{S}) \in \mathbb{E}^{(6,4)}$ and has 6 triples $t_1, t_2, t_3, t_4, t_5, t_6$. Recall that Theorem 4.4 says that if $\mathscr{S}$ has a triple $t_7 \in$

$\mathbb{T}^{(3)} \backslash \{t_1, t_2, t_3, t_4, t_5, t_6\}$, then $c(\mathscr{S}) \in \mathbb{E}^{(10,10)}$ or $c(\mathscr{S}) \in \mathbb{E}^{(18,10)}$. So $c(\mathscr{S}) \in \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}$. It follows that

$$\mathbb{S}^{(7)} \cap \mathbb{E}^{(6,4)} \subseteq \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}.$$

Consequently, we get

$$(\mathbb{S}^{(7)} \cap \mathbb{E}^{(6,4)}) \cup \mathbb{E}^{(9,4)} \subseteq \mathbb{E}^{(9,4)} \cup \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}.$$

Thus

$$\begin{aligned} \mathbb{S}^{(7)} &\subseteq (\mathbb{S}^{(7)} \cap \mathbb{E}^{(6,4)}) \cup \mathbb{E}^{(9,4)} \subseteq \mathbb{E}^{(9,4)} \cup \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)} \\ &\subseteq \mathbb{S}^{(9)} \subseteq \mathbb{S}^{(8)} \subseteq \mathbb{S}^{(7)}. \end{aligned}$$

□

Proposition 4.10. $\mathbb{S}^{(10)} = \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}$.

Proof. Assume that $c(\mathscr{S}) \in \mathbb{S}^{(10)}$. By the definition of $\mathbb{S}^{(9)}$, we get $c(\mathscr{S}) \in \mathbb{S}^{(9)}$. Now according to Proposition 4.9, we have $c(\mathscr{S}) \in \mathbb{E}^{(9,4)} \cup \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}$. It follows that

$$\mathbb{S}^{(10)} \subseteq \mathbb{E}^{(9,4)} \cup \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}.$$

Consequently, we get

$$\begin{aligned} \mathbb{S}^{(10)} &= \mathbb{S}^{(10)} \cap (\mathbb{E}^{(9,4)} \cup \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}) \\ &= (\mathbb{S}^{(10)} \cap \mathbb{E}^{(9,4)}) \cup (\mathbb{S}^{(10)} \cap \mathbb{E}^{(10,10)}) \cup (\mathbb{S}^{(10)} \cap \mathbb{E}^{(18,10)}) \\ &= (\mathbb{S}^{(10)} \cap \mathbb{E}^{(9,4)}) \cup (\mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}) \text{ since } \mathbb{E}^{(10,10)}, \mathbb{E}^{(18,10)} \subseteq \mathbb{S}^{(10)}. \end{aligned}$$

Now if $c(\mathscr{S}) \in \mathbb{S}^{(10)} \cap \mathbb{E}^{(9,4)}$ then the corresponding non-singular cubic surface $\mathscr{S}$ has 9 Eckardt points which correspond to the triples t_i; $i \in \{1, \ldots, 9\}$. Up to permutations and quadratic transformations, we can assume that (see Proposition 4.6)

$$t_1 = (c_{12}c_{34}c_{56}),$$
$$t_2 = (c_{15}c_{24}c_{36}),$$
$$t_3 = (c_{13}c_{26}c_{45}),$$
$$t_4 = (a_4b_1c_{14}),$$
$$t_5 = (a_2b_3c_{23}),$$
$$t_6 = (a_5b_2c_{25}),$$
$$t_7 = (a_3b_5c_{35}),$$
$$t_8 = (a_1b_6c_{16}),$$
$$t_9 = (a_6b_4c_{46}).$$

Moreover, the previous 9 triples contain all the 27 lines on $\mathscr{S}$ (see Remark 4.6). However, if $c(\mathscr{S}) \in \mathbb{S}^{(10)}$ then there is another Eckardt point which corresponds to some triple on $\mathscr{S}$, namely $t_{10} \in \mathbb{T}^{(3)} \backslash \{t_1, \ldots, t_9\}$. So all lines of t_{10} must belong to some triad among the 12 triads on $\mathscr{S}$. Without loss of generality, we can assume all the line of t_{10} are in common with the triad $T_5 = \{t_1, t_2, t_3\}$ (see Proposition 4.6). Up to permutations and quadratic transformations, we can take $t_{10} = (c_{12}c_{36}c_{45})$. Now if we consider the triad $T_1 = \{t_1, t_4, t_5\}$ in Proposition 4.6, then we get that t_{10} has only one common line with T_1. Consequently, according to Corollary 4.6, the set $T_1 \vee t_{10}$ will generate 6 Eckardt points on $\mathscr{S}$ and 4 triads so that T_1 is one of them. Hence $c(\mathscr{S}) \in \mathbb{E}^{(6,4)}$. On the other hand, the lines of the triple t_2 are not all in common with T_1. Therefore, according to Theorem 4.4(II), we deduce that the set of triples $T_1 \vee t_{10} \vee t_2$ generate the 18 Eckardt points on $\mathscr{S}$ and hence $c(\mathscr{S}) \in \mathbb{E}^{(18,10)}$. It follows that

$$\mathbb{S}^{(10)} \subseteq \mathbb{E}^{(18,10)} \cup \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)} = \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)} \subseteq \mathbb{S}^{(10)}.$$

$\square$

Proposition 4.11. $\mathbb{S}^{(11)} = \mathbb{S}^{(12)} = \mathbb{S}^{(13)} = \mathbb{S}^{(14)} = \mathbb{S}^{(15)} = \mathbb{S}^{(16)} = \mathbb{S}^{(17)} = \mathbb{S}^{(18)} = \mathbb{E}^{(18,10)}$.

Proof. It obvious that

$$\mathbb{E}^{(18,10)} \subseteq \mathbb{S}^{(18)} \subseteq \mathbb{S}^{(17)} \subseteq \mathbb{S}^{(16)} \subseteq \mathbb{S}^{(15)} \subseteq \mathbb{S}^{(14)} \subseteq \mathbb{S}^{(13)} \subseteq \mathbb{S}^{(12)} \subseteq \mathbb{S}^{(11)} \subseteq \mathbb{S}^{(10)}.$$

Now according to Proposition 4.10, we get

$$\mathbb{S}^{(10)} = \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}.$$

More precisely, we have $\mathbb{S}^{(11)} \subseteq \mathbb{E}^{(10,10)} \cup \mathbb{E}^{(18,10)}$. However, $\mathbb{S}^{(11)} \not\subseteq \mathbb{E}^{(10,10)}$ and $\mathbb{E}^{(10,10)} \cap \mathbb{E}^{(18,10)} = \emptyset$. It follows that $\mathbb{S}^{(11)} \subseteq \mathbb{E}^{(18,10)}$. Thus

$$\mathbb{E}^{(18,10)} \subseteq \mathbb{S}^{(18)} \subseteq \mathbb{S}^{(17)} \subseteq \mathbb{S}^{(16)} \subseteq \mathbb{S}^{(15)} \subseteq \mathbb{S}^{(14)} \subseteq \mathbb{S}^{(13)} \subseteq \mathbb{S}^{(12)} \subseteq \mathbb{S}^{(11)} \subseteq \mathbb{E}^{(18,10)}.$$

So we can replace the inclusion relation by equality. □

Corollary 4.13. For every $k > 18$, we have $\mathbb{S}^{(k)} = \emptyset$.

Proof. Let $\mathscr{S}$ be a non-singular cubic surface that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(18)}$. We know that $\mathscr{S}$ has 18 Eckardt points which correspond to the 18 triples $t_i; i \in \{1, \ldots, 18\}$ [see Theorem 4.4(II)]. Furthermore, applying our operation on the previous triples does not give any new triples.

Now if there is a non-singular cubic surface $\mathscr{S}$ that corresponds to $c(\mathscr{S}) \in \mathbb{S}^{(k)}$ for some $k > 18$, then

$$\mathbb{S}^{(k)} \subseteq \mathbb{S}^{(k-1)} \subseteq \ldots \subseteq \mathbb{S}^{(18)} = \mathbb{E}^{(18,10)}$$

which is a contradiction.

□

The following diagram (see Figure 4.10) illustrates most properties of the classes $\mathbb{S}^{(k)}$ and $\mathbb{E}^{(m,k)}$, where the end of dotted arrows is the union of their beginnings. The hook line represents the inclusion relation while the double line refers to equality relation.

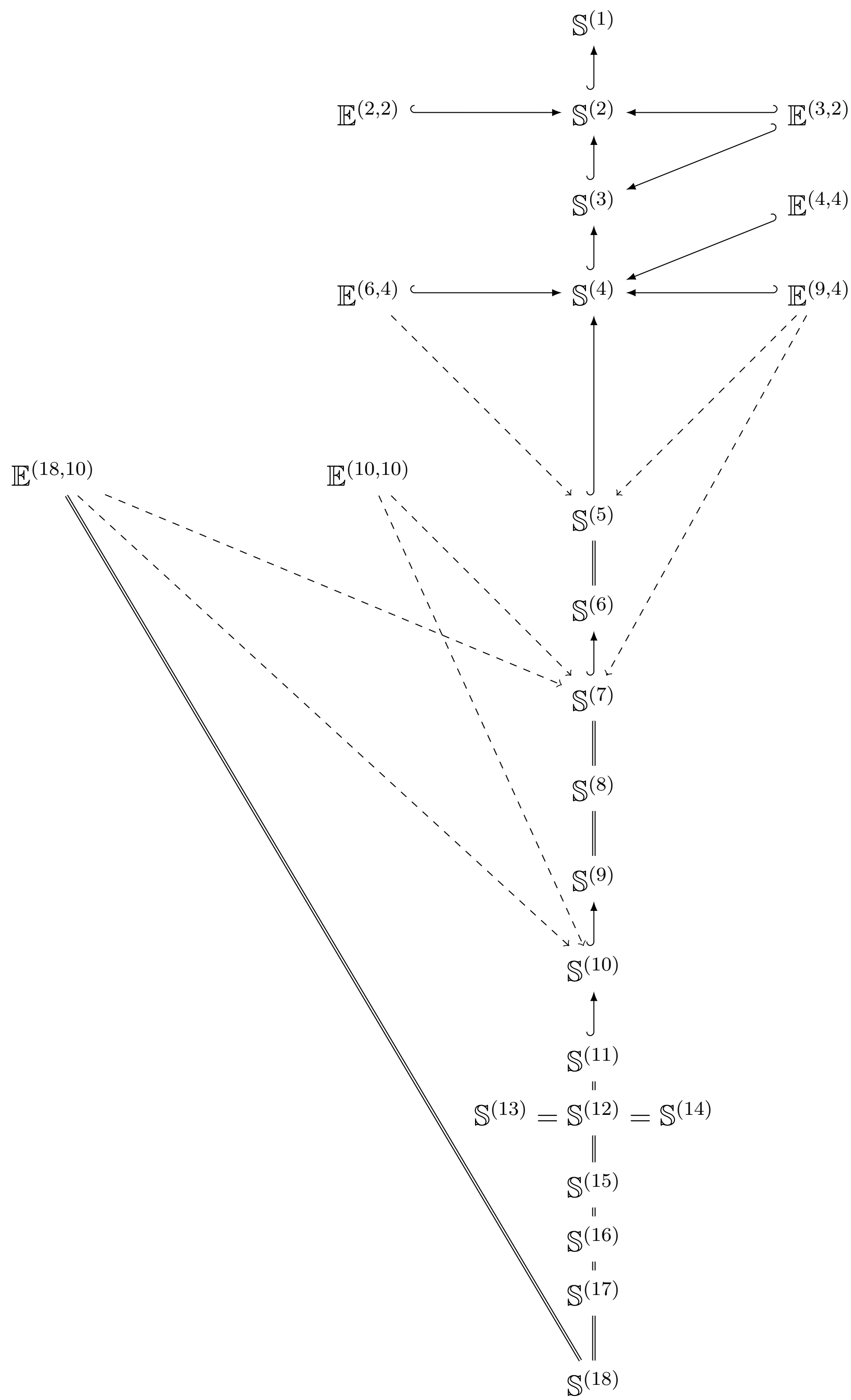

Figure 4.10: The main diagram for $\mathbb{S}^{(k)}$ and $\mathbb{E}^{(m,k)}$.

4.3 Irreducibility and codimension of classes of smooth cubic surfaces over the complex field

In this section, we discuss the irreducibility and codimension of classes of smooth cubic surfaces as a subvarieties of $\mathbb{P}^{19}_{\mathbb{C}}$. Let $\varphi : X \to Y$ be a morphism of affine algebraic sets [see [25], Page 80], defined over an algebraically closed field k. For $y \in Y$, the set $\varphi^{-1}(y)$ is called the fiber over y. Recall from ([13], Page 23) that X is an affine variety if it is an irreducible closed subset of $\mathbb{A}^n$, and a morphism $\varphi : X \to Y$ between two varieties is said to be dominant if the image of φ is dense in Y.

Theorem 4.5. ([5], Page 62) Let $\varphi : X \to Y$ be a finite dominant morphism of affine varieties. Then

1. φ is surjective.
2. If $y \in Y$, then the fiber $\varphi^{-1}(y)$ is finite.
3. Let $Z \subset X$ be a closed subvariety. Then $\varphi(Z)$ is closed, $\dim(Z) = \dim(\varphi(Z))$ and $\varphi : Z \to \varphi(Z)$ is finite.

Theorem 4.6. ([5], Page 64) Let $\varphi : X \to Y$ be a dominant morphism of affine varieties. There exists a nonempty open set $\mathcal{U} \subset \varphi(X)$ such that for all $y \in \mathcal{U}$ and any irreducible component Z of $\varphi^{-1}(y)$, $\dim(Z) = r := \dim(X) - \dim(Y)$.

Theorem 4.7. ([29], Page 76) Let $\varphi : X \to Y$ be a regular map between irreducible varieties. Suppose that φ is surjective, $\varphi(X) = Y$, and that $\dim X = n, \dim Y = m$. Then $m \leq n$, and

1. $\dim F \geq n - m$ for any $y \in Y$ and for any component F of the fibre $\varphi^{-1}(y)$;
2. there exists a nonempty open subset $\mathcal{U} \subset Y$ such that $\varphi^{-1}(y) = n - m$ for $y \in \mathcal{U}$.

Recall that $\mathbb{S}^{(k)}$ denotes the set of all smooth cubic surfaces with at least k Eckardt points (as a subvariety of $\mathbb{P}^{19}_{\mathbb{C}}$). It is clear that $\mathbb{S}_6$ is an open subvariety of $(\mathbb{P}^2)^6$ and hence $\dim \mathbb{S}_6 = 12$. Furthermore, $\dim \mathbb{W}_s = 4$. Let $s := \kappa_{123456} \in \mathbb{S}_6$ and assume that $\widehat{w_{1234}}$ forms a basis of $\mathbb{W}_s$, where $\mathbb{W}_s$ is the space of all plane cubics passing through the points of $\widehat{s} := \{P_1, P_2, P_3, P_4, P_5, P_6\}$. Then we have the following birational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^3 : P \mapsto \Big(w_1(P) : w_2(P) : w_3(P) : w_4(P)\Big),$$

which induces a morphism

$$\psi : \mathbb{P}^2 - \widehat{\kappa_{123456}} \longrightarrow \mathbb{P}^3 : P \mapsto \Big(w_1(P) : w_2(P) : w_3(P) : w_4(P)\Big).$$

In this case, we get $\operatorname{im}\psi := \operatorname{blw}_s \mathbb{P}^2 = \mathscr{S}$ which is a non-singular cubic surface with 27 lines. As we said previously, $\mathscr{S}$ corresponds to $c(\mathscr{S}) \in \mathbb{S}_{sm}$. Moreover, the 27 lines are on $\mathscr{S}$:

1. the exceptional curves $\widetilde{P_i} := a_i$ is defined to be the total transform of P_i in $\widehat{s}$,
2. the curve $\widetilde{l_{ij}} := c_{ij}$ is the strict transform of $l_{ij} = \overline{P_i P_j}$,
3. the curve $\widetilde{\mathscr{C}_j} := b_j$ is the strict transform of the conic $\mathscr{C}_j$ passing through all points of $\widehat{s}$ except P_j.

