

Spring 2019

## On The Generators of Quantum Dynamical Semigroups

Alexander Wiedemann

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ON THE GENERATORS OF QUANTUM DYNAMICAL SEMIGROUPS

by

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Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2019

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## DEDICATION

*To my parents, without whom none of my success would be possible.*

## ABSTRACT

In recent years, digraph induced generators of quantum dynamical semigroups have been introduced and studied, particularly in the context of unique relaxation and invariance. We define the class of pair block diagonal generators, which allows for additional interaction coefficients but preserves the main structural properties. Namely, when the basis of the underlying Hilbert space is given by the eigenbasis of the Hamiltonian (for example the generic semigroups), then the action of the semigroup leaves invariant the diagonal and off-diagonal matrix spaces. In this case, we explicitly compute all invariant states of the semigroup.

In order to define this class we provide a characterization of when the Gorini-Kossakowski-Sudarshan-Lindblad (GKSL) equation defines a proper generator when arbitrary Lindblad operators are allowed (in particular, they do not need to be traceless as demanded by the GKSL Theorem). Moreover, we consider the converse construction to show that every generator naturally gives rise to a digraph, and that under certain assumptions the properties of this digraph can be exploited to gain knowledge of both the number and the structure of the invariant states of the corresponding semigroup.

We also consider more general constructions on the von Neumann algebra of all bounded linear operators on a Hilbert space, perhaps infinite dimensional. In particular, we prove that for every semigroup of Schwarz maps on such an algebra which has a subinvariant faithful normal state there exists an associated semigroup of contractions on the space of Hilbert-Schmidt operators of the Hilbert space. Moreover, we show that if the original semigroup is weak\* continuous then the associated semi-

group is strongly continuous. We introduce the notion of the extended generator of a semigroup on the bounded operators of a Hilbert space with respect to an orthonormal basis of the Hilbert space. We describe this form of the generator of a quantum Markov semigroup on the von Neumann algebra of all bounded linear operators on a Hilbert space which has an invariant faithful normal state under the assumption that the generator of the associated semigroup has compact resolvent, or under the assumption that the minimal unitary dilation of the associated semigroup of contractions is compact.

## PREFACE

The principle objects of study in this work are the generators of semigroups acting on operator spaces. For topological reasons, there is a clear distinction between those generators which act on finite dimensional spaces and those which act on infinite dimensional spaces. This work is thus presented in two Parts, each of which is independent of the other. The details of what is proved in each Part is summarized in the appropriate introduction.

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# CHAPTER 1

## INTRODUCTION TO PART I

The Schrödinger picture time evolution of an open quantum system with finitely many degrees of freedom is, under certain limiting conditions, described in terms of a quantum dynamical semigroup (QDS)  $(T_t)_{t \geq 0} : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  (see e.g. [5, 6]), where  $M_N(\mathbb{C})$  denotes the  $N \times N$  matrices with complex entries. Each such QDS can be written as  $T_t = e^{t\mathcal{L}} = \sum_{k=0}^{\infty} t^k \mathcal{L}^k / k!$  for some  $\mathcal{L}$  called the generator of the QDS. Famously, simultaneous results of Gorini-Kossakowski-Sudarshan in [42] and Lindblad in [53] show that every QDS generator can be written as  $\mathcal{L}(\rho) = -i[H, \rho] + \frac{1}{2} \sum c_{ij}([F_i, \rho F_j^*] + [F_i \rho, F_j^*])$ , the now-called GKSL form (see Theorem 2.1). We call  $H$  the Hamiltonian of the QDS.

Of particular interest are the digraph induced generators (where digraph means directed, positively weighted graph; see Section 4.2), which we define as those of the form

$$\mathcal{L}(\rho) = -i[H, \rho] + \frac{1}{2} \sum_{i \neq j} \gamma_{ij} \left( [E_{ij}, \rho E_{ij}^*] + [E_{ij} \rho, E_{ij}^*] \right), \quad (1.1)$$

where  $E_{ij}$  are the standard basis elements of  $M_N(\mathbb{C})$  which have entry 1 in the  $i$ th row and  $j$ th column and all other entries are zero. We choose this terminology as given an digraph  $G$  on  $N$  vertices with weights  $\gamma_{ij}$  one can consider the induced generator acting on  $M_N(\mathbb{C})$  given by (1.1) for some appropriately chosen Hamiltonian  $H$ . Indeed, Rodríguez-Rosario, Whitfield, and Aspuru-Guzik in [70] introduced such an example in the graph case (i.e.  $\gamma_{ij} = \gamma_{ji}$ ) with  $H = 0$  to recover the classical random walk on  $G$ . Liu and Balu in [54], also in the graph case, set  $H$  to be the corresponding graph Laplacian (defined in Section 4.1) to give an alternate definition for a continuous-time

open quantum random walk on  $G$  (the original owing to Pellegrini in [58], and yet another by Sinayskiy and Petruccione in [62]); further, they show connected graphs induce uniquely relaxing semigroups. Glos, Miszczak, and Ostaszewski in [41] extend this definition to digraphs by allowing  $\gamma_{ij} \neq \gamma_{ji}$ , and show  $\mathcal{L}$  generates a uniquely relaxing semigroup for arbitrary  $H$  if the digraph has strictly one terminal strongly connected component (defined in Section 4.2).

In the case  $H = \sum_{n=1}^N h_n E_{nn}$  in (1.1) we recover the generic generators, which were introduced (in the infinite dimensional case) by Accardi and Kozyrev in [2] as the stochastic limit of a discrete system with generic free Hamiltonian interacting with a mean zero, gauge invariant, 0-temperature, Gaussian field (and later generalized to positive temperature in [1]). The finite-dimensional class of generic generators contain many well known and physically important models, such as coherent quantum control of a three-level atom in  $\Lambda$ -configuration interacting with two laser fields [3]. Though the physical models require relations between the coefficients beyond what we write here, e.g. that  $H$  is generic (hence the name), we ignore such restrictions and consider more generally any generator of form (1.1) with  $H = \sum_{n=1}^N h_n E_{nn}$  a generic generator.

The generic generators are well studied and, though typically parsed in the language of Markov chains, some relations to digraph theory are known. Notably, from Accardi, Fagnola, and Hachica in [1] it is known that given any matrix its diagonal and off-diagonal evolve independently of each other under the QDS arising from a generic generator, and in fact the action on diagonal operators describes the evolution of a classical continuous time Markov chain (with rates  $\gamma_{ij}$ ) and the action on off-diagonal operators is given by conjugation with a contraction semigroup and its adjoint. With this relationship to Markov chains, Carbone, Sasso, and Umanita in [17] find the general structure of the states fixed by the QDS, which can be computed given the kernel of the generator of the associated Markov chain. In that paper, these authors also examine the related problem of fixed points for the dual semigroup

(Heisenberg picture) in context of the decoherence-free subalgebra (see also [31, 25, 16, 14] and references therein).

The purpose of this work is twofold: First, we generalize the digraph induced generators given by (1.1) in such a way that the results mentioned above remain true. We accomplish this generalization by allowing additional interaction coefficients, such as  $\gamma_{ii}$ , which preserve the main structural properties (notably, that if the Hamiltonian is diagonal then the diagonal and off-diagonal of a matrix evolve independently). We call such generators ‘pair block diagonal’ generators, for reasons which will be made clear, and compute explicitly all invariant states in the diagonal Hamiltonian case. Second, we consider the converse construction to show that every QDS generator naturally gives rise to a digraph, and that under certain assumptions the properties of this digraph can be exploited to gain knowledge of both the number and the structure of the invariant states of the corresponding semigroup.

## 1.1 STRUCTURE OF PART I

The structure of this Part is as follows:

- In Section 2.1 we establish formal definitions and notation for QDSs, and then provide a characterization of when the GKSL form defines a proper generator when allowed arbitrary orthonormal Lindblad operators. A physical three-level system is discussed to highlight some differences between the forms. In Section 2.2 we note the equivalence between identity preservation and contractivity of a QDS in some, equivalently all Schatten  $p$ -norms for  $p > 1$ .
- In Section 3.1 we establish the bulk of our notation and examine the structural properties of a generator when written with respect to the standard basis, which allows us to motivate and define the class of pair block diagonal generators (which contains the aforementioned digraph induced generators). Whereas the digraph induced generators can be used to model jumps between vector states, we remark that

the pair block diagonal generators can be used to model jumps between superpositions of states. In Section 3.2 we rephrase this notation and definition in terms of the Gell-Mann basis.

- In Sections 4.1 and 4.2 we establish the necessary graph and digraph terminology, as well as recall the necessary results.
- In Chapter 5 we define our main digraph of interest and show explicitly that every generator is naturally associated to a digraph through restriction to the diagonal subalgebra of  $M_N(\mathbb{C})$ . We explicitly give the kernel of such restrictions.
- In Section 6.1 we consider the action of pair block diagonal generators on the off-diagonal subspace, and compute explicitly the eigenvalues and eigenmatrices of such. In Section 6.2 we combine these kernel representations of the diagonal and off-diagonal restrictions to give an explicit formula for the kernel of a pair block diagonal generator, and thereby an explicit formula for all invariant states of the corresponding QDS.
- In Section 7.1 we examine QDSs which are contractive for Schatten norms  $p > 1$  and show all invariant states of such QDSs are invariant for a naturally associated graph induced QDS. In Section 7.2 we define the notion of consistent generators as those which have Hamiltonian consistent with the naturally associated digraph, and show such generators have a lower bound on the number of invariant states for the corresponding QDS based on the connectedness of the digraph.

## CHAPTER 2

### GENERAL PROPERTIES OF QDSs

#### 2.1 THE FORM OF $\mathcal{L}$

Formally, a QDS (in the Schrödinger picture) on  $M_N(\mathbb{C})$  is a one-parameter family of linear operators  $(T_t)_{t \geq 0}$  of  $M_N(\mathbb{C})$  satisfying:

- $T_0$  is the identity on  $M_N(\mathbb{C})$ ,
- $T_{t+s} = T_t T_s$  for all  $t, s \geq 0$ ,
- $t \mapsto T_t(A)$  is (weakly) continuous for all  $A \in M_N(\mathbb{C})$ ,
- $\text{Tr}(T_t(A)) = \text{Tr}(A)$  for all  $A \in M_N(\mathbb{C})$  and all  $t \geq 0$ , and
- $T_t$  is completely positive for all  $t \geq 0$ .

Let  $D_N(\mathbb{C})$  denote the set of  $N \times N$  states (i.e. positive semidefinite matrices of unit trace). When restricted to  $D_N(\mathbb{C})$  the QDS describes the Schrödinger dynamics of a quantum system with finitely many degrees of freedom. Every QDS on  $M_N(\mathbb{C})$  can be written in the form  $T_t = e^{t\mathcal{L}} := \sum_{k=0}^{\infty} t^k \mathcal{L}^k / k!$ , where  $\mathcal{L}(x) = \lim_{t \downarrow 0} \frac{1}{t}(T_t(x) - x)$  is called the generator of the QDS. Let  $S_2^N$  denote  $M_N(\mathbb{C})$  endowed with the norm  $\|A\|_2 = (\text{Tr}(|A|^2))^{1/2}$ , which is induced by the Hilbert-Schmidt inner product  $\langle A, B \rangle = \text{Tr}(A^* B)$ . The following characterization of such  $\mathcal{L}$  is the renowned GKSL form:

**Theorem 2.1** ([42, 53]). *Let  $\{F_i | 1 \leq i \leq N^2 - 1\}$  be a set of  $N \times N$  traceless orthonormal matrices (w.r.t. the Hilbert-Schmidt inner product). An operator  $\mathcal{L} :$*

$M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  is the generator of a QDS on  $M_N(\mathbb{C})$  if and only if it can be expressed in the form

$$\mathcal{L}(\rho) = -i[H, \rho] + \frac{1}{2} \sum_{i,j=1}^{N^2-1} c_{ij}([F_i, \rho F_j^*] + [F_i \rho, F_j^*]), \quad (2.1)$$

with  $H$  Hermitian and  $C = (c_{ij})$  an  $(N^2 - 1) \times (N^2 - 1)$  positive semidefinite matrix. Given  $\mathcal{L}$  the Hamiltonian  $H$  is uniquely determined by  $\text{Tr}(H) = 0$ ; given  $\mathcal{L}$  the coefficient matrix  $C$  is uniquely determined by the choice of  $F_i$ 's.

If  $H = 0$  we say  $\mathcal{L}$  is Hamiltonian-free. We note that  $H$  describes the reversible dynamics of the system, and that all physically important information pertaining to the irreversible dynamics is contained in the positive semidefinite matrix  $C$ .

We are particularly interested in characterizing invariant states of a given QDS  $(T_t)_{t \geq 0}$ ; that is, states  $\rho \in D_N(\mathbb{C})$  satisfying  $T_t(\rho) = \rho$  for all  $t \geq 0$ . To this end, notice that if  $T_t(x) = x$  for all  $t \geq 0$  then  $\mathcal{L}(x) = \lim_{t \downarrow 0} \frac{1}{t}(T_t(x) - x) = 0$ , and if  $\mathcal{L}(x) = 0$  then certainly  $T_t(x) = \sum_{k=0}^{\infty} t^k \mathcal{L}^k(x)/k! = x$ . Hence a  $T_t(x) = x$  for all  $t \geq 0$  if and only if  $\mathcal{L}(x) = 0$ . Recalling Lemma 17 of [9], which states that  $\ker \mathcal{L}$  is spanned by states, we have

$$\ker \mathcal{L} = \text{Span}\{\rho \in D_N(\mathbb{C}) : T_t(\rho) = \rho \text{ for all } t \geq 0\}. \quad (2.2)$$

Note that  $\dim \ker \mathcal{L} \geq 1$  since  $\mathcal{L}$  has traceless range, and so every QDS possesses at least one invariant state.

Let  $M_N^0(\mathbb{C})$  denote the set of  $N \times N$  traceless matrices. Given two orthonormal bases  $\{F_i | 1 \leq i \leq N^2 - 1\}$  and  $\{G_i | 1 \leq i \leq N^2 - 1\}$  of  $M_N^0(\mathbb{C})$  there is an  $(N^2 - 1) \times (N^2 - 1)$  unitary matrix  $U$  such that  $[G_1, G_2, \dots, G_{N^2-1}] = [F_1, F_2, \dots, F_{N^2-1}]U$ , representing the change of basis from  $G_i$ 's to  $F_i$ 's; that is, for  $U = (u_{ij})$ , we have  $G_i = \sum_{k=1}^{N^2-1} u_{ki} F_k$  and contrariwise  $F_i = \sum_{k=1}^{N^2-1} \bar{u}_{ik} G_k$  for all  $1 \leq i \leq N^2 - 1$ .



Considering (2.1), we have  $\mathcal{L}(\rho) + i[H, \rho] =$

$$\begin{aligned}
&= \frac{1}{2} \sum_{i,j=1}^{N^2-1} c_{ij} ([F_i, \rho F_j^*] + [F_i \rho, F_j^*]) \\
&= \frac{1}{2} \sum_{i,j=1}^{N^2-1} c_{ij} \left( \left[ \sum_{k=1}^{N^2-1} \bar{u}_{ik} G_k, \rho \left( \sum_{\ell=1}^{N^2-1} \bar{u}_{j\ell} G_\ell \right)^* \right] + \left[ \sum_{k=1}^{N^2-1} \bar{u}_{ik} G_k \rho, \left( \sum_{\ell=1}^{N^2-1} \bar{u}_{j\ell} G_\ell \right)^* \right] \right) \\
&= \frac{1}{2} \sum_{i,j,k,\ell=1}^{N^2-1} \bar{u}_{ik} c_{ij} u_{j\ell} ([G_k, \rho G_\ell^*] + [G_k \rho, G_\ell^*]) \\
&= \frac{1}{2} \sum_{k,\ell=1}^{N^2-1} \tilde{c}_{k\ell} ([G_k, \rho G_\ell^*] + [G_k \rho, G_\ell^*]),
\end{aligned}$$

where  $\tilde{c}_{k\ell} = \sum_{i,j=1}^{N^2-1} \bar{u}_{ik} c_{ij} u_{j\ell}$  are the entries of  $\tilde{C} = U^* C U$ . Thus, the  $(N^2-1) \times (N^2-1)$  matrix  $C$  when viewed as an operator  $C : M_N^0(\mathbb{C}) \rightarrow M_N^0(\mathbb{C})$  is uniquely determined by  $\mathcal{L}$ , with the choice of  $F_i$ 's being nothing but a choice of which orthonormal basis of  $M_N^0(\mathbb{C})$  for the matrix form of  $C$  to be represented in.

This operator viewpoint allows us to view every QDS generator  $\mathcal{L}$  as the pair  $H$  and  $C$  uniquely determined by Theorem 2.1. If we drop the traceless requirement from Theorem 2.1 so that the coefficient matrix acts on all of  $M_N(\mathbb{C})$  instead of just  $M_N^0(\mathbb{C})$ , then we need to require stronger operator level properties (i.e., properties that do not rely on the choice of basis) to guarantee  $\mathcal{L}$  is a QDS generator.

**Theorem 2.2.** *Let  $\{F_i | 1 \leq i \leq N^2\}$  be a set of  $N \times N$  orthonormal matrices (w.r.t. the Hilbert-Schmidt inner product). An operator  $\mathcal{L} : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  is the generator of a QDS on  $M_N(\mathbb{C})$  if and only if it can be expressed in the form*

$$\mathcal{L}(\rho) = -i[\tilde{H}, \rho] + \frac{1}{2} \sum_{i,j=1}^{N^2} \gamma_{ij} ([F_i, \rho F_j^*] + [F_i \rho, F_j^*]), \quad (2.3)$$

with  $\tilde{H}$  Hermitian and  $\Gamma = (\gamma_{ij})$  an  $N^2 \times N^2$  matrix, regarded as acting on  $M_N(\mathbb{C})$  equipped with basis  $\{F_i\}$ , satisfying

- $P\Gamma|_{M_N^0(\mathbb{C})} \geq 0$ , where  $P$  is the orthogonal projection from  $M_N(\mathbb{C})$  onto  $M_N^0(\mathbb{C})$ ,
- and

- $\text{Re Tr}(\Gamma(A)) = \text{Re Tr}(\Gamma(I_N)A)$  for all Hermitian  $A \in M_N(\mathbb{C})$ .

The operator  $P\Gamma|_{M_N^0(\mathbb{C})}$  is uniquely determined by  $\mathcal{L}$ . These conditions are satisfied if  $\Gamma \geq 0$ .

We remark that Theorem 2.2 is a natural extension of Theorem 2.1, in that the latter can be recovered by defining operator  $\Gamma : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  by  $\Gamma|_{M_N^0(\mathbb{C})} = C$  and  $\Gamma(I_N) = 0$ . Indeed, in this case  $P\Gamma|_{M_N^0(\mathbb{C})} = C \geq 0$  and  $\text{Tr}(\Gamma(A)) = 0$  for all  $A \in M_N(\mathbb{C})$  simply because  $C$  has traceless range.

*Proof.* As (2.1) is a special case of (2.3), it suffices to prove that (2.3) always defines as QDS generator. Since the preceding argument for converting bases did not rely on any properties of the  $F_i$ 's or  $G_i$ 's beyond orthonormality, it will suffice to prove this for a fixed orthonormal basis  $\{F_i\}$ . To this end, we assume without loss of generality that  $F_{N^2} = I_N/\sqrt{N}$  and that each  $F_i$  is Hermitian (e.g., the Gell-Mann basis defined in Section 3.2). First note that the value of  $\gamma_{N^2 N^2}$  has no effect on the action of  $\mathcal{L}$ , since  $\gamma_{N^2 N^2}([I_N/\sqrt{N}, \rho I_N/\sqrt{N}] + [I_N/\sqrt{N} \rho, I_N/\sqrt{N}]) = 0$ . We thus assume that  $\gamma_{N^2 N^2} = 0$ . Next, we compute

$$\begin{aligned} & \gamma_{iN^2} \left( \left[ F_i, \rho \frac{I_N}{\sqrt{N}} \right] + \left[ F_i \rho, \frac{I_N}{\sqrt{N}} \right] \right) + \gamma_{N^2 i} \left( \left[ \frac{I_N}{\sqrt{N}}, \rho F_i \right] + \left[ \frac{I_N}{\sqrt{N}} \rho F_i \right] \right) = \\ & = \frac{\gamma_{iN^2}}{\sqrt{N}} [F_i, \rho] + \frac{\gamma_{N^2 i}}{\sqrt{N}} [\rho, F_i] = \frac{\gamma_{iN^2} - \gamma_{N^2 i}}{\sqrt{N}} [F_i, \rho] = -i \left[ \frac{\text{Im}(\gamma_{N^2 i} - \gamma_{iN^2})}{\sqrt{N}} F_i, \rho \right] \end{aligned}$$

where the last equality follows since

$$\text{Re } \gamma_{iN^2} = \text{Re Tr} \left( F_i \Gamma \left( \frac{I_N}{\sqrt{N}} \right) \right) = \text{Re Tr} \left( \Gamma(F_i) \frac{I_N}{\sqrt{N}} \right) = \text{Re } \gamma_{N^2 i}$$

by assumption. Thus the real parts of these coefficients have no effect on the action of  $\mathcal{L}$ , so we may assume  $\text{Re } \gamma_{iN^2} = \text{Re } \gamma_{N^2 i} = 0$  for all  $i = 1, \dots, N^2 - 1$ . Further, since the imaginary parts act as a commutator, we may write

$$\mathcal{L} = -i \left[ \widetilde{H} + \sum_{i=1}^{N^2-1} \frac{\text{Im}(\gamma_{N^2 i} - \gamma_{iN^2})}{2\sqrt{N}} F_i, \rho \right] + \frac{1}{2} \sum_{i,j=1}^{N^2-1} \gamma_{ij} ([F_i, \rho F_j^*] + [F_i \rho, F_j^*]), \quad (2.4)$$

which is of GKSL form (2.1) since  $P\Gamma|_{M_N^0(\mathbb{C})} = (\gamma_{ij})_{i,j=1}^{N^2-1} \geq 0$  and each  $F_i$  Hermitian implies  $H = \widetilde{H} + \sum_{i=1}^{N^2-1} \frac{\text{Im}(\gamma_{N^2-i} - \gamma_{iN^2})}{2\sqrt{N}} F_i$  is Hermitian. Uniqueness of the operator  $P\Gamma|_{M_N^0(\mathbb{C})}$  also follows from Theorem 2.1.

It remains to show that these conditions are satisfied if  $\Gamma \geq 0$ . That  $P\Gamma|_{M_N^0(\mathbb{C})} \geq 0$  follows immediately since every principal submatrix of a positive semidefinite matrix is positive semidefinite (consider the quadratic form  $\text{Tr}(A^*\Gamma(A)) \geq 0$  restricted to traceless  $A$ ). That  $\text{Re Tr}(\Gamma(A)) = \text{Re Tr}(\Gamma(I_N)A)$  for  $A$  Hermitian (in  $S_2^N$ ) follows since  $\Gamma$  is Hermitian (on  $S_2^N$ ). Explicitly,

$$\text{Tr}(\Gamma(I_N)A) = \text{Tr}(A\Gamma(I_N)) = \langle A, \Gamma(I_N) \rangle = \langle \Gamma(A), I_N \rangle = \overline{\langle I_N, \Gamma(A) \rangle} = \overline{\text{Tr}(\Gamma(A))},$$

and so  $\text{Re Tr}(\Gamma(I_N)A) = \text{Re } \overline{\text{Tr}(\Gamma(A))} = \text{Re Tr}(\Gamma(A))$ .  $\square$

We ward here against the thought that allowing the matrices  $F_i$  to have trace in GKSL form (2.1) equates to ‘shifting’ some of the action of  $-\imath[H, \cdot]$  to the dissipative part (i.e.,  $\mathcal{L} + \imath[H, \cdot]$ ). That indeed is the case in the previous proof, but this relied on our choice of  $F_i$ ’s being both traceless and Hermitian. For general  $F_i$ ’s the interaction is more subtle, and indeed it is easy to construct examples of Hamiltonian-free  $\mathcal{L}$  written in GKSL form (2.1) which are equivalent to Hamiltonian-free form (2.3) with only  $F_i$ ’s of unit trace appearing ( $\mathcal{L}_d$  defined in Example 2.4 at the end of this subsection is one such example).

What is true, however, is that one can disallow any ‘shifting’ of the action of  $-\imath[H, \cdot]$  to the dissipative part by choosing  $\widetilde{H}$  to be  $H$  uniquely determined by Theorem 2.1, and  $\Gamma$  to be the natural dilation of the operator  $C$  uniquely determined by Theorem 2.1.

**Theorem 2.3.** *Let  $\{F_i | 1 \leq i \leq N^2\}$  be a set of  $N \times N$  orthonormal matrices (w.r.t. the Hilbert-Schmidt inner product). An operator  $\mathcal{L} : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  is the generator of a QDS on  $M_N(\mathbb{C})$  if and only if it can be expressed in the form*

$$\mathcal{L}(\rho) = -\imath[H, \rho] + \frac{1}{2} \sum_{i,j=1}^{N^2} \gamma_{ij} ([F_i, \rho F_j^*] + [F_i \rho, F_j^*]), \quad (2.5)$$

with  $H$  traceless and Hermitian, and  $\Gamma = (\gamma_{ij})$  an  $N^2 \times N^2$  matrix, regarded as acting on the basis  $\{F_i\}$ , satisfying

- $\Gamma \geq 0$ ,
- $\Gamma(I_N) = 0$ , and
- $\text{Tr}(\Gamma(A)) = 0$  for all  $A \in M_N(\mathbb{C})$ .

Given  $\mathcal{L}$  the Hamiltonian  $H$  is uniquely determined by  $\text{Tr}(H) = 0$  (and is the same as  $H$  as Theorem 2.1); given  $\mathcal{L}$  the coefficient matrix  $\Gamma$  is uniquely determined by the choice of  $F_i$ 's.

*Proof.* As before, given QDS generator  $\mathcal{L}$  we may write it in form (2.1) with any traceless orthonormal basis  $\{\tilde{F}_i\}$  and define  $\Gamma : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  by  $\Gamma|_{M_N^0(\mathbb{C})} = C$  and  $\Gamma(I_N) = 0$ . Changing the basis from  $\{\tilde{F}_i\}$  to the desired  $\{F_i\}$  preserves the operator properties  $\Gamma \geq 0$ ,  $\Gamma(I_N) = 0$ , and  $\text{Tr}(\Gamma(A)) = 0$ , and the coefficients of the resulting matrix are uniquely determined by this basis change. The converse is a special case of Theorem 2.2.  $\square$

Though easier to check as compared to Theorem 2.2, the disadvantage of Theorem 2.3 is that one may fail to detect if a given equation represents a QDS generator in the case  $\Gamma$  fails to satisfy these stronger properties. The following example illustrates this, as well as the importance of allowing the  $F_i$ 's to have trace when considering phenomenological operators.

**Example 2.4.** We follow [40, 60, 44], and consider a single three-level atom with ground, excited, and Rydberg states

$$|g\rangle = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad |e\rangle = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad |r\rangle = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

interacting with two laser fields: a probe laser field which drives the transition from the ground to the excited state, and a coupling laser field which drives the transition

from the excited to the Rydberg state. In this regime there are two decay modes: one from  $|e\rangle$  to  $|g\rangle$  at rate  $\Gamma_{eg}$ , and another from  $|r\rangle$  to  $|e\rangle$  at rate  $\Gamma_{re}$ . The spontaneous emission from  $|a\rangle$  to  $|b\rangle$  is described by setting  $F_i = F_j = \sqrt{\Gamma_{ab}}|b\rangle\langle a|$  in (2.1); that is, by the GKSL operator

$$\mathcal{L}_{ab}(\rho) = \Gamma_{ab}([|b\rangle\langle a|\rho, |a\rangle\langle b|] + [|b\rangle\langle a|, \rho|a\rangle\langle b|]).$$

Due to the finite linewidths of the laser fields, there are additional dephasing mechanisms which lead to additional decay of the coherences between states. The linewidth of the laser driving a transition from  $|a\rangle$  to  $|b\rangle$  can be taken into account by phenomenological operator

$$\mathcal{L}_{ab}^d(\rho) = -\frac{\Gamma_{ab}^d}{2}(|a\rangle\langle a|\rho|b\rangle\langle b| + |b\rangle\langle b|\rho|a\rangle\langle a|),$$

where  $\Gamma_{ab}^d$  is the full width of the spectral laser profile. Note that such operators are not of GKSL type, but they can be written as a linear combination of GKSL operators via

$$\mathcal{L}_{ab}^d = \frac{\Gamma_{ab}^d}{2}(\mathcal{L}_{aa} + \mathcal{L}_{bb} - \mathcal{L}_{cc}),$$

where  $(a, b, c)$  are permutations of  $(g, e, r)$  and  $\Gamma_{aa} = \Gamma_{bb} = \Gamma_{cc} = 1$ . In total, the master equation describing the system is given by

$$\partial_t \rho = \mathcal{L}(\rho) = -i[H, \rho] + \mathcal{L}_{eg}(\rho) + \mathcal{L}_{re}(\rho) + \mathcal{L}_{ge}^d(\rho) + \mathcal{L}_{er}^d(\rho) + \mathcal{L}_{gr}^d(\rho),$$

where  $H$  describes the time evolution in the absence of decoherence. We focus on the extra dephasing terms, and define

$$\begin{aligned} \mathcal{L}_d &= \mathcal{L}_{ge}^d + \mathcal{L}_{er}^d + \mathcal{L}_{gr}^d \\ &= \frac{1}{2} \left( (\Gamma_{ge}^d + \Gamma_{gr}^d - \Gamma_{er}^d) \mathcal{L}_{gg} + (\Gamma_{ge}^d + \Gamma_{er}^d - \Gamma_{gr}^d) \mathcal{L}_{ee} + (\Gamma_{gr}^d + \Gamma_{er}^d - \Gamma_{ge}^d) \mathcal{L}_{rr} \right). \end{aligned}$$

Consider the diagonal subalgebra  $\mathcal{D} = \text{Span}(|g\rangle\langle g|, |e\rangle\langle e|, |r\rangle\langle r|)$  of  $M_N(\mathbb{C})$ . Since  $\mathcal{L}_d|_{\mathcal{D}} = 0$  it is tempting to write that  $\mathcal{L}_d$  cannot be written in GKSL form (2.1)

(see e.g. section 4.1.1 of [60]). Regarding the coefficient matrix  $\Gamma$  of  $\mathcal{L}_d$  as acting of  $M_N(\mathbb{C})$ , however, we have that  $\Gamma|_{\mathcal{D}^\perp} = 0$  and  $\Gamma|_{\mathcal{D}} : \mathcal{D} \rightarrow \mathcal{D}$  acts by

$$\Gamma|_{\mathcal{D}} = \frac{1}{2} \begin{pmatrix} \Gamma_{ge}^d + \Gamma_{gr}^d - \Gamma_{er}^d & & \\ & \Gamma_{ge}^d + \Gamma_{er}^d - \Gamma_{gr}^d & \\ & & \Gamma_{gr}^d + \Gamma_{er}^d - \Gamma_{ge}^d \end{pmatrix}.$$

This matrix is Hermitian and under mild conditions positive semidefinite (e.g. consider independent lasers, so that  $\Gamma_{gr} = \Gamma_{ge} + \Gamma_{er}$ ). In such a case it is immediate that  $\Gamma$  satisfies the conditions of Theorem 2.2, and so  $\mathcal{L}_d$  is indeed a GKSL generator. Because the summation of operators of form (2.1) returns another operator of that form, this implies  $\mathcal{L}$  itself is a GKSL operator.

Note that  $\mathcal{L}_d$  is a Hamiltonian-free QDS generator in form (2.3) with only  $F_i$ 's of unit trace appearing. The given representation is not of form (2.5), however, as  $\Gamma$  has not been chosen properly to satisfy the stronger conditions of Theorem 2.3. To write  $\mathcal{L}$  in form (2.5) we replace  $\Gamma|_{\mathcal{D}}$  above by

$$\widetilde{\Gamma}|_{\mathcal{D}} = \frac{1}{18} \begin{pmatrix} 4\Gamma_{ge} + 4\Gamma_{gr} - 2\Gamma_{er} & \Gamma_{gr} - 5\Gamma_{ge} + \Gamma_{er} & \Gamma_{ge} - 5\Gamma_{gr} + \Gamma_{er} \\ \Gamma_{gr} - 5\Gamma_{ge} + \Gamma_{er} & 4\Gamma_{ge} - 2\Gamma_{gr} + 4\Gamma_{er} & \Gamma_{ge} + \Gamma_{gr} - 5\Gamma_{er} \\ \Gamma_{ge} - 5\Gamma_{gr} + \Gamma_{er} & \Gamma_{ge} + \Gamma_{gr} - 5\Gamma_{er} & 4\Gamma_{gr} - 2\Gamma_{ge} + 4\Gamma_{er} \end{pmatrix},$$

which can be found by writing  $\Gamma$  in terms of a Hermitian orthonormal basis  $\{F_i | 1 \leq i \leq 9\}$  with  $F_1, \dots, F_8$  traceless and  $F_9 = I_3/\sqrt{3}$  as in the proof of Theorem 2.2, setting equal to zero the non-contributing terms (i.e., setting  $\gamma_{99} = \text{Re } \Gamma_{i9} = \text{Re } \gamma_{9i} = 0$  for all  $i = 1, \dots, 8$ ), and then rewriting  $\Gamma$  again back in terms of the original basis. Because  $H = 0$ , and forms (2.1) and (2.5) use the same Hamiltonian, any representation of  $\mathcal{L}_d$  in form (2.1) is Hamiltonian-free. In particular, allowing the matrices  $F_i$  to have trace in GKSL form (2.1) is not equivalent to ‘shifting’ some of the action of  $-\imath[H, \cdot]$  to the dissipative part (i.e.,  $\mathcal{L} + \imath[H, \cdot]$ ).

## 2.2 CONTRACTIVITY OF $T_t$

For  $1 \leq p \leq \infty$ , we call  $M_N(\mathbb{C})$  endowed with the Schatten  $p$ -norm  $\|A\|_p = (\text{Tr}(|A|^p))^{1/p}$  for  $p < \infty$  and  $\|A\|_\infty = \sup_{\|v\|=1} \|Av\|$  the  $p$ -Schatten space  $S_p^N$ . In particular,  $S_2^N$  is the Hilbert-Schmidt space defined previously and  $S_1^N$  is the usual trace class space. For  $T : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$ , let  $\|T\|_{p \rightarrow p}$  denote the operator norm  $\|T\|_{p \rightarrow p} = \sup_{x \in M_N(\mathbb{C})} \frac{\|T(x)\|_p}{\|x\|_p}$ .

It is well known that every QDS  $(T_t)_{t \geq 0}$  is a contraction semigroup on  $S_1^N$  (i.e., satisfies  $\|T_t\|_{1 \rightarrow 1} \leq 1$  for all  $t \geq 0$ ). Indeed, if  $T$  is trace preserving and positive then its trace-dual  $T^\dagger$  is unital and positive, and hence achieves its norm at the identity. Thus,  $\|T\|_{1 \rightarrow 1} = \|T^\dagger\|_{\infty \rightarrow \infty} = \|T^\dagger(I_N)\|_\infty = \|I_N\|_\infty = 1$  (actually, if  $T$  is trace preserving then  $\|T\|_{1 \rightarrow 1} \leq 1$  if and only if  $T$  is positive; see Proposition 2.11 of [57]). We wish to take advantage of the Hilbert space properties of  $S_2^N$ , however, so we seek QDSs which are contractions on  $S_2^N$ . The Lumer-Phillips Theorem states that  $\|T_t\|_{2 \rightarrow 2} \leq 1$  for all  $t$  if and only if the generator  $\mathcal{L}$  satisfies  $\text{Re Tr}(x^* \mathcal{L}(x)) \leq 0$  for all  $x \in M_N(\mathbb{C})$  (see e.g. Corollary II.3.20 of [27]). We particularize a result of Pérez-García, Wolf, Petz, and Ruskai [59] to offer the following characterization, and compare it to this well known Lumer-Phillips result:

**Corollary 2.5.** *Suppose  $(T_t)_{t \geq 0}$  is a QDS with generator  $\mathcal{L}$ . The following are equivalent:*

- $\|T_t\|_{p \rightarrow p} \leq 1$  for some  $1 < p \leq \infty$  and all  $t \geq 0$ ,
- $\|T_t\|_{p \rightarrow p} \leq 1$  for all  $1 \leq p \leq \infty$  and all  $t \geq 0$ ,
- $\mathcal{L}(I_N) = 0$ .

*In this case  $\text{Tr}(x \mathcal{L}(x)) \leq 0$  for all Hermitian matrices  $x \in M_N(\mathbb{C})$ .*

*Proof.* Considering fixed  $t$ , we have that  $\|T_t\|_{p \rightarrow p} \leq 1$  for some, equivalently all  $1 < p \leq \infty$  if and only if  $T_t(I_N) = I_N$  by Theorem II.4 of [59]. The result then follow from (2.2), which shows  $T_t(I_N) = I_N$  for all  $t \geq 0$  if and only if  $\mathcal{L}(I_N) = 0$ , as desired.

For the second statement, since the Lumer-Phillips Theorem gives that  $\operatorname{Re} \operatorname{Tr}(x^* \mathcal{L}(x)) \leq 0$  for all  $x \in M_N(\mathbb{C})$ , it suffices to prove that  $\operatorname{Tr}(x \mathcal{L}(x)) \in \mathbb{R}$  for Hermitian  $x$ . This follows immediately from

$$\overline{\operatorname{Tr}(x \mathcal{L}x)} = \operatorname{Tr}((x \mathcal{L}(x))^*) = \operatorname{Tr}(\mathcal{L}(x)^* x^*) = \operatorname{Tr}(x^* \mathcal{L}(x)^*) = \operatorname{Tr}(x^* \mathcal{L}(x^*)) = \operatorname{Tr}(x \mathcal{L}(x)),$$

where we use that  $\mathcal{L}(x)^* = \mathcal{L}(x^*)$  since  $T(x)^* = T(x^*)$  (as a positive linear map).  $\square$

One may read the previous Corollary as saying a QDS is contractive for all Schatten  $p$ -norms if and only if the maximally mixed state  $I_N/N$  is invariant. Calling an operator  $T : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  Hermitian if it is Hermitian when regarded as  $T : S_2^N \rightarrow S_2^N$ , the next result describes potential invariant states of such a QDS given a Hermitian ‘part’ of its generator.

