2018

States and the Numerical Range in the Regular Algebra

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by

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Bachelor of Arts
Coker College 2013

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2018

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Abstract

In this dissertation we investigate the algebra numerical range defined by the Banach algebra of regular operators on a Dedekind complete complex Banach lattice, i.e., $V(\mathcal{L}_r(E), T) = \{\Phi(T) : \Phi \in \mathcal{L}_r(E)^*, ||\Phi|| = 1 = \Phi(I)\}$. For T in the center $\mathcal{Z}(E)$ of E we prove that $V(\mathcal{L}_r(E), T) = \mathrm{co}(\sigma(T))$. For $T \perp I$ we prove that $V(\mathcal{L}_r(E), T)$ is a disk centered at the origin. We then consider the part of $V(\mathcal{L}_r(E), T)$ obtained by restricting ourselves to positive states $\Phi \in \mathcal{L}_r(E)^*$. In this case we show that we get a closed interval on the real line.

Next we consider the problem of characterizing the linear maps on $\mathcal{L}_r(E)$ which preserve $V(\mathcal{L}_r(E), T)$. For this we first describe the regular states on $\mathcal{L}_r(E)$, in particular for the case $E = \ell_p(n)$ for $1 \leq p \leq \infty$. This description allows us to show that any map Ψ on $\mathcal{L}_r(\ell_p(n))$ preserving $V(\mathcal{L}_r(\ell_p(n)), T)$ for all $T \in \mathcal{L}_r(\ell_p(n))$ is of the form $\Psi(T) = U * (P^t \mathcal{Q} T P)$ where U consists of elements of modulus 1, $(*)$ represents Hadamard multiplication, P is a permutation, and $\mathcal{Q}$ is a map that permutes off-diagonal entries of T. Furthermore, special conditions are given for $\mathcal{Q}$ for the cases $p = 1$, $p = \infty$ and $p = 2$.

Finally, some extensions of these results to more general finite dimensional Banach lattices and infinite dimensional ℓ_p's are considered.

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Chapter 1

Introduction

To begin we clarify some notation that will be used throughout this dissertation as some notations are not standard. Let E be a normed space. We will denote the space of linear operators from E to E as $\mathcal{L}(E)$. The dual of E will be denoted E^*. The unit ball of E will be denoted by $B(E)$. Let A be a set. We denote the the convex hull of A as $\mathrm{co}(A)$ and the closed convex hull of A as $\overline{\mathrm{co}}(A)$. In general, the closure of the set A will be denoted by $\overline{A}$. Also let $\mathcal{E}(A)$ represent the extreme points of the set A. Further definitions and notations will be given throughout the introductory chapter.

1.1 Banach Lattices

The early study of Riesz spaces and Banach Lattices is attributed to G. Birkhoff, H. Freudenthal, L.V. Kantorovič, and F. Riesz. The following definitions, properties, and theorems are well known and further references can be found in [12] and [1]

Definition 1.1. An ordered set $(M, \leq)$ is called a *lattice* if any two elements $x, y \in M$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$.

Definition 1.2. A real vector space E which is also an ordered set is called an *ordered vector space* if the order and the vector space structure are compatible in the following sense:

If $x, y \in E$ such that $x \leq y$, then $x + z \leq y + z$ for all $z \in E$ and $ax \leq ay$ for all real a with $a \geq 0$.

Definition 1.3. If $(E, \leq)$ a real ordered vector space is in addition a lattice, then E is called a *Riesz space*.

Example 1.4. $\mathbb{R}^n$ with the standard order $(x_1, x_2, ..., x_n) \leq (y_1, y_2, ..., y_n)$ if and only if $x_k \leq y_k$ for all $k \in [1, n]$ is a lattice.

Definition 1.5. A norm $||\cdot||$ on E satisfying $||x|| \leq ||y||$ whenever $x \vee (-x) =: |x| \leq |y|$ is called a *lattice norm*. $(E, ||\cdot||)$ is called a *normed Riesz space*. If in addition E is complete, it is called a *Banach lattice*.

