Spring 2019

On the Characteristic Polynomial of a Hypergraph

Gregory J. Clark

On the Characteristic Polynomial of a Hypergraph

by

Gregory J. Clark

Bachelor of Science
Westminster College, 2014

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2019

Accepted by:

Joshua Cooper, Major Professor

Éva Czabarka, Committee Member

Stephen Dilworth, Committee Member

Linyuan Lu, Committee Member

Stephen Fenner, Committee Member

Cheryl L. Addy, Vice Provost and Dean of the Graduate School

Dedication

To my parents.

Acknowledgments

There are several people I would like to thank for their role in shaping this document. I would first like to thank my advisor, Joshua Cooper, for presenting me with this problem and using his expertise to direct this research. His guidance has proven to be invaluable and it was a distinct honor to be his pupil. Through a joint effort we were able to produce the research found within this dissertation. There are a number of other people who are indirectly responsible for the creation of this manuscript that I would like to acknowledge.

I would first like to thank my family, both those by blood and those of choice, who have supported me in every venture. In particular, I have the deepest gratitude to my parents, Jeffrey and Shari Clark, for their undeniable role in my personal and academic development. Their ceaseless love and support have enabled me to chase my every dream without looking back.

There are several faculty members who deserve a special mention for their mentorship over the years. David Offner, for presenting me with my first research problem; Pamela Richardson, for teaching me "the classics"; Sean Yee, for helping me develop as an educator; and Éva Czabarka, for her spot-on career advice and world-famous cooking.

As I begin the next chapter of my life, I would like to thank Felipe Thomaz for recognizing the potential of this work and providing a future direction to my research. Finally, I would like to thank Jessie Thomaz for her hospitality and much needed reminders that I should get back to work.

Abstract

We consider the computation of the adjacency characteristic polynomial of a uniform hypergraph. Computing the aforementioned polynomial is intractable, in general; however, we present two mechanisms for computing partial information about the spectrum of a hypergraph as well as a methodology (and in particular, an algorithm) for combining this information to determine complete information about said spectrum. The first mechanism is a generalization of the Harary-Sachs Theorem for hypergraphs which yields an explicit formula for each coefficient of the characteristic polynomial of a hypergraph as a weighted sum over a special family of its subgraphs. The second is a mechanism to obtain the eigenvalues of a hypergraph in terms of certain induced subgraphs. We further provide an efficient and numerically stable algorithm which combines this information, the set-spectrum of a hypergraph and its leading coefficients, to determine the multi-set spectrum of the hypergraph. We demonstrate our algorithm by computing the characteristic polynomial of numerous hypergraphs which could not be computed using previous methods.

Table of Contents

List of Tables

List of Figures

Chapter 1

Introduction

Spectral graph theory is a well-studied topic which seeks to relate the structure of a graph with its spectrum, and vice versa. Results from spectral graph theory have found application in combinatorics, computer science, and the social sciences. Most notably, in questions regarding the analysis of networks (a famous example being Google's PageRank algorithm). In recent years, there has been increasing interest from both academics and practitioners to analyze non-binary interactions between agents in a network. Such multi-relational data can be modeled with a hypergraph using higher dimensional edges. It is not obvious *a priori* what definition of the spectrum of a hypergraph correctly analogizes the spectrum of a graph. There have been various approaches to establish such a definition and we summarize several of these approaches below.

1. **Matrices and linear spectral theory.** A natural approach is to encode the incidence structure of a hypergraph into a matrix and relate the eigenvalues thereof back to the structure of the hypergraph. This approach has been widely used in applied sciences (e.g., social networks, reaction and metabolic networks, protein complex networks, etc. [16]). A benefit of this approach is that one can immediately apply linear algebra to study hypergraphs, but considerable loss of information about the structure may be incurred when shoehorning the hypergraph into a two-dimensional array.

2. **Hypermatrices and low-rank decompositions.** One can define a hypermatrix to be a higher dimensional array which maintains desirable properties of matrices. In particular, one might seek to analogize the property that the array can be decomposed into rank one hypermatrices (i.e., outer products of vectors with themselves) as in the classical Spectral Theorem from linear algebra. Important contributions to this approach include a generalization of the expander mixing lemma [9] and the considerable literature on rank-1 decomposition of tensors (e.g., [18])

3. **Hypermatrices and polynomial maps.** Similar to the previous approach one considers the entries of a hypermatrix as coefficients of multilinear forms. In the case when the underlying hypermatrix is *symmetric* (i.e., the entries are invariant under permutation of the indices) these multilinear forms define homogeneous polynomials. The study of symmetric tensors has led to a useful definition of the spectrum of a hypergraph as the roots of a characteristic polynomial which is the resultant of a certain polynomial system whose coefficients are drawn from the adjacency hypermatrix, as described in the following section.

We address spectral hypergraph theory from the perspective of hypermatrices and polynomial maps. While this approach is admittedly more complex than its two predecessors, its merit is that it retains all information of the underlying system. The cost of this faithful analysis is paid for in computational complexity: the characteristic polynomial of a hypergraph is the *resultant* of a system of homogeneous multi-linear equations. The resultant is an ubiquitous tool in mathematics and has been a concept of intense study. Despite this, computing the resultant (and thus, the characteristic polynomial of a hypergraph) is a NP-hard in general. In this dissertation we provide a new approach for computing the characteristic polynomial of a hypergraph as well as a new paradigm for analyzing multi-dimensional networks.

We begin by providing an analogue of the Harary-Sachs Theorem which yields an explicit combinatorial formula for the codegree-d coefficient of a uniform hypergraph which is a function of d and the hypergraph's structure. This formula allows us to compute partial information about the characteristic polynomial of a hypergraph which is novel as computing the entire polynomial can be intractable. This motivates a numerically stable algorithm which allows us to determine the multiplicity of each eigenvalue given the set spectrum of a hypergraph and an appropriate number of leading coefficients (i.e., an analogue of the Newton Identities). Finally, we prove a remarkable property of the spectrum of a hypergraph: the eigenvalues of a hypergraph can be written as a union of eigenvalues arising from its induced subgraphs. This result is unique to spectral hypergraph theory as it is not true for graphs (c.f. Cauchy Interlacing Theorem). We can make further use of this result by appealing to the Lu-Man Method to determine the set spectrum of a hypergrpah. It is our hope that this dissertation will provide researchers with new tools for analyzing the set and multi-set spectrum of a hypergraph.

The paper is arranged as follows. In Chapter 1 we present the necessary background concerning the normalized adjacency spectrum of a uniform hypergraph. We then prove an analogue of the Harary-Sachs Theorem in Chapter 2. Using this theorem we provide an explicit combinatorial formula for low-codegree coefficients of the charactersitic polynomial of a hypergraph in Chapter 3. In Chapter 4 we consider the computation of the characteristic polynomial given its low-codegree coefficients. In doing so we motivate the computation of the set spectrum of a hypergraph which we present in Chapter 5. In Chapter 6 we provide several examples of this computation as well as examples which highlight the current limits of this approach. We leave the reader with open problems in Chapter 7.

Chapter 2

Preliminaries

Here we present requisite background maintaining much of the notation of [12]. A *(cubical) hypermatrix* $\mathcal{A}$ over a set $\mathbb{S}$ of *dimension* n and *order* k is a collection of n^k elements $a_{i_1 i_2 \ldots i_k} \in \mathbb{S}$ where $i_j \in [n]$. A hypermatrix is said to be *symmetric* if entries with identical multisets of indices are the same. That is, $\mathcal{A}$ is symmetric if $a_{i_1 i_2 \ldots i_k} = a_{i_{\sigma(1)} i_{\sigma(2)} \ldots i_{\sigma(k)}}$ for all permutations σ of $[k]$. An order k dimension n symmetric hypermatrix $\mathcal{A}$ uniquely defines a homogeneous degree k polynomial in n variables (a.k.a. a "k-form") by

$$F_{\mathcal{A}}(\mathbf{x}) = \sum_{i_1, i_2, \ldots, i_k = 1}^{n} a_{i_1 i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k}.$$

If we write $\mathbf{x}^{\otimes r}$ for the order r dimension n hypermatrix with $i_1, i_2, \ldots, i_k$ entry $x_{1_1} x_{i_2} \ldots x_{i_r}$ and x^r for the vector with i-th entry x_i^r then the above expression can be written as

$$\mathcal{A}\mathbf{x}^{\otimes k-1} = \lambda \mathbf{x}^{k-1}$$

where the multiplication denoted by concatenation is tensor contraction. Call $\lambda \in \mathbb{C}$ an *eigenvalue* of $\mathcal{A}$ if there is a non-zero vector $\mathbf{x} \in \mathbb{C}^n$, which we call an *eigenvector*, satisfying

$$\sum_{i_2, i_3, \ldots, i_k = 1}^{n} a_{j i_2 \ldots i_k} x_{i_1} x_{i_2} \ldots x_{i_k} = \lambda x_j^{k-1}.$$

Next we offer an important result from commutative algebra to proceed the definition of the adjacency characteristic polynomial of a hypergraph.

Theorem 2.1. *(The Resultant, [20]) Fix degrees $d_1, d_2, \ldots, d_n$. For $i \in [n]$, consider all monomials $\boldsymbol{x}^\alpha$ (where α is itself a vector) of total degree d_i in $x_1, \ldots, x_n$. For* *each such monomial, define a variable* $u_{i,\alpha}$*. Then there is a unique polynomial* $\textsc{res} \in \mathbb{Z}[\{u_{i,\alpha}\}]$ *with the following three properties:*

1. *If* $F_1, \ldots, F_n \in \mathbb{C}[x_1, \ldots, x_n]$ *are homogeneous polynomials of degrees* $d_1, \ldots, d_n$ *respectively, then the polynomials have a non-trivial common root in* $\mathbb{C}^n$ *exactly when* $\textsc{res}(F_1, \ldots, F_n) = 0$*. Here,* $\textsc{res}(F_1, \ldots, F_n)$ *is interpreted to mean substituting the coefficient of* $\boldsymbol{x}^\alpha$ *in* F_i *for the variable* $u_{i,\alpha}$ *in* $\textsc{res}$*.*

2. $\textsc{res}(x_1^{d_1}, \ldots, x_n^{d_n}) = 1$*.*

3. $\textsc{res}$ *is irreducible, even in* $\mathbb{C}[\{u_{i,\alpha}\}]$*.*

Moreover, for $i \in [n]$*,* $\textsc{res}$ *is homogeneous in the variable* $\{u_{i,\alpha}\}$ *with degree*

$$\prod_{j \in [n], j \neq i} d_i.$$

Definition 2.2. ([30]) The symmetric hyperdeterminant of $\mathcal{A}$, denoted $\det(\mathcal{A})$ is the resultant of the polynomials which comprise the coordinates of $\mathcal{A}x^{\otimes k-1}$. Let λ be an indeterminate. The characteristic polynomial $\phi_{\mathcal{A}}(\lambda)$ of a hypermatrix $\mathcal{A}$ is $\phi_{\mathcal{A}}(\lambda) = \det(\lambda\mathcal{I} - \mathcal{A})$.

We consider the normalized adjacency matrix of a k-uniform hypergraph, $\mathcal{H} = (V, E)$. We refer to such hypergraphs as *k-graphs* and we reserve the language of *graph* for the case of $k = 2$. For a k-graph $\mathcal{H} = ([n], E)$ we denote the *(normalized) adjacency hypermatrix* $\mathcal{A}_{\mathcal{H}}$ to be the order k dimension n hypermatrix with entries

$$a_{i_1,i_2,\ldots,i_k} = \frac{1}{(k-1)!}\begin{cases} 1 & : \{i_1, i_2, \ldots, i_k\} \in E(H) \\ 0 & : \text{otherwise.} \end{cases}$$

For simplicity, we denote $\phi(\mathcal{H}) = \phi_{\mathcal{A}_{\mathcal{H}}}(\lambda)$ and write

$$\phi(\mathcal{H}) = \sum_{i=0}^{t} c_i \lambda^{t-i}$$

where $t = n(k-1)^{n-1}$ by Theorem 2.1. Throughout, we make use of the notation $\phi_d(\mathcal{H}) = c_d$ for the codegree-d coefficient of $\phi(\mathcal{H})$. We refer to $\sigma(\mathcal{H}) = \{r : \phi_{\mathcal{H}}(r) = 0\} \subset \mathbb{C}$ as the *spectrum* of $\mathcal{H}$ and each $\lambda \in \sigma(\mathcal{H})$ is an *eigenvalue* of $\mathcal{H}$.

Chapter 3

The Harary-Sachs Theorem for Hypergraphs

An early, seminal result in spectral graph theory of Harary [23] (and later, more explicitly, Sachs [33]) showed how to express the coefficients of a graph's characteristic polynomial as a certain weighted sum of the counts of various subgraphs of G (a thorough treatment of the subject is given in [3], Chapter 7).

Theorem 3.1. *([23],[33]) Let G be a labeled simple graph on n vertices. If H_i denotes the collection of i-vertex graphs whose components are edges or cycles, and c_i denotes the coefficient of λ^{n-i} in the characteristic polynomial of G, then*

$$c_i = \sum_{H \in H_i} (-1)^{c(H)} 2^{z(H)} [\#H \subseteq G]$$

where $c(H)$ is the number of components of H, $z(H)$ is the number of components which are cycles, and $[\#H \subseteq G]$ denotes the number of (labeled) subgraphs of G which are isomorphic to H.

The goal of this chapter is to provide an analogous result for the characteristic polynomial of a hypergraph. The full result is given in Theorem 3.27, but to state it here simply: fix $k \geq 2$ and let H_d denote the set of *k-valent* (i.e., k divides the degree of each vertex) k-uniform multi-hypergraphs on d edges. For a k-uniform hypergraph $\mathcal{H}$ the *codegree-d* coefficient (i.e., the coefficient of $x^{\deg - d}$) of the characteristic polynomial of the n-vertex hypergraph $\mathcal{H}$ can be written

$$c_d = \sum_{H \in H_d} (-(k-1)^n)^{c(H)} C_H (\#H \subseteq \mathcal{H})$$

where $c(H)$ is the number of components of H, C_H is a constant depending only on H, and $(\#H \subseteq \mathcal{H})$ is the number of times H occurs (in a certain sense that is a minor generalization of the subgraph relation) in $\mathcal{H}$.

The quantity $(\#H \subseteq \mathcal{H})$ is straightforward to compute. However, computing the *associated coefficient of* H, C_H, is more complicated. This notion of an associated coefficient of a hypergraph first appeared in [12], where Cooper and Dutle provide a combinatorial description of the codegree k and codegree $k+1$ coefficient, denoted c_k and c_{k+1} respectively, for the normalized adjacency characteristic polynomial of a k-uniform hypergraph.

Theorem 3.2. *[12] Let $\mathcal{H}$ be a k-uniform hypergraph. Then*

$$c_k = -k^{k-2}(k-1)^{n-k}|E(\mathcal{H})|$$

and

$$c_{k+1} = -C_k(k-1)^{n-k}(\#\textit{ of simplices in }\mathcal{H}),$$

where C_k is some constant depending on k.

This idea was further studied by Shao, Qi, and Hu where the authors prove (restating Theorem 4.1 of [35]),

$$c_d = (k-1)^{n-1} \sum_{D \in \mathbf{D}} f_D |\mathfrak{E}(D)|$$

where $\mathbf{D}$ is a certain large family of digraphs, f_D is a function of D and $\mathfrak{E}(D)$ is the set of Euler circuits in D. The authors then use their formula to provide a general description of $\mathrm{Tr}_2(T)$ and $\mathrm{Tr}_3(T)$ for a general tensor T. Our first few results are similar to that of [35] (as described in more detail herein), and we use them to provide an explicit combinatorial description of H_D and the resulting C_H which yields a Harary-Sachs type formula for hypergraphs which is amenable to computation.

3.1 Background

Our approach relies on the following trace formula for the hyperdeterminant of a tensor. In [28], Morozov and Shakirov give a formula for calculating $\det(\mathcal{I} - \mathcal{A})$ using Schur polynomials in the generalized traces of the order k, dimension n hypermatrix $\mathcal{A}$. Let $f : \mathbb{C}^n \to C^n$ be a linear map and let I be the unity map, $I = (x_1, x_2, \dots, x_n)^T \to (x_1, x_2, \dots, x_n)^T$. Famously,

$$\log\det(I - f) = \operatorname{tr}\log(I - f) = -\sum_{k=1}^{\infty} \frac{\operatorname{tr}(f^k)}{k}.$$

The characteristic polynomial is defined as the resultant of a certain system of equations by definition, so calculating the characteristic polynomial requires computation of the resultant. Moroz and Shakirov give a formula for calculating $\det(\mathcal{I} - \mathcal{A})$ using *Schur polynomials* in the generalized traces of the order k, dimension n hypermatrix $\mathcal{A}$.

Definition 3.3. Define the d-th Schur polynomial $P_d \in \mathbb{Z}[t_1, \dots, t_d]$ by $P_0 = 1$ and, for $d > 0$,

$$P_d(t_1, \dots, t_d) = \sum_{m=1}^{d} \sum_{d_1 + \cdots + d_m = d} \frac{t_{d_1} \cdots t_{d_m}}{m!}.$$

More compactly, one may define P_d by

$$\exp\left(\sum_{d=1}^{\infty} t_d z^d\right) = \sum_{d=1}^{\infty} P_d(t_1, \dots, t_d) z^d.$$

Let f_i denote the ith coordinate of $\mathcal{A}\mathbf{x}^{\otimes k-1}$. Define A to be an auxiliary $n \times n$ matrix with distinct variables A_{ij} as entries. For each i, we define the differential operator

$$\hat{f}_i = f_i\left(\frac{\partial}{\partial A_{i1}}, \frac{\partial}{\partial A_{i2}}, \dots, \frac{\partial}{\partial A_{in}}\right)$$

in the natural way. In [12], Cooper and Dutle use the aforementioned Morozov-Shakirov formula to show that the d-th trace of $\mathcal{A}_H$,

$$\operatorname{Tr}_d(\mathcal{A}_H) = (k-1)^{n-1} \sum_{d_1 + \cdots + d_n = d} \left(\prod_{i=1}^{n} \frac{\hat{f}_i^{\,d_i}}{(d_i(k-1))!} \operatorname{tr}(A^{d(k-1)})\right) \tag{3.1}$$

where $\operatorname{tr}(A^{d(k-1)})$ is the standard matrix trace (for a more detailed explanation, see [12]). We prove our main theorem with the aid of the following reformulation of Equation 3.1:

$$\operatorname{Tr}_d(\mathcal{A}_H) = (k-1)^n \sum_{H \in H_d} C_H(\#H \subseteq \mathcal{H}).$$

3.2 The Associated Digraph of an Operator

Recall the d-th trace of $\mathcal{A}_H$ from Equation 3.1,

$$\operatorname{Tr}_d(\mathcal{A}_H) = (k-1)^{n-1} \sum_{d_1+\cdots+d_n=d} \left(\prod_{i=1}^{n} \frac{\hat{f}_i^{\,d_i}}{(d_i(k-1))!} \operatorname{tr}(A^{d(k-1)}) \right)$$

where $\operatorname{tr}(A^{d(k-1)})$ is the standard matrix operation. Let $\hat{f}_{d_1,d_2,\ldots,d_n}$ be an addend of $\prod_{i=1}^{n} \hat{f}_i^{d_i}$ in $\operatorname{Tr}_d(\mathcal{A}_H)$. When the context is clear we suppress the subscript and simply write $\hat{f}$. Given $\alpha = (i_1, i_2, \ldots, i_{d(k-1)})$ let

$$A_\alpha := A_{i_1,i_2} A_{i_2,i_3} \ldots A_{i_{d(k-1)-1}, i_{d(k-1)}} A_{i_{d(k-1)}, i_1} \tag{3.2}$$

and recall that

$$\operatorname{tr}(A^{d(k-1)}) = \sum_{\alpha} A_\alpha$$

where the factors of A_α are commutative. Adhering to the terminology of [12] we say A_α is *k-valent* if k divides the number of times i occurs in a subscript of A_α. We utilize divisibility notation for monomials in $\operatorname{tr}(A^{d(k-1)})$, e.g., using $g|h$ to denote that g occurs as a factor of the formal product h. We say that A_α *survives* $\hat{f}$ if $\hat{f} A_\alpha \neq 0$.

Definition 3.4. For a differential operator $\hat{f}_{d_1,d_2,\ldots,d_n}$ the *associated digraph* of $\hat{f}$, denoted $D_{\hat{f}}$, is the directed multigraph where there are d_i distinguishable edges directed from i to j given $\left(\frac{\partial}{\partial A_{i,j}}\right)^{d_i} \mid \hat{f}$ and isolated vertices are ignored.

We suppress the subscript and write D when $\hat{f}$ is understood. We recall the following graph theoretic definitions according to [14].

Definition 3.5. A *walk* in a graph G is a non-empty alternating sequence

$$v_0 e_0 v_1 e_1 \dots e_{k-1} v_k$$

of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. A walk is *closed* if $v_0 = v_k$. A closed walk in a graph is an *Euler tour* if it traverses every edge of the graph exactly once. An *Euler circuit* is an Euler tour up to cyclic permutation of its edges, i.e., an Euler tour with no distinguished beginning.

We denote the set of Euler tours of a graph G which begin at the edge $e \in E(G)$ by $\mathfrak{E}_e(G)$ and we denote the set of Euler circuits of G by $\mathfrak{E}(G)$. Recall that a digraph D has an Euler circuit if and only if $\deg^+(v) = \deg^-(v)$ for all $v \in V(D)$ and D is weakly connected.

Definition 3.6. Let $w = (v_i)_{i=0}^m$ be a sequence of (not necessarily distinct) vertices of D. We say that w *describes an Euler tour in* D if there exist distinct edges $e_0, \dots, e_{m-1}$ such that $v_0 e_0 v_1 e_1 \dots e_{m-1} v_m$ is an Euler tour in D. Moreover we say that such Euler tours are *described by* w.

Note that the use of Euler tour in the previous definition is well-founded as e_0 is distinguished as the first edge.

Lemma 3.7. *Consider* $\mathrm{Tr}_d(\mathcal{H})$*,* $\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ *if and only if* D *is Eulerian. In this case*

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) = |E(D)||\mathfrak{E}(D)|. \tag{3.3}$$

Proof. Consider $\mathrm{Tr}_d(\mathcal{H})$. Fix a term A_α of $\mathrm{tr}(A^{d(k-1)})$ and a differential operator $\hat{f}$ of $\prod_{i=1}^n \hat{f}_i^{d_i}$. Suppose $\hat{f} A_\alpha \neq 0$. Whence $\hat{f} A_\alpha \neq 0$ the factors of $\hat{f}$ are in one-to-one correspondence with the factors of A_α. It follows that the edges of $D_{\hat{f}}$ are in one-to-one correspondence with the factors of A_α. Notice that for $i \in V(D)$, $\deg^+(i) = \deg^-(i)$ by Equation 3.2. Further, D is strongly connected as the sequence of indices from

i to j (cyclically, if necessary) is a walk from vertex i to vertex j. Therefore, D is Eulerian.