Following ([24], Page 173), if $\mathscr{S}(c) := \mathbb{V}(\sum c_\alpha x^\alpha)$ represents a cubic surface with coefficients $c(\mathscr{S}) = (c_\alpha)$, then we can consider $\mathscr{S}$ as a subvariety of $\mathbb{P}^{19}$. It follows that

$$\mathscr{H} := \bigcup_{c \in \mathbb{P}^{19}} \mathscr{S}(c) \times \{c\} \subseteq \mathbb{P}^3 \times \mathbb{P}^{19}$$

is defined by the equation $\sum z_\alpha x^\alpha = 0$ where z_α denotes the homogeneous coordinates of the projective space $\mathbb{P}^{19}$. Now if $S \subseteq \mathscr{H}$ is the set of points where $p_2 : \mathscr{H} \to \mathbb{P}^{19}_z$ is not smooth or equivalently where the fiber $\mathscr{S}(c)$ is not smooth, then let $\mathbb{S}_{sn} := p_2(S)$. Consequently, $\mathbb{P}^{19}_z - \mathbb{S}_{sn}$ parameterize the smooth cubics. In fact, there is a homogeneous polynomial $d \in k[z_0, \ldots, z_{19}]$ such that $\mathbb{S}_{sn} = \mathbb{V}(d)$. Thus $\mathbb{P}^{19}_z - \mathbb{S}_{sn}$ is an open subset of $\mathbb{P}^{19}_z$ and hence it is irreducible and dense. Therefore, $\dim \mathbb{S}_{sm} = 19$.

Let $P_1 = (1:0:0) = Q_1, P_2 = (0:1:0) = Q_2, P_3 = (0:0:1) = Q_3$, and

$$\pi_{S(P)} \text{ represents the blowing up of } \mathbb{P}^2_X \text{ at } S(P) := \{P_1, P_2, P_3\},$$

$$\pi_{S(Q)} \text{ represents the blowing up of } \mathbb{P}^2_Y \text{ at } S(Q) := \{Q_1, Q_2, Q_3\},$$

$$\mathcal{V} := \operatorname{blw}_{S(P)} \mathbb{P}^2_X,$$

then we have the following proposition.

Proposition 4.12. Let $s = \kappa_{123456} \in \mathbb{S}_6$ and

$$t = \{Q_1, Q_2, Q_3, \varphi_{123}(P_4), \varphi_{123}(P_5), \varphi_{123}(P_6)\}.$$

Then

$$\operatorname{blw}_s \mathbb{P}^2_X \cong \operatorname{blw}_t \mathbb{P}^2_Y.$$

In particular, if $s' \in \mathbb{S}_6$ is obtained from s via quadratic transformation, then

$$\operatorname{blw}_s \mathbb{P}^2_X \cong \operatorname{blw}_{s'} \mathbb{P}^2_X.$$

Proof. Let $\rho_1 : \mathcal{X} \to \mathcal{V}$ represent the blowing up of $\mathcal{V}$ at $\pi^{-1}_{S(P)}(P_4)$, $\pi^{-1}_{S(P)}(P_5)$, $\pi^{-1}_{S(P)}(P_6)$ and let

$$\rho_2 : \mathcal{Y} \to \mathcal{V}$$

represent the blowing up of $\mathcal{V}$ at $\{\pi^{-1}_{S(Q)}(\varphi_{123}(P_4)), \pi^{-1}_{S(Q)}(\varphi_{123}(P_5), \pi^{-1}_{S(Q)}(\varphi_{123}(P_6)\}$.

Since $\pi_{S(Q)} \circ \pi^{-1}_{S(P)} = \varphi_{123}$, we have $\pi^{-1}_{S(P)}(P_k) = \pi^{-1}_{S(Q)}(\varphi_{123}(P_k))$ for $k \in \{4, 5, 6\}$. Thus

$$\mathcal{X} = \operatorname{blw}_s \mathbb{P}^2_X \cong \operatorname{blw}_t \mathbb{P}^2_Y = \mathcal{Y}.$$

See the diagram in Figure 4.11.

□

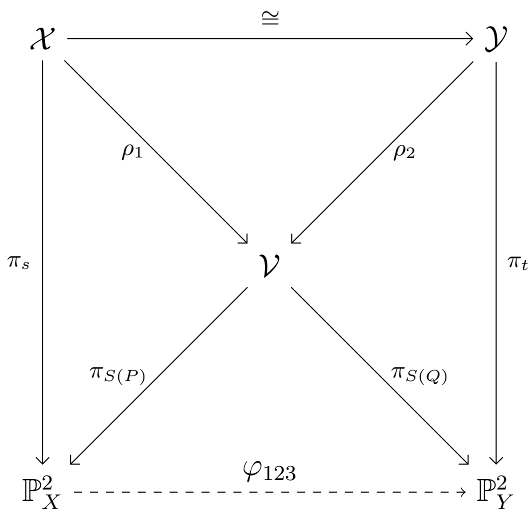

Figure 4.11: $\operatorname{blw}_s \mathbb{P}^2_X \cong \operatorname{blw}_t \mathbb{P}^2_Y$.

Let $s = \kappa_{123456} \in \mathbb{S}_6$ and $E_s = (a_1, \ldots, a_6)$ be the exceptional curves corresponding to s. Let $\mathscr{S} = \operatorname{blw}_s \mathbb{P}^2$. Define

$$\mathbb{B}_1 := \{(s, c(\mathscr{S}), E_s) : s \in \mathbb{S}_6\},$$

$$\mathbb{B}_2 := \{(c(\mathscr{S}), \lambda_{123456}) : l_i \cap l_j = \emptyset\}.$$

Consider the following mappings:

$$\begin{array}{ccccccc} & & (\mathbb{P}^2)^6 \times \mathbb{B}_2 & & \mathbb{P}^{19} \times \mathbb{G}^6_{2,4} & & \\ & & \cup & & \cup & & \\ \mathbb{S}_6 & \xleftarrow{\ \Psi\ } & \mathbb{B}_1 & \xrightarrow{\qquad \Xi \qquad} & \mathbb{B}_2 & \xrightarrow{\ \Gamma\ } & \mathbb{S}_{sm} \\ s & \longleftarrow\!\shortmid & (s, c(\mathscr{S}), E_s) & \longmapsto & (c(\mathscr{S}), \lambda_{123456}) & \longmapsto & c(\mathscr{S}) \end{array}$$

By David Mumford ([24], Page 174) we have $\mathbb{B}_2 \subset \mathbb{P}^{19} \times \mathbb{G}^6_{2,4}$. According to the projection Γ we get the fiber $\Gamma^{-1}(c(\mathscr{S}))$ has 51840 elements which is a finite set, and $\dim \mathbb{B}_2 = 19$. In fact, $\dim \Gamma^{-1}(c(\mathscr{S})) = 0$ and hence

$$\dim \mathbb{B}_2 = \dim \mathbb{S}_{sm} + \dim \Gamma^{-1}(c(\mathscr{S})) = 19 + 0 = 19.$$

Furthermore, the fiber

$$\Psi^{-1}(s) \cong \operatorname{Aut}(\mathbb{P}^3) \cong \operatorname{PGL}(3, k) \cong \operatorname{GL}(4, k)/k^*$$

has dimension equal to $4^2 - 1 = 15$. Similarly, the fiber

$$\Xi^{-1}(c(\mathscr{S}), \lambda_{123456}) \cong \operatorname{Aut}(\mathbb{P}^2) \cong \operatorname{PGL}(2, k) \cong \operatorname{GL}(3, k)/k^*$$

has dimension equal to $3^2 - 1 = 8$.

Assume that $\mathbb{K}$ is a closed subset of $\mathbb{S}_6$. Define the closed subset of $\mathbb{K} \times (\mathbb{P}^9)^4$ as follows:

$$\mathcal{W} := \Big\{(s, w_{1234}) : s \in \mathbb{K}, \widehat{w_{1234}} \subset \mathbb{W}_s \backslash \{\mathbf{0}\}\Big\}.$$

Let us define an open subset $\mathcal{W}^*$ of $\mathcal{W}$ as follows:

$$\mathcal{W}^* := \Big\{w = (s, w_{1234}) \in \mathcal{W} : \widehat{w_{1234}} \text{ forms a basis for } \mathbb{W}_s\Big\},$$

Then we have the following maps:

$$\begin{array}{ccccccc}
(\mathbb{P}^2)^6 \times \mathbb{W}_s & & (\mathbb{P}^2)^6 \times \mathbb{W}_s & & \mathbb{S}_6 \times (\mathbb{P}^9)^4 & & \mathbb{S}_6 \\
\cup & & \cup & & \cup & & \cup \\
\mathcal{W}^* & \xhookrightarrow{\iota_1} & \mathcal{W} & \xhookrightarrow{\iota_2} & \mathbb{K} \times (\mathbb{P}^9)^4 & \xrightarrow{p_{\mathbb{K}}} & \mathbb{K}
\end{array}$$

Where $p_{\mathbb{K}}$ is the projection to the first coordinate, and ι_1,ι_2 represent the embedding maps. The following map, namely

$$\omega : \mathcal{W} \longrightarrow \mathbb{K} : (s, w_{1234}) \overset{\omega}{\longmapsto} s,$$

is surjective and $\omega = p_{\mathbb{K}} \circ \iota_2$. Furthermore, the fiber is $\omega^{-1}(s) \cong (\mathbb{P}^3)^4$ and has dimension equal to 12. It follows that $\mathcal{W}$ is an irreducible closed subset of $\mathbb{K} \times (\mathbb{P}^9)^4$ with

$$\dim \mathcal{W} = \dim \mathbb{K} + 12. \tag{4.3.1}$$

Since $\mathcal{W}^*$ is an irreducible open subset of $\mathcal{W}$, it follows that $\mathcal{W}^*$ is irreducible and dense in $\mathcal{W}$ (i.e, $\overline{\mathcal{W}^*} = \mathcal{W}$) and hence

$$\dim \mathcal{W}^* = \dim \overline{\mathcal{W}^*} = \dim \mathcal{W}.$$

Let $B_1 := \psi^{-1}(\mathbb{K})$, and $B_2 := \xi(B_1)$. Then $\gamma(B_2)$ is a closed subset of $\mathbb{S}_{sm}$ where

$$\xi := \Xi\Big|_{B_1}, \;\; \gamma := \Gamma\Big|_{B_2} \;\; \textit{and} \;\; \psi := \Psi\Big|_{B_1}.$$

Note that for any basis $\widehat{w_{1234}}$ of $\mathbb{W}_s$; $s \in \mathbb{S}_6$ there exists an embedding in $\mathbb{P}^3$ of the blow-up of $\mathbb{P}^2$ at six points in general position, namely the embedding coming from the birational map

$$\mathbb{P}^2 \dashrightarrow \mathbb{P}^3 : P \mapsto (\alpha_1 w_1(P) : \alpha_2 w_2(P) : \alpha_3 w_3(P) : \alpha_4 w_4(P)); \alpha_j \in k^*,$$

where $\{\alpha_1 w_1, \alpha_2 w_2, \alpha_3 w_3, \alpha_4 w_4\}$ forms a basis of $\mathbb{W}_s$. The above birational map induces a morphism, namely

$$\mathbb{P}^2 - s \to \mathbb{P}^3 : P \mapsto (w_1(P) : w_2(P) : w_3(P) : w_4(P)),$$

Thus we deduced that B_1 isomorphic to an open subset $\mathcal{U}$ of $\mathcal{W}^* \times \mathbb{P}^3$ (see Figure 4.12). Hence B_1 is irreducible with

$$\dim B_1 = \dim \overline{\mathcal{U}} = \dim(\mathcal{W}^* \times \mathbb{P}^3) = \dim \mathcal{W} + 3.$$

According to Equation (4.4), we get

$$\dim B_1 = \dim \mathbb{K} + 15. \tag{4.3.2}$$

All the above results are illustrated in the main diagram (Figure 4.12), where

$$\omega : \mathcal{W} \longrightarrow \mathbb{K} : (s, w_{1234}) \overset{\omega}{\longmapsto} s,$$

is the surjective map which is defined previously. In fact, $\omega = p_{\mathbb{K}} \circ \iota_2$.

Lemma 4.3.1. Let $\mathbb{K}^{(1)} := \Big\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(12, 34, 56) \neq \emptyset \Big\}$. Then $\mathbb{K}^{(1)}$ is an irreducible subset of $\mathbb{S}_6$ and $\operatorname{codim} \mathbb{K}^{(1)} = 1$.

Proof. Let $\mathbb{L}_1$ be a subset of $\mathbb{P}^2 \times \mathbb{G}^3_{2,3}$ defined by

$$\mathbb{L}_1 := \{(Q; \lambda_{123}) : \wedge(\lambda_{123}) = Q\}.$$

Let $\mathbb{L}_2$ be the subset of $\mathbb{L}_1 \times (\mathbb{P}^2)^6$ defined by

$$\mathbb{L}_2 := \Big\{ (t; s) : s = \kappa_{123456} \in \mathbb{S}_6,\ t \in \mathbb{L}_1 \text{ such that } \lambda_{123} = (l_{12}, l_{34}, l_{56}) \Big\}.$$

Then we have the following projections:

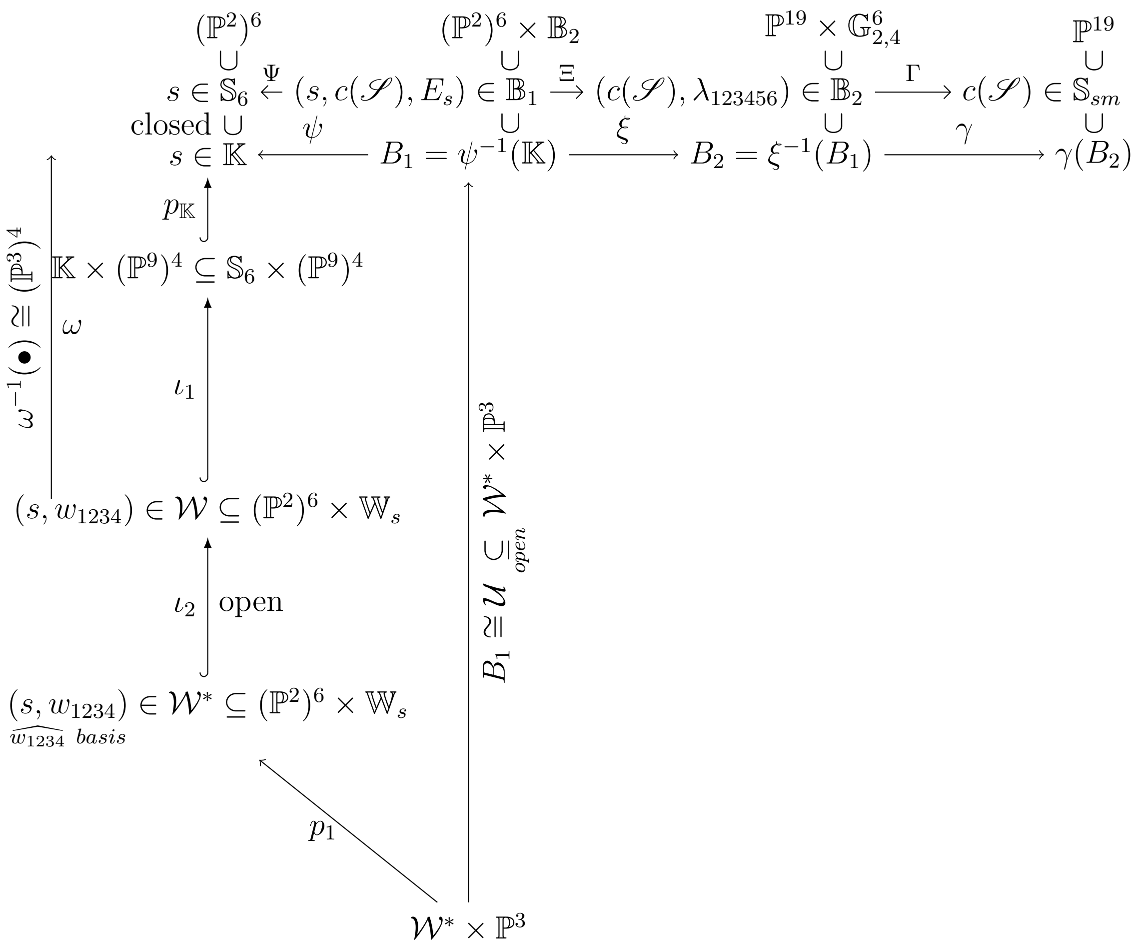

Figure 4.12: Main diagram.

$$\begin{array}{ccccccc}
\mathbb{S}_6 & & \mathbb{L}_1 \times \mathbb{S}_6 & & \mathbb{P}^2 \times \mathbb{G}^3_{2,3} & & \\
\cup & & \cup & & \cup & & \\
\mathbb{K}^{(1)} & \xleftarrow{p_3} & \mathbb{L}_2 & \xrightarrow{p_2} & \mathbb{L}_1 & \xrightarrow{p_1} & \mathbb{P}^2 \\
s & \longleftarrow & (t; s) & \longmapsto & t = (Q; \lambda_{123}) & \longmapsto & Q
\end{array}$$

Let us consider the fiber of each map (see Figure 4.13).