Example 1.6. All of the classical (real) Banach spaces $\ell_p, c_0, C(K), L^p(\mu)$ are Banach lattices for their usual norm and the pointwise (almost everywhere) order.

The previous definitions are defined for a real vector space E. In this paper it will be necessary to use a complex vector space. To this end we must define the complexification of a Banach lattice.

Definition 1.7. The *complexification* of a real Banach lattice E is the complex Banach lattice given by

$$E_{\mathbb{C}} = E \oplus iE = \{x + iy : x, y \in E\}$$

Let $z = x + iy \in E_{\mathbb{C}}$. The modulus on E can be extended as

$$|z| = \sup_{\theta \in [0, 2\pi]} \{x \cos\theta + y \sin\theta\}.$$

We can then define the norm of $\mathbb{E}_{\mathbb{C}}$ as

$$||z|| = ||\,|z|\,||.$$

It is a further necessity that the spaces we consider in this paper have the following property.

Definition 1.8. A Banach lattice, E, is called *Dedekind complete* if every non-empty order bounded set has a supremum and an infimum in E.

With the lattice structure comes several properties and sets related to positivity. For the following let E and F be real or complex Banach lattices (unless otherwise stated).

Definition 1.9. The *positive cone*, denoted E_+ is the set of all $x \in E$ such that $x \geq 0$.

Definition 1.10. The *disjoint complement* A^d of $A \subset E$ is defined by

$$A^d = \{x \in E : |x| \wedge |y| = 0 \text{ for all } y \in A\}$$

Definition 1.11. A subset A of E is called *solid* if $|x| \leq |y|$ for some $y \in A$ implies that $x \in A$. Every solid subspace I of E is called an *ideal* in E. An ideal B of E is called a *band* if $B = B^{dd}$.

Definition 1.12. A band B of E is called a *projection band* if there is a linear projection $P : E \to B$ such that $0 \leq Px \leq x$ for all $x \in E_+$. The linear projection P will be called a *band projection*. A *principal band* B_x is the smallest (with respect to inclusion) band that contains x. The band projection of E onto B_x will be denoted by P_x.

It is well known that given a projection band B of E then $E = B \oplus B^d$. Furthermore, if the space E is Dedekind complete then every band is a projection band and $P(x) = \sup\{y : 0 \leq y \leq x : y \in B\}$ for each $x \geq 0$.

Definition 1.13. An operator, $T : E \to F$ is called *positive* if $T(E_+) \subset F_+$. The operator T is called *regular* if T is a linear combination of positive linear operators. We will denote the collection of regular operators by $\mathcal{L}_r(E, F)$.

Definition 1.14. Let F be Dedekind complete. A regular operator, $T : E \to F$, has a *modulus* defined by

$$|T| = \sup\{(\cos\theta)\mathrm{Re}\, T + (\sin\theta)\mathrm{Im}\, T : 0 \leq \theta \leq 2\pi\}.$$

For positve elements one can prove,

$$|T|(x) = \sup\{|Ty| : y \in E, |y| \leq x\}$$

for all $x \in E_+$.

Definition 1.15. Let F be Dedekind complete. For every $T \in \mathcal{L}_r(E,F)$, we can define the *r-norm* of T by

$$||T||_r = ||\,|T|\,||$$

Remark 1.16. By definition we have that for all $T \in \mathcal{L}_r(\ell_1(n))$, $||T||_r = ||T||_1$. Also for all $S \in \mathcal{L}_r(\ell_\infty(n))$ we have $||S||_r = ||S||_\infty$.

Theorem 1.17. *If F is Dedekind complete, then $(\mathcal{L}_r(E,F), ||\cdot||_r)$ is a Banach lattice under the ordering $T \geq S$ if an only if $T - S \geq 0$.*

The following theorem due to Kakutani [9] states that Banach lattices are locally like $C(K)$ spaces for some compact Hausdorff space K.