Suppose now that D is Eulerian and let $\alpha = (v_i)_{i=0}$ describe an Euler tour in D. We claim that $\hat{f}A_\alpha$ is equal to the number of Euler tours in $D_{\hat{f}}$ described by α. Let $m(i,j)$ denote the number of edges from i to j in D. Notice that

$$A_\alpha = \prod_{i,j \in V(D)} A_{i,j}^{m(i,j)} \text{ and } \hat{f} = \prod_{i,j \in V(D)} \frac{\partial^{m(i,j)}}{\partial A_{i,j}^{m(i,j)}}.$$

Clearly

$$\hat{f}A_\alpha = \prod_{i,j \in V(D)} m(i,j)!.$$

Moreover, α describes $\prod_{i,j \in V(D)} m(i,j)!$ Euler tours in D by straightforward enumeration. Observe that there are $|E(D)||\mathfrak{E}(D)|$ Euler tours in D as we may distinguish any edge from D as the first edge of an Euler circuit. Thus,

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) = \sum_\alpha \hat{f}A_\alpha = |E(D)||\mathfrak{E}(D)|$$

as every Euler tour is described by exactly one α. □

We conclude this section with a remark about the evaluation of Equation 3.3. Conveniently, $|\mathfrak{E}(D)|$ can be computed using the BEST theorem, originally appearing in [1] as a variation of a result of [38].

Theorem 3.8. *(BEST Theorem) The number of Euler circuits in a connected Eulerian graph G is*

$$|\mathfrak{E}(G)| = \tau(G) \prod_{v \in V} (\deg(v) - 1)!$$

where $\tau(G)$ is the number of arborescences (i.e., the number of rooted subtrees of G with a specified root).

For simplicity we abbreviate $\tau(f) = \tau(D_{\hat{f}})$. Combining the BEST theorem and the observation that $|E(D)| = d(k-1)$ yields the following.

Corollary 3.9.

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) = d(k-1)\tau(f) \prod_{v \in V(D)} (\deg^-(v) - 1)!$$

As a final note, recall that $\tau(G)$ can be computed using the Matrix Tree Theorem, which makes the computation of the right-hand side of the equality in Corollary 3.9 efficient.

Theorem 3.10. *(Matrix Tree Theorem/Kirchhoff's Theorem) For a given connected graph G with n labeled vertices, let $\lambda_1, \lambda_2, \dots \lambda_{n-1}$ be the nonzero eigenvalues of $\mathcal{L}(G) = D(G) - A(G)$. Then*

$$\tau(G) = \frac{\lambda_1 \lambda_2 \dots \lambda_{n-1}}{n}.$$

3.2.1 Euler operators and Veblen hypergraphs

In Lemma 3.7 we showed that the only differential operators $\hat{f}$ for which

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$$

are the operators whose associated digraphs are Eulerian. The question remains: which $\hat{f}$ have an Eulerian associated digraph? We answer this question with the following graph decoration.

Definition 3.11. We define the *u-rooted directed star of a k-uniform edge e* to be

$$S_e(u) = (e, \{uv : v \in e, u \neq v\}).$$

A *rooting* of a k-graph $\mathcal{H}$ is an ordering $R = (S_{e_1}(v_1), S_{e_2}(v_2), \dots, S_{e_m}(v_m))$ such that $E(\mathcal{H}) = \{e_1, \dots, e_m\}$ and $v_i \leq v_{i+1}$. Given a rooting of $\mathcal{H}$ we define the *rooted multi-digraph* of R to be

$$D_R = \bigcup_{i=1}^{m} S_{e_i}(v_i)$$

where the union sums edge multiplicities. We say that a rooting R is an *Euler rooting* if D_R is Eulerian. We denote the mutli-set of rooted digraphs of $\mathcal{H}$ as $S(\mathcal{H})$.

Note that two distinct rootings can yield the same rooted digraph. We suppress the subscript D_R and write D when the context is clear. We further refer to D as a *rooted digraph* of $\mathcal{H}$ for convenience.

Definition 3.12. Given a rooted digraph $D \in S(\mathcal{H})$, we define the *rooted operator of* D to be

$$\hat{f}_D = \prod_{uv \in E(D)} \frac{\partial}{\partial A_{u,v}}.$$

Moreover, we denote

$$\hat{S}(\mathcal{H}) = \{\hat{f}_D : D \in R(\mathcal{H})\}.$$

In the case when D is Eulerian we refer to $\hat{f}_D$ as an *Euler operator*.

The notation of $\hat{f}_D$ is consistent with our usage of $\hat{f}$ whence

$$\hat{f}_D \mid \prod_{i=1}^{n} \hat{f}_i^{d_i}$$

where d_i is equal to the number of times vertex i is appears as a root of D. If $\hat{f}$ is a rooted operator then it is understood that there exists a (not necessarily unique) rooting R such that, with a slight abuse of notation, $\hat{f} = \hat{f}_{D_R}$. We call such a rooting an *underlying rooting* of a differential operator.

Lemma 3.13. *The associated digraph of an operator $\hat{f}$ is Eulerian if and only if $\hat{f}$ is an Euler operator.*

Remark 3.14. By Lemma 3.13 the only operators which have nonzero contribution to $\mathrm{Tr}_d(\mathcal{H})$ are Euler operators. We denote $R(\mathcal{H}) \subseteq S(\mathcal{H})$ to be the multi-set of *Euler rooted digraphs of* $\mathcal{H}$. We further denote

$$\hat{R}(\mathcal{H}) = \{\hat{f}_D : D \in R(\mathcal{H})\}.$$

Remark 3.15. One can deduce Theorem 4.1 of [35] from Lemma 3.13 by a change of notation: our $\hat{f}_D$ is their F, our set of Eulerian associated digraphs arising from $\hat{f}_R$ is their $\mathbf{E}_{d,m-1}(n)$, and our $\mathfrak{E}(D)$ is their $\mathbf{W}(E)$.

We now show that an Euler rooting is a rooting of a special type of hypergraph.

Definition 3.16. A *Veblen hypergraph*[1] is a k-uniform, k-valent multi-hypergraph.

Lemma 3.17. *An Euler rooting R is a rooting of precisely one labeled Veblen hypergraph.*

Proof. Suppose $S = (S_i)_{i=1}^m$ is a rooting of a connected k-graph $\mathcal{H}$. Since D_S is Eulerian we have for all $j \in V(\mathcal{H})$

$$\deg^+(j) = (k-1)|\{i : v_i = j\}| = |\{i : v_i \neq j, j \in e_i\}| = \deg^-(j).$$

Fix a vertex $v \in V(\mathcal{H})$. We compute

$$\begin{aligned}\deg_{\mathcal{H}}(v) &= \deg^+_D(v) + \deg^-_D(v) \\ &= |\{i : v_i = v\}| + |\{i : v_i \neq v, v \in e_i\}| \\ &= |\{i : v_i = v\}| + (k-1)|\{i : v_i = v\}| \\ &= k|\{i : v_i = v\}|.\end{aligned}$$

Observe that $k \mid \deg_{\mathcal{H}}(v)$; it follows that $\mathcal{H}$ is Veblen by definition. Now suppose that $\mathcal{H}_0$ is a connected Veblen graph such that S is an Euler rooting of $\mathcal{H}_0$. As S is a rooting of H_0, $E(\mathcal{H}_0) = E(\mathcal{H})$ and since both hypergraphs are connected $V(\mathcal{H}_0) = V(\mathcal{H})$. It follows that $\mathcal{H}$ is unique. □

We combine Lemmas 3.13 and 3.17 into the following Lemma.

Lemma 3.18. *We have*

$$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$$

if and only if $\hat{f} = \hat{f}_D$ is a rooted operator. Moreover, the underlying rooting of $\hat{f}$ is necessarily an Euler rooting of precisely one connected, labeled Veblen hypergraph.

---

[1] The nomenclature is a reference to Oswald Veblen (1880-1960) who proved an extension of Euler's theorem in 1912. We present a brief note about Veblen's namesake theorem at the conclusion of this section.

In the following section we use Lemma 3.18 to express the codegree-d coefficient of a k-graph as a function of Veblen hypergraphs. Here we conclude with a note about Veblen's theorem.

Theorem 3.19. *(Veblen's theorem [39]) The set of edges of a finite graph can be written as a union of disjoint simple cycles if and only if every vertex has even degree.*

Unfortunately, Veblen's theorem does not extend to higher uniformity: the set of edges of a finite k-graph $\mathcal{H}$ can not always be written as a union of disjoint simple k-regular k-graphs if and only if $\mathcal{H}$ is k-valent. Consider the Veblen 3-graph $\mathcal{T}$ which consists of three bottomless tetrahedrons each sharing a common base. To be precise,

$$\mathcal{T} = \left(\{a, b, c, 1, 2, 3\}, \bigcup_{i=1}^{3} \left(\binom{\{a, b, c, i\}}{3} \setminus \{a, b, c\} \right) \right).$$

A drawing of $\mathcal{T}$ is given in Figure 3.1. Since there are only three edges containing

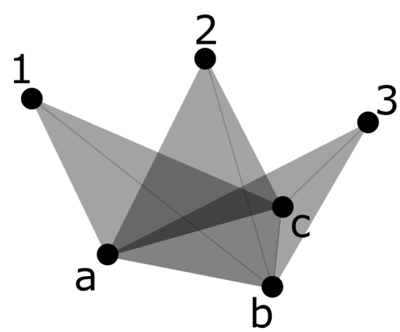

Figure 3.1 The Veblen 3-graph $\mathcal{T}$ where edges are drawn as triangular faces.

$i \in [3]$ it must be the case that any partition into Veblen graphs places each edge containing i into the same class. Observe that for each i the vertices a, b, and c each have degree 2. Therefore, the only 3-valent edge partition is the trivial one.

3.3 The Associated Coefficient of a Veblen Hypergraph

We now turn our attention to computing the codegree-d coefficient of a k-graph $\mathcal{H}$ via Equation 3.1. From Lemma 3.18 we know that the only operators which satisfy

$\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ are rooted operators. Furthermore, as a differential operator of $\mathrm{Tr}_d(\mathcal{A}_{\mathcal{H}})$ is of degree d, the underlying Euler rooting of $\hat{f}$ is a rooting of precisely one connected, labeled Veblen hypergraph with d edges. We equate $\mathrm{Tr}_d(\mathcal{H})$ to a weighted sum over Euler rootings of connected Veblen graphs with d edges which "appear" in $\mathcal{H}$. Consider the following generalization of the notion of subgraph.

Definition 3.20. For a labeled multi-hypergraph H, we call the simple k-graph formed by removing duplicate edges of H the *flattening* of H and denote it $\underline{H}$. We say that H is an *infragraph* of $\mathcal{H}$ if $\underline{H} \subseteq \mathcal{H}$. Let $\mathcal{V}_d(\mathcal{H})$ denote the set of isomorphism classes of connected, labeled Veblen infragraphs with d edges of $\mathcal{H}$.

Definition 3.21. The *associated coefficient of a connected Veblen hypergraph* H is

$$C_H = \sum_{D \in R(H)} \left(\frac{\tau_D}{\prod_{v \in V(D)} \deg^-(v)} \right).$$

The *associated coefficient* of a (possibly disconnected) Veblen hypergraph is

$$C_H = \prod_{i=1}^{m} C_{G_i}, \text{ for } H = \bigcup_{i=1}^{m} G_i.$$

We present the associated coefficient of Veblen 3-graphs with six or fewer edges in Table 3.3.

Definition 3.22. For a k-graph $\mathcal{H}$ and a Veblen k-graph $H = \bigcup_{i=1}^{m} G_i$ we define

$$(\#H \subseteq \mathcal{H}) = \frac{1}{|\mathrm{Aut}(H)|} \prod_{i=1}^{m} |\mathrm{Aut}(\underline{G_i})||\{S \subseteq \mathcal{H} : S \cong \underline{G_i}\}|.$$

In the case when H is connected this simplifies to

$$(\#H \subseteq \mathcal{H}) = \frac{|\mathrm{Aut}(\underline{H})|}{|\mathrm{Aut}(H)|}|\{S \subseteq \mathcal{H} : S \cong \underline{H}\}| = |\mathrm{Aut}(\underline{H})/\mathrm{Aut}(H)| \cdot |\{S \subseteq \mathcal{H} : S \cong \underline{H}\}|.$$

Note that for $H = \bigcup_{i=1}^{m} G_i$, $(H \subseteq \mathcal{H})$ is not multiplicative over the components of H as

$$\prod_{i=1}^{m} (\#G_i \subseteq \mathcal{H}) = \frac{(\#H \subseteq \mathcal{H})|\mathrm{Aut}(H)|}{\prod_{i=1}^{m} |\mathrm{Aut}(G_i)|}.$$

However, we have the following identity.

Table 3.1 The associated coefficient of Veblen 3-graphs with six or fewer edges.

$E(H)$	C_H
[0, 1, 2], [0, 1, 2], [0, 1, 2]	3/8
[0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 2, 3]	21/8
[0, 1, 2], [0, 1, 4], [0, 3, 4], [1, 2, 3], [2, 3, 4]	51/16
[0, 1, 2], [0, 3, 4], [0, 3, 4], [1, 2, 3], [1, 2, 4]	27/16
[0, 1, 2], [0, 1, 2], [0, 1, 2], [0, 1, 2], [0, 1, 2], [0, 1, 2]	3/16
[0, 1, 2], [0, 1, 2], [0, 1, 2], [0, 1, 3], [0, 1, 3], [0, 1, 3]	9/8
[0, 1, 2], [0, 1, 2], [0, 1, 2], [0, 3, 4], [0, 3, 4], [0, 3, 4]	9/32
[0, 1, 2], [0, 1, 2], [0, 1, 3], [0, 2, 4], [0, 3, 4], [0, 3, 4]	99/32
[0, 1, 2], [0, 1, 3], [0, 1, 4], [0, 2, 3], [0, 2, 4], [0, 3, 4]	213/16
[0, 1, 2], [0, 1, 3], [0, 4, 5], [1, 4, 5], [2, 3, 4], [2, 3, 5]	69/16
[0, 1, 2], [0, 1, 3], [0, 3, 4], [1, 3, 5], [2, 4, 5], [2, 4, 5]	63/32
[0, 1, 2], [0, 2, 3], [0, 3, 4], [1, 3, 5], [1, 4, 5], [2, 4, 5]	129/32
[0, 1, 2], [0, 1, 2], [0, 1, 3], [2, 4, 5], [3, 4, 5], [3, 4, 5]	27/32
[0, 1, 2], [0, 1, 3], [0, 2, 3], [1, 4, 5], [2, 4, 5], [3, 4, 5]	63/16
[0, 1, 2], [0, 1, 3], [0, 2, 4], [1, 3, 5], [2, 4, 5], [3, 4, 5]	117/32

Lemma 3.23. *Let $H = \bigcup_{i=1}^{m} G_i$ be a Veblen k-graph. If μ_H denotes the number of linear orderings of the components of H (where two components are indistinguishable if they are isomorphic) then*

$$(\#H \subseteq \mathcal{H}) = \frac{\mu_H}{m!} \prod_{i=1}^{m} (\#G_i \subseteq \mathcal{H}).$$

Proof. Suppose there are t isomorphism classes of components of H with representatives $H_1, H_2, \dots H_t$. Denote the number of components of H which are isomorphic to H_i as μ_i. Fix an ordering of the components which are isomorphic to H_i, $\{G_{1,i}, G_{2,i}, \dots, G_{\mu_i,i}\}$. The number of distinct linear orderings of the components of H where G_i and G_j are indistinguishable when $G_{r,i} \cong G_{s,i}$ is

$$\mu_H = \binom{m}{\mu_1, \mu_2, \dots, \mu_t}$$

so that

$$\frac{m!}{\mu_H} = \prod_{i=1}^{t} \mu_i!.$$

Note that for $a \in \mathrm{Aut}(H)$, there exists $\sigma \in \mathfrak{S}_{\mu_i}$ such that $a(G_{j,i}) = G_{\sigma(j),i}$. In this way, $\mathrm{Aut}(H)$ induces a permutation on the isomorphism classes of the components of H (note that this map is well-defined since the components are labeled). Let $\psi : \mathrm{Aut}(H) \to \mathfrak{S}_{\mu_1} \times \mathfrak{S}_{\mu_2} \times \cdots \times \mathfrak{S}_{\mu_t}$ be such a map. Notice $\ker(\psi)$ is the group of automorphisms of H which maps each component to itself. Appealing to the First Isomorphism Theorem we have

$$\frac{|\mathrm{Aut}(H)|}{\prod_{i=1}^{m} |\mathrm{Aut}(G_i)|} = \prod_{i=1}^{t} \mu_i!.$$

The desired equality follows by substitution. □

Remark 3.24. The equation in Lemma 3.23 implies that $(\#H \subseteq \mathcal{H})$ is multiplicative over its components if and only if the components of H are pairwise non-isomorphic.

Let

$$A(d,n) = \left\{(d_1, \ldots, d_n) : \sum d_i = d, d_i \geq 0\right\}$$

be the set of *arrangements of d into n non-negative parts* and further let $A^+(d,n) \subset A(d,n)$ be the set of *arrangements of d into n positive parts*. For a k-graph $\mathcal{H}$ and $a \in A^+(d,|E|)$ let $R^a(\mathcal{H})$ be the set of Euler rootings of all labeled, connected Veblen infragraphs of $\mathcal{H}$ which have the property that vertex v_i is the root of exactly d_i edges. (N.B. We take $a \in A^+(d,|E|)$ as it is necessary that $d_i > 0$ for $D \in R^a(\mathcal{H})$ to be Eulerian.)

Remark 3.25. Let $\mathcal{V}_d^*(\mathcal{H})$ denote the set of (possibly disconnected) Veblen infragraphs of $\mathcal{H}$ with d edges up to isomorphism. Further let $\mathcal{V}_d(\mathcal{H}) \subseteq \mathcal{V}_d^*(\mathcal{H})$ denote the set of connected Veblen infragraphs of $\mathcal{H}$ with d edges up to isomorphism.

We now present a formula for $\mathrm{Tr}_d(\mathcal{H})$ as a weighted sum over its Veblen infragraphs.

Lemma 3.26. *For a k-graph $\mathcal{H}$*

$$\operatorname{Tr}_d(\mathcal{H}) = d(k-1)^n \sum_{H \in \mathcal{V}_d(\mathcal{H})} C_H(\#H \subseteq \mathcal{H}).$$

Proof. For convenience let $|E| = |E(\mathcal{H})|$. We equate

$$\begin{aligned}\sum_{H \in \mathcal{V}_d(\mathcal{H})} C_H(\#H \subseteq \mathcal{H}) &= \sum_{H \in \mathcal{V}_d(\mathcal{H})} \left(\left(\sum_{D \in R(H)} \frac{\tau_D}{\prod_{v \in V(D))} \deg^-(v)} \right) (\#H \subseteq \mathcal{H}) \right) \\ &= \sum_{a \in A^+(d,|E|)} \left(\sum_{D \in R^a(\mathcal{H})} \frac{\tau_D}{\prod_{v \in V(D)} \deg^-(v)} \right).\end{aligned}$$

Recall

$$\operatorname{Tr}_d(\mathcal{H}) = (k-1)^{n-1} \sum_{d_1+\cdots+d_n=d} \left(\prod_{i=1}^{n} \frac{\hat{f}_i^{d_i}}{(d_i(k-1))!} \operatorname{tr}(A^{d(k-1)}) \right).$$

Applying Lemma 3.18 we have

$$\operatorname{Tr}_d(\mathcal{H}) = (k-1)^{n-1} \sum_{a \in A^+(d,|E|)} \left(\sum_{D \in R^a(\mathcal{H})} \frac{\hat{f}_D \operatorname{tr}(A^{d(k-1)})}{\prod_{i=1}^{n}(d_i(k-1))!} \right).$$

By Corollary 3.9 ,

$$\hat{f}_D \operatorname{tr}(A^{d(k-1)}) = d(k-1)\tau_D \prod_{v \in V(D_R)} (\deg^-(v) - 1)!$$

When $D \in R^a(\mathcal{H})$ with $a = (d_1, \ldots, d_n)$, we have $\deg^-(v_i) = d_i(k-1)$. By substitution we have

$$\begin{aligned}\operatorname{Tr}_d(\mathcal{H}) &= d(k-1)^n \sum_{a \in A^+(d,|E|)} \left(\sum_{D \in R^a(\mathcal{H})} \frac{\tau_D}{\prod_{v \in V(D)} \deg^-(v)} \right) \\ &= d(k-1)^n \sum_{H \in \mathcal{V}_d(\mathcal{H})} C_H(\#H \subseteq \mathcal{H}).\end{aligned}$$

□

We are now prove a generalization of the Harary-Sachs formula for k-graphs.