The fiber $p_1^{-1}(Q)$ is an irreducible subset $\mathbb{L}_1$ with $\dim p_1^{-1}(Q) = 3$. Hence $\mathbb{L}_1$ is an irreducible subset of $\mathbb{P}^2 \times \mathbb{G}^3_{2,3}$ with dimension equal to 5.

The fiber $p_2^{-1}(t)$ is an irreducible subset of $\mathbb{L}_2$ with dimension equal to 6. So $\mathbb{L}_2$ is an irreducible subset of $\mathbb{L}_1 \times (\mathbb{P}^2)^6$ with

$$\dim \mathbb{L}_2 = \dim \mathbb{L}_1 + 6 = 5 + 6 = 11.$$

If we assume $l_{12} := l_1, l_{34} := l_2$ and $l_{56} := l_3$ so that $\wedge(12, 34, 56) = Q$, we get the

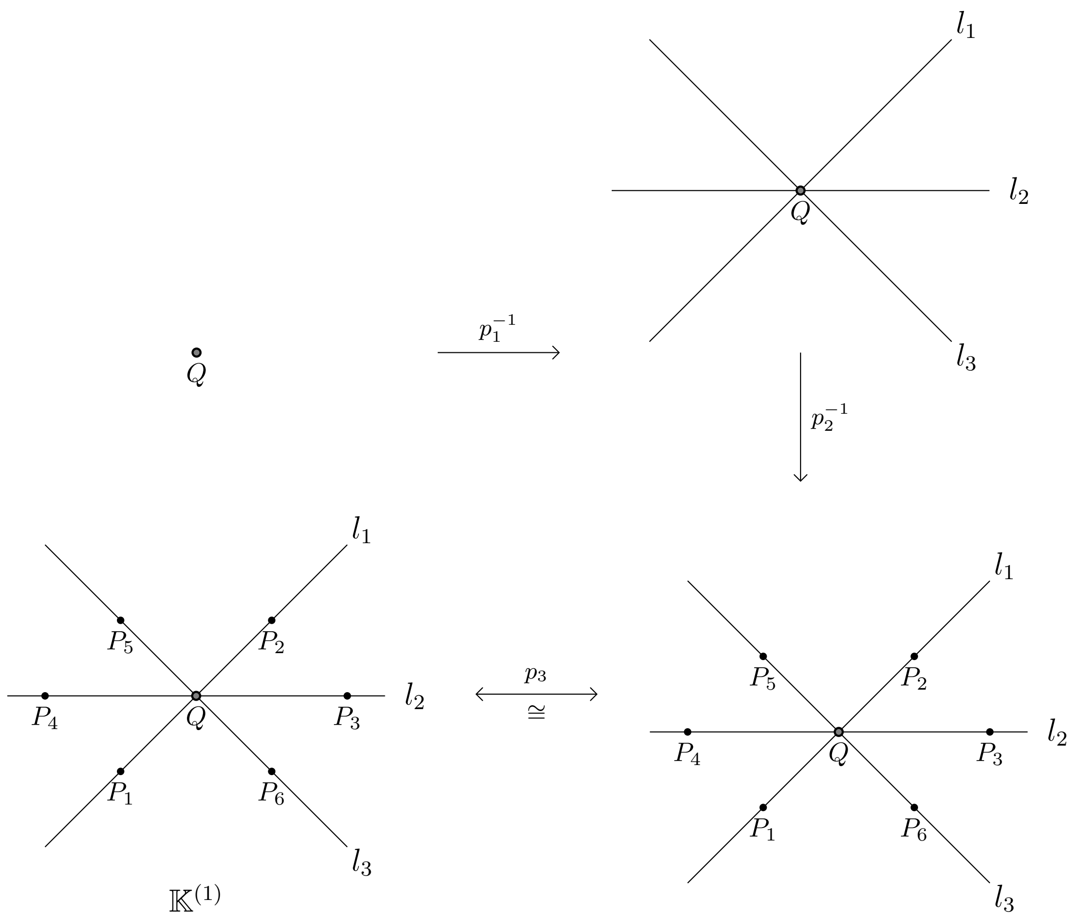

Figure 4.13: $\mathbb{K}^{(1)}$ and configurations of members of the fibers.

following isomorphism

$$\mathbb{L}_2 \xrightarrow{p_3} \mathbb{K}^{(1)} : (t; s) \mapsto s.$$

Thus $\mathbb{K}^{(1)}$ is an irreducible subset of $\mathbb{S}_6$ with codimension equal to 1. □

Theorem 4.8. $\mathbb{S}^{(1)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 1.

Proof. Assume that $\mathbb{K} := \mathbb{K}^{(1)}$, where $\mathbb{K}^{(1)}$ is the set defined in Lemma 4.3.1. Let us consider the following diagram (see Figure 4.14):

$$\begin{array}{ccccccc}
\mathbb{K}^{(1)} & \xleftarrow{\quad\psi\quad} & B_1 & \xrightarrow{\quad\xi\quad} & B_2 & \xrightarrow{\quad\gamma\quad} & \mathbb{S}^{(1)} \\
\cap & & \cap & & \cap & & \cap \\
\mathbb{S}_6 & & \mathbb{B}_1 & & \mathbb{B}_2 & & \mathbb{S}_{sm}
\end{array}$$

Figure 4.14: $\mathbb{S}^{(1)} = \gamma \circ \xi(B_1)$.

where $B_1 := \psi^{-1}(\mathbb{K}^{(1)})$, and $B_2 := \xi(B_1)$. Then $\mathbb{S}^{(1)} = \gamma(B_2)$ where

$$\xi := \Xi\Big|_{B_1},\ \ \gamma := \Gamma\Big|_{B_2}\ \text{ and }\ \psi := \Psi\Big|_{B_1}.$$

Then according to Equation (4.3.2), we have:

$$\dim B_1 = \dim \mathbb{K}^{(1)} + 15 = 11 + 15 = 26.$$

Consequently, $\mathbb{S}^{(1)} = \gamma \circ \xi(B_1)$ is an irreducible subset of $\mathbb{S}_{sm}$ with

$$\begin{aligned}\dim \mathbb{S}^{(1)} &= \dim B_1 - \dim \xi^{-1} \circ \gamma^{-1}(x) = \dim B_1 - \dim \xi^{-1}(b_2) \\ &= 26 - 8 = 18,\ \text{where}\ b_2 = \gamma^{-1}(x) \text{ and } x \in \mathbb{S}^{(1)}.\end{aligned}$$

Hence $\mathbb{S}^{(1)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 1. □

Lemma 4.3.2. Let

$$\mathbb{K}^{(2)} := \Big\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(12, 34, 56) = \{P_7\} \text{ and } \wedge(12, 35, 46) = \{P_8\} \Big\}.$$

Then $\mathbb{K}^{(2)}$ is an irreducible subset of $\mathbb{S}_6$ and $\operatorname{codim} \mathbb{K}^{(2)} = 2$.

Proof. Let $\mathbb{L}_1$ be the subset of $\mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_1 := \Big\{ x = (P_7, P_8) \in \mathbb{P}^2 \times \mathbb{P}^2 : P_7 \neq P_8 \Big\}.$$

Let $\mathbb{L}_2$ be the subset of $\mathbb{L}_1 \times \mathbb{G}^4_{2,3}$ defined by

$$\mathbb{L}_2 := \Big\{ (x; \lambda_{1234}) : x \in \mathbb{L}_1 \text{ and } \wedge(1, 2) = \{P_7\}, \wedge(3, 4) = \{P_8\} \Big\}.$$

Let $\mathbb{L}_3$ be the subset of $\mathbb{L}_2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_3 := \Big\{ (y; P_1, P_2) : y \in \mathbb{L}_2 \text{ and } P_1, P_2 \in l_{78} \Big\}.$$

Then we have the following projections:

$$\begin{array}{ccccc} \mathbb{L}_2 \times \mathbb{P}^2 \times \mathbb{P}^2 & & \mathbb{L}_1 \times \mathbb{G}^4_{2,3} & & \mathbb{P}^2 \times \mathbb{P}^2 \\ \cup & & \cup & & \cup \\ \mathbb{L}_3 & \xrightarrow{\quad p_1 \quad} & \mathbb{L}_2 & \xrightarrow{\quad p_2 \quad} & \mathbb{L}_1 \\ (y; P_1, P_2) & \longmapsto & y = (x; \lambda_{1234}) & \longmapsto & x = (P_7, P_8) \end{array}$$

It is evident that the set $\mathbb{L}_1$ is an open subset of $\mathbb{P}^2 \times \mathbb{P}^2$ and hence

$$\dim \mathbb{L}_1 = \dim \overline{\mathbb{L}_1} = \dim(\mathbb{P}^2 \times \mathbb{P}^2) = 4.$$

Let us consider the fiber of each map (see Figure 4.15).

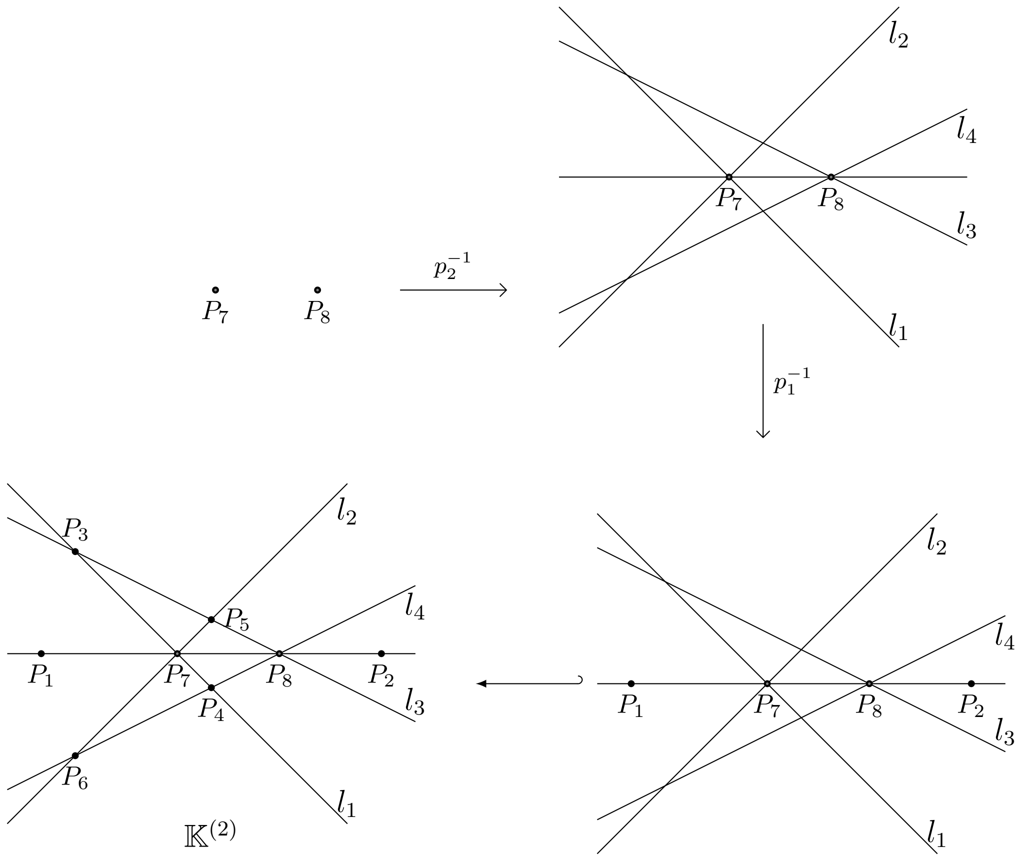

Figure 4.15: $\mathbb{K}^{(2)}$ and configurations of members of the fibers.

The fiber $p_2^{-1}(x)$ is an irreducible subset $\mathbb{L}_2$ with $\dim p_2^{-1}(x) = 4$. Since p_2 is surjective with irreducible target of dimension 4, $\mathbb{L}_2$ is an irreducible with dimension $4 + 4 = 8$. Furthermore, $\mathbb{L}_3$ is irreducible with $\dim \mathbb{L}_3 = 8 + 2 = 10$ since p_1 is surjective and the fiber $p_1^{-1}((x; \lambda_{1234}))$ is irreducible with $\dim p_1^{-1}((x; \lambda_{1234})) = 2$. If we assume that $\wedge(1,3) = \{P_3\}$, $\wedge(1,4) = \{P_4\}$, $\wedge(2,3) = \{P_5\}$ and $\wedge(2,4) = \{P_6\}$, then $\mathbb{K}^{(2)}$ becomes an open subset of $\mathbb{L}_3$. Thus $\mathbb{K}^{(2)}$ is an irreducible subset of $\mathbb{S}_6$ with dimension

$$\dim \mathbb{K}^{(2)} = \dim \overline{\mathbb{K}^{(2)}} = \dim \mathbb{L}_3 = 10.$$

$\square$

Theorem 4.9. $\mathbb{E}^{(2)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 2.

Proof. Let $\mathbb{K} := \mathbb{K}^{(2)}$, where $\mathbb{K}^{(2)}$ is the set defined in Lemma 4.3.2. Assume that Ξ, Ψ, and Γ are the maps defined in the main diagram (Figure 4.12). Let $B_1 := \psi^{-1}(\mathbb{K}^{(2)})$, and $B_2 := \xi(B_1)$. Then $\mathbb{E}^{(2)} = \gamma(B_2)$ where

$$\xi := \Xi\Big|_{B_1}, \ \ \gamma := \Gamma\Big|_{B_2} \text{ and } \psi := \Psi\Big|_{B_1}.$$

Then up to the diagram (see Figure 4.16), we have

$$\begin{array}{ccccccc} \mathbb{K}^{(2)} & \xleftarrow{\psi} & B_1 & \xrightarrow{\xi} & B_2 & \xrightarrow{\gamma} & \mathbb{E}^{(2)} \\ \cap & & \cap & & \cap & & \cap \\ \mathbb{S}_6 & & \mathbb{B}_1 & & \mathbb{B}_2 & & \mathbb{S}_{sm} \end{array}$$

Figure 4.16: $\mathbb{E}^{(2)} = \gamma \circ \xi(B_1)$.

$\mathbb{E}^{(2)} = \gamma \circ \xi(B_1)$ which is a subset of $\mathbb{S}_{sm}$. According to Equation (4.3.2), we have:

$$\dim B_1 = \dim \mathbb{K}^{(2)} + 15 = 10 + 15 = 25.$$

Consequently, $\mathbb{E}^{(2)} = \gamma \circ \xi(B_1)$ is an irreducible subset of $\mathbb{S}_{sm}$ with

$$\begin{aligned} \dim \mathbb{E}^{(2)} &= \dim B_1 - \dim \xi^{-1} \circ \gamma^{-1}(x) = \dim B_1 - \dim \xi^{-1}(b_2) \\ &= 25 - 8 = 17, \text{ where } b_2 = \gamma^{-1}(x) \text{ and } x \in \mathbb{E}^{(2)}. \end{aligned}$$

Therefore $\mathbb{E}^{(2)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 2. $\square$

Lemma 4.3.3. Let

$$\mathbb{K}^{(3)} := \Big\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(12, 34, 56) = \{P_7\} \text{ and } \wedge(13, 45, 26) = \{P_8\} \Big\}.$$

Then $\mathbb{K}^{(3)}$ is an irreducible subset of $\mathbb{S}_6$ and $\operatorname{codim} \mathbb{K}^{(3)} = 2$.