Theorem 1.18 (Kakutani)**.** *Let E be a Banach lattice with an order unit e. Then there is a compact Hausdorff space K and a linear mapping $J : E \to C(K)$ such that*

1. *J is a lattice isomophism.*

2. *$J(E)$ is equal to $C(K)$.*

3. *$Je = 1_K$*

4. *$||Jx||_\infty = ||x||_e$ for all $x \in E$*

In other words, every order ideal in a Dedekind complete Banach lattice is always lattice isomorphic to some $C(K)$. This theorem will be useful when considering the following set of operators in a Banach lattice.

Definition 1.19. The *center* of E, denoted $\mathcal{Z}(E)$, consists of all linear operators T such that T is dominated by a multiple of the identity operator. That is, there exists some $0 < \lambda \in \mathbb{R}$ such that $|Tx| \leq \lambda|x|$ for all $x \in E$.

It is worth noting that $\mathcal{Z}(E) \subseteq \mathcal{L}_r(E)$.

Theorem 1.20. *The center of a Banach lattice $\mathcal{Z}(E)$ coincides with the band generated by the identity operator I in $\mathcal{L}_r(E)$. Furthermore, if E is Dedekind complete then $\mathcal{Z}(E)$ is a projection band, and so $\mathcal{L}_r(E) = \mathcal{Z}(E) \oplus I^d$*

Example 1.21. Let (E, Σ, μ) be a finite measure space. For every $1 \leq p \leq \infty$ the center $\mathcal{Z}(L^p(\mu))$ of $L^p(\mu)$ is isomorphic to $L^\infty(\mu)$ where the isomorphism is given by $h \to T_h : T_h f = h \cdot f$ for every $h \in L^\infty(\mu)$. In other words the centeral operators are given by a multiplication operator.

Example 1.22. Let $E = \ell_p(n)$. Then the center is given by the set of diagonal matrices. This is clear when you consider the definition given in 1.19

The final proposition is given as an exercise in [1]. The exercise is wrong as stated, but under the assumption that one of your operators is the identity, the statement is correct.

Proposition 1.23. *Let E be a Dedekind complete Banach lattice and let $T : E \to E$ such that $T \perp I$. For each $0 < x \in E$ and each $0 < \varepsilon < 1$ there exists a non-zero component a of x (i.e. $a \wedge (x - a) = 0$) such that $P_a(|T|a) \leq \varepsilon a$.*

Proof. A proof can be found in [2]. As mentioned above, the proof does not prove the exercise as stated in [1], but does prove the special case here. $\square$

1.2 Classical Numerical Ranges

For the second half of this introductory chapter we introduce the other central topic in this paper. Similar to the spectrum of a linear operator, the numerical range of a linear operator is a subset of the scalar field. The numerical range differs from the spectrum in that it is dependent on both the algebraic structure and the norm. The numerical range, $W(T)$, was first defined on a Hilbert space. The following definitions and theorems are well known and can be found in [6] and [5]

Definition 1.24. Let $(\mathcal{H}, <,>)$ be a Hilbert space. Let $T : \mathcal{H} \to \mathcal{H}$ be a linear operator. The *numerical range* of T is defined to be

$$W(T) := \{< Tx, x >: ||x|| = 1\}$$

Numerical ranges on Hilbert spaces have been studied in-depth but we provide a few well-known motivating theorems.

Theorem 1.25. *The numerical range of every bounded linear operator $T : \mathcal{H} \to \mathcal{H}$ is convex.*

Theorem 1.26. *If T is a bounded linear operator on $\mathcal{H}$, then $\sigma(T) \subset \overline{W(T)}$. Where $\sigma(T)$ is the spectrum of T.*

Theorem 1.27. *Assume $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_2$. Then $W(T_1 \oplus T_2) = co(V(T_1), V(T_2))$*

Naturally the study of numerical ranges continued onto Banach spaces. The obvious problem being that a Banach space does not have an inner product, a new definition for the numerical range was considered.

Definition 1.28. The *spatial numerical range* for a linear operator T on a Banach space E is defined to be

$$V(T) := \{f(Tx) : x \in E, f \in E^*, ||x|| = ||f|| = 1 = f(x)\}$$

Remark 1.29. Pairs (f, x) as above will be known as vector states.