**Lemma 2.6.** *Suppose  $\mathcal{L}$  is a QDS generator satisfying  $\mathcal{L}(I_N) = 0$  which can be written  $\mathcal{L} = \mathcal{A} + \mathcal{B}$  with  $\mathcal{A}$  and  $\mathcal{B}$  each a QDS generator. If  $\mathcal{A}$  is Hermitian and  $\mathcal{A}(I_N) = 0$  then  $\ker \mathcal{L} \subseteq \ker \mathcal{A}$ .*

*Proof.* Since (2.2) shows that  $\ker \mathcal{L}$  is spanned by states, it suffices to show that if  $\mathcal{L}(\rho) = 0$  for some state  $\rho$  then  $\mathcal{A}(\rho) = 0$ . To this end, notice that  $\mathcal{A}(I_N) = \mathcal{L}(I_N) = 0$  implies  $\mathcal{B}(I_N) = 0$ , and so  $\operatorname{Tr}(x \mathcal{A}(x)) \leq 0$  and  $\operatorname{Tr}(x \mathcal{B}(x)) \leq 0$  for all Hermitian  $x$  by Corollary 2.5. Fixing state  $\rho$  such that  $\mathcal{L}(\rho) = 0$ , equivalently  $\mathcal{A}(\rho) = -\mathcal{B}(\rho)$ , we must then have  $\operatorname{Tr}(\rho \mathcal{A}(\rho)) = 0$ . Thus,

$$-\operatorname{Tr}(\rho \mathcal{A}(\rho)) = \langle \rho, -\mathcal{A} \rho \rangle = \langle (-\mathcal{A})^{1/2} \rho, (-\mathcal{A})^{1/2} \rho \rangle = 0$$

implies  $(-\mathcal{A})^{1/2} \rho = 0$ , and hence  $\mathcal{A} \rho = 0$ .  $\square$



## CHAPTER 3

### THE MATRIX REPRESENTATION OF $\mathcal{L}$

#### 3.1 THE STANDARD BASIS

Our proofs rely on exact calculations and the ability to move between two well-known bases of  $M_N(\mathbb{C})$ : the standard basis and the (generalized) Gell-Mann basis (introduced in Section 3.2). Recall that the standard basis consists of the  $N \times N$  matrices  $E_{ij}$  that have entry 1 in the  $i$ th row and  $j$ th column and all other entries are zero. It is easy to see that the standard basis satisfies  $E_{ij}E_{k\ell} = \delta_{jk}E_{i\ell}$ , where  $\delta_{jk}$  is the standard Kronecker delta.

By way of Theorem 2.2, every QDS generator  $\mathcal{L}$  can be written with respect to the standard basis; that is,

$$\mathcal{L}(\rho) = -i[\widetilde{H}, \rho] + \frac{1}{2} \sum_{i,j,k,\ell=1}^N \gamma_{ijk\ell} ([E_{ij}, \rho E_{k\ell}^*] + [E_{ij}\rho, E_{k\ell}^*]). \quad (3.1)$$

We henceforth reserve  $\Gamma$  to denote the  $N^2 \times N^2$  coefficient matrix  $\Gamma := (\gamma_{ijk\ell})$  for  $\mathcal{L}$  written with respect to the standard basis, and so always assume  $\Gamma$  satisfies the criteria of Theorem 2.2. We use

$$D_{ijk\ell} := [E_{ij}, \cdot E_{\ell k}] + [E_{ij} \cdot, E_{\ell k}]$$

to denote the individual **Lindblad operators** written with respect to the standard basis. For  $(i, j) = (k, \ell)$ , the so-called **diagonal Lindblad operators**, we use the simplified notation

$$D_{ij} := [E_{ij}, \cdot E_{ji}] + [E_{ij} \cdot, E_{ji}].$$

We are interested in matrix representations for  $\Gamma$  and  $\mathcal{L}$  with respect to the standard basis, and to this end we order the standard basis of  $M_N(\mathbb{C})$  by pairing together  $E_{ij}$  and  $E_{ji}$  for  $i \neq j$ , then adjoining the diagonal  $E_{nn}$ . For example, for  $N = 3$  we may take the natural ordering  $E_{12}, E_{21}, E_{13}, E_{31}, E_{23}, E_{32}, E_{11}, E_{22}, E_{33}$ , but the exact ordering of the  $E_{ij}, E_{ji}$  pairs or the  $E_{nn}$  is immaterial.

With this ordering, consider  $\Gamma : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  written as an  $N^2 \times N^2$  matrix. Denote by  $\Gamma^\mathcal{O}$  the  $N(N-1)$  order leading principal submatrix of  $\Gamma$ ; that is,  $\Gamma^\mathcal{O} : \mathcal{O} \rightarrow \mathcal{O}$  is the submatrix formed by the rows and columns corresponding to the **off-diagonal subspace**  $\mathcal{O} := \text{Span}\{E_{ij}\}_{i,j=1,i \neq j}^N$  of  $M_N(\mathbb{C})$ . Further, denote by  $\Gamma^\mathcal{D} : \mathcal{D} \rightarrow \mathcal{D}$  the complementary submatrix formed by the rows and columns corresponding to the **diagonal subalgebra**  $\mathcal{D} := \text{Span}\{E_{nn}\}_{n=1}^N$  of  $M_N(\mathbb{C})$ . Then

$$\Gamma = \begin{pmatrix} \Gamma^\mathcal{O} & * \\ * & \Gamma^\mathcal{D} \end{pmatrix}.$$

Since  $\Gamma$  satisfies  $P\Gamma|_{M_N^0(\mathbb{C})} \geq 0$  we have  $\Gamma^\mathcal{O} \geq 0$ , as every principal submatrix of a positive semidefinite matrix is itself positive semidefinite. For each fixed pair  $i, j$ , with  $i < j$ , we call the  $2 \times 2$  sub-matrix of  $\Gamma^\mathcal{O}$  consisting of the rows and columns corresponding to  $E_{ij}$  and  $E_{ji}$  the  **$ij$  block**. Note that each  $ij$  block is positive semidefinite. Similar to the language used when referring to the diagonal of a matrix or when a matrix is diagonal, we refer to the collection of all  $ij$  blocks of  $\Gamma^\mathcal{O}$  as the **pair block diagonal** of  $\Gamma^\mathcal{O}$ , and if  $\Gamma^\mathcal{O}$  has no nonzero entries outside of its pair block diagonal we say  $\Gamma^\mathcal{O}$  is pair block diagonal. We denote the upper-right entry of the  $ij$  block by  $\gamma_{ijji} =: \alpha_{ij} + \imath\beta_{ij}$  (and thus the lower-left by  $\gamma_{jiij} =: \alpha_{ij} - \imath\beta_{ij}$ ), where  $\alpha_{ij}, \beta_{ij} \in \mathbb{R}$ . Denote the diagonal entries of  $\Gamma$  by  $\gamma_{ijij} =: \gamma_{ij}$ ,  $\gamma_{jiji} =: \gamma_{ji}$ , and  $\gamma_{nnnn} =: \gamma_{nn}$  in the natural way, noting  $\gamma_{ij}, \gamma_{ji} \geq 0$  since  $\Gamma^\mathcal{O} \geq 0$ .

To illustrate these notations, the following is an example of a matrix  $\Gamma$  in dimen-

sion  $N = 3$  for which  $\Gamma^{\mathcal{O}}$  is pair block diagonal and  $\Gamma^{\mathcal{D}}$  is diagonal:

$$\Gamma = \begin{pmatrix} \gamma_{12} & \alpha_{12} + i\beta_{12} & & & & & & \\ \alpha_{12} - i\beta_{12} & \gamma_{21} & & & & & & \\ & & \gamma_{13} & \alpha_{13} + i\beta_{13} & & & & \\ & & \alpha_{13} - i\beta_{13} & \gamma_{31} & & & & \\ & & & & \gamma_{23} & \alpha_{23} + i\beta_{23} & & \\ & & & & \alpha_{23} - i\beta_{23} & \gamma_{32} & & \\ & & & & & & \gamma_{11} & \gamma_{22} \\ & & & & & & & \gamma_{33} \end{pmatrix}$$

**Remark 3.1.** Fix orthogonal vector states  $|i\rangle$  and  $|j\rangle$  and consider a system which transfers superposition state  $|\psi\rangle = a|i\rangle + b|j\rangle$  to superposition state  $|\phi\rangle = c|i\rangle + d|j\rangle$  with probability  $\gamma$  over a very short evolution time  $dt$ . To construct a model for such a system we make use of a short time expansion of the Kraus operator sum representation  $\rho' = \sum_{\alpha} K_{\alpha}(dt)\rho K_{\alpha}^{*}(dt)$  (see e.g. section IX of [52]). Setting

$$F_{ij} := \frac{c}{b}E_{ij} + \frac{d}{a}E_{ji}$$

so that  $F_{ij}|\psi\rangle = |\phi\rangle$ , we take Kraus operator

$$K_1(dt) = \sqrt{\gamma dt} F_{ij}$$

to represent the transition. Normalization  $\sum_{\alpha} K_{\alpha}^{*}(dt)K_{\alpha}(dt) = I_N$  up to order  $O(dt)$  (to ensure the evolution is trace preserving) requires a second Kraus operator

$$K_2(dt) = I_N - \frac{1}{2}K_1^{*}(dt)K_1(dt).$$

Thus, we have that

$$\rho' = K_1(dt)\rho K_1^{*}(dt) + K_2(dt)\rho K_2^{*}(dt) = \rho + \gamma dt([F_{ij}, \rho F_{ij}^{*}] + [F_{ij}\rho, F_{ij}^{*}]).$$

Assuming the same Kraus representation works over all time, we arrive at the GKSL equation

$$\mathcal{L}(\rho) = \lim_{dt \rightarrow 0} \frac{\rho' - \rho}{dt} = \gamma([F_{ij}, \rho F_{ij}^{*}] + [F_{ij}\rho, F_{ij}^{*}]).$$

Rewriting  $\mathcal{L}$  in terms of the standard basis (3.1), the coefficient matrix  $\Gamma$  has nonzero entries only in the  $ij$  block, which is given by

$$\Gamma_{ij} = \gamma \begin{pmatrix} \frac{c\bar{c}}{bb} & \frac{c\bar{d}}{ab} \\ \frac{\bar{c}d}{ab} & \frac{\bar{d}d}{aa} \end{pmatrix}.$$

Thus, while diagonal coefficient matrices can be interpreted as describing jumps between states  $|i\rangle$  and  $|j\rangle$  (as with the graph induced generators (1.1)), the pair block diagonal coefficient matrices can describe jumps between two superpositions of states  $|i\rangle$  and  $|j\rangle$ . A main result of this work is to characterize invariant states of QDS generators with such coefficient matrices (see Theorem 6.9 and Example 6.13).

Extending the submatrix notations to  $\mathcal{L} : M_N(\mathbb{C}) \rightarrow M_N(\mathbb{C})$  in the natural way, we write

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}^{\mathcal{O}} & * \\ * & \mathcal{L}^{\mathcal{D}} \end{pmatrix}. \quad (3.2)$$

We note that Havel considered the entries of  $\mathcal{L}$  when written as such an  $N^2 \times N^2$  matrix to recover the coefficients of  $\Gamma$  in terms of Choi matrices (Proposition 12 of [43]). We are interested in the other direction, however: how the coefficients of  $\Gamma$  affect the action of  $\mathcal{L}$ .

Per the introduction, we seek generators  $\mathcal{L}$  which gives rise to QDSs which **evolve independently** on  $\mathcal{D}$  and  $\mathcal{O}$  in the sense that

$$T_t(A) = T_t^{\mathcal{O}}(\text{diag}(A)) + T_t^{\mathcal{D}}(A - \text{diag}(A))$$

for all  $A \in M_N(\mathbb{C})$ . Since exponentiation preserves block diagonal structure, if  $\mathcal{D}$  and  $\mathcal{O}$  are each invariant for  $\mathcal{L}$  (equivalently  $*$  = 0 in (3.2)), then  $e^{t\mathcal{L}} = T_t = \begin{pmatrix} T_t^{\mathcal{O}} & 0 \\ 0 & T_t^{\mathcal{D}} \end{pmatrix}$ , where  $T_t^{\mathcal{O}} := e^{t\mathcal{L}^{\mathcal{O}}}$  and  $T_t^{\mathcal{D}} := e^{t\mathcal{L}^{\mathcal{D}}}$ . Conversely, if  $(T_t)_{t \geq 0}$  evolves independently on  $\mathcal{D}$  and  $\mathcal{O}$ , then necessarily  $\mathcal{D}$  and  $\mathcal{O}$  are each invariant for  $T_t$  for all  $t \geq 0$ , and hence invariant for  $\mathcal{L}$ . We are thus seeking generators for which  $*$  = 0 in (3.2).

As each entry of  $\mathcal{L}$ 's matrix representation is a linear combination of entries of  $\widetilde{H}$  and  $\Gamma$  as determined by (3.1), we can consider how each entry of  $\Gamma$  contributes to various entries of  $\mathcal{L}$ . Explicitly, we compute

$$\begin{aligned}
D_{ijk\ell}(E_{st}) &= [E_{ij}, E_{st}E_{\ell k}] + [E_{ij}E_{st}, E_{\ell k}] \\
&= 2E_{ij}E_{st}E_{\ell k} - E_{st}E_{\ell k}E_{ij} - E_{\ell k}E_{ij}E_{st} \\
&= 2\delta_{js}\delta_{\ell t}E_{ik} - \delta_{\ell t}\delta_{ik}E_{sj} - \delta_{ik}\delta_{js}E_{\ell t}.
\end{aligned} \tag{3.3}$$

In particular,

$$D_{ij}(E_{k\ell}) = -(\delta_{jk} + \delta_{j\ell})E_{k\ell}, \quad D_{ijji}(E_{k\ell}) = 2\delta_{jk}\delta_{i\ell}E_{k\ell}$$

and

$$D_{iijj}(E_{k\ell}) = (2\delta_{ik}\delta_{j\ell} - \delta_{ij}\delta_{ik} - \delta_{ij}\delta_{j\ell})E_{k\ell}.$$

Notably, entries of  $\Gamma^{\mathcal{D}}$  and of the pair block diagonal of  $\Gamma^{\mathcal{O}}$  contribute only to  $\mathcal{L}^{\mathcal{D}}$  and to the pair block diagonal of  $\mathcal{L}^{\mathcal{O}}$ . If we assume the Hamiltonian is diagonal, that is  $\widetilde{H} = \sum_{n=1}^N h_n E_{nn}$ , then we compute

$$-\imath[\widetilde{H}, E_{k\ell}] = -\imath \sum_{n=1}^N h_n [E_{nn}, E_{k\ell}] = -\imath(h_k - h_\ell)E_{k\ell},$$

and see that entries of  $\widetilde{H}$  contribute only to the diagonal of  $\mathcal{L}^{\mathcal{D}}$ . This gives us the following:

**Remark 3.2.** Let  $\mathcal{L}$  be a QDS generator written with respect to the standard basis (3.1) with Hamiltonian  $\widetilde{H} = \sum_{n=1}^N h_n E_{nn}$ . If  $\Gamma = \begin{pmatrix} \Gamma^{\mathcal{O}} & 0 \\ 0 & \Gamma^{\mathcal{D}} \end{pmatrix}$  with  $\Gamma^{\mathcal{O}}$  pair block diagonal, then

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}^{\mathcal{O}} & 0 \\ 0 & \mathcal{L}^{\mathcal{D}} \end{pmatrix} \tag{3.4}$$

with  $\mathcal{L}^{\mathcal{O}}$  pair block diagonal; in this case, if  $\Gamma^{\mathcal{O}}$  is diagonal then  $\mathcal{L}^{\mathcal{O}}$  is diagonal.

A partial converses are also true: no entry of  $\widetilde{H}$  outside its diagonal and no entry of  $\Gamma$  outside both  $\Gamma^{\mathcal{D}}$  and the pair block diagonal of  $\Gamma^{\mathcal{O}}$  contributes to the pair block diagonal of  $\mathcal{L}^{\mathcal{O}}$  or to  $\mathcal{L}^{\mathcal{D}}$ .

**Definition 3.3.** We call QDS generator  $\mathcal{L}$  **pair block diagonal with respect to the standard basis** if  $\mathcal{L}$  is of form (3.1) with

$$\Gamma = \begin{pmatrix} \Gamma^{\mathcal{O}} & 0 \\ 0 & \Gamma^{\mathcal{D}} \end{pmatrix}$$

and  $\Gamma^{\mathcal{O}}$  pair block diagonal.

Note that a generator which is pair block diagonal with respect to the standard basis with  $\widetilde{H} = \sum_{n=1}^N h_n E_{nn}$  satisfies (3.4), with  $\mathcal{L}^{\mathcal{O}}$  diagonal if  $\Gamma^{\mathcal{O}}$  is. Also note that every digraph induced generator (1.1) is pair block diagonal with respect to the standard basis with  $\Gamma^{\mathcal{O}}$  diagonal and  $\Gamma^{\mathcal{D}} = 0$ .

As noted before,  $\gamma_{ij} \geq 0$  since these are diagonal entries of positive semidefinite  $\Gamma^{\mathcal{O}}$ . It is not true in general, however, that  $\gamma_{ii} \geq 0$ , or that  $\gamma_{ii}$  is even real. Indeed, considering the simple case of  $\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma^{\mathcal{D}} \end{pmatrix}$ , the criteria of Theorem 2.2 are satisfied for both  $\Gamma^{\mathcal{D}} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix}$  and  $\Gamma^{\mathcal{D}} = \begin{pmatrix} -i & -i & -i \\ i & i & i \\ 0 & 0 & 0 \end{pmatrix}$ .

Some things can still be said in our case of interest, though, as  $\Gamma = \begin{pmatrix} \Gamma^{\mathcal{O}} & 0 \\ 0 & \Gamma^{\mathcal{D}} \end{pmatrix}$  satisfies the conditions of Theorem 2.2 if and only if  $\Gamma = \begin{pmatrix} \Gamma^{\mathcal{O}} & 0 \\ 0 & 0 \end{pmatrix}$  and  $\Gamma = \begin{pmatrix} 0 & 0 \\ 0 & \Gamma^{\mathcal{D}} \end{pmatrix}$  do. In particular, since  $E_{ii} - E_{jj}$  is traceless it follows that

$$\begin{aligned} \langle E_{ii} - E_{kk}, \Gamma^{\mathcal{D}}(E_{ii} - E_{jj}) \rangle &= \text{Tr} \left( (E_{ii} - E_{jj}) \left( \sum_{k=1}^N (\gamma_{ki} - \gamma_{kj}) E_{kk} \right) \right) \\ &= \gamma_{ii} + \gamma_{jj} - \gamma_{iijj} - \gamma_{jjii} \geq 0. \end{aligned}$$

We will recall this later as the following:

**Remark 3.4.** If  $\Gamma = \begin{pmatrix} \Gamma^{\mathcal{O}} & 0 \\ 0 & \Gamma^{\mathcal{D}} \end{pmatrix}$  then  $\gamma_{ii} + \gamma_{jj} - \gamma_{iijj} - \gamma_{jjii} \geq 0$  for all  $1 \leq i, j \leq N$ .

### 3.2 THE GELL-MANN BASIS

By the Gell-Mann basis we mean the collection consisting of the normalized  $N \times N$  identity matrix  $\frac{1}{\sqrt{N}} I_N$  and three other sets of matrices:

1) The  $\frac{N(N-1)}{2}$  many symmetric matrices defined by

$$\lambda_{ij} := \frac{1}{\sqrt{2}}(E_{ij} + E_{ji}) \quad \text{for } 1 \leq i < j \leq N,$$

2) the  $\frac{N(N-1)}{2}$  many antisymmetric matrices defined by

$$\lambda_{ji} := \frac{-i}{\sqrt{2}}(E_{ij} - E_{ji}) \quad \text{for } 1 \leq i < j \leq N,$$

3) and the  $N - 1$  many diagonal matrices defined by

$$\lambda_{nn} := \frac{1}{\sqrt{n(n+1)}} \left( \sum_{m=1}^n E_{mm} - nE_{n+1,n+1} \right) \quad \text{for } 1 \leq n \leq N - 1.$$

Each  $\lambda_{ij}$  is Hermitian and traceless by construction, and they are orthonormal and orthogonal to  $\frac{1}{\sqrt{N}}I_N$  in the Hilbert-Schmidt inner product [10]. By dimension count, we see that  $\text{Span}(\lambda_{ij}, \frac{1}{\sqrt{N}}I_N) = M_N(\mathbb{C})$ .

Given a matrix written in the Gell-Mann basis, it is immediate how to write it in the standard basis. For the opposite direction, we use the formula given in [10]:

$$E_{ij} = \begin{cases} \frac{1}{\sqrt{2}}(\lambda_{ij} + i\lambda_{ji}) & \text{for } i < j \\ \frac{1}{\sqrt{2}}(\lambda_{ij} - i\lambda_{ji}) & \text{for } j < i \\ -\sqrt{\frac{j-1}{j}}\lambda_{j-1,j-1} + \sum_{m=j}^{N-1} \frac{1}{\sqrt{m(m+1)}}\lambda_{mm} + \frac{1}{N}I_N & \text{for } i = j \end{cases} \quad (3.5)$$

where the summation is interpreted as vacuously zero for  $j = N$  and we take  $\lambda_{00} := 0$ .

Since the Gell-Mann basis without  $I_N/\sqrt{N}$  is a complete set of traceless orthonormal matrices, given any QDS  $T_t$  we may use Theorem 2.1 to write its generator  $\mathcal{L}$  with respect to the Gell-Mann basis:

$$\mathcal{L}(\rho) = -i[H, \rho] + \frac{1}{2} \sum c_{ijkl} ([\lambda_{ij}, \rho\lambda_{kl}] + [\lambda_{ij}\rho, \lambda_{kl}]) \quad (3.6)$$

Note that no adjoints appear since each  $\lambda_{ij}$  is Hermitian, and the sum is over all valid choices of  $i, j, k, \ell$ ; specifically,  $i, j \in \{1, \dots, N\}$  for  $i \neq j$  and  $i, j \in \{1, \dots, N - 1\}$  for  $i = j$ , and similarly  $k, \ell \in \{1, \dots, N\}$  for  $k \neq \ell$  and  $k, \ell \in \{1, \dots, N - 1\}$  for

$k = \ell$ . We henceforth reserve  $C$  to denote the  $(N^2 - 1) \times (N^2 - 1)$  coefficient matrix  $C := (c_{ijk\ell})$  for  $\mathcal{L}$  written with respect to the Gell-Mann basis, and

$$D_{ijk\ell}^\lambda := [\lambda_{ij}, \cdot \lambda_{k\ell}] + [\lambda_{ij}, \lambda_{k\ell}]$$

to denote the individual **Gell-Mann basis Lindblad operators**.

Order the Gell-Mann basis as we did the standard basis, by pairing together  $\lambda_{ij}$  and  $\lambda_{ji}$  for  $i \neq j$ , then adjoining the diagonal  $\lambda_{nn}$ , and finally  $I_N/\sqrt{N}$ . Define  $C^\mathcal{O}$  and  $C^{\mathcal{D}_0}$  analogously as well, where now  $\mathcal{D}_0 := \text{Span}(\lambda_{ii})_{i=1}^{N-1}$  is the **traceless diagonal subspace** of  $M_N(\mathbb{C})$ , so that  $C^{\mathcal{D}_0} : \mathcal{D}_0 \rightarrow \mathcal{D}_0$  is an  $(N-1) \times (N-1)$  matrix. We use  $a_{ij}, b_{ij}$  and  $c_{ij}$  for entries of  $C$  as we used the notations  $\alpha_{ij}, \beta_{ij}$  and  $\gamma_{ij}$  for entries of  $\Gamma$ .

To illustrate these notations, the following is an example of a matrix  $C$  in dimension  $N = 3$  for which  $C^\mathcal{O}$  is pair block diagonal and  $C^{\mathcal{D}_0}$  is diagonal:

$$C = \begin{pmatrix} c_{12} & a_{12}+ib_{12} & & & & \\ a_{12}-ib_{12} & c_{21} & & & & \\ & & c_{13} & a_{13}+ib_{13} & & \\ & & a_{13}-ib_{13} & c_{31} & & \\ & & & & c_{23} & a_{23}+ib_{23} \\ & & & & a_{23}-ib_{23} & c_{32} \\ & & & & & c_{11} & c_{22} \end{pmatrix}$$

Motivated by the distinction between  $\mathcal{D}$  and  $\mathcal{D}_0$ , let us denote by  $\mathcal{L}^{\mathcal{D}_0}$  the submatrix of  $\mathcal{L}$  formed by the rows and columns corresponding to diagonal  $\lambda_{nn}$  for  $1 \leq n \leq N-1$ . Explicitly,

$$\mathcal{L}^{\mathcal{D}} = \begin{pmatrix} \mathcal{L}^{\mathcal{D}_0} & * \\ 0 & 0 \end{pmatrix},$$

where the last row is zero since  $\mathcal{L}$  has traceless range.

Under certain restrictions the matrix representations for  $C$  and  $\mathcal{L}$  with respect to the Gell-Mann basis (3.6) are unsurprisingly similar to those of  $\Gamma$  and  $\mathcal{L}$  with respect to the standard basis (3.1). Indeed, consider the basis change from the standard basis to the Gell-Mann basis represented by unitary matrix  $U$ , so that  $\Gamma = U^* \tilde{C} U$ , where  $\tilde{C}$  is the matrix  $C$  extended to act on all of  $M_N(\mathbb{C})$  by setting  $\tilde{C}(I_N) = 0$  (i.e.,  $\tilde{C} = \begin{pmatrix} C & 0 \\ 0 & 0 \end{pmatrix}$ ). Then (3.5) implies  $U = \begin{pmatrix} U^\mathcal{O} & 0 \\ 0 & U^\mathcal{D} \end{pmatrix}$  where  $U^\mathcal{O}$  is pair block diagonal with



each  $ij$  block given by  $\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ \imath & -\imath \end{pmatrix}$  by (3.5). We have general  $ij$  blocks of the two forms are related via

$$\begin{pmatrix} c_{ij} & a_{ij} + \imath b_{ij} \\ a_{ij} - \imath b_{ij} & c_{ji} \end{pmatrix}^C \equiv \frac{1}{2} \begin{pmatrix} c_{ij} + c_{ji} - 2b_{ij} & c_{ij} - c_{ji} - 2\imath a_{ij} \\ c_{ij} - c_{ji} + 2\imath a_{ij} & c_{ij} + c_{ji} + 2b_{ij} \end{pmatrix}^\Gamma$$

$$\begin{pmatrix} \gamma_{ij} & \alpha_{ij} + \imath \beta_{ij} \\ \alpha_{ij} - \imath \beta_{ij} & \gamma_{ji} \end{pmatrix}^\Gamma \equiv \frac{1}{2} \begin{pmatrix} \gamma_{ij} + \gamma_{ji} + 2\alpha_{ij} & -2\beta_{ij} - \imath(\gamma_{ij} - \gamma_{ji}) \\ -2\beta_{ij} + \imath(\gamma_{ij} - \gamma_{ji}) & \gamma_{ij} + \gamma_{ji} - 2\alpha_{ij} \end{pmatrix}^C, \quad (3.7)$$

where  $\equiv$  denotes equal contribution to  $\mathcal{L}$ . This shows that for every  $C = \begin{pmatrix} C^\mathcal{O} & 0 \\ 0 & C^{\mathcal{D}_0} \end{pmatrix}$  with  $C^\mathcal{O}$  pair block diagonal there is some  $\Gamma = \begin{pmatrix} \Gamma^\mathcal{O} & 0 \\ 0 & \Gamma^\mathcal{D} \end{pmatrix}$  with  $\Gamma^\mathcal{O}$  pair block diagonal such that  $C \equiv \Gamma$  (and vice-versa, up to Hamiltonian). Thus, assuming  $H = \sum_{n=1}^N h_n E_{nn}$ , so that for  $k < \ell$  we have

$$-\imath[H, \lambda_{k\ell}] = \frac{-\imath}{\sqrt{2}} \sum_{n=1}^N h_n [E_{nn}, E_{k\ell} + E_{\ell k}] = (h_k - h_\ell) \lambda_{\ell k}$$

and similarly  $-\imath[H, \lambda_{\ell k}] = -(h_k - h_\ell) \lambda_{k\ell}$ , from Remark 3.2 we have the following:

**Remark 3.5.** Let  $\mathcal{L}$  be a QDS generator written with respect to the Gell-Mann basis (3.6) with Hamiltonian  $H = \sum_{n=1}^N h_n E_{nn}$ . If  $C = \begin{pmatrix} C^\mathcal{O} & 0 \\ 0 & C^{\mathcal{D}_0} \end{pmatrix}$  with  $C^\mathcal{O}$  pair block diagonal then

$$\mathcal{L} = \begin{pmatrix} \mathcal{L}^\mathcal{O} & 0 & 0 \\ 0 & \mathcal{L}^{\mathcal{D}_0} & * \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} \mathcal{L}^\mathcal{O} & 0 \\ 0 & \mathcal{L}^\mathcal{D} \end{pmatrix}, \quad (3.8)$$

with  $\mathcal{L}^\mathcal{O}$  pair block diagonal; in this case, if  $C^\mathcal{O}$  is diagonal and  $H = 0$  then  $\mathcal{L}^\mathcal{O}$  diagonal.

A partial converse is also true, in the sense that no entry of  $H$  outside its diagonal and no entry of  $C$  outside both  $C^{\mathcal{D}_0}$  and the pair block diagonal of  $C^\mathcal{O}$  contributes to the pair block diagonal of  $\mathcal{L}^\mathcal{O}$ , to  $\mathcal{L}^{\mathcal{D}_0}$ , or to the portion of the  $\mathcal{L}$  marked by  $*$  in (3.8). We also note that if  $C^\mathcal{O}$  is diagonal and  $C^{\mathcal{D}_0}$  is arbitrary then  $\mathcal{L}(I_N) = 0$  (and hence  $*$  = 0) is easily verified.

**Definition 3.6.** We call QDS generator  $\mathcal{L}$  **pair block diagonal with respect to the Gell-Mann basis** if  $\mathcal{L}$  is of form (3.6) with

$$C = \begin{pmatrix} C^{\mathcal{O}} & 0 \\ 0 & C^{\mathcal{D}_0} \end{pmatrix}$$

and  $C^{\mathcal{O}}$  pair block diagonal.

Note that a QDS generator can be written as pair block diagonal with respect to the Gell-Mann basis if and only if it can be written as pair block diagonal with respect to the standard basis.

For basis-free definitions one may define  $\mathcal{L}^{\mathcal{D}} := P_{\mathcal{D}}\mathcal{L}|_{\mathcal{D}}$ , where  $P_{\mathcal{D}}$  is orthogonal projection onto  $\mathcal{D}$ , and similarly  $\mathcal{L}^{\mathcal{D}_0} := P_{\mathcal{D}_0}\mathcal{L}|_{\mathcal{D}_0}$ . In the case  $\mathcal{L}$  is of the form (3.8), it follows from (2.2) that  $\ker \mathcal{L}^{\mathcal{D}}$  is nonempty, spanned by diagonal states (i.e., diagonal as  $N \times N$  matrices), and it is natural to view  $\ker \mathcal{L}^{\mathcal{D}_0} \subseteq \ker \mathcal{L}^{\mathcal{D}}$ . It turns out this is true for arbitrary generators.

**Proposition 3.7.** *Let  $\mathcal{L}$  be a QDS generator. Then  $\ker \mathcal{L}^{\mathcal{D}}$  is nonempty, spanned by diagonal states, and*

$$\ker \mathcal{L}^{\mathcal{D}} = \ker \mathcal{L}^{\mathcal{D}_0} \oplus \mathbb{C}\{\rho\}$$

for any  $\rho \in \ker \mathcal{L}^{\mathcal{D}}$  with nonzero trace. In particular,  $\dim \ker \mathcal{L}^{\mathcal{D}} = \dim \ker \mathcal{L}^{\mathcal{D}_0} + 1$ .

*Proof.* Without loss of generality assume  $\mathcal{L}$  is written in Gell-Mann form (3.6), and consider the matrix  $\overline{C}$  obtained by setting equal to zero all entries of  $C$  except those in the pair block diagonal of  $C^{\mathcal{O}}$ . Then the operator  $\overline{\mathcal{L}}$  defined via (3.6) (with  $H = 0$ ) is a QDS generator, since  $\overline{C}$  is positive semidefinite as each  $ij$  block of  $C$  is. Further, Remark 3.5 and the partial converse thereof imply  $\overline{\mathcal{L}}^{\mathcal{D}} = \mathcal{L}^{\mathcal{D}}$ , and so we may assume without loss of generality that  $C = \overline{C}$ . From (2.2) we conclude  $\ker \mathcal{L}$  is nonempty and spanned by states. The block form (3.8) of  $\mathcal{L}$  then implies  $\ker \mathcal{L}^{\mathcal{D}}$  is nonempty and spanned by diagonal states. We now only need remark that given diagonal states

$\rho_1, \rho_2 \in \ker \mathcal{L}^{\mathcal{D}}$  we have that  $\rho_1 - \rho_2$  is diagonal, traceless, and in  $\ker \mathcal{L}$ , and hence  $\rho_1 - \rho_2 \in \ker \mathcal{L}^{\mathcal{D}_0}$ ; that is, given fixed diagonal state  $\rho_0 \in \ker \mathcal{L}^{\mathcal{D}}$  we have that for any diagonal state  $\rho \in \ker \mathcal{L}^{\mathcal{D}}$  there exists some diagonal traceless  $A \in \ker \mathcal{L}^{\mathcal{D}_0}$  such that  $\rho = \rho_0 + A$ . The dimensionality statement follows since every element in  $\ker \mathcal{L}^{\mathcal{D}_0}$  is traceless but  $\rho_0 \in \ker \mathcal{L}^{\mathcal{D}}$  has unit trace.  $\square$

# CHAPTER 4

## GRAPH THEORY BACKGROUND

In this chapter we establish notation and background for the needed graph theoretical notions; see [22] or any comparable text on elementary graph theory.

### 4.1 GRAPHS

A **graph** consists of a set of vertices, labeled  $1, \dots, N$ , together with a set of weighted edges, which are 2-element sets  $ij := \{i, j\}$  of vertices each with an associated weight  $w_{ij} > 0$ . A graph is called **connected** if there is a path between every pair of vertices, and called a **tree** if there is a unique path between every pair of vertices. Each maximal connected subgraph is called a **connected component**. If  $G$  is a graph on  $N$  vertices, by its **graph Laplacian**  $L(G)$  we mean the  $N \times N$  matrix whose  $(i, j)$  entry is given by

$$(L(G))_{ij} = \begin{cases} w_{ij} & i \neq j \\ -\sum_{k \neq j} w_{kj} & i = j \end{cases},$$

where we take  $w_{ij} = 0$  if  $ij$  is not an edge of  $G$ .

It is easy to see that  $x^* L(G) x = \frac{-1}{2} \sum_{i,j=1}^N w_{ij} |x_i - x_j|^2 \leq 0$  for all vectors  $x \in \mathbb{C}^N$ , and so  $L(G)$  is negative semidefinite. Notice that this quadratic form is zero if and only if  $w_{ij} = 0$  whenever  $x_i \neq x_j$ . Hence, if  $G$  is connected the only vectors satisfying  $x^* L(G) x = 0$  are multiples of  $\vec{1}$ , the all ones vector, and so  $\ker L(G) = \mathbb{C} \vec{1}$ . If  $G$  is not connected, then given connected components  $G^1, \dots, G^k$  of  $G$  one may permute

the underlying basis so that  $L(G)$  is block diagonal of the form

$$L(G) = \begin{pmatrix} L(G^1) & 0 & \cdots & 0 \\ 0 & L(G^2) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & L(G^k) \end{pmatrix},$$

from which we establish the following well-known fact:

**Remark 4.1.** For each connected component  $G^n$  of a graph  $G$  let  $\gamma^{G^n}$  be the vector with one at each entry corresponding to a vertex in  $G^n$  and zero elsewhere. Then  $\text{Span}(\gamma^{G^n})_{n=1}^k = \ker(L(G))$ .

## 4.2 DIGRAPHS

A **digraph**  $G$  consists of a set  $V(G)$  of vertices, labeled  $1, \dots, N$ , together with a set  $E(G)$  of weighted edges, which are ordered pairs  $ij := (i, j)$  of vertices each with an associated weight  $w_{ji} > 0$  (note the reversal of the indices). We regard edges  $ij$  as the arrow from vertex  $i$  to vertex  $j$ . A digraph is called a **directed tree** if the graph obtained by ignoring the directedness of the edges is a tree. The **weight of a directed tree**  $T$  is given by  $\prod_{k\ell \in E(T)} w_{\ell k}$ . We say  $T$  is a **directed spanning subtree** if  $T$  is a subdigraph of  $G$  which is a directed tree and  $V(T) = V(G)$ ; we say further that  $T$  is **rooted at**  $i \in V(T)$  if  $i$  is the only vertex of  $T$  with no out-edges (in  $T$ ). Denote by  $\mathcal{T}_i(G)$  the collection of all directed spanning subtrees of  $G$  rooted at  $i$ . If  $G$  is a digraph on  $N$  vertices, by **digraph Laplacian**  $L(G)$  we mean the  $N \times N$  matrix whose  $(i, j)$  entry is given by

$$(L(G))_{ij} = \begin{cases} w_{ij} & i \neq j \\ -\sum_{k \neq j} w_{kj} & i = j \end{cases},$$

where we take  $w_{ji} = 0$  if  $ij$  is not an edge of  $G$ . By  $L_k(G)$  we mean the  $(N-1) \times (N-1)$  matrix obtained by deleting row  $k$  and column  $k$  from  $L(G)$ .

**Theorem 4.2** ([68]). *Let  $G$  be a weighted digraph on  $N$  vertices and let  $L(G)$  be the corresponding digraph Laplacian. Then the total weight of all directed spanning subtrees of  $G$  rooted at  $i$  is given by*

$$\sum_{T \in \mathcal{T}_i(G)} \prod_{k \ell \in E(T)} w_{\ell k} = (-1)^{N-1} \det(L_i(G)).$$

A digraph is called **strongly connected** if between any two distinct vertices  $i$  and  $j$  there is a path from  $i$  to  $j$  and a path from  $j$  to  $i$ . Each maximal strongly connected subdigraph is called a **strongly connected component (SCC)**. Following Mirzaev and Gunawardena in [55], we denote the SCC containing vertex  $i$  as  $[i]$ , and write  $[i] \preceq [j]$  if there is a path from  $i'$  to  $j'$  for some  $i' \in [i]$  and  $j' \in [j]$ . If  $[i] \preceq [j]$  implies  $[i] = [j]$  for any  $[j]$ , we say  $[i]$  is a **terminal SCC (TSCC)**.

For each TSCC  $G^n$  of  $G$  define vector  $\tilde{\rho}^{G^n} \in \mathbb{R}^N$  (where  $N = |V(G)|$ ) by setting  $\tilde{\rho}_i^{G^n}$  to be the total weight of directed spanning subtrees of  $G^n$  rooted at  $i$ ; that is,

$$\tilde{\rho}_i^{G^n} = \sum_{T \in \mathcal{T}_i(G^n)} \prod_{k \ell \in E(T)} w_{\ell k} = (-1)^{N-1} \det(L_i(G^n)),$$

where this quantity is taken to be zero if  $i \notin G^n$ . We define

$$\rho^{G^n} = \frac{1}{\lambda} \tilde{\rho}^{G^n},$$

where the normalization factor  $\lambda > 0$  is chosen so that  $\sum_{i=1}^N \rho_i^{G^n} = 1$  (explicitly,  $\lambda = (-1)^{N-1} \sum_i \det(L_i(G^n))$ ).

**Proposition 4.3** ([55]). *Let  $G$  be a digraph (with all positive weights). Then*

$$\ker L(G) = \text{Span}(\rho^{G^n})_{n=1}^k,$$

where  $G^1, \dots, G^k$  are the TSCCs of  $G$ .

By a **sink** of a digraph we mean a single vertex which forms a TSCC; i.e., a vertex from which no edges originate. In a similar fashion, we call a pair of vertices  $k$  and  $\ell$  a **2-sink** if they form a TSCC; that is, there is an edge from  $k$  to  $\ell$  and vice versa, but no other edges originate from  $k$  or  $\ell$ . If the context is clear, we denote a 2-sink on vertices  $k$  and  $\ell$  simply by the edge notation  $k\ell$ .