Theorem 3.27. *For a simple k-graph $\mathcal{H}$,*

$$\phi_d(\mathcal{H}) = \sum_{H \in \mathcal{V}_d^*(\mathcal{H})} (-(k-1)^n)^{c(H)} C_H(\#H \subseteq \mathcal{H}).$$

Proof. Fix a k-graph $\mathcal{H}$ and $d \geq 1$. From [12] we have by Equation 3.1

$$\phi_d(\mathcal{H}) = P_d\left(\frac{-\operatorname{Tr}_1(\mathcal{H})}{1}, \frac{-\operatorname{Tr}_2(\mathcal{H})}{2}, \ldots, \frac{-\operatorname{Tr}_d(\mathcal{H})}{d}\right)$$

where

$$P_d(t_1, t_2, \ldots, t_d) = \sum_{m=1}^{d} \sum_{d_1+\cdots+d_m=d} \frac{t_{d_1} t_{d_2} \cdots t_{d_m}}{m!}.$$

Fix an $1 \leq m \leq d$ and an arrangement $a = (d_1, d_2, \ldots, d_m) \in A^+(d, m)$. By Lemma 3.26,

$$\frac{-\operatorname{Tr}_{d_i}(\mathcal{H})}{d_i} = -(k-1)^n \sum_{G \in \mathcal{V}_{d_i}(\mathcal{H})} C_G(\#G \subseteq \mathcal{H}).$$

Let $\mathcal{V}^a(\mathcal{H})$ denote the set of m-tuples of connected unlabeled Veblen infragraphs of $\mathcal{H}$ whose i-th coordinate has d_i edges for $i \in [m]$, such that $\sum_i d_i = d$. We have

$$\begin{aligned}
\prod_{i=1}^{m} \frac{-\operatorname{Tr}_{d_i}(\mathcal{H})}{d_i} &= (-1)^m (k-1)^{mn} \prod_{i=1}^{m} \sum_{G \in \mathcal{V}_{d_i}(\mathcal{H})} C_G(\#G \subseteq \mathcal{H}) \\
&= (-(k-1)^n)^m \sum_{\substack{H = G_1 \cup \cdots \cup G_m \\ (G_1, G_2, \ldots, G_m) \in \mathcal{V}^a(\mathcal{H})}} C_H \prod_{i=1}^{m} (\#G_i \subseteq \mathcal{H})
\end{aligned}$$

For $m \in \mathbb{N}$, let $\mathcal{V}_d^m(\mathcal{H})$ be the set of unlabeled Veblen infragraphs of $\mathcal{H}$ with d edges and m components. Appealing to Lemma 3.23 we may write

$$\begin{aligned}
\phi_d &= P_d\left(-\frac{\operatorname{Tr}_1(\mathcal{H})}{1}, -\frac{\operatorname{Tr}_2(\mathcal{H})}{2}, \ldots, -\frac{\operatorname{Tr}_d(\mathcal{H})}{d}\right) \\
&= \sum_{m=1}^{d} \sum_{d_1+\cdots+d_m=d} \frac{1}{m!} \prod_{i=1}^{m} \frac{-\operatorname{Tr}_{d_i}(\mathcal{H})}{d_i} \\
&= \sum_{m=1}^{d} \left(\sum_{a \in A^+(m,d)} (-(k-1)^n)^m \sum_{H \in \mathcal{V}^a(\mathcal{H})} C_H \frac{\left(\prod_{i=1}^{m} (\#G_i \subseteq \mathcal{H})\right)}{m!} \right) \\
&= \sum_{m=1}^{d} (-(k-1)^n)^m \sum_{H \in \mathcal{V}_d^m(\mathcal{H})} C_H \left(\frac{\mu_H}{m!} \prod_{i=1}^{m} (\#G_i \subseteq \mathcal{H}) \right) \\
&= \sum_{H \in \mathcal{V}_d^*(\mathcal{H})} (-(k-1)^n)^{c(H)} C_H (\#H \subseteq \mathcal{H}).
\end{aligned}$$

□

3.4 Deducing the Harary-Sachs Theorem for Graphs

With Theorem 3.27 in hand we can express the codegree-d coefficient of the normalized adjacency characteristic polynomial of a hypergraph as a weighted sum of Veblen infragraphs with *d edges*. In the case when G is a 2-graph our theorem simplifies to

$$\phi_d(G) = \sum_{H \in \mathcal{V}_d^*} (-1)^{c(H)} C_H (\# H \subseteq G).$$

Recall that the Harary-Sachs theorem (Theorem 3.1) expresses the codegree-d coefficient as a weighted sum over certain subgraphs on *d vertices*, whereas Theorem 3.27 expresses the same quantity as a weighted sum over certain subgraphs with *d edges*. We now argue that these two sums are equal.

An *elementary subgraph* of a graph G, is a simple subgraph of G whose components are edges or cycles (see [3] for further details). In keeping with their notation, let $\Lambda_d(G)$ be the set of elementary subgraphs of G with d vertices. Notice that a connected elementary graph is the flattening of a cycle (e.g., the flattening of a 2-cycle is an edge). Recall that cycles (and disjoint unions of cycles) are the only two regular non-empty graphs which have an equal number of vertices and edges. Indeed

$$\Lambda_d(G) \subseteq \{\underline{H} : H \in \mathcal{V}_d^*(G)\}.$$

By straightforward computation we have that the associated coefficient of a 2-cycle is 1 and the associated coefficient of a simple cycle (i.e., any cycle which is not a 2-cycle) is 2. Restricting our attention to $\Lambda_d(G)$, we have by Theorem 3.27

$$\sum_{H \in \Lambda_d(G)} (-1) C_H (\# H \subseteq G) = (-1)^{c(H)} 2^{z(H)} (\# H \subseteq G) \qquad (3.4)$$

where $z(H)$ is the number of cycles in H. Note that Equation 3.4 is the conclusion of the Harary-Sachs theorem. We deduce the Harary-Sachs theorem from Theorem 3.27 by showing that the summands of

$$\phi_d(G) = \sum_{H \in \mathcal{V}_d^*} (-1)^{c(H)} C_H (\# H \subseteq G)$$

which do not arise from elementary graphs sum to zero. We make this statement precise with the following.

Definition 3.28. For a multigraph G and an edge $e \in E(G)$, write $m(e) = m_G(e)$ for the multiplicity of e in $E(G)$. Let G be a connected, labeled Veblen graph with distinguishable multi-edges. Given a multiset P of multigraphs whose multi-edges are indistinguishable, each on the vertex set $V(G)$, we write $P \vdash G$ if, for each $e \in E(G)$

$$\sum_{P_i \in P} m_{P_i}(e) = m_G(e).$$

Lemma 3.29.

$$\sum_{P \vdash G} (-1)^{c(P)} C_P = \begin{cases} 1 & : G \textit{ is a 2-cycle} \\ 2 & : G \textit{ is a simple cycle} \\ 0 & : \textit{otherwise.} \end{cases}$$

Remark 3.30. We refer to Veblen 2-graphs as Veblen graphs. An *Euler orientation* of a graph G is an orientation of the edges of G such that the resulting digraph is Eulerian.

We first provide a combinatorial formula for the associated coefficient of a Veblen graph.

Lemma 3.31. *Let G be a connected Veblen graph. We have*

$$C_G = \frac{|\mathfrak{E}(G)|}{\prod_{e \in E(G)} m(e)!}.$$

Proof. Let G be a Veblen graph. Since G is Eulerian, we write $\deg(v) = 2d_v$ for convenience. Because $\deg^-(v) = d_v$ for all $D \in R(G)$ we have

$$C_G = \frac{1}{\prod_{v \in V(G)} d_v} \sum_{D \in R(G)} \tau_D.$$

By the BEST theorem (i.e, Theorem 3.8) we have

$$\tau_D = \frac{|\mathfrak{E}(D)|}{\prod_{v \in V(G)} (d_v - 1)!}.$$

Let $N_D(v)$ denote the out-neighborhood of v in D and let $\deg_D(v,u)$ denote the number of edges directed from v to u in D. We denote

$$\binom{d_v}{N_D(v)} = \frac{d_v!}{\prod_{u\in N_D(v)} \deg_D(v,u)!}$$

which is the number of linear orderings of out-edges of v in an Euler orientation D. Consider the equivalence relation $\sim_{R(G)}$ where $R \sim R'$ if and only if $D_R = D_{R'}$. Note that $\sim$ identifies two Euler rootings if their associated digraphs are the same Euler orientation. Let $[R]$ denote the equivalence class of R under $\sim$. Suppose $R(G) = \bigcup_{i=1}^{t}[R_i]$ and note

$$|[R]| = \prod_{v\in V(G)} \binom{d_v}{N_{D_R}(v)}$$

as two rootings in $[R]$ differ only in the ordering of the u-rooted stars for $u \in V(G)$. Let $\mathcal{O}(G) = \{D_1, \ldots, D_t\}$, where $D_i = D_{R_i}$, denote the Euler orientations of G. For convenience, we write N_i for N_{D_i}. We equate

$$\sum_{D\in R(G)} \tau_D = \sum_{i=1}^{t} \tau_i |[R_i]| = \sum_{i=1}^{t} \left(\tau_i \prod_{v\in V(G)} \binom{d_v}{N_i(v)} \right).$$

Substitution and simplification yields

$$C_G = \sum_{i=1}^{t} \frac{|\mathfrak{E}(D_i)|}{\prod_{v\in V(G)} \left(\prod_{u\in N_i(v)} \deg_i(v,u)!\right)}.$$

Since $\deg_i(u,v) + \deg_i(v,u) = m(uv)$ we have

$$\binom{m(uv)}{\deg_i(u,v)} = \frac{m(uv)!}{\deg_i(u,v)!\,\deg_i(v,u)!}$$

so that

$$\frac{\prod_{e\in E(G)} m(e)!}{\prod_{v\in V(G)} \prod_{u\in V(G)} \deg_i(v,u)!} = \frac{\prod_{e\in E(G)} m(e)!}{\prod_{u<v\in V(G)} \deg_i(u,v)!\,\deg_i(v,u)!} = \prod_{uv\in E(G), u<v} \binom{m(uv)}{\deg_i(u,v)}.$$

Then

$$
\begin{aligned}
C_G &= \sum_{i=1}^{t} \frac{|\mathfrak{E}(D_i)|}{\prod_{v\in V(G)}\left(\prod_{u\in N_i(v)} \deg_i(v,u)!\right)} \\
&= \frac{\prod_{e\in E(G)} m(e)!}{\prod_{e\in E(G)} m(e)!} \sum_{i=1}^{t} \frac{|\mathfrak{E}(D_i)|}{\prod_{v\in V(G)}\left(\prod_{u\in N_i(v)} \deg_i(v,u)!\right)} \\
&= \frac{1}{\prod_{e\in E(G)} m(e)!} \sum_{i=1}^{t} \left(\prod_{uv\in E(G), u<v} \binom{m_G(uv)}{\deg_i(u,v)} \right) |\mathfrak{E}(D_i)| \\
&= \frac{|\mathfrak{E}(G)|}{\prod_{e\in E(G)} m(e)!}
\end{aligned}
$$

where the last equality follows from the observation that D_i has indistinguishable multi-edges. □

We now prove Lemma 3.29.

Proof. Let G be a connected Veblen graph which is not a cycle. Further assume that the multi-edges of G are distinguishable. We aim to show

$$\sum_{P\vdash G} (-1)^{c(P)} C_P = 0.$$

By Lemma 3.31 we have for connected G,

$$C_G = \frac{|\mathfrak{E}(G)|}{\prod_{e\in E(G)} m(e)!}.$$

Let $P = \bigcup_{i=1}^{t} P_i$ be a disjoint union of Veblen graphs. We denote

$$\binom{m_G(e)}{P(e)} = \frac{m_G(e)!}{\prod_{i=1}^{t} m_{P_i}(e)!}$$

and

$$|\mathfrak{E}(P)| = \prod_{i=1}^{t} |\mathfrak{E}(P_i)|.$$

We equate

$$\left(\prod_{e\in E(G)} m(e)!\right) \sum_{P\vdash G} (-1)^{c(P)} C_P = \sum_{P\vdash G} (-1)^{c(P)} |\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)}.$$

Notice that $|\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)}$ counts the number of partitions of $E(G)$ into edge-disjoint Euler circuits of graphs on $V(G)$ with unlabelled edges which are precisely the elements of P.

We say that an Euler circuit is *decomposable* if it can be written as a union of (at least) two edge disjoint Euler circuits, and *indecomposable* otherwise. The number of decompositions of Euler circuits of G into exactly t parts is

$$\sum_{P\vdash G,} \left(|\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)} \right).$$

By Inclusion/Exclusion, the number of indecomposable Euler circuits of G is

$$\sum_{t=1} (-1)^t \sum_{P\vdash G,} \left(|\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)} \right).$$

We assumed that G is a Veblen graph which is not a cycle. We have by Veblen's theorem (i.e., Theorem 3.19) that every Euler circuit in G is decomposable. It follows that G has no indecomposable Euler circuits; that is to say,

$$\sum_{t=1} (-1)^t \sum_{P\vdash G,} \left(|\mathfrak{E}(P)| \prod_{e\in E(G)} \binom{m_G(e)}{P(e)} \right) = 0,$$

from which the desired conclusion follows. □

Chapter 4

Low-Codegree Coefficients

From the Harary-Sachs Theorem (i.e., Theorem 3.1) we know that for a 2-graph G, if $\phi(G) = \sum_{i=0}^{n} c_i x^{n-i}$ then

$$c_0 = 1, c_1 = 0, c_2 = -|E(G)|, \text{ and } c_3 = -2(\# \text{ of triangles}).$$

The goal of this chapter is to provide a similar description for the low-codegree coefficients of a k-graph. We begin by providing such a description for the first k proper coefficients of $\phi(\mathcal{H})$. The following is a collection of results from [12] but we provide a new proof via Theorem 3.27.

Lemma 4.1. *For a k-graph $\mathcal{H}$ and $d \in [k-1]$, $\phi(\mathcal{H})_d = 0$. Moreover*

$$\phi_k(\mathcal{H}) = -(k-1)^{n-k} k^{k-2} |E(\mathcal{H})|.$$

Proof. Recall from Theorem 3.27 we have

$$\phi_d(\mathcal{H}) = \sum_{H \in \mathcal{V}_d^*(\mathcal{H})} (-(k-1)^n)^{c(H)} C_H (\# H \subseteq \mathcal{H}).$$

Notice that for $d \in [k-1]$, $\mathcal{V}_d^* = \emptyset$ so that $\phi_d(\mathcal{H}) = 0$. Further, notice that $\mathcal{V}_k^* = \{E_k\}$ where E_k is the k-uniform edge. By Lemma 4.2 we have that

$$R(E_k) = \{((1, [k]), (2, [k]), \ldots, (k, [k]))\}.$$

For $R \in R(E_k)$ we have $D_R \cong K_k$ so that $\mathcal{L}(D_R) = kI_k - J_k$ whose multi-set spectrum is $\{0, k, k, \ldots, k\}$. By Kirchoff's Theorem we have $\tau_R = k^{k-2}$ so that

$$C_{E_k} = \frac{k^{k-2}}{(k-1)^k}.$$

Substitution yields

$$c_k = (-(k-1)^n)C_{E_k}(\#E_k \subseteq \mathcal{H}) = -(k-1)^{n-k}k^{k-2}|E(\mathcal{H})|.$$

$\square$

We begin by providing insight into the computation of C_H. Then we turn our attention to an explicit formula for $\phi_{k+1}(\mathcal{H})$. In the final section we provide an explicit combinatorial formula for the first six proper leading coefficients of a 3-graph.

4.1 Computational Notes

For an arbitrary Veblen k-graph we compute C_H by first determining a list of all Euler rootings. Then for each Euler rooting we compute the corresponding summand via Kirchoff's Theorem. The difficulty in this computation is in determining the set of all Euler rootings efficiently. The following lemma imposes a necessary condition on rootings.

Lemma 4.2. *Let H be a Veblen k-graph. Fix $t \in \mathbb{N}$ and d such that $tk \leq d < (t+1)k$. Let $\hat{f}_{d_1,d_2,\ldots,d_n}$ be a differential operator of $\operatorname{Tr}_d(H)$. If $d_i > t$ then*

$$\hat{f}\,\operatorname{tr}(A^{d(k-1)}) = 0.$$

In particular, if $\hat{f}\,\operatorname{tr}(A^{d(k-1)}) \neq 0$ then $d_i \leq \lfloor \frac{d}{k} \rfloor$ and this bound is sharp.

Proof. Consider $\operatorname{Tr}_d(H)$ and fix a differential operator $\hat{f}_{d_1,d_2,\ldots,d_n}$. Observe that for $D = D_{\hat{f}}$ we have $\deg^+_D(i) = (k-1)d_i$ and $\deg^-_D(i) \leq d - d_i$. Without loss of generality, suppose $d_1 > t$. Then

$$\deg^-_D(1) \leq d - d_1 \leq d - (t+1) < (t+1)k - (t+1) = (t+1)(k-1) \leq \deg^+_D(1).$$

By Euler's theorem D does not have an Euler circuit. Appealing to Lemma 3.7,

$$\hat{f}\,\operatorname{tr}(A^{d(k-1)}) = 0$$

and the first statement follows.

Fix $tk \le d < (t+1)k$ and suppose A_α survives $\hat{f}_{d_1,d_2,\dots,d_n}$. As $\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ we have by our first statement $d_i \le t \le \lfloor d/k \rfloor$. To see that this bound is sharp, fix $t \in \mathbb{N}$ and $k \ge 2$. Set $d := kt$ and consider $\mathrm{Tr}_d(H)$ where $H = ([k], \{d \times [k]\})$ is the k-uniform edge with multiplicity d. Choosing $d_i = t$ for each $i \in [k]$ yields precisely one differential operator $\hat{f}$ as

$$f_i = d\frac{x_1 x_2 \dots x_k}{x_i}.$$

Observe that D is the complete multi-digraph on k vertices where each pair of vertices has t edges oriented in both directions. By Euler's theorem D has an Euler circuit. Let α describe such a circuit. By our previous claim A_α survives $\hat{f}$ so $\hat{f}\,\mathrm{tr}(A^{d(k-1)}) \neq 0$ as desired. □

Note that, for the classical Harary-Sachs Theorem, the number of elementary graphs (explained in the next section) one needs to sum over for the codegree-d coefficient is simply the number of partitions of d into positive parts. The number of Veblen 3-graphs is exponentially larger.

Remark 4.3. Lemma 4.2 is most useful when d is close to k in value. In particular, if $d < 2k$ then $d_i = 1$ for all i.

Recall that $\mathcal{V}_d$ and $\mathcal{V}_d^*$ denote the number of connected and (possibly) disconnected Veblen 3-graphs with d edges. We have computed

$$(|\mathcal{V}_d|)_{d=1}^{\infty} = (0, 0, 1, 1, 2, 11, 26, 122, 781, \dots)$$

and further

$$(|\mathcal{V}_d^*|)_{d=1}^{\infty} = (0, 0, 1, 1, 2, 12, 27, 125, 795, \dots)$$

see A320648 [36].

4.2 The Codegree-$(k+1)$ Coefficient of a k-graph

Fix $k \geq 2$ and let

$$K_{k+1}^{(k)} = \left([k+1], \binom{[k+1]}{k}\right)$$

be the k-uniform simplex. It was shown in [12] that the k-uniform simplex is the only connected Veblen hypergraph with $k+1$ edges, up to isomorphism. Moreover, it was shown for a k-graph $\mathcal{H}$

$$\phi_{k+1}(K_{k+1}^{(k)}) = -(k-1)^{n-k} C_k(\#H \subseteq \mathcal{H})$$

for some constant C_k depending only on k. The authors of [12] were able to show that $C_2 = 2$, $C_3 = 21$, $C_4 = 588$, $C_5 = 28230$ via laborious use of resultants. In this section we provide an explicit, efficient formula for C_k and use it to compute C_{100}. To that end, we first describe the set of rootings of the k-uniform simplex.

For the remainder of this section let H denote the k-uniform simplex $K_k^{(k+1)}$. Observe that the i-th coordinate polynomial of $\mathcal{A}_H \mathbf{x}^{\otimes k-1}$ can be written

$$f_i = \sum_{j \neq i} \frac{x_1 x_2 \ldots x_{k+1}}{x_i x_j}$$

for $i \in [k+1]$. Let $\mathfrak{D}_{k+1}$ be the set of derangements of $[k+1]$, i.e., permutations without any fixed points. Recall that $S_e(u)$ is the u-rooted directed star of e.

Lemma 4.4. *For $\sigma \in \mathfrak{D}_{k+1}$ define*

$$D_\sigma = \bigcup_{i=1}^{k+1} S_{[k+1]\setminus\{\sigma(i)\}}(i)$$

Then

$$R(H) = \{D_\sigma : \sigma \in \mathfrak{D}_{k+1}\}.$$

Proof. Let $\sigma \in \mathfrak{D}_{k+1}$ and consider D_σ. We suppress the subscript and write D for convenience. Note that D is Eulerian whence

$$\deg_D^+(j) = (k-1)|\{i : v_i = j\}| = k-1 = |\{i : v_i \neq j, j \in e_i\}| = \deg_D^-(j)$$

for all $j \in [k+1]$ as there are exactly k edges which contain j and σ is a derangement.

Suppose $D \in R(H)$. As D is Eulerian we have by Lemma 4.2

$$D = \bigcup_{i=1}^{k+1} S_{[k+1]\setminus\{v_i\}}(i).$$

In particular, $\{v_1, \ldots, v_{k+1}\} = [k+1]$. Observe that $\sigma = \{(i, v_i)\}_{i=1}^{k+1} \in \mathfrak{D}_{k+1}$ is a derangement. The conclusion follows from the fact that $D = D_\sigma$. $\square$

This immediately implies the following.

Lemma 4.5. *For the k-uniform simplex H,*

$$C_H = \sum_{\sigma \in \mathfrak{D}_{k+1}} \frac{\tau_\sigma}{\prod_{v \in V(D_\sigma)} \deg^-(v)}.$$

We now show that a summand in the aformentioned formula of C_H depends only on the cycle type of the derangement.

Theorem 4.6. *Let $\sigma \in \mathfrak{D}_{k+1}$ with cycle decomposition $c_1 c_2 \ldots c_t$ where cycle c_i has length ℓ_i. Then*

$$C_H = \frac{1}{(k-1)^{k+1}(k+1)} \sum_{\sigma = c_1 c_2 \ldots c_t \in \mathfrak{D}_{k+1}} \prod_{i=1}^{t} \left(k^{\ell_i} + (-1)^{\ell_i + 1}\right).$$

We first prove a technical lemma. The notation spec(M) denotes the ordinary (multiset) spectrum of a matrix M.

Lemma 4.7. *For $\sigma \in S_{n+1}$, spec($M_\sigma - J$) = (spec(M_σ) $\setminus \{1\}) \cup \{-n\}$ where M_σ is the permutation matrix associated with σ.*

Proof. Let σ be a permutation of $[n+1]$ with cycles $c_1, c_2, \ldots, c_t$ of length $l_1, l_2, \ldots, l_t$, respectively. Recall that the spectrum of M_σ is given by

$$\text{spec}(M_\sigma) = \bigcup_{i=1}^{t} \{\zeta_{l_i}^0, \zeta_{l_i}^1, \ldots, \zeta_{l_i}^{l_i - 1}\}$$

and note that the spectrum of M_σ depends only on the cycle type of σ. Without loss of generality, suppose that the cycles of σ are increasing (i.e., $[1, \ell_1], [\ell_1+1, \ell_2+\ell_1], \dots$). Consider the following block partition

$$M_\sigma - J_{n+1} = \begin{pmatrix} B_1 & -J_{\ell_2} & \dots & -J_{\ell_t} \\ -J_{\ell_1} & B_2 & \dots & -J_{\ell_n} \\ \vdots & \vdots & \ddots & \vdots \\ -J_{\ell_1} & -J_{\ell_2} & \dots & B_t \end{pmatrix}$$

where B_i is the $l_i \times l_i$ square circulant matrix corresponding to c_i,

$$B_i = \begin{pmatrix} -1 & 0 & -1 & \dots & -1 \\ -1 & -1 & 0 & \dots & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & -1 & \dots & 0 \\ 0 & -1 & -1 & \dots & -1 \end{pmatrix}.$$

Let $j > 0$ and consider the eigenpair $(\zeta_{\ell_i}^j, \mathbf{x})$ of M_σ. We compute

$$(M_\sigma - J)\mathbf{x} = M_\sigma \mathbf{x} - J\mathbf{x} = \zeta_{\ell_i}^j \mathbf{x} - \mathbf{0} = \zeta_{\ell_i}^j \mathbf{x}$$

where $J\mathbf{x} = 0$ because the coordinates of x corresponding to C_i are the complete set of ℓ_i-th roots of unity.