Proof. Let $\mathbb{L}_1$ be the subset of $\mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_1 := \Big\{ x = (P_7, P_8) \in \mathbb{P}^2 \times \mathbb{P}^2 : P_7 \neq P_8 \Big\}.$$

Let $\mathbb{L}_2$ be the subset of $\mathbb{L}_1 \times \mathbb{G}^4_{2,3}$ defined by

$$\mathbb{L}_2 := \Big\{(x;\lambda_{1234}) : x \in \mathbb{L}_1 \text{ and } \wedge(1,2) = \{P_7\}, \wedge(3,4) = \{P_8\}\Big\}.$$

Let $\mathbb{L}_3$ be the subset of $\mathbb{L}_2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_3 := \Big\{(y;P_1,P_2) : y \in \mathbb{L}_2 \text{ and } P_7 \in \overline{P_1P_2}, P_8 \in \overline{P_1P_3}\Big\}.$$

Then we have the following projections:

$$\begin{array}{ccccc}
\mathbb{L}_2 \times \mathbb{P}^2 \times \mathbb{P}^2 & & \mathbb{L}_1 \times \mathbb{G}^4_{2,3} & & \mathbb{P}^2 \times \mathbb{P}^2 \\
\cup & & \cup & & \cup \\
\mathbb{L}_3 & \xrightarrow{\quad p_1 \quad} & \mathbb{L}_2 & \xrightarrow{\quad p_2 \quad} & \mathbb{L}_1 \\
(y;P_1,P_2) & \longmapsto & y = (x;\lambda_{1234}) & \longmapsto & x = (P_7,P_8)
\end{array}$$

Let us consider the fiber of each projection (see Figure 4.17).

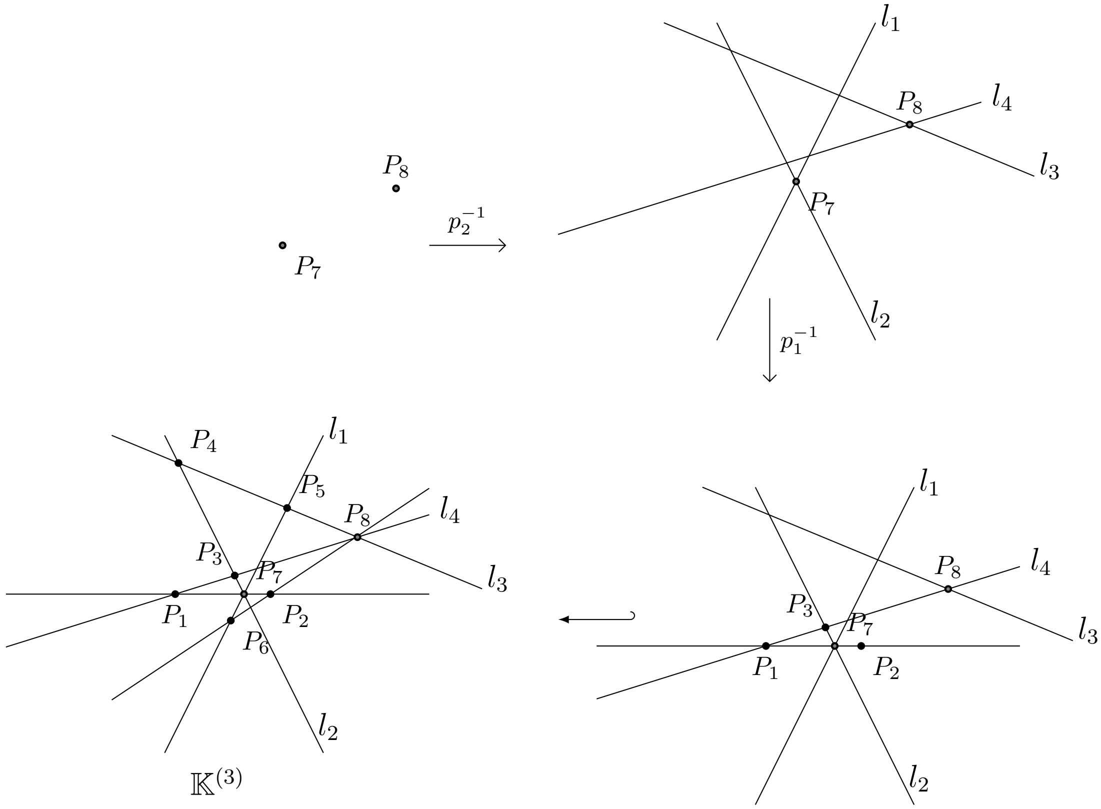

Figure 4.17: $\mathbb{K}^{(3)}$ and configurations of members of the fibers.

As we illustrated in Lemma 4.3.2, the set $\mathbb{L}_1$ is an open subset of $\mathbb{P}^2 \times \mathbb{P}^2$ with

$$\dim \mathbb{L}_1 = \dim \overline{\mathbb{L}_1} = \dim \mathbb{P}^2 \times \mathbb{P}^2 = 4.$$

Let us consider the following projections:

$$\begin{array}{ccccc} \mathbb{L}_2 \times \mathbb{P}^2 \times \mathbb{P}^2 & & \mathbb{L}_1 \times \mathbb{G}^4_{2,3} & & \mathbb{P}^2 \times \mathbb{P}^2 \\ \cup & & \cup & & \cup \\ \mathbb{L}_3 & \xrightarrow{p_1} & \mathbb{L}_2 & \xrightarrow{p_2} & \mathbb{L}_1 \\ (y; P_1, P_2) & \longmapsto & y = (x; \lambda_{1234}) & \longmapsto & x = (P_7, P_8) \end{array}$$

By the same argument used in the proof of Lemma 4.3.2, the surjectivity of p_2 implies the irreducibility of the fiber $p_2^{-1}(x)$. Moreover, we have $\dim p_2^{-1}(x) = 4$. Since p_2 is surjective with irreducible target of dimension 4, it follows that the set $\mathbb{L}_2$ is an irreducible with dimension $4+4=8$. Furthermore, $\mathbb{L}_3$ is an irreducible set with $\dim \mathbb{L}_3 = 8+2 = 10$ since p_1 is surjective and the fiber $p_1^{-1}(y)$ is irreducible with $\dim p_1^{-1}((x;\lambda_{1234})) = 2$. If we assume that $\wedge(2,4) = \{P_3\}, \wedge(2,3) = \{P_4\}, \wedge(1,3) = \{P_5\}, \wedge(1,2) = \{P_7\}, \wedge(3,4) = \{P_8\}$ and $l_1 \cap \overline{P_2P_8} = \{P_6\}$ (see Figure 4.17), we conclude that $\mathbb{K}^{(3)}$ is an open subset of $\mathbb{L}_3$. Thus $\mathbb{K}^{(3)}$ becomes an irreducible subset of $\mathbb{S}_6$ with dimension

$$\dim \mathbb{K}^{(3)} = \dim \overline{\mathbb{K}^{(3)}} = \dim \mathbb{L}_3 = 10.$$

□

Theorem 4.10. $\mathbb{E}^{(3)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 2.

Proof. Let $\mathbb{K} := \mathbb{K}^{(3)}$ be the set defined as in Lemma 4.3.3. Let us consider the same procedure used in the proof of Theorem 4.9, and assume that Ξ, Ψ, and Γ are the maps defined as in the main diagram (Figure 4.12). Let $B_1 := \psi^{-1}(\mathbb{K}^{(3)})$, and $B_2 := \xi(B_1)$. Then $\mathbb{E}^{(3)} = \gamma(B_2)$ where

$$\xi := \Xi\Big|_{B_1}, \ \gamma := \Gamma\Big|_{B_2} \text{ and } \psi := \Psi\Big|_{B_1}.$$

Then according to the following diagram (see Figure 4.18), we have

$\mathbb{E}^{(3)} = \gamma \circ \xi(B_1)$ which is a subset of $\mathbb{S}_{sm}$. According to Equation (4.3.2), we obtain:

$$\dim B_1 = \dim \mathbb{K}^{(3)} + 15 = 10 + 15 = 25.$$

$$\begin{array}{ccccccc} \mathbb{K}^{(3)} & \xleftarrow{\quad\psi\quad} & B_1 & \xrightarrow{\quad\xi\quad} & B_2 & \xrightarrow{\quad\gamma\quad} & \mathbb{E}^{(3)} \\ \cap & & \cap & & \cap & & \cap \\ \mathbb{S}_6 & & \mathbb{B}_1 & & \mathbb{B}_2 & & \mathbb{S}_{sm} \end{array}$$

Figure 4.18: $\mathbb{E}^{(3)} = \gamma \circ \xi(B_1)$.

Consequently, $\mathbb{E}^{(3)} = \gamma \circ \xi(B_1)$ is an irreducible subset of $\mathbb{S}_{sm}$ with

$$\begin{aligned} \dim \mathbb{E}^{(3)} &= \dim B_1 - \dim \xi^{-1} \circ \gamma^{-1}(x) = \dim B_1 - \dim \xi^{-1}(b_2) \\ &= 25 - 8 = 17, \text{ where } b_2 = \gamma^{-1}(x) \text{ and } x \in \mathbb{E}^{(3)}. \end{aligned}$$

Therefore $\mathbb{E}^{(3)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 2. □

Corollary 4.14. $\mathbb{S}^{(1)}$ and $\mathbb{S}^{(2)}$ are closed subset of $\mathbb{S}_{sm}$. Moreover, $\mathbb{S}^{(2)}$ has two irreducible components $\mathbb{E}^{(2)}$ and $\mathbb{E}^{(3)}$ in $\mathbb{S}_{sm}$ with codimension 2.

Proof. Let $\mathbb{L}_{\mathscr{S}} := (l_1, \ldots, l_{27})$ be the 27 lines on the non-singular cubic surface $\mathscr{S}$. Define

$$\begin{aligned} \mathbb{G} &:= \Big\{(c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) : c(\mathscr{S}) \in \mathbb{S}_{sm}\Big\} \subset \mathbb{S}_{sm} \times \mathbb{G}_{2,4}^{27}, \\ \mathbb{G}_1 &:= \Big\{(c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) \in \mathbb{G} : \lambda_{123} \in \mathbb{T}^{(3)}\Big\} \subseteq \mathbb{G}, \\ \mathbb{G}_2 &:= \Big\{(c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) \in \mathbb{G} : \lambda_{123}, \lambda_{145} \in \mathbb{T}^{(3)}\Big\} \subseteq \mathbb{G}, \\ \mathbb{G}_3 &:= \Big\{(c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) \in \mathbb{G} : \lambda_{123}, \lambda_{456} \in \mathbb{T}^{(3)}\Big\} \subseteq \mathbb{G}. \end{aligned}$$

We know that $\mathbb{T}^{(3)} \subset \mathbb{G}_{2,4}^3$ and it is evident that the set

$$\mathbb{G}^* := \Big\{\lambda_{12} \in \mathbb{G}_{2,4} \times \mathbb{G}_{2,4} : \wedge(\lambda_{12}) \neq \emptyset\Big\},$$

is a closed subset of $\mathbb{G}_{2,4} \times \mathbb{G}_{2,4}$ (see [2] and [24]). Let

$$\mathbb{G} \xrightarrow{\;p\;} \mathbb{S}_{sm} : (c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) \mapsto c(\mathscr{S})$$

be the projection map, and consider the mappings in Figure 4.19.

Note that

$$\mathbb{T}^{(3)} = \eta_{12}^{-1}(\mathbb{G}^*) \cap \eta_{13}^{-1}(\mathbb{G}^*) \cap \eta_{23}^{-1}(\mathbb{G}^*) \subset \mathbb{G}_{2,4} \times \mathbb{G}_{2,4} \times \mathbb{G}_{2,4}.$$

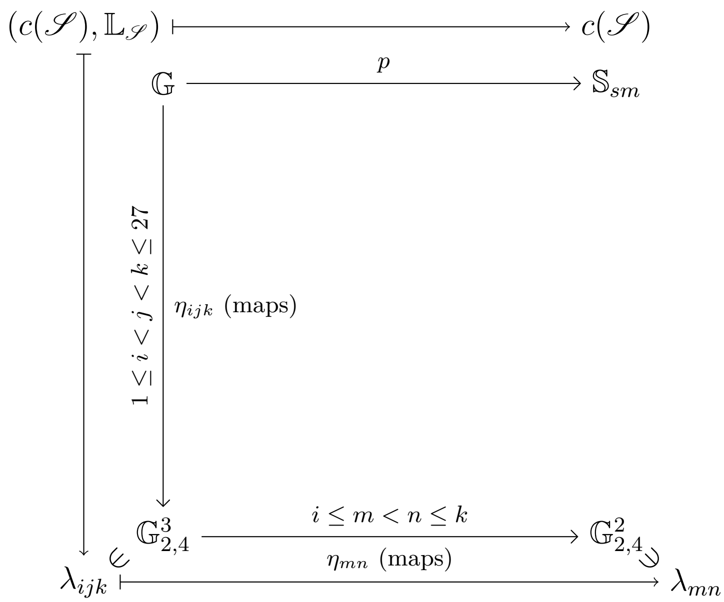

Figure 4.19: Irreducible components of $\mathbb{S}^{(2)}$.

Hence $\mathbb{T}^{(3)}$ is a closed subset of $\mathbb{G}_{2,4} \times \mathbb{G}_{2,4} \times \mathbb{G}_{2,4}$. Furthermore, $\mathbb{G}_1 = \eta_{123}^{-1}(\mathbb{T}^{(3)})$. It follows that $\mathbb{G}_1$ is a closed subset of $\mathbb{G}$. But we know that $\mathbb{S}^{(1)} = p(\mathbb{G}_1)$. This means that the set $\mathbb{S}^{(1)}$ is a closed subset of $\mathbb{S}_{sm}$. Moreover, the subsets

$$\mathbb{G}_2 = \eta_{123}^{-1}(\mathbb{T}^{(3)}) \cap \eta_{145}^{-1}(\mathbb{T}^{(3)}),$$

$$\mathbb{G}_3 = \eta_{123}^{-1}(\mathbb{T}^{(3)}) \cap \eta_{456}^{-1}(\mathbb{T}^{(3)}),$$

are both closed subsets of $\mathbb{S}_{sm}$. Consequently, we get

$$\mathbb{E}^{(2)} = p(\mathbb{G}_2),$$

$$\mathbb{E}^{(3)} = p(\mathbb{G}_3),$$

are closed subsets of $\mathbb{S}_{sm}$. Thus $\mathbb{S}^{(2)}$ is a closed subset of $\mathbb{S}_{sm}$ with two irreducible components, namely $\mathbb{E}^{(2)}$ and $\mathbb{E}^{(3)}$. According to Theorem 4.9 and Theorem 4.10, we know that both $\mathbb{E}^{(2)}$ and $\mathbb{E}^{(3)}$ have codimension 2. Thus $\mathbb{S}^{(2)}$ has two irreducible components, namely $\mathbb{E}^{(2)}$ and $\mathbb{E}^{(3)}$ in $\mathbb{S}_{sm}$ with codimension 2. $\square$

Lemma 4.3.4. Let

$$\mathbb{K}^{(4)} := \Big\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(13, 24, 56) = \{P_7\} \text{ and } l_{12}, l_{13} \text{ tangents to } \mathscr{C}_1 \Big\}.$$

Then $\mathbb{K}^{(4)}$ is an irreducible subset of $\mathbb{S}_6$ and $\operatorname{codim} \mathbb{K}^{(4)} = 3$.

Proof. Let $\mathbb{L}_1$ be the subset of $\mathbb{C}_{sm} \times (\mathbb{P}^2)^3$ defined by

$$\mathbb{L}_1 := \Big\{ (c(\mathscr{C}_1), \kappa_{123}) : l_{12}, l_{13} \text{ tangents to } \mathscr{C}_1 \Big\}.$$

Let $\mathbb{L}_2$ be the subset of $\mathbb{L}_1 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_2 := \Big\{ (x; P_7) : x \in \mathbb{L}_1 \text{ and } P_7 \in l_{13} \backslash \{P_1, P_3\} \Big\}.$$

Let $\mathbb{L}_3$ be the subset of $\mathbb{L}_2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_3 := \Big\{ (y; \kappa_{456}) : y \in \mathbb{L}_2 \text{ and } P_4 \in \mathscr{C}_1 \cap l_{27} \text{ and } P_5, P_6 \in \mathscr{C}_1, l_{13} \cap l_{56} = \{P_7\} \Big\}.$$

Then we have the projection mapping as in Figure 4.20.

$$\begin{array}{ccccccc}
(y; \kappa_{456}) & \longmapsto & (x; P_7) & \longmapsto & (c(\mathscr{C}_1), \kappa_{123}) & \longmapsto & c(\mathscr{C}_1) \\
\mathbb{L}_3 & \xrightarrow{p_3} & \mathbb{L}_2 & \xrightarrow{p_2} & \mathbb{L}_1 & \xrightarrow{p_1} & \mathbb{C}_{sm} \\
\wr\!\| \big\downarrow p_4 & & & & & & \\
s \quad \mathbb{K}^{(4)} & & & & & &
\end{array}$$

Figure 4.20: Irreducibility of $\mathbb{K}^{(4)}$.