## CHAPTER 5

### RELATING GENERATORS TO DIGRAPHS

Given a QDS generator  $\mathcal{L}$ , we define our main digraph of interest  $G_{\mathcal{L}}$  to be the weighted digraph on  $N$  vertices (labeled  $1, 2, \dots, N$ ) with weight of edge from  $j$  to  $i$  (with  $i \neq j$ ) given by  $\gamma_{ij}$ , where  $\gamma_{ij}$  are the (uniquely determined by Theorem 2.2) entries of  $\Gamma^{\mathcal{O}}$  when  $\mathcal{L}$  is written with respect to standard basis (3.1). Equivalently, (3.7) reveals that one may write  $\mathcal{L}$  with respect to the Gell-Mann basis (3.6) and define  $G_{\mathcal{L}}$  to be the weighted digraph on  $N$  vertices (labeled  $1, 2, \dots, N$ ) with weight of edge from  $j$  to  $i$  given by

$$\gamma_{ij} = \frac{1}{2} \begin{cases} c_{ij} + c_{ji} - 2b_{ij} & i < j \\ c_{ji} + c_{ij} + 2b_{ji} & i > j \end{cases}.$$

We note that

$$\frac{c_{ij} + c_{ji}}{2} \geq \sqrt{c_{ij}c_{ji}} \geq \sqrt{a_{ij}^2 + b_{ij}^2} \geq |b_{ij}|, \quad (5.1)$$

where the first inequality is a comparison of arithmetic and geometric means, and the second follows since the  $ij$  block of  $C$  is positive semidefinite (as  $C$  itself is). Further, these inequalities are equality only in the case  $c_{ij} = c_{ji} = |b_{ij}|$  and  $a_{ij} = 0$ . Hence the following:

**Remark 5.1.** The weights of graph  $G_{\mathcal{L}}$  are nonnegative. Fix  $i < j$ . Then  $\gamma_{ij} = 0$  if

and only if the  $ij$  block of  $C$  is given by  $c_{ij} \begin{pmatrix} 1 & \imath \\ -\imath & 1 \end{pmatrix}$ , and  $\gamma_{ji} = 0$  if and only if the

$ij$  block of  $C$  is given by  $c_{ij} \begin{pmatrix} 1 & -\imath \\ \imath & 1 \end{pmatrix}$ .

The following proposition shows that every QDS is naturally associated to a digraph.

**Theorem 5.2.** *Let  $\mathcal{L}$  be a QDS generator written in matrix form with respect to the standard basis (3.2). Then  $\mathcal{L}^{\mathcal{D}} = L(G_{\mathcal{L}})$ .*

*Proof.* Consider  $\mathcal{L}$  given by form (3.1). The Hamiltonian part  $i[H, \cdot]$  does not contribute to  $\mathcal{L}^{\mathcal{D}}$  since evaluating  $[H, E_{nn}]$  yields a matrix with null diagonal (explicitly, the  $n$ th column of  $H$  minus the  $n$ th row of  $H$ ). To find the contribution of the dissipative part, from (3.3) we find

$$D_{ijkl}(E_{nn}) = 2\delta_{jn}\delta_{\ell n}E_{ik} - \delta_{\ell n}\delta_{ik}E_{nj} - \delta_{ik}\delta_{jn}E_{\ell n}.$$

Hence,  $D_{ijkl}(E_{nn})$  has diagonal output if and only if  $j = \ell = n$  and  $i = k$ , in which case  $D_{ijij}(E_{jj}) = 2E_{ii} - 2E_{jj}$ . We have that  $\mathcal{L}(E_{jj})$  has diagonal given by  $\sum_{i \neq j} \gamma_{ij}(E_{ii} - E_{jj})$ , and thus  $\mathcal{L}^{\mathcal{D}}$  is given by

$$(\mathcal{L}^{\mathcal{D}})_{iijj} = \begin{cases} \gamma_{ij} & i \neq j \\ -\sum_{k \neq j} \gamma_{kj} & i = j \end{cases}.$$

□

**Remark 5.3.** If  $G_{\mathcal{L}}$  satisfies  $\gamma_{ij} = \gamma_{ji}$  for all pairs  $i, j$ , then  $\mathcal{L}^{\mathcal{D}}$  is negative semidefinite (since undirected graph Laplacians are always negative semidefinite, as shown in Section 4.1).

Recall Proposition 4.3, which states that vectors  $\rho^{G_{\mathcal{L}}^n}$  give rise to a natural basis of  $\ker L(G_{\mathcal{L}})$ . Considering TSCCs  $G_{\mathcal{L}}^1, \dots, G_{\mathcal{L}}^k$  of  $G_{\mathcal{L}}$ , we write these vectors as matrices by defining

$$d^{G_{\mathcal{L}}^n} := \sum_{i=1}^N \rho_i^{G_{\mathcal{L}}^n} E_{ii} = \sum_{i=1}^{N-1} \left( \sum_{j=1}^i \rho_j^{G_{\mathcal{L}}^n} - i \rho_{i+1}^{G_{\mathcal{L}}^n} \right) \frac{\lambda_{ii}}{\sqrt{i(i+1)}} + \frac{I_N}{N} \quad 1 \leq n \leq k, \quad (5.2)$$

where the second equality can be checked using (3.5). From Proposition 5.2 and Proposition 4.3 follows the analogous result:



**Corollary 5.4.** *Let  $\mathcal{L}$  be a QDS generator. Let  $G_{\mathcal{L}}^1, \dots, G_{\mathcal{L}}^k$  denote the TSCCs of  $G_{\mathcal{L}}$ . Then*

$$\ker \mathcal{L}^{\mathcal{D}} = \text{Span} \left( d^{G_{\mathcal{L}}^n} \right)_{n=1}^k.$$

In the case  $\gamma_{ij} = \gamma_{ji}$  for all pairs  $i, j$  (for example, if  $\mathcal{L}$  arises from diagonal  $C$ ), then a basis for  $\ker \mathcal{L}^{\mathcal{D}}$  is easier to compute. Indeed, considering the digraph  $G_{\mathcal{L}}$  as an undirected graph  $H_{\mathcal{L}}$ , for each connected component  $H_{\mathcal{L}}^1, \dots, H_{\mathcal{L}}^k$  of  $H_{\mathcal{L}}$  we may use the simpler vectors  $\gamma^{H_{\mathcal{L}}^n}$  given in Remark 4.1 to define

$$d^{H_{\mathcal{L}}^n} = \sum_{i=1}^N \gamma_i^{H_{\mathcal{L}}^n} E_{ii} = \sum_{i=1}^{N-1} \left( \sum_{j=1}^i \gamma_j^{H_{\mathcal{L}}^n} - i \gamma_{i+1}^{H_{\mathcal{L}}^n} \right) \frac{\lambda_{ii}}{\sqrt{i(i+1)}} \quad 1 \leq n \leq k, \quad (5.3)$$

and establish the following result:

**Proposition 5.5.** *Let  $\mathcal{L}$  be a QDS generator such that  $\gamma_{ij} = \gamma_{ji}$  for all pairs  $i \neq j$ . Let  $H_{\mathcal{L}}^1, \dots, H_{\mathcal{L}}^k$  denote the connected components of  $H_{\mathcal{L}}$ . Then*

$$\ker \mathcal{L}^{\mathcal{D}} = \text{Span}(d^{H_{\mathcal{L}}^n})_{n=1}^k.$$

## CHAPTER 6

### PAIR BLOCK DIAGONAL $\mathcal{L}$

#### 6.1 THE $\mathcal{L}^\mathcal{O}$ PART OF $\mathcal{L}$

The previous chapter revealed that  $\ker \mathcal{L}^\mathcal{D}$  is characterized by the TSCCs of  $G_\mathcal{L}$ . The aim of this section is to establish a similar result for  $\mathcal{L}^\mathcal{O}$  when  $\mathcal{L}$  is pair block diagonal. The type of TSCCs we require here is more precise, however, and we must begin by establishing a few definitions.

We call a 2-sink  $k\ell$  of  $G_\mathcal{L}$  a **singular 2-sink** if  $\gamma_{k\ell} = \gamma_{\ell k}$  and the  $k\ell$  block of  $\Gamma^\mathcal{O}$  is singular. Rephrased in terms of  $C$ , a 2-sink  $k\ell$  of  $G_\mathcal{L}$  is a singular 2-sink if  $c_{k\ell}c_{\ell k} - a_{k\ell}^2 = 0$ , as this equality implies  $b_{k\ell} = 0$  (equivalently  $\gamma_{k\ell} = \gamma_{\ell k}$ ) by (5.1). We use  $S_{G_\mathcal{L}}$  to denote the set of sinks of  $G_\mathcal{L}$  and  $S_{G_\mathcal{L}}^2$  to denote the set of singular 2-sinks of  $G_\mathcal{L}$ .

Notably, in the definition of singular 2-sinks we require information beyond the weights of  $G_\mathcal{L}$ , namely  $\alpha_{k\ell}$  and  $\beta_{k\ell}$ . It follows that graph induced generators (1.1) satisfy  $S_{G_\mathcal{L}}^2 = \emptyset$ , as in this case the  $k\ell$  block of  $\Gamma^\mathcal{O}$  is always nonsingular unless it is identically zero, precluding the possibility of  $k\ell$  to be a 2-sink. The next lemma shows further coefficients which are not graph induced, such as the entries of  $\Gamma^\mathcal{D}$ , also affect  $\ker \mathcal{L}^\mathcal{O}$ . Here we assume for simplicity that  $\Gamma \geq 0$  as in Theorem 2.3, but we note after Theorem 6.3 how one may produce the statement for  $\Gamma \not\geq 0$ .

**Lemma 6.1.** *Let  $\mathcal{L}$  be a QDS generator which is pair block diagonal with respect to the standard basis (3.1) with  $\widetilde{H} = \sum_{n=1}^N h_n E_{nn}$  and  $\Gamma \geq 0$ . Then the  $k\ell$  block  $\mathcal{L}_{k\ell}$  of  $\mathcal{L}^\mathcal{O}$  is singular if and only if  $h_k = h_\ell$ ,  $\gamma_{kk} = \gamma_{\ell\ell} = \gamma_{kk\ell\ell}$ , and either*

- $k, \ell \in S_{G_{\mathcal{L}}}$ , in which case  $\ker \mathcal{L}_{k\ell} = \text{Span}(E_{k\ell}, E_{\ell k})$ , or
- $k\ell \in S_{G_{\mathcal{L}}}^2$ , in which case

$$\ker \mathcal{L}_{k\ell} = \mathbb{C} \{ (\gamma_{k\ell} + \alpha_{k\ell} + \imath\beta_{k\ell})E_{k\ell} + (\gamma_{k\ell} + \alpha_{k\ell} - \imath\beta_{k\ell})E_{\ell k} \}.$$

*Proof.* We fix  $k < \ell$  and calculate the exact matrix form of  $\mathcal{L}_{k\ell}$  by evaluating  $\mathcal{L}$  at  $E_{k\ell}$  and  $E_{\ell k}$ . From (3.3) we have

$$\sum_{n,m=1}^N \gamma_{nnmm} D_{nnmm}(E_{k\ell}) = (2\gamma_{kk\ell\ell} - \gamma_{kk} - \gamma_{\ell\ell})E_{k\ell}$$

and

$$\sum_{n,m=1}^N \gamma_{nnmm} D_{nnmm}(E_{\ell k}) = (2\gamma_{\ell\ell kk} - \gamma_{kk} - \gamma_{\ell\ell})E_{\ell k},$$

which is to say  $\Gamma^{\mathcal{D}}$  contributes to  $\mathcal{L}_{k\ell}$  the  $2 \times 2$  matrix

$$D := \frac{1}{2} \begin{pmatrix} 2\gamma_{kk\ell\ell} - \gamma_{kk} - \gamma_{\ell\ell} & 0 \\ 0 & 2\gamma_{\ell\ell kk} - \gamma_{kk} - \gamma_{\ell\ell} \end{pmatrix} = \begin{pmatrix} d_{k\ell} & 0 \\ 0 & \bar{d}_{k\ell} \end{pmatrix},$$

where we define  $d_{k\ell} := \gamma_{kk\ell\ell} - \frac{1}{2}(\gamma_{kk} + \gamma_{\ell\ell})$  for future notational convenience (and hence  $\bar{d}_{k\ell} = \gamma_{\ell\ell kk} - \frac{1}{2}(\gamma_{kk} + \gamma_{\ell\ell})$  since  $\Gamma \geq 0$ ). Remark 3.4 gives that  $\text{Re } d_{k\ell} \leq 0$ , and so  $D$  has eigenvalues in the closed right hand plane.

Considering  $\Gamma^{\mathcal{O}}$ , from (3.3) we have, for  $i \neq j$ ,

$$D_{ij}(E_{k\ell}) = -(\delta_{jk} + \delta_{j\ell})E_{k\ell}, \quad D_{ij}(E_{\ell k}) = -(\delta_{j\ell} + \delta_{jk})E_{\ell k}$$

and

$$D_{ijji}(E_{k\ell}) = 2\delta_{jk}\delta_{i\ell}E_{\ell k}, \quad D_{ijji}(E_{\ell k}) = 2\delta_{j\ell}\delta_{ik}E_{k\ell}.$$

Thus, an  $ij$  block of  $\Gamma^{\mathcal{O}}$  for which  $|\{i, j\} \cap \{k, \ell\}| = 0$  contributes nothing to  $\mathcal{L}_{k\ell}$ , and an  $ij$  block of  $\Gamma^{\mathcal{O}}$  for which  $|\{i, j\} \cap \{k, \ell\}| = 1$  contributes to  $\mathcal{L}_{k\ell}$  the  $2 \times 2$  matrix

$$IJ := \frac{1}{2} \begin{cases} -\gamma_{ji}I_2 & i \in \{k, \ell\} \not\ni j \\ -\gamma_{ij}I_2 & i \notin \{k, \ell\} \ni j \end{cases}.$$

Note that  $IJ$  is negative semidefinite since  $\gamma_{ij}, \gamma_{ji} \geq 0$  (see Remark 5.1). Note also that  $IJ$  is singular if and only if  $\gamma_{ji} = 0$  when  $i \in \{k, \ell\}$  or  $\gamma_{ij} = 0$  when  $j \in \{k, \ell\}$ , in which case  $IJ = 0$ .

Similarly, the above equations show that the  $k\ell$  block  $\begin{pmatrix} \gamma_{k\ell} & \alpha_{k\ell} + \imath\beta_{k\ell} \\ \alpha_{k\ell} - \imath\beta_{k\ell} & \gamma_{\ell k} \end{pmatrix}$  of  $\Gamma^\mathcal{O}$  contributes to  $\mathcal{L}_{k\ell}$  the  $2 \times 2$  matrix

$$KL := \frac{1}{2} \begin{pmatrix} -\gamma_{k\ell} - \gamma_{\ell k} & 2(\alpha_{k\ell} + \imath\beta_{k\ell}) \\ 2(\alpha_{k\ell} - \imath\beta_{k\ell}) & -\gamma_{k\ell} - \gamma_{\ell k} \end{pmatrix}.$$

Note that the  $k\ell$  of  $\Gamma^\mathcal{O}$  block is positive semidefinite, as it is a principal submatrix of positive semidefinite  $\Gamma^\mathcal{O}$ . Thus  $KL$  is negative semidefinite, as it is the negated sum of the  $k\ell$  block of  $\Gamma^\mathcal{O}$  and its anti-diagonal transpose, both positive semidefinite matrices. Also note that  $KL$  is singular if and only if  $\det(KL) = 0$ .

Finally, we compute

$$-\imath[\widetilde{H}, E_{k\ell}] = -\imath \sum_{n=1}^N h_n [E_{nn}, E_{k\ell}] = -\imath(h_k - h_\ell)E_{k\ell}$$

and similarly  $-\imath[\widetilde{H}, E_{\ell k}] = -\imath(h_\ell - h_k)E_{\ell k}$ , which is to say  $\widetilde{H}$  contributes to  $\mathcal{L}_{k\ell}$  the  $2 \times 2$  matrix

$$\overline{H} := \begin{pmatrix} -\imath(h_k - h_\ell) & 0 \\ 0 & \imath(h_k - h_\ell) \end{pmatrix}.$$

In total, we now have that

$$\mathcal{L}_{k\ell} = KL + \overline{H} + D + \sum_{|\{i,j\} \cap \{k\ell\}|=1} IJ.$$

We claim that  $KL + \overline{H} + D$  has eigenvalues all in the closed left-hand plane. Indeed, if we consider the matrix  $\tilde{C}$  obtained by setting equal to zero all entries of  $\Gamma$  except those in  $\Gamma^\mathcal{D}$  and the  $k\ell$  block of  $\Gamma^\mathcal{O}$ , then  $\tilde{\Gamma} \geq 0$  and so  $\tilde{\mathcal{L}}$  is a QDS generator by Theorem 2.2. Moreover, this has the affect of setting  $IJ = 0$  for all  $IJ$  but leaving the other calculations unchanged above, and so we have  $\tilde{\mathcal{L}}_{k\ell} = KL + \overline{H} + D$ . The block form (3.4) implies every eigenvalue of  $\tilde{\mathcal{L}}_{k\ell}$  is an eigenvalue of  $\tilde{\mathcal{L}}$  and so

must lie in the closed left-hand plane (if  $\tilde{\mathcal{L}}(x) = \lambda x$  then  $\tilde{T}_t(x) = e^{t\lambda}x$ , and so  $\|T_t(x)\|_1 = |e^{t\lambda}| \operatorname{Tr}(|x|) \leq \operatorname{Tr}(|x|) = \|x\|_1$  implies  $\operatorname{Re} \lambda \leq 0$  since  $\|T_t\|_{1 \rightarrow 1} \leq 1$  as remarked in Section 2.2).

Since  $KL + \overline{H} + D$  and all  $IJ$  pairwise commute (every  $IJ$  is a multiple of  $I_2$ ), every eigenvalue of  $\mathcal{L}_{k\ell}$  is the sum of eigenvalues  $KL + \overline{H} + D$  and each  $IJ$ . Since each  $IJ$  is negative semidefinite and  $KL + \overline{H} + D$  has eigenvalues in the closed left-hand plane,  $\mathcal{L}_{k\ell}$  is singular (has eigenvalue 0) if and only if  $KL + \overline{H} + D$  and each of the  $IJ$  are singular; that is,  $\mathcal{L}_{k\ell}$  is singular if and only if each of the following hold:

- (i)  $\det(KL + \overline{H} + D) = 0$
- (ii)  $\gamma_{ji} = 0$  for all  $i < j$  with  $i \in \{k, \ell\} \not\ni j$
- (iii)  $\gamma_{ij} = 0$  for all  $i < j$  with  $i \notin \{k, \ell\} \ni j$

We claim that condition (i) can be rewritten as

- (i)  $\gamma_{k\ell} = \gamma_{\ell k}$ , the  $k\ell$  block of  $\Gamma^\mathcal{O}$  is singular,  $\gamma_{kk} = \gamma_{\ell\ell} = \gamma_{kk\ell\ell}$ , and  $h_k = h_\ell$ .

Indeed, using  $d_{k\ell} = \gamma_{kk\ell\ell} - \frac{1}{2}(\gamma_{kk} + \gamma_{\ell\ell})$ ,  $h_{k\ell} = h_k - h_\ell$ , and  $y_{k\ell} = \frac{1}{2}(\gamma_{k\ell} + \gamma_{\ell k})$  for notational convenience, we have  $\det(KL + \overline{H} + D) =$

$$\begin{aligned} &= (-y_{k\ell} + d_{k\ell} - \imath h_{k\ell})(-y_{k\ell} + \bar{d}_{k\ell} + \imath h_{k\ell}) - (\alpha_{k\ell} + \imath \beta_{k\ell})(\alpha_{k\ell} - \imath \beta_{k\ell}) \\ &= y_{k\ell}^2 + (d_{k\ell} - \imath h_{k\ell})(\bar{d}_{k\ell} + \imath h_{k\ell}) - y_{k\ell}(d_{k\ell} + \bar{d}_{k\ell}) - \alpha_{k\ell}^2 - \beta_{k\ell}^2. \end{aligned}$$

We understand this equation as three nonnegative parts:

First, since the  $k\ell$  block of  $C$  is positive semidefinite, we have that

$$\begin{aligned} P_1 &:= y_{k\ell}^2 - \alpha_{k\ell}^2 - \beta_{k\ell}^2 = (y_{k\ell} + \alpha_{ij})(y_{k\ell} - \alpha_{ij}) - (-\beta_{ij})^2 \\ &= c_{k\ell}c_{\ell k} - a_{k\ell}^2 \geq 0 \end{aligned}$$

using conversion (3.7). It follows that  $P_1 = 0$  if and only if  $\gamma_{k\ell} = \gamma_{\ell k}$  and the  $k\ell$  block of  $\Gamma^\mathcal{O}$  is singular, as remarked in the equivalent definitions of singular 2-sinks in the preamble of this section.

Second,

$$P_2 := (d_{k\ell} - \imath h_{k\ell})(\bar{d}_{k\ell} + \imath h_{k\ell}) = (d_{k\ell} - \imath h_{k\ell})(\overline{d_{k\ell} - \imath h_{k\ell}}) \geq 0.$$

Since  $\Gamma$  is positive semidefinite the submatrix  $\begin{pmatrix} \gamma_{kk} & \gamma_{kk\ell\ell} \\ \gamma_{\ell\ell kk} & \gamma_{\ell\ell} \end{pmatrix}$  is as well, from which it follows that

$$-2 \operatorname{Re}(d_{k\ell} - \imath h_{k\ell}) = -2 \operatorname{Re}(d_{k\ell}) = -(d_{k\ell} + \bar{d}_{k\ell}) = \gamma_{kk} + \gamma_{\ell\ell} - 2 \operatorname{Re}(\gamma_{kk\ell\ell}) \geq 0,$$

with equality if and only if  $\gamma_{kk} = \gamma_{\ell\ell} = \gamma_{kk\ell\ell}$  (this follows identically as (5.1)). In particular,  $\operatorname{Re}(d_{k\ell}) = 0$  implies  $\operatorname{Im}(d_{k\ell}) = 0$ , so we have that  $P_2 = 0$  if and only if  $\gamma_{kk} = \gamma_{\ell\ell} = \gamma_{kk\ell\ell}$  and  $h_{k\ell} = 0$ .

Finally,

$$P_3 := -y_{k\ell}(d_{k\ell} + \bar{d}_{k\ell}) = \frac{1}{2}(\gamma_{k\ell} + \gamma_{\ell k})(\gamma_{kk} + \gamma_{\ell\ell} - 2 \operatorname{Re}(\gamma_{kk\ell\ell})) \geq 0,$$

with  $P_3 = 0$  if and only if  $\gamma_{kk} = \gamma_{\ell\ell} = \gamma_{kk\ell\ell}$  or  $\gamma_{k\ell} = \gamma_{\ell k} = 0$ , with similar reasoning as above.

Thus, we have that  $\det(KL + \bar{H} + D) = P_1 + P_2 + P_3 = 0$  if and only if  $P_1 = P_2 = P_3 = 0$ . By the arguments above, this happens if and only if the rephrased (i) holds.

The next two conditions (ii) and (iii) simply say that vertices  $k$  and  $\ell$  have no out edges, except possibly to each other. Thus, if (i) holds, this means either  $\gamma_{k\ell} = \gamma_{\ell k} \neq 0$  and  $k\ell$  is a singular 2-sink of  $G_{\mathcal{L}}$ , or  $\gamma_{k\ell} = \gamma_{\ell k} = 0$  and  $k$  and  $\ell$  are sinks of  $G_{\mathcal{L}}$ .

It remains to note that if  $\mathcal{L}_{k\ell}$  is singular, and hence (i), (ii), and (iii) hold, then  $\mathcal{L}_{k\ell} = KL$ , as  $\bar{H}$ ,  $D$ , and all  $IJ$  are necessarily zero. Thus, if  $\mathcal{L}_{k\ell}$  is singular then

$$\ker \mathcal{L}_{k\ell} = \begin{cases} \mathbb{C}\{(\gamma_{k\ell} + \alpha_{k\ell} + \imath\beta_{k\ell})E_{k\ell} + (\gamma_{k\ell} + \alpha_{k\ell} - \imath\beta_{k\ell})E_{\ell k}\} & \text{if } k\ell \in S_{G_{\mathcal{L}}}^2 \\ \operatorname{Span}(E_{k\ell}, E_{\ell k}) & \text{if } k, \ell \in S_{G_{\mathcal{L}}} \end{cases}, \quad (6.1)$$

as can either be directly verified or obtained as a corollary of Theorem 6.3 (see Remark 6.4).  $\square$

**Corollary 6.2.** *Let  $\mathcal{L}$  be a Hamiltonian-free QDS generator which is pair block diagonal with respect to the standard basis (3.1) with  $\Gamma^{\mathcal{D}}$  diagonal. Then  $\mathcal{L}^{\mathcal{O}}$  is negative semidefinite.*

*Proof.* Considering a  $k\ell$  block  $\mathcal{L}_{k\ell}$  of  $\mathcal{L}$  computed as in the proof of Lemma 6.1, we have  $\mathcal{L}_{k\ell} = KL + D + \sum_{|\{i,j\} \cap \{k\ell\}|=1} IJ$ . As before,  $KL$  and each  $IJ$  is negative semidefinite, so it suffices to show that  $D$  is negative semidefinite if  $\Gamma^{\mathcal{D}}$  diagonal. This is indeed the case, since  $D = \frac{1}{2} \begin{pmatrix} -\gamma_{kk} & -\gamma_{\ell\ell} \\ 0 & -\gamma_{kk} - \gamma_{\ell\ell} \end{pmatrix}$  and  $\gamma_{kk} + \gamma_{\ell\ell} \geq 0$  by Remark 3.4.  $\square$

**Theorem 6.3.** *Let  $\mathcal{L}$  be a QDS generator which is pair block diagonal with respect to the standard basis (3.1) with  $\widetilde{H} = \sum_{n=1}^N h_n E_{nn}$ . Then, setting*

$$\lambda = \sqrt{\alpha_{k\ell}^2 + \beta_{k\ell}^2 + \left( \frac{\gamma_{kk\ell\ell} - \gamma_{\ell\ell kk}}{2} - \imath(h_k - h_{\ell}) \right)^2},$$

*the  $k\ell$  block  $\mathcal{L}_{k\ell}$  of  $\mathcal{L}^{\mathcal{O}}$  has eigenmatrices*

$$\begin{aligned} A^{\pm} = & \left[ \alpha_{k\ell} + \imath\beta_{k\ell} + \frac{\gamma_{kk\ell\ell} - \gamma_{\ell\ell kk}}{2} - \imath(h_k - h_{\ell}) \pm \lambda \right] E_{k\ell} \\ & + \left[ \alpha_{k\ell} - \imath\beta_{k\ell} - \frac{\gamma_{kk\ell\ell} - \gamma_{\ell\ell kk}}{2} + \imath(h_k - h_{\ell}) \pm \lambda \right] E_{\ell k} \end{aligned}$$

*corresponding to eigenvalues*

$$\mu^{\pm} = -\frac{1}{2} \left( \gamma_{k\ell} + \gamma_{\ell k} + \gamma_{kk} + \gamma_{\ell\ell} - \gamma_{kk\ell\ell} - \gamma_{\ell\ell kk} + \sum_{i \notin \{k, \ell\} \ni j} \gamma_{ij} + \sum_{i \in \{k, \ell\} \nexists j} \gamma_{ji} \right) \pm \lambda.$$

*In particular,  $E_{k\ell}$  and  $E_{\ell k}$  are eigenmatrices of  $\mathcal{L}^{\mathcal{O}}$  if and only if  $\alpha_{k\ell} = \beta_{k\ell} = 0$ , in which case they have eigenvalues  $\gamma_{kk\ell\ell} - \imath(h_k - h_{\ell}) - \mu$  and  $\gamma_{\ell\ell kk} - \imath(h_k - h_{\ell}) - \mu$ , respectively, where*

$$\mu = \frac{1}{2} \left( \gamma_{k\ell} + \gamma_{\ell k} + \gamma_{kk} + \gamma_{\ell\ell} + \sum_{i \notin \{k, \ell\} \ni j} \gamma_{ij} + \sum_{i \in \{k, \ell\} \nexists j} \gamma_{ji} \right).$$

*Proof.* It is well known that given a  $2 \times 2$  matrix  $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$  its eigenvectors are given by  $\begin{pmatrix} \mu^{\pm} + b - d \\ \mu^{\pm} + c - a \end{pmatrix}$ , where  $\mu^{\pm} = \text{Tr}(M)/2 \pm (\text{Tr}^2(M)/4 - \det(M))^{1/2}$  are the corresponding eigenvalues, as can be verified by simply evaluating  $M$  at the proposed eigenvectors. This fact applied to  $KL + \widetilde{H} + D$  (as compute in the proof of Lemma 6.1), along with the shift from adding  $\sum IJ$  (multiple of  $I_2$ ) immediately gives the above formula.  $\square$

**Remark 6.4.** If  $k\ell \in S_{G_{\mathcal{L}}}^2$  then  $\gamma_{k\ell}^2 - \alpha_{k\ell}^2 - \beta_{k\ell}^2 = 0$  since the  $k\ell$  block of  $\Gamma^{\mathcal{O}}$  is singular. Hence,  $\gamma_{k\ell} = \gamma_{\ell k} = \sqrt{\alpha_{k\ell}^2 + \beta_{k\ell}^2}$  in this case. If we further assume  $h_k = h_\ell$  and  $\gamma_{kk} = \gamma_{\ell\ell} = \gamma_{kk\ell\ell}$ , then we have that  $A^+ = (\gamma_{k\ell} + \alpha_{k\ell} + \imath\beta_{k\ell})E_{k\ell} + (\gamma_{k\ell} + \alpha_{k\ell} - \imath\beta_{k\ell})E_{\ell k}$  corresponding to  $\mu^+ = 0$  generates  $\ker \mathcal{L}_{k\ell}$ , as given before in (6.1).

We note two facts: First,  $\Gamma \geq 0$  was not assumed in Theorem 6.3, as the calculations needed did not rely on this fact. Hence, one may set  $\mu^\pm = 0$  to write Lemma 6.1 without the  $\Gamma \geq 0$  assumption. Second, Theorem 6.3 provides an explicit formula for  $N^2 - N$  of  $\mathcal{L}$ 's  $N^2$  many eigenpairs, but since the digraph Laplacian  $\mathcal{L}^{\mathcal{D}}$  is not diagonalizable in general the entire matrix  $\mathcal{L}$  may not be diagonalizable. Finding the eigenvalues of a digraph Laplacian is historically difficult, but much work has been done on finding the spectral gap, as this controls the rate of convergence of  $e^{t\mathcal{L}}$ . Though we do not explore such applications in this work, we note that, together with the eigenvalues given by Theorem 6.3, the spectral gap of  $\mathcal{L}^{\mathcal{D}}$  gives the rate of convergence for  $T_t = e^{t\mathcal{L}}$ . We refer the interested reader to the seminal work of Wu [71] for more on the eigenvalues of digraph Laplacians.

Having established results for the standard basis, we now consider the Gell-Mann basis. Certainly one may use (3.7) and the corresponding equivalence for converting  $C^{\mathcal{D}_0}$  into  $\Gamma^{\mathcal{D}}$  to translate Theorem 6.3 immediately into the corresponding general statement for the Gell-Mann basis. As we will only consider the Gell-Mann basis in specialized cases, we avoid writing this tedious conversion here and instead prove the needed statement directly.

**Lemma 6.5.** *Let  $\mathcal{L}$  be a QDS generator which is pair block diagonal with respect to the Gell-Mann basis (3.6) with  $H = \sum_{n=1}^N h_n E_{nn}$  and  $C^{\mathcal{D}_0}$  diagonal. Then the  $k\ell$  block  $\mathcal{L}_{k\ell}$  of  $\mathcal{L}$  is singular if and only if  $h_k = h_\ell$ ,  $c_{nn} = 0$  for all  $k - 1 \leq n \leq \ell - 1$ , and*

- $k, \ell \in S_{G_{\mathcal{L}}}$ , in which case  $\ker \mathcal{L}_{k\ell} = \text{Span}(\lambda_{k\ell}, \lambda_{\ell k})$ , or



- $k\ell \in S_{G_L}^2$ , in which case  $\ker \mathcal{L}_{k\ell} = \mathbb{C}\{(c_{k\ell} + a_{k\ell})\lambda_{k\ell} + (c_{\ell k} + a_{\ell k})\lambda_{\ell k}\}$ .

*Proof.* As in the proof of Lemma 6.1, we calculate  $\mathcal{L}_{k\ell}$  explicitly. Indeed, the only difference here is the contribution of  $C^{\mathcal{D}_0}$ , since the contribution of  $H$  and  $C^{\mathcal{O}}$  can be recovered from the formula for  $\overline{H}$ ,  $IJ$ , and  $KL$  calculated there. Using the same basis change as in the derivation of (3.7), these matrices are represented in the Gell-Mann basis as

$$\overline{H} = \begin{pmatrix} 0 & h_\ell - h_k \\ h_k - h_\ell & 0 \end{pmatrix}, \quad KL = \begin{pmatrix} -c_{\ell k} & a_{k\ell} \\ a_{k\ell} & -c_{k\ell} \end{pmatrix},$$

$$\text{and } IJ = -\frac{1}{4} \begin{cases} (c_{ij} + c_{ji} + 2b_{ij})I_2 & i \in \{k, \ell\} \not\ni j \\ (c_{ij} + c_{ji} - 2b_{ij})I_2 & i \notin \{k, \ell\} \ni j \end{cases}.$$

Use  $\delta_{i \leq j}$  to denote the indicator

$$\delta_{i \leq j} = \begin{cases} 1 & \text{if } i \leq j \\ 0 & \text{otherwise} \end{cases},$$

and similarly for  $\delta_{i \leq j \leq k}$ . For  $k < \ell$ , we have

$$D_{nn}^\lambda(\lambda_{k\ell}) = [\lambda_{nn}, \lambda_{k\ell}\lambda_{nn}] + [\lambda_{nn}\lambda_{k\ell}, \lambda_{nn}] = 2\lambda_{nn}\lambda_{k\ell}\lambda_{nn} - \lambda_{k\ell}\lambda_{nn}\lambda_{nn} - \lambda_{nn}\lambda_{nn}\lambda_{k\ell},$$

where  $2\lambda_{nn}\lambda_{k\ell}\lambda_{nn} =$

$$\begin{aligned} &= \frac{2}{\sqrt{2n(n+1)}} \left( \sum_{m=1}^n E_{mm} - nE_{n+1,n+1} \right) (E_{k\ell} + E_{\ell k}) \left( \sum_{m=1}^n E_{mm} - nE_{n+1,n+1} \right) \\ &= \frac{2}{\sqrt{2n(n+1)}} (\delta_{k \leq n} E_{k\ell} + \delta_{\ell \leq n} E_{\ell k} - n\delta_{k,n+1} E_{k\ell} - n\delta_{\ell,n+1} E_{\ell k}) \left( \sum_{m=1}^n E_{mm} - nE_{n+1,n+1} \right) \\ &= \frac{2}{\sqrt{2n(n+1)}} \left( \delta_{k \leq n} \delta_{\ell \leq n} E_{k\ell} + \delta_{\ell \leq n} \delta_{k \leq n} E_{\ell k} - n\delta_{k,n+1} \delta_{\ell \leq n} E_{k\ell} - n\delta_{\ell,n+1} \delta_{k \leq n} E_{\ell k} \right. \\ &\quad \left. - n\delta_{k \leq n} \delta_{\ell,n+1} E_{k\ell} - n\delta_{\ell \leq n} \delta_{k,n+1} E_{\ell k} + n^2 \delta_{k,n+1} \delta_{\ell,n+1} E_{k\ell} + n^2 \delta_{\ell,n+1} \delta_{k,n+1} E_{\ell k} \right) \\ &= \frac{2}{\sqrt{2n(n+1)}} (\delta_{\ell \leq n} E_{k\ell} + \delta_{\ell \leq n} E_{\ell k} - n\delta_{\ell,n+1} E_{\ell k} - n\delta_{\ell,n+1} E_{k\ell}) \quad \text{using that } k < \ell \\ &= \frac{2}{n(n+1)} (\delta_{\ell \leq n} \lambda_{k\ell} - n\delta_{\ell,n+1} \lambda_{k\ell}) \\ &= \frac{2}{n(n+1)} (\delta_{\ell \leq n} - n\delta_{\ell,n+1}) \lambda_{k\ell} \end{aligned}$$

and  $\lambda_{k\ell}\lambda_{nn}\lambda_{nn} + \lambda_{nn}\lambda_{nn}\lambda_{k\ell} =$

$$\begin{aligned}
&= \frac{1}{\sqrt{2n(n+1)}} \left( (E_{k\ell} + E_{\ell k}) \left( \sum_{m=1}^n E_{mm} + n^2 E_{n+1,n+1} \right) \right. \\
&\quad \left. + \left( \sum_{m=1}^n E_{mm} + n^2 E_{n+1,n+1} \right) (E_{k\ell} + E_{\ell k}) \right) \\
&= \frac{1}{\sqrt{2n(n+1)}} \left( (\delta_{\ell \leq n} E_{k\ell} + \delta_{k \leq n} E_{\ell k} + n^2 \delta_{\ell, n+1} E_{k\ell} + n^2 \delta_{k, n+1} E_{\ell k}) \right. \\
&\quad \left. + (\delta_{k \leq n} E_{k\ell} + \delta_{\ell \leq n} E_{\ell k} + n^2 \delta_{k, n+1} E_{k\ell} + n^2 \delta_{\ell, n+1} E_{\ell k}) \right) \\
&= \frac{1}{n(n+1)} (\delta_{\ell \leq n} \lambda_{k\ell} + \delta_{k \leq n} \lambda_{k\ell} + n^2 \delta_{\ell, n+1} \lambda_{k\ell} + n^2 \delta_{k, n+1} \lambda_{k\ell}) \\
&= \frac{1}{n(n+1)} (\delta_{\ell \leq n} + \delta_{k \leq n} + n^2 \delta_{\ell, n+1} + n^2 \delta_{k, n+1}) \lambda_{k\ell}.
\end{aligned}$$

Thus,

$$\begin{aligned}
D_{nn}^\lambda(\lambda_{k\ell}) &= \frac{1}{n(n+1)} \left( 2(\delta_{\ell \leq n} - n\delta_{\ell, n+1}) - (\delta_{\ell \leq n} + \delta_{k \leq n} + n^2 \delta_{\ell, n+1} + n^2 \delta_{k, n+1}) \right) \lambda_{k\ell} \\
&= \frac{1}{n(n+1)} (-n^2 \delta_{k, n+1} - \delta_{k \leq n \leq \ell-2} - (n+1)^2 \delta_{\ell, n+1}) \lambda_{k\ell} \\
&= \begin{cases} \frac{-n}{(n+1)} \lambda_{k\ell} & n = k-1 \\ \frac{-1}{n(n+1)} \lambda_{k\ell} & k \leq n \leq \ell-2 \\ \frac{-(n+1)}{n} \lambda_{k\ell} & n = \ell-1 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Similarly,

$$\begin{aligned}
D_{nn}^\lambda(\lambda_{\ell k}) &= \frac{1}{n(n+1)} (-n^2 \delta_{k, n+1} - \delta_{k \leq n \leq \ell-2} - (n+1)^2 \delta_{\ell, n+1}) \lambda_{\ell k} \\
&= \begin{cases} \frac{-n}{(n+1)} \lambda_{\ell k} & n = k-1 \\ \frac{-1}{n(n+1)} \lambda_{\ell k} & k \leq n \leq \ell-2 \\ \frac{-(n+1)}{n} \lambda_{\ell k} & n = \ell-1 \\ 0 & \text{otherwise} \end{cases}.
\end{aligned}$$

Thus,

$$\sum_{n=1}^{N-1} c_{nn} D_{nn}^\lambda(\lambda_{k\ell}) = - \left( \frac{k-1}{k} c_{k-1, k-1} + \sum_{m=k}^{\ell-2} \frac{1}{m(m+1)} c_{mm} + \frac{\ell}{(\ell-1)} c_{\ell-1, \ell-1} \right) \lambda_{k\ell},$$

$$\sum_{n=1}^{N-1} c_{nn} D_{nn}^\lambda(\lambda_{\ell k}) = - \left( \frac{k-1}{k} c_{k-1,k-1} + \sum_{m=k}^{\ell-2} \frac{1}{m(m+1)} c_{mm} + \frac{\ell}{(\ell-1)} c_{\ell-1,\ell-1} \right) \lambda_{\ell k},$$

which is to say  $C^{\mathcal{D}_0}$  contributes to  $\mathcal{L}_{k\ell}$  the  $2 \times 2$  matrix

$$D^C := -\frac{1}{2} \left( \frac{k-1}{k} c_{k-1,k-1} + \sum_{m=k}^{\ell-2} \frac{1}{m(m+1)} c_{mm} + \frac{\ell}{\ell-1} c_{\ell-1,\ell-1} \right) I_2.$$

Note that  $D^C$  is negative semidefinite (each  $c_{nn} \geq 0$  since  $C \geq 0$ ). Furthermore,  $D^C$  is singular if and only if  $c_{nn} = 0$  for all  $k-1 \leq n \leq \ell-1$ , in which case  $D^C = 0$ .