We now show $1 \in \text{spec}(M_\sigma - J)$ has a geometric multiplicity of $t-1$. Fix $1 \le i \le t-1$ and consider $\mathbf{x} \in \mathbb{C}^{n+1}$ where

$$\mathbf{x}_j = \begin{cases} 1 & : j \in C_i \\ -\frac{\ell_i}{\ell_{i+1}} & : j \in C_{i+1} \\ 0 & : \text{otherwise.} \end{cases}$$

For $j \in C_i$ we have,

$$((M_\sigma - J)\mathbf{x})_j = -(\ell_i - 1) + \frac{\ell_i}{\ell_{i+1}}(\ell_{i+1}) = 1 = \mathbf{x}_j,$$

for $j \in C_{i+1}$ we have,

$$((M_\sigma - J)\mathbf{x})_j = -\ell_i + \frac{\ell_i}{\ell_{i+1}}(\ell_{i+1} - 1) = -\frac{\ell_i}{\ell_{i+1}} = \mathbf{x}_j,$$

and for $j \notin C_i, C_{i+1}$ we have

$$((M_\sigma - J)\mathbf{x})_j = -\ell_i + \frac{\ell_i}{\ell_{i+1}}(\ell_{i+1}) = 0 = \mathbf{x}_j.$$

Therefore, $(1, \mathbf{x})$ is an eigenpair for $1 \leq i \leq t$; moreover, these vectors are linearly independent.

Finally, consider the all-ones vector $\mathbf{1} \in \mathbb{C}^{n+1}$ where

$$(M_\sigma - J)\mathbf{1} = -n\mathbf{1}$$

so $(-n, \mathbf{1})$ is an eigenpair of $M_\sigma - J$. We have shown

$$\operatorname{spec}(M_\sigma - J) \supseteq (\operatorname{spec}(M_\sigma) \setminus \{1\}) \cup \{-n\}$$

and the reverse inclusion follows from the observation that the multiplicities on the right-hand side add up to (at least, and therefore exactly) $n + 1$. □

We now prove Theorem 4.6.

Proof. Consider

$$C_H = \sum_{\sigma \in \mathfrak{D}_{k+1}} \frac{\tau_\sigma}{\prod_{v \in V(D_\sigma)} \deg^-(v)}.$$

Applying Lemma 4.2 (choosing $d = k + 1$) implies that each vertex of H is a root of exactly one edge in an Euler rooting of H. It follows that, for all $\sigma \in \mathfrak{D}_{k+1}$,

$$\prod_{v \in V(D_\sigma)} \deg^-(v) = (k - 1)^{k+1}.$$

Thus,

$$C_H = \frac{1}{(k-1)^{k+1}} \sum_{\sigma \in \mathfrak{D}_{k+1}} \tau_\sigma.$$

Consider $\sigma = c_1 c_2 \ldots c_t$ where cycle c_i has length ℓ_i. Observe that

$$\mathcal{L}(D_\sigma) = kI + M_\sigma - J.$$

Since $\operatorname{spec}(kI + M_\sigma - J) = k + \operatorname{spec}(M_\sigma - J)$ we have by Lemma 4.7 and Kirchoff's theorem

$$\tau_\sigma = \frac{\prod_{i=1}^{t}\left(k^{l_i} + (-1)^{l_i+1}\right)}{k+1}.$$

The desired equality follows by substitution. □

Remark 4.8. It was shown in [12] that

$$\phi_{k+1}(H) = -C_k(k-1)^{n-k}$$

and by Theorem 3.27 we have

$$\phi_{k+1}(H) = -C_H(k-1)^n.$$

In our notation, we can write $C_k = (k-1)^k C_H$.

For ease of computation we consider C_k instead of C_H. As it is stated, Theorem 4.6 is slow to compute as $|\mathfrak{D}_{n+1}| \sim n!/e$. However, summing over all derangements of $[k+1]$ is wasteful, as we have shown C_k is a function only of the cycle structure of σ. In fact,

$$\prod_{i=1}^{t}\left(k^{l_i} + (-1)^{l_i+1}\right)$$

is constant for derangements with the same cycle type. We present a reformulation of Theorem 4.6 which has the advantage of considering a smaller search space.

Definition 4.9. Let $P(n)$ be the *set of partitions of* n and let $P_{\geq 2}(n) \subseteq P(n)$ be the *set of partitions of* n *into parts of size at least* 2. For $p \in P_{\geq 2}(k+1)$, let $\mathfrak{D}_{k+1}(p) \subseteq \mathfrak{D}_{k+1}$ be the set of derangements whose cycle lengths agree with the parts of p. Further, for a partition $p \in P(n)$ let $V_p : [n] \to [0, n]$ be the map $V_p(i) = |\{j : p_j = i\}|$.

We reformulate Theorem 4.6 (for C_k) as follows.

Corollary 4.10.

$$C_k = \frac{1}{(k-1)(k+1)} \sum_{p=(p_1,\ldots,p_t)\in P_{\geq 2}(k+1)} \left(|\mathfrak{D}_{k+1}(p)| \prod_{i=1}^{t} \left(k^{p_i} + (-1)^{p_i+1} \right) \right)$$

where

$$|\mathfrak{D}_{k+1}(p)| = \frac{(k+1)!}{\left(\prod_{i=1}^{t} p_i\right)\left(\prod_{i=2}^{k+1} V_p(i)\right)}.$$

Proof. It is sufficient to show

$$|\mathfrak{D}_{k+1}(p)| = \frac{(k+1)!}{\left(\prod_{i=1}^{t} p_i\right)\left(\prod_{i=2}^{k+1} V_p(i)!\right)}$$

for $p = (p_1, \ldots, p_t) \in P_{\geq 2}(k+1)$. Let $\Delta : \mathfrak{S}_{k+1} \to \mathfrak{D}_{k+1}(p)$ by

$$\Delta(\sigma) = (\sigma(1), \sigma(2), \ldots, \sigma(p_1))(\sigma(p_1+1), \ldots, \sigma(p_1+p_2)) \ldots.$$

Note that Δ is surjective. Since a cycle of $\Delta(\sigma)$ can be written with any one of its p_i elements first and there are $\prod_{i=2}^{k+1} V_p(i)!$ linear orderings of the cycles by non-increasing length we have for $\delta \in \mathfrak{D}_{k+1}$

$$|\Delta^{-1}(\delta)| = \left(\prod_{i=1}^{t} p_i\right)\left(\prod_{i=2}^{k+1} V_p(i)!\right).$$

In particular, $|\Delta^{-1}(\delta)|$ is constant for $\delta \in \mathfrak{D}_{k+1}(p)$ so that

$$|\mathfrak{S}_{k+1}| = \sum_{\delta \in \mathfrak{D}_{k+1}(p)} |\Delta^{-1}(\delta)|$$

which implies

$$|\mathfrak{D}_{k+1}(p)| = \frac{|\mathfrak{S}_{k+1}|}{|\Delta^{-1}(\delta)|}.$$

□

Remark 4.11. Corollary 4.10 reduces the number of summands in the computation of C_k exponentially because

$$\log |P_{\geq(2)}(n)| \leq \log |P(n)| \approx \pi\sqrt{2n/3},$$

but $\log |\mathfrak{D}_{n+1}| \approx n \log n$.

We have computed the first few values of C_k to be

$$
\begin{aligned}
C_6 &= 2092206 \\
C_7 &= 220611384 \\
C_8 &= 31373370936 \\
C_9 &= 5785037767440 \\
C_{10} &= 1342136211324090 \\
&\vdots \\
C_{100} &= 343345241982479590844776717586346303452689609890358711139013913758 \\
&\quad 779957888170716788865639598053642953208929209278848309297069686374206 \\
&\quad 618031496101898485314300253248855334075609527915686375386625810970778 \\
&\quad 814182546067369319275314946445603388115577892354872286012782651661555 \\
&\quad 310652736903712206018666865354152426390366852479991417222805659546614 5 \\
&\quad 2080249009900
\end{aligned}
$$

so that $C_{100} \approx 3.433... \cdot 10^{343}$, see A320653 [36]. Note that $C_2 = 2$ gives the well-known result that, for a graph G

$$\phi_3(G) = -2(\# \text{ of triangles in } G).$$

We conclude by presenting the asymptotics of C_k.

Theorem 4.12. $C_k \sim (k+1)!k^{k+1}$ *so* $C_k = \exp(k \log k(2 + o(1)))$.

Proof. As σ is a derangement it follows that the length of each cycle is at least 2. Notice

$$|\mathfrak{D}_{k+1}|(k^2-1)^{\frac{k+1}{2}} \leq \sum_{\sigma = c_1 c_2 \ldots c_t \in \mathfrak{D}_{k+1}} \left(\prod_{i=1}^{t} k^{l_i} + (-1)^{l_i+1} \right) \leq |\mathfrak{D}_{k+1}|(k^3-1)^{\frac{k+1}{e}}$$

which implies

$$\sum_{\sigma = c_1 c_2 \ldots c_t \in \mathfrak{D}_{k+1}} \left(\prod_{i=1}^{t} k^{l_i} + (-1)^{l_i+1} \right) \sim \frac{(k+1)!k^{k+1}}{e}.$$

Thus,

$$\lim_{k\to\infty} \frac{C_k}{(k+1)!k^{k+1}} = \lim_{k\to\infty} \frac{|\mathfrak{D}_{k+1}|k^{k+1}}{(k+1)!k^{k+1}} = \frac{1}{e}$$

and we have $C_k \sim (k+1)!k^{k+1}$. □

4.3 Low-Codegree Coefficients of 3-Graphs

We provide an explicit formula for the first six proper leading coefficients of the characteristic polynomial of a 3-graph. Let $\mathcal{H}$ be a simple 3-graph with n vertices. We write $(\# H \in \mathcal{H}) = |\{S \subseteq \mathcal{H} : S \cong \underline{H})|$. From [12] we have that $c_1 = 0, c_2 = 0$,

$$c_3 = -3 \cdot 2^{n-3}(\# e \in \mathcal{H}),$$

where $e = K_3^{(3)}$ is the single-edge hypergraph and

$$c_4 = 21 \cdot 2^{n-3}(\# K_4^{(3)} \in \mathcal{H}).$$

We use Theorem 3.27 to situation these results and further provide an analogous description of c_5 and c_6. Clearly there are no Veblen 3-graphs with one or two edges as each vertex must have degree at least three. It follows that $c_1, c_2 = 0$. There is only one Veblen 3-graph with three edges, the single edge with multiplicity three. By Theorem 3.27, then, we have

$$c_3 = -2^n \cdot \frac{3}{8}(\# e \in \mathcal{H}).$$

Similarly, [12] observed that the only Veblen 3-graph with four edges is the complete graph, which implies

$$c_4 = -2^n \cdot \frac{21}{8}(\# K_4^{(3)} \in \mathcal{H}).$$

For the case of c_5 we have

$$c_5 = -2^n \left(\frac{51}{16}(\# \Gamma_{5,1} \in \mathcal{H}) + \frac{27}{16}(\# \Gamma_{5,2} \in \mathcal{H})\right)$$

where $\Gamma_{5,1}$ is the tight 5-cycle and $\Gamma_{5,2}$ is the 3-pointed crown (given explicitly in Figure 4.1). We further compute

$$c_6 = 2^{2n} \cdot \frac{9}{64}\left(\frac{(\# e \in \mathcal{H})^2}{2}\right) - 2^n \left(\frac{3}{16}(\# e \in \mathcal{H}) + \frac{9}{8}(\# \Gamma_{6,1} \in \mathcal{H}) + \frac{9}{32}(\# \Gamma_{6,2} \in \mathcal{H})\right.$$
$$+ \frac{99}{32} \cdot 2(\# \Gamma_{6,3} \in \mathcal{H}) + \frac{213}{16}(\# \Gamma_{6,4} \in \mathcal{H}) + \frac{69}{16}(\# \Gamma_{6,5} \in \mathcal{H}) + \frac{63}{32}(\# \Gamma_{6,6} \in \mathcal{H})$$
$$\left.+ \frac{129}{32}(\# \Gamma_{6,7} \in \mathcal{H}) + \frac{27}{32} \cdot 2(\# \Gamma_{6,8} \in \mathcal{H}) + \frac{63}{16}(\# \Gamma_{6,9} \in \mathcal{H}) + \frac{117}{32}(\# \Gamma_{6,10} \in \mathcal{H})\right)$$

where $\Gamma_{6,i}$ are provided in Figure 4.1.

Table 4.1 Some connected Veblen 3-graphs and corresponding values.

Γ	$E(\Gamma)$	C_Γ	$\lvert\mathrm{Aut}(\underline{\Gamma})\rvert / \lvert\mathrm{Aut}(\Gamma)\rvert$
$\Gamma_{5,1}$	$(123)(125)(145)(234)(345)$	51/16	1
$\Gamma_{5,2}$	$(123)(145)(145)(234)(235)$	27/16	1
$\Gamma_{6,1}$	$(123)^3(124)^3$	9/8	1
$\Gamma_{6,2}$	$(123)^3(145)^3$	9/32	1
$\Gamma_{6,3}$	$(123)^2(124)(135)(145)^2$	99/32	2
$\Gamma_{6,4}$	$(123)(124)(125)(134)(135)(145)$	213/16	1
$\Gamma_{6,5}$	$(123)(124)(156)(256)(345)(346)$	69/16	1
$\Gamma_{6,6}$	$(123)(124)(145)(246)^3$	63/32	1
$\Gamma_{6,7}$	$(123)(134)(145)(246)(256)^2$	129/32	1
$\Gamma_{6,8}$	$(123)^2(124)(356)(456)^2$	27/32	2
$\Gamma_{6,9}$	$(123)(124)(134)(256)(356)(456)$	63/16	1
$\Gamma_{6,10}$	$(123)(124)(135)(246)(356)(456)$	117/16	1
$\Gamma_{9,2}$	$(123)^6(145)^3$	9/32	2
$\Gamma_{9,3}$	$(123)^3(145)^3(246)^3$	9/8	1
$\Gamma_{9,4}$	$(123)^3(145)^3(167)^3$	81/128	1
$\Gamma_{12,1}$	$(123)^9(145)^3$	9/32	2
$\Gamma_{12,2}$	$(123)^6(145)^6$	27/64	1
$\Gamma_{12,3}$	$(123)^6(145)^3(167)^3$	81/128	3
$\Gamma_{12,4}$	$(123)^6(145)^3(246)^3$	63/32	3
$\Gamma_{12,5}$	$(123)^3(145)^3(167)^3(246)^3$	459/64	1
$\Gamma_{12,6}$	$(123)^3(145)^3(246)^3(356)^3$	255/16	1

Remark 4.13. The Fano Plane is a Veblen 3-graph with seven edges. The associated coefficient of the Fano Plane is $87/16 = 5.4375$.

Chapter 5

Stably Computing the Multiplicity of Known Roots

Given a k-graph $\mathcal{H}$ we aim to compute $\phi(\mathcal{H})$. Unfortunately, the resultant is known to be NP-hard to compute [22]. An efficient method for computing the resultant, even in special cases, would impact several fields of mathematics, perhaps nowhere more so than computational algebraic geometry. Performing this computation directly using built-in routines has proven to be intractable even for 3-uniform hypergraphs with few vertices. Nonetheless, one can attempt to imitate classical approaches to computing characteristic polynomials of ordinary graphs.

A notable example of this is the work of Harary [23] (and Sachs [33]) who showed that the coefficients of $\phi(G)$ can be expressed as a certain weighted sum of the counts of subgraphs of G. This formula allows one to compute many low codegree coefficients – i.e., the coefficients of x^{d-k} for k small and $d = \deg(\phi_{\mathcal{H}})$ – by a certain linear combination of subgraph counts in $\mathcal{H}$. Alas, this computation becomes exponentially harder as the codegree increases, making computation of the entire (often extremely high degree) characteristic polynomial impossible for all but the simplest cases.

We could instead view the leading coefficients as the result of mixing minimal polynomials of $\phi(\mathcal{H})$ according to their multiplicity. We make this problem precise as follows.

Problem 5.1. Let K be a field of characteristic zero. Is it true that a monic polynomial $p \in K[x]$ of degree n with exactly k distinct, known roots is determined by its k proper leading coefficients?

We answer Problem 5.1 via an algorithm which allows us to stably compute the multiplicity of each root exactly. The algorithm is stable in the sense that if an eigenvalue is approximated by an ε-disk, where ε depends "reasonably" on the parameters of the problem, the resulting disk approximating its multiplicity contains exactly one integer. Furthermore, the computational complexity of the algorithm is quasi-linear in nb where $b = O(\ln \varepsilon)$ and n is the number of roots. We address the problem of computing $\sigma(\mathcal{H})$ in the following chapter.

5.1 Determining the Multiplicities of Known Roots

Given the set of roots of a polynomial without multiplicity and an appropriate number of leading coefficients, one can determine the multiplicity of its roots using the Faddeev-LeVerrier algorithm, a matrix form of the Newton Identities. We analogize this result as follows.

Theorem 5.2. *Let $p = \sum_{i=0}^{n} c_i x^{n-i} \in K[x]$ be a monic polynomial with known degree n, $k+1$ distinct roots, $r_0 = 0, r_1, r_2, \ldots, r_k$, with multiplicity $m_0, m_1, m_2, \ldots m_k$, respectively. Then the multiplicities are uniquely determined by $c_0 = 1, c_1, \ldots, c_k$.*

Furthermore, p may be determined by fewer than k proper coefficients when K is not algebraically closed.

Theorem 5.3. *Let $p = \sum_{i=0}^{n} c_i x^{n-i} \in K[x]$ be a monic polynomial such that $p(0) \neq 0$. Suppose $p = \prod_{i=1}^{t} q_i^{m_i}$ for $q_i \in K[x]$. The multiplicity vector $\mathbf{m} = \langle m_1, \ldots, m_t \rangle^T$ is uniquely determined by the t proper leading coefficients if and only if $V \in \overline{K}^{t \times t}$ is non-singular where*

$$V_{i,j} = \sum_{r : q_j(r) = 0} r^i.$$

Remark 5.4. Observe that when $q_i = x - r_i$ (i.e., p splits over K) Theorem 5.3 provides the same conclusion as Theorem 5.2.

We begin with a proof of Theorem 5.2.

Proof of Theorem 5.2. Fix such a monic polynomial p with distinct roots $r_0 = 0$, r_1, …, r_k with respective multiplicities m_0, m_1, …, m_k. Ignoring r_0 for a moment, let $\mathbf{r} = \langle r_1, \ldots, r_k \rangle^T$ and $\mathbf{m} = \langle m_1, \ldots, m_k \rangle^T$. We denote the Vandermonde matrix

$$V = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ r_1 & r_2 & \ldots & r_k \\ r_1^2 & r_2^2 & \ldots & r_k^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_1^{k-1} & r_2^{k-1} & \ldots & r_k^{k-1} \end{pmatrix}$$

and consider

$$V_0 = \begin{pmatrix} r_1 & r_2 & \ldots & r_k \\ r_1^2 & r_2^2 & \ldots & r_k^2 \\ \vdots & \vdots & \vdots & \vdots \\ r_1^k & r_2^k & \ldots & r_n^k \end{pmatrix}.$$

Let $\mathbf{p} \in \overline{K}^n$ where

$$\mathbf{p}_i := \sum_{j=1}^{k} r_j^i m_j$$

then

$$V_0 \mathbf{m} = \mathbf{p}.$$

Notice that $V_0 = V \operatorname{diag}(\mathbf{r})$ is non-singular as it is the product of two non-singular matrices. We have then

$$\mathbf{m} = V_0^{-1} \mathbf{p}. \tag{5.1}$$

We present a formula for $\mathbf{p}$ which is a function of only the leading $k+1$ coefficients of p.

Let A be the diagonal matrix where r_i occurs m_i times and note

$$p(x) = \det(xI - A).$$

By the Faddeev-LeVerrier algorithm (aka the Method of Faddeev [19]) we have for $j \geq 1$

$$c_j = -\frac{1}{j}\sum_{i=1}^{j} c_{j-i}\operatorname{tr}(A^i) = -\frac{1}{j}\sum_{i=1}^{j} c_{j-i}\mathbf{p}_i. \tag{5.2}$$

Let $\mathbf{c} = \langle c_1, c_2, \ldots, c_k\rangle^T$, $\Lambda = -\operatorname{diag}(1, 1/2, \ldots, 1/k)$, and

$$C = \begin{pmatrix} c_0 = 1 & 0 & \ldots & 0 \\ c_1 & c_0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ c_{k-1} & c_{k-2} & \ldots & c_0 \end{pmatrix}.$$

By Equation 5.2, $\mathbf{c} = \Lambda C\mathbf{p}$. Moreover, as Λ and C are invertible we have $\mathbf{p} = (\Lambda C)^{-1}\mathbf{c}$. It follows from Equation 5.1 that

$$\mathbf{m} = (\Lambda C V_0)^{-1}\mathbf{c}. \tag{5.3}$$

Furthermore, $m_0 = n - \mathbf{1} \cdot \mathbf{m}$. □

We briefly remark about the proof of Theorem 5.2. Problem 5.1 has a flavor of polynomial interpolation: given k points, how many (univariate) polynomials of degree n go through each of the k points? If $n \leq k - 1$ the polynomial is known to be unique and is relatively expensive to compute (as any standard text in numerical analysis will attest). Our proof technique mimics this approach as the classical problem of determining $p = \sum_{i=0}^{k-1} c_i x^{(k-1)-i}$, which resembles k distinct points $\{(x_i, y_i)\}_{i=1}^{k}$, can be solved by computing

$$V^T\mathbf{c} = \mathbf{y}$$

where $\mathbf{c} = \langle c_{k-1}, c_{k-2}, \ldots, c_0\rangle^T$, V is as previously defined given $r_i = x_i$, and $\mathbf{y} = \langle y_1, \ldots, y_k\rangle^T$. Suppose for a moment that each root is distinct so that

$$p(x) = \prod_{i=1}^{k}(x - r_i).$$

Then c_j, the codegree-j coefficient, is precisely the jth elementary symmetric polynomial in the variables $r_1, \dots, r_k$. In the case of repeated roots we have that c_j can be expressed using modified symmetric polynomials in the distinct roots where r_i is replaced with $\binom{m_i}{j} r_i^j$. The expression for each coefficient via these modified symmetric polynomials is given by Equation 5.2.