Remark 1.30. This definition for a numerical range in a Banach space coincides with the Hilbert space definition in that if E is in fact a Hilbert space, $V(T) = W(T)$.

Definition 1.31. The *numerical radius* of a linear operator T on a Banach space E is defined to be

$$v(T) = \sup\{|\lambda| : \lambda \in V(T)\}.$$

Some properties of $W(T)$ are also present when studying $V(T)$.

Theorem 1.32. *For a linear operator $T \in \mathcal{L}(E)$ we have $\sigma(T) \subset \overline{V(T)}$.*

Although in some instances $W(T)$ and $V(T)$ can be similar, one distinguishing difference is that $V(T)$ need not be convex, even when considering common Banach spaces such as the ℓ_p spaces.

Example 1.33. This example was given in [5]. Let $T = \begin{bmatrix} 0 & \frac{1}{3} \\ \frac{1}{3} & 1 \end{bmatrix}$ and $||(z, w)|| = \max\{||(z, w)||_\infty, \frac{3}{\sqrt{10}}||(z, w)||_2\}$
Figure 1.1 represents $V(T)$.

For this reason another numerical range was introduced. In this second definition we will require our space to be a Banach algebra. This new definition gives us a numerical range that is both closed and convex.

Definition 1.34. The *algebraic numerical range* for an element a in a Banach algebra $\mathcal{A}$ is defined to be

$$V(\mathcal{A}, a) := \{\Phi(a) : \Phi \in \mathcal{A}^*, \Phi(I) = 1 = ||\Phi||\}$$

Remark 1.35. Such Φ as above are called states and will be discussed further in this dissertation.

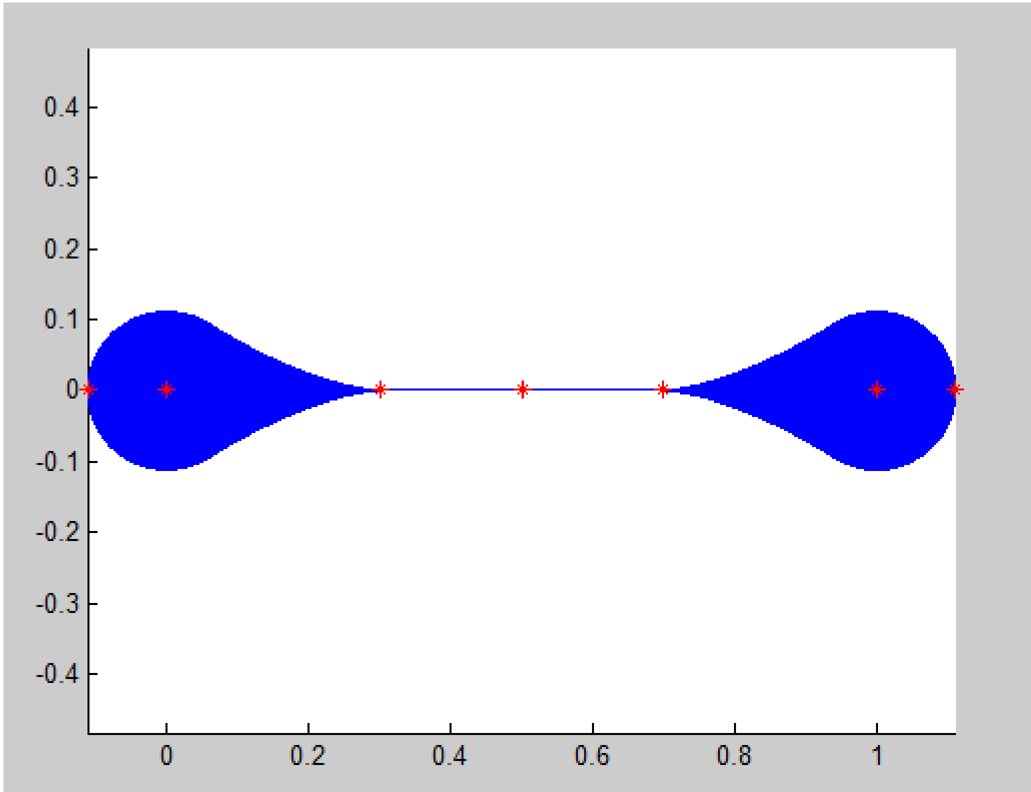

Figure 1.1 The spatial numerical range of T

In [13] Pellegrini gives the following example of an algebraic numerical range.