In total, we now have that

$$\mathcal{L}_{k\ell} = KL + \overline{H} + D^C + \sum_{|\{i,j\} \cap \{k\ell\}|=1} IJ,$$

so  $\mathcal{L}_{k\ell}$  is singular if and only if  $KL + \overline{H}$  is singular and  $D^C = \sum IJ = 0$ , as  $KL + \overline{H}$  has eigenvalues in the closed left-hand plane (by the same argument as before) and  $D^C$  and each  $IJ$  is negative semidefinite. The same logic as before shows this happens if and only if  $h_k = h_\ell$ ,  $c_{nn} = 0$  for all  $k-1 \leq n \leq \ell-1$ , and either  $k\ell \in S_{G_\mathcal{L}}^2$  or  $k, \ell \in S_{G_\mathcal{L}}$ , in which case

$$\ker \mathcal{L}_{k\ell} = \ker KL = \begin{cases} \mathbb{C}\{(c_{k\ell} + a_{k\ell})\lambda_{k\ell} + (c_{\ell k} + a_{\ell k})\lambda_{\ell k}\} & \text{if } k\ell \in S_{G_\mathcal{L}}^2 \\ \text{Span}(\lambda_{k\ell}, \lambda_{\ell k}) & \text{if } k, \ell \in S_{G_\mathcal{L}} \end{cases}.$$

□

The next two statements follow similarly to Corollary 6.2 and Theorem 6.3.

**Corollary 6.6.** *Let  $\mathcal{L}$  be a Hamiltonian-free QDS generator which is pair block diagonal with respect to the Gell-Mann basis (3.6) with  $C^{\mathcal{D}_0}$  diagonal. Then  $\mathcal{L}^\mathcal{O}$  is negative semidefinite.*

**Remark 6.7.** If  $\mathcal{L}$  is a QDS generator which is pair block diagonal with respect to the Gell-Mann basis (3.6) with  $H = \sum_{n=1}^N h_n E_{nn}$  and  $C^{\mathcal{D}_0}$  diagonal, then the  $k\ell$  block

$\mathcal{L}_{k\ell}$  of  $\mathcal{L}$  has eigenmatrices

$$A^\pm = \begin{bmatrix} a_{k\ell} + \frac{c_{k\ell} - c_{\ell k}}{2} - (h_k - h_\ell) \pm \sqrt{\left(\frac{c_{k\ell} - c_{\ell k}}{2}\right)^2 + a_{k\ell}^2 - (h_k - h_\ell)^2} \\ a_{k\ell} - \frac{c_{k\ell} - c_{\ell k}}{2} + (h_k - h_\ell) \pm \sqrt{\left(\frac{c_{k\ell} - c_{\ell k}}{2}\right)^2 + a_{k\ell}^2 - (h_k - h_\ell)^2} \end{bmatrix} \begin{matrix} \lambda_{k\ell} \\ \lambda_{\ell k} \end{matrix}$$

corresponding to eigenvalues  $\mu^\pm =$

$$\begin{aligned} & -\frac{1}{2} \left( c_{k\ell} + c_{\ell k} + \frac{k-1}{k} c_{k-1,k-1} + \sum_{m=k}^{\ell-2} \frac{1}{m(m+1)} c_{mm} + \frac{\ell}{\ell-1} c_{\ell-1,\ell-1} \right. \\ & \left. + \frac{1}{2} \sum_{i \notin \{k,\ell\} \ni j} (c_{ij} + c_{ji} - 2b_{ij}) + \frac{1}{2} \sum_{i \in \{k,\ell\} \not\ni j} (c_{ij} + c_{ji} + 2b_{ij}) \right) \\ & \pm \sqrt{\left(\frac{c_{k\ell} - c_{\ell k}}{2}\right)^2 + a_{k\ell}^2 - (h_k - h_\ell)^2}. \end{aligned}$$

In particular, both of  $\lambda_{k\ell}$  and  $\lambda_{\ell k}$  are eigenmatrices of  $\mathcal{L}^\mathcal{O}$  if and only if  $h_k - h_\ell = a_{k\ell} = 0$ , in which case they have eigenvalues  $-c_{\ell k} - \mu$  and  $-c_{k\ell} - \mu$ , respectively, where

$$\begin{aligned} 2\mu = & \frac{1}{2} \sum_{i \notin \{k,\ell\} \ni j} (c_{ij} + c_{ji} - 2b_{ij}) + \frac{1}{2} \sum_{i \in \{k,\ell\} \not\ni j} (c_{ij} + c_{ji} + 2b_{ij}) + \frac{k-1}{k} c_{k-1,k-1} \\ & + \sum_{m=k}^{\ell-2} \frac{1}{m(m+1)} c_{mm} + \frac{\ell}{\ell-1} c_{\ell-1,\ell-1}. \end{aligned}$$

One might compare this last remark to Theorem 5 of [64], where Siudzińska determines the eigenvalues of a QDS generator  $\mathcal{L}$  which is written in Gell-Mann form (3.6) with  $H = 0$  and  $C$  diagonal, and for which every  $\lambda_{ij}$  (including  $i = j$ ) is an eigenmatrix of  $\mathcal{L}$ .

In the case  $C^\mathcal{O}$  is diagonal the digraph  $G_\mathcal{L}$  satisfies  $\gamma_{ij} = \gamma_{ji}$  for all vertices  $i$  and  $j$ , and hence  $G_\mathcal{L}$  may be regarded as an (undirected) graph  $H_\mathcal{L}$ . Let  $I_{H_\mathcal{L}}$  denote the set of isolated vertices of  $H_\mathcal{L}$ , and let  $I_{H_\mathcal{L}}^2$  denote the set of isolated edges  $k\ell$  of  $H_\mathcal{L}$  for which  $c_{k\ell}c_{\ell k} = 0$  (i.e., the set singular 2-sinks ignoring direction). The statement of Lemma 6.5 is simplified to the following:

**Corollary 6.8.** *Let  $\mathcal{L}$  be a QDS generator written with respect to the Gell-Mann basis (3.6) such that  $H = \sum_{n=1}^N h_n E_{nn}$  and  $C$  is diagonal. Then the  $k\ell$  block  $\mathcal{L}_{k\ell}$  of  $\mathcal{L}$  is singular if and only if  $h_k = h_\ell$ ,  $c_{nn} = 0$  for  $k-1 \leq n \leq \ell-1$ , and*

- $k, \ell \in I_{H_{\mathcal{L}}}$ , in which case  $\ker \mathcal{L}_{k\ell} = \text{Span}(\lambda_{k\ell}, \lambda_{\ell k})$ , or
- $k\ell \in I_{H_{\mathcal{L}}}^2$ , in which case
  - $\ker \mathcal{L}_{k\ell} = \mathbb{C}\{\lambda_{k\ell}\}$  if  $c_{\ell k} = 0$ ,
  - $\ker \mathcal{L}_{k\ell} = \mathbb{C}\{\lambda_{\ell k}\}$  if  $c_{k\ell} = 0$ .

## 6.2 EXAMINING THE FULL GENERATOR $\mathcal{L}$

To establish the final kernel results for this section, we need only recall that pair block diagonal generators are of form (3.4). From Corollary 5.4 and Lemma 6.1, we have the following:

**Theorem 6.9.** *Let  $\mathcal{L}$  be a QDS generator which is pair block diagonal with respect to the standard basis (3.1) with  $\widetilde{H} = \sum_{n=1}^N h_n E_{nn}$  and  $\Gamma \geq 0$ . Then*

$$\ker \mathcal{L} = \bigoplus_{k,\ell} \ker \mathcal{L}_{k\ell} \oplus \text{Span} \left( d^{G_{\mathcal{L}}^n} \right)_{n=1}^k,$$

where  $d^{G_{\mathcal{L}}^n}$  are given by (5.2) and  $\ker \mathcal{L}_{k\ell}$  are as in Lemma 6.1.

**Theorem 6.10.** *Let  $\mathcal{L}$  be a QDS generator which is pair block diagonal with respect to the Gell-Mann basis (3.6) with  $H = \sum_{n=1}^N h_n E_{nn}$  and  $C^{\mathcal{D}_0}$  diagonal. Then*

$$\ker \mathcal{L} = \bigoplus_{k,\ell} \ker \mathcal{L}_{k\ell} \oplus \text{Span} \left( d^{G_{\mathcal{L}}^n} \right)_{n=1}^k,$$

where  $d^{G_{\mathcal{L}}^n}$  are given by (5.2) and  $\ker \mathcal{L}_{k\ell}$  are as in Lemma 6.5.

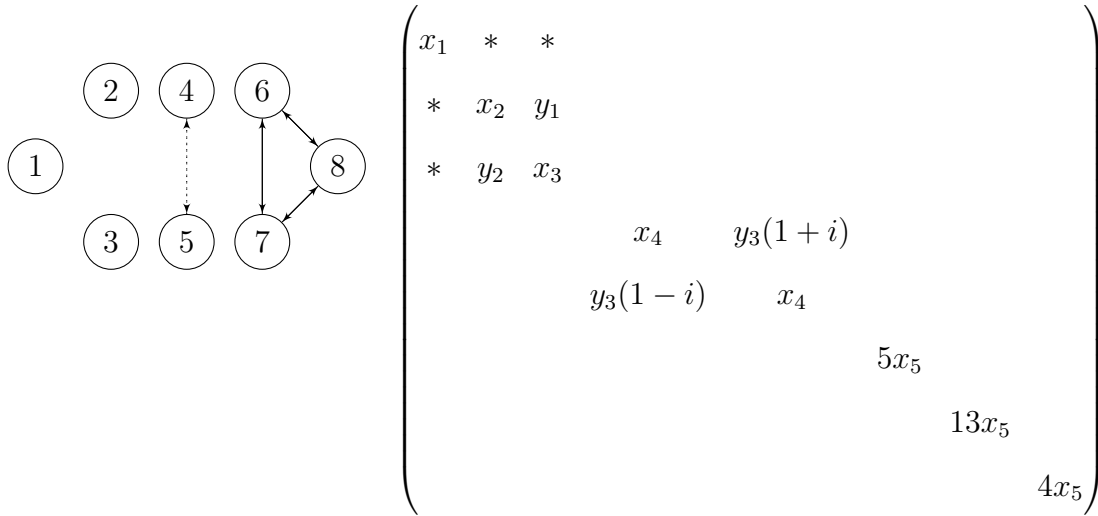
**Corollary 6.11.** *Let  $\mathcal{L}$  be a QDS generator written with respect to the Gell-Mann basis (3.6) such that  $H = \sum_{n=1}^N h_n E_{nn}$  and  $C$  is diagonal. Then*

$$\ker \mathcal{L} = \bigoplus_{k,\ell} \ker \mathcal{L}_{k\ell} \oplus \text{Span} \left( d^{H_{\mathcal{L}}^n} \right)_{n=1}^k,$$

where  $d^{H_{\mathcal{L}}}$  are given by (5.3) and  $\ker \mathcal{L}_{k\ell}$  are as in Corollary 6.8.

Recalling (2.2), these Theorems allow us to compute exactly the invariant states for pair block diagonal generators with diagonal Hamiltonian from statistics of the underlying graph. Namely, the diagonal entries are computed from the total weight of spanning trees rooted at each vertex, and the off-diagonal entries arise from the presence of sinks and singular 2-sinks. Examples 6.12 and 6.13 below illustrate how these various structures in the associated digraph  $G_{\mathcal{L}}$  affect the structure of the invariant states.

**Example 6.12.** In dimension  $N = 8$ , consider QDS generator  $\mathcal{L}$  given by (3.1) with Hamiltonian  $H = \sum_{i=1}^8 h_i E_{ii}$  with  $h_2 = h_3$  and  $h_4 = h_5$ , and coefficient matrix  $\Gamma$  whose entries are all zero except the 45 block given by  $\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}$  and the 67, 68, and 78 blocks given by  $\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ ,  $\begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$ , and  $\begin{pmatrix} 4 & 0 \\ 0 & 1 \end{pmatrix}$  respectively. The graph  $G_{\mathcal{L}}$  is drawn below, where the dashed edge is a singular 2-sink.



The kernel of  $\mathcal{L}$  can be computed via Theorem 6.9, where each pair of  $(k, \ell)$  and  $(\ell, k)$  entries are given by  $\ker \mathcal{L}_{k\ell}$ . The displayed matrix represents an arbitrary element in  $\ker \mathcal{L}$  where missing entries are zero. Specifically, the five  $x_n$ 's represent multiples of  $d^{G_{\mathcal{L}}}$  for each of the five TSCCs, computed as in (5.2), and the  $y_n$ 's represent multiples of the off-diagonal kernel elements described in Lemma 6.1. The

entries denoted by  $*$  represent zero if  $h_1 \neq h_2, h_3$ , or additional free variables if  $h_1 = h_2 = h_3$ . Notice that one may create both non-faithful and/or non-diagonal invariant states. Notice also that the presence of a singular 2-sink puts relations on the real and imaginary parts of certain off-diagonal coordinates of the kernel elements, a phenomenon that does not happen in the graph induced case (1.1).

**Example 6.13.** Consider a system with three states:  $|1\rangle$ ,  $|2\rangle$ , and  $|3\rangle$ . Consider the jump between  $\frac{1}{\sqrt{2}}(|1\rangle + i|2\rangle) \mapsto \frac{1}{\sqrt{2}}(i|1\rangle + |2\rangle)$  at rate  $a > 0$  together with the jumps  $|3\rangle \mapsto |1\rangle$  at rate  $b > 0$  and  $|3\rangle \mapsto |2\rangle$  at rate  $c > 0$ . Following Remark 3.1, we model this by setting the entries of coefficient matrix  $\Gamma$  all zero except the 12 block given by  $\begin{pmatrix} a & a \\ a & a \end{pmatrix}$ , 13 block given by  $\begin{pmatrix} b & 0 \\ 0 & 0 \end{pmatrix}$ , and the 23 block given by  $\begin{pmatrix} c & 0 \\ 0 & 0 \end{pmatrix}$ . Applying Theorem 6.9, we have that

$$\ker \mathcal{L} = \text{Span} \left( \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right),$$

and so the invariant states of this system are given by

$$\frac{1}{2} \begin{pmatrix} 1 & x & 0 \\ x & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

for any  $-1 \leq x \leq 1$ . In particular,  $\frac{1}{\sqrt{2}}(|1\rangle + |2\rangle)$  is an invariant state. In the graph induced case (1.1), i.e. if only jumps between vector states  $|i\rangle \mapsto |j\rangle$  had been allowed, this could only happen in the trivial case that the jump rates for  $|1\rangle \mapsto |i\rangle$  and  $|2\rangle \mapsto |i\rangle$  were identically zero for all  $i$ . Allowing jumps between superpositions thus enables the system to maintain coherence despite nontrivial evolution.

## CHAPTER 7

### OTHER GENERATORS

#### 7.1 IDENTITY PRESERVING QDSs

In this section we examine QDSs whose generators satisfy  $\mathcal{L}(I_N) = 0$ ; that is, QDSs for which the maximally mixed state  $I_N/N$  is invariant, or, equivalently by Corollary 2.5, QDSs which are contractive for some/all  $p$ -Schatten norm with  $p > 1$ . We prove that the kernel of such a QDS generator is contained in the kernel of a second, naturally induced QDS generator which is characterized by Corollary 6.11. To define this second generator we first consider the kernel of the coefficient matrix  $C$  for  $\mathcal{L}$  written in Gell-Mann form (3.6).

**Lemma 7.1.** *Let  $C : M_N^0(\mathbb{C}) \rightarrow M_N^0(\mathbb{C})$  with  $C \geq 0$ , and let  $x_1, \dots, x_n \in M_N^0(\mathbb{C})$  be orthonormal in  $S_2^N$ . Then  $C - \epsilon \sum_{i=1}^n |x_i\rangle\langle x_i| \geq 0$  for some  $\epsilon > 0$  if and only if  $\{x_1, \dots, x_n\} \subseteq (\ker C)^\perp$ .*

*Proof.* Let  $\epsilon = \inf_{y \in (\ker C)^\perp, \|y\|=1} \langle y, Cy \rangle$ . That  $\epsilon \geq 0$  is clear since  $C \geq 0$ . We claim that  $\epsilon > 0$ . Indeed, the unit ball of  $(\ker C)^\perp$  is compact (being finite dimensional) and so the infimum is achieved at some  $y_0 \in (\ker C)^\perp$ . Since  $Cy_0 \neq 0$  we have  $\sqrt{C}y_0 \neq 0$ , and hence  $\langle y_0, Cy_0 \rangle = \langle \sqrt{C}y_0, \sqrt{C}y_0 \rangle = \|\sqrt{C}y_0\|^2 \neq 0$ .

Now, suppose  $\{x_1, \dots, x_n\}$  is an orthonormal subset of  $(\ker C)^\perp$  and let  $\{k_1, \dots, k_m\}$  be an orthonormal basis of  $\ker C$ . Then there exist  $x_{n+1}, \dots, x_\ell \in M_N^0(\mathbb{C})$  such that  $\{k_1, \dots, k_m, x_1, \dots, x_\ell\}$  is an orthonormal basis of  $M_N(\mathbb{C})$ . Letting  $z \in M_N^0(\mathbb{C})$  we aim to show  $\langle z, (C - \epsilon \sum_{i=1}^n |x_i\rangle\langle x_i|)z \rangle \geq 0$ . Indeed, writing  $z = \sum_{s=1}^m a_s k_s + \sum_{t=1}^\ell b_t x_t$  we may define  $\tilde{z} := \sum_{t=1}^\ell b_t x_t$  and assume  $\|\tilde{z}\|^2 = \sum_{t=1}^\ell |b_t|^2 = 1$



without loss of generality. Then  $C = C^*$  and  $Cz = C\tilde{z}$  imply

$$\langle z, Cz \rangle = \langle z, C\tilde{z} \rangle = \langle Cz, \tilde{z} \rangle = \langle C\tilde{z}, \tilde{z} \rangle = \langle \tilde{z}, C\tilde{z} \rangle \geq \epsilon,$$

and so

$$\begin{aligned} \langle z, (C - \epsilon \sum_{i=1}^n |x_i\rangle\langle x_i|)z \rangle &= \langle z, Cz \rangle - \epsilon \sum_{i=1}^n \langle z, |x_i\rangle\langle x_i|z \rangle \\ &= \langle z, Cz \rangle - \epsilon \sum_{t=1}^{\ell} |b_t|^2 = \langle z, Cz \rangle - \epsilon \geq 0. \end{aligned}$$

Conversely, suppose  $\{x_1, \dots, x_n\} \not\subseteq (\ker C)^\perp$  so there is some  $k \in \ker C$  such that  $k \not\perp x_j$  for some  $1 \leq j \leq n$ . Then  $|\langle k, x_j \rangle|^2 > 0$ , and so for all  $\epsilon > 0$  we have

$$\langle k, (C - \epsilon \sum_{i=1}^n |x_i\rangle\langle x_i|)k \rangle = \langle k, Ck \rangle - \epsilon \sum_{i=1}^n \langle k, |x_i\rangle\langle x_i|k \rangle = -\epsilon \sum_{i=1}^n |\langle k, x_i \rangle|^2 < 0.$$

□

**Remark 7.2.** Let  $\mathcal{L}$  be a QDS generator written in Gell-Mann form (3.6) with coefficient matrix  $C$ , and define  $K : M_N^0 \rightarrow M_N^0$  by  $K = \sum |\lambda_{ij}\rangle\langle \lambda_{ij}|$ , where the sum is over all  $\lambda_{ij}$  perpendicular to  $\ker C$ . Then  $C - \epsilon K \geq 0$  for some  $\epsilon > 0$ . Further,  $K \geq 0$  and so taking  $K$  to be the coefficient matrix in Gell-Mann form (3.6) defines a QDS generator  $\mathcal{K}$  by Theorem 2.1. Since  $K$  is diagonal we have  $\mathcal{K}$  is of form (3.8),  $\mathcal{K}(I_N) = 0$ , and further  $\mathcal{K}$  is negative semidefinite by Remark 5.3 and Corollary 6.6.

**Proposition 7.3.** *Let  $\mathcal{L}$  be a QDS generator satisfying  $\mathcal{L}(I_N) = 0$ . Then*

$$\ker \mathcal{L} \subseteq \ker \mathcal{K},$$

where  $\ker \mathcal{K}$  is given by Corollary 6.11.

*Proof.* Fix  $\epsilon > 0$  such that  $C - \epsilon K \geq 0$ . It is easy to see that using  $C - \epsilon K$  as the coefficient matrix in Gell-Mann form (3.6) gives rise to the QDS generator  $\mathcal{L} - \epsilon \mathcal{K}$ , and that  $\mathcal{L} = (\mathcal{L} - \epsilon \mathcal{K}) + \mathcal{K}$ . The result then follows from Lemma 2.6. □

We note that  $\mathcal{L}$  does not need to be written in Gell-Mann form (3.6) to define  $\mathcal{K}$ , as our definition relies only on the kernel of the coefficient matrix  $C$ . Recalling that Theorem 2.1 uniquely defines  $C$  (as an operator), or more generally that Theorem 2.3 uniquely defines  $\Gamma$ , this kernel is uniquely defined regardless of basis  $\{F_i\}$ .

## 7.2 CONSISTENT GENERATORS.

In this section we examine those generators for which the Hamiltonian  $H$  is ‘well-behaved’. More precisely, let  $H_{\mathcal{L}}$  denote the graph obtained from  $G_{\mathcal{L}}$  by ignoring weights and directedness of the edges, and for each connected component  $H_{\mathcal{L}}^k$  of  $H_{\mathcal{L}}$  let  $P_k$  be the orthogonal projection onto  $\text{Span}(E_{ij})_{i,j \in V(H_{\mathcal{L}}^k)}$ . We call  $H$  **consistent** if  $P_k H P_{\ell} = 0$  for all  $\ell \neq k$ . We provide a lower bound for the dimension of the kernel of a QDS generator for which  $H$  is consistent.

Recall that the definition of a QDS immediately implies  $\text{Tr}(\mathcal{L}(A)) = 0$  for all  $A \in M_N(\mathbb{C})$ . The next result says that certain submatrices of  $\mathcal{L}(A)$  are also traceless if we assume the Hamiltonian  $H$  is consistent.

**Theorem 7.4.** *Let  $\mathcal{L}$  be a QDS generator. Considering fixed  $k$ , if  $P_k H P_{\ell} = 0$  for all  $\ell \neq k$ , then  $\text{Tr}(P_k \mathcal{L}(A)) = 0$  for all  $A \in M_N(\mathbb{C})$ .*

*Proof.* Consider  $\mathcal{L}$  written with respect to the standard basis (3.1) such that  $\Gamma$  satisfies the conditions of Theorem 2.3. If  $H_{\mathcal{L}}$  is connected then the statement is obvious since  $\mathcal{L}$  has traceless range, so assume that  $H_{\mathcal{L}}$  is not connected and  $H_{\mathcal{L}}^n, H_{\mathcal{L}}^m$  are distinct connected components. Then for any  $i \in V(H_{\mathcal{L}}^n)$  and  $j \in V(H_{\mathcal{L}}^m)$  we have that weights  $\gamma_{ij} = \gamma_{ji} = 0$ . Further, positive semidefiniteness of  $\Gamma$  implies that each entry of  $\Gamma$  which shares a row or column with  $\gamma_{ijij}$  or  $\gamma_{jiji}$  is also zero (for if not the  $2 \times 2$  submatrix formed by removing all other rows and columns would have negative

determinant, contradicting positive semidefiniteness). Hence

$$\mathcal{L} = -\imath[H, \cdot] + \sum_{n,m} \sum_{\substack{i,j \in V(H_{\mathcal{L}}^n) \\ k,\ell \in V(H_{\mathcal{L}}^m)}} \gamma_{ijkl} D_{ijkl}. \quad (7.1)$$

By linearity of  $\mathcal{L}$  it suffices to show  $\text{Tr}(P_k \mathcal{L}(E_{st})) = 0$  for arbitrary  $1 \leq s, t \leq N$ . To this end, we claim that every output  $\mathcal{L}(E_{st})$  which has nonzero diagonal is traceless with its nonzero diagonal in  $\text{Span}(E_{nn})_{n \in V(H_{\mathcal{L}}^m)}$  for some  $m$ . Since each output of  $\mathcal{L}$  is a linear combination of outputs of  $[H, \cdot]$  and of the  $D_{ijkl}$  appearing in (7.1), it suffices to show this for  $[H, \cdot]$  and those  $D_{ijkl}$  separately.

For the Hamiltonian part we write  $H = \sum h_{ij} E_{ij}$  so that  $[H, \cdot] = \sum h_{ij} [E_{ij}, \cdot]$ . Note that if  $P_k H P_\ell = 0$  for  $k \neq \ell$  then for any  $i \in V(H_{\mathcal{L}}^k)$  and  $j \in V(H_{\mathcal{L}}^\ell)$  we have  $h_{ij} = 0$ . That is, if  $h_{ij} \neq 0$  then  $i, j \in V(H_{\mathcal{L}}^m)$  for some  $m$ . From this the claim is clear, as  $[E_{ij}, E_{st}]$  has nonzero diagonal output if and only if  $i = t$  and  $j = s$ , in which case  $[E_{ij}, E_{st}] = E_{ii} - E_{jj}$ .

For the operators  $D_{ijkl}$  we recall (3.3), which reads

$$D_{ijkl}(E_{st}) = 2\delta_{js}\delta_{\ell t}E_{ik} - \delta_{ik}\delta_{js}E_{\ell t} - \delta_{\ell t}\delta_{ik}E_{sj}.$$

Thus,  $D_{ijkl}(E_{st})$  has nonzero diagonal if and only if  $i = k$ ,  $j = s$ , and  $\ell = t$ , in which case  $D_{ijkl}(E_{st}) = 2E_{ii} - E_{jj} - E_{\ell\ell}$ . If  $D_{ijkl}$  appears in (7.1), then these equalities imply  $i, j, \ell \in V(H_{\mathcal{L}}^m)$  for some  $m$ .

□

**Corollary 7.5.** *Let  $\mathcal{L}$  be a QDS generator such that  $H$  is consistent. Then*

$$cc(H_{\mathcal{L}}) \leq \dim \ker \mathcal{L},$$

where  $cc(H_{\mathcal{L}})$  is the number of connected components of  $H_{\mathcal{L}}$ .

*Proof.* Consider the connected components  $H_{\mathcal{L}}^1, \dots, H_{\mathcal{L}}^\ell$  of  $H_{\mathcal{L}}$  ordered so that  $|V(H_{\mathcal{L}}^n)| \geq 2$  for  $n \leq m$  and  $|V(H_{\mathcal{L}}^n)| = 1$  for  $n > m$  for some  $m \geq 0$ . It

suffices to find  $\ell$  many pairwise orthogonal matrices not in  $\text{Range}(\mathcal{L})$ . Since  $H$  is consistent, by Theorem 7.4 we have  $\text{Tr}(P_k \mathcal{L}(A)) = 0$  for all  $A \in M_N(\mathbb{C})$  and all  $1 \leq k \leq \ell$ . In the boundary case of  $m = 0$  we have that  $E_{ii} \notin \text{Range}(\mathcal{L})$  for all  $1 \leq i \leq N$  and so  $\ell = N \leq \dim \ker \mathcal{L}$ . Otherwise, if  $m > 1$ , fixing  $i_1 \in V(H_{\mathcal{L}}^1)$  and  $j_2 \in V(H_{\mathcal{L}}^2)$  we have  $E_{i_1 i_1} - E_{j_2 j_2} \notin \text{Range}(\mathcal{L})$ . Similarly, fixing some  $i_2 \in V(H_{\mathcal{L}}^2) \setminus \{j_2\}$  and  $j_3 \in V(H_{\mathcal{L}}^3)$  we have  $E_{i_2 i_2} - E_{j_3 j_3} \notin \text{Range}(\mathcal{L})$ . We continue until we find  $E_{i_m i_m} - E_{j_{m+1} j_{m+1}} \notin \text{Range}(\mathcal{L})$ , for a total of  $m$  simple differences  $E_{ii} - E_{jj}$  not in  $\text{Range}(\mathcal{L})$ . Further, writing  $V(H_{\mathcal{L}}^n) = \{i_n\}$  for all  $n \geq m+2$  we have  $E_{i_n i_n} \notin \text{Range}(\mathcal{L})$ , for a total of  $\ell - m - 1$  distinct  $E_{ii}$  not in  $\text{Range}(\mathcal{L})$ . Because these chosen matrices are all diagonal and we have no repeated indices, we have a set of  $\ell - 1$  pairwise orthogonal matrices. It is clear that  $I_N - \sum_{m+2 \leq n \leq \ell} E_{i_n i_n}$  is nonzero and orthogonal to the above matrices, and is not in  $\text{Range}(\mathcal{L})$  since  $\mathcal{L}$  has traceless range, and so we have found a set of  $\ell$  many orthogonal matrices not in  $\text{Range}(\mathcal{L})$ , as desired.  $\square$

Since certainly a QDS is not uniquely relaxing if it has multiple invariant states, we immediately have the following.

**Corollary 7.6.** *Let  $\mathcal{L}$  be a QDS generator such that  $H$  is consistent. If  $T_t$  is uniquely relaxing then  $H_{\mathcal{L}}$  is connected.*

We note that it is not true that the number of TSCCs of  $G_{\mathcal{L}}$  lower bounds  $\dim \ker \mathcal{L}$  in general, even with consistent  $H$ ; for example, see the example of section 2 of aforementioned [41] for which  $G_{\mathcal{L}}$  has two TSCCs yet the QDS has a single invariant state.

## CHAPTER 8

### CONCLUSION TO PART I

We began this Part by determining when the famed GKSL form (2.1) would define a QDS generator when allowed not necessarily traceless operators  $F_i$  (Theorem 2.2). Along the way, we identified that the coefficient matrix  $C$  of the classical GKSL form (2.1) is uniquely determined by  $\mathcal{L}$  when viewed as an operator (discussion above Theorem 2.2), but this is not necessarily true for the coefficient matrix  $\Gamma$  of the more general form (2.3) unless stronger assumptions are met (Theorem 2.3). In any case, these theorems offer criteria for when  $\mathcal{L}$  written with respect to the standard basis (3.1) defines a QDS generator, a form whose simplicity is advantageous for both calculation and understanding.

With this easy to work with form, we established the class of pair block diagonal generators (Definition 3.3) to generalize the graph induced generators given by (1.1) while preserving the important properties, such as leaving the diagonal subalgebra  $\mathcal{D}$  and off-diagonal subspace  $\mathcal{O}$  invariant in the case of diagonal Hamiltonian  $H$ . We also established the synonymous definition in terms of the Gell-Mann basis (Definition 3.6), which is often used due to its traceless construction when dealing with the GKSL form (2.1).

For the class of pair block diagonal generators, we found explicit formula for all invariant states when the Hamiltonian is diagonal (Theorem 6.9), and furthermore all eigenmatrices which belong to the off-diagonal subspace  $\mathcal{O}$  and their corresponding eigenvalues (Theorem 6.3). In particular, the invariant states depend on the structure of a naturally induced digraph. Though we do not explore such applications in this

work, we note that these results allow for exact computation of rates of convergence of such QDSs, given the Laplacian spectral gap of the induced digraph.

We have also shown explicitly that, when written in matrix form, every QDS generator contains as a submatrix a naturally associated digraph Laplacian (Theorem 5.2). In the case the Hamiltonian is consistent with this digraph, connectedness properties of the digraph identify submatrices of elements in the range of  $\mathcal{L}$  as traceless (Theorem 7.4), and hence we have established lower bounds on the number of invariant states of the QDS based on the connectedness properties of the digraph (Corollary 7.5). In the case the maximally mixed state is invariant, which happens if and only if the QDS is contraction in some/all  $p$ -Schatten norms with  $p > 1$  (Corollary 2.5), we have shown that the structure of the invariant states can be inferred from the digraph naturally associated to the kernel of the coefficient matrix (Proposition 7.3).

## CHAPTER 9

### INTRODUCTION TO PART II

It is known that, under certain assumptions, semigroups on von Neumann algebras or their preduals give rise to associated semigroups on Hilbert spaces. Moreover, these associated semigroups often have stronger continuity properties than the original semigroups. For example, in [51, Equation (2.1)] it is stated that if  $(T_t)_{t \geq 0}$  is a quantum Markov semigroup on a von Neumann algebra  $\mathcal{A}$  which has an invariant faithful normal state, and if  $(\mathcal{K}, \pi, \Omega)$  is the GNS triple associated to that state, then there exists a strongly continuous semigroup  $(\overline{T}_t)_{t \geq 0}$  of contractions on  $\mathcal{K}$  such that

$$\overline{T}_t(\pi(A)\Omega) = \pi(T_t(A))\Omega \quad \text{for all } A \in \mathcal{A} \text{ and } t \geq 0. \quad (9.1)$$

Since the proof of this statement is not included in [51] we provide a proof here (see Remarks 11.6 and 12.15). Other results which give rise to semigroups on Hilbert spaces starting from semigroups defined on spaces of operators can be found in literature. For example, in [50, Footnote of Theorem 6] it is proved that every strongly continuous semigroup  $(T_t)_{t \geq 0}$  of positive isometries on the real Banach space of self-adjoint trace-class operators on a Hilbert space gives rise to a strongly continuous semigroup  $(V_t)_{t \geq 0}$  of isometries on the Hilbert space such that  $T_t$  is given as a conjugation by  $V_t$  for all  $t \geq 0$ . In [32, Theorem 3] dilation theory is used to prove that under appropriate assumptions weakly continuous semigroups on  $\mathcal{B}(\mathcal{H})$  (where  $\mathcal{H}$  is a separable Hilbert space) give rise to corresponding semigroups of unitaries on some associated Hilbert space. Dilation theory has also been used in [47, Theorem 3.3.7] in order to produce a strongly continuous group of unitaries associated with a norm continuous semigroup on the space of trace-class operators on a related Hilbert space.

In this Part we prove a result similar to the result stated above in Equation (9.1) (Theorem 12.14). More precisely, we prove that every semigroup of Schwarz maps on  $\mathcal{B}(\mathcal{H})$  (where  $\mathcal{H}$  is a Hilbert space) which has an invariant faithful state gives rise to an associated semigroup  $(\widetilde{T}_t)_{t \geq 0}$  of contractions on the space of Hilbert-Schmidt operators on  $\mathcal{H}$ . Our map is “more symmetric” than the one provided by Equation (9.1) (see the comments following Remark 11.6). Moreover, we explicitly describe how the generators of  $(T_t)_{t \geq 0}$  and  $(\widetilde{T}_t)_{t \geq 0}$  are related. Further, we use the dilation theory by Foias and Sz.-Nagy in order to obtain a minimal unitary dilation of  $(\widetilde{T}_t)_{t \geq 0}$ . We introduce the notion of the extended generator of a semigroup on bounded operators on a Hilbert space with respect to an orthonormal basis of the Hilbert space. Finally, under the assumption that the semigroup  $(T_t)_{t \geq 0}$  is a quantum Markov semigroup having an invariant faithful normal state and that either the generator of the minimal unitary dilation of  $(\widetilde{T}_t)_{t \geq 0}$  is compact or the generator of  $(\widetilde{T}_t)_{t \geq 0}$  itself has compact resolvent, we describe the form of the extended generator of the semigroup  $(T_t)_{t \geq 0}$  with respect to an orthonormal basis (see Theorems 13.9 and 13.14).

**Acknowledgments:** I would like to thank Franco Fagnola and Matthew Ziemke. Their contributions to the results in this Part were vital from its conception to the final touches. Without their help the existence of this Part would not be possible.

## 9.1 STRUCTURE OF PART II

- In Chapter 10 we establish the formal notation and definitions required for this Part, and give some historical notes on the terminology.
- In Chapter 11 we consider several constructions arising from faithful, positive, normal functionals. In particular, in Section 11.1 we prove that every faithful positive normal functional on  $\mathcal{B}(\mathcal{H})$  induces a canonical bounded linear map from  $\mathcal{B}(\mathcal{H})$  to  $\mathcal{S}_2(\mathcal{H})$ . This map is used in Theorem 11.5 to prove that for every bounded linear Schwarz map on  $\mathcal{B}(\mathcal{H})$ , which has a subinvariant faithful positive functional, there



exists a corresponding contraction on  $\mathcal{S}_2(\mathcal{H})$ . In Section 11.2 we consider an alternate construction for such induced maps using the GNS construction, and then compare and contrast the two methods.

- In Chapter 12 we recall the basic notions of continuity for semigroups, as well as formalize the definition of a semigroup's generator and its generator's domain. In Section 12.1 we introduce the notion of an extended generator, which can be defined on a larger domain while still agreeing with the usual generator on all finite subspaces. Theorem 12.14 relates the domains and actions of the generator, the extended generator, and the generator of the semigroup induced on  $\mathcal{S}_2(\mathcal{H})$  (from Section 11.1). In Section 12.2 we apply the dilation theory of Foias and Sz.-Nagy to the semigroup induced on  $\mathcal{S}_2(\mathcal{H})$  in order to obtain a minimal semigroup of unitaries on a larger Hilbert space, as well as Stone's Theorem in order to give a description of its generator.

- In Chapter 13 we investigate the applications of Theorem 12.14 in the study of Quantum Markov semigroups (QMSs), for which the exact form of the generator is known that if the generator is bounded (see [42] and [53]). In section 13.1, we describe the generator of the QMS under the assumption that the generator of the minimal semigroup of unitary dilations of the associated semigroup of contractions is compact. In section 13.2, we describe the generator of the QMS under the assumption that the generator of the semigroup induced on  $\mathcal{S}_2(\mathcal{H})$  has compact resolvent.