Note that if the roots of p are known, it is possible to determine p with fewer coefficients than the number of distinct roots (e.g., when p is a non-linear minimal polynomial). We modify Theorem 5.2 to include the case when some of the roots are known to occur with the same multiplicity. We now prove Theorem 5.3.

Proof of Theorem 5.3. The proof follows similarly to that of Theorem 5.2. First suppose that V is non-singular. Let $\mathbf{m} = \langle m_1, \dots, m_t \rangle^T$ and

$$\mathbf{p}_i = \sum_{j=1}^{t} \left(\sum_{r : q_j(r)=0} r^i \right) m_j = (V\mathbf{m})_i$$

so that if V is non-singular, $\mathbf{m} = V^{-1}\mathbf{p}$. Let A be defined as in Theorem 5.2: the diagonal matrix where the roots of q_i occur m_i times and note

$$p(x) = \det(xI - A).$$

We have by the the Faddeev-LeVerrier algorithm, for $j \geq 1$

$$c_j = -\frac{1}{j} \sum_{i=1}^{j} c_{j-i} \operatorname{tr}(A^i) = -\frac{1}{j} \sum_{i=1}^{j} c_{j-i} \mathbf{p}_i$$

so that for $\mathbf{c} = \langle c_1, \dots, c_t \rangle^T$, $\Lambda = -\operatorname{diag}(1, 1/2, \dots, 1/t)$ we have

$$\mathbf{m} = (\Lambda C V)^{-1} \mathbf{c}.$$

If instead $\mathbf{m}$ is uniquely determined by the first t proper coefficients then $V\mathbf{m} = (\Lambda C)^{-1}\mathbf{c}$ has exactly one solution, hence V is non-singular. □

As a non-example, consider the minimal polynomial of $\alpha = \sqrt{2}$, $q_\alpha(x) = x^2 - 2 \in \mathbb{Z}[x]$ and suppose

$$p = q_\alpha^d = x^{2d} + 0x^{2d-1} - 2dx^{2d-2} + \cdots \in \mathbb{Z}[x].$$

Observe that we cannot determine d given $c_1 = 0$; moreover, this conclusion is unsurprising, given the hypotheses from Theorem 5.3, since

$$V = [\sqrt{2} - \sqrt{2}] = [0] \in \mathbb{C}^{1,1}$$

is singular. However, by inspection we could determine d given $c_2 = -2d$ and in fact the matrix $[\sqrt{2}^2 + (-\sqrt{2})^2] = [4] \in \mathbb{C}^{1\times 1}$ is non-singular. Indeed we could determine d with a simple change of variable: apply Theorem 5.3 to $p_0 = (y-2)^d$ where $y = x^2$.

5.2 The Stability of Computing Multiplicities

We now consider the feasibility of computing $\mathbf{m}$ over $\mathbb{C}$. In general, the matrix V_0 in Theorem 5.2 may be poorly conditioned, so this calculation is often difficult to carry out even for modest values of k. The goal of this section is to show that if each root of a monic polynomial $p(x) \in \mathbb{Z}[x]$ is approximated by a disk of radius at most ε, a "reasonable" precision, then the interval approximating $\mathbf{m}_i$, resulting from a particular algorithm, contains exactly one integer. That is, we provide an algorithm for exactly computing $\mathbf{m}$ via SageMath [10] with substantially improved numerical stability over simply applying the Newton Identities.

Theorem 5.5. *Let $p(x) = \sum_{i=0}^{t} c_i x^{t-i} \in \mathbb{Z}[x]$ be a monic polynomial with distinct nonzero roots $r_1, \ldots, r_n$ such that $|r_1| \geq |r_2| \geq \cdots \geq |r_n| > 0$. If each root is approximated by a disk of radius ε such that*

$$\varepsilon < \frac{m^2 r}{2^{2n+7} n^5} \left(\frac{m}{MRc}\right)^n = \left(\frac{m}{MRc}\right)^{n(1+o(1))}$$

where

- $M = \max\{\max\{|r_i - r_j|\}, 1\}$ *and* $m = \min\{\min\{|r_i - r_j| : i \neq j\}, 1\}$
- $R = \max\{|r_1|, 1\}$ *and* $r = \min\{|r_n|, 1\}$
- $c = \max\{\max\{|c_i|\}_{i=1}^{n}, 1\}$.

then the resulting disk approximating $\mathbf{m}_i = (\Lambda C V_0)^{-1}\mathbf{c}_i \in \mathbb{Z}$ *contains exactly one integer (i.e., the computation of* $\mathbf{m}$ *is exact).*

Notice that $MRc \geq 1$ and $MRc = 2$ for $x^n - 1$ when n is even. Roots of unity occur frequently in the spectrum of hypergraphs; see Section 3. In particular, k-cylinders – essentially k-colorable k-graphs – have a spectrum which is invariant under multiplication by the kth roots of unity. Consider now $p(x) = x^n - 1$. We have $m = \sqrt{2 - 2\cos(\frac{2\pi}{n})}$ so that

$$\varepsilon < \frac{(2 - 2\cos(\frac{2\pi}{n}))^{\frac{n+2}{2}}}{2^{3n+7}n^5}.$$

While ε may seem small, we are chiefly concerned with the number of bits of precision needed to approximate each root. Indeed for $x^n - 1$ we need $|\ln \varepsilon| = O(n \ln n)$ bits of precision by the small-angle approximation.

Remark 5.6. The bound on ε is "reasonable", as the number of bits required to approximate each root is proportional to the number of distinct roots of p and the logarithms of the ratio of the smallest difference of the roots with the largest difference of roots, the largest root, and the largest coefficient.

For the algorithm we describe below, the computational complexity of computing $\mathbf{m}$ is quasi-linear in nb where $b = O(\ln \varepsilon)$ is the requisite number of bits of precision given. More precisely, the complexity is the maximum of $O(nb \ln nb)$ and $O(n^\delta)$ where δ is the best known exponent of matrix multiplication, currently [25] at $\delta \approx 2.373$ but widely believed to be $2 + o(1)$. As we will see, $\mathbf{m}$ can be written as a product of $n \times n$ matrices and length n vectors populated by entries with $O(nb)$ digits. The complexity follows as the number of digits needed to approximate each entry in the product of two such matrices/vectors is $O(nb)$.

In practice, the difficulty of computing $\mathbf{m}$ as described in Theorem 5.2 is in computing the inverse of the Vandermonde matrix, whose entries may vary widely in magnitude and which may be very poorly conditioned. The task of inverting Vandermonde matrices has been studied extensively. Eisenberg and Fedele [15] provide a brief history of the topic as well results concerning the accuracy and effectiveness of several known algorithms. However, these algorithms provide good approximations for the entries of V^{-1}, whereas we seek to express them exactly as elements of the field of algebraic complex numbers, since $\mathbf{m}$ is a vector of integers. Soto-Eguibar and Moya-Cessa [37] showed that $V^{-1} = \Delta W L$ where Δ is the diagonal matrix

$$\Delta_{i,j} = \begin{cases} \prod_{k=1,k\neq i}^{n} \frac{1}{r_i - r_k} & : i = j \\ 0 & : i \neq j, \end{cases}$$

W is the lower triangular matrix

$$W_{i,j} = \begin{cases} 0 & : i > j \\ \prod_{k=j+1,k\neq i}^{n} (r_i - r_k) & : \text{ otherwise}, \end{cases}$$

and L is the upper triangular matrix

$$L_{i,j} = \begin{cases} 0 & : i < j \\ 1 & : i = j \\ L_{i-1,j-1} - L_{i-1,j} r_{i-1} & : i \in [2, n], j \in [2, i-1]. \end{cases}$$

Using this decomposition, it is possible to compute $\mathbf{m}$ exactly. To prove Theorem 5.5 we first provide an upper bound for the diameter of the disk approximating an entry of Δ, W, and L, respectively; to do so, we extensively employ computations of Petković and Petković [29] found in Chapter 1.3. We present the necessary background here.

Let $D(z, \varepsilon)$ be the open disk in the complex plane centered at z of radius ε. For $A = D(a, r_1)$, $B = D(b, r_2)$ complex open disks, we have

1. $A \pm B = D(a \pm b, r_1 + r_2)$
2. $1/B = D\left(\frac{\bar{b}}{|b|^2 - r_2^2}, \frac{r_2}{|b|^2 - r_2^2}\right)$
3. $AB = D(ab, |a|r_2 + |b|r_2 + r_1 r_2)$

In particular, for the special case of A^n we have

$$D(a, r_1)^n = D(a^n, (|a| - r_1)^n - |a|^n). \tag{5.4}$$

Moreover, given $0 < r_1 < 1 \leq |a|$

$$(|a| - r_1)^n - |a|^n \leq r_1(2|a|)^n \tag{5.5}$$

since

$$(|a| - r_1)^n - |a|^n \leq \sum_{k=1}^{n} \binom{n}{k} r_1^k |a|^{n-k} \leq r_1(2|a|)^n.$$

Finally, let $d(A) = 2r_1$ denote the diameter of A and let

$$|A| = |a| + r_1$$

be the absolute value of A. Then for $u \in \mathbb{C}$ we have

1. $d(A \pm B) = d(A) + d(B)$
2. $d(uA) = |u|d(A)$
3. $d(AB) \leq |B|d(A) + |A|d(B)$

For the remainder of this paper some numbers will be exact (e.g., rational numbers) while others will be approximated by a disk. The non-exact entries of a matrix $M \in \mathbb{C}^{n \times n}$ will be referred to as disks; this will be clear from the problem formulation or derived from the computations. With a slight abuse of notation we use $d(M_{i,j})$ and $|M_{i,j}|$ to denote the diameter and absolute value of the disk approximating the entry $M_{i,j}$. Moreover, we write

$$d(M) = \max\{d(M_{i,j}) : i, j \in [n]\} \text{ and } |M| = \max\{|M_{i,j}| : i, j \in [n]\}.$$

In the case when the entry is exact, the diameter is zero and the absolute value (of the disk) is simply the modulus.

Theorem 5.7. *Assume the notations of Theorem 5.3, let V denote the Vandermonde matrix from the proof of Theorem 5.2, and let $V^{-1} = \Delta W L$ by Soto-Eguibar and Moya-Cessa [37]. Then*

$$d(V^{-1}) \leq \frac{2^{2n+4}n}{m^2}\left(\frac{MR}{m}\right)^n \varepsilon.$$

and

$$|V^{-1}| \leq 2n\left(\frac{RM}{m}\right)^n.$$

Proof. Let

$$D_i := D(r_i, \varepsilon)$$

denote the disk centered at r_i with radius ε. By Equation 5.5 we have for $s \neq t$

$$\begin{aligned} d(\Delta) &\leq d\left(\left(\frac{1}{D_s - D_t}\right)^n\right) \\ &\leq 2^n \left(\frac{2\varepsilon}{m^2 - (2\varepsilon)^2}\right)\left(\frac{m}{m^2 - (2\varepsilon)^2}\right)^n \\ &\leq \frac{2^{2n+2}}{m^{n+2}} \cdot \varepsilon, \end{aligned}$$

since $\varepsilon < m/4$,

$$d(W) \leq d((D_s - D_t)^n) \leq 2^{n+1} M^n \varepsilon,$$

and

$$d(L) \leq d(D_s^n) \leq (2R)^n \varepsilon.$$

We first consider $d(\Delta W)$. Observe that ΔW is upper triangular and each nonzero entry of ΔW is a product of exactly one nonzero entry of Δ and W. In this way

$$d((\Delta W)_{i,j}) \leq |W_{i,j}| d(\Delta_{i,i}) + |\Delta_{i,i}| d(W_{i,j}) \leq \frac{2^{2n+3}}{m^2}\left(\frac{M}{m}\right)^n \varepsilon$$

and

$$|\Delta W| \leq 2\left(\frac{M}{m}\right)^n.$$

We now determine $d(\Delta W L)$ by first computing

$$d((\Delta W)_{i,k} L_{k,j}) \leq |L_{k,j}| d(\Delta W_{i,k}) + |\Delta W_{i,k}| d(L_{k,j}) \leq \frac{2^{2n+4}}{m^2}\left(\frac{RM}{m}\right)^n \varepsilon.$$

Hence

$$d(V^{-1}) = d(\Delta W L) \leq \max_{i,j} \sum_{k=1}^{n} d((\Delta W)_{i,k} L_{k,j}) \leq \frac{2^{2n+4} n}{m^2} \left(\frac{RM}{m} \right)^n \varepsilon$$

and

$$|V^{-1}| \leq 2n \left(\frac{RM}{m} \right)^n .$$

□

In our computations we are concerned with $V_0 = V \cdot \operatorname{diag}(\mathbf{r})$ where $\operatorname{diag}(\mathbf{r}) = \operatorname{diag}(r_1, \ldots, r_n)$ so that

$$V_0^{-1} = \operatorname{diag}(\mathbf{r})^{-1} V^{-1}.$$

The following Corollary is immediate from the observation that

$$d(\operatorname{diag}(\mathbf{r})^{-1}) \leq \frac{2}{r}.$$

Corollary 5.8.

$$d(V_0^{-1}) \leq \frac{2^{2n+6} n}{m^2 r} \left(\frac{MR}{m} \right)^n \varepsilon$$

and

$$|V_0^{-1}| \leq \frac{2n}{r} \left(\frac{RM}{m} \right)^n .$$

We are now able to prove Theorem 5.5.

Proof of Theorem 5.5. Recall $\mathbf{m} = V_0^{-1} C^{-1} \Lambda^{-1} \mathbf{c}$ as defined in the proof of Theorem 5.2. Fortunately, the remainder of the computations are straightforward as C^{-1}, Λ^{-1}, and $\mathbf{c}$ have integer, and thus exact, entries. As

$$C_{i,j}^{-1} = \begin{cases} 0 & : i < j \\ 1 & : i = j \\ -\sum_{k=1}^{i-1} c_{i-k} C_{k,j}^{-1} & : i > j \end{cases}$$

we have

$$d(V_0^{-1} C^{-1}) \leq n(nc^{n-1}) \frac{2^{2n+6} n}{m^2 r} \left(\frac{MR}{m} \right)^n \varepsilon = \frac{2^{2n+6} n^3}{m^2 r c} \left(\frac{MRc}{m} \right)^n \varepsilon$$

Further, since $\Lambda^{-1} = -\operatorname{diag}(1, 2, \ldots, n)$ we have

$$d(V_0^{-1}C^{-1}\Lambda^{-1}) \leq |-n| d(V_0^{-1}C^{-1}) = \frac{2^{2n+6}n^4}{m^2rc}\left(\frac{MRc}{m}\right)^n \varepsilon$$

and, finally,

$$\begin{aligned} d(V_0^{-1}C^{-1}\Lambda^{-1})\mathbf{c}) &\leq nc \cdot d(V_0^{-1}C^{-1}\Lambda^{-1}) \\ &\leq \frac{2^{2n+6}n^5}{m^2r}\left(\frac{MRc}{m}\right)^n \varepsilon < \frac{1}{2}. \end{aligned}$$

Thus each interval will contain at most one integer as desired. □

Chapter 6

Determining the Set Spectrum of a Hypergraph

In the previous chapter we showed that the characteristic polynomial of a hypergraph could be stably computed given its set spectrum and an appropriate number of leading coefficients. We now consider the problem of determining the set spectrum of $\mathcal{H}$. In this chapter we summarize the Lu-Man Method which allows us to compute the *totally nonzero eigenvalues* (defined below) of a hypergraph. We then prove that the set spectrum of a hypergraph is contained in the union of the totally nonzero eigenvalues of all of its subgraphs. We further provide a partial characterization of precisely which of these eigenvalues extend to eigenvalues of the host hypergraph. This methodology is novel to spectral hypergraph theory as the same statement is not true for graphs (c.f. Cauchy Interlacing Theorem).

6.1 The Lu-Man Method

The Lu-Man Method was introduced in [27] to study the spectral radius of a hypergraph and was further developed in [2], [40], etc. The method is based on the concept of an "α-consistent labeling" which is an assignment of complex values to every vertex-edge pair which satisfy certain conditions.

Definition 6.1. ([40]) A generalized weighted incidence matrix B of a hypergraph H is a $|V| \times |E|$ matrix such that for any vertex v and any edge e, the entry $B(v,e) = 0$ if $v \notin e$ and $B(v,e) \neq 0$ if $v \in e$.

Definition 6.2. ([40]) A hypergraph is called consistently generalized α-normal if there exists a generalized weighted incidence matrix B satisfying

1. $\sum_{e:v\in e} B(v,e) = 1$, for any $v \in V(H)$

2. $\prod_{v\in e} B(v,e) = \alpha$ for any $e \in E(H)$

3. for any cycle $v_0 e_1 v_1 e_2 \dots v_\ell$ ($v_\ell = v_0$),

$$\prod_{i=1}^{\ell} \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1.$$

A *totally nonzero eigenvalue* of a hypergraph is a nonzero eigenvalue which corresponds to an eigenvector with all nonzero entries. We denote the set of all totally nonzero eigenvalues of $\mathcal{H}$ as $\sigma^+(\mathcal{H}) \subseteq \sigma(\mathcal{H})$.

Lemma 6.3. *([40]) Let H be a connected k-uniform hypergraph and $\lambda \neq 0$. Then λ is a totally nonzero eigenvalue of H if and only if H is consistently generalized α-normal with $\alpha = \lambda^{-k}$.*

As the totally nonzero eigenvalues of a hypergraph are roots of its characteristic polynomial it is natural to define the *α-polynomial* of a hypergraph so that $\sigma^+(\mathcal{H}) = \{\lambda : \alpha_{\mathcal{H}}(\lambda) = 0\}$. When appropriate, we write $\alpha(\mathcal{H})$ for simplicity. We refer to the process of determining the α-polynomial of a hypergraph and all of its subgraphs via α-consistent labelings as the *Lu-Man Method*.

Computing $\sigma(\mathcal{H})$ by way of $\sigma^+(H)$ for $H \subseteq \mathcal{H}$ involves solving smaller multi-linear systems than the one involved in computing $\alpha(\mathcal{H})$. Generally speaking, $|\sigma(\mathcal{H})|$ is considerably smaller than the degree of $\phi(\mathcal{H})$. In practice, this approach has yielded $\phi(\mathcal{H})$ when other approaches of computing $\phi(\mathcal{H})$ via the resultant have failed. In general, using the Lu-Man Method to determine the set of all totally nonzero eigenvalues of a hypergraph can be difficult as it involves determining a Gröbner basis a multi-linear system of (possibly non-homogeneous) equations with $k|E| + 1$ unknowns; however, this method has been fruitful in practice.

We begin by providing a partial characterization of eigenvalues which are inherited from subhypergraphs. This characterization allows us to fully describe the spectrum of hypertrees. We conclude by providing a spectral characterization of power trees.

6.1.1 Inheriting Eigenvalues from Subgraphs

Recall that a vector is *totally nonzero* if each coordinate is nonzero and an eigenpair $(\lambda, \mathbf{x})$ is totally nonzero if $\lambda \neq 0$ and $\mathbf{x}$ is a totally nonzero vector. Given a vector $\mathbf{x} \in \mathbb{C}^n$ the *support* $\operatorname{supp}(\mathbf{x})$ is the set of all indices of nonzero coordinates of $\mathbf{x}$. Let $\mathbf{x}^\circ$ denote the totally nonzero projection (by restriction) of $\mathbf{x}$ onto $\mathbb{C}^{|\operatorname{supp}(\mathbf{x})|}$. For ease of notation, we assume that the coordinate indices of vectors agree with the vertex labeling of the hypergraph under consideration. We denote the *induced subgraph* of $\mathcal{H}$ on $U \subseteq V(\mathcal{H})$ by

$$\mathcal{H}[U] = (U, \{v_1 \dots v_k \in E(\mathcal{H}) : v_i \in U\})$$

and write $H \sqsubseteq \mathcal{H}$ to mean $H = \mathcal{H}[U]$ for some $U \subseteq V(\mathcal{H})$. We begin by proving that an eigenpair of $\mathcal{H}$ can be projected onto its support.

Lemma 6.4. *Let $(\lambda, \mathbf{x})$ be a nonzero eigenpair of the normalized adjacency matrix of a k-uniform hypergraph $\mathcal{H}$. Then $(\lambda, \mathbf{x}^\circ)$ is a totally nonzero eigenpair of $\mathcal{H}[\operatorname{supp}(\mathbf{x})]$.*

Proof. As $(\lambda, \mathbf{x})$ is an eigenpair of $\mathcal{H}$,

$$\sum_{i_2.i_2,\dots,i_k=1}^{n} a_{ji_2i_3\dots i_k} x_{i_2} x_{i_3} \dots x_{i_k} = \lambda x_j^{k-1}$$

for $j \in [n]$ by definition. Let $m = |\operatorname{supp}(\mathbf{x})|$ and suppose without loss of generality that supp($\mathbf{x}$) = $[m]$. For $j \in [m]$ we have

$$\begin{aligned}\lambda(x^\circ)_j^{k-1} &= \lambda x_j^{k-1} \\ &= \sum_{i_2.i_2,\ldots,i_k=1}^{n} a_{ji_2i_3\ldots i_k} x_{i_2} x_{i_3} \ldots x_{i_k} \\ &= \sum_{i_2.i_2,\ldots,i_k=1}^{m} a_{ji_2i_3\ldots i_k} x_{i_2} x_{i_3} \ldots x_{i_k}.\end{aligned}$$

Thus, $(\lambda, \mathbf{x}^\circ)$ is an eigenpair of $\mathcal{H}[m]$ by definition; moreover, $(\lambda, \mathbf{x}^\circ)$ is totally nonzero, as each coordinate of $\mathbf{x}^\circ$ is nonzero by construction. □

It is not hard to see that if $(\lambda, \mathbf{x})$ is a nonzero eigenpair of $\mathcal{H}$ then λ is an eigenvalue of $\mathcal{H}[\text{supp}(\mathbf{x})] \subseteq \mathcal{H}$, the subgraph of $\mathcal{H}$ induced by the support of $\mathbf{x}$ [8]. This implies that the nonzero set spectrum of a hypergraph is contained in the union of the totally nonzero eigenvalues of all of its subgraphs (c.f. Cauchy's Interlacing Theorem). That is for $k > 2$,

$$\sigma(\mathcal{H}) \subseteq \bigcup_{H \subseteq \mathcal{H}} \sigma^+(H).$$

We now consider the inverse problem: when can we extend a totally nonzero eigenpair of an induced subgraph of $\mathcal{H}$ to an eigenpair of $\mathcal{H}$?