Let us consider the fiber of each projection in previous diagram (see Figure 4.21).

The surjectivity of p_1 implies to the irreducibility of the fiber $p_1^{-1}(c(\mathscr{C}_1))$ as a subset of $\mathbb{L}_1$. Moreover, we have $\dim p_1^{-1}(c(\mathscr{C}_1)) = 2$. Since p_1 has an irreducible target of dimension 5, $\mathbb{L}_1$ is irreducible with dimension $5 + 2 = 7$. Furthermore, $\mathbb{L}_2$ is an irreducible with $\dim \mathbb{L}_2 = 7 + 1 = 8$ since every fiber $p_2^{-1}(c(\mathscr{C}_1), \kappa_{123})$ has dimension 1 under the surjective map p_2.

Note that the fiber $p_3^{-1}(x; P_7)$ is a subset of $\mathscr{C}_1 \times \mathscr{C}_1$ and isomorphic to the set S, where $S := \{(P_5, P_6) \in \mathscr{C}_1 \times \mathscr{C}_1 : P_7 \in l_{56}\}$. Define the surjective map p_5 as follows:

$$p_5 : S \longrightarrow \mathscr{C}_1 \text{ by } (P_5, P_6) \mapsto P_5.$$

We see that every fiber under the map p_5 is a singleton set and hence $\dim S = 1$. Therefore $\dim S = \dim p_3^{-1}(x; P_7) = 1$. Consequently, $\mathbb{L}_3$ is irreducible with

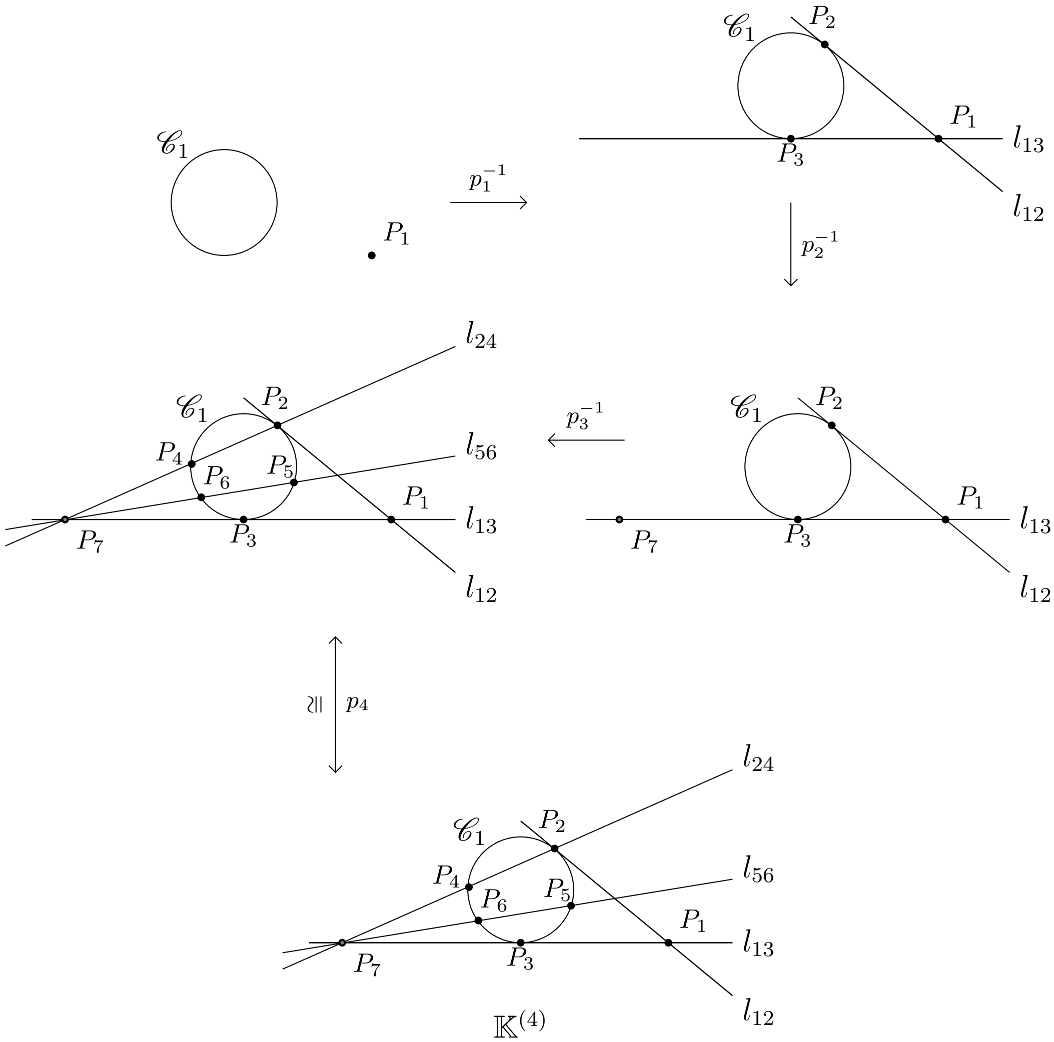

Figure 4.21: $\mathbb{K}^{(4)}$ and configurations of members of the fibers.

dimension $8 + 1 = 9$. Finally, $\mathbb{L}_3$ isomorphic to $\mathbb{K}^{(4)}$ via the projection map p_4. So $\mathbb{K}^{(4)}$ is an irreducible subset of $\mathbb{S}_6$ with $\operatorname{codim} \mathbb{K}^{(4)} = 3$. $\square$

Theorem 4.11. $\mathbb{E}^{(4)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 3.

Proof. Let $\mathbb{K} := \mathbb{K}^{(4)}$ be the set defined as in Lemma 4.3.4. Assume that $B_1 := \psi^{-1}(\mathbb{K}^{(4)})$, and $B_2 := \xi(B_1)$. Then $\mathbb{E}^{(4)} = \gamma(B_2)$ where $\xi := \Xi\Big|_{B_1}$, $\gamma := \Gamma\Big|_N$, and $\psi := \Psi\Big|_M$ (see Figure 4.12). Consequently, we have $\mathbb{E}^{(4)} = \gamma \circ \xi(B_1)$ which is a subset of $\mathbb{S}_{sm}$ (see Figure 4.22).

$$\begin{array}{ccccccc} \mathbb{K}^{(4)} & \xleftarrow{\ \psi\ } & B_1 & \xrightarrow{\ \xi\ } & B_2 & \xrightarrow{\ \gamma\ } & \mathbb{E}^{(4)} \\ \cap & & \cap & & \cap & & \cap \\ \mathbb{S}_6 & & \mathbb{B}_1 & & \mathbb{B}_2 & & \mathbb{S}_{sm} \end{array}$$

Figure 4.22: $\mathbb{E}^{(4)} = \gamma \circ \xi(B_1)$.

According to Equation (4.3.2), we have:

$$\dim B_1 = \dim \mathbb{K}^{(4)} + 15 = 9 + 15 = 24.$$

It follows that $\mathbb{E}^{(4)} = \gamma \circ \xi(B_1)$ is an irreducible subset of $\mathbb{S}_{sm}$ with

$$\begin{aligned} \dim \mathbb{E}^{(4)} &= \dim B_1 - \dim \xi^{-1} \circ \gamma^{-1}(x) = \dim B_1 - \dim \xi^{-1}(b_2) \\ &= 24 - 8 = 16, \text{ where } b_2 = \gamma^{-1}(x) \text{ and } x \in \mathbb{E}^{(4)}. \end{aligned}$$

Therefore we conclude that $\mathbb{E}^{(4)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 3. □

Lemma 4.3.5. Let

$$\mathbb{K}^{(6)} := \Big\{ s = \kappa_{123456} \in \mathbb{S}_6 : \wedge(14, 23, 56) = \{P_7\} \text{ and } l_{12}, l_{13} \text{ tangents to } \mathscr{C}_1 \Big\}.$$

Then $\mathbb{K}^{(6)}$ is an irreducible subset of $\mathbb{S}_6$ with $\operatorname{codim} \mathbb{K}^{(6)} = 3$.

Proof. Let $\mathbb{L}_1$ be the subset of $\mathbb{C}_{sm} \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_1 := \Big\{ (c(\mathscr{C}_1), \kappa_{123}) : l_{12}, l_{13} \text{ tangents to } \mathscr{C}_1 \Big\}.$$

Let $\mathbb{L}_2$ be the subset of $\mathbb{L}_1 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_2 := \Big\{ (x; P_4) : x \in \mathbb{L}_1 \text{ and } P_4 \in \mathscr{C}_1 \backslash \{P_2, P_3\} \Big\}.$$

Let $\mathbb{L}_3$ be the subset of $\mathbb{L}_2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_3 := \left\{ \begin{array}{c} (y; P_5, P_6) : \kappa_{123456} \in \mathbb{S}_6, y \in \mathbb{L}_2 \text{ and } P_5, P_6 \in \mathscr{C}_1 \\ \text{and } \wedge(14, 23, 56) = \{P_7\} \end{array} \right\}.$$

Then we have the projections as in Figure 4.23.

$$
\begin{array}{ccccccc}
(y;P_5,P_6) & \longmapsto & (x;P_4) & \longmapsto & (c(\mathscr{C}_1),\kappa_{123}) & \longmapsto & c(\mathscr{C}_1) \\
\mathbb{L}_3 & \xrightarrow{p_3} & \mathbb{L}_2 & \xrightarrow{p_2} & \mathbb{L}_1 & \xrightarrow{p_1} & \mathbb{C}_{sm} \\
\wr\!\| \big\downarrow p_4 & & & & & & \\
\mathbb{K}^{(6)} & & & & & &
\end{array}
\qquad (y;P_5,P_6) \longmapsto s
$$

Figure 4.23: Irreducibility of $\mathbb{K}^{(6)}$.

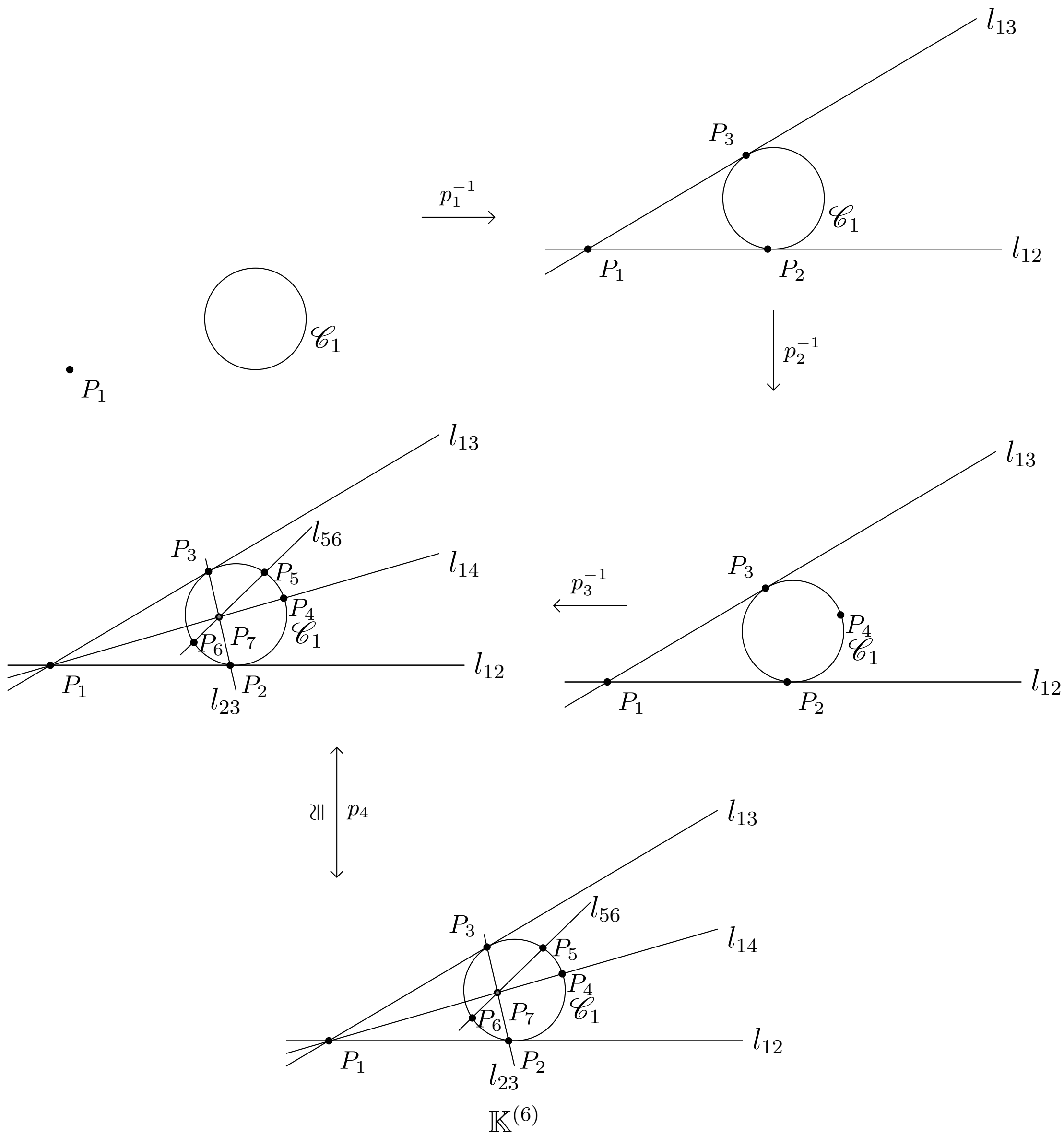

Figure 4.24: $\mathbb{K}^{(6)}$ and configurations of members of the fibers.

Let us consider the fiber of each projection in previous diagram (see Figure 4.24). As in Lemma 4.3.4, the surjectivity of p_1 implies to the irreducibility of the fiber

$p_1^{-1}(c(\mathscr{C}_1))$ as a subset of $\mathbb{L}_1$. Moreover, we have $\dim p_1^{-1}(c(\mathscr{C}_1)) = 2$. Since p_2 has an irreducible target of dimension 5, $\mathbb{L}_1$ is an irreducible with dimension $5 + 2 = 7$. Furthermore, $\mathbb{L}_2$ is irreducible with $\dim \mathbb{L}_2 = 7 + 1 = 8$ because every fiber $p_2^{-1}(c(\mathscr{C}_1), \kappa_{123})$ has dimension 1 under the surjective map p_2.

Note that the fiber $p_3^{-1}(x; P_4)$ consists of all points P_5, P_6 on the non-singular conic $\mathscr{C}_1$, so that $P_7 \in l_{56}$. More precisely, the fiber $p_3^{-1}(x; P_4)$ is a subset of $\mathscr{C}_1 \times \mathscr{C}_1$ isomorphic to the set S, where

$$S := \{(P_5, P_6) \in \mathscr{C}_1 \times \mathscr{C}_1 : P_7 \in l_{56}\}.$$

Define the surjective map p_5 as follows:

$$p_5 : S \longrightarrow \mathscr{C}_1 \text{ by } (P_5, P_6) \mapsto P_5.$$

Now the fiber under the map p_5 consists of one point, and hence it has dimension equal to 1. Thus $\dim S = \dim p_3^{-1}(x; P_4) = 1$. Consequently, $\mathbb{L}_3$ is irreducible with dimension $8 + 1 = 9$. Finally, $\mathbb{L}_3$ is isomorphic to $\mathbb{K}^{(6)}$ via the projection map p_4. So $\mathbb{K}^{(6)}$ is an irreducible subset of $\mathbb{S}_6$ with $\operatorname{codim} \mathbb{K}^{(6)} = 3$. $\square$

Theorem 4.12. $\mathbb{E}^{(6)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 3.

Proof. Let $\mathbb{K} := \mathbb{K}^{(6)}$ be the set defined as in Lemma 4.3.5. Let us assume $B_1 := \psi^{-1}(\mathbb{K}^{(6)})$, and $B_2 := \xi(B_1)$. Then $\mathbb{E}^{(6)} = \gamma(B_2)$ where $\xi := \Xi\big|_{B_1}$, $\gamma := \Gamma\big|_{B_2}$, and $\psi := \Psi\big|_{B_1}$ (see the main diagram in Figure 4.12). Consequently, we have $\mathbb{E}^{(6)} = \gamma \circ \xi(B_1)$ which is a subset of $\mathbb{S}_{sm}$ (see Figure 4.25).

$$\begin{array}{ccccccc}
\mathbb{K}^{(6)} & \xleftarrow{\quad\psi\quad} & B_1 & \xrightarrow{\quad\xi\quad} & B_2 & \xrightarrow{\quad\gamma\quad} & \mathbb{E}^{(6)} \\
\cap & & \cap & & \cap & & \cap \\
\mathbb{S}_6 & & \mathbb{B}_1 & & \mathbb{B}_2 & & \mathbb{S}_{sm}
\end{array}$$

Figure 4.25: $\mathbb{E}^{(6)} = \gamma \circ \xi(B_1)$.