## CHAPTER 10

### PRELIMINARIES

We first fix some notation. If  $\mathcal{H}$  is a Hilbert space, let  $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$  denote the space of all bounded linear operators on  $\mathcal{H}$ . For  $1 \leq p < \infty$ , let  $(\mathcal{S}_p(\mathcal{H}), \|\cdot\|_p)$  denote the Schatten- $p$  space of operators. In particular,  $(\mathcal{S}_2(\mathcal{H}), \|\cdot\|_2)$  denotes the space of Hilbert-Schmidt operators on  $\mathcal{H}$  and  $(\mathcal{S}_1(\mathcal{H}), \|\cdot\|_1)$  denotes the space of trace-class operators on  $\mathcal{H}$ . Let  $\langle \cdot, \cdot \rangle_{\mathcal{S}_2(\mathcal{H})}$  denote the inner product in  $\mathcal{S}_2(\mathcal{H})$ . If  $L$  is a linear operator which is not necessarily bounded, then  $D(L)$  will denote the domain of  $L$ .

We adopt the convention that **functional** will always mean bounded linear functional. Usually the functionals that we will consider will be faithful, positive, and normal, so this convention will help us to cut down the number of adjectives.

We would like to recall the Schwarz inequality and define the Schwarz maps. The classical Cauchy-Schwarz inequality states that  $|\langle y, x \rangle| \leq \|y\| \|x\|$  for all vectors  $x, y$  in a Hilbert space. This inequality is extended to  $|\phi(y^*x)| \leq \sqrt{\phi(y^*y)} \sqrt{\phi(x^*x)}$  for all  $x, y$  in a  $C^*$ -algebra  $\mathcal{A}$ , where  $\phi$  is a positive functional on  $\mathcal{A}$  (see [49, Theorem 4.3.1]). The last inequality can be further extended to  $(T(y^*x))^*T(y^*x) \leq \|T(y^*y)\|T(x^*x)$  if  $T$  is a completely positive map from a  $C^*$ -algebra  $\mathcal{A}$  to the  $C^*$ -algebra  $\mathcal{B}(\mathcal{H})$  of all bounded operators on a Hilbert space  $\mathcal{H}$  (see [11, Lemma 2.6]). If in the last inequality one assumes that  $\mathcal{A}$  is unital and  $T$  is unital, then by replacing  $y$  by the unit we obtain

$$T(x)^*T(x) \leq T(x^*x) \quad \text{for all } x \in \mathcal{A}. \quad (10.1)$$

A similar inequality was proved by Choi [20, Corollary 2.8] who proved that if  $\mathcal{A}$  is a unital  $C^*$ -algebra and  $T$  is a 2-positive unital map from  $\mathcal{A}$  to  $\mathcal{A}$  then  $T(x^*)T(x) \leq$

$T(x^*x)$  for all  $x \in \mathcal{A}$ . Choi calls the last inequality “Schwarz inequality”. Similar inequalities appear in [48, Theorem 1] and [65, Theorem 7.4]. Since a positive linear map  $T$  on a  $C^*$ -algebra  $\mathcal{A}$  satisfies  $T(x^*) = T(x)^*$  for all  $x \in \mathcal{A}$ , the last inequality is equivalent to (10.1). Following [66, page 14], we say that a bounded linear operator  $T$  on a  $C^*$ -algebra  $\mathcal{A}$  is a **Schwarz map** if Inequality (10.1) is satisfied. The advantage of Inequality (10.1) versus the inequality proved by Choi, is that Inequality (10.1) implies that  $T$  is positive. Be warned that Inequality (10.1) is not homogeneous for  $T$ , and therefore by scaling the operator  $T$  by a positive constant the above inequality is affected.

Next we recall the definition of invariant functionals and we define the notion of subinvariant positive functionals on a  $C^*$ -algebra. If  $X$  is a Banach space,  $T : X \rightarrow X$  is a bounded linear operator, and  $\omega$  is a functional on  $X$ , then  $\omega$  is called **invariant for T** if

$$\omega(Tx) = \omega(x) \quad \text{for all } x \in X.$$

If  $\mathcal{A}$  is a  $C^*$ -algebra,  $T : \mathcal{A} \rightarrow \mathcal{A}$  is a positive bounded linear operator, and  $\omega$  is a positive functional on  $\mathcal{A}$ , then we will say that  $\omega$  is **subinvariant** for  $T$  if

$$\omega(Ta) \leq \omega(a) \quad \text{for all } a \in \mathcal{A} \text{ with } a \geq 0.$$

If  $\mathcal{H}$  is a Hilbert space, then a functional  $\omega$  on  $\mathcal{B}(\mathcal{H})$  is called **normal** if and only if it is continuous in the weak operator topology. This is equivalent to the fact that there exists a unique positive operator  $\rho \in \mathcal{S}_1(\mathcal{H})$  such that

$$\omega(x) = \text{Tr}(\rho x) \quad \text{for all } x \in \mathcal{B}(\mathcal{H}) \tag{10.2}$$

where  $\text{Tr}$  denotes the trace. The positive functional  $\omega$  associated to the positive trace-class operator  $\rho$  via Equation (10.2) is denoted by  $\omega_\rho$ . If  $\omega$  is a state (i.e. unital positive functional) on  $\mathcal{B}(\mathcal{H})$  then  $\omega$  is normal if and only if the positive trace-class operator  $\rho$  which satisfies Equation (10.2) has trace equal to 1. Note that if  $\mathcal{H}$  is a

Hilbert space and  $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is a bounded linear operator, then a normal positive functional  $\omega_\rho$  (for some positive trace-class operator  $\rho$ ) is invariant for  $T$  if and only if

$$T^\dagger(\rho) = \rho,$$

where  $T^\dagger$  denotes the Banach dual operator of  $T$  restricted to  $\mathcal{S}_1(\mathcal{H})$  (viewed as a subspace of the dual of  $\mathcal{B}(\mathcal{H})$ ). Also, if  $\mathcal{H}$  is a Hilbert space and  $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is a positive bounded linear operator, then a normal positive functional  $\omega_\rho$  (for some positive trace-class operator  $\rho$ ) is subinvariant for  $T$  if and only if

$$T^\dagger(\rho) \leq \rho.$$

If  $\mathcal{H}$  is a Hilbert space, recall that a positive functional  $\omega$  on  $\mathcal{B}(\mathcal{H})$  is **faithful** provided that  $\omega(x) > 0$  for all  $x > 0$ . It is worth noting that  $\mathcal{B}(\mathcal{H})$  has a faithful normal functional if and only if  $\mathcal{H}$  is separable (see [12, Example 2.5.5]).

# CHAPTER 11

## CONSTRUCTIONS USING FAITHFUL, POSITIVE, NORMAL FUNCTIONALS

We extensively use the next proposition so we want to give it along with a proof.

**Proposition 11.1.** *Let  $\mathcal{H}$  be a Hilbert space and  $\rho \in \mathcal{S}_1(\mathcal{H})$  be positive. Then the following are equivalent:*

- (i) *the positive normal functional  $\omega_\rho$  is faithful,*
- (ii) *the operator  $\rho$  is injective,*
- (iii) *the operator  $\rho$  has dense range.*

*Proof.* [(i)  $\Rightarrow$  (ii)]. Suppose  $\omega_\rho$  is faithful. Let  $h$  be a nonzero element of  $\mathcal{H}$  and  $P_h$  be the orthogonal projection onto the span of  $h$ . Then  $P_h$  is a positive non-zero operator on  $\mathcal{H}$ . Hence, since  $\omega_\rho$  is faithful,

$$0 < \omega_\rho(P_h) = \text{Tr}(\rho P_h) = \text{Tr}(\rho^{1/2} P_h P_h \rho^{1/2}) = \|P_h \rho^{1/2}\|_2^2 = \frac{1}{\|h\|^2} \|\rho^{1/2} h\|^2$$

and so  $\rho^{1/2} h \neq 0$  and hence, by using the same argument with  $h$  replaced by  $\rho^{1/2} h$ , we have that  $\rho h \neq 0$ . Therefore  $\rho$  is injective.

[(i)  $\Rightarrow$  (iii)]. If we assume that  $\rho$  does not have dense range then if we let  $P$  be the orthogonal projection to  $\text{Range}(\rho)^\perp$  then  $P$  is a positive non-zero operator on  $\mathcal{H}$  and so  $\omega_\rho(P) > 0$ . However,  $P\rho = 0$  and so

$$\omega_\rho(P) = \text{Tr}(\rho P) = \text{Tr}(P\rho) = \text{Tr}(0) = 0$$

which is a contradiction, so  $\rho$  has dense range.

[(iii)  $\Rightarrow$  (i)]. Let  $A \in \mathcal{B}(\mathcal{H})$  and suppose  $\omega_\rho(A^*A) = 0$ . Then

$$0 = \omega_\rho(A^*A) = \text{Tr}(\rho A^*A) = \text{Tr}(\rho^{1/2} A^* A \rho^{1/2}) = \|A \rho^{1/2}\|_2^2. \quad (11.1)$$

So we have that  $A \rho^{1/2} = 0$  and therefore  $A \rho = 0$ . Since  $\rho$  has dense range, we then have that  $A = 0$  and so  $\omega_\rho$  is faithful.

[(ii)  $\Rightarrow$  (i)]. Assume that  $\rho$  is injective and let  $A \in \mathcal{B}(\mathcal{H})$  such that  $\omega_\rho(A^*A) = 0$ . Equation (11.1) implies that  $A \rho^{1/2} = 0$  and so  $\rho^{1/2} A^* = 0$  which gives that  $\rho A^* = 0$ . Hence, for any  $x \in \mathcal{H}$ , we have that  $\rho A^* x = 0$  and since  $\rho$  is injective we have that  $A^* x = 0$  for all  $x \in \mathcal{H}$  and therefore  $A = 0$  and so  $\rho$  is faithful.  $\square$

**Remark 11.2.** Note that in the proof of [(i)  $\Rightarrow$  (ii)] of the above proposition, we proved that (i) implies that  $\rho^{1/2}$  is injective. Since  $\rho^{1/2} = \rho^{1/4} \rho^{1/4}$  we immediately obtain that  $\rho^{1/4}$  is injective. Since  $\rho^{3/4} = \rho^{1/2} \rho^{1/4}$  we obtain that  $\rho^{3/4}$  is injective as it is a composition of two injective maps. Further, since an operator is injective if and only if its adjoint has dense range, and  $\rho^{1/4}$ ,  $\rho^{1/2}$ , and  $\rho^{3/4}$  are self-adjoint, we have that  $\rho^{1/4}$ ,  $\rho^{1/2}$ , and  $\rho^{3/4}$  have dense range.

### 11.1 INDUCING MAPS ON $S_2(\mathcal{H})$

Let  $\mathcal{H}$  be a Hilbert space and fix  $\rho \in \mathcal{S}_1(\mathcal{H})$  which is positive. Define

$$i_\rho : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H}) \quad \text{by} \quad i_\rho(x) = \rho^{1/4} x \rho^{1/4}.$$

The next proposition summarizes the properties of the map  $i_\rho$ . First recall that for any Hilbert space  $\mathcal{H}$ , we have the following set inclusions

$$\mathcal{S}_1(\mathcal{H}) \subseteq \mathcal{S}_2(\mathcal{H}) \subseteq \mathcal{B}(\mathcal{H}).$$

**Proposition 11.3.** *Let  $\rho \in \mathcal{S}_1(\mathcal{H})$  be positive such that  $\omega_\rho$  is a faithful positive functional. Then the following statements are valid:*

(a) The map  $i_\rho$  is injective.

(b) The map  $i_\rho$  preserves positivity.

(c) The restriction  $i_\rho|_{\mathcal{S}_2(\mathcal{H})}$  of  $i_\rho$  to  $\mathcal{S}_2(\mathcal{H})$ , maps  $\mathcal{S}_2(\mathcal{H})$  into  $\mathcal{S}_1(\mathcal{H})$  and  $\|i_\rho|_{\mathcal{S}_2(\mathcal{H})} : \mathcal{S}_2(\mathcal{H}) \rightarrow \mathcal{S}_1(\mathcal{H})\| \leq 1$ .

(d) The map  $i_\rho$  maps  $\mathcal{B}(\mathcal{H})$  onto a dense subset of  $\mathcal{S}_2(\mathcal{H})$  and  $\|i_\rho : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{S}_2(\mathcal{H})\| \leq 1$ .

*Proof.* In order to see part (a), let  $x \in \mathcal{B}(\mathcal{H})$  and suppose  $i_\rho(x) = 0$ . By Remark 11.2 we have that  $\rho^{1/4}$  is injective. Therefore, since  $\rho^{1/4}x\rho^{1/4} = 0$ , we obtain that  $x\rho^{1/4} = 0$ . Further, since  $\rho^{1/4}$  has dense range, (by Remark 11.2 again), we obtain that  $x = 0$ . Thus  $i_\rho$  is injective.

In order to see (b), let  $x \in \mathcal{B}(\mathcal{H})$  where  $x \geq 0$ . Let  $h \in \mathcal{H}$ . Then

$$\langle h, i_\rho(x)h \rangle = \langle h, \rho^{1/4}x\rho^{1/4}h \rangle = \langle \rho^{1/4}h, x\rho^{1/4}h \rangle \geq 0$$

since  $x \geq 0$ . Thus  $i_\rho$  maps positive operators to positive operators.

In order to see (c), first note that for  $p, q, r \geq 1$  where  $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$  and for  $x \in \mathcal{S}_p(\mathcal{H})$ ,  $y \in \mathcal{S}_q(\mathcal{H})$ , and  $z \in \mathcal{S}_r(\mathcal{H})$ , two applications of Holder's inequality give that  $\|xyz\|_1 \leq \|x\|_p \|y\|_q \|z\|_r$ . From this we obtain that for  $y \in \mathcal{S}_2(\mathcal{H})$  with  $\|y\|_2 \leq 1$  we have that

$$\|i_\rho(y)\|_1 = \|\rho^{1/4}y\rho^{1/4}\|_1 \leq \|\rho^{1/4}\|_4 \|y\|_2 \|\rho^{1/4}\|_4 = \|\rho\|_1^{1/4} \|y\|_2 \|\rho\|_1^{1/4} \leq \|\rho\|_1^{1/4},$$

which finishes the proof of (c).

For the proof of (d), first notice that  $i_\rho(x) \in \mathcal{S}_2(\mathcal{H})$  for all  $x \in \mathcal{B}(\mathcal{H})$  since

$$\begin{aligned} \|i_\rho(x)\|_2^2 &= \|\rho^{1/4}x\rho^{1/4}\|_2^2 = \text{Tr}(\rho^{1/2}x^*\rho^{1/2}x) \\ &\leq \|\rho^{1/2}x^*\rho^{1/2}x\|_1 \leq \|\rho^{1/2}x^*\|_2 \|\rho^{1/2}x\|_2 \\ &= \text{Tr}((\rho^{1/2}x^*)^*(\rho^{1/2}x^*))^{1/2} \text{Tr}((\rho^{1/2}x)^*(\rho^{1/2}x))^{1/2} \\ &= \text{Tr}(x\rho x^*)^{1/2} \text{Tr}(x^*\rho x)^{1/2} < \infty. \end{aligned}$$

Let  $y \in \mathcal{S}_2(\mathcal{H})$  such that  $y \perp i_\rho(x)$  for all  $x \in \mathcal{B}(\mathcal{H})$ , (where the orthogonality is taken with respect to the Hilbert-Schmidt inner product). Then, for all  $x \in \mathcal{B}(\mathcal{H})$ , we have

$$0 = \langle i_\rho(x), y \rangle_{\mathcal{S}_2(\mathcal{H})} = \langle \rho^{1/4} x \rho^{1/4}, y \rangle_{\mathcal{S}_2(\mathcal{H})} = \text{Tr}(\rho^{1/4} x^* \rho^{1/4} y) = \langle x, \rho^{1/4} y \rho^{1/4} \rangle_{\mathcal{S}_2(\mathcal{H})}.$$

Therefore  $\rho^{1/4} y \rho^{1/4} = 0$ . Since  $\rho^{1/4}$  is injective, we have that  $y \rho^{1/4} = 0$  and, since  $\rho^{1/4}$  has dense range, we have that  $y = 0$ . Therefore  $i_\rho$  has dense range.

To see that  $\|i_\rho : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{S}_2(\mathcal{H})\| \leq 1$ , let  $x \in \mathcal{B}(\mathcal{H})$  and notice that

$$\begin{aligned} \|i_\rho(x)\|_2 &= \sup_{\substack{y \in \mathcal{S}_2(\mathcal{H}) \\ \|y\|_2 \leq 1}} |\langle i_\rho(x), y \rangle_{\mathcal{S}_2(\mathcal{H})}| = \sup_{\substack{y \in \mathcal{S}_2(\mathcal{H}) \\ \|y\|_2 \leq 1}} |\text{Tr}(i_\rho(x)^* y)| \\ &= \sup_{\substack{y \in \mathcal{S}_2(\mathcal{H}) \\ \|y\|_2 \leq 1}} |\text{Tr}(\rho^{1/4} y \rho^{1/4} x^*)| \leq \sup_{\substack{y \in \mathcal{S}_2(\mathcal{H}) \\ \|y\|_2 \leq 1}} \|\rho^{1/4} y \rho^{1/4}\|_1 \|x\|_\infty \\ &= \|i_\rho|_{\mathcal{S}_2(\mathcal{H})} : \mathcal{S}_2(\mathcal{H}) \rightarrow \mathcal{S}_1(\mathcal{H})\| \|x\|_\infty \leq \|x\|_\infty, \end{aligned}$$

where we used part (c) for the last inequality.  $\square$

**Definition 11.4.** Let  $\mathcal{H}$  be a Hilbert space and  $\rho \in \mathcal{S}_1(\mathcal{H})$  be a positive operator. If  $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is a bounded linear operator, we define the operator  $\tilde{T} : i_\rho(\mathcal{B}(\mathcal{H})) \rightarrow i_\rho(\mathcal{B}(\mathcal{H}))$  by

$$\tilde{T}(\rho^{1/4} x \rho^{1/4}) = \rho^{1/4} T(x) \rho^{1/4} \quad \text{for all } x \in \mathcal{B}(\mathcal{H}).$$

Note that  $\tilde{T}$  depends on  $\rho$  but, for simplicity, we chose notation which does not reflect this dependence.

The following theorem was first proven in [13]. For the convenience of the reader we provide a proof of it here.

**Theorem 11.5.** *Suppose  $\mathcal{H}$  is a Hilbert space and  $\rho \in \mathcal{S}_1(\mathcal{H})$  be a positive operator such that  $\omega_\rho$  is a faithful positive functional on  $\mathcal{B}(\mathcal{H})$ . Let  $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is a bounded linear operator which is a Schwarz map such that  $\omega_\rho$  is a subinvariant*



functional for  $T$ . Then the corresponding operator  $\hat{T}$  can be extended to all of  $\mathcal{S}_2(\mathcal{H})$  as a contraction from  $\mathcal{S}_2(\mathcal{H})$  to  $\mathcal{S}_2(\mathcal{H})$ .

*Proof.* Since  $\omega_\rho$  is a faithful normal functional on  $\mathcal{B}(\mathcal{H})$ , we have that  $\mathcal{H}$  must be separable (see the comment above Proposition 11.1), so let  $(e_k)_{k \geq 0}$  be an orthonormal basis for  $\mathcal{H}$  which diagonalizes  $\rho$  and let  $P_n = \sum_{k=0}^n |e_k\rangle\langle e_k|$ . Note that  $\rho$  and its positive powers commute with each  $P_n$ . Define the linear subspace  $\mathcal{A} = \{x\rho^{1/2} : x \in \mathcal{B}(\mathcal{H})\}$  and the map  $\hat{T} : \mathcal{A} \rightarrow \mathcal{A}$  by  $\hat{T}(x\rho^{1/2}) = T(x)\rho^{1/2}$ . Further, for  $n \in \mathbb{N}$ , define the map  $\Delta_n : \mathcal{S}_2(\mathcal{H}) \rightarrow \mathcal{S}_2(\mathcal{H})$  by

$$\Delta_n(x) = P_n \rho^{1/2} x \rho^{-1/2} P_n \quad \text{for all } x \in \mathcal{S}_2(\mathcal{H})$$

(note that  $\rho^{1/2}$  is not invertible but  $\rho^{-1/2} P_n$  is a bounded operator). Then, for any  $x \in \mathcal{B}(\mathcal{H})$ , we have

$$\begin{aligned} \|\hat{T}(i_\rho(x))\|_2^2 &= \|\rho^{1/4} T(x) \rho^{1/4}\|_2^2 = \text{Tr} \left( \rho^{1/4} T(x)^* \rho^{1/2} T(x) \rho^{1/4} \right) \\ &= \lim_{n \rightarrow \infty} \text{Tr} \left( \rho^{1/2} T(x)^* P_n \rho^{1/2} T(x) \rho^{1/2} \rho^{-1/2} P_n \right) \\ &= \lim_{n \rightarrow \infty} \left\langle T(x) \rho^{1/2}, \Delta_n(T(x) \rho^{1/2}) \right\rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \lim_{n \rightarrow \infty} \left\langle \hat{T}(x \rho^{1/2}), \Delta_n \hat{T}(x \rho^{1/2}) \right\rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \lim_{n \rightarrow \infty} \left\langle x \rho^{1/2}, \hat{T}^* \Delta_n \hat{T}(x \rho^{1/2}) \right\rangle_{\mathcal{S}_2(\mathcal{H})} \end{aligned} \tag{11.2}$$

where we will see later on why  $\hat{T}^*$  is well-defined.

Define  $\Delta : \mathcal{A} \rightarrow \mathcal{A}$  by  $\Delta(x\rho^{1/2}) = \rho^{1/2} x$ , which is well-defined since  $\rho^{1/2}$  has dense range (hence, for  $x, y \in \mathcal{B}(\mathcal{H})$ ,  $x\rho^{1/2} = y\rho^{1/2}$  implies  $x = y$ ). Let  $\mathcal{B} = \{x\rho : x \in \mathcal{B}(\mathcal{H})\}$ . We make the following three claims:

- (i)  $\hat{T}$  is a contraction on  $\mathcal{A}$ . Therefore  $\hat{T}$  can be extended to a contraction on  $\mathcal{S}_2(\mathcal{H})$  since  $\mathcal{A}$  is dense in  $\mathcal{S}_2(\mathcal{H})$ .
- (ii)  $\Delta_n^2$  is positive. Therefore, by [56, Lemma 1.2], we have

$$\hat{T}^* \Delta_n \hat{T} \leq \left( \hat{T}^* \Delta_n^2 \hat{T} \right)^{1/2}. \tag{11.3}$$

(iii)  $\widehat{T}^* \Delta_n^2 \widehat{T} \leq \Delta^2$  on  $\mathcal{B}$ . Thus,

$$\left(\widehat{T}^* \Delta_n^2 \widehat{T}\right)^{1/2} \leq (\Delta^2)^{1/2} = \Delta. \quad (11.4)$$

Hence, by combining (11.3) and (11.4), we obtain  $\widehat{T}^* \Delta_n \widehat{T} \leq \Delta$  on  $\mathcal{B}$ .

Assume for the moment that the above claims (i), (ii), and (iii) are true in order to finish the proof and we will prove the claims afterwards. By replacing  $x$  by  $x\rho^{1/2}$ , in Equation (11.2) we obtain that

$$\begin{aligned} \|\widetilde{T}(i_\rho(x\rho^{1/2}))\|_2^2 &= \lim_{n \rightarrow \infty} \langle x\rho, \widehat{T}^* \Delta_n \widehat{T}(x\rho) \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &\leq \langle x\rho, \Delta(x\rho) \rangle_{\mathcal{S}_2(\mathcal{H})} = \langle x\rho, \rho^{1/2} x\rho^{1/2} \rangle_{\mathcal{S}_2(\mathcal{H})} \\ &= \text{Tr} \left( \rho x^* \rho^{1/2} x\rho^{1/2} \right) = \text{Tr} \left( \rho^{3/4} x^* \rho^{1/4} \rho^{1/4} x\rho^{3/4} \right) \\ &= \langle \rho^{1/4} x\rho^{3/4}, \rho^{1/4} x\rho^{3/4} \rangle_{\mathcal{S}_2(\mathcal{H})} = \|i_\rho(x\rho^{1/2})\|_2^2 \end{aligned}$$

and so  $\widetilde{T}$  is a contraction on  $i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$ . We now show that  $i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$  is dense in  $\mathcal{S}_2(\mathcal{H})$ . Let  $y \in \mathcal{S}_2(\mathcal{H})$  such that  $y \perp i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$ . Then, for any  $x \in \mathcal{B}(\mathcal{H})$  we have that

$$0 = \langle i_\rho(x\rho^{1/2}), y \rangle_{\mathcal{S}_2(\mathcal{H})} = \text{Tr}(i_\rho(x\rho^{1/2})^* y) = \text{Tr}(\rho^{1/4} \rho^{1/2} x^* \rho^{1/4} y) = \langle x, \rho^{1/4} y \rho^{3/4} \rangle_{\mathcal{S}_2(\mathcal{H})}$$

and hence  $\rho^{1/4} y \rho^{3/4} = 0$ . Since  $\rho^{1/4}$  is injective, we then have that  $y \rho^{3/4} = 0$  and, since  $\rho^{3/4}$  has dense range, we obtain that  $y = 0$ . Therefore  $i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$  is dense in  $\mathcal{S}_2(\mathcal{H})$ . Since  $\widetilde{T}$  is a contraction on  $i_\rho(\mathcal{B}(\mathcal{H})\rho^{1/2})$ , we can extend it to a contraction on  $\mathcal{S}_2(\mathcal{H})$ . This finishes the proof of the theorem as long as we verify above claims (i), (ii), and (iii) as well as the fact that  $\widehat{T}^*$  is well-defined which was used above.

First, we need to prove claim (i), i.e., that  $\widehat{T}$  is a contraction on  $\mathcal{A}$ . Let  $x \in \mathcal{B}(\mathcal{H})$ . Then

$$\begin{aligned} \|\widehat{T}(x\rho^{1/2})\|_2^2 &= \|T(x)\rho^{1/2}\|_2^2 = \langle T(x)\rho^{1/2}, T(x)\rho^{1/2} \rangle_{\mathcal{S}_2(\mathcal{H})} = \text{Tr}(\rho^{1/2} T(x)^* T(x) \rho^{1/2}) \\ &\leq \text{Tr}(\rho^{1/2} T(x^* x) \rho^{1/2}) \end{aligned}$$

since  $T$  is a Schwarz map. Further

$$\begin{aligned}\mathrm{Tr}(\rho^{1/2}T(x^*x)\rho^{1/2}) &= \mathrm{Tr}(\rho T(x^*x)) \leq \mathrm{Tr}(\rho x^*x) = \mathrm{Tr}(\rho^{1/2}x^*x\rho^{1/2}) \\ &= \langle x\rho^{1/2}, x\rho^{1/2} \rangle_{\mathcal{S}_2(\mathcal{H})} = \|x\rho^{1/2}\|_2^2.\end{aligned}$$

Therefore  $\|\widehat{T}(x\rho^{1/2})\|_2^2 \leq \|x\rho^{1/2}\|_2^2$  and so  $\widehat{T}$  is a contraction on  $\mathcal{A}$ . Hence,  $\widehat{T}$  can be extended to a contraction on  $\mathcal{S}_2(\mathcal{H})$ , since  $\mathcal{A}$  is dense in  $\mathcal{S}_2(\mathcal{H})$  (which also shows that  $\widehat{T}^*$  is well-defined).

For claim (ii), i.e., that  $\Delta_n^2$  is positive, first note that since  $\rho$  commutes with  $P_n$  we have that

$$\Delta_n^2 x = P_n \rho x P_n \rho^{-1} P_n \quad \text{for all } x \in \mathcal{S}_2(\mathcal{H})$$

(note that  $\rho$  is not invertible but  $\rho^{-1}P_n$  is a bounded operator). Indeed, if  $x \in \mathcal{S}_2(\mathcal{H})$  then

$$\begin{aligned}\langle x, \Delta_n^2 x \rangle_{\mathcal{S}_2(\mathcal{H})} &= \langle x, P_n \rho x P_n \rho^{-1} P_n \rangle_{\mathcal{S}_2(\mathcal{H})} = \mathrm{Tr} \left( x^* P_n \rho x P_n \rho^{-1} P_n \right) \\ &= \mathrm{Tr} \left( \rho^{1/2} P_n x P_n \rho^{-1/2} P_n P_n \rho^{-1/2} P_n x^* P_n \rho^{1/2} \right) \\ &= \mathrm{Tr} \left( (\rho^{1/2} P_n x P_n \rho^{-1/2} P_n) (\rho^{1/2} P_n x P_n \rho^{-1/2} P_n)^* \right) \geq 0\end{aligned}$$

and so  $\Delta_n^2$  is positive. Then by [56, Lemma 1.2], we have that

$$\widehat{T}^* \Delta_n \widehat{T} \leq (\widehat{T}^* \Delta_n^2 \widehat{T})^{1/2}. \tag{11.5}$$

It is left to prove claim (iii), i.e., that  $\widehat{T}^* \Delta_n^2 \widehat{T} \leq \Delta^2$  on  $\mathcal{B}$ . Indeed,

$$\begin{aligned}
\langle x\rho, \widehat{T}^* \Delta_n^2 \widehat{T}(x\rho) \rangle_{\mathcal{S}_2(\mathcal{H})} &= \langle T(x\rho^{1/2})\rho^{1/2}, \Delta_n^2 T(x\rho^{1/2})\rho^{1/2} \rangle_{\mathcal{S}_2(\mathcal{H})} \\
&= \langle T(x\rho^{1/2})\rho^{1/2}, P_n \rho T(x\rho^{1/2})\rho^{1/2} P_n \rho^{-1} P_n \rangle_{\mathcal{S}_2(\mathcal{H})} \\
&= \text{Tr} \left( \rho^{1/2} T(x\rho^{1/2})^* P_n \rho T(x\rho^{1/2}) \rho^{1/2} P_n \rho^{-1} P_n \right) \\
&= \text{Tr} \left( \rho T(x\rho^{1/2}) P_n T(x\rho^{1/2})^* P_n \right) \\
&\leq \text{Tr} \left( \rho T(x\rho^{1/2}) T(x\rho^{1/2})^* \right) \quad (\text{see below}) \tag{11.6} \\
&\leq \text{Tr} \left( \rho T \left( (x\rho^{1/2})(x\rho^{1/2})^* \right) \right) \quad (\text{since } T \text{ is a Schwarz map}) \\
&\leq \text{Tr} \left( \rho (x\rho^{1/2})(x\rho^{1/2})^* \right) \quad (\text{since } \omega_\rho \text{ is subinvariant for } T) \\
&= \text{Tr} (\rho x \rho x^*) = \text{Tr} (\rho x (x\rho)^*) = \text{Tr} ((x\rho)^* \rho x) \tag{11.7} \\
&= \text{Tr} \left( (x\rho)^* \Delta^2 (x\rho) \right) \quad (\text{since } \Delta^2 (x\rho) = \rho x) \tag{11.8} \\
&= \langle x\rho, \Delta^2 (x\rho) \rangle_{\mathcal{S}_2(\mathcal{H})}.
\end{aligned}$$

This completes the proof as long as we justify the inequality (11.6). In general we have that for any  $A \in \mathcal{S}_2(\mathcal{H})$ , the inequality  $\text{tr}(P_n A^* P_n A) \leq \text{tr}(A^* A)$  holds. Indeed, if  $(e_k)_{k \geq 1}$  is the orthonormal basis of  $\mathcal{H}$  used to define each  $P_n$ , then

$$\text{Tr}(P_n A^* P_n A) = \sum_{k=1}^{\infty} \langle e_k, P_n A^* P_n^2 A e_k \rangle = \sum_{k=1}^{\infty} \langle P_n A P_n e_k, P_n A e_k \rangle = \sum_{k=1}^n \langle P_n A e_k, P_n A e_k \rangle.$$

Further,

$$\sum_{k=1}^n \langle P_n A e_k, P_n A e_k \rangle = \sum_{k=1}^n \|P_n A e_k\|^2 \leq \sum_{k=1}^n \|P_n\|^2 \|A e_k\|^2 \leq \sum_{k=1}^{\infty} \|A e_k\|^2 = \text{Tr}(A^* A)$$

and so the proof of the inequality is complete.  $\square$

## 11.2 AN ALTERNATE CONSTRUCTION

There is another situation where a bounded operator on a  $C^*$ -algebra gives rise to a corresponding operator on a Hilbert space, and we would like to mention this in the next remark.

**Remark 11.6.** Let  $\mathcal{A}$  be a unital  $C^*$ -algebra and  $\omega$  be a faithful state on  $\mathcal{A}$ . Consider the GNS construction of  $\mathcal{A}$  associated with  $\omega$ . Let  $\mathcal{K}$  be the Hilbert space associated with the GNS construction,  $\pi : \mathcal{A} \rightarrow \mathcal{B}(\mathcal{K})$  be the  $*$ -representation of  $\mathcal{A}$  into the  $C^*$ -algebra of all bounded operators on  $\mathcal{K}$ , and  $\Omega$  denote the cyclic element of the Hilbert space  $\mathcal{K}$  for the representation  $\pi$ , (i.e. the subspace  $\{\pi(a)(\Omega) : a \in \mathcal{A}\}$  is norm dense in  $\mathcal{K}$ ) which is equal to the unit of  $\mathcal{A}$  viewed as an element of  $\mathcal{K}$ . Let  $T$  be a bounded operator on  $\mathcal{A}$  which is a Schwarz map. Assume that  $\omega$  is subinvariant for  $T$ . Define an operator  $\bar{T}$  on the dense subspace  $\{\pi(a)(\Omega) : a \in \mathcal{A}\}$  of  $\mathcal{K}$  with values in  $\mathcal{K}$ , by

$$\bar{T}(\pi(a)(\Omega)) = \pi(T(a))(\Omega) \quad \text{for all } a \in \mathcal{A}.$$

Then  $\bar{T}$  is a contraction (hence it extends to  $\mathcal{K}$ ).

*Proof.* Since  $\omega$  is faithful, the quotient that is usually associated with the GNS construction does not take place, and the elements of  $\mathcal{A}$  belong to  $\mathcal{K}$ . Let  $\langle \cdot, \cdot \rangle_\omega$  denote the inner product in  $\mathcal{K}$  and  $\| \cdot \|_\omega$  denote the norm of  $\mathcal{K}$ . Then since  $\omega$  is faithful, we have that for  $a, b \in \mathcal{A}$ ,  $\langle a, b \rangle_\omega = \omega(a^*b)$  and hence  $\|(\pi(a))(\Omega)\|_\omega^2 = \omega(a^*a)$ .

For every  $a \in \mathcal{A}$  we have

$$\begin{aligned} \|\bar{T}(\pi(a)(\Omega))\|_\omega^2 &= \|\pi(T(a))(\Omega)\|_\omega^2 = \omega(T(a)^*T(a)) \leq \omega(T(a^*a)) \\ &\quad (\text{since } \omega \text{ is positive and } T \text{ is a Schwarz map}) \\ &\leq \omega(a^*a) \quad (T \geq 0 \text{ is a Schwarz map and } \omega \text{ is subinvariant for } T) \\ &= \|\pi(a)(\Omega)\|_\omega^2, \end{aligned}$$

which finishes the proof. □

Notice the similarities between Theorem 11.5 and Remark 11.6. Both refer to a bounded operator on some  $C^*$ -algebra where a positive linear functional is fixed, and they conclude the existence of an associated contraction on some Hilbert space. There are three differences between Theorem 11.5 and Remark 11.6. First, Theorem 11.5

refers to an operator on  $\mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$  (which is necessarily separable since  $\mathcal{B}(\mathcal{H})$  is assumed to admit a faithful normal state), while Remark 11.6 assumes that the operator is defined on a general  $C^*$ -algebra. Second, the state  $\omega_\rho$  which is mentioned in Theorem 11.5 is normal since it is defined via the trace-class operator  $\rho$ , while there is no such assumption in Remark 11.6 (the normality of the state  $\omega$  in Remark 11.6 does not make sense in general since  $\mathcal{A}$  is simply assumed to be a  $C^*$ -algebra and not a von Neumann algebra as it is assumed in [51, Equation (2.1)]). Third, the Hilbert space that is used in Theorem 11.5 is the space  $\mathcal{S}_2(\mathcal{H})$  which does not depend on the positive linear functional, while the map  $i_\rho$  which maps  $\mathcal{B}(\mathcal{H})$  to  $\mathcal{S}_2(\mathcal{H})$ , does depend on the positive linear functional. On the other hand, the Hilbert space that is used in Remark 11.6 (i.e. the GNS construction associated to the faithful state  $\omega$  of the  $C^*$ -algebra  $\mathcal{A}$ ), depends on the state, while the  $*$ -representation  $\pi$  of the von Neumann algebra which is associated with the GNS construction does not depend on the state. Notice that the combinations of the Hilbert spaces with the representations in Theorem 11.5 and Remark 11.6 are very similar. More precisely, for  $a, b \in \mathcal{B}(\mathcal{H})$  we have that  $i_\rho(a), i_\rho(b) \in \mathcal{S}_2(\mathcal{H})$  hence

$$\langle i_\rho(a), i_\rho(b) \rangle_{\mathcal{S}_2(\mathcal{H})} = \text{Tr}(i_\rho(a)^* i_\rho(b)) = \text{Tr}(\rho^{1/4} a^* \rho^{1/4} \rho^{1/4} b \rho^{1/4}) = \text{Tr}(a^* \rho^{1/2} b \rho^{1/2}).$$

On the other hand, if we assume for the moment that the  $C^*$ -algebra  $\mathcal{A}$  that appears in Remark 11.6 is equal to  $\mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ , and the faithful state  $\omega$  on the  $C^*$ -algebra  $\mathcal{A}$  is given by  $\omega(a) = \text{Tr}(\rho a)$  for some positive trace-class operator  $\rho$  on  $\mathcal{H}$ , then the inner product of two elements  $a, b \in \mathcal{A}$  via the GNS construction is given by

$$\langle a, b \rangle_\omega = \omega(a^* b) = \text{Tr}(\rho a^* b).$$

Thus the combination of the inner product with the representation that is used in Theorem 11.5 is slightly more "symmetric" than the combination of the inner product with the representation that is used in Remark 11.6. The reader of course will notice

the difference between the complexity of the proof of Theorem 11.5 and that of Remark 11.6. The extra intricacies in the proof of Theorem 11.5 is the price we pay in order to achieve the extra symmetry in the combination of the inner product and the representation as discussed above.