Definition 6.5. An induced subgraph $\mathcal{H}[U] \subseteq \mathcal{H}$ is *isolated* if $E(\mathcal{H}[U \cup \{v\}]) = E(\mathcal{H}[U])$ for all $v \in V(\mathcal{H})$. Moreover, a hypergraph is *insular* if every connected induced subgraph is isolated.

Intuitively, an isolated induced subgraph has the property that including any one additional vertex to its vertex set does not induce additional edges.

Lemma 6.6. *Suppose $H = \mathcal{H}[U] \subseteq \mathcal{H}$ is an isolated induced subgraph, then for $(\lambda, \mathbf{x}) \in \sigma^+(H)$ we have $(\lambda, \mathbf{x}') \in \sigma(\mathcal{H})$ where $\mathbf{x}'_v = x_v$ for $v \in V(H)$ and is zero otherwise.*

Proof. Let $H = \mathcal{H}[U]$. Recall that $(\lambda, \mathbf{x})$ must satisfy

$$p_i = \lambda x_i^{k-1} - \sum_{\{i,j_1,j_2,\ldots,j_{k-1}\} \in E(H)} x_{j_1} x_{j_2} \cdots x_{j_{k-1}} = 0$$

for $i \in V(\mathcal{H})$. Let $i \in U$ and note that for $\{i, j_1, j_2, \ldots, j_{k-1}\} \notin E(H)$ there is some $t \in [k-1]$ such that $j_t \notin U$ which implies $x_{j_t} = 0$. Indeed

$$x_{j_1} x_{j_2} \cdots x_{j_{k-1}} = 0.$$

Whence $(\lambda, \mathbf{x}^\circ) \in \sigma^+(H)$,

$$\begin{aligned} p_i &= \lambda x_i^{k-1} - \sum_{\{i,j_1,j_2,\ldots,j_{k-1}\} \in E(\mathcal{H})} x_{j_1} x_{j_2} \cdots x_{j_{k-1}} \\ &= \lambda x_i^{k-1} - \sum_{\{i,j_1,j_2,\ldots,j_{k-1}\} \in E(H)} x_{j_1} x_{j_2} \cdots x_{j_{k-1}} = 0. \end{aligned}$$

Instead suppose $i \notin U$. Note that any edge containing i is not an edge in H. Moreover, as H is isolated in $\mathcal{H}$, every edge $\{i, j_1, j_2, \ldots, j_{k-1}\} \notin E(H)$ has at least two vertices not in U. Indeed, there exists a $t \in [k-1]$ such that $x_{j_t} = 0$. In particular,

$$x_{j_1} x_{j_2} \cdots x_{j_{k-1}} = 0$$

so that $p_i = 0$. □

Observe that Lemma 6.6 yields a sufficient condition for extending totally nonzero eigenvalues of subgraphs to eigenvalues of the 'host' hypergraph. We refer to totally nonzero eigenpairs which have this property as *extendable in* $\mathcal{H}$. We have shown that all totally nonzero eigenvalues of an isolated subgraph are extendable; however, the converse of this statement is not always true. Consider the 3-uniform simplex, $K_3^{(4)}$. We claim that the simplex inherits eigenvalues from the single-edge despite the single-edge not being an isolated induced subgraph. In the case when a hypergraph is insular, its spectrum is exactly the union of the totally nonzero eigenvalues of its subgraphs.

The previous argument can be applied to an arbitrary induced subgraph of $\mathcal{H}[U]$ if one can show that $p_i = 0$ for $i \notin U$. This yields the following.

Corollary 6.7. *A totally nonzero eigenpair of $\mathcal{H}[U]$ extends to an eigenpair of $\mathcal{H}$ if and only if $p_i = 0$ for $i \notin U$ as in the proof of Lemma 6.6.*

Remark 6.8. Recall that Theorem 3.27 expresses $\phi_d(\mathcal{H})$ as a weighted sum of *graph* statistics. It is natural to question if our approach to spectral hypergraph theory, via hypermatrices and polynomial maps as discussed in the Introduction, is equivalent to that of a linear spectral theory in some way. Corollary 6.7 suggests that these two approaches are inherently different as demonstrated by this spectral inheritance property.

In the following section we characterize the totally nonzero eigenvalues of a hypertree.

6.2 Spectra of Hypertrees

The following beautiful result was shown in [40]: the set of roots of a certain matching polynomial of a k-uniform hypertree (an acyclic k-uniform hypergraph) is a subset of its homogeneous adjacency spectrum.

Theorem 6.9. *([40]) Let $\mathcal{H}$ be a k-tree. Then*

$$\alpha(\mathcal{H}) = \sum_{i=0}^{m} (-1)^i |\mathcal{M}_i| x^{(m-i)k}$$

where $\mathcal{M}_i$ is the collection of all i-matchings of H.

We present the α-polynomials of 3-trees with six or fewer edges in Table 6.1 and the number of 3-trees is given by A003081 [36]. We now show how to obtain *all* of the eigenvalues of a hypertree, and use this description to give a spectral characterization of "power" hypertrees (defined below). We extend Theorem 6.9, as follows, to describe the spectrum of a hypertree, answering the main open question in [40].

Lemma 6.10. *A 3-graph $\mathcal{H}$ is insular if and only if it is a* hypertree *(i.e., $\mathcal{H}$ is connected and every nontrivial walk $v_1e_1 \dots e_{n-1}v_1$ must have a repeated edge). In particular,*

$$\sigma(\mathcal{H}) = \{0\} \cup \bigcup_{H \sqsubseteq \mathcal{H}} \sigma^+(H)$$

where the union is taken over all connected induced subgraphs, if and only if $\mathcal{H}$ is a hypertree.

Proof. Let $\mathcal{H}$ be a hypertree and let $U \subseteq V(\mathcal{H})$ such that $H = \mathcal{H}[U]$ is a subtree of $\mathcal{H}$. Suppose $e = \{u, v, w\} \in E(\mathcal{H})$ for $u, v \in U$ and $w \in U'$. Note that a hypertree is *linear* (i.e., two edges intersect in at most one vertex). It follows that e is the only edge of $\mathcal{H}$ which contains both u and v. Since $u, v \in U$ and H is connected, there exists a nontrivial walk from $u, e_1, v_2, \dots, e_{t-1}, v$ in H where all the edges are distinct. Note that $u, e_1, v_2, \dots, e_{t-1}, v, e, u$ is a cycle in $\mathcal{H}$, a contradiction. Indeed no such $e \in E(\mathcal{H})$ exists so that H is isolated in $\mathcal{H}$.

Let $\mathcal{H}$ be an insular 3-graph. Note that $\mathcal{H}$ is *linear*, for if this were not the case the hypergraph would contain an edge which was not isolated. Moreover, $\mathcal{H}$ is acyclic as a linear 3-uniform hypercycle (a.k.a. a loose 3-cycle) is not insular as the path formed by removing one vertex is not isolated. □

6.2.1 Spectra of Power Trees

The following generalizes the definition of powers of a hypergraph from [24].

Definition 6.11. Let H be an r-graph for $r \geq 2$. For any $k \geq r$, the k^{th} power of G, denoted H^k, is a k-uniform hypergraph with edge set

$$E(H^k) = \{e \cup \{v_{e,1}, \dots, v_{e,k-r}\} : e \in E(G)\},$$

Table 6.1 The α-polynomial of all 3-trees with six or fewer edges.

$E(T)$	$\alpha(T)$
$[1,2,3]$	x^3-1
$[1,2,3],[1,4,5]$	x^3-2
$[1,2,3],[1,4,5],[1,6,7]$	x^3-3
$[1,2,3],[1,4,5],[2,6,7]$	x^6-3x^3+1
$[1,2,3],[1,4,5],[1,6,7],[1,8,9]$	x^3-4
$[1,2,3],[1,4,5],[1,8,9],[2,6,7]$	x^6-4x^3+2
$[1,2,3],[1,4,5],[2,6,7],[3,8,9]$	$x^9-4x^6+3x^3-1$
$[1,2,3],[1,4,5],[2,6,7],[4,8,9]$	x^6-4x^3+3
$[1,2,3],[1,4,5],[1,6,7],[1,8,9],[1,10,11]$	x^3-5
$[1,2,3],[1,4,5],[1,8,9],[1,10,11],[2,6,7]$	x^6-5x^3+3
$[1,2,3],[1,4,5],[1,8,9],[2,6,7],[2,10,11]$	x^6-5x^3+4
$[1,2,3],[1,4,5],[1,8,9],[2,6,7],[3,10,11]$	$x^9-5x^6+5x^3-2$
$[1,2,3],[1,4,5],[1,8,9],[2,6,7],[4,10,11]$	$x^9-5x^6+5x^3-1$
$[1,2,3],[1,4,5],[2,6,7],[2,10,11],[4,8,9]$	x^6-5x^3+5
$[1,2,3],[1,4,5],[2,6,7],[3,10,11],[4,8,9]$	$x^9-5x^6+6x^3-2$
$[1,2,3],[1,4,5],[2,6,7],[4,8,9],[6,10,11]$	$x^9-5x^6+6x^3-1$
$[1,2,3],[1,4,5],[1,6,7],[1,8,9],[1,10,11],[1,12,13]$	x^3-6
$[1,2,3],[1,4,5],[1,8,9],[1,10,11],[1,12,13],[2,6,7]$	x^6-6x^3+4
$[1,2,3],[1,4,5],[1,8,9],[1,10,11],[2,6,7],[2,12,13]$	x^6-6x^3+6
$[1,2,3],[1,4,5],[1,8,9],[1,10,11],[2,6,7],[4,12,13]$	$x^9-6x^6+7x^3-2$
$[1,2,3],[1,4,5],[1,8,9],[1,12,13],[2,6,7],[3,10,11]$	$x^9-6x^6+7x^3-3$
$[1,2,3],[1,4,5],[1,8,9],[2,6,7],[2,12,13],[3,10,11]$	$x^9-6x^6+8x^3-4$
$[1,2,3],[1,4,5],[1,8,9],[2,6,7],[2,12,13],[4,10,11]$	$x^9-6x^6+8x^3-2$
$[1,2,3],[1,4,5],[1,8,9],[2,6,7],[3,12,13],[4,10,11]$	$x^{12}-6x^9+9x^6-5x^3+1$
$[1,2,3],[1,4,5],[1,8,9],[2,6,7],[4,10,11],[6,12,13]$	$x^9-6x^6+9x^3-3$
$[1,2,3],[1,4,5],[1,8,9],[2,6,7],[4,10,11],[8,12,13]$	$x^9-6x^6+9x^3-4$
$[1,2,3],[1,4,5],[1,8,9],[1,12,13],[2,6,7],[6,10,11]$	x^6-6x^3+7
$[1,2,3],[1,4,5],[1,8,9],[2,6,7],[6,10,11],[6,12,13]$	x^6-6x^3+8
$[1,2,3],[1,4,5],[2,6,7],[2,12,13],[3,10,11],[4,8,9]$	$x^9-6x^6+9x^3-4$
$[1,2,3],[1,4,5],[2,6,7],[3,10,11],[4,8,9],[4,12,13]$	$x^9-6x^6+9x^3-3$
$[1,2,3],[1,4,5],[2,6,7],[3,10,11],[4,8,9],[5,12,13]$	$x^{12}-6x^9+10x^6-6x^3+1$
$[1,2,3],[1,4,5],[2,6,7],[3,10,11],[4,8,9],[6,12,13]$	$x^9-6x^6+10x^3-5$
$[1,2,3],[1,4,5],[2,6,7],[3,10,11],[4,8,9],[8,12,13]$	$x^{12}-6x^9+10x^6-5x^3+1$
$[1,2,3],[1,4,5],[2,6,7],[4,8,9],[4,12,13],[6,10,11]$	$x^9-6x^6+9x^3-2$
$[1,2,3],[1,4,5],[2,6,7],[4,8,9],[6,10,11],[8,12,13]$	$x^9-6x^6+10x^3-4$

and vertex set

$$V(H^k) = V(G) = V(G) \cup \{i_{e,j} : e \in E(G), j \in [k-r]\}.$$

In other words, one adds exactly enough new vertices (each of degree 1) to each edge of H so that H^k is k-uniform. Note that, if $k = r$, then $H^k = H$. Adhering to this nomenclature we refer to a power of a 2-tree simply as a *power tree*. In this section we prove the following characterization of power trees.

Theorem 6.12. *Let $\mathcal{H}$ be a k-tree and let $\sigma(\mathcal{H})$ denote its (multiset) spectrum. Then $\sigma(\mathcal{H}) \subseteq \mathbb{R}[\zeta_k]$ if and only if $\mathcal{H}$ is a power tree, where ζ_k is a principal k^{th} root of unity.*

We recall the following Theorem from Cooper and Dutle.

Theorem 6.13. *[12] The (multiset) spectrum of a k-cylinder is invariant under multiplication by any k^{th} root of unity.*

One can show by straightforward induction that a k-tree is a k-cylinder, so its spectrum is symmetric in the above sense. The following result, from [41], shows that power trees have spectra which satisfy a much more stringent condition: they are cyclotomic, in the sense that they belong to $\mathbb{R}[\zeta_k]$.

Theorem 6.14. *[41] If $\lambda \neq 0$ is an eigenvalue of any subgraph of G, then $\lambda^{2/k}$ is an eigenvalue of G^k for $k \geq 4$.*

We restate Theorem 6.14 with the additional assumption that the underlying graph is a tree; the proof is easily obtained by applying Lemma 6.10 to the proof of Theorem 6.14 appearing in [41].

Corollary 6.15. *If $\lambda \neq 0$ is an eigenvalue of any subgraph of a tree T, then $\lambda^{2/k}$ is an eigenvalue of T^k for $k \geq 3$.*

Note that Theorem 6.12 provides a converse to Corollary 6.15 in the case of power trees. In particular, appealing to Lemma 6.10, Corollary 6.15, and the fact that the spectrum of a graph is real-valued, we have that the spectrum of a power tree is a subset of $\mathbb{R}[\zeta_k]$. All that remains to be shown is that if a k-tree is not a power tree then it has a root in $\mathbb{C} \setminus \mathbb{R}[\zeta_k]$. To that end, we introduce the k-comb.

Let Comb_k be the k-graph where

$$\text{Comb}_k = ([k^2], \{[k] \cup \{\{i + tk : 0 \le t \le k-1\} : i \in [k]\}\}).$$

We refer to Comb_k as the k-comb. By the definition of power tree, a non-power tree H must contain an edge e incident to a family $\mathcal{F}$ consisting of at least three other edges which are mutually disjoint. This edge e, together with $\mathcal{F}$, form a connected induced subgraph H' of H which is the k^{th} power of a t-comb for $t = |\mathcal{F}| \ge 3$. It is straightforward to see that

$$\alpha_{H'}(x) = \alpha_{\text{Comb}_t^k}(x) = \alpha_{\text{Comb}_t}(x^{k/t}),$$

since matchings in H' are simply k^{th} powers of matchings in Comb_t; therefore, roots of $\alpha(H')$ are k^{th} roots of reals if and only if the roots of $\alpha(\text{Comb}_t)$ are t^{th} roots of reals. We presently show that the spectrum of the k-comb is not contained within the k^{th} cyclotomic extension of $\mathbb{R}$, completing the proof of Theorem 6.12.

Lemma 6.16. *There exists a root λ of $\alpha(\text{Comb}_k)$ for $k \ge 3$ such that $\lambda \in \mathbb{C} \setminus \mathbb{R}[\zeta_i]$.*

Proof. Let $\mathcal{H}$ be a k-comb where $k \ge 3$. By a simple counting argument,

$$\alpha(\mathcal{H}) = \left(\sum_{i=0}^{k} (-1)^i \binom{k}{i} \alpha^{k-i} \right) - \alpha^{k-1}$$

where $\alpha = x^k$. Appealing to the binomial theorem we have

$$\alpha(\mathcal{H}) = (1 - \alpha)^k - \alpha^{k-1}.$$

Let $\beta = \alpha^{-1}$. Setting $\alpha(\mathcal{H}) = 0$ yields

$$(\beta - 1)^k = \beta. \tag{6.1}$$

It is easy to see that (6.1) has precisely one solution when $k \geq 3$ is odd and precisely two solutions when it is even. In either case, as the number of solutions is strictly less than k it follows that there must be a non-real solution and the claim follows. $\square$

Chapter 7

Computational Notes and Examples

We consider the problem of computing the characteristic polynomial of a uniform hypergraph. For a k-graph on n vertices the characteristic polynomial is the resultant of n homogeneous multi-linear polynomials of degree $k-1$. While there are numerous methods for solving such a system, computers must be employed even for the simplest of cases. In practice, built-in methods for solving such systems, via Sage [10] and Macaulay2 [21] for example, have proven to be insufficient. This motivates the need for a new approach which we now present by invoking our aforementioned results.

1. Determine S, the set of induced subgraphs of $\mathcal{H}$ up to isomorphism.

2. We say that $H \in S$ is a *spectral candidate in* $\mathcal{H}$ if there exists an $H' \sqsubseteq \mathcal{H}$ such that $H' \cong H$ and H' satisfies Corollary 6.7 (this includes the case when it cannot be determined if H' satisfies the Corollary). Let $S' \subseteq S$ be the set of spectral candidates of $\mathcal{H}$.

3. Compute $\alpha(H)$ for $H \in S'$ by way of the Lu-Man Method. The authors use Sage [10] to create the system of equations and solve the system using Macaulay2 [21] by computing the generators of the Gröbner basis of the ideal generated by the aforementioned equations.

4. Let $\{p_i\}_{i=1}^t$ be the collection of minimal polynomials of $\alpha(H)$ for $H \in S'$. Note that $x^{-m_0}\phi(\mathcal{H}) = \prod_{i=1}^t p_i^{m_i}$. Furthermore, it may be the case that $m_i = 0$ if p_i is a factor of $\alpha(H)$ for $H \in S'$ where Corollary 6.7 could not be verified.

5. In the case when $\sigma(\mathcal{H})$ is symmetric by some principle root of unity make an appropriate change of variable to simplify further computations. This symmetry will be clear from the minimal polynomials.

6. Choose to apply Theorem 5.2 or Theorem 5.3. Generally speaking, Theorem 5.3 is more efficient in the case when there are few minimal polynomials of larger degree.

7. Determine an appropriate number of leading coefficients of $\phi(\mathcal{H})$ (with the change of variable, if applicable) using Theorem 3.27. Note that the coefficients can be computed more efficiently by taking the structure of the hypergraph into account in order to reduce the search space of the Euler rootings.

8. Apply Theorem 5.2 (or 5.3) to compute the multiplicities of each root (or minimal polynomial, respectively).

We now demonstrate this algorithm by computing the characteristic polynomial of the Hummingbird hypergraph and the Rowling hypergraph.

7.1 The Hummingbird Hypergraph

Consider the *hummingbird hypergraph* $\mathcal{B} = ([13], E)$ where

$$E = \{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,8,9\},\{3,10,11\},\{3,12,13\}\}.$$

We present a drawing of $\mathcal{B}$ in Figure 7.1 where the edges are drawn as shaded in triangles. Note that

$$\deg(\phi(\mathcal{B})) = n(k-1)^{n-1} = 13 \cdot 2^{12} = 53248$$

and, since $\mathcal{B}$ is a hypertree (and thus a 3-cylinder), its spectrum is invariant under multiplication by any third root of unity [12]. Appealing to Lemma 6.10 we have that

Table 7.1 Subgraphs of $\mathcal{B}$ and their α-polynomial.

$T \sqsubseteq \mathcal{B}$	$\alpha(T)$
$P_1 = S_1$	$x^3 - 1$
$P_2 = S_2$	$x^3 - 2$
S_3	$x^3 - 3$
$\{\{1,2,3\},\{1,4,5\},\{1,6,7\},\{2,8,9\}\}$	$x^6 - 4x^3 + 2$
$\{\{1,2,3\},\{1,4,5\},\{2,8,9\},\{3,10,11\}$	$x^9 - 4x^6 + 3x^3 - 1$
$\mathcal{B} - \{2,8,9\}$	$x^6 - 5x^3 + 4$
$\mathcal{B} - \{3,12,13\}$	$x^9 - 5x^6 + 5x^3 - 2$
$\mathcal{B}$	$x^9 - 6x^6 + 8x^3 - 4$

S' is the set of all subtrees of $\mathcal{B}$, up to isomorphism. We compute the α-polynomials of subtrees of $\mathcal{B}$ via the Lu-Man Method.

$$\begin{aligned}\phi(\mathcal{B}) =& x^{m_0}(x^9 - 6x^6 + 8x^3 - 4)^{m_1}(x^9 - 5x^6 + 5x^3 - 2)^{m_2} \\ & \cdot (x^3 - 1)^{m_3}(x^6 - 4x^3 + 2)^{m_4}(x^9 - 4x^6 + 3x^3 - 1)^{m_5} \\ & \cdot (x^6 - 3x^3 + 1)^{m_6}(x^3 - 3)^{m_7}(x^3 - 2)^{m_8}.\end{aligned}$$

With the intent of applying Theorem 5.2 to $\phi(\mathcal{B})$ we consider the change of variable $y = x^3$ and observe that we need to determine $c_3, c_6, \ldots, c_{48}$ as there are sixteen distinct nonzero cube roots. Appealing to Theorem 3.27 we compute

$$\begin{aligned}
c_3 &= -18432 \\
c_6 &= 169843968 \\
c_9 &= -1043209971456 \\
c_{12} &= 4804960103034624 \\
c_{15} &= -17702435302276375440 \\
c_{18} &= 54341319772238850901668 \\
c_{21} &= -142960393819753656566577552 \\
c_{24} &= 329036832924106136747171871042 \\
c_{27} &= -673063350744784559041302787109576 \\
c_{30} &= 1238925078774563882036470496247467682 \\
c_{33} &= -2072891735870949695930286542580991559916 \\
c_{36} &= 3178738418917825954994865036362341584776658 \\
c_{39} &= -4498896549573513724009044022281523093964642496 \\
c_{42} &= 5911636016042739623328802656744094043553245557890 \\
c_{45} &= -7249053168113446546908444934275696322928768819713512 \\
c_{48} &= 8332230937213678426258491158832963153453272812465162851
\end{aligned}$$

Using Theorem 5.5 we have $M < 3, m > .14, R < 4.39, r > .38$, and $c = c_{48}$ so that each root of $\phi(\mathcal{B})$ needs to be approximated to at most 3091 bits of precision. Using SageMath [10], we obtain

$$\begin{aligned}
\phi_{\mathcal{B}} =& x^{20983}(x^9 - 6x^6 + 8x^3 - 4)^{729}(x^9 - 5x^6 + 5x^3 - 2)^{972} \\
&\cdot (x^3 - 1)^{1782}(x^6 - 4x^3 + 2)^{486}(x^9 - 4x^6 + 3x^3 - 1)^{324} \\
&\cdot (x^6 - 3x^3 + 1)^{216}(x^3 - 3)^{54}(x^3 - 2)^{119}.
\end{aligned}$$

In Figure 7.1 we provide a plot of $\sigma(\mathcal{B})$ drawn in the complex plane where a disk is centered at each root and each disk's area is proportional to the algebraic multiplicity of the underlying root in $\phi(\mathcal{B})$.