According to Equation (4.3.2), we have:

$$\dim B_1 = \dim \mathbb{K}^{(6)} + 15 = 9 + 15 = 24.$$

Since $\mathbb{E}^{(6)} = \gamma \circ \xi(B_1)$, it follows that $\mathbb{E}^{(6)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with

$$\begin{aligned}\dim \mathbb{E}^{(6)} &= \dim B_1 - \dim \xi^{-1} \circ \gamma^{-1}(x) = \dim B_1 - \dim \xi^{-1}(n) \\ &= 24 - 8 = 16, \text{ where } n = \gamma^{-1}(x) \text{ and } x \in \mathbb{E}^{(6)}.\end{aligned}$$

Therefore we conclude that $\mathbb{E}^{(6)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 3. □

Lemma 4.3.6. Let $s = \kappa_{123456} \in \mathbb{S}_6$ and define

$$\mathbb{K}^{(9)} := \left\{ \begin{array}{c} s \in \mathbb{S}_6 : \wedge(12,34,56) = \{P_8\}, \wedge(15,24,36) = \{P_7\} \text{ and} \\ l_{14} \text{ tangent to } \mathscr{C}_1 \text{ at } P_4 \end{array} \right\}.$$

Then $\mathbb{K}^{(9)}$ is an irreducible subset of $\mathbb{S}_6$ with codimension equal 3.

Proof. Let $\mathbb{L}_1$ be the subset of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_1 := \Big\{ \kappa_{1234} : \text{there is no three points in } \widehat{\kappa_{1234}} \text{ on the same line} \Big\}.$$

Let $\mathbb{L}_2$ be the subset of $\mathbb{L}_1 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_2 := \Big\{ (x; P_7) : x \in \mathbb{L}_1 \text{ and } P_7 \in l_{24} \backslash l_{23} \cup l_{14} \cup l_{13} \Big\}.$$

Let $\mathbb{L}_3$ be the subset of $\mathbb{L}_2 \times \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_3 := \left\{ \begin{array}{c} (y; P_5, P_6) : \kappa_{123456} \in \mathbb{S}_6, y \in \mathbb{L}_2, \wedge(15,24,36) = \{P_7\} \text{ and} \\ l_{14} \text{ tangent to } \mathscr{C}_1 \text{ at } P_4 \end{array} \right\}.$$

Then we have the following projections (see Figure 4.26).

Now let us consider the fiber of each projection in the diagram (see Figure 4.27).

$$
\begin{array}{ccccc}
(y;P_5,P_6) & \longmapsto & (x;P_7) & \longmapsto & \kappa_{1234} \\
\mathbb{L}_3 & \xrightarrow{\;p_2\;} & \mathbb{L}_2 & \xrightarrow{\;p_1\;} & \mathbb{L}_1 \\
\cong \Big\downarrow p_3 & & & & \\
s \quad \mathbb{K}^{(9)} & & & &
\end{array}
$$

Figure 4.26: Irreducible components of $\mathbb{K}^{(9)}$.

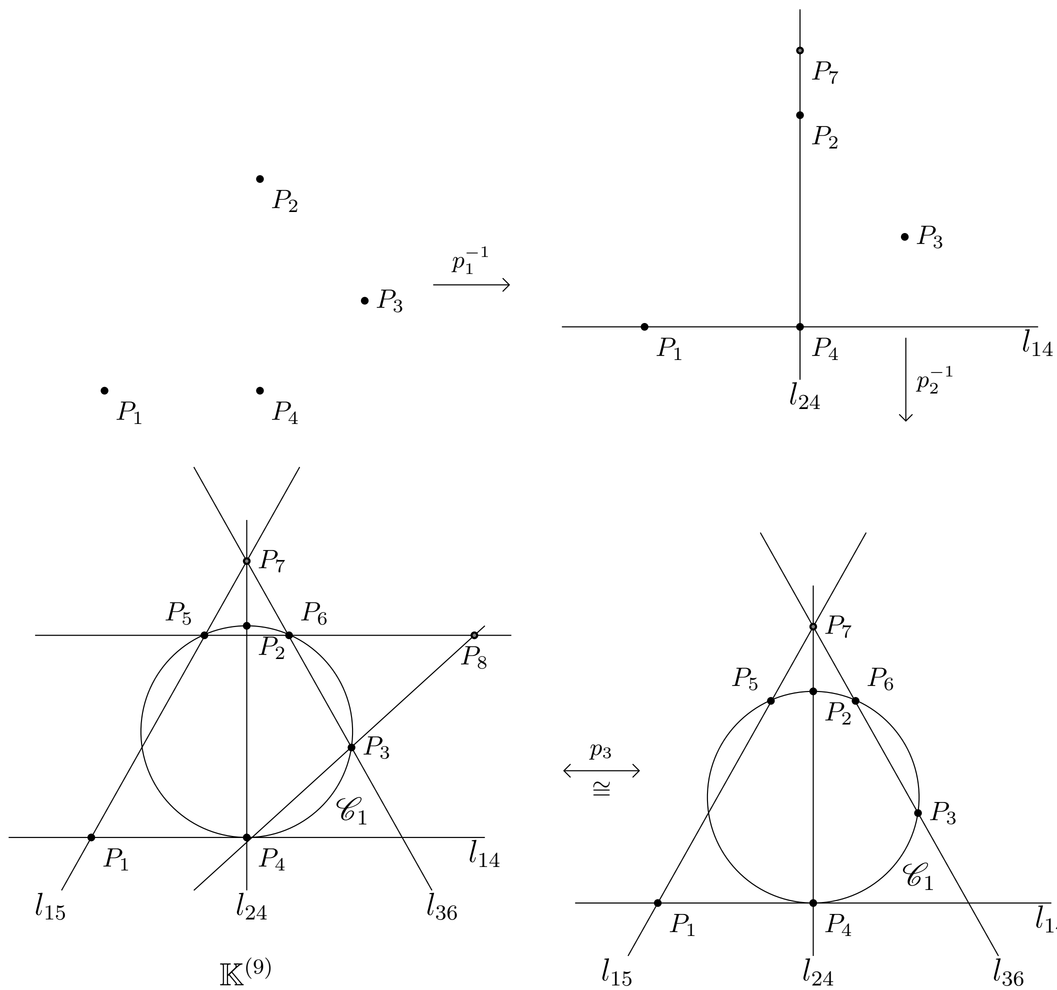

Figure 4.27: $\mathbb{K}^{(9)}$ and configurations of members of the fibers.

It is evident that $\mathbb{L}_1$ is an open subset of $\mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2 \times \mathbb{P}^2$. Thus $\dim \mathbb{L}_1 = 8$. Furthermore, the surjectivity of p_1 implies the irreducibility of the fiber $p_1^{-1}(\kappa_{1234})$ as a subset of $\mathbb{L}_2$. Moreover, we have $\dim p_1^{-1}(\kappa_{1234}) = 1$. Consequently, $\mathbb{L}_2$ is an irreducible with dimension equal to $8 + 1 = 9$.

More precisely, Let

$$P_1 = (1:0:1),\ P_2 = (1:1:0),\ P_3 = (-1:1:0), \text{ and } P_4 = (0:0:1).$$

Then

$$\begin{aligned} l_{13} &= \mathbb{V}(x_0 + x_1 + x_2), \\ l_{14} &= \mathbb{V}(x_1), \\ l_{23} &= \mathbb{V}(x_2), \\ l_{24} &= \mathbb{V}(x_0 - x_1). \end{aligned}$$

Note that $P_7 = (\alpha : \alpha : 1) \in l_{24} \backslash l_{23} \cup l_{14} \cup l_{13}$ where $\alpha \in (k\backslash\{0\}) \cup \{\infty\}$. Therefore, except for finite number of values of α, the fiber $p_2^{-1}((x; P_7))$ consists of two distinct points. It follows that $\dim p_2^{-1}(x; P_8) = 0$, and the set $\mathbb{L}_3$ has dimension equal to 9.

More precisely, we have

$$\dim \mathbb{L}_3 = \dim \mathbb{L}_2 + 0 = 9.$$

But $\mathbb{L}_3$ isomorphic to $\mathbb{K}^{(9)}$ via the projection map p_3. It follows that $\mathbb{K}^{(9)}$ is an irreducible subset of $\mathbb{S}_6$ with codimension equal 3. □

Theorem 4.13. $\mathbb{E}^{(9)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 3.

Proof. First, Let $\mathbb{K} := \mathbb{K}^{(9)}$ (see Lemma 4.3.6). Let us assume $B_1 := \psi^{-1}(\mathbb{K}^{(9)})$, and $B_2 := \xi(B_1)$. Then $\mathbb{E}^{(9)} = \gamma(B_2)$ where $\xi := \Xi\big|_{B_1}$, $\gamma := \Gamma\big|_{B_2}$, and $\psi := \Psi\big|_{B_1}$ (see the main diagram in Figure 4.12). Consider the diagram (see Figure 4.28).

$$\begin{array}{ccccccc} \mathbb{K}^{(9)} & \xleftarrow{\ \psi\ } & B_1 & \xrightarrow{\ \xi\ } & B_2 & \xrightarrow{\ \gamma\ } & \mathbb{E}^{(9)} \\ \cap & & \cap & & \cap & & \cap \\ \mathbb{S}_6 & & \mathbb{B}_1 & & \mathbb{B}_2 & & \mathbb{S}_{sm} \end{array}$$

Figure 4.28: $\mathbb{E}^{(9)} = \gamma \circ \xi(B_1)$.

According to Equation (4.3.2), we have:

$$\dim B_1 = \dim \mathbb{K}^{(9)} + 15 = 9 + 15 = 24.$$

Since $\mathbb{E}^{(9)} = \gamma \circ \xi(B_1)$, it follows that $\mathbb{E}^{(9)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with

$$\begin{aligned}\dim \mathbb{E}^{(9)} &= \dim B_1 - \dim \xi^{-1} \circ \gamma^{-1}(x) = \dim B_1 - \dim \xi^{-1}(n) \\ &= 24 - 8 = 16, \text{ where } n = \gamma^{-1}(x), x \in \mathbb{E}^{(9)}.\end{aligned}$$

□

Corollary 4.15. $\mathbb{E}^{(4)}, \mathbb{E}^{(6)}$ and $\mathbb{E}^{(9)}$ are closed subset of $\mathbb{S}_{sm}$.

Proof. Let $\mathbb{L}_{\mathscr{S}} := (l_1, \ldots, l_{27})$ be the 27 lines on a non-singular cubic surface $\mathscr{S}$ with coefficients $c(\mathscr{S})$. Define

$$\begin{aligned}\mathbb{G} &:= \Big\{(c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) : c \in \mathbb{S}_{sm}\Big\} \subset \mathbb{S}_{sm} \times \mathbb{L}_{27}^{27}, \\ \mathbb{G}_1 &:= \Big\{(c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) \in \mathbb{G} : \lambda_{123}, \lambda_{145}, \lambda_{267} \in \mathbb{T}^{(3)}\Big\} \subset \mathbb{G}, \\ \mathbb{G}_2 &:= \Big\{(c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) \in \mathbb{G} : \lambda_{123}, \lambda_{145}, \lambda_{678} \in \mathbb{T}^{(3)}\Big\} \subset \mathbb{G}, \\ \mathbb{G}_3 &:= \Big\{(c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) \in \mathbb{G} : \lambda_{123}, \lambda_{456}, \lambda_{789} \in \mathbb{T}^{(3)}\Big\} \subset \mathbb{G}.\end{aligned}$$

We know that $\mathbb{T}^{(3)} \subset \mathbb{G}_{2,4} \times \mathbb{G}_{2,4} \times \mathbb{G}_{2,4}$. Moreover, it is evident that the set

$$\mathbb{G}^* := \Big\{\lambda_{12} \in \mathbb{G}_{2,4} \times \mathbb{G}_{2,4} : \wedge(\lambda_{12}) \neq \emptyset\Big\},$$

is closed subset of $\mathbb{G}_{2,4} \times \mathbb{G}_{2,4}$ (see [2] and [24]). Let

$$\mathbb{G} \xrightarrow{\ p\ } \mathbb{S}_{sm} : (c(\mathscr{S}), \mathbb{L}_{\mathscr{S}}) \mapsto c(\mathscr{S})$$

be the projection map, and consider the mappings in Figure 4.29.

Note that

$$\mathbb{T}^{(3)} = \eta_{12}^{-1}(\mathbb{G}^*) \cap \eta_{13}^{-1}(\mathbb{G}^*) \cap \eta_{23}^{-1}(\mathbb{G}^*) \subset \mathbb{G}_{2,4} \times \mathbb{G}_{2,4} \times \mathbb{G}_{2,4}.$$

Hence $\mathbb{T}^{(3)}$ is a closed subset of $\mathbb{G}_{2,4} \times \mathbb{G}_{2,4} \times \mathbb{G}_{2,4}$. Furthermore,

$$\mathbb{G}_1 = \eta_{123}^{-1}(\mathbb{T}^{(3)}) \cap \eta_{145}^{-1}(\mathbb{T}^{(3)}) \cap \eta_{267}^{-1}(\mathbb{T}^{(3)}).$$

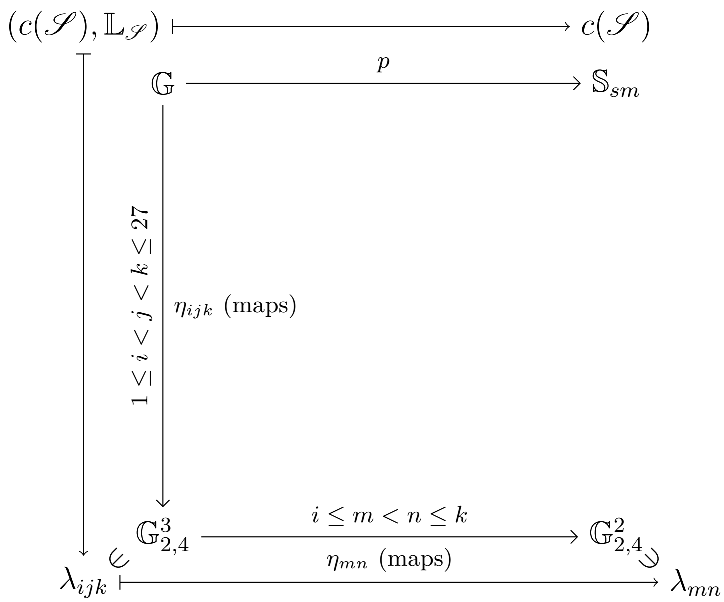

Figure 4.29: $\mathbb{E}^{(4)}, \mathbb{E}^{(6)}$ and $\mathbb{E}^{(9)}$ are closed subsets of $\mathbb{S}_{sm}$.

So $\mathbb{G}_1$ is closed subset of $\mathbb{G}$. Moreover, we have

$$\mathbb{G}_2 = \eta^{-1}_{123}(\mathbb{T}^{(3)}) \cap \eta^{-1}_{145}(\mathbb{T}^{(3)}) \cap \eta^{-1}_{678}(\mathbb{T}^{(3)}),$$

$$\mathbb{G}_3 = \eta^{-1}_{123}(\mathbb{T}^{(3)}) \cap \eta^{-1}_{456}(\mathbb{T}^{(3)}) \cap \eta^{-1}_{789}(\mathbb{T}^{(3)}).$$

It follows that both $\mathbb{G}_2$ and $\mathbb{G}_3$ are closed subsets of $\mathbb{G}$. Finally, since

$$\mathbb{E}^{(4)} = p(\mathbb{G}_1), \mathbb{E}^{(6)} = p(\mathbb{G}_2) \text{ and } \mathbb{E}^{(9)} = p(\mathbb{G}_3),$$

we have $\mathbb{E}^{(4)}, \mathbb{E}^{(6)}$ and $\mathbb{E}^{(9)}$ are closed subsets of $\mathbb{S}_{sm}$. □

Lemma 4.3.7. Let $s = \kappa_{123456} \in \mathbb{S}_6$ and define

$$\mathbb{K}^{(10)} := \left\{ \begin{array}{c} s \in \mathbb{S}_6 : \wedge(12,34,56) = \{P_7\}; \wedge(14,23,56) = \{P_8\} \text{ and} \\ l_{12}, l_{13} \text{ tangents to } \mathscr{C}_1 \end{array} \right\}.$$

Then $\mathbb{K}^{(10)}$ is an irreducible subset of $\mathbb{S}_6$ with codimension equal 4.