**Remark 11.7.** The assumption that “ $\omega$  is subinvariant for  $T$ ” cannot be omitted in Remark 11.6.

*Proof.* We present an example where all the assumptions of Remark 11.6 are valid, except  $\omega$  is not a subinvariant functional for  $T$ . In this example, the operator  $\bar{T}$  is not bounded from  $\{\pi(a)(\Omega) : a \in \mathcal{A}\}$  to  $\mathcal{K}$ . This shows that the assumption that “ $\omega$  is subinvariant for  $T$ ” cannot be omitted. Consider the Hilbert space  $\mathcal{H} = \ell_2(\mathbb{N}) \otimes L_2[0, 1]$  (with the Lebesgue measure  $d\lambda$  used on  $[0, 1]$ ). The elements of  $\mathcal{H}$  can be represented as column vectors  $(f_1, f_2, \dots)^t$  where  $f_n \in L_2[0, 1]$  for every  $n \in \mathbb{N}$  and

$$\|(f_1, f_2, \dots)^t\| = \left( \sum_{n=1}^{\infty} \int_0^1 |f_n|^2 d\lambda \right)^{1/2} < \infty.$$

Consider the von Neumann subalgebra  $\mathcal{A} = \mathcal{B}(\ell_2(\mathbb{N})) \otimes L_{\infty}[0, 1]$  of  $\mathcal{B}(\mathcal{H})$  where we view the von Neumann algebra  $L_{\infty}[0, 1]$  as multiplication operators on  $L_2[0, 1]$ . A generic element of  $\mathcal{A}$  can be written as an infinite by infinite matrix  $x = (x_{i,j})_{i,j \in \mathbb{N}}$  where  $x_{i,j} \in L_{\infty}[0, 1]$  for every  $i, j \in \mathbb{N}$  and in order to make sure that  $x$  represents a bounded operator on  $\mathcal{H}$ , we assume that for every  $(f_1, f_2, \dots)^t \in \mathcal{H}$  we have that  $(\sum_{j=1}^{\infty} x_{1,j} f_j, \sum_{j=1}^{\infty} x_{2,j} f_j, \dots)^t \in \mathcal{H}$ , where for every  $i \in \mathbb{N}$ , the infinite series  $\sum_{j=1}^{\infty} x_{i,j} f_j$  converges  $\lambda$ -almost everywhere on  $[0, 1]$ , (the boundedness of the operator  $x$  follows from the Uniform Boundedness Principle applied to the sequence of bounded linear operators on  $\mathcal{H}$  indexed by  $n \in \mathbb{N}$  and represented by the infinite by infinite matrices whose first  $n$  columns agree with the first  $n$  columns of  $x$  and the rest of their columns are equal to zero). Let  $\omega$  be a state on  $\mathcal{A}$  defined by

$$\omega(x) = \sum_{m=1}^{\infty} \frac{1}{2^m} \int_0^1 x_{m,m} d\lambda \quad \text{for } x = (x_{i,j})_{i,j \in \mathbb{N}} \in \mathcal{A}.$$

Obviously  $\omega$  is a faithful state on  $\mathcal{A}$ . If for every  $j \in \mathbb{N}$  vectors  $b_j = (b_{1,j}, b_{2,j}, \dots)^t \in \mathcal{H}$  are chosen so that the sequence  $(b_j)_{j \in \mathbb{N}}$  is an orthonormal sequence in  $\mathcal{H}$ , then the infinite by infinite matrix  $B = (b_{i,j})_{i,j \in \mathbb{N}}$  represents an isometry on  $\mathcal{H}$ . In particular, for  $i \in \mathbb{N}$  let

$$b_{i,1} = \frac{1}{2^{i/2}} \chi_{[0,1]}, \quad b_{i,2} = \frac{1}{2^{i/2}} (\chi_{[0, \frac{1}{2})} - \chi_{[\frac{1}{2}, 1]}),$$

$$b_{i,3} = \frac{1}{2^{i/2}} (\chi_{[0, \frac{1}{2^2})} - \chi_{[\frac{1}{2^2}, \frac{1}{2})} + \chi_{[\frac{1}{2}, \frac{3}{2^2})} - \chi_{[\frac{3}{2^2}, 1]}), \dots$$

(where  $\chi_A$  denotes the characteristic function of  $A \subseteq [0, 1]$ ). Then the column vectors  $b_j = (b_{i,j})_{i \in \mathbb{N}} \in \mathcal{H}$  form an orthonormal sequence in  $\mathcal{H}$  and hence  $B = (b_{i,j})_{i,j \in \mathbb{N}}$  represents an isometry on  $\mathcal{H}$ . We have that the adjoint operator  $B^*$  is represented by the matrix  $(b_{j,i})_{j,i \in \mathbb{N}}$ . Define  $T : \mathcal{A} \rightarrow \mathcal{A}$  by  $T(x) = Bx B^*$ . Then  $T$  is written in Kraus representation, so it is a completely positive map hence a bounded Schwarz map. We claim that the map  $\bar{T}$  is not bounded on  $\{\pi(a)(\Omega) : a \in \mathcal{A}\}$ . Indeed, let the sequence  $a_n \in \mathcal{A}$  for  $n \in \mathbb{N}$ , such that each  $a_n$  is represented with the infinite by infinite matrix  $a_n = (a_{n,i,j})_{i,j \in \mathbb{N}}$ , where  $a_{n,n,n} = \chi_{[0,1]}$  and  $a_{n,i,j} = 0$  if  $(i,j) \neq (n,n)$ . Then  $a_n^* = a_n$  and  $a_n^* a_n = a_n a_n^* = a_n$ , hence

$$\|a_n\|_\omega^2 = \omega(a_n^* a_n) = \omega(a_n) = \frac{1}{2^n} \rightarrow 0 \quad \text{as } n \rightarrow \infty,$$

but on the other hand,

$$T(a_n) = B a_n B^* = B a_n a_n^* B^* = (B a_n)(B a_n)^* = |b_n\rangle\langle b_n|$$

since  $B a_n$  is the infinite by infinite matrix whose  $n$ th column is equal to  $b_n$  (i.e. the  $n$ th column of  $B$ ) and the other columns are equal to zero. Thus

$$(T(a_n))^* T(a_n) = |b_n\rangle\langle b_n| |b_n\rangle\langle b_n| = \langle b_n, b_n \rangle |b_n\rangle\langle b_n| = |b_n\rangle\langle b_n|.$$

Note that for  $m \in \mathbb{N}$ , the  $m$ th entry of  $|b_n\rangle\langle b_n|$  is equal to

$$\left( \frac{1}{2^{m/2}} \left( \chi_{[0, \frac{1}{2^{m-1}})} - \chi_{[\frac{1}{2^{m-1}}, \frac{2}{2^{m-1}})} + \dots - \chi_{[\frac{2^{m-1}-1}{2^{m-1}}, 1]} \right) \right)^2 = \frac{1}{2^m} \chi_{[0,1]}.$$



Thus

$$\|T(a_n)\|_\omega^2 = \omega((T(a_n))^*T(a_n)) = \sum_{m=1}^{\infty} \frac{1}{2^m} \int_0^1 \frac{1}{2^m} \chi_{[0,1]} d\lambda = \sum_{m=1}^{\infty} \frac{1}{2^{2m}} = \frac{1}{3}$$

which is positive and independent of  $n$ , showing that  $\overline{T}$  is not bounded.  $\square$

**Remark 11.8.** Note that if  $\mathcal{H}$  is a Hilbert space,  $T : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is a bounded positive linear operator and  $\omega$  is a subinvariant positive faithful functional for  $T$ , then  $\omega/\omega(1)$  is a subinvariant faithful state for  $T$  (here 1 denotes the identity operator on  $\mathcal{H}$ ). Thus from now on, instead of assuming the existence of subinvariant positive faithful functionals, we simply assume the existence of subinvariant faithful states. Hence our subsequent results remain valid if the assumptions of the existence of subinvariant faithful states are replaced by the assumptions of the existence of subinvariant positive faithful functionals.

# CHAPTER 12

## SEMIGROUPS OF SCHWARZ MAPS

We first recall some basic definitions about semigroups.

**Definition 12.1.** Let  $X$  be a Banach space. A one-parameter family  $(T_t)_{t \geq 0}$  of bounded operators on  $X$  is a **semigroup** on  $X$  if  $T_{t+s} = T_t T_s$  for all  $t, s \geq 0$ , and  $T_0 = I$  where  $I$  is the identity operator on  $X$ . We say the semigroup  $(T_t)_{t \geq 0}$  on a Banach space  $X$  is

- **uniformly continuous** if the map  $t \mapsto T_t$  is continuous with respect to the operator norm.
- **strongly continuous** provided that, for all  $x \in X$ , the map  $t \mapsto T_t x$  is continuous with respect to the norm on  $X$ .
- **weakly continuous** if, for all  $x \in X$  and all  $x^* \in X^*$ , we have that the map  $t \mapsto x^*(T_t x)$  is continuous.
- **weak\* continuous** if  $X$  is a dual Banach space  $X = Y^*$  and for all  $y \in Y$  and  $x \in X$  we have that the map  $t \mapsto (T_t(x))(y)$  is continuous.

If  $\mathcal{H}$  is a Hilbert space and  $X = \mathcal{B}(\mathcal{H})$  then the semigroup  $(T_t)_{t \geq 0}$  on the Banach space  $X$  is **WOT continuous** (where this acronym stands as usually for the weak operator topology) if for all  $h_1, h_2 \in \mathcal{H}$  and  $x \in \mathcal{B}(\mathcal{H})$  we have that the map  $t \mapsto \langle h_1, (T_t(x))(h_2) \rangle$  is continuous.

It can be shown that a semigroup on a Banach space is strongly continuous if and only if it is weakly continuous (see [8, Thm. 3.31]). Next, we would like to define the

generator of a semigroup. If  $(T_t)_{t \geq 0}$  is a uniformly continuous semigroup on a Banach space  $X$  then its **generator** is defined as the operator norm limit

$$L = \lim_{t \rightarrow 0} \frac{T_t - I}{t}.$$

This limit exists and it defines a bounded operator on  $X$ . If we do not assume the uniform continuity of the semigroup, then the definition of the generator is given next:

**Definition 12.2.** Let  $(T_t)_{t \geq 0}$  be a strongly continuous semigroup (resp. weakly continuous, resp. weak\* continuous), on a Banach space  $X$  (of course, when we assume that the semigroup is weak\* continuous we assume that  $X$  is a dual Banach space). We say an element  $x \in X$  belongs to the **domain**  $D(L)$  of the generator  $L$  of  $(T_t)_{t \geq 0}$ , if

$$\lim_{t \rightarrow 0} \frac{T_t(x) - x}{t} \quad (12.1)$$

converges in norm (resp. weakly, resp. weak\*) and, in this case, define the **generator** to be the generally unbounded operator  $L$  such that

$$L(x) = \lim_{t \rightarrow 0} \frac{T_t(x) - x}{t} \quad \text{for all } x \in D(L) \quad (12.2)$$

where the last limit is taken in the norm (resp. weak, resp. weak\*) topology of  $X$ .

Since a semigroup on a Banach space is strongly continuous if and only if it is weakly continuous, it is natural to ask whether the limits (12.1) and (12.2) can be replaced by weak limits and end up with the same  $D(L)$  and  $L$ . It turns out that this is indeed the case (see [8, Proposition 3.36]). We will make use of this fact in the proof of Theorem 12.14.

## 12.1 THE EXTENDED GENERATOR $L_{(h_n)}$ OF $(T_t)_{t \geq 0}$

We now wish to extend the definition of the generator to include some cases where the limit (12.2) does not exist.

**Definition 12.3.** Let  $\mathcal{H}$  be a Hilbert space and  $(h_n)_{n \in N}$  be a (countable or uncountable) orthonormal basis of  $\mathcal{H}$ . We let  $M_\infty^{(h_n)}$  denote the set of all complex  $\infty \times \infty$  matrices with rows and columns indexed by  $N$ . We view a matrix  $L \in M_\infty^{(h_n)}$  as a linear map  $L : D(L) \rightarrow \mathbb{C}^N$  acting on  $\mathcal{H}$  as follows: denote  $L = (L_{n,m})_{n,m \in N}$ , and define  $D(L) \subset \mathcal{H}$  as the set of all vectors  $h = \sum_{m \in N} \langle h_m, h \rangle h_m \in \mathcal{H}$  such that the series  $\sum_{m \in N} L_{n,m} \langle h_m, h \rangle$  converges for all  $n \in N$ . Then

$$L(h) = \left( \sum_{m \in N} L_{n,m} \langle h_m, h \rangle \right)_{n \in N}.$$

This is in particular the natural of matrix multiplication of  $L$  against  $h$  written as a column vector.

**Definition 12.4.** Let  $\mathcal{H}$  be a Hilbert space and  $(h_n)_{n \in N}$  be a (countable or uncountable) orthonormal basis of  $\mathcal{H}$ . Let  $(T_t)_{t \geq 0}$  be a semigroup of bounded operators on  $\mathcal{B}(\mathcal{H})$ . We will define the extended generator  $L_{(h_n)}$  of  $(T_t)_{t \geq 0}$  with respect to the basis  $(h_n)_{n \in N}$ , but first we must define its domain as the linear subspace of all  $x \in \mathcal{B}(\mathcal{H})$  such that the function

$$[0, \infty) \ni t \mapsto \langle h_n, T_t(x) h_m \rangle$$

is differentiable at 0 for every  $n, m \in N$ ; that is,  $D(L_{(h_n)})$  is the linear subspace of all  $x \in \mathcal{B}(\mathcal{H})$  such that the limit

$$\lim_{t \rightarrow 0} \langle h_n, \frac{T_t(x) - x}{t} h_m \rangle$$

exists for every  $n, m \in N$ . In general  $D(L_{(h_n)})$  can be the zero subspace, but if the semigroup is WOT continuous then  $D(L_{(h_n)})$  is WOT dense in  $\mathcal{B}(\mathcal{H})$ . Define the **extended generator  $L_{(h_n)}$  of  $(T_t)_{t \geq 0}$  (with respect to the orthonormal basis  $(h_n)_{n \in N}$ )** to be the map with domain  $D(L_{(h_n)})$  whose range elements are matrices  $L_{(h_i)}(x) \in M_\infty^{(h_n)}$  with entries given by

$$[L_{(h_i)}(x)]_{n,m} = \lim_{t \rightarrow 0} \langle h_n, \frac{T_t(x) - x}{t} h_m \rangle.$$

Next we want to compare the generator of a semigroup on  $\mathcal{B}(\mathcal{H})$  with respect to an orthonormal basis of  $\mathcal{H}$  to the usual generator. The following notation, which is commonly used in dilation theory, will be helpful for that purpose.

**Notation 12.5.** Let  $\mathcal{H}$  and  $\mathcal{K}$  be Hilbert spaces with  $\mathcal{H} \subseteq \mathcal{K}$  and let  $A \in \mathcal{B}(\mathcal{H})$  and  $B \in \mathcal{B}(\mathcal{K})$ . We shall denote by

$$A = \text{pr}_{\mathcal{H}}(B)$$

the fact that

$$A = P_{\mathcal{H}}B|_{\mathcal{H}}$$

where  $|_{\mathcal{H}}$  denotes the restriction to  $\mathcal{H}$  and  $P_{\mathcal{H}} : \mathcal{K} \rightarrow \mathcal{H}$  denotes the orthogonal projection from  $\mathcal{K}$  onto  $\mathcal{H}$ . The operator  $B$  is called a **dilation** of the operator  $A$  and the operator  $A$  is called **compression** of the operator  $B$ .

**Notation 12.6.** If  $N$  is a non empty set, then we denote by  $\mathcal{P}_{\text{fin}}(N)$  the set of all finite subsets of  $N$ .

Since the definition of the generator depends on the continuity of the semigroup, in the next remark we will consider a weak\* continuous semigroup on  $\mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ . The reason that we choose the weak\* continuity versus any other continuity assumption is because it is the weakest among all continuity assumptions that appear in Definition 12.1.

**Remark 12.7.** Let  $\mathcal{H}$  be a Hilbert space,  $(T_t)_{t \geq 0}$  be a weak\* continuous semigroup of bounded operators on  $\mathcal{B}(\mathcal{H})$ , and let  $L$  denote its generator. Let  $(h_n)_{n \in N}$  be a (countable or uncountable) orthonormal basis of  $\mathcal{H}$  and let  $L_{(h_n)}$  denote the generator of  $(T_t)_{t \geq 0}$  with respect to  $(h_n)_{n \in N}$ . Then  $D(L) \subseteq D(L_{(h_n)})$ , and for every  $x \in D(L)$  and any  $F \in \mathcal{P}_{\text{fin}}(N)$  we have  $L_{(h_n)}(x)_F = \text{pr}_{\text{Span}(h_n)_{n \in F}}(L(x))$ .

Indeed, for fixed  $x \in D(L)$  and every  $h, h' \in \mathcal{H}$  we have that

$$\langle h, \frac{T_t(x) - x}{t} h' \rangle \rightarrow \langle h, L(x) h' \rangle \quad \text{as } t \rightarrow 0. \quad (12.3)$$

In particular, the limit

$$\lim_{t \rightarrow 0} \langle h_n, \frac{T_t(x) - x}{t} h_m \rangle$$

exists for every  $n, m \in N$ . Thus  $x \in D(L_{(h_n)})$ . Now fix  $F \in \mathcal{P}_{\text{fin}}(N)$ . From Definition 12.4,  $L_{(h_n)}(x)_F$  is the operator  $L_{(h_n)}(x)_F : \text{Span}(x_n)_{n \in F} \rightarrow \text{Span}(x_n)_{n \in F}$  defined by

$$[L_{(h_n)}(x)_F](h) = \sum_{n, m \in F} \lim_{t \rightarrow 0} \langle h_n, \frac{T_t(x) - x}{t} h_m \rangle \langle h_m, h \rangle h_n \quad \text{if } h = \sum_{m \in F} \langle h_m, h \rangle h_m \quad (12.4)$$

**Remark 12.8.** Let  $\mathcal{H}$  be a Hilbert space with (countable or uncountable) dimension  $N$ ,  $(h_n)_{n \in N}$  be an orthonormal basis of  $\mathcal{H}$ ,  $(T_t)_{t \geq 0}$  be a semigroup of bounded operators on  $\mathcal{B}(\mathcal{H})$ , and let  $L_{(h_n)}$  denote its generator with respect to  $(h_n)_{n \in N}$ . For  $x \in D(L_{(h_n)})$  and  $F \in \mathcal{P}_{\text{fin}}(N)$  we have that  $L_{(h_n)}(x)_F$  is the unique operator

$$L_{(h_n)}(x)_F : \text{Span}(h_n)_{n \in F} \rightarrow \text{Span}(h_n)_{n \in F}$$

satisfying

$$\lim_{t \rightarrow 0} \langle h, \frac{T_t(x) - x}{t} h' \rangle = \langle h, L_{(h_n)}(x)_F h' \rangle \quad \text{for all } h, h' \in \text{Span}(h_n)_{n \in F}. \quad (12.5)$$

or equivalently

$$\left\| \text{pr}_{\text{Span}(h_n)_{n \in F}} \left( \frac{T_t(x) - x}{t} \right) - L_{(h_n)}(x)_F \right\|_{\mathcal{B}(\text{Span}(h_n)_{n \in F})} \rightarrow 0 \quad \text{as } t \rightarrow 0. \quad (12.6)$$

Indeed, (12.5) is obvious from Definition 12.4 and (12.4). The equivalence of (12.5) and (12.6) this follows since for any finite subset  $F$  of  $N$ , all linear Hausdorff topologies on the space of linear operators on  $\text{Span}(h_n)_{n \in F}$  are equivalent. Thus the WOT on  $\text{Span}(h_n)_{n \in F}$  in (12.5) can be replaced by the  $\mathcal{B}(\text{Span}(h_n)_{n \in F})$  topology.

**Remark 12.9.** Let  $\mathcal{H}$  be a Hilbert space with (countable or uncountable) dimension  $N$ ,  $(h_n)_{n \in N}$  be an orthonormal basis of  $\mathcal{H}$ ,  $(T_t)_{t \geq 0}$  be a semigroup of bounded operators on  $\mathcal{B}(\mathcal{H})$ , and let  $L_{(h_n)_n}$  denote its generator with respect to  $(h_n)_{n \in N}$ . Fix

$x \in D(L_{(h_n)})$ . Then the family  $(L_{(h_n)}(x)_F)_{F \in \mathcal{P}_{\text{fin}}(N)}$  is compatible in the following sense: If  $G \subset F$  are two finite subsets of  $N$  then  $\text{pr}_{\text{Span}(h_n)_{n \in G}}(L_{(h_n)}(x)_F) = L_{(h_n)}(x)_G$ .

Indeed, this is obvious from (12.4).

**Remark 12.10.** The generator of a semigroup with respect to an orthonormal basis that we defined above is related to the **form generator** which was defined by Davies [24] and was further studied in [45, 19, 63, 46, 18, 33, 7, 61]. If  $(T_t)_{t \geq 0}$  is a weak\* continuous semigroup on  $\mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ , then a form generator is the map  $\phi : \mathcal{K} \times \mathcal{B}(\mathcal{H}) \times \mathcal{K} \rightarrow \mathbb{C}$  where  $\mathcal{K}$  is a dense linear subspace of  $\mathcal{H}$ , defined by

$$\phi(h, x, h') = \langle h, \lim_{t \rightarrow 0} \frac{T_t(x) - x}{t} h' \rangle \quad \text{for every } h, h' \in \mathcal{K} \text{ and every } x \in \mathcal{B}(\mathcal{H}).$$

Note that if  $(h_n)_{n \in N}$  is an orthonormal basis of  $\mathcal{H}$  and  $\mathcal{K}$  denotes the linear span of  $(h_n)_{n \in N}$  then the form generator coincides with the generator with respect to  $(h_n)_{n \in N}$  if the domain of the generator with respect to  $(h_n)_{n \in N}$  is equal to  $\mathcal{B}(\mathcal{H})$ . Here we assume that the domain of the generator with respect to an orthonormal basis is a linear subspace of  $\mathcal{B}(\mathcal{H})$ , not necessarily equal to  $\mathcal{B}(\mathcal{H})$ .

We require a few more definitions in order to state the next result.

**Definition 12.11.** Let  $\mathcal{H}$  be a Hilbert space,  $\omega$  be a state on  $\mathcal{B}(\mathcal{H})$  and  $(T_t)_{t \geq 0}$  be a semigroup of positive operators on  $\mathcal{B}(\mathcal{H})$ . We say that  $\omega$  is a subinvariant state for the semigroup  $(T_t)_{t \geq 0}$ , if and only if  $\omega$  is subinvariant for  $T_t$  for every  $t \geq 0$ .

**Definition 12.12.** The **Moore-Penrose inverse** or **pseudoinverse**  $x^{(-1)}$  of  $x \in \mathcal{B}(\mathcal{H})$  is defined as the unique linear extension of  $(x|_{\mathcal{N}(x)^\perp})^{-1}$ , the inverse as a function, to

$$D(x^{(-1)}) := \mathcal{R}(x) + \mathcal{R}(x)^\perp$$

with  $\mathcal{N}(x^{(-1)}) = \mathcal{R}(x)^\perp$ , where  $\mathcal{N}(x)$  and  $\mathcal{R}(x)$  denote the nullspace and range of  $x$ , respectively. Letting  $P$  and  $Q$  denote the orthogonal projections onto  $\mathcal{N}(x)$  and

$\overline{\mathcal{R}(x)}$ , respectively, it can be shown (see e.g. [28]) that  $x^{(-1)}$  is uniquely determined by the relations

$$x^{(-1)}x = I - P \quad \text{and} \quad xx^{(-1)} = Q|_{D(x^{(-1)})}.$$

**Notation 12.13.** By  $i_{\rho^{(-1)}}$  we mean the map from  $\mathcal{B}(\mathcal{H})$  to the space of linear maps on  $\mathcal{H}$  defined via

$$i_{\rho^{(-1)}}(x) = (\rho^{1/4})^{(-1)}x(\rho^{1/4})^{(-1)}.$$

Now we are ready to state the next result.

**Theorem 12.14.** *Let  $\mathcal{H}$  be a Hilbert space,  $(T_t)_{t \geq 0}$  be a semigroup of Schwarz maps on  $\mathcal{B}(\mathcal{H})$  and let  $\rho \in \mathcal{S}_1(\mathcal{H})$  be such that  $\omega_\rho$  is a faithful state on  $\mathcal{B}(\mathcal{H})$  which is subinvariant for the semigroup  $(T_t)_{t \geq 0}$ . Then there exists a unique semigroup  $(\widetilde{T}_t)_{t \geq 0}$  of contractions on  $\mathcal{S}_2(\mathcal{H})$  such that*

$$\widetilde{T}_t(i_\rho(x)) = i_\rho(T_t(x)) \quad \text{for all } x \in \mathcal{B}(\mathcal{H}). \quad (12.7)$$

Moreover, if  $(T_t)_{t \geq 0}$  is weak\*-continuous then  $(\widetilde{T}_t)_{t \geq 0}$  is strongly continuous. Let  $L$  denote the generator of  $(T_t)_{t \geq 0}$ , let  $\widetilde{L}$  denote the generator of  $(\widetilde{T}_t)_{t \geq 0}$ , and let  $L_{(h_n)}$  denote the generator of  $(T_t)_{t \geq 0}$  with respect to  $(h_n)_{n \in \mathbb{N}}$ , where  $(h_n)_{n \in \mathbb{N}}$  is an orthonormal basis of  $\mathcal{H}$  consisting of eigenvectors of  $\rho$  (guaranteed by the Spectral Theorem). Then we have that for each  $x \in \mathcal{B}(\mathcal{H})$ , if  $x \in D(L)$  then  $i_\rho(x) \in D(\widetilde{L})$ , and moreover

$$\widetilde{L}(i_\rho(x)) = i_\rho(L(x));$$

conversely, if  $i_\rho(x) \in D(\widetilde{L})$  then  $x \in D(L_{(h_n)})$ , and moreover

$$L_{(h_n)}(x) = i_{\rho^{(-1)}}(\widetilde{L}(i_\rho(x))). \quad (12.8)$$

*Proof.* The operators  $\widetilde{T}_t$  are well-defined by Theorem 11.5. Uniqueness comes from Equation (12.7) and the fact that  $i_\rho(\mathcal{B}(\mathcal{H}))$  is dense in  $\mathcal{S}_2(\mathcal{H})$ . It is easy to see that  $\widetilde{T}_{t+s} = \widetilde{T}_t \widetilde{T}_s$  and that  $\widetilde{T}_0 = 1$  on  $i_\rho(\mathcal{B}(\mathcal{H}))$  and the density of  $i_\rho(\mathcal{B}(\mathcal{H}))$  imply they hold on all of  $\mathcal{S}_2(\mathcal{H})$ .



It is now left to prove that if  $(T_t)_{t \geq 0}$  is weak\*-continuous then  $(\widetilde{T}_t)_{t \geq 0}$  is strongly continuous. To this end, it suffices to show  $(\widetilde{T}_t)_{t \geq 0}$  is strongly continuous on  $i_\rho(\mathcal{B}(\mathcal{H}))$  since  $i_\rho(\mathcal{B}(\mathcal{H}))$  is dense in  $\mathcal{S}_2(\mathcal{H})$  and  $\widetilde{T}_t$  is a contraction on  $\mathcal{S}_2(\mathcal{H})$  for all  $t \geq 0$ . Let  $x \in \mathcal{B}(\mathcal{H})$ . Then we have

$$\begin{aligned}
\|\widetilde{T}_t(i_\rho(x)) - i_\rho(x)\|_2^2 &= \|\rho^{1/4}T_t(x)\rho^{1/4} - \rho^{1/4}x\rho^{1/4}\|_2^2 = \|\rho^{1/4}T_t(x)\rho^{1/4}\|_2^2 + \|\rho^{1/4}x\rho^{1/4}\|_2^2 \\
&\quad - \langle \rho^{1/4}x\rho^{1/4}, \rho^{1/4}T_t(x)\rho^{1/4} \rangle_{\mathcal{S}_2(\mathcal{H})} - \langle \rho^{1/4}T_t(x)\rho^{1/4}, \rho^{1/4}x\rho^{1/4} \rangle_{\mathcal{S}_2(\mathcal{H})} \\
&= \|\widetilde{T}_t(i_\rho(x))\|_2^2 + \|i_\rho(x)\|_2^2 - 2\Re \langle \rho^{1/4}x\rho^{1/4}, \rho^{1/4}T_t(x)\rho^{1/4} \rangle_{\mathcal{S}_2(\mathcal{H})} \\
&\leq 2\|i_\rho(x)\|_2^2 - 2\Re \left( \text{tr}(\rho^{1/4}x^*\rho^{1/4}\rho^{1/4}T_t(x)\rho^{1/4}) \right) \\
&= 2\Re \left( \text{tr}(\rho^{1/4}x^*\rho^{1/4}\rho^{1/4}x\rho^{1/4} - \rho^{1/4}x^*\rho^{1/4}\rho^{1/4}T_t(x)\rho^{1/4}) \right) \\
&= 2\Re \left( \text{tr} \left( \rho^{1/2}x^*\rho^{1/2}(x - T_t(x)) \right) \right) \rightarrow 0
\end{aligned}$$

since  $\rho^{1/2}x^*\rho^{1/2}$  is trace-class. Therefore  $(\widetilde{T}_t)_{t \geq 0}$  is a strongly continuous semigroup of contractions on  $\mathcal{S}_2(\mathcal{H})$ .

Now, let  $x \in \mathcal{B}(\mathcal{H})$  and we wish to show that if  $x \in D(L)$  then  $\rho^{1/4}x\rho^{1/4} \in D(\widetilde{L})$  and  $\rho^{1/4}L(x)\rho^{1/4} = \widetilde{L}(\rho^{1/4}x\rho^{1/4})$ . First, assume that  $x \in D(L)$ . Then

$$\text{weak}^* - \lim_{t \rightarrow 0} \frac{T_t(x) - x}{t} = L(x). \quad (12.9)$$

Notice that for every  $y \in \mathcal{S}_2(\mathcal{H})$  we obtain, by Proposition 11.3(c) that  $\rho^{1/4}y^*\rho^{1/4} \in \mathcal{S}_1(\mathcal{H})$  and therefore the map  $\mathcal{B}(\mathcal{H}) \ni z \mapsto \text{tr}(z\rho^{1/4}y^*\rho^{1/4}) \in \mathbb{C}$  is weak\* continuous.

Thus Equation (12.9) implies

$$\text{Tr} \left( \rho^{1/4}y^*\rho^{1/4} \frac{T_t(x) - x}{t} \right) \xrightarrow{t \rightarrow 0} \text{Tr} \left( \rho^{1/4}y^*\rho^{1/4}L(x) \right),$$

that is,

$$\left\langle y, \rho^{1/4} \frac{T_t(x) - x}{t} \rho^{1/4} \right\rangle_{\mathcal{S}_2(\mathcal{H})} \xrightarrow{t \rightarrow 0} \left\langle y, \rho^{1/4}L(x)\rho^{1/4} \right\rangle_{\mathcal{S}_2(\mathcal{H})},$$

and hence,

$$\left\langle y, \frac{\widetilde{T}_t(\rho^{1/4}x\rho^{1/4}) - \rho^{1/4}x\rho^{1/4}}{t} \right\rangle_{\mathcal{S}_2(\mathcal{H})} \xrightarrow{t \rightarrow 0} \left\langle y, \rho^{1/4}L(x)\rho^{1/4} \right\rangle_{\mathcal{S}_2(\mathcal{H})}. \quad (12.10)$$

By [8, Proposition 3.36], we obtain that  $\rho^{1/4}x\rho^{1/4} \in D(\tilde{L})$  and  $\tilde{L}(\rho^{1/4}x\rho^{1/4}) = \rho^{1/4}L(x)\rho^{1/4}$ .

Conversely, by the Spectral Theorem there exists an orthonormal basis  $(h_n)_{n \in \mathbb{N}}$  of  $\mathcal{H}$  formed by eigenvectors of  $\rho$ . Let  $L_{(h_n)}$  denote the generator of  $(T_t)_{t \geq 0}$  with respect to  $(h_n)_{n \in \mathbb{N}}$ . Let  $x \in \mathcal{B}(\mathcal{H})$  and assume that  $\rho^{1/4}x\rho^{1/4} \in D(\tilde{L})$ . Then we have that

$$\frac{\tilde{T}_t(\rho^{1/4}x\rho^{1/4}) - \rho^{1/4}x\rho^{1/4}}{t} \xrightarrow{t \rightarrow 0} \tilde{L}(\rho^{1/4}x\rho^{1/4}) \quad \text{in } \mathcal{S}_2(\mathcal{H}).$$

Hence

$$\rho^{1/4} \frac{T_t(x) - x}{t} \rho^{1/4} \xrightarrow{t \rightarrow 0} \tilde{L}(\rho^{1/4}x\rho^{1/4}) \quad \text{in } \mathcal{S}_2(\mathcal{H}). \quad (12.11)$$

We will prove that  $x \in D(L_{(h_n)})$ . Indeed, we have that

$$\langle h, \rho^{1/4} \frac{T_t(x) - x}{t} \rho^{1/4} h' \rangle \xrightarrow{t \rightarrow 0} \langle h, \tilde{L}(\rho^{1/4}x\rho^{1/4}) h' \rangle$$

for all  $h, h' \in \mathcal{H}$ , so for any  $n, m \in \mathbb{N}$  we may set  $h = (\rho^{1/4})^{(-1)}h_n$  and  $h' = (\rho^{1/4})^{(-1)}h_m$  to obtain

$$\begin{aligned} & \langle (\rho^{1/4})^{(-1)}h_n, \rho^{1/4} \frac{T_t(x) - x}{t} \rho^{1/4} (\rho^{1/4})^{(-1)}h_m \rangle \\ & \xrightarrow{t \rightarrow 0} \langle (\rho^{1/4})^{(-1)}h_n, \tilde{L}(\rho^{1/4}x\rho^{1/4}) (\rho^{1/4})^{(-1)}h_m \rangle. \end{aligned}$$

Noting that  $(\rho^{1/4})^* = \rho^{1/4}$ ,  $((\rho^{1/4})^{(-1)})^* = (\rho^{1/4})^{(-1)}$ , and  $\rho^{1/4}(\rho^{1/4})^{(-1)}h_k = h_k$  for all  $k \in \mathbb{N}$ , this implies

$$\langle h_n, \frac{T_t(x) - x}{t} h_m \rangle \xrightarrow{t \rightarrow 0} \langle h_n, (\rho^{1/4})^{(-1)} \tilde{L}(\rho^{1/4}x\rho^{1/4}) (\rho^{1/4})^{(-1)} h_m \rangle.$$

Because this limit exists for all  $n, m \in \mathbb{N}$  we have  $x \in D(L_{(h_n)})$ , and moreover

$$L_{(h_n)}(x) = (\rho^{1/4})^{(-1)} \tilde{L}(\rho^{1/4}x\rho^{1/4}) (\rho^{1/4})^{(-1)}.$$

□

**Remark 12.15.** Since the proof of Equation (9.1) is not included in [51], we want to mention that its proof follows from our Remark 11.6 in a similar way that our Theorem 12.14 followed from our Theorem 11.5 (even the proof of the strong continuity

of the semigroup  $(\overline{T}_t)_{t \geq 0}$  follows the exact same argument as the proof of the strong continuity of the semigroup  $(\widetilde{T}_t)_{t \geq 0}$  that appeared in Theorem 12.14). Moreover, the assumptions that the faithful state is normal and invariant for the semigroup and that the operators of the semigroup are completely positive that are mentioned in [51] for Equation (9.1) are not needed for its proof, because such assumptions were not used in Remark 11.6. Instead, for the validity of Equation (9.1), one merely needs to assume that the faithful state is subinvariant for the semigroup of Schwarz maps. Note also that, unlike Equation (9.1), Theorem 12.14 relates the generators of the two semigroups.

## 12.2 DILATING SEMIGROUPS OF CONTRACTIONS TO SEMIGROUPS OF UNITARY OPERATORS

Since Theorem 12.14 provides a semigroup of contractions on a Hilbert space, there is a natural way to improve the contraction property to the unitary property. The trick is to use the theory of dilations of contraction semigroups on Hilbert spaces given in [67, Theorem 8.1 on page 31]. For other uses of the dilation theory to semigroups see [32, 29, 26]. The theory of dilations of contraction semigroups on Hilbert spaces due to Foias and Sz.-Nagy can be stated as follows:

**Theorem 12.16.** *[67, Theorem 8.1 on page 31] For every strongly continuous semigroup  $(T_t)_{t \geq 0}$  of contractions on a Hilbert space  $\mathcal{S}$ , there exists a Hilbert space  $\mathcal{K}$  which contains  $\mathcal{S}$ , and a strongly continuous semigroup  $(U_t)_{t \in \mathbb{R}}$  of unitary operators on  $\mathcal{K}$  such that*

$$T_t = pr_{\mathcal{S}}(U_t) \quad \text{for all } t \geq 0$$

and

$$\mathcal{K} = \overline{\text{Span} \bigcup_{t \in \mathbb{R}} U_t(\mathcal{S})}.$$

Further, these conditions determine  $(U_t)_{t \geq 0}$  up to an isomorphism.

Since the dilation theory of Foias and Sz.-Nagy provides us with a semigroup of unitaries, naturally Stone's Theorem is applicable and gives information about the generator of the semigroup. The next result does exactly that: it combines the dilation theory with Stone's Theorem.

**Proposition 12.17.** *Let  $(\mathcal{T}_t)_{t \geq 0}$  be a strongly continuous semigroup of contractions on a Hilbert space  $\mathcal{S}$ . Then there exists a (unique up to isomorphism) Hilbert space  $\mathcal{K}$  which contains  $\mathcal{S}$  and a unique self-adjoint (not necessarily bounded) operator  $A$  on  $\mathcal{K}$  such that  $\{e^{itA}(s) : s \in \mathcal{S}, t \geq 0\}$  is dense in  $\mathcal{K}$  and*

$$\mathcal{T}_t(s) = Pe^{itA}(s) \quad \text{for all } s \in \mathcal{S} \text{ and } t \geq 0 \quad (12.12)$$

where  $P$  is the orthogonal projection from  $\mathcal{K}$  onto  $\mathcal{S}$ . Further, if  $\mathcal{L}$  is the generator of  $(\mathcal{T}_t)_{t \geq 0}$  then  $\mathcal{S} \cap D(A) \subseteq D(\mathcal{L})$  and  $\mathcal{L}(s) = iPA(s)$  for all  $s \in \mathcal{S} \cap D(\mathcal{L})$ .

Note that for the self-adjoint (not necessarily bounded) operator  $A$  with a dense domain in  $\mathcal{K}$  and  $t > 0$ , the operator  $e^{itA}$  is defined via functional calculus on a dense subspace of  $\mathcal{K}$ . It turns out that the operator  $e^{itA}$  is bounded, and in fact can be extended to a unitary operator on  $\mathcal{K}$ . Hence, Equation (12.12) is valid for all  $s \in \mathcal{S}$ .

*Proof.* Let  $(\mathcal{T}_t)_{t \geq 0}$  be a strongly continuous semigroup of contractions on a Hilbert space  $\mathcal{S}$ . From Theorem 12.16 there exists a strongly continuous semigroup  $(U_t)_{t \geq 0}$  of unitary operators on a Hilbert space  $\mathcal{K} \supseteq \mathcal{S}$  such that  $\mathcal{T}_t = \text{pr}_{\mathcal{S}}(U_t)$  for all  $t \geq 0$ . From Stone's Theorem, there exists a unique self-adjoint operator  $A$  on a dense domain in  $\mathcal{K}$  so that  $U_t = e^{itA}$  for all  $t \geq 0$ , where  $iA$  is the generator of  $(U_t)_{t \geq 0}$ . So, we have that  $\mathcal{T}_t(s) = Pe^{itA}(s)$  for all  $s \in \mathcal{S}$  and  $t \geq 0$  where  $P$  is the orthogonal projection from  $\mathcal{K}$  onto  $\mathcal{S}$ .