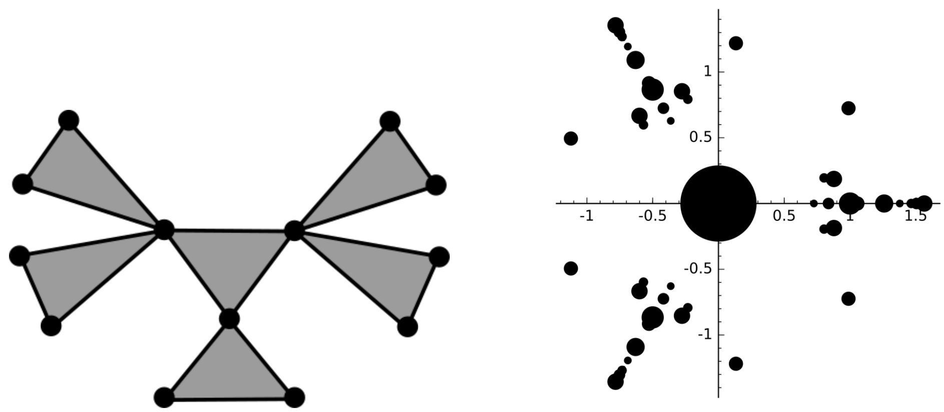

Figure 7.1 The hummingbird hypergraph and its spectrum.

7.2 The Fano Plane and its Subgraphs

We compute the characteristic polynomial of the Rowling Hypergraph, that is, the Fano Plane with two edges removed. The characteristic polynomial of the Fano Plane and the Fano Plane with one edge removed remain to be determined. In this chapter we present our progress towards computing the characteristic polynomial of the Fano Plane with one edge removed and then the Fano Plane with the intent of demonstrating the current computational limits of this method. We encourage the eager reader to improve upon them. Before we proceed, we present the first few leading coefficients of the aforementioned hypergraphs in Table 7.2.

Table 7.2 Leading coefficients of the Fano plane and its subgraphs.

	FP-2	FP-1	FP
c_0	1	1	1
c_1	0	0	0
c_2	0	0	0
c_3	-240	-288	-336
c_4	0	0	0
c_5	0	0	0
c_6	28320	40788	55524
c_7	0	0	-696
c_8	0	0	0
c_9	-2190860	-3788016	-6017746
c_{10}	0	0	220038
c_{11}	0	0	0
c_{12}	125012034	259553826	481293561
c_{13}	0	0	-34237560
c_{14}	0	0	-122004
c_{15}	-5612445168	-13997317932	-30303162330
c_{16}	0	0	?
c_{17}	0	0	?
c_{18}	206518037092	618904026740	?

7.2.1 The Fano Plane With Two Edges Removed

Consider the *Rowling hypergraph,*[1]

$$\mathcal{R} = ([7], \{\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,5,6\}, \{3,5,7\}\}).$$

Note that $\mathcal{R}$ is the Fano Plane with two edges removed. A drawing of $\mathcal{R}$ is given in Figure 7.2 where the edges are drawn as arcs and its multi-spectrum is drawn such that each disk is centered on an eigenvalue of $\mathcal{R}$ and the area of each disk is proportional to the multiplicity of the approximated root. We have

$$\deg(\phi(\mathcal{R})) = n(k-1)^{n-1} = 7 \cdot 2^6 = 448.$$

[1]The name was chosen for its resemblance to an important narrative device[32].

There are four induced subgraphs of $\mathcal{R}$, up to isomorphism, and we pick a representative of each class so that $S = \{H_i\}_{i=1}^4$:

$$H_1 = \mathcal{R}[[3]] \cong P_1, H_2 = \mathcal{R}[5] \cong P_2, H_3 = \mathcal{R}[[7] \setminus 4], H_4 = \mathcal{R}.$$

Again we have that H_1 and H_4 are both isolated subgraphs of $\mathcal{R}$ so that by Lemma 6.6 their totally nonzero eigenvalues extend to $\mathcal{R}$.

We claim that the totally nonzero eigenvalues of H_2 and H_3 are not extendable in $\mathcal{R}$. Adhering to the language of Lemma 6.6 we set $U = [5]$. Consider $\mathcal{R}[U \cup \{6\}]$ and observe

$$p_6 = \lambda x_6^2 - x_2 x_5 = -x_2 x_5$$

since $x_6 = 0$ as $6 \notin U$. However, because $2, 5 \in U$ we have that $p_6 \neq 0$. By Corollary 6.7 the eigenvalues of H_2 are not extendable to $\mathcal{R}$. This argument can be generalized to prove the following.

Lemma 7.1. *If $\mathcal{H}$ is a k-graph and $\mathcal{H}[U]$ has the property that there exists $v \notin U$ such that $|E(\mathcal{H}[U \cup v])| - |E(\mathcal{H}[U])| = 1$ then the eigenvalues of $\mathcal{H}[U]$ are not extendable in $\mathcal{H}$.*

Lemma 7.1 implies that the totally nonzero eigenvalues of H_3 are not extendable in $\mathcal{R}$ with the choice of $v = 4$.

Indeed, $S' = \{H_1, H_4\}$ so that $\sigma(\mathcal{R}) = \sigma^+(P_1) \cup \sigma^+(\mathcal{R})$. We have previously shown that $\alpha(P_1) = x^3 - 1$ and we have by the Lu-Man Method that

$$\alpha(\mathcal{R}) = (x^{15} - 13x^{12} + 65x^9 - 147x^6 + 157x^3 - 64)(x^6 - x^3 + 2)(x^6 - 17x^3 + 64).$$

Notice that $\mathcal{R}$ is not a 3-cylinder; however, its spectrum, like that of 3-cylinders [12], is invariant under multiplication by any third root of unity (see Lemma 3.11 of Fan, et al. [17]). We have then that

$$\begin{aligned}\phi(\mathcal{R}) =& x^{m_0}(x^3 - 1)^{m_1}(x^{15} - 13x^{12} + 65x^9 - 147x^6 + 157x^3 - 64)^{m_2} \\ & \cdot (x^6 - x^3 + 2)^{m_3}(x^6 - 17x^3 + 64)^{m_4}.\end{aligned}$$

With the intent of applying Theorem 5.3 to the minimal polynomials of

$$x^{-m_0/3}\phi_{\mathcal{R}}(x^{1/3}),$$

we verify that

$$V = \begin{pmatrix} 1 & 13 & 1 & 17 \\ 1 & 39 & -3 & 161 \\ 1 & 103 & -5 & 1649 \\ 1 & 87 & 1 & 17729 \end{pmatrix}$$

is non-singular. Indeed, we need to determine the first four proper leading coefficients of $x^{-m_0/3}\phi_{\mathcal{R}}(x^{1/3})$, or equivalently c_3, c_6, c_9, c_{12} of $\phi_{\mathcal{R}}$. We have

$$\begin{aligned} c_3 &= -240 \\ c_6 &= 28320 \\ c_9 &= -2190860 \\ c_{12} &= 125012034. \end{aligned}$$

By Theorem 5.5 we have $M < 4.5, m > .69, R < 2.25$, and $r = 1$ so that at most 252 digits of precision are required to approximate each root. We compute

$$\begin{aligned} \phi_{\mathcal{R}} =& x^{133}(x^3-1)^{27}(x^{15}-13x^{12}+65x^9-147x^6+157x^3-64)^{12} \\ &\cdot (x^6-x^3+2)^6(x^6-17x^3+64)^3 \end{aligned}$$

7.2.2 The Fano Plane with one edge removed

Let

$$F = ([7], \{\{1,2,3\},\{1,6,7\},\{2,4,7\},\{2,5,6\},\{3,4,6\},\{3,5,7\}\}).$$

be the Fano Plane with one edge removed. There are four spectral candidates we need to consider:

$$H_1 = F[[3]] \cong P_1, H_2 = F[\{1,2,3,5,7\}] \cong P_2, H_3 = F[[7]\setminus 4], H_4 = F.$$

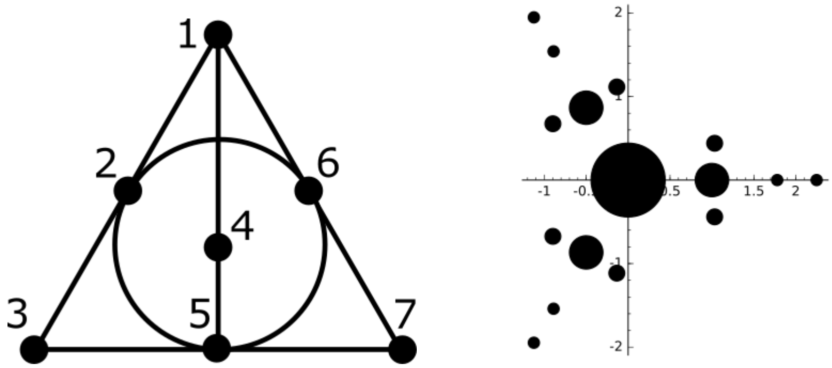

Figure 7.2 The Rowling hypergraph and its spectrum.

Again we have that H_1 and H_4 are isolated subgraphs so that their totally nonzero eigenvalues extend to F. Observe that H_2 and H_3 are *not* isolated subgraphs and further the assumptions of Lemma 7.1 are not satisfied. Thus, we cannot conclude whether or not the totally nonzero eigenvalues of H_2 or H_3 extend to F. In this case, we assume H_2 and H_3 are spectral candidates of F so that $S' = S$. By the Lu-Man Method we have

$$\begin{aligned}
\alpha(H_1) =& x^3 - 1\\
\alpha(H_2) =& x^3 - 2\\
\alpha(H_3) =& (x^3 - 8)(x^6 - 5x^3 + 8)\\
\alpha(H_4) =& (x^3 - 2)(x^3 - 18)(x^6 + x^3 + 1)\\
& (x^6 - 17x^3 + 80)(x^{12} - 6x^9 + 19x^6 - 19x^3 + 8).
\end{aligned}$$

Observe that $\alpha(H_2)$ and $\alpha(H_4)$ have a common root. We have then that

$$\begin{aligned}
\phi_F(x) =& x^{m_0}(x^3 - 1)^{m_1}(x^3 - 2)^{m_2}(x^3 - 8)^{m_3}(x^6 - 5x^3 + 8)^{m_4}\\
& \cdot (x^3 - 18)^{m_5}(x^6 + x^3 + 1)^{m_6}(x^6 - 17x^3 + 80)^{m_7}\\
& \cdot (x^{12} - 6x^9 + 19x^6 - 19x^3 + 8)^{m_8}.
\end{aligned}$$

In order to determine $m_0, \ldots, m_8$ we would apply Theorem 5.3 which requires knowing the first eight proper leading coefficients of $x^{-m_0/3}\phi_F(x^{1/3})$. As shown above, we know only the first six proper leading coefficients. Unfortunately, our current routines are not able to compute the remaining two coefficients due to run time.

7.2.3 The Fano Plane

Consider the finite projective plane of order 2, or more affectionately, the Fano Plane

$$\mathcal{F} = ([7], \{\{1,2,3\}, \{1,4,5\}, \{1,6,7\}, \{2,4,7\}, \{2,5,6\}, \{3,4,6\}, \{3,5,7\}\}).$$

Once again, there are four spectral candidates of $\mathcal{F}$,

$$H_1 = \mathcal{F}[[3]] \cong P_1, H_2 = \mathcal{F}[[5]] \cong P_2, H_3 = \mathcal{F}[[7] \setminus 4], H_4 = \mathcal{F}.$$

In this case, we have been unable to compute $\alpha(\mathcal{F})$. We have attempted to use several different built-in methods in Macaulay2, all of which failed due to run time. Once one determines $\alpha(\mathcal{F})$ the task of determining the multiplicities remains. It is unclear if we will be able to apply Theorem 5.3 or if we must use Theorem 5.2. If we apply Theorem 5.2, the first fifteen coefficients are required before considering $\alpha(\mathcal{F})$! This means we will need to compute a number of coefficients equal to the degree of $\alpha(\mathcal{F})$. It appears that considerable work must be done in this direction. Not just in terms of being able to compute a sufficient number of coefficients, but also in computing the α-polynomial of an arbitrary hypergraph.

Chapter 8

Open Problems

We conclude by presenting open problems related to our work.

8.1 Cospectral Hypergraphs

Two graphs are *cospectral*, or *isospectral*, if they have the same multi-set spectrum (i.e., they have the same characteristic polynomial). In 1973, Schwenk showed that almost all trees have a cospectral mate [34]. Thirty years later, van Dam and Haemers published a survey paper on graphs which are uniquely determined by their spectrum, abbreviated *DS* [13]. In it, they suggest that it is conceivable that almost all graphs are DS. In [4], Bu, Zhou, and Wei show that complete k-uniform hypergraphs are DS (so the complete graph is DS for any uniformity). Determining which hypergraphs are DS appears to be an interesting question and it is unclear *a priori* if this question has a different fate than the graph case.

To demonstrate this, recall that the smallest pair of cospectral simple graphs is the star on five vertices and the 4-cycle with an isolated vertex. We show that the power graphs of these two graphs are no longer cospectral. Let $\mathcal{S}$ and $\mathcal{C}$ be the 3-graphs formed by adding a unique vertex to each edge in S_5 and $C_4 \cup \{v\}$, respectively. We will show that $\mathcal{S}$ and $\mathcal{C}$ are not cospectral despite their base graphs being so. From the Lu-Man Method we have

$$\phi(\mathcal{S}) = x^{m_0}(x^3-1)^{m_1}(x^3-2)^{m_2}(x^3-3)^{m_3}(x^3-4)^{m_4}$$

where $m_i > 0$ and further

$$\phi(\mathcal{C}) = x^{n_0}(x^3 - 1)^{n_1}(x^3 - 2)^{n_2}(x^3 - 4)^{n_3}$$

where $n_i > 0$. Observe that $\sigma(\mathcal{S})$ and $\sigma(\mathcal{C})$ are not equal as *sets*. Indeed $\mathcal{S}$ and $\mathcal{C}$ are not cospectral mates. This begs the question:

Question 8.1. What are the smallest cospectral k-graphs for $k \geq 3$?

In order to answer this question, one would first determine two hypergraphs which have the same set spectrum and then one would compute the characteristic polynomial of each hypergraph, which is intractable in general. This motivates the following variants of cospectral hypergraphs. Two hypergraphs are *weakly cospectral* if their spectra are equal as sets. Moreover, if H_1 and H_2 are hypergraphs we say that H_1 is *subspectral* to H_2 if $\phi(H_1) \mid \phi(H_2)$. Under these definitions we can restate a conjecture of [8] as follows.

Conjecture 8.2. *A hypertree is subspectral to any hypertree which contains it as a subtree.*

We conclude this section by presenting an example supporting this conjecture. Consider the 3-uniform hypergraphs

$$\begin{aligned}
\mathcal{H}_1 &= ([9], \{\{1,2,3\}, \{1,4,7\}, \{2,5,8\}, \{3,6,9\}\}) = \mathrm{Comb}_3 \\
\mathcal{H}_2 &= ([11], \{\{1,2,3\}, \{1,4,7\}, \{3,6,9\}, \{1,10,11\}\}) \\
\mathcal{H}_3 &= ([11], E(\mathcal{H}_1) \cup E(\mathcal{H}_2)).
\end{aligned}$$

Figure 8.1 gives a drawing of $\mathcal{H}_3$ (the striped subgraph is $\mathcal{H}_1$) and a plot of the roots of $\phi(\mathcal{H}_3)$, with a circle centered at each root in the complex plane whose area is proportional to the multiplicity of the root. We have computed

$$\phi(\mathcal{H}_1) = x^{567}(x^9 - 4x^6 + 3x^3 - 1)^{81}(x^6 - 3x^3 + 1)^{81}(x^3 - 2)^{27}(x^3 - 1)^{147}$$

$$\phi(\mathcal{H}_2) = x^{999}(x^6 - 4x^3 + 2)^{81}(x^6 - 3x^3 + 1)^{54}(x^3 - 3)^{27}(x^3 - 2)^{63}(x^3 - 1)^{75}$$

$$\phi(\mathcal{H}_3) = x^{3767}(x^9 - 5x^6 + 5x^3 - 2)^{243}(x^9 - 4x^6 + 3x^3 - 1)^{162}(x^6 - 4x^3 + 2)^{162} \cdot (x^6 - 3x^3 + 1)^{135}(x^3 - 3)^{27}(x^3 - 2)^{180}(x^3 - 1)^{483}.$$

Note that $\mathcal{H}_1$ and $\mathcal{H}_2$ are subspectral to $\mathcal{H}_3$.

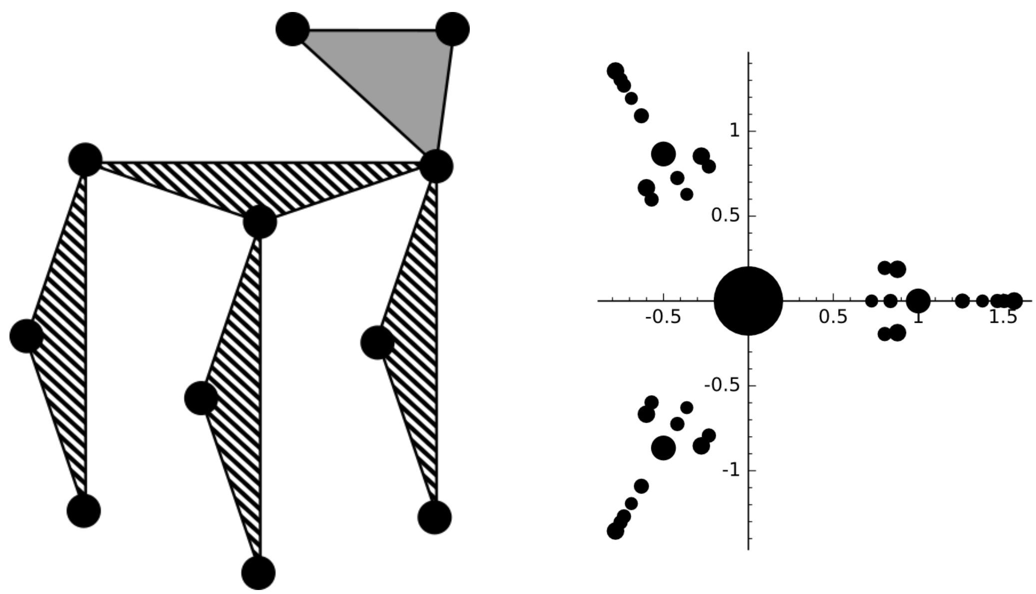

Figure 8.1 $\mathcal{H}_3$ and its spectrum.

8.2 Multiplicity of the Zero Eigenvalue

A characterization of the multiplicity of the zero eigenvalue for the adjacency characteristic polynomial of a graph remains open. For convenience let m_0 denote the multiplicity of the zero eigenvalue for a given polynomial. Notice that one can provide an upper bound on m_0 by showing that a particular coefficient of $\phi(\mathcal{H})$ is non-zero. Combining this idea with the Harary-Sachs Theorem gives the only known result in this direction for the adjacency characteristic polynomial of a graph: if T is a (2-uniform) tree then m_0 is the size of the largest matching of T. Note that the same statement for hypertrees cannot be true (consider Lemma 6.10); however, we know that the spectrum of a hypertree can be written in terms of the matching polynomials of all of its subtrees. Indeed, if one could characterize the multiplicities of the matching polynomials of a given subtree of a hypertree one could answer the following question.

Question 8.3. Express the multiplicity of the zero eigenvalue of a hypertree in terms of the sizes of its matchings over all of its subtrees.

In our proof of Lemma 3.29 we showed that, for 2-graphs, the summands in $\phi_d(G)$ arising from Veblen graphs which are not elementary graphs necessarily summed to zero. We define the *coefficient threshold* of $\phi(\mathcal{H})$ as the least codegree at which the coefficients of $\phi(\mathcal{H})$ cancel thusly.

Definition 8.4. For an integer $v \geq 0$, the *coefficient v-threshold* of a k-graph $\mathcal{H}$, denoted $\mathrm{Th}_v(\mathcal{H})$, is the least integer such that for $d > \mathrm{Th}_v(\mathcal{H})$

$$\sum_{H \in \mathcal{V}_d^*(\mathcal{H})} (-((k-1)^v)^{c(H)}) C_H(\#H \subseteq \mathcal{H}) = 0.$$

Note that the contribution of $\mathcal{H}$ to the codegree-d coefficient of $\phi(\mathcal{G})$ for $\mathcal{H} \subseteq \mathcal{G}$ is zero if $d > \mathrm{Th}_v(\mathcal{H})$ where $v = |V(\mathcal{G})|$. As an example, we show that the v-threshold of the 3-uniform edge is $9 \cdot 2^{v-3}$.