Proof. Let $\mathbb{L}_1$ be the subset of $\mathbb{C}_{sm} \times (\mathbb{P}^2)^3$ defined by

$$\mathbb{L}_1 := \Big\{ c(\mathscr{C}_1), \kappa_{123}) : l_{12}, l_{13} \text{ are tangents to } \mathscr{C}_1 \text{ at } P_2, P_3 \text{ respectively} \Big\}.$$

Let $\mathbb{L}_2$ be the subset of $\mathbb{L}_1 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_2 := \Big\{(x; P_8) : x \in \mathbb{L}_1 \text{ and } P_8 \in l_{23} \backslash \{P_2, P_3\}\Big\}.$$

Let $\mathbb{L}_3$ be the subset of $\mathbb{L}_2 \times (\mathbb{P}^2)^3$ defined by

$$\mathbb{L}_3 := \left\{ \begin{array}{c} (y; \kappa_{456}) : \kappa_{123456} \in \mathbb{S}_6, y \in \mathbb{L}_2 \text{ and } P_5, P_6 \in \mathscr{C}_1, \wedge(23, 56) = \{P_8\}, \\ \wedge(12, 34, 56) \neq \emptyset \text{ and } \mathscr{C}_1 \cap l_{18} = \{P_4\} \end{array} \right\}.$$

Let us consider the following projections (Figure 4.30).

$$\begin{array}{ccccccc}
(y; \kappa_{456}) & \longmapsto & (x; P_8) & \longmapsto & (c(\mathscr{C}_1), \kappa_{123}) & \longmapsto & c(\mathscr{C}_1) \\
\mathbb{L}_3 & \xrightarrow{p_3} & \mathbb{L}_2 & \xrightarrow{p_2} & \mathbb{L}_1 & \xrightarrow{p_1} & \mathbb{C}_{sm} \\
s \downarrow \quad \wr\!\!\parallel \Big\downarrow p_4 & & & & & & \\
\mathbb{K}^{(10)} & & & & & &
\end{array}$$

Figure 4.30: Irreducible components of $\mathbb{K}^{(10)}$.

The fibers of each projection are illustrated in Figure 4.31.

As in Lemma 4.3.5, the surjectivity of p_1 implies to the irreducibility of the fiber $p_1^{-1}(c(\mathscr{C}_1))$ as a subset of $\mathbb{L}_1$. Moreover, we have $\dim p_1^{-1}(c(\mathscr{C}_1)) = 2$. Since p_2 has an irreducible target of dimension 5, $\mathbb{L}_1$ is irreducible with dimension $5 + 2 = 7$. Furthermore, $\mathbb{L}_2$ is an irreducible with $\dim \mathbb{L}_2 = 7 + 1 = 8$ because every fiber $p_2^{-1}(c(\mathscr{C}_1), \kappa_{123})$ has dimension 1 under the surjective map p_2.

Note that the fiber $p_3^{-1}(x; P_8)$ consists of all points P_5, P_6 on the non-singular conic $\mathscr{C}_1$, so that $P_7 \in l_{56}$. More precisely, the fiber $p_3^{-1}(x; P_8)$ is a subset of $\mathscr{C}_1 \times \mathscr{C}_1$ isomorphic to the set S, where

$$S := \Big\{(P_5, P_6) \in \mathscr{C}_1 \times \mathscr{C}_1 : P_7 \in l_{56}\} = \{(P_5, P_6), (P_6, P_5)\Big\}.$$

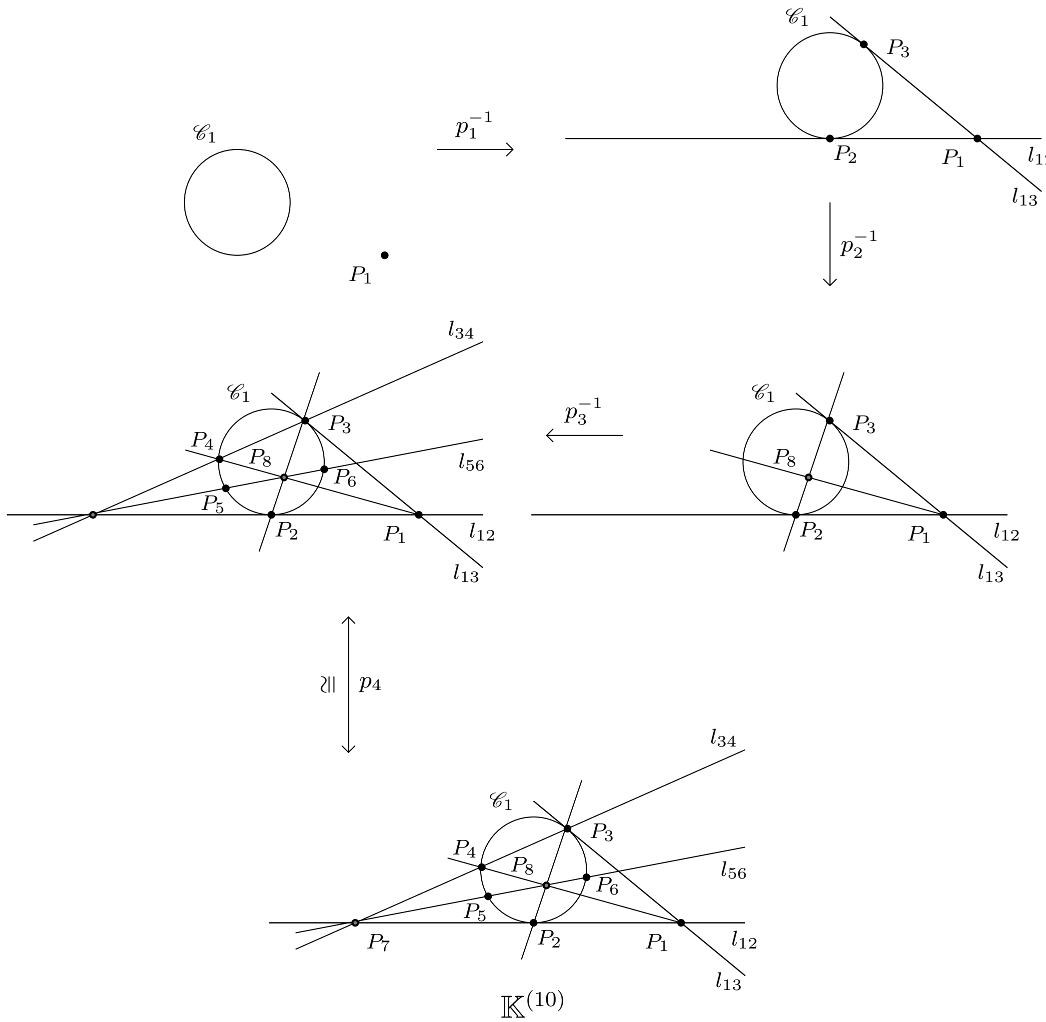

Figure 4.31: $\mathbb{K}^{(10)}$ and configurations of members of the fibers.

Since S is finite, we get $\dim S = 0 = \dim p_3^{-1}(x; P_8)$. Consequently, $\mathbb{L}_3$ is irreducible with dimension $8 + 0 = 8$. Finally, $\mathbb{L}_3$ isomorphic to $\mathbb{K}^{(10)}$ via the projection map p_4. So we conclude that $\mathbb{K}^{(10)}$ is irreducible subset of $\mathbb{S}_6$ with $\operatorname{codim} \mathbb{K}^{(10)} = 4$. $\square$

Theorem 4.14. $\mathbb{E}^{(10)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 4.

Proof. Let $\mathbb{K} := \mathbb{K}^{(10)}$ be the set defined as in Lemma 4.3.7. As in the proof of Theorem 4.11. Let us assume $B_1 := \psi^{-1}(\mathbb{K}^{(10)})$, and $B_2 := \xi(B_1)$. Then $\mathbb{E}^{(10)} = \gamma(B_2)$ where $\xi := \Xi\big|_{B_1}$, $\gamma := \Gamma\big|_{B_2}$, and $\psi := \Psi\big|_{B_1}$ (see the main diagram in Figure 4.12).

Then we have $\mathbb{E}^{(10)} = \gamma \circ \xi(B_1)$ which is a subset of $\mathbb{S}_{sm}$ (see Figure 4.32).

$$\begin{array}{ccccccc} \mathbb{K}^{(10)} & \xleftarrow{\ \psi\ } & B_1 & \xrightarrow{\ \xi\ } & B_2 & \xrightarrow{\ \gamma\ } & \mathbb{E}^{(10)} \\ \cap & & \cap & & \cap & & \cap \\ \mathbb{S}_6 & & \mathbb{B}_1 & & \mathbb{B}_2 & & \mathbb{S}_{sm} \end{array}$$

Figure 4.32: $\mathbb{E}^{(10)} = \gamma \circ \xi(B_1)$.

According to Equation (4.3.2), we have:

$$\dim B_1 = \dim \mathbb{K}^{(10)} + 15 = 8 + 15 = 23.$$

Since $\mathbb{E}^{(10)} = \gamma \circ \xi(B_1)$, it follows that $\mathbb{E}^{(10)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with

$$\begin{aligned} \dim \mathbb{E}^{(10)} &= \dim B_1 - \dim \xi^{-1} \circ \gamma^{-1}(x) = \dim B_1 - \dim \xi^{-1}(n) \\ &= 23 - 8 = 15, \text{ where } n = \gamma^{-1}(x), x \in \mathbb{E}^{(10)}. \end{aligned}$$

Therefore we conclude that $\mathbb{E}^{(10)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 4. □

Lemma 4.3.8. Let $s = \kappa_{123456} \in \mathbb{S}_6$ and define

$$\mathbb{K}^{(18)} := \left\{ \begin{array}{c} s \in \mathbb{S}_6 : \wedge(15, 24, 36) = \{P_7\};\ \wedge(14, 23, 56) = \{P_8\} \text{ and} \\ l_{12}, l_{13} \text{ tangents to } \mathscr{C}_1 \end{array} \right\}.$$

Then $\mathbb{K}^{(18)}$ is an irreducible subset of $\mathbb{S}_6$ with codimension equal 4.

Proof. Let $\mathbb{L}_1$ be the subset of $\mathbb{C}_{sm} \times (\mathbb{P}^2)^3$ defined by

$$\mathbb{L}_1 := \Big\{ c(\mathscr{C}_1), \kappa_{123}) : l_{12}, l_{13} \text{ are tangents to } \mathscr{C}_1 \text{ at } P_2, P_3 \text{ respectively} \Big\}.$$

Let $\mathbb{L}_2$ be the subset of $\mathbb{L}_1 \times \mathbb{P}^2$ defined by

$$\mathbb{L}_2 := \Big\{ (x; P_8) : x \in \mathbb{L}_1 \text{ and } P_8 \in l_{23} \backslash \{P_2, P_3\} \Big\}.$$

Let $\mathbb{L}_3$ be the subset of $\mathbb{L}_2 \times (\mathbb{P}^2)^3$ defined by

$$\begin{array}{ccccccc}
(y;\kappa_{456}) & \longmapsto & (x;P_8) & \longmapsto & (c(\mathscr{C}_1),\kappa_{123}) & \longmapsto & c(\mathscr{C}_1) \\
\mathbb{L}_3 & \xrightarrow{p_3} & \mathbb{L}_2 & \xrightarrow{p_2} & \mathbb{L}_1 & \xrightarrow{p_1} & \mathbb{C}_{sm} \\
\big\downarrow \; \wr\!\!\downarrow p_4 & & & & & & \\
s \;\; \mathbb{K}^{(18)} & & & & & &
\end{array}$$

Figure 4.33: Irreducible components of $\mathbb{K}^{(18)}$.

$$\mathbb{L}_3 := \left\{ \begin{array}{c} (y;\kappa_{456}) : \kappa_{123456} \in \mathbb{S}_6, y \in \mathbb{L}_2 \text{ and } P_5, P_6 \in \mathscr{C}_1, \wedge(23,56) = \{P_8\}, \\ \wedge(15,24,36) \neq \emptyset \text{ and } \mathscr{C}_1 \cap l_{18} = \{P_4\} \end{array} \right\}.$$

Let us consider the following projections (Figure 4.33):

As in Lemma 4.3.7, the surjectivity of p_1 implies the irreducibility of the fiber $p_1^{-1}(c(\mathscr{C}_1))$ as a subset of $\mathbb{L}_1$. Moreover, we have $\dim p_1^{-1}(c(\mathscr{C}_1)) = 2$. Since p_2 has an irreducible target of dimension 5, $\mathbb{L}_1$ is irreducible with dimension $5 + 2 = 7$. Furthermore, $\mathbb{L}_2$ is irreducible with $\dim \mathbb{L}_2 = 7 + 1 = 8$ because every fiber $p_2^{-1}(c(\mathscr{C}_1), \kappa_{123})$ has dimension 1 under the surjective map p_2.

Note that the fiber $p_3^{-1}(x; P_8)$ consists of all points P_5, P_6 on the non-singular conic

$$\mathscr{C}_1 = \mathbb{V}(\alpha_1 x_0^2 + \alpha_2 x_1^2 + \alpha_3 x_2^2 + \alpha_4 x_0 x_1 + \alpha_5 x_0 x_2 + \alpha_6 x_1 x_2),$$

so that $P_5, P_6 \in \mathscr{C}_1$. First of all, we need to find the equation of the non-singular conic $\mathscr{C}_1$ so that $\mathscr{C}_1$ passing through P_2, P_3, P_4, P_5, P_6 and has two tangents l_{12}, l_{13} at P_2, P_3 respectively. Now by change of coordinates, we can assume

$$P_1 = (0:1:0), P_2 = (-1:0:1), P_3 = (1:0:1), P_4 = (0:1:1).$$

It is clear that $l_{12} = \mathbb{V}(x_0 + x_2)$ and $l_{13} = \mathbb{V}(x_0 - x_2)$. Being the curve $\mathscr{C}_1$ passing through the points P_2, P_3, P_4 implies $\alpha_5 = 0$ and $\alpha_3 = -\alpha_1$. Furthermore, l_{12} and l_{13} being tangents to $\mathscr{C}_1$ at P_2, P_3 respectively implies that $\alpha_1 = -\frac{1}{2}, \alpha_3 = \frac{1}{2}, \alpha_6 - \alpha_4 = 0$

and $\alpha_6 + \alpha_4 = 0$. Thus $\alpha_2 = -\alpha_3 = \frac{1}{2}$ and $\alpha_6 = \alpha_4 = 0$. It follows that

$$\mathscr{C}_1 = \mathbb{V}(x_0^2 + x_1^2 - x_2^2).$$

By some algebraic computations, if we assume ξ is a cubic root of -1 then we can choose

$$P_5 = (1-\xi^2 : 2\xi : 1+\xi^2) \text{ and } P_6 = (\xi^2 - 1 : -2\xi : 1+\xi^2)$$

or

$$P_5 = (1+\xi : -2\xi^2 : 1-\xi) \text{ and } P_6 = (-1-\xi : 2\xi^2 : 1-\xi).$$

The fiber $p_3^{-1}(x; P_8)$ consists of two distinct points. It follows that $\dim p_3^{-1}(x; P_8) = 0$, and the set $\mathbb{L}_3$ has dimension equal to 8.

More precisely, we have

$$\dim \mathbb{L}_3 = \dim \mathbb{L}_2 + 0 = 8.$$

But $\mathbb{L}_3$ isomorphic to $\mathbb{K}^{(18)}$ via the projection map p_4. It follows that $\mathbb{K}^{(18)}$ has dimension equal 8. Therefore we conclude that $\mathbb{K}^{(18)}$ is an irreducible subset of $\mathbb{S}_6$ with codimension equal 4.. □

Theorem 4.15. $\mathbb{E}^{(18)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 4.