Example 1.36. Let K be a compact Hausdorff space. Then

$$V(C(K), f) = \operatorname{co} f(K)$$

for all $f \in C(K)$.

Although you can define the algebraic numerical range for a linear operator on any Banach algebra, the most common use of this definition is to let $\mathcal{A}$ be the set of bounded operators on a Banach space, which will be denoted $\mathcal{L}(E)$. In this form the definition for the algebraic numerical range is

$$V(\mathcal{L}(E), T) = \{\Phi(T) : \Phi \in \mathcal{L}(E)^*, \Phi(1) = 1 = ||\Phi||\}.$$

Similar to $V(T)$, if E is in fact a Hilbert space we have $\overline{W(T)} = V(\mathcal{L}(E), T)$. In the non-Hilbert space case there is still a relationship between the spatial numerical range and algebraic numerical range.

Proposition 1.37. *For a linear operator $T \in \mathcal{L}(E)$ we have $V(T) \subset V(\mathcal{L}(E), T)$*

Proof. Consider a pair of vector states (f, x) such that $f(x) = 1 = ||f|| = ||x||$. Define $\Phi \in \mathcal{L}(E)^*$ by

$$\Phi(T) = f(Tx) \text{ for all } T \in \mathcal{L}(E).$$

It is clear that $\Phi(I) = 1 = ||\Phi||$. The inclusion follows immediately. □

Theorem 1.38. *For a linear operator $T \in \mathcal{L}(E)$ we have $\overline{co}(V(T)) = V(\mathcal{L}(E), T)$*

Thus we have that the algebraic numerical range is the closed convex hull of the spatial numerical range. This fact leads to the following property regarding the numerical radius.

Proposition 1.39. *For a linear operator $T \in \mathcal{L}(E)$ we have $v(T) = \sup\{|\lambda| : \lambda \in V(\mathcal{L}(E), T)\}$.*

The following property of the numerical radius is a well-known fact.

Theorem 1.40. *For a complex Banach space the numerical radius is equivalent to the norm.*

$$\frac{1}{e}||T|| \leq v(T) \leq ||T||$$

In particular if $v(T) = 0$ then $T = 0$.

The final property given in this introduction gives a criteria for the set of complex numbers in the numerical range. We use the Banach algebra definition in order to be able to apply this theorem later in the dissertation.

Theorem 1.41. *Let $\mathcal{A}$ be a Banach algebra with $a \in \mathcal{A}$. Then $V(\mathcal{A}, a)$ is the set of all complex numbers λ such that*

$$|z - \lambda| \leq ||z - a|| \text{ for all } z \in \mathbb{C}$$

When considering $\mathcal{A} = \mathcal{L}(E)$ for some complex Banach space E and $T \in \mathcal{L}(E)$ the above theorem gives us

$$V(\mathcal{L}(E), T) = \bigcap_{z \in \mathbb{C}} \{\lambda : |z - \lambda| \leq ||T - zI||\}$$

When our space E is a Banach lattice we have another possible algebra we could use instead of $\mathcal{L}(E)$ to define a numerical range. The rest of this dissertation will focus on choosing the regular operators, $\mathcal{L}_r(E)$, as the given algebra.

Chapter 2

Numerical Ranges in Banach Lattices

2.1 Regular Algebra Numerical Range

As seen in the previous section you can define a numerical range using any Banach algebra. When in a Banach lattice you not only have the choice of using the bounded linear operators, but also the regular operators. We define the regular algebra numerical range using this Banach algebra. For this section, Banach lattices will be assumed to be both complex and Dedekind complete.

Definition 2.1. The *regular algebra numerical range* of a regular operator T on a Banach lattice E is defined to be

$$V(\mathcal{L}_r(E), T) := \{\Phi(T) : \Phi \in \mathcal{L}_r(E)^*, ||\Phi|| = 1 = \Phi(I)\}$$

Remark 2.2. Such Φ as above will be called regular states to differentiate them from the states defined in remark 1.35.