Lemma 8.5. *Let e be the 3-uniform edge. Then $Th_v(e) = 9 \cdot 2^{v-3}$ for $v \geq 3$.*

Proof. For $n \geq 0$ define $f_n(t) : \mathbb{Z}^+ \to \mathbb{Q}$ by $f_n(0) = 1$ and

$$f_n(t) = \sum_{H \in \mathcal{V}_{3t}^*(e)} (-(2^n)^{c(H)}) C_H(\#H \subseteq e), t > 0.$$

Observe that $f_3(t) = \phi_{3t}(e)$ by Theorem 3.27. We conclude by showing $f_n(t) = (-1)^t \binom{3 \cdot 2^{n-3}}{t}$. Considering the characteristic polynomial of a single 3-uniform edge,

$$f_3(0) = 1,\ f_3(1) = -3,\ f_3(2) = 3,\ f_3(3) = -1,\ f_3(4) = 0$$

and $f_3(t) = 0$ for $t > 4$. Indeed $f_3(t) = (-1)^t\binom{3}{t}$ for all t. Suppose that for all n, up to some fixed n, we have $f_n(t) = (-1)^t\binom{3\cdot 2^{n-3}}{t}$ for all t. Consider $f_{n+1}(t)$. We claim

$$f_{n+1}(t) = \sum_{j=0}^{t} f_n(j) f_n(t-j).$$

Recall that the associated coefficient is multiplicative over components. It follows that

$$\begin{aligned}\sum_{j=0}^{t} f_n(j) f_n(t-j) &= \sum_{H\in\mathcal{V}_3^*(e)}\left(\sum_{\substack{H=H_1\cup H_2\\ H_1\cap H_2=\emptyset}} (-(2^n))^{c(H_1)}C_{H_1}\cdot(-(2^n))^{c(H_2)}C_{H_2}\right)\\ &= \sum_{H\in\mathcal{V}_3^*(e)} 2^n(-(2^n))^{c(H)}C_H = f_{n+1}(t).\end{aligned}$$

We have

$$f_{n+1}(t) = (-1)^t\sum_{j=0}^{t}\binom{3\cdot 2^{n-3}}{j}\binom{3\cdot 2^{n-3}}{t-j} = (-1)^t\binom{3\cdot 2^{n-2}}{t}$$

where the first equality follows from the inductive hypothesis and the second equality is given by the Chu-Vandermonde identity [6]. As $\mathrm{Th}_v(e) = f_v(t)$ we have that $\mathrm{Th}_v(e) = 9\cdot 2^{n-3}$ as $f_v(3\cdot 2^{n-3}) = \pm 1$ for $t = 3\cdot 2^{n-2}$ and $f_v = 0$ for $t > 3\cdot 2^{n-2}$. $\square$

Conjecture 8.6. *If $\mathcal{H} \subseteq \mathcal{G}$ are k-graphs, with $k > 2$ where $|V(\mathcal{G})| = n$ then $Th_n(\mathcal{H}) \leq Th_n(\mathcal{G})$.*

This conjecture implies

$$m_0 \leq \deg(\phi(\mathcal{G})) - \mathrm{Th}_n(\mathcal{H}).$$

Note that the restriction of $k > 2$ is necessary as the conjecture is not true for graphs. For example, $C_6 \subseteq K_{6,6}$ and $\mathrm{Th}(K_{6,6}) = 2 < \mathrm{Th}(C_6) = 6$. More generally, it would help our understanding of the multiplicity of 0 to have a better understanding of $\mathrm{Th}_v(\mathcal{H})$, and so we ask:

Question 8.7. Show how to compute or estimate $\mathrm{Th}_v(\mathcal{H})$ for various hypergraphs $\mathcal{H}$.

8.3 Computing the α-polynomial of a Hypergraph

The Lu-Man Method is quintessential to our algorithm for computing the characteristic polynomial of a hypergraph. For this reason it is of great interest to the authors to be able to compute the α-polynomial of a given hypergraph. At the present, the authors have had mixed success in this direction. Recall that computing $\alpha(\mathcal{H})$ involves solving a system of equations with $k|E(\mathcal{H})| + 1$ unknowns with a number of equations equal to the sum of the sizes of the vertex set, edge set, and cycle basis. In general, we have found that the α-polynomial of a hypergraph with a vertex (or vertices) of degree 1 can be computed efficiently. For example, the α-polynomial of a tree is its matching polynomial, but the α-polynomial of the Fano Plane remains unknown.

Question 8.8. Show how to compute the α-polynomial of a hypergraph efficiently.

Another approach would be to characterize the α-polynomial of a certain family of hypergraphs. A natural family to consider would be that of loose-cycles. We present the α-polynomials of loose 3-cycles with 12 or fewer edges in Table 8.1. Notice that, for the values given, $\alpha(C_i^{(3)}) \mid \alpha(C_j^{(3)})$ given $i \mid j$.

Table 8.1 The α-polynomial of loose 3-cycles with i edges where $y = x^3$.

i	$\alpha(C_i^{(3)})$
3	$(y-1)(y-4)$
4	$(y-2)(y-4)$
5	$(y-4)(y^2-3y+1)$
6	$(y-1)(y-3)(y-4)$
7	$(y-4)(y^3-5y^2+6y-1)$
8	$(y-2)(y-4)(y^2-4y+2)$
9	$(y-1)(y-4)(y^3-6y^2+9y-1)$
10	$(y-4)(y^2-5y+5)(y^2-3y+1)$
11	$(y-4)(y^5-9y^4+28y^3-35y^2+15y-1)$
12	$(y-1)(y-2)(y-3)(y-4)(y^2-4y+1)$

Question 8.9. Characterize the α-polynomial of loose 3-cycles.

Aside from computing the α-polynomial of a hypergraph we would like a better test for determining if $\mathcal{H}[U] \sqsubseteq \mathcal{H}$ is a spectral candidate of $\mathcal{H}$. In particular, in the case when $|E(\mathcal{H})| - |E(\mathcal{H}[U \cup \{v\}])| > 1$. Such a test would permit the study of hypergraphs with few eigenvalues. Famously, strongly regular graphs are characterized by having few distinct eigenvalues. A natural question to ask is whether the power of a strongly regular graph retains this property and an answer to this question would serve as a basis for answering the following.

Question 8.10. Characterize k-graphs which have few distinct eigenvalues.

8.4 The Spectra of a Random Hypergraph

A central topic in spectral graph theory is the spectra of random graphs. This area addresses questions concerning quasirandomness, graph expansion, and mixing time of Markov chains. The spectrum of an (Erdős-Rényi) random graph is well understood; however, understanding the spectrum of a random hypergraph is a more subtle matter. We present two approaches to understanding the spectrum of a random hypergraph. We first address the problem given an analogue of the Weyl inequality, then we address the problem from the perspective of quasirandom hypergraphs.

Consider for a moment the simpler boundary case of this question where each edge is chosen with probability 1. Here, the resulting hypergraph is the complete k-graph on n vertices, denoted $K_n^{(k)}$. We could approximate the spectrum of $K_n^{(k)}$ by comparing it to a suitable hypermatrix if we had an analogue of the Weyl inequality (as stated below) for symmetric hypermatrices. Consider the normalized all-ones order n dimension k hypermatrix, $(k-1)!^{-1}\mathbb{J}_n^k$ and observe that this hypermatrix agrees with $\mathcal{A}(K_n^{(k)})$ except for tuples of indices whose coordinates are not all distinct. In [11], Cooper showed that there is an approximate bijection between the set of eigenvalues of $K_n^{(k)}$ and the set of eigenvalues of $(k-1)!^{-1}\mathbb{J}_n^k$ for the case of $k = 2, 3$. More precisely, the set of eigenvalues L of $K_n^{(k)}$ and the set of eigenvalues M of $(k-1)!^{-1}\mathbb{J}_n^k$ satisfy $\delta(L, M) = o(n^{k-1})$ where δ is the Minkowski distance, given $k = 2, 3$. The author has further conjectured the following analogue of the Weyl inequality for hypergraphs, that is, if one can show that if two hypermatrices are "close" then they will have spectra which are "close". Such a result would imply that the eigenvalues of $K_n^{(k)}$ and $(k-1)!^{-1}\mathbb{J}_n^k$ are asymptotically the same, because the difference of the entries of the aforementioned hypermatrices have norm converging to 0.

Conjecture 8.11. *([11]) Suppose A and B are hypermatrices so that $||A - B|| \leq \varepsilon$ for some norm $||\cdot||$ (or spectral radius) and $\varepsilon > 0$. Then there is a bijection ρ between the eigenvalues (with multiplicity) of A and B and a function f with $\lim_{\varepsilon \to 0} f(\varepsilon) = 0$ so that $|\lambda - \rho(\lambda)| < f(\varepsilon)$ for each eigenvalue λ.*

We present another approach to determining the eigenvalues of a random k-graph, by way of quasirandomness, which circumvents the need for a Weyl inequality. Intuitively, a hypergraph is *quasirandom* if it has the same number of copies of any fixed subgraph as one would expect in a random hypergraph (where, in its simplest form, each edge is taken with probability 1/2). This idea was first introduced for graphs in [6] and was later extended to hypergraphs in [7]. In [7], Chung shows that a hypergraph is quasirandom if it has approximately the expected number of even partial octahedra (as described therein). One can restate this condition in terms of the coefficients, and perhaps the spectrum itself, by appealing to Theorem 3.27 to show that the linear combinations of subgraph counts appearing in the result are indeed *forcing families* for quasirandomness (see [26]). We can address this problem using the following definition.

Definition 8.12. Let

$$\phi_{\mathcal{H}}^*(z) = (x^{-D_n}\phi_{H_n}(x))|_{z=n/x} = \sum_{n=0}^{\infty} c_n^* z^n$$

where $D = \deg(\phi(\mathcal{H}))$. The *characteristic power series of a random k-graph* is

$$\phi_k^*(z) = \lim_{n\to\infty} \mathbb{E}_{|\mathcal{H}|=n}[\phi_{\mathcal{H}}^*(z)].$$

The characteristic power series has a flavor of flag algebras (which should not come as a surprise as flag algebras are integral to the study of extremal combinatorics). We recommend [5] and [31] for readers interested in learning more about this topic. With this in mind, we adhere to the notation of flag algebras and denote $P(H)$ simply as H and further write G^n to mean n disjoint copies of G. For example,

$$\begin{aligned}\phi_2^*(z) &= 1 + 0 + (-P_2)z^2 + (-2C_3)z^3 + (P_2^2 - 2C_4)z^4 \\ &\quad + (-2C_5 + 2P_2C_3)z^5 + (-2C_6 + 2C_4P_2 + 4C_3^2 - P_2^3)z^6 + \dots \\ &= 1 + 0 - \frac{1}{2}z^2 - \frac{1}{4}z^3 - \frac{3}{64}z^4 + \frac{1}{64}z^5 - \frac{175}{16384}z^6 + \dots.\end{aligned}$$

As an aside, we ask the following question:

Question 8.13. Find a closed formula for ϕ_2^*.

Famously, a forcing family of a graph is the edge and the 4-cycle so that, for $\phi_G^*(z)$, if c_2^* is 'close' to $-1/2$ and c_4^* is 'close' to $-1/4$ then G is quasirandom. In particular, this implies (or more appropriately, forces) the coefficients of ϕ_G^* to be 'close' to the coefficients of ϕ_2^*. We have from [6] (and to some extent [26]) that the eigenvalues of a quasirandom hypergraph are asymptotically equal to those of the random graph so that the roots of ϕ_G^* are asymptotically equal to ϕ_2^*. The goal then is to apply this approach to ϕ_k^* for $k \geq 3$ (notice that the multi-set spectrum of the random k-graph is precisely the roots of ϕ_k^*). We can do so by answering the following two questions.

Question 8.14. What are the forcing families of quasirandom hypergraphs?

More specifically, what is a *small* forcing family for quasirandom 3-graphs? An analogue of the graph case (i.e., edges and 4-cycles) is desirable. Such a result is achievable using flag algebras, and further, ϕ_k^* could be computed explicitly using flagmatic software for low-values of k. Finally, we need to guarantee that the roots of ϕ_k^* behave as expected under this limits.

Question 8.15. If $\{\mathcal{H}_n\}_{n=1}^{\infty}$ is a sequence of k-graphs so that $\phi_{\mathcal{H}_n}^*(z) \to \phi_k^*(z)$ then does their spectrum similarly converge?

Recall that this question is true for the graph case as the limit object of the sequence is the uniform graphon. The authors feel that a similar result would hold for the hypergraphon but the details will need to be verified.

Bibliography

[1] T. van Aardenne-Ehrenfest and N.G. de Bruijn. "Circuits and Trees in Oriented Linear Graphs". In: *Classic Papers in Combinatorics*. Ed. by Ira Gessel and Gian-Carlo Rota. Boston, MA: Birkhäuser Boston, 1987, pp. 149–163. ISBN: 978-0-8176-4842-8. DOI: 10.1007/978-0-8176-4842-8_12. URL: https://doi.org/10.1007/978-0-8176-4842-8_12.

[2] Shuliang Bai and Linyuan Lu. "A bound on the spectral radius of hypergraphs with e edges". In: *Linear Algebra Appl.* 549 (2018), pp. 203–218. ISSN: 0024-3795. DOI: 10.1016/j.laa.2018.03.030. URL: https://doi.org/10.1016/j.laa.2018.03.030.

[3] Norman Biggs. *Algebraic graph theory*. Second. Cambridge Mathematical Library. Cambridge University Press, Cambridge, 1993, pp. viii+205. ISBN: 0-521-45897-8.

[4] Changjiang Bu, Jiang Zhou, and Yimin Wei. "E-cospectral hypergraphs and some hypergraphs determined by their spectra". In: *Linear Algebra and its Applications* 459 (2014), pp. 397–403. ISSN: 0024-3795. DOI: https://doi.org/10.1016/j.laa.2014.07.020. URL: http://www.sciencedirect.com/science/article/pii/S0024379514004704.

[5] Marcel K. de Carli Silva, Fernando Mário de Oliveira Filho, and Cristiane Maria Sato. "Flag algebras: a first glance". In: *Nieuw Arch. Wiskd. (5)* 17.3 (2016), pp. 193–199. ISSN: 0028-9825.

[6] F. R. K. Chung, R. L. Graham, and R. M. Wilson. "Quasi-random graphs". In: *Combinatorica* 9.4 (Dec. 1989), pp. 345–362.

[7] Fan Chung. "Quasi-random hypergraphs revisited". In: *Random Structures & Algorithms* 40.1 (2011), pp. 39–48. DOI: 10.1002/rsa.20388. eprint: https://onlinelibrary.wiley.com/doi/pdf/10.1002/rsa.20388. URL: https://onlinelibrary.wiley.com/doi/abs/10.1002/rsa.20388.

[8] Gregory J. Clark and Joshua Cooper. "On the Adjacency Spectra of Hypertrees". In: *Electronic Journal of Combinatorics* 25 (2017).

[9] Emma Cohen et al. "Inverse expander mixing for hypergraphs". In: *Electron. J. Combin.* 23.2 (2016), Paper 2.20, 24. ISSN: 1077-8926.

[10] The Sage-Combinat community. *Sage-Combinat: enhancing Sage as a toolbox for computer exploration in algebraic combinatorics.* http://sagemath.org. 2008.

[11] J. Cooper. "Adjacency Spectra of Random and Uniform Hypergraphs". In: *ArXiv e-prints* (July 2015). https://arxiv.org/abs/1507.07118. arXiv: 1507.07118 [math.CO].

[12] Joshua Cooper and Aaron Dutle. "Spectra of uniform hypergraphs". In: *Linear Algebra Appl.* 436.9 (2012), pp. 3268–3292. ISSN: 0024-3795. DOI: 10.1016/j.laa.2011.11.018. URL: https://doi.org/10.1016/j.laa.2011.11.018.

[13] Edwin R. van Dam and Willem H. Haemers. "Which graphs are determined by their spectrum?" In: *Linear Algebra Appl.* 373 (2003). Special issue on the Combinatorial Matrix Theory Conference (Pohang, 2002), pp. 241–272. ISSN: 0024-3795. DOI: 10.1016/S0024-3795(03)00483-X. URL: https://doi.org/10.1016/S0024-3795(03)00483-X.

[14] Reinhard Diestel. *Graph theory.* Third. Vol. 173. Graduate Texts in Mathematics. Springer-Verlag, Berlin, 2005, pp. xvi+411. ISBN: 978-3-540-26182-7; 3-540-26182-6; 978-3-540-26183-4.

[15] A. Eisinberg and G. Fedele. "On the inversion of the Vandermonde matrix". In: *Appl. Math. Comput.* 174.2 (2006), pp. 1384–1397. ISSN: 0096-3003. DOI: 10.1016/j.amc.2005.06.014. URL: https://doi.org/10.1016/j.amc.2005.06.014.

[16] E. Estrada and J. A. Rodríguez-Velázquez. "Subgraph centrality and clustering in complex hyper-networks". In: *Physica A Statistical Mechanics and its Applications* 364 (May 2006). http://adsabs.harvard.edu/abs/2006PhyA..364..581E, pp. 581–594. DOI: 10.1016/j.physa.2005.12.002. eprint: physics/0505137.

[17] Y.-Z. Fan et al. "The spectral symmetry of weakly irreducible nonnegative tensors and connected hypergraphs". In: *ArXiv e-prints* (Apr. 2017). arXiv: 1704.08799 [math.CO].

[18] S. Friedland et al. "On best rank one approximation of tensors". In: *Numer. Linear Algebra Appl.* 20.6 (2013), pp. 942–955. ISSN: 1070-5325. DOI: 10.1002/nla.1878. URL: https://doi.org/10.1002/nla.1878.

[19] F. R. Gantmacher. *The theory of matrices. Vol. 1.* Translated from the Russian by K. A. Hirsch, Reprint of the 1959 translation. AMS Chelsea Publishing, Providence, RI, 1998, pp. x+374. ISBN: 0-8218-1376-5.

[20] I. M. Gelfand, M. M. Kapranov, and A. V. Zelevinsky. *Discriminants, resultants and multidimensional determinants*. Modern Birkhäuser Classics. Reprint of the 1994 edition. Birkhäuser Boston, Inc., Boston, MA, 2008, pp. x+523. ISBN: 978-0-8176-4770-4.

[21] Daniel R. Grayson and Michael E. Stillman. *Macaulay2, a software system for research in algebraic geometry*. Available at `http://www.math.uiuc.edu/Macaulay2/`.

[22] Bruno Grenet, Pascal Koiran, and Natacha Portier. "The multivariate resultant is NP-hard in any characteristic". In: *Mathematical foundations of computer science 2010*. Vol. 6281. Lecture Notes in Comput. Sci. Springer, Berlin, 2010, pp. 477–488. DOI: `10.1007/978-3-642-15155-2_42`. URL: `https://doi.org/10.1007/978-3-642-15155-2_42`.

[23] Frank Harary. "The determinant of the adjacency matrix of a graph". In: *SIAM Rev.* 4 (1962), pp. 202–210. ISSN: 0036-1445. DOI: `10.1137/1004057`. URL: `https://doi.org/10.1137/1004057`.

[24] Shenglong Hu and Ke Ye. "Multiplicities of tensor eigenvalues". In: *Commun. Math. Sci.* 14.4 (2016), pp. 1049–1071. ISSN: 1539-6746. DOI: `10.4310/CMS.2016.v14.n4.a9`. URL: `https://doi.org/10.4310/CMS.2016.v14.n4.a9`.

[25] François Le Gall. "Powers of tensors and fast matrix multiplication". In: *ISSAC 2014—Proceedings of the 39th International Symposium on Symbolic and Algebraic Computation*. ACM, New York, 2014, pp. 296–303. DOI: `10.1145/2608628.2608664`. URL: `https://doi.org/10.1145/2608628.2608664`.

[26] L. Lovász and B. Szegedy. "Finitely forcible graphons". In: *J. Combin. Theory Ser. B* 101.5 (2011), pp. 269–301. ISSN: 0095-8956. DOI: `10.1016/j.jctb.2011.03.005`. URL: `https://doi.org/10.1016/j.jctb.2011.03.005`.

[27] Linyuan Lu and Shoudong Man. "Connected hypergraphs with small spectral radius". In: *Linear Algebra Appl.* 509 (2016), pp. 206–227. ISSN: 0024-3795. DOI: `10.1016/j.laa.2016.07.013`. URL: `https://doi.org/10.1016/j.laa.2016.07.013`.

[28] A. Morozov and Sh. Shakirov. "Analogue of the identity Log Det = Trace Log for resultants". In: *J. Geom. Phys.* 61.3 (2011), pp. 708–726. ISSN: 0393-0440. DOI: `10.1016/j.geomphys.2010.12.001`. URL: `https://doi.org/10.1016/j.geomphys.2010.12.001`.

[29] Miodrag S. Petković and Ljiljana D. Petković. *Complex interval arithmetic and its applications*. Vol. 105. Mathematical Research. Wiley-VCH Verlag Berlin GmbH, Berlin, 1998, p. 284. ISBN: 3-527-40134-2.

[30] Liqun Qi. “Eigenvalues of a real supersymmetric tensor”. In: *J. Symbolic Comput.* 40.6 (2005), pp. 1302–1324. ISSN: 0747-7171. DOI: 10.1016/j.jsc.2005.05.007. URL: https://doi.org/10.1016/j.jsc.2005.05.007.

[31] Alexander A. Razborov. “What is. . .a flag algebra?” In: *Notices Amer. Math. Soc.* 60.10 (2013), pp. 1324–1327. ISSN: 0002-9920. DOI: 10.1090/noti1051. URL: https://doi.org/10.1090/noti1051.

[32] Peter Rowlinson. “On the maximal index of graphs with a prescribed number of edges”. In: *Linear Algebra Appl.* 110 (1988), pp. 43–53. ISSN: 0024-3795. DOI: 10.1016/0024-3795(83)90131-3. URL: https://doi.org/10.1016/0024-3795(83)90131-3.

[33] Horst Sachs. “Über Teiler, Faktoren und charakteristiche Polynome von Graphen”. In: *Teil I* 12 (1966), pp. 7–12.

[34] Allen J. Schwenk. “Almost all trees are cospectral”. In: *New directions in the theory of graphs (Proc. Third Ann Arbor Conf., Univ. Michigan, Ann Arbor, Mich., 1971).* Academic Press, New York, 1973, pp. 275–307.

[35] Jia-Yu Shao, Liqun Qi, and Shenglong Hu. “Some new trace formulas of tensors with applications in spectral hypergraph theory”. In: *Linear Multilinear Algebra* 63.5 (2015), pp. 971–992. ISSN: 0308-1087. DOI: 10.1080/03081087.2014.910208. URL: https://doi.org/10.1080/03081087.2014.910208.

[36] N. J. A. Sloane. *The On-Line Encyclopedia of Integer Sequences*. published electronically at https://oeis.org. 2018.

[37] F. Soto-Eguibar and H. Moya-Cessa. “Inverse of the Vandermonde and Vandermonde confluent matrices”. In: *Appl. Math. Inf. Sci.* 5.3 (2011), pp. 361–366. ISSN: 1935-0090.

[38] W. T. Tutte and C. A. B. Smith. “On Unicursal Paths in a Network of Degree 4”. In: *Amer. Math. Monthly* 48.4 (1941), pp. 233–237. ISSN: 0002-9890. DOI: 10.2307/2302716. URL: https://doi.org/10.2307/2302716.

[39] Oswald Veblen. “An application of modular equations in analysis situs”. In: *Ann. of Math. (2)* 14.1-4 (1912/13), pp. 86–94. ISSN: 0003-486X. DOI: 10.2307/1967604. URL: https://doi.org/10.2307/1967604.

[40] Wei Zhang et al. “The spectra of uniform hypertrees”. In: *Linear Algebra Appl.* 533 (2017), pp. 84–94. ISSN: 0024-3795. DOI: 10.1016/j.laa.2017.07.018. URL: https://doi.org/10.1016/j.laa.2017.07.018.

[41] Jiang Zhou et al. “Some spectral properties of uniform hypergraphs”. In: *Electron. J. Combin.* 21.4 (2014), Paper 4.24, 14. ISSN: 1077-8926.