For the second statement of the proposition, let  $\mathcal{L}$  be the generator of  $(\mathcal{T}_t)_{t \geq 0}$  and let  $s \in \mathcal{S}$  be in the domain of  $A$ . Since  $s \in D(A)$  we have that

$$\frac{1}{t} (U_t(s) - s) \rightarrow iA(s) \quad \text{as } t \rightarrow 0$$

and so

$$P\left(\frac{1}{t}(U_t(s) - s)\right) \rightarrow iPA(s) \quad \text{as } t \rightarrow 0.$$

Since  $P(s) = s$ , we then have that

$$\frac{1}{t}(\mathcal{T}_t(s) - s) \rightarrow iPA(s) \quad \text{as } t \rightarrow 0$$

and so  $s \in D(\mathcal{L})$  and  $\mathcal{L}(s) = iPA(s)$ . Therefore  $\mathcal{L}(s) = iPA(s)$  for all  $s \in \mathcal{S} \cap D(A)$ .

This completes the proof.  $\square$

An easy consequence of Theorem 12.14 and Proposition 12.17 will be the following:

**Corollary 12.18.** *Let  $\mathcal{H}$  be a Hilbert space,  $(T_t)_{t \geq 0}$  be a weak\*-continuous semigroup of Schwarz maps on  $\mathcal{B}(\mathcal{H})$  and let  $\rho \in \mathcal{S}_1(\mathcal{H})$  be such that  $\omega_\rho$  is a faithful state which is subinvariant for  $(T_t)_{t \geq 0}$ . By the Spectral Theorem there exists an orthonormal basis  $(h_n)_{n \in \mathbb{N}}$  of  $\mathcal{H}$  formed by eigenvectors of  $\rho$ . Let  $L_{(h_n)}$  denote the generator of  $(T_t)_{t \geq 0}$  with respect to  $(h_n)_{n \in \mathbb{N}}$ . Then there exists a Hilbert space  $\mathcal{K}$  which contains  $\mathcal{S}_2(\mathcal{H})$ , and a self-adjoint (not necessarily bounded) operator  $A$  on  $\mathcal{K}$ , so that if  $x \in \mathcal{B}(\mathcal{H})$  and  $\rho^{1/4}x\rho^{1/4} \in D(A)$  then  $x \in D(L_{(h_n)})$  and*

$$L_{(h_n)}(x)_F = i_{\rho(-1)}(PA(i_\rho(x)))$$

where  $P$  is the orthogonal projection from  $\mathcal{K}$  onto  $\mathcal{S}_2(\mathcal{H})$ .

*Proof.* First apply Theorem 12.14 to obtain  $(\widetilde{T}_t)_{t \geq 0}$  and  $\widetilde{L}$  satisfying the conclusion of Theorem 12.14. In particular, we obtain that if  $x \in \mathcal{B}(\mathcal{H})$  and  $\rho^{1/4}x\rho^{1/4} \in D(\widetilde{L})$  then  $x \in D(L_{(h_n)})$ , (where  $L_{(h_n)}$  denotes the generator of  $(T_t)_{t \geq 0}$  with respect to the orthonormal sequence  $(h_n)_{n \in \mathbb{N}}$  of the eigenvectors of  $\rho$ ), equation (12.8) is satisfied. Then apply Proposition 12.17 for  $\mathcal{T}_t = \widetilde{T}_t$ ,  $\mathcal{L} = \widetilde{L}$ , and  $\mathcal{S} = \mathcal{S}_2(\mathcal{H})$ , to obtain a Hilbert space  $\mathcal{K}$  which contains  $\mathcal{S}_2(\mathcal{H})$  and a unique self-adjoint (not necessarily bounded) operator  $A$  on  $\mathcal{K}$  satisfying the conclusion of Proposition 12.17. Thus we have  $\mathcal{S}_2(\mathcal{H}) \cap D(A) \subseteq D(\widetilde{L})$  and  $\widetilde{L}(s) = iPA(s)$  for all  $s \in \mathcal{S}_2(\mathcal{H}) \cap D(A)$  where

$P$  is the orthogonal projection from  $\mathcal{K}$  onto  $\mathcal{S}_2(\mathcal{H})$ . Thus for  $x \in \mathcal{B}(\mathcal{H})$ , if  $s = \rho^{1/4}x\rho^{1/4} \in D(A)$  then  $s \in \mathcal{S}_2(\mathcal{H}) \cap D(A)$ , hence  $s \in D(\tilde{L})$  and  $\tilde{L}(s) = \tilde{L}(\rho^{1/4}x\rho^{1/4}) = iPA(\rho^{1/4}x\rho^{1/4})$ . Therefore if  $\rho^{1/4}x\rho^{1/4} \in D(A)$ , equation (12.8) finishes the proof of Corollary 12.18.  $\square$

# CHAPTER 13

## APPLICATIONS TO QUANTUM MARKOV SEMIGROUPS

### AND THEIR GENERATORS

Since Quantum Markov semigroups (QMSs) are semigroups of completely positive maps on some von Neumann algebra, and hence 2-positive maps, and hence Schwarz maps, we naturally obtain applications of Theorem 12.14 in the study of QMSs. The existence of invariant normal states for QMSs has been discussed in [35] and [36]. Sufficient conditions for a semigroup to be decomposable into a sequence of irreducible semigroups each of them having an invariant normal state are given in [69] (see top half of page 608, Theorem 5 on page 608, and Proposition 5 on page 609). There are many results in the literature of semigroups which depend on the existence of invariant faithful normal states (for example, see [38], [39], [37], [34], and [15]) and this assumption is often taken for granted as being physically reasonable. QMSs have been extensively studied since the 1970s with the exact form for the generators being one of the topics which has garnered a fair amount of attention. See for example [53], [42], [21], [23], [46], [4], [7], and [61]. The generator of a QMS is a generally unbounded operator defined on a weak\* dense linear subspace of  $\mathcal{B}(\mathcal{H})$ . If the generator is bounded then the semigroup is uniformly continuous and the exact form of the generator was found in [42] and [53]. In this Chapter, given a Hilbert space  $\mathcal{H}$  and a QMS on  $\mathcal{B}(\mathcal{H})$  having an invariant faithful normal state we study the associated semigroup of contractions on  $\mathcal{S}_2(\mathcal{H})$ . In particular, in Theorems 13.14 and 13.9 we describe the generator of the QMS on  $\mathcal{B}(\mathcal{H})$  having an invariant faithful

normal state, under the assumption that the minimal semigroup of unitary dilations of the associated semigroup of contractions is compact, or under the assumption that the generator of the associated semigroup on  $\mathcal{S}_2(\mathcal{H})$  has compact resolvent, respectively.

**Definition 13.1.** A **quantum Markov semigroup (QMS)** on  $\mathcal{B}(\mathcal{H})$ , (for some Hilbert space  $\mathcal{H}$ ), is a weak\*-continuous one-parameter semigroup of bounded linear operators acting on  $\mathcal{B}(\mathcal{H})$ , such that each member of the semigroup is completely positive, weak\*-continuous, and preserves the identity.

**Remark 13.2.** If  $\mathcal{H}$  is a Hilbert space and  $(T_t)_{t \geq 0}$  is a QMS on  $\mathcal{B}(\mathcal{H})$ , which has a subinvariant normal state  $\omega_\rho$  for some  $\rho \in \mathcal{S}_1(\mathcal{H})$  then  $\omega_\rho$  is in fact an invariant state for  $(T_t)_{t \geq 0}$ . Indeed for every  $t \geq 0$ ,

$$\mathrm{Tr}(T_t^\dagger(\rho)) = \mathrm{Tr}(T_t^\dagger(\rho)1) = \mathrm{Tr}(\rho T_t(1)) = \mathrm{Tr}(\rho 1) = \mathrm{Tr}(\rho),$$

which together with  $T_t^\dagger(\rho) \leq \rho$  implies that  $T_t^\dagger(\rho) = \rho$ .

If  $(T_t)_{t \geq 0}$  is a quantum Markov semigroup with an invariant faithful normal state then Corollary 12.18 can be applied. This result however does not use the fact that the semigroup  $(T_t)_{t \geq 0}$  is a QMS but merely that it is a semigroup of Schwarz maps. Addressing this issue is the main goal of this section. Usually the notion of complete positivity applies to maps on  $C^*$ -algebras. In particular, if the  $C^*$ -algebra is equal to  $\mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ , then the notion of complete positivity becomes equivalent to the following: A map  $\mathcal{T} : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{H})$  is completely positive if and only if for every  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in \mathcal{B}(\mathcal{H})$  and  $h_1, \dots, h_n \in \mathcal{H}$ ,

$$\sum_{i,j=1}^n \langle h_i, \mathcal{T}(x_i^* x_j) h_j \rangle \geq 0. \quad (13.1)$$

Note that Equation (13.1) makes perfect sense even if the map  $\mathcal{T}$  is not defined on a  $C^*$ -algebra, as long as  $\mathcal{T}$  is defined on a Banach \*-algebra  $\mathcal{S}$  of operators on a Hilbert space  $\mathcal{H}$ . For example,  $\mathcal{S}$  can be equal to  $\mathcal{S}_2(\mathcal{H})$  and  $\mathcal{T}$  can be a bounded linear



operator from  $\mathcal{S}$  to  $\mathcal{S}$ . We make this extension of the notion of complete positivity in the next definition.

**Definition 13.3.** Let  $\mathcal{H}$  be a Hilbert space and  $\mathcal{S}$  be a Banach  $*$ -algebra of bounded linear operators on  $\mathcal{H}$ . A bounded linear map  $\mathcal{T} : \mathcal{S} \rightarrow \mathcal{S}$  will be called **completely positive** if for every  $n \in \mathbb{N}$ ,  $x_1, \dots, x_n \in \mathcal{S}$  and  $h_1, \dots, h_n \in \mathcal{H}$ , Equation (13.1) holds.

This terminology will be used in the next result.

**Proposition 13.4.** Let  $(T_t)_{t \geq 0}$  be a QMS on  $\mathcal{B}(\mathcal{H})$  for some Hilbert space  $\mathcal{H}$ , having an invariant faithful normal state  $\omega_\rho$  for some  $\rho \in \mathcal{S}_1(\mathcal{H})$ . Then the operators  $\widetilde{T}_t$  constructed in Theorem 12.14 are completely positive for all  $t \geq 0$ .

*Proof.* Let  $t \geq 0$ ,  $x_1, x_2, \dots, x_n \in \mathcal{B}(\mathcal{H})$  and  $h_1, h_2, \dots, h_n \in \mathcal{H}$ . Then,

$$\begin{aligned} \sum_{i,j=1}^n \langle h_i, \widetilde{T}_t \left( (\rho^{1/4} x_i \rho^{1/4})^* (\rho^{1/4} x_j \rho^{1/4}) \right) h_j \rangle &= \sum_{i,j=1}^n \langle \rho^{1/4} h_i, T_t \left( (\rho^{1/4} x_i)^* (\rho^{1/4} x_j) \right) \rho^{1/4} h_j \rangle \\ &\geq 0 \end{aligned}$$

since  $T_t$  is completely positive. Further, since the map  $i_\rho$  from Proposition 11.3 has dense range,  $\widetilde{T}_t$  is completely positive on  $\mathcal{S}_2(\mathcal{H})$ . Therefore,  $\widetilde{T}_t$  is completely positive for all  $t \geq 0$ .  $\square$

For the next result, recall the notion of conditionally completely positive maps introduced by Lindblad in [53]. A linear operator  $L : D(L) (\subseteq \mathcal{B}(\mathcal{H})) \rightarrow \mathcal{B}(\mathcal{H})$  is called **conditionally completely positive** if for all  $n \in \mathbb{N}$ , for all  $a_1, a_2, \dots, a_n \in \mathcal{B}(\mathcal{H})$  such that  $a_i^* a_j \in D(L)$  for all  $i, j = 1, 2, \dots, n$ , that for all  $h_1, h_2, \dots, h_n \in \mathcal{H}$  with  $\sum_{i=1}^n a_i(h_i) = 0$ , we have that

$$\sum_{i,j=1}^n \langle h_i, L(a_i^* a_j) h_j \rangle \geq 0.$$

The next result is known for uniformly continuous semigroups. For example, see [33, Proposition 3.12 and Lemma 3.13], or see [30, Proposition 2.9]. In fact the known proof works for a more general setting as the next result indicates.

**Theorem 13.5.** *Let  $\mathcal{S}$  be a Banach  $*$ -algebra of operators acting on a Hilbert space  $\mathcal{H}$ .*

1. *Let  $(\mathcal{T}_t)_{t \geq 0}$  be a WOT continuous semigroup on  $\mathcal{S}$  and let  $\mathcal{L}$  be its generator. If  $\mathcal{T}_t$  is completely positive for all  $t \geq 0$  then  $\mathcal{L}(a^*) = \mathcal{L}(a)^*$  for all  $a \in D(\mathcal{L})$  and  $\mathcal{L}$  is conditionally completely positive.*
2. *Let  $(\mathcal{T}_t)_{t \geq 0}$  be a uniformly continuous semigroup on  $\mathcal{S}$  with generator  $\mathcal{L}$ . If  $\mathcal{L}(a^*) = \mathcal{L}(a)^*$  for all  $a \in \mathcal{S}$  and  $\mathcal{L}$  is conditionally completely positive, then  $\mathcal{T}_t$  is completely positive for all  $t \geq 0$ .*

*Proof.* The proof of (2) immediately follows from [33, Proposition 3.12 and Lemma 3.13]. To prove (1), suppose  $a_1, a_2, \dots, a_n \in \mathcal{S}$  such that  $a_i^* a_j \in D(L)$  for all  $i, j = 1, \dots, n$  and  $h_1, h_2, \dots, h_n \in \mathcal{H}$  such that  $\sum_{i=1}^n a_i(h_i) = 0$ . Then,

$$\begin{aligned} \sum_{i,j=1}^n \langle h_i, \mathcal{L}(a_i^* a_j) h_j \rangle &= \lim_{t \rightarrow 0^+} \sum_{i,j=1}^n \frac{1}{t} \langle h_i, (T_t - 1)(a_i^* a_j) h_j \rangle \\ &= \lim_{t \rightarrow 0^+} \sum_{i,j=1}^n \frac{1}{t} \langle h_i, T_t(a_i^* a_j) h_j \rangle \quad (\text{since } \sum_{i=1}^n a_i(h_i) = 0) \\ &\geq 0 \end{aligned}$$

since  $T_t$  is completely positive for all  $t \geq 0$ . □

**Corollary 13.6.** *Let  $\mathcal{H}$  be a Hilbert space and  $(T_t)_{t \geq 0}$  be a QMS on  $\mathcal{B}(\mathcal{H})$  having an invariant faithful normal state  $\omega_\rho$  for some  $\rho \in \mathcal{S}_1(\mathcal{H})$ . Let  $(\tilde{T}_t)_{t \geq 0}$  be the strongly continuous semigroup of contractions on  $\mathcal{S}_2(\mathcal{H})$  defined in Theorem 12.14 and let  $\tilde{L}$  be the generator of  $(\tilde{T}_t)_{t \geq 0}$ . Then  $\tilde{L}(a^*) = \tilde{L}(a)^*$  for all  $a \in D(\tilde{L})$  and  $\tilde{L}$  is conditionally completely positive.*

*Proof.* The proof follows immediately from Proposition 13.4 and Theorem 13.5(1).  $\square$

**Corollary 13.7.** *Let  $\mathcal{H}$  be a Hilbert space and  $(T_t)_{t \geq 0}$  be a QMS on  $\mathcal{B}(\mathcal{H})$  having an invariant faithful normal state  $\omega_\rho$  for some  $\rho \in \mathcal{S}_1(\mathcal{H})$ . Let  $(\tilde{T}_t)_{t \geq 0}$  be the strongly continuous semigroup of contractions on  $\mathcal{S}_2(\mathcal{H})$  defined in Theorem 12.14 and let  $\tilde{L}$  be its generator. Then there exists a Hilbert space  $\mathcal{K}$  which contains  $\mathcal{S}_2(\mathcal{H})$  and a self-adjoint (not necessarily bounded) operator  $A$  on  $\mathcal{K}$  such that  $\mathcal{S}_2(\mathcal{H}) \cap D(A) \subseteq D(\tilde{L})$ ,  $\tilde{L}|_{\mathcal{S}_2(\mathcal{H}) \cap D(A)} = iPA|_{\mathcal{S}_2(\mathcal{H})}$  (where  $P$  is the orthogonal projection from  $\mathcal{K}$  to  $\mathcal{S}_2(\mathcal{H})$ ),  $iPA(a^*) = (iPA(a))^*$  for all  $a \in \mathcal{S}_2(\mathcal{H}) \cap D(A)$ , and the operator  $iPA|_{\mathcal{S}_2(\mathcal{H})}$  is conditionally completely positive.*

*Proof.* First apply Proposition 12.17 for  $(\mathcal{T}_t)_{t \geq 0} = (\tilde{T}_t)_{t \geq 0}$  and  $\mathcal{S} = \mathcal{S}_2(\mathcal{H})$  to obtain the existence of the Hilbert space  $\mathcal{K} \supseteq \mathcal{S}_2(\mathcal{H})$ , and the self-adjoint (not necessarily bounded) operator  $A$  on  $\mathcal{K}$  such that

$$\tilde{T}_t(x) = Pe^{itA}(x) \quad \text{for all } x \in \mathcal{S}_2(\mathcal{H}) \text{ and } t \geq 0,$$

where  $P$  is the orthogonal projection from  $\mathcal{K}$  onto  $\mathcal{S}_2(\mathcal{H})$ . Moreover the generator  $\tilde{L}$  of  $(\tilde{T}_t)_{t \geq 0}$  satisfies  $\mathcal{S}_2(\mathcal{H}) \cap D(A) \subseteq D(\tilde{L})$  and

$$\tilde{L}(x) = iPA(x) \quad \text{for all } x \in \mathcal{S}_2(\mathcal{H}) \cap D(\tilde{L}).$$

Then apply Corollary 13.6 to obtain that  $\tilde{L}$  respects adjoints and it is conditionally completely positive.  $\square$

Corollary 13.7 has two disadvantages: First, the intersection  $\mathcal{S}_2(\mathcal{H}) \cap D(A)$  can potentially contain nothing but zero! Second, the conditional complete positivity of  $iPA|_{\mathcal{S}_2(\mathcal{H})}$  can be very hard to be recognized in practice! Indeed, the conditional complete positivity of  $iPA|_{\mathcal{S}_2(\mathcal{H})}$  means that for every  $n \in \mathbb{N}$ ,  $a_1, \dots, a_n \in \mathcal{B}(\mathcal{H})$  such that  $a_k^* a_j \in \mathcal{S}_2(\mathcal{H}) \cap D(A)$  for  $k, j \in \{1, \dots, n\}$ , and for every  $h_1, \dots, h_n \in \mathcal{H}$  such

that  $\sum_{i=1}^n a_i(h_i) = 0$ , we have that

$$\sum_{k,j=1}^n \langle h_k, iPA(a_k^* a_j) h_j \rangle \geq 0, \quad (13.2)$$

or

$$\sum_{k,j=1}^n \langle h_k, \tilde{L}(a_k^* a_j) h_j \rangle \geq 0. \quad (13.3)$$

The large number of arbitrary test sequences  $(h_i)_{i=1}^n$  and  $(a_i)_{i=1}^n$  satisfying  $\sum_{i=1}^n a_i(h_i) = 0$ , makes the conditional complete positivity of  $\tilde{L}$  hard to be recognized. In the following sections we will get rid of both of these two disadvantages of Corollary 13.7. This will be achieved by adding the additional assumption of compactness of the generator of the minimal unitary dilation of Foias and Sz.-Nagy, and by carefully analyzing the notion of conditional complete positivity under this assumption.

### 13.1 THE FORM OF $L_{(h_n)}$ WHEN THE GENERATOR OF THE MINIMAL UNITARY DILATION OF $(\tilde{T}_t)_{t \geq 0}$ IS COMPACT

In this section we consider the form of  $L_{(h_n)}$  when the generator of the minimal unitary dilation of  $(\tilde{T}_t)_{t \geq 0}$ , as defined in the previous section, is compact. First, we establish two notations:

**Notation 13.8.** Let  $\mathcal{H}$  be a Hilbert space and  $w, z \in \mathcal{S}_2(\mathcal{H})$ . Define  $M_w \otimes z : \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H} \rightarrow \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$  by

$$M_w \otimes z \left( \sum_{i=1}^k x_i \otimes h_i \right) = \sum_{i=1}^k x_i w \otimes z(h_i).$$

Let  $e \in \mathcal{H}$  such that  $\|e\| = 1$ . Define  $T_e : \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H} \rightarrow \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$  by

$$T_e \left( \sum_{i=1}^k x_i \otimes h_i \right) = \sum_{i=1}^k |x_i(h_i)\rangle \langle e| \otimes e.$$

**Theorem 13.9.** Let  $\mathcal{H}$  be a Hilbert space,  $(T_t)_{t \geq 0}$  be a QMS on  $\mathcal{B}(\mathcal{H})$  having an invariant faithful normal state  $\omega_\rho$  for some  $\rho \in \mathcal{S}_1(\mathcal{H})$ , and  $L$  be the generator of

$(T_t)_{t \geq 0}$ . Let  $(\tilde{T}_t)_{t \geq 0}$  be the strongly continuous semigroup of contractions on  $\mathcal{S}_2(\mathcal{H})$  defined in Theorem 12.14 and let  $\tilde{L}$  be its generator. Assume that the generator of the minimal unitary dilation of  $(\tilde{T}_t)_{t \geq 0}$  is compact. Then the following assertions are valid:

(a)  $\tilde{L} : \mathcal{S}_2(\mathcal{H}) \rightarrow \mathcal{S}_2(\mathcal{H})$  is bounded.

(b) There exist families  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  of self-adjoint elements in  $\mathcal{S}_2(\mathcal{H})$ , and sequence  $(\lambda_n)_{n \in \mathbb{N}} \subseteq \mathbb{R}$  such that

$$\tilde{L} = \sum_{n=1}^{\infty} \lambda_n (|a_n\rangle\langle b_n| + |b_n\rangle\langle a_n|) \quad (13.4)$$

where the sums converge in the SOT (if it is infinite).

(c) By the Spectral Theorem there is an orthonormal basis  $(h_n)_{n \in \mathbb{N}}$  of  $\mathcal{H}$  which consists of eigenvectors of  $\rho$ . Let  $L_{(h_n)}$  denote the generator of  $(T_t)_{t \geq 0}$  with respect to  $(h_n)_{n \in \mathbb{N}}$ . Then  $D(L_{(h_n)}) = \mathcal{B}(\mathcal{H})$ .

(d) We have

$$L_{(h_n)} = \sum_{n=1}^{\infty} \lambda_n (|i_{\rho(-1)}(a_n)\rangle\langle i_{\rho}(b_n)| + |i_{\rho(-1)}(b_n)\rangle\langle i_{\rho}(a_n)|) \quad (13.5)$$

where the sums converge in the SOT (if it is infinite). We note that, despite the Hilbert space notation,  $|i_{\rho(-1)}(a_n)\rangle\langle i_{\rho}(b_n)|$  has domain  $\mathcal{B}(\mathcal{H})$  for all  $n$ , since

$$|i_{\rho(-1)}(a_n)\rangle\langle i_{\rho}(b_n)|x = \langle i_{\rho}(b_n), x \rangle i_{\rho(-1)}(a_n) = \langle b_n, i_{\rho}(x) \rangle i_{\rho(-1)}(a_n)$$

and  $b_n, i_{\rho}(x) \in \mathcal{S}_2(\mathcal{H})$ . Similarly  $|i_{\rho(-1)}(b_n)\rangle\langle i_{\rho}(a_n)|$  has domain  $\mathcal{B}(\mathcal{H})$  for all  $n$ .

(e) For all  $e \in \mathcal{H}$  with  $\|e\| = 1$  we have that the operator  $\tilde{L}_{\otimes, e} : \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H} \rightarrow \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$  is positive, where the operator  $\tilde{L}_{\otimes, e}$  is defined by

$$\tilde{L}_{\otimes, e} = (Id + T_e^*) \left( \sum_{n=1}^{\infty} \lambda_n (M_{b_n} \otimes a_n - M_{a_n} \otimes b_n) \right) (Id + T_e) \quad (13.6)$$

where  $Id$  stands for the identity operator on  $\mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$  and the sum converges in the SOT (if it is infinite).

In order to prove Theorem 13.9, we need the following two lemmas:

**Lemma 13.10.** *Let  $\mathcal{H}$  be a Hilbert space,  $\mathcal{K}$  be a Hilbert space containing  $\mathcal{S}_2(\mathcal{H})$  and  $P : \mathcal{K} \rightarrow \mathcal{S}_2(\mathcal{H})$  be the orthogonal projection. Let  $A$  be a compact self-adjoint operator on  $\mathcal{K}$  and let  $\tilde{L} : \mathcal{S}_2(\mathcal{H}) \rightarrow \mathcal{S}_2(\mathcal{H})$  be given by  $\tilde{L} = iPA|_{\mathcal{S}_2(\mathcal{H})}$ . Then  $\tilde{L}(a^*) = (\tilde{L}(a))^*$  for all  $a \in \mathcal{S}_2(\mathcal{H})$  if and only if  $\tilde{L}$  has the form in Equation (13.4).*

**Lemma 13.11.** *Let  $\mathcal{H}$  be a Hilbert space and  $\tilde{L}$  be a bounded linear operator on  $\mathcal{S}_2(\mathcal{H})$  which has the form (13.4). Then  $\tilde{L}$  is conditionally completely positive if and only if for some (equivalently all) normalized vector  $e \in \mathcal{H}$ , the operator  $\tilde{L}_{\otimes, e}$  defined on the Hilbert space  $\mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$ , by Equation (13.6), is positive.*

**Remark 13.12.** Theorem 13.9 provides the form of the generator  $L_{(h_n)}$  of  $(T_t)_{t \geq 0}$  with respect to the orthonormal basis  $(h_n)_{n \in \mathbb{N}}$ , but of course the assumption that the generator of the minimal unitary dilation of the associated semigroup of contractions is compact cannot be easily verified. Notice though, that if we restrict our attention to quantum Markov semigroups which have an invariant faithful normal state, then Theorem 13.5 and Lemmas 13.10 and 13.11 imply that the form of the generator  $L_{(h_n)}$  of the semigroup  $(T_t)_{t \geq 0}$  with respect to the orthonormal basis  $(h_n)_{n \in \mathbb{N}}$ , which is provided by Theorem 13.9, is “almost” equivalent to the assumptions of Theorem 13.9 (namely that the generator of the minimal unitary dilation of the associated semigroup of contractions is compact).

Assume for the moment the validity of Lemmas 13.10 and 13.11 in order to see the proof of Theorem 13.9.

*Proof of Theorem 13.9.* Since the generator  $iA$  of the unitary dilation of the semigroup  $(\tilde{T}_t)_{t \geq 0}$  of contractions is compact, we have that  $A$  is bounded and  $D(A) = \mathcal{K}$ . Corollary 13.7 implies that the generator  $\tilde{L}$  of  $(\tilde{T}_t)_{t \geq 0}$  satisfies

$$\tilde{L} = iPA|_{\mathcal{S}_2(\mathcal{H})}.$$

Since  $A$  is bounded, we obtain that  $\tilde{L}$  is bounded and hence  $D(\tilde{L}) = \mathcal{S}_2(\mathcal{H})$ . By Theorem 13.5(1) we have that  $\tilde{L}(a^*) = \tilde{L}(a)^*$  for all  $a \in \mathcal{S}_2(\mathcal{H})$ , and  $\tilde{L}$  is conditionally completely positive. Since  $\tilde{L}(a^*) = \tilde{L}(a)^*$  for all  $a \in \mathcal{S}_2(\mathcal{H})$ , Lemma 13.10 implies that  $\tilde{L}$  has the form of Equation (13.4). Then, since  $\tilde{L}$  is conditionally completely positive, Lemma 13.11 implies that  $\tilde{L}_{\otimes, e} \geq 0$  for all normalized vectors  $e \in \mathcal{H}$ . Finally, since  $D(\tilde{L}) = \mathcal{S}_2(\mathcal{H})$ , we have that  $\rho^{1/4}x\rho^{1/4} \in D(\tilde{L})$  for every  $x \in \mathcal{B}(\mathcal{H})$ . Thus if  $(h_n)_{n \in \mathbb{N}}$  is an orthonormal basis of  $\mathcal{H}$  which consists of eigenvectors of  $\rho$  and  $L_{(h_n)}$  denotes the generator of  $(T_t)_{t \geq 0}$ , then Theorem 12.14 gives that  $D(L_{(h_n)}) = \mathcal{B}(\mathcal{H})$  and Equations (12.8) and (12.5) give Equation (13.5).  $\square$

We now present the

*Proof of Lemma 13.10.* Since the generator  $iA$  of the unitary dilation of the semi-group  $(\tilde{T}_t)_{t \geq 0}$  of contractions is compact, we have that  $A$  is bounded,  $D(A) = \mathcal{K}$  and the spectrum  $\sigma(A)$  of  $A$  is discrete. Let  $\sigma(A) \setminus \{0\} = (\lambda_n)_n \subseteq \mathbb{R}$  listed according to multiplicity, and for every  $n$  let  $x_n$  be a normalized eigenvector of  $A$  corresponding to  $\lambda_n$ . Then, by the Spectral Theorem for compact self-adjoint operators, we have that

$$A = \sum_n \lambda_n |x_n\rangle\langle x_n|,$$

where the series converges in the SOT on  $\mathcal{K}$ . Since  $D(A) = \mathcal{K}$ , Corollary 13.7 implies that

$$\tilde{L} = iPA \Big|_{\mathcal{S}_2(\mathcal{H})} = iP \sum_n \lambda_n |x_n\rangle\langle x_n| \Big|_{\mathcal{S}_2(\mathcal{H})} = i \sum_n \lambda_n |Px_n\rangle\langle x_n| \Big|_{\mathcal{S}_2(\mathcal{H})}. \quad (13.7)$$

Since  $x_n \in \mathcal{K}$ , the bra  $\langle x_n|$  in Equation (13.7) uses the inner product of  $\mathcal{K}$ . On the other hand,  $\tilde{L}$  is defined on  $\mathcal{S}_2(\mathcal{H})$  hence, without loss of generality, the bra  $\langle x_n|$  in Equation (13.7) can be replaced by  $\langle x_n|P = \langle P^*x_n| = \langle Px_n|$ . Thus, we obtain

$$\tilde{L} = i \sum_n \lambda_n |Px_n\rangle\langle Px_n|$$

where the bra  $\langle Px_n|$  uses the inner product of  $\mathcal{S}_2(\mathcal{H})$  instead of the inner product of  $\mathcal{K}$ .

For every  $n$  decompose the operator  $Px_n \in \mathcal{S}_2(\mathcal{H})$  as  $Px_n = \Re(Px_n) + i\Im(Px_n)$  where  $\Re(Px_n)$  and  $\Im(Px_n)$  stand for the real and the imaginary parts of  $Px_n$  respectively. Then we obtain

$$\begin{aligned} \tilde{L} = & i \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Re(Px_n)| - i \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Im(Px_n)| \\ & - \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Im(Px_n)| - \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Re(Px_n)|. \end{aligned} \quad (13.8)$$

Notice that if  $b$  is a self-adjoint operator in  $\mathcal{S}_2(\mathcal{H})$  and  $c \in \mathcal{S}_2(\mathcal{H})$  then for every  $a \in \mathcal{S}_2(\mathcal{H})$  we have

$$(|b\rangle \langle c| a^*)^* = (\langle c, a^* \rangle_{\mathcal{S}_2(\mathcal{H})} b)^* = (\text{Tr}(c^* a^*) b)^* = (\overline{\text{Tr}(ac)} b)^* = \text{Tr}(ac) b = |b\rangle \langle c| a.$$

Applying this to Equation (13.8) we obtain that for every  $a \in \mathcal{S}_2(\mathcal{H})$ ,

$$\begin{aligned} (\tilde{L}(a^*))^* = & \left\{ -i \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Re(Px_n)| + i \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Im(Px_n)| \right. \\ & \left. - \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Im(Px_n)| - \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Re(Px_n)| \right\} (a). \end{aligned} \quad (13.9)$$

By Corollary 13.6 we have that  $(\tilde{L}(a^*))^* = \tilde{L}(a)$  for all  $a \in \mathcal{S}_2(\mathcal{H})$ . Therefore, from Equations (13.8) and (13.9) we obtain

$$\begin{aligned} & -i \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Re(Px_n)| + i \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Im(Px_n)| \\ & - \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Im(Px_n)| - \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Re(Px_n)| \\ = & i \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Re(Px_n)| - i \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Im(Px_n)| \\ & - \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Im(Px_n)| - \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Re(Px_n)|. \end{aligned}$$

Therefore,

$$i \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Re(Px_n)| - i \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Im(Px_n)| = 0. \quad (13.10)$$

By replacing Equation (13.10) in Equation (13.8) we obtain

$$\tilde{L} = - \sum_n \lambda_n |\Re(Px_n)\rangle \langle \Im(Px_n)| - \sum_n \lambda_n |\Im(Px_n)\rangle \langle \Re(Px_n)|.$$

This proves that  $\tilde{L}$  is of form Equation (13.4).  $\square$



Finally we present the

*Proof of Lemma 13.11.* We will start with the forward direction so suppose  $\tilde{L}$  is conditionally completely positive. Let  $e \in \mathcal{H}$  with  $\|e\| = 1$ . Since  $W = \{\sum_{i=1}^k y_i \otimes h'_i : y_i \in \mathcal{S}_2(\mathcal{H}), h'_i \in \mathcal{H}\}$  is dense in  $\mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$ , in order to verify that  $\tilde{L}_{\otimes, e} \geq 0$  it is enough to consider an element  $w = \sum_{i=1}^k y_i \otimes h'_i \in W$  and verify that  $\langle w, \tilde{L}_{\otimes, e} w \rangle_{\otimes} \geq 0$ , where  $\langle \cdot, \cdot \rangle_{\otimes}$  will denote the inner product of  $\mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$ . (The reason that we chose  $h'_i$  to denote a generic element of  $\mathcal{H}$  is because we have used  $h_n$  to denote the orthonormal eigenvectors of  $\rho$  in the statement of Theorem 13.9). We will denote the inner product of  $\mathcal{H}$  by  $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ . Fix  $w = \sum_{i=1}^k y_i \otimes h'_i \in W$  and let  $v = -\sum_{i=1}^k y_i(h'_i)$ . Define  $y_{k+1} = |v\rangle\langle e|$  and  $h'_{k+1} = e$ . Then  $\sum_{i=1}^{k+1} y_i(h'_i) = 0$  and, since  $\tilde{L}$  is conditionally completely positive, we have that

$$\begin{aligned}
0 &\leq \sum_{i,j=1}^{k+1} \langle h'_i, \tilde{L}(y_i^* y_j) h'_j \rangle_{\mathcal{H}} \\
&= \sum_{i,j=1}^{k+1} \sum_{n=1}^{\infty} \left( \text{tr}(y_i^* y_j b_n) \langle h'_i, a_n(h'_j) \rangle_{\mathcal{H}} - \text{tr}(y_i^* y_j a_n) \langle h'_i, b_n(h'_j) \rangle_{\mathcal{H}} \right) \\
&= \sum_{i,j=1}^{k+1} \sum_{n=1}^{\infty} \left( \langle y_i \otimes h'_i, y_j b_n \otimes a_n(h'_j) \rangle_{\otimes} - \langle y_i \otimes h'_i, y_j a_n \otimes b_n(h'_j) \rangle_{\otimes} \right) \\
&= \sum_{i,j=1}^{k+1} \sum_{n=1}^{\infty} \left( \left\langle y_i \otimes h'_i, M_{b_n} \otimes a_n(y_j \otimes h'_j) \right\rangle_{\otimes} - \left\langle y_i \otimes h'_i, M_{a_n} \otimes b_n(y_j \otimes h'_j) \right\rangle_{\otimes} \right) \\
&= \left\langle \sum_{i=1}^{k+1} y_i \otimes h'_i, \sum_{n=1}^{\infty} (M_{b_n} \otimes a_n - M_{a_n} \otimes b_n) \left( \sum_{j=1}^{k+1} y_j \otimes h'_j \right) \right\rangle_{\otimes}.
\end{aligned}$$

Notice that

$$\begin{aligned}
\sum_{i=1}^{k+1} y_i \otimes h'_i &= \sum_{i=1}^k y_i \otimes h'_i + y_{k+1} \otimes h'_{k+1} = w - \sum_{i=1}^k |y_i(h'_i)\rangle\langle e| \otimes e \\
&= w - T_e \left( \sum_{i=1}^k y_i \otimes h'_i \right) = (Id - T_e)(w)
\end{aligned}$$

where  $Id$  denotes the identity operator on  $\mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$ , which finishes the proof of the forward direction.

For the other direction, suppose that  $\tilde{L}_{\otimes, e} \geq 0$  for some  $e \in \mathcal{H}$  with  $\|e\| = 1$ . Let  $k \in \mathbb{N}$ ,  $y_1, \dots, y_k \in \mathcal{S}_2(\mathcal{H})$  and  $h'_1, \dots, h'_k \in \mathcal{H}$  such that  $\sum_{i=1}^k y_i(h'_i) = 0$ . Let

$w = \sum_{i=1}^k y_i \otimes h'_i \in \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$ . Then,

$$0 \leq \langle w, \tilde{L}_{\otimes, e}(w) \rangle_{\otimes} \quad (13.11)$$

$$\begin{aligned} &= \left\langle w, (Id - T_e)^* \left[ \sum_{n \in N} (M_{b_n} \otimes a_n - M_{a_n} \otimes b_n) \right] (Id - T_e)(w) \right\rangle_{\otimes} \\ &= \left\langle (Id - T_e)w, \sum_{n \in N} (M_{b_n} \otimes a_n - M_{a_n} \otimes b_n) (Id - T_e)(w) \right\rangle_{\otimes}. \end{aligned} \quad (13.12)$$

Notice that

$$T_e(w) = T_e \left( \sum_{i=1}^k y_i \otimes h'_i \right) = \left| \sum_{i=1}^k y_i(h'_i) \right\rangle \langle e| \otimes e = |0\rangle \langle e| \otimes e = 0.$$

Hence Inequality (13.11) gives

$$\begin{aligned} 0 &\leq \left\langle Id(w), \sum_{n \in N} (M_{b_n} \otimes a_n - M_{a_n} \otimes b_n) Id(w) \right\rangle_{\otimes} \\ &= \sum_{i,j=1}^k \sum_{n \in N} \left( \langle y_i \otimes h'_i, M_{b_n} \otimes a_n(y_j \otimes h'_j) \rangle_{\otimes} - \langle y_i \otimes h'_i, M_{a_n} \otimes b_n(y_j \otimes h'_j) \rangle_{\otimes} \right) \\ &= \sum_{i,j=1}^k \sum_{n \in N} \left( \langle y_i \otimes h'_i, y_j b_n \otimes a_n(h'_j) \rangle_{\otimes} - \langle y_i \otimes h'_i, y_j a_n \otimes b_n(h'_j) \rangle_{\otimes} \right) \\ &= \sum_{i,j=1}^k \sum_{n \in N} \left( \text{tr}(y_i^* y_j b_n) \langle h'_i, a_n(h'_j) \rangle_{\mathcal{H}} - \text{tr}(y_i^* y_j a_n) \langle h'_i, b_n(h'_j) \rangle_{\mathcal{H}} \right) \\ &= \sum_{i,j=1}^{k+1} \langle h_i, \tilde{L}(y_i^* y_j) h'_j \rangle_{\mathcal{H}}. \end{aligned}$$

Therefore  $\tilde{L}$  is conditionally completely positive. This completes the proof.  $\square$

The proof of Lemma 13.11 reveals the following:

**Remark 13.13.** Let  $\mathcal{A} = \{ \sum_{i=1}^k y_i \otimes h'_i \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{H} : \sum_{i=1}^k y_i(h'_i) = 0 \}$ . Then

- For every  $w = \sum_{i=1}^k y_i \otimes h'_i \in \mathcal{B}(\mathcal{H}) \otimes \mathcal{H}$  there exists  $y_{k+1} \in \mathcal{B}(\mathcal{H})$  and  $h'_{k+1} \in \mathcal{H}$  such that  $\sum_{i=1}^{k+1} y_i \otimes h'_i \in \mathcal{A}$  and  $(Id - T_{h'_{k+1}})(w) = \sum_{i=1}^{k+1} y_i \otimes h'_i$ .
- If a bounded operator  $\tilde{L}$  on  $\mathcal{H}$  has form (13.4) then  $\tilde{L}$  is completely positive if and only if the operator  $\sum_{n=1}^{\infty} \lambda_n (M_{b_n} \otimes a_n - M_{a_n} \otimes b_n) : S_2(\mathcal{H}) \otimes \mathcal{H} \rightarrow S_2(\mathcal{H}) \otimes \mathcal{H}$  is positive.
- For every  $e \in \mathcal{H}$  we have  $\mathcal{A} \subseteq \ker T_e$ .