First we must relate this new numerical range to the classical numerical ranges defined in Section 1.2.

Proposition 2.3. *Let T be a linear operator on a Banach lattice E. Then we have*

$$V(T) \subseteq V(\mathcal{L}(E), T) \subset V(\mathcal{L}_r(E), T)$$

Proof. The first inclusion is proposition 1.37.

Now consider $\Phi \in \mathcal{L}(E)^*$ such that $||\Phi|| = 1 = \Phi(I)$. Let $\Phi_r = \Phi\mid_{\mathcal{L}_r(E)}$. Since $||\Phi|| = 1$ we have that for all $S \in \mathcal{L}_r(E)$, $|\Phi_r(S)| = |\Phi(S)| \leq ||S|| \leq ||S||_r$. Hence

$||\Phi_r||_r \leq 1$. However, we have that $1 = \Phi(I) = \Phi_r(I) \leq ||\Phi_r||_r ||I||_r = ||\Phi_r||_r$ so $||\Phi_r||_r = 1$.

Thus if $\lambda = \Phi(T) \in V(\mathcal{L}(E), T)$ we also have that $\lambda = \Phi_r(T) \in V(\mathcal{L}_r(E), T)$ giving the desired result. $\square$

The next question is whether the regular algebra numerical range is distinct from the algebra numerical range for a regular operator T. This question will be answered later in this section, but first we must consider the regular algebra numerical range for specific types of operators. The following proof uses the regular algebra application of Theorem 1.41 that states

$$V(\mathcal{L}_r(E), T) = \bigcap_{z \in \mathbb{C}} \{\lambda : |z - \lambda| \leq ||T - zI||_r\}.$$

Theorem 2.4. *Let E be a Banach lattice and let $T \in \mathcal{L}_r(E)$ with $T \perp I$. Then $V(\mathcal{L}_r(E), T)$ is a disk centered around $z = 0$, i.e. if $\lambda \in V(\mathcal{L}_r(E), T)$, then $e^{i\theta}\mu \in V(\mathcal{L}_r(E), T)$ for all $0 \leq \theta \leq 2\pi$, and for all $|\mu| \leq |\lambda|$.*

Proof. Assume $T \perp I$ and let $\lambda \in V(\mathcal{L}_r(E), T)$ and $\theta \in [0, 2\pi)$. Consider $\alpha := \lambda e^{i\theta}$. Since $\lambda \in V(\mathcal{L}_r(E), T)$ and by theorem 1.41 we have that $|\lambda - \omega| \leq ||(T - \omega I)||_r \, \forall \omega \in \mathbb{C}$. Now we have

$$|\alpha - \omega| = |e^{i\theta}\lambda - \omega| = |e^{i\theta}(\lambda - e^{-i\theta}\omega)| = |\lambda - e^{-i\theta}\omega| \leq ||(T - e^{-i\theta}\omega I)||_r$$

$$= || \, |T - e^{-i\theta}\omega I| \, ||_r = || \, |T| - |e^{-i\theta}\omega I| \, ||_r = || \, |T| - |\omega| I||_r = ||T - \omega I||_r$$

for all $\omega \in \mathbb{C}$. By Theorem 1.41 we have that $\alpha \in V(\mathcal{L}_r(E), T)$. $\square$

We can now provide an example where the algebra numerical range is strictly contained in the regular algebra numerical range.

Example 2.5. Let $T = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ over $\ell_2(\mathbb{C})$. Since E is a Hilbert space we can use the fact that $V(T) = W(T) \subseteq \mathbb{R}$ because T is a Hermitian matrix. Furthermore,

$V(\mathcal{L}(E), T) = \overline{\text{co}}\, V(T) \subseteq \mathbb{R}$.

However, $T \perp I$, so $V(\mathcal{L}_r(E), T)$ must be a disk and thus not a subset of $\mathbb{R}$.

Figure 2.1 gives a complete description of these two numerical ranges.