Proof. First, let $\mathbb{K} := \mathbb{K}^{(18)}$ (see Lemma 4.3.8). Let us assume $B_1 := \psi^{-1}(\mathbb{K}^{(18)})$, and $B_2 := \xi(B_1)$. Then $\mathbb{E}^{(18)} = \gamma(B_2)$ where $\xi := \Xi\big|_{B_1}$, $\gamma := \Gamma\big|_{B_2}$, and $\psi := \Psi\big|_{B_1}$ (see the main diagram in Figure 4.12). According to following diagram (see Figure 4.34) and Equation (4.3.2), we have

$$\begin{array}{ccccccc}
\mathbb{K}^{(18)} & \xleftarrow{\ \psi\ } & B_1 & \xrightarrow{\ \xi\ } & B_2 & \xrightarrow{\ \gamma\ } & \mathbb{E}^{(18)} \\
\cap & & \cap & & \cap & & \cap \\
\mathbb{S}_6 & & \mathbb{B}_1 & & \mathbb{B}_2 & & \mathbb{S}_{sm}
\end{array}$$

Figure 4.34: $\mathbb{E}^{(18)} = \gamma \circ \xi(B_1)$.

$$\dim B_1 = \dim \mathbb{K}^{(18)} + 15 = 8 + 15 = 23.$$

Since $\mathbb{E}^{(18)} = \gamma \circ \xi(B_1)$, it follows that $\mathbb{E}^{(18)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with

$$\begin{aligned}\dim \mathbb{E}^{(18)} &= \dim B_1 - \dim \xi^{-1} \circ \gamma^{-1}(x) = \dim B_1 - \dim \xi^{-1}(n) \\ &= 23 - 8 = 15, \text{ where } n = \gamma^{-1}(x), x \in \mathbb{E}^{(18)}.\end{aligned}$$

Thus $\mathbb{E}^{(18)}$ is an irreducible subset of $\mathbb{S}_{sm}$ with codimension 4. □

Almost all the main results of this section are summarized in Table 4.25.

Table 4.25: Closed subsets of $\mathbb{S}_{sm}$ with their codimension

Closed subsets of $\mathbb{S}_{sm}$	Irreducible components	Codimension
$\mathbb{S}^{(1)}$	$\mathbb{S}^{(1)}$	1
$\mathbb{S}^{(2)}$	$\mathbb{E}^{(2)}$	2
	$\mathbb{E}^{(3)}$	2
$\mathbb{E}^{(4)}$	$\mathbb{E}^{(4)}$	3
$\mathbb{E}^{(6)}$	$\mathbb{E}^{(4)}$	3
$\mathbb{E}^{(9)}$	$\mathbb{E}^{(9)}$	3
$\mathbb{E}^{(10)}$	$\mathbb{E}^{(10)}$	4
$\mathbb{E}^{(18)}$	$\mathbb{E}^{(18)}$	4

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Appendix A

Main Algorithms

Algorithm 1 The classification of 5-arcs in $PG(2,q)$

This algorithm is used for the classification of 5-arcs in $PG(2,q)$. The algorithm's input is

$$\text{projectively distinct 4-arcs } \mathcal{K} \text{ in } PG(2,q)$$

and the algorithm's output is

$$\text{projectively distinct 5-arcs } \mathcal{F} \text{ in } PG(2,q).$$

The algorithm proceeds in five stages.

Stage 1: For each 4-arc $\mathcal{K}$, calculate all bisecants $\overline{P_iP_j}$ of $\mathcal{K}$, where $i,j = 1,2,3,4$.

Stage 2: For each 4-arc $\mathcal{K}$, calculate all points not on any bisecants of $\mathcal{K}$, namely $O(\mathcal{K})$.

Stage 3: For each 4-arc $\mathcal{K}$, add one point from $O(\mathcal{K})\backslash\mathcal{K}$ to $\mathcal{K}$ in order to get 5-arc. The number of 5-arcs $\mathcal{F}$ that produced in this stage is $\sum |O(\mathcal{K})|$ where the sum is taken over all 4-arcs $\mathcal{K}$.

Stage 4: For each 5-arc $\mathcal{F}$, find the corresponding group of projectivities $G(\mathcal{F})$ (see the procedure used in Section 2.5).

Stage 5: Find the projectively distinct 5-arcs in $PG(2,q)$. Two 5-arcs $\mathcal{F}_1$ and $\mathcal{F}_2$ are projectively equivalent if and only if $G(\mathcal{F}_1) \cong G(\mathcal{F}_2)$.

Algorithm 2 The classification of 6-arcs in $PG(2,q)$

This algorithm is used for the classification of 6-arcs in $PG(2,q)$. The algorithm's input is

$$\text{projectively distinct 5-arcs } \mathcal{F} \text{ in } PG(2,q)$$

and the algorithm's output is

$$\text{projectively distinct 6-arcs } \mathcal{S} \text{ in } PG(2,q).$$

The algorithm proceeds in five stages.

Stage 1: For each 5-arc $\mathcal{F}$, calculate all bisecants $\overline{P_iP_j}$ of $\mathcal{F}$, where $i,j = 1,2,3,4,5$.

Stage 2: For each 5-arc $\mathcal{F}$, calculate all points not on any bisecants of $\mathcal{F}$, namely $O(\mathcal{F})$.

Stage 3: For each 5-arc $\mathcal{F}$, add one point from $O(\mathcal{F})\backslash\mathcal{F}$ to $\mathcal{F}$ in order to get 6-arc. The number of 6-arcs $\mathcal{F}$ that produced in this stage is $\sum |O(\mathcal{F})|$ where the sum taken over all 5-arcs $\mathcal{F}$.

Stage 4: For each 6-arc $\mathcal{F}$, find the corresponding group of projectivities $G(\mathcal{F})$ (see the procedure used in Section 2.5).

Stage 5: Find the projectively distinct 6-arcs in $PG(2,q)$. Two 6-arcs $\mathcal{S}_1$ and $\mathcal{S}_2$ are projectively equivalent if and only if $G(\mathcal{S}_1) \cong G(\mathcal{S}_2)$.

Algorithm 3 Blowing-up $PG(2,q)$ at six points in general position

---

This algorithm is used for the construction of the cubic surface with twenty-seven lines in $PG(3,q)$ arising from a six points in general position in $PG(2,q)$. The algorithm's input is

$$\mathcal{S} = \text{6-arc not on a conic in } PG(2,q)$$

and the algorithm's output is

$$\mathscr{S} = \text{cubic surface with twenty-seven lines in } PG(3,q) \text{ associated to the blowing-up of } \mathcal{S}.$$

The algorithm proceeds in eight stages.

Stage 1: For each 6-arc $\mathcal{S}$, calculate six different conics $\mathscr{C}_j$ through $\mathcal{S}\backslash P_j$, where $j = 1, \ldots, 6$.

Stage 1: For each 6-arc $\mathcal{S}$, calculate six different conics $\mathscr{C}_j$ through $\mathcal{S}\backslash P_j$, where $j = 1, \ldots, 6$.

Stage 2: For each 6-arc $\mathcal{S}$, calculate 15 different bisecants $\overline{P_iP_j}$ of $\mathcal{S}$.

Stage 3: For each 5-arc $\mathcal{S}$, calculate 30 different cubic curves through $\mathcal{S}$ of the form $\mathscr{C}_j \times \overline{P_iP_j}$.

Stage 4: For each 6-arc $\mathcal{S}$, calculate 15 different cubic curves through $\mathcal{S}$ of the form $\overline{P_iP_j} \times \overline{P_kP_l} \times \overline{P_mP_n}$.

Stage 5: For each 6-arc $\mathcal{S}$, fix a four linearly independent plane cubic curves as base curves. For example, for a 6-arc

$\mathcal{S}_3 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{33}\}$ in $PG(2, 19)$ we have

$$\omega_1 = \mathbb{V}(\overline{P_1P_3} \times \mathscr{C}_3), \qquad \omega_1' = \mathbb{V}(\overline{P_2P_3} \times \mathscr{C}_3),$$
$$\omega_2 = \mathbb{V}(\overline{P_2P_6} \times \mathscr{C}_6), \qquad \omega_2' = \mathbb{V}(\overline{P_1P_3} \times \overline{P_2P_6} \times \overline{P_4P_5}).$$

Stage 6: For each 6-arc $\mathcal{S}$, find the tritangent planes of the corresponding $\mathcal{S}$ in $PG(3, q)$.

The tritangent plane $(c_{ij}c_{kl}c_{mn})$ is the image of the cubic curve $\overline{P_iP_j} \times \overline{P_kP_l} \times \overline{P_mP_n}$ and the tritangent plane $(a_ib_jc_{ij})$ is the image of the cubic curve $\mathscr{C}_j \times \overline{P_iP_j}$. Consider four base cubic plane curves, namely

$$\omega_1 = \mathbb{V}(W_1),$$
$$\omega_2 = \mathbb{V}(W_2),$$
$$\omega_3 = \mathbb{V}(W_3),$$
$$\omega_4 = \mathbb{V}(W_4),$$

the four associated base tritangent planes can be written as follows:

$$\pi_{\omega_1} = \mathbb{V}(\Pi_{W_1}),$$

$$\pi_{\omega_2} = \mathbb{V}(\Pi_{W_2}),$$

$$\pi_{\omega_3} = \mathbb{V}(\Pi_{W_3}),$$

$$\pi_{\omega_4} = \mathbb{V}(\Pi_{W_4}),$$

where Π_{W_j} is a linear form defining π_{ω_j} and corresponding to the cubic form W_j defining ω_j. Every tritangent plane on $\mathscr{S}$ can be written as a linear combination of $\Pi_{W_1}, \Pi_{W_2}, \Pi_{W_3}$ and Π_{W_4}. In this stage, 45 tritangent planes have to be calculated as the linear combination of $\Pi_{W_1}, \Pi_{W_2}, \Pi_{W_3}$ *and* Π_{W_4}.

Stage 7: For each 6-arc $\mathcal{S}$, find the trihedral pairs of the corresponding non-singular cubic surface $\mathscr{S}$ in $PG(3, q)$.

Pick a trihedral pair related to four base tritangent planes $\mathbb{V}(y_0)$, $\mathbb{V}(y_1)$, $\mathbb{V}(y_2)$, $\mathbb{V}(y_3)$ and four base cubic curves $\omega_1,\ \omega_2,\ \omega_1',\ \omega_2'$. For example, for the non-singular cubic surface $\mathscr{S}^{(10)}(19)$ that corresponds to the 6-arc

$$\mathcal{S}_3 = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{6}, \mathbf{33}\}$$

in $PG(2, 19)$ we have the following trihedral pairs

$$T_{12,36}, T_{13,26}, T_{34,52} \ \textit{and} \ T_{14,56}.$$

Pick one of them, namely

$$\begin{array}{lllll} T_{12,36}: & a_1 & b_3 & c_{13} & \rightsquigarrow \omega_1 = \mathbb{V}(W_1) \leftrightarrow \pi_{\omega_1} = \mathbb{V}(\Pi_{W_1}) \\ & b_6 & a_2 & c_{26} & \rightsquigarrow \omega_2 = \mathbb{V}(W_2) \leftrightarrow \pi_{\omega_2} = \mathbb{V}(\Pi_{W_2}) \\ & c_{16} & c_{23} & c_{45} & \rightsquigarrow \omega_3 = \mathbb{V}(W_3) \leftrightarrow \pi_{\omega_3} = \mathbb{V}(\Pi_{W_3}) \\ & \wr\downarrow & \wr\downarrow & \wr\downarrow & \\ & \omega_3' & \omega_1' & \omega_2' & \end{array}$$

where

$$\omega_3' = \mathbb{V}(W_3') \leftrightarrow \pi_{\omega_3'} = \mathbb{V}(\Pi_{W_3'})$$

$$\omega_1' = \mathbb{V}(W_1') \leftrightarrow \pi_{\omega_1'} = \mathbb{V}(\Pi_{W_1'})$$

$$\omega_2' = \mathbb{V}(W_2') \leftrightarrow \pi_{\omega_2'} = \mathbb{V}(\Pi_{W_2'})$$

Stage 8: For each 6-arc $\mathcal{S}$, find the equation of the corresponding cubic surface with 27 lines $\mathscr{S}$ where

$$\mathscr{S} = \mathbb{V}(\Pi_{W_1}\Pi_{W_2}\Pi_{W_3} - \lambda\Pi_{W_1'}\Pi_{W_2'}\Pi_{W_3'}) \text{ and } \lambda \in GF(q)\backslash\{0\}. \tag{A.0.1}$$

It is known that every cubic surface can be written in 120 ways in the form above. The parameter λ can be found in the following way. The Clebsch map is surjective; that is, every point in $PG(2,q)$ is an image of points on the cubic surface $\mathscr{S}$. Now, pick any point not lying on any of the 4 base plane cubic, namely $P^*(x_0,x_1,x_2)$. It follows that

$$\Big(\omega_1(P^*) : \omega_2(P^*) : \omega_1'(P^*) : \omega_2'(P^*)\Big) \mapsto (y_0 : y_1 : y_2 : y_3),$$

where $\omega_1, \omega_2, \omega_1', \omega_2'$ are the base cubic curves in the plane passing through the 6-arc $\mathcal{S}$. Consequently, the parameter λ is found by evaluating (y_0, y_1, y_2, y_3) in Equation (4.6)

Algorithm 4 e-invariants of a smooth cubic surface with 27 lines

---

This algorithm is used for the classification of cubic surfaces with twenty-seven lines in $PG(3,q)$ up to e-invariants. The algorithm's input is

$\mathscr{S}$ = a cubic surface with twenty-seven lines in $PG(3,q)$ associated to the blowing-up of a 6-arcs $\mathcal{S} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$ not on a conic in $PG(2,q)$.

and the algorithm's output is

e-invariants which correspond to $\mathscr{S}$, namely e_3, e_2, e_1, e_0.

The algorithm proceeds in three stages.

Stage 1: For each a cubic surface $\mathscr{S}$ associated to the blowing-up of a 6-arcs

$$\mathcal{S} = \{P_1, P_2, P_3, P_4, P_5, P_6\}$$

not on a conic in $PG(2, q)$, find all possible Eckardt points.

If $C_j \cap \overline{P_iP_j} = \{P_i\}$ then

print "E_{ij} is an Eckardt point";

if else $\overline{P_iP_j} \cap \overline{P_kP_l} \cap \overline{P_mP_n} = \{P\}$ where $P \in PG(2, q)$ then

print "$E_{ij,kl,mn}$ is an Eckardt point";

else print failed;

end

In this stage we get the number of Eckardt points, namely e_3.

Stage 2: For each cubic surface $\mathscr{S}$, calculate e-invariants which correspond to $\mathscr{S}$, namely e_3, e_2, e_1, e_0 where

$$e_2 = 135 - 3e_3,$$

$$e_1 = 27(q - 4) - e_2,$$

$$e_0 = (q - 10)^2 - e_3 + 9.$$

Stage 3: Calculate the distinct non-singular cubic surfaces with 27 lines up to e-invariants in $PG(3, q)$.

Two cubic surfaces $\mathscr{S}_1$ and $\mathscr{S}_2$ are equivalent if and only if they have the same e-invariants.

Algorithm 5 Elliptic and hyperbolic lines on a non-singular cubic surface

---

This algorithm is used to determine the elliptic and hyperbolic lines on a non-singular cubic surface with 27 lines in $PG(3, q)$. The algorithm's input is

$$\mathscr{S} = \text{a cubic surface with twenty-seven lines in } PG(3, q).$$

and the algorithm's output is

$$\text{elliptic and hyperbolic lines on } \mathscr{S}.$$

The algorithm proceeds in three stages.

Stage 1: Find all the 27 lines on $\mathscr{S} = \mathbb{V}(f)$.

Stage 2: For each line l on $\mathscr{S}$, namely

$$l = \Big\{(a_0\lambda + b_0\mu : a_1\lambda + b_1\mu : a_2\lambda + b_2\mu : a_3\lambda + a_3\mu) : (\lambda : \mu) \in PG(2, q)\Big\},$$

pick a basis $\beta_3, \beta_2, \beta_1, \beta_0$ for $GF(q)^{\oplus 4}$ so that

$$\beta_3 = (a_0, a_1, a_2, a_3),$$

$$\beta_2 = (b_0, b_1, b_2, b_3).$$

Stage 3: For each line l on $\mathscr{S}$, compute the resultant of the partial derivative of the homogeneous polynomial defining $\mathscr{S}$ with respect to β_0 and β_1, and then restrict the result to the line l:

$$R_{0,1} = R\left(\frac{\partial f}{\partial \beta_0}\bigg|_l, \frac{\partial f}{\partial \beta_1}\bigg|_l\right).$$

If $R_{0,1} = \zeta^2 \pmod q$ for some $\zeta \in GF(q)$ then

print "l is a hyperbolic line";

else if $R_{0,1} \neq \zeta^2 \pmod q$ for all $\zeta \in GF(q)$ then

print "l is an elliptic line";

end if