### 13.2 THE FORM OF $L_{(h_n)}$ WHEN THE RESOLVENT OF $\tilde{L}$ IS COMPACT

Finally, we consider the form of extended generator  $L_{(h_n)}$  when the resolvent  $\tilde{L}$  is compact, by which we mean that  $(\tilde{L} - \lambda)^{-1}$  is compact for some  $\lambda$  in the resolvent set of  $\tilde{L}$  (equivalently all  $\lambda$  in the resolvent set, by the resolvent identity):

**Theorem 13.14.** *Let  $\mathcal{H}$  be a Hilbert space,  $(T_t)_{t \geq 0}$  be a QMS on  $\mathcal{B}(\mathcal{H})$  having an invariant faithful normal state  $\omega_\rho$  for some  $\rho \in \mathcal{S}_1(\mathcal{H})$ , and  $L$  be the generator of  $(T_t)_{t \geq 0}$ . Let  $(\tilde{T}_t)_{t \geq 0}$  be the strongly continuous semigroup of contractions on  $\mathcal{S}_2(\mathcal{H})$  defined in Theorem 12.14 and let  $\tilde{L}$  be its generator. Assume that the generator  $\tilde{L}$  has compact resolvent. Then the following assertions are valid:*

- (a) *There exist complete orthonormal families  $(a_n)_{n \in \mathbb{N}}$  and  $(b_n)_{n \in \mathbb{N}}$  of self-adjoint elements in  $\mathcal{S}_2(\mathcal{H})$  and a sequence of positive scalars  $(\lambda_n)_{n \in \mathbb{N}}$  with  $\lambda_n \rightarrow \infty$  as  $n \rightarrow \infty$  (if  $\mathcal{H}$  is infinite dimensional) such that*

$$\tilde{L} = I + \sum_{n=1}^{\infty} \lambda_n |a_n\rangle\langle b_n| \quad (13.13)$$

*where the sums converge in the SOT (if it is infinite).*

- (b) *By the Spectral Theorem there is an orthonormal basis  $(h_n)_{n \in \mathbb{N}}$  of  $\mathcal{H}$  which consists of eigenvectors of  $\rho$ . Let  $L_{(h_n)}$  denote the generator of  $(T_t)_{t \geq 0}$  with respect to  $(h_n)_{n \in \mathbb{N}}$ . Then*

$$L_{(h_n)} = I + \sum_{n=1}^{\infty} \lambda_n |i_{\rho^{(-1)}}(a_n)\rangle\langle i_\rho(b_n)| \quad (13.14)$$

*where the sum converges in the SOT (if it is infinite). We note that  $|i_{\rho^{(-1)}}(a_n)\rangle\langle i_\rho(b_n)|$  has domain  $\mathcal{B}(\mathcal{H})$  for all  $n$ , since*

$$|i_{\rho^{(-1)}}(a_n)\rangle\langle i_\rho(b_n)|x = \langle i_\rho(b_n), x \rangle i_{\rho^{(-1)}}(a_n) = \langle b_n, i_\rho(x) \rangle i_{\rho^{(-1)}}(a_n)$$

*and  $b_n, i_\rho(x) \in \mathcal{S}_2(\mathcal{H})$ .*

(c) For all  $e \in \mathcal{H}$  with  $\|e\| = 1$  we have that the operator  $\tilde{L}_{\otimes, e} : \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H} \rightarrow \mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$  is positive, where the operator  $\tilde{L}_{\otimes, e}$  is defined by

$$\tilde{L}_{\otimes, e} = (Id + T_e^*) \left( \sum_{n=1}^{\infty} \lambda_n M_{b_n} \otimes a_n \right) (Id + T_e) \quad (13.15)$$

where  $Id$  stands for the identity operator on  $\mathcal{S}_2(\mathcal{H}) \otimes \mathcal{H}$  and the sum converges in the SOT (if it is infinite).

In order to prove Theorem 13.14, we need the following two results:

**Lemma 13.15.** *Let  $\mathcal{H}$  be a separable Hilbert space and  $A$  be an invertible linear operator on  $\mathcal{S}_2(\mathcal{H})$  with dense domain which is closed under adjoints. If  $A$  satisfies  $A(a^*) = (A(a))^*$  for all  $a \in D(A)$ , then  $D(A^\dagger)$  and  $D(A^{-1})$  are closed under adjoints,  $A^\dagger(b^*) = (A^\dagger(b))^*$  for all  $b \in D(A^\dagger)$ , and  $A^{-1}(c^*) = (A^{-1}(c))^*$  for all  $c \in D(A^{-1})$ .*

*Proof.* Let  $a \in D(A)$  and  $b \in D(A^\dagger)$ . Then

$$\begin{aligned} |\langle A(a), b^* \rangle| &= |\langle (A(a^*))^*, b^* \rangle| = |\langle b, A(a^*) \rangle| \\ &= |\langle A^\dagger(b), a^* \rangle| = |\langle a, (A^\dagger(b))^* \rangle| \leq \|a\| \|(A^\dagger(b))^*\|, \end{aligned}$$

and so  $b^* \in D(A^\dagger)$  by definition. As before,

$$\langle a, A^\dagger(b^*) \rangle = \langle A(a), b^* \rangle = \langle a, (A^\dagger(b))^* \rangle,$$

and since  $D(A)$  is dense this implies  $A^\dagger(b^*) = (A^\dagger(b))^*$  for all  $b \in D(A^\dagger)$ . Further, for every  $c \in D(A^{-1})$  there exists an  $a \in D(A)$  such that  $A(a) = c$ . Since  $A$  is star-preserving we have that  $A(a^*) = c^*$ . Then, by definition,  $(A^{-1}(c))^* = a^* = A^{-1}(c^*)$ .  $\square$

**Lemma 13.16.** *Let  $\mathcal{H}$  be a separable Hilbert space and  $A$  be an compact and self-adjoint linear operator on  $\mathcal{S}_2(\mathcal{H})$  which satisfies  $A(a^*) = (A(a))^*$  for all  $a \in \mathcal{S}_2(\mathcal{H})$ . Then*

$$A = \sum_{n=1}^{\infty} \lambda_n |x_n\rangle \langle x_n|$$

with  $(\lambda_n)_{n=1}^\infty \subseteq \mathbb{R}$  and  $(x_n)_{n=1}^\infty$  an orthonormal basis of  $S_2(\mathcal{H})$  consisting of self-adjoint operators.

*Proof.* If  $A$  is compact and self-adjoint, then the Spectral Theorem implies there is an eigensystem decomposition

$$A = \sum_{n=1}^{\infty} \lambda_n |y_n\rangle \langle y_n|,$$

with  $(\lambda_n)_{n=1}^\infty \subseteq \mathbb{R}$  and  $(x_n)_{n=1}^\infty$  an orthonormal basis of  $S_2(\mathcal{H})$ . Because  $A$  is self-adjoint and star-preserving, we have that  $A(y_n) = \lambda_n y_n$  implies  $A(y_n^*) = \lambda_n y_n^*$ . Thus, every eigenspace of  $A$  is self-adjoint. For eigenspace  $E$  of  $A$ , consider the orthonormal basis  $(y_{n_j})_{j=1}^N \subseteq (y_n)_{n=1}^\infty$  of  $E$ . Because  $E$  is self-adjoint, we also have that  $(y_{n_j}^*)_{j=1}^N \subseteq (y_n)_{n=1}^\infty$  is an orthonormal basis of  $E$ . Define self-adjoint operators  $a_j = y_{n_j} + y_{n_j}^*$  and  $a_{N+j} = i(y_{n_j} - y_{n_j}^*)$  for each  $1 \leq j \leq N$  so that  $E = \text{Span}(a_j)_{j=1}^{2N}$ . From  $\langle y_{n_j}, y_{n_k} \rangle = \langle y_{n_j}^*, y_{n_k}^* \rangle = \delta_{jk}$ , straight forward calculation reveals that  $\langle a_j, a_k \rangle$  is real for every  $1 \leq j \leq 2N$ . We follow the Gram-Schmidt process and set  $b_1 = a_1$  and recursively define

$$b_k = a_k - \sum_{j=1}^{k-1} \frac{\langle b_k, a_j \rangle}{\langle b_k, b_j \rangle} b_j$$

to produce a sequence of  $N$  many orthogonal operators which span  $E$  (the remaining  $N$  many operators produced by the Gram-Schmidt process become zero). As a real combination of self-adjoint operators, each  $b_k$  is self-adjoint, and thus can be normalized to a set of self-adjoint orthonormal operators  $(x_j)_{j=1}^N$  which span  $E$ . Replacing  $y_n$  with  $x_n$  in the original eigensystem decomposition for each eigenspace  $E$ , we have

$$A = \sum_{n=1}^{\infty} \lambda_n |x_n\rangle \langle x_n|,$$

as desired. □

Finally, we present the

*Proof of Theorem 13.14.* Since  $\tilde{L}$  generates a strongly continuous semigroup of contractions, we have that  $\lambda \in \rho(\tilde{L})$  for all  $\lambda > 0$  by the Hille-Yosida Generation Theorem

(e.g. Theorem 3.5 of [27]). Further,  $D(\tilde{L})$  is dense in  $S_2(\mathcal{H})$  by Theorem 3.1.16 of [12] and  $\tilde{L}$  is star-preserving by Corollary 13.6, and so  $K := (\tilde{L} - I)^{-1}$  is star-preserving by Lemma 13.15 as the inverse of a star-preserving map with dense domain. Because  $\tilde{L}$  has compact resolvent by assumption, we have that  $K$  is furthermore compact. Thus,  $K^\dagger K$  is compact, self-adjoint, and star-preserving, and so Lemma 13.16 implies

$$K^\dagger K = \sum_{n=1}^{\infty} \sigma_n^2 |v_n\rangle\langle v_n|,$$

where  $\{\sigma_n^2\}_{n \in \mathbb{N}}$  are the nonzero eigenvalues of  $K^\dagger K$  corresponding to the system  $\{v_n\}_{n \in \mathbb{N}}$  of self-adjoint orthonormal eigenoperators. This notation is chosen so that, following Section 2.2 of [28], the singular value expansion of  $K$  can be written

$$K = \sum_{n=1}^{\infty} \sigma_n |u_n\rangle\langle v_n|,$$

where  $\{u_n\}_{n \in \mathbb{N}}$  are self-adjoint orthonormal eigenoperators of  $KK^\dagger$  given by the relation  $\sigma_n u_n := K v_n$ . By Theorem 2.8 of [28] we have that

$$\tilde{L} - I = K^{(-1)} = \sum_{n=1}^{\infty} \frac{1}{\sigma_n} |v_n\rangle\langle u_n|,$$

and hence

$$\tilde{L} = I + \sum_{n=1}^{\infty} \frac{1}{\sigma_n} |v_n\rangle\langle u_n|,$$

proving (13.13). Equation (13.14) follows from (13.13) and (12.8). Part (c) follows similarly as the proof of Lemma 13.11, with the note that  $I + A$  is conditionally completely positive if and only if  $A$  is (as is easily verified).  $\square$

## CHAPTER 14

### CONCLUSION TO PART II

We began this Part by considering several constructions arising from faithful, positive, normal functionals, such as how every such functional on  $\mathcal{B}(\mathcal{H})$  induces a contraction from  $\mathcal{B}(\mathcal{H})$  to  $\mathcal{S}_2(\mathcal{H})$ . This allowed us to prove in Theorem 11.5 that bounded linear Schwarz maps on  $\mathcal{B}(\mathcal{H})$  which have a subinvariant faithful positive functional naturally induce contractions on  $\mathcal{S}_2(\mathcal{H})$ . In Section 11.2 we considered alternate GNS construction which can be used to induce a contraction from a Schwarz map which has a subinvariant faithful state acting on a general  $C^*$ -algebra. We remarked that while both constructions induce a contraction on a Hilbert space using a Schwarz map on a  $C^*$ -algebra, the former construction works only for the  $C^*$ -algebra  $\mathcal{B}(\mathcal{H})$  but is more symmetric and always induces a contraction on the Hilbert space  $\mathcal{S}_2(\mathcal{H})$ , whereas the latter works on general  $C^*$  algebras but induces a contraction on a Hilbert space which depends on the subinvariant functional of the original map.

In Chapter 12 we recalled the basic notions of semigroup generators and their domains. In particular, the domain of a generator is defined via an appropriate limit which may not always exist. In Section 12.1 we introduced the notion of an extended generator using weaker limits, and the so extended generator is defined on a larger domain. True to its name, we showed that the extended generator agrees with the usual generator on all finite subspaces. This new definition was useful in stating one of the main theorems of this work, Theorem 12.14, which states that every semigroup of Schwarz maps on  $\mathcal{B}(\mathcal{H})$  with a subinvariant faithful state induces a semigroup of contractions on  $\mathcal{S}_2(\mathcal{H})$ . Moreover, if the original semigroup is  $w^*$ -continuous then the

induced semigroup is strongly continuous. The domains and actions of the generator, the extended generator, and the generator of the semigroup induced on  $\mathcal{S}_2(\mathcal{H})$  are related explicitly, and in particular the image of the domain of the generator under natural contraction is contained in the domain of the induced generator, whereas the preimage of the domain of the induced generator under that natural contraction is contained in the domain of the extended generator. Because the induced semigroup acts on a Hilbert space, in Section 12.2 we were able to apply the dilation theory of Foias and Sz.-Nagy to obtain a minimal semigroup of unitaries on a larger Hilbert space. From there we applied Stone's Theorem to give a description of its generator in terms of the extended generator of the original semigroup.

In Chapter 13 we applied Theorem 12.14 in the study of Quantum Markov semigroups (QMSs), which are dual to the QDSs examined in the finite dimensional case of Part I. The exact form of a QMS generator is known that if the generator is bounded (see [42] and [53]), so we were particularly interested in the unbounded case. To this end, we show that many properties of a QMS generator are inherited by the generator of the contraction semigroup it induces on  $\mathcal{S}_2(\mathcal{H})$ , such as conditional complete positivity (Corollary 13.6). We then examined two particular instances of compactness to provide a form of the induced generator and the extended generator, because it agrees with the original generator on all finite subspaces, in the unbounded case: First, in Theorem 13.9 we assumed that the generator of the minimal semigroup of unitary dilations of the induced semigroup of contractions is compact. This assumption allowed for an explicit eigensystem decomposition of the compact generator, which was traced back to a form for the extended generator. In Theorem 13.14, we described the generator of the QMS under the assumption that the generator of the semigroup induced on  $\mathcal{S}_2(\mathcal{H})$  has compact resolvent. In this case, compactness of the resolvent operator allows for an explicit singular value decomposition, which can then be traced back to a form for the extended generator using Moore-Penrose inverses. This we view



as a weaker assumption, since in particular it does not imply the induced semigroup generator is bounded (as the former case did).

## BIBLIOGRAPHY

- [1] Luigi Accardi, Franco Fagnola, and Skander Hachicha. “Generic  $q$ -Markov semigroups and speed of convergence of  $q$ -algorithms”. In: *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 9.4 (2006), pp. 567–594. ISSN: 0219-0257. DOI: 10.1142/S0219025706002548. URL: <https://doi.org/10.1142/S0219025706002548>.
- [2] Luigi Accardi and Sergei Kozyrev. “Lectures on quantum interacting particle systems”. In: *Quantum interacting particle systems (Trento, 2000)*. Vol. 14. QP-PQ: Quantum Probab. White Noise Anal. World Sci. Publ., River Edge, NJ, 2002, pp. 1–195. DOI: 10.1142/9789812776853\_0001. URL: [https://doi.org/10.1142/9789812776853\\_0001](https://doi.org/10.1142/9789812776853_0001).
- [3] Luigi Accardi, Sergei Kozyrev, and Alexander Pechen. “Coherent quantum control of  $\Lambda$ -atoms through the stochastic limit”. In: *Quantum Information and Computing*. Ed. by L. Accardi, M. Ohya, and N. Watanabe. Mar. 2006, p. 1. DOI: 10.1142/9789812774491\_0001.
- [4] Sabina Alazzawi and Bernhard Baumgartner. “Generalized Kraus operators and generators of quantum dynamical semigroups”. In: *Rev. Math. Phys.* 27.7 (2015), pp. 1550016, 19. ISSN: 0129-055X. DOI: 10.1142/S0129055X15500166. URL: <https://doi.org/10.1142/S0129055X15500166>.
- [5] Robert Alicki. “Invitation to Quantum Dynamical Semigroups”. In: *Dynamics of Dissipation*. Ed. by Piotr Garbaczewski and Robert Olkiewicz. Springer Berlin Heidelberg, 2002, pp. 239–264. ISBN: 978-3-540-46122-7. DOI: 10.1007/3-540-46122-1\_10. URL: [https://doi.org/10.1007/3-540-46122-1\\_10](https://doi.org/10.1007/3-540-46122-1_10).
- [6] Robert Alicki and Karl Lendi. *Quantum dynamical semigroups and applications*. Second. Vol. 717. Lecture Notes in Physics. Springer, Berlin, 2007, pp. xii +128. ISBN: 978-3-540-70860-5; 3-540-70860-X.
- [7] George Androulakis and Matthew Ziemke. “Generators of quantum Markov semigroups”. In: *J. Math. Phys.* 56.8 (2015), pp. 083512, 16. ISSN: 0022-2488. DOI: 10.1063/1.4928936. URL: <https://doi.org/10.1063/1.4928936>.
- [8] Stephane Attal. “Lecture 3: Operator Semigroups”. In: *Lectures in Quantum Noise Theory*. Retrieved from <http://math.univ-lyon1.fr/~attal/>.

- [9] Bernhard Baumgartner and Heide Narnhofer. “Analysis of quantum semigroups with GKS-Lindblad generators. II. General”. In: *J. Phys. A* 41.39 (2008), pp. 395 303, 26. ISSN: 1751-8113. DOI: 10.1088/1751-8113/41/39/395303. URL: <https://doi.org/10.1088/1751-8113/41/39/395303>.
- [10] Reinhold A. Bertlmann and Philipp Krammer. “Bloch vectors for qudits”. In: *J. Phys. A* 41.23 (2008), pp. 235303, 21. ISSN: 1751-8113. DOI: 10.1088/1751-8113/41/23/235303. URL: <https://doi.org/10.1088/1751-8113/41/23/235303>.
- [11] S. J. Bhatt. “Stinespring representability and Kadison’s Schwarz inequality in non-unital Banach star algebras and applications”. In: *Proc. Indian Acad. Sci. Math. Sci.* 108.3 (1998), pp. 283–303. ISSN: 0253-4142. DOI: 10.1007/BF02844483. URL: <https://doi.org/10.1007/BF02844483>.
- [12] Ola Bratteli and Derek W. Robinson. *Operator algebras and quantum statistical mechanics. 1*. Second. Texts and Monographs in Physics.  $C^*$ - and  $W^*$ -algebras, symmetry groups, decomposition of states. Springer-Verlag, New York, 1987, pp. xiv+505. ISBN: 0-387-17093-6. DOI: 10.1007/978-3-662-02520-8. URL: <https://doi.org/10.1007/978-3-662-02520-8>.
- [13] Raffaella Carbone and Franco Fagnola. “Exponential convergence in  $L^2$  of quantum Markov semigroups on  $B(h)$ ”. In: *Mat. Zametki* 68.4 (2000), pp. 523–538. ISSN: 0025-567X. DOI: 10.1007/BF02676724. URL: <https://doi.org/10.1007/BF02676724>.
- [14] Raffaella Carbone, Emanuela Sasso, and Veronica Umanità. “Decoherence for positive semigroups on  $M_2(\mathbb{C})$ ”. In: *J. Math. Phys.* 52.3 (2011), pp. 032202, 17. ISSN: 0022-2488. DOI: 10.1063/1.3560478. URL: <https://doi.org/10.1063/1.3560478>.
- [15] Raffaella Carbone, Emanuela Sasso, and Veronica Umanità. “Environment induced decoherence for Markovian evolutions”. In: *J. Math. Phys.* 56.9 (2015), pp. 092704, 22. ISSN: 0022-2488. DOI: 10.1063/1.4931479. URL: <https://doi.org/10.1063/1.4931479>.
- [16] Raffaella Carbone, Emanuela Sasso, and Veronica Umanità. “On the asymptotic behavior of generic quantum Markov semigroups”. In: *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 17.1 (2014), pp. 1450001, 18. ISSN: 0219-0257. DOI: 10.1142/S0219025714500015. URL: <https://doi.org/10.1142/S0219025714500015>.
- [17] Raffaella Carbone, Emanuela Sasso, and Veronica Umanità. “Structure of generic quantum Markov semigroup”. In: *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 20.2 (2017), pp. 1750012, 19. ISSN: 0219-0257. URL: <https://doi.org/10.1142/S0219025717500126>.

- [18] Alexandr M. Chebotarev and Franco Fagnola. “Sufficient conditions for conservativity of minimal quantum dynamical semigroups”. In: *J. Funct. Anal.* 153.2 (1998), pp. 382–404. ISSN: 0022-1236. DOI: 10.1006/jfan.1997.3189. URL: <https://doi.org/10.1006/jfan.1997.3189>.
- [19] Alexandr M. Chebotarev and Franco Fagnola. “Sufficient conditions for conservativity of quantum dynamical semigroups”. In: *J. Funct. Anal.* 118.1 (1993), pp. 131–153. ISSN: 0022-1236. DOI: 10.1006/jfan.1993.1140. URL: <https://doi.org/10.1006/jfan.1993.1140>.
- [20] Man Duen Choi. “A Schwarz inequality for positive linear maps on  $C^*$ -algebras”. In: *Illinois J. Math.* 18 (1974), pp. 565–574. ISSN: 0019-2082. URL: <http://projecteuclid.org/euclid.ijm/1256051007>.
- [21] Erik Christensen and David E. Evans. “Cohomology of operator algebras and quantum dynamical semigroups”. In: *J. London Math. Soc. (2)* 20.2 (1979), pp. 358–368. ISSN: 0024-6107. DOI: 10.1112/jlms/s2-20.2.358. URL: <https://doi.org/10.1112/jlms/s2-20.2.358>.
- [22] Fan R. K. Chung. *Spectral graph theory*. Vol. 92. CBMS Regional Conference Series in Mathematics. Published for the Conference Board of the Mathematical Sciences, Washington, DC; by the American Mathematical Society, Providence, RI, 1997, pp. xii+207. ISBN: 0-8218-0315-8.
- [23] E. B. Davies. “Generators of dynamical semigroups”. In: *J. Funct. Anal.* 34.3 (1979), pp. 421–432. ISSN: 0022-1236. DOI: 10.1016/0022-1236(79)90085-5. URL: [https://doi.org/10.1016/0022-1236\(79\)90085-5](https://doi.org/10.1016/0022-1236(79)90085-5).
- [24] E. B. Davies. “Quantum dynamical semigroups and the neutron diffusion equation”. In: *Rep. Mathematical Phys.* 11.2 (1977), pp. 169–188. ISSN: 0034-4877. DOI: 10.1016/0034-4877(77)90059-3. URL: [https://doi.org/10.1016/0034-4877\(77\)90059-3](https://doi.org/10.1016/0034-4877(77)90059-3).
- [25] Ameer Dhahri, Franco Fagnola, and Rolando Rebolledo. “The decoherence-free subalgebra of a quantum Markov semigroup with unbounded generator”. In: *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 13.3 (2010), pp. 413–433. ISSN: 0219-0257. DOI: 10.1142/S0219025710004176. URL: <https://doi.org/10.1142/S0219025710004176>.
- [26] Frederik vom Ende and Gunther Dirr. *Unitary Dilations of Discrete-Time Quantum Dynamical Semigroups*. 2018. eprint: arXiv:1804.00918.
- [27] Klaus-Jochen Engel and Rainer Nagel. *One-parameter semigroups for linear evolution equations*. Vol. 194. Graduate Texts in Mathematics. With contributions by S. Brendle, M. Campiti, T. Hahn, G. Metafune, G. Nickel, D. Pallara,

- C. Perazzoli, A. Rhandi, S. Romanelli and R. Schnaubelt. Springer-Verlag, New York, 2000, pp. xxii+586. ISBN: 0-387-98463-1.
- [28] Heinz W. Engl, Martin Hanke, and Andreas Neubauer. *Regularization of inverse problems*. Vol. 375. Mathematics and its Applications. Kluwer Academic Publishers Group, Dordrecht, 1996, pp. viii+321. ISBN: 0-7923-4157-0.
  - [29] D. E. Evans and John T. Lewis. “Dilations of irreversible evolutions in algebraic quantum theory”. In: *Comm. Dublin Inst. Adv. Studies Ser. A* 24 (1977), pp. v+104.
  - [30] David E. Evans. “Conditionally completely positive maps on operator algebras”. In: *Quart. J. Math. Oxford Ser. (2)* 28.111 (1977), pp. 271–283. ISSN: 0033-5606. DOI: 10.1093/qmath/28.3.271. URL: <https://doi.org/10.1093/qmath/28.3.271>.
  - [31] David E. Evans. “Irreducible quantum dynamical semigroups”. In: *Comm. Math. Phys.* 54.3 (1977), pp. 293–297. ISSN: 0010-3616. URL: <http://projecteuclid.org/euclid.cmp/1103900873>.
  - [32] David E. Evans and John T. Lewis. “Dilations of dynamical semi-groups”. In: *Comm. Math. Phys.* 50.3 (1976), pp. 219–227. ISSN: 0010-3616. URL: <http://projecteuclid.org/euclid.cmp/1103900230>.
  - [33] Franco Fagnola. “Quantum Markov semigroups and quantum flows”. In: *Proyecciones* 18.3 (1999), p. 144. ISSN: 0716-0917. DOI: 10.22199/s07160917.1999.0003.00006. URL: <https://doi.org/10.22199/s07160917.1999.0003.00006>.
  - [34] Franco Fagnola and Rolando Rebolledo. “Algebraic conditions for convergence of a quantum Markov semigroup to a steady state”. In: *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 11.3 (2008), pp. 467–474. ISSN: 0219-0257. DOI: 10.1142/S0219025708003142. URL: [doi.org/10.1142/S0219025708003142](https://doi.org/10.1142/S0219025708003142).
  - [35] Franco Fagnola and Rolando Rebolledo. “On the existence of stationary states for quantum dynamical semigroups”. In: *J. Math. Phys.* 42.3 (2001), pp. 1296–1308. ISSN: 0022-2488. DOI: 10.1063/1.1340870. URL: <https://doi.org/10.1063/1.1340870>.
  - [36] Franco Fagnola and Rolando Rebolledo. “Quantum Markov semigroups and their stationary states”. In: *Stochastic analysis and mathematical physics II*. Trends Math. Birkhäuser, Basel, 2003, pp. 77–128.
  - [37] Franco Fagnola and Rolando Rebolledo. “The approach to equilibrium of a class of quantum dynamical semigroups”. In: *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 1.4 (1998), pp. 561–572. ISSN: 0219-0257. DOI: 10.1142/S0219025798000302. URL: <https://doi.org/10.1142/S0219025798000302>.

- [38] Alberto Frigerio. “Quantum dynamical semigroups and approach to equilibrium”. In: *Lett. Math. Phys.* 2.2 (1977/78), pp. 79–87. ISSN: 0377-9017. DOI: 10.1007/BF00398571. URL: <https://doi.org/10.1007/BF00398571>.
- [39] Alberto Frigerio. “Stationary states of quantum dynamical semigroups”. In: *Comm. Math. Phys.* 63.3 (1978), pp. 269–276. ISSN: 0010-3616. URL: <http://projecteuclid.org/euclid.cmp/1103904568>.
- [40] Martin Gärttner. “Many-Body Effects in Rydberg Gases: Coherent Dynamics of Strongly Interacting Two-Level Atoms and Nonlinear Optical Response of a Rydberg Gas in EIT Configuration”. In: *PhD Thesis, Universität Library Heidelberg* (2013). DOI: 10.11588/heidok.00015992. URL: <http://www.ub.uni-heidelberg.de/archiv/15992>.
- [41] Adam Glos, Jarosław Adam Mischczak, and Mateusz Ostaszewski. “Limiting properties of stochastic quantum walks on directed graphs”. In: *J. Phys. A* 51.3 (2018), pp. 035304, 16. ISSN: 1751-8113. DOI: 10.1088/1751-8121/aa9a4a. URL: <https://doi.org/10.1088/1751-8121/aa9a4a>.
- [42] Vittorio Gorini, Andrzej Kossakowski, and E. C. G. Sudarshan. “Completely positive dynamical semigroups of  $N$ -level systems”. In: *J. Mathematical Phys.* 17.5 (1976), pp. 821–825. ISSN: 0022-2488. DOI: 10.1063/1.522979. URL: <https://doi.org/10.1063/1.522979>.
- [43] Timothy F. Havel. “Robust procedures for converting among Lindblad, Kraus and matrix representations of quantum dynamical semigroups”. In: *J. Math. Phys.* 44.2 (2003), pp. 534–557. ISSN: 0022-2488. DOI: 10.1063/1.1518555. URL: <https://doi.org/10.1063/1.1518555>.
- [44] Christoph Hofmann. “Emergence of correlations in strongly interacting ultra-cold Rydberg gases”. In: *PhD Thesis, Universität Library Heidelberg* (2013). DOI: 10.11588/heidok.00014629. URL: <http://www.ub.uni-heidelberg.de/archiv/14629>.
- [45] Alexander S. Holevo. “On conservativity of covariant dynamical semigroups”. In: *Proceedings of the XXV Symposium on Mathematical Physics (Toruń, 1992)*. Vol. 33. 1-2. 1993, pp. 95–110. DOI: 10.1016/0034-4877(93)90044-F. URL: [https://doi.org/10.1016/0034-4877\(93\)90044-F](https://doi.org/10.1016/0034-4877(93)90044-F).
- [46] Alexander S. Holevo. “On the structure of covariant dynamical semigroups”. In: *J. Funct. Anal.* 131.2 (1995), pp. 255–278. ISSN: 0022-1236. DOI: 10.1006/jfan.1995.1089. URL: <https://doi.org/10.1006/jfan.1995.1089>.
- [47] Alexander S. Holevo. *Statistical structure of quantum theory*. Vol. 67. Lecture Notes in Physics. Monographs. Springer-Verlag, Berlin, 2001, pp. x+159. ISBN:

- 3-540-42082-7. DOI: 10.1007/3-540-44998-1. URL: <https://doi.org/10.1007/3-540-44998-1>.
- [48] Richard V. Kadison. “A generalized Schwarz inequality and algebraic invariants for operator algebras”. In: *Ann. of Math. (2)* 56 (1952), pp. 494–503. ISSN: 0003-486X. DOI: 10.2307/1969657. URL: <https://doi.org/10.2307/1969657>.
  - [49] Richard V. Kadison and John R. Ringrose. *Fundamentals of the theory of operator algebras. Vol. I*. Vol. 15. Graduate Studies in Mathematics. Elementary theory, Reprint of the 1983 original. American Mathematical Society, Providence, RI, 1997, pp. xvi+398. ISBN: 0-8218-0819-2.
  - [50] Andrzej Kossakowski. “On quantum statistical mechanics of non-Hamiltonian systems”. In: *Rep. Mathematical Phys.* 3.4 (1972), pp. 247–274. ISSN: 0034-4877. DOI: 10.1016/0034-4877(72)90010-9. URL: [https://doi.org/10.1016/0034-4877\(72\)90010-9](https://doi.org/10.1016/0034-4877(72)90010-9).
  - [51] Andrzej Kossakowski et al. “Quantum detailed balance and KMS condition”. In: *Comm. Math. Phys.* 57.2 (1977), pp. 97–110. ISSN: 0010-3616. URL: <http://projecteuclid.org/euclid.cmp/1103901281>.
  - [52] Daniel Lidar. *Lecture Notes on the Theory of Open Quantum Systems*. Feb. 2019. URL: <https://arxiv.org/abs/1902.00967>.
  - [53] Göran Lindblad. “On the generators of quantum dynamical semigroups”. In: *Comm. Math. Phys.* 48.2 (1976), pp. 119–130. ISSN: 0010-3616. URL: <http://projecteuclid.org/euclid.cmp/1103899849>.
  - [54] Chaobin Liu and Radhakrishnan Balu. “Steady states of continuous-time open quantum walks”. In: *Quantum Inf. Process.* 16.7 (2017), Art. 173, 11. ISSN: 1570-0755. DOI: 10.1007/s11128-017-1625-8. URL: <https://doi.org/10.1007/s11128-017-1625-8>.
  - [55] Inomzhon Mirzaev and Jeremy Gunawardena. “Laplacian dynamics on general graphs”. In: *Bull. Math. Biol.* 75.11 (2013), pp. 2118–2149. ISSN: 0092-8240. DOI: 10.1007/s11538-013-9884-8. URL: <https://doi.org/10.1007/s11538-013-9884-8>.
  - [56] Masanori Ohya and Dénes Petz. *Quantum entropy and its use*. Texts and Monographs in Physics. Springer-Verlag, Berlin, 1993, pp. viii+335. ISBN: 3-540-54881-5. DOI: 10.1007/978-3-642-57997-4. URL: <https://doi.org/10.1007/978-3-642-57997-4>.
  - [57] Vern Paulsen. *Completely bounded maps and operator algebras*. Vol. 78. Cambridge Studies in Advanced Mathematics. Cambridge University Press, Cambridge, 2002, pp. xii+300. ISBN: 0-521-81669-6.

- [58] Clément Pellegrini. “Continuous time open quantum random walks and non-Markovian Lindblad master equations”. In: *J. Stat. Phys.* 154.3 (2014), pp. 838–865. ISSN: 0022-4715. DOI: 10.1007/s10955-013-0910-x. URL: <https://doi.org/10.1007/s10955-013-0910-x>.
- [59] David Pérez-García et al. “Contractivity of positive and trace-preserving maps under  $L_p$  norms”. In: *J. Math. Phys.* 47.8 (2006), pp. 083506, 5. ISSN: 0022-2488. DOI: 10.1063/1.2218675. URL: <https://doi.org/10.1063/1.2218675>.
- [60] Jonathan Pritchard. *Cooperative Optical Non-Linearity in a Blockaded Rydberg Ensemble*. Springer-Verlag Berlin Heidelberg, 2012, pp. xiv+158.
- [61] Inken Siemon, Alexander S. Holevo, and Reinhard F. Werner. “Unbounded generators of dynamical semigroups”. In: *Open Syst. Inf. Dyn.* 24.4 (2017), pp. 1740015, 24. ISSN: 1230-1612. DOI: 10.1142/S1230161217400157. URL: <https://doi.org/10.1142/S1230161217400157>.
- [62] Ilya Sinayskiy and Francesco Petruccione. “Microscopic derivation of open quantum walks”. In: *Phys. Rev. A* (3) 92.3 (2015), pp. 032105, 11. ISSN: 1050-2947. DOI: 10.1103/PhysRevA.92.032105. URL: <https://doi.org/10.1103/PhysRevA.92.032105>.
- [63] Kalyan B. Sinha. “Quantum dynamical semigroups”. In: *Mathematical results in quantum mechanics*. Vol. 70. Oper. Theory Adv. Appl. Birkhäuser, Basel, 1994, pp. 161–169. DOI: 10.1007/978-3-0348-8545-4\_22. URL: [https://doi.org/10.1007/978-3-0348-8545-4\\_22](https://doi.org/10.1007/978-3-0348-8545-4_22).
- [64] Katarzyna Siudzińska. “Two definitions of the Gell-Mann channels – a comparative analysis”. In: *Rep. Math. Phys.* 81.3 (2018), pp. 321–345. ISSN: 0034-4877. DOI: 10.1016/S0034-4877(18)30051-X. URL: [https://doi.org/10.1016/S0034-4877\(18\)30051-X](https://doi.org/10.1016/S0034-4877(18)30051-X).
- [65] Erling Størmer. “Positive linear maps of operator algebras”. In: *Acta Math.* 110 (1963), pp. 233–278. ISSN: 0001-5962. DOI: 10.1007/BF02391860. URL: <https://doi.org/10.1007/BF02391860>.
- [66] Erling Størmer. *Positive linear maps of operator algebras*. Springer Monographs in Mathematics. Springer, Heidelberg, 2013, pp. viii+134. ISBN: 978-3-642-34368-1; 978-3-642-34369-8. DOI: 10.1007/978-3-642-34369-8. URL: <https://doi.org/10.1007/978-3-642-34369-8>.
- [67] Béla Sz.-Nagy and Ciprian Foiaş. *Harmonic analysis of operators on Hilbert space*. Translated from the French and revised. North-Holland Publishing Co., Amsterdam-London; American Elsevier Publishing Co., Inc., New York; Akadémiai Kiadó, Budapest, 1970, pp. xiii+389.



- [68] William T. Tutte. “The dissection of equilateral triangles into equilateral triangles”. In: *Proc. Cambridge Philos. Soc.* 44 (1948), pp. 463–482.
- [69] Veronica Umanità. “Classification and decomposition of quantum Markov semi-groups”. In: *Probab. Theory Related Fields* 134.4 (2006), pp. 603–623. ISSN: 0178-8051. DOI: 10.1007/s00440-005-0450-7. URL: <https://doi.org/10.1007/s00440-005-0450-7>.
- [70] James D. Whitfield, César A. Rodríguez-Rosario, and Alán Aspuru-Guzik. “Quantum stochastic walks: A generalization of classical random walks and quantum walks”. In: *Phys. Rev. A* 81 (2 2010), p. 022323. DOI: <https://10.1103/PhysRevA.81.022323>. URL: <https://link.aps.org/doi/10.1103/PhysRevA.81.022323>.
- [71] Chai Wah Wu. “Algebraic connectivity of directed graphs”. In: *Linear and Multilinear Algebra* 53.3 (2005), pp. 203–223. DOI: 10.1080/03081080500054810. eprint: <https://doi.org/10.1080/03081080500054810>. URL: <https://doi.org/10.1080/03081080500054810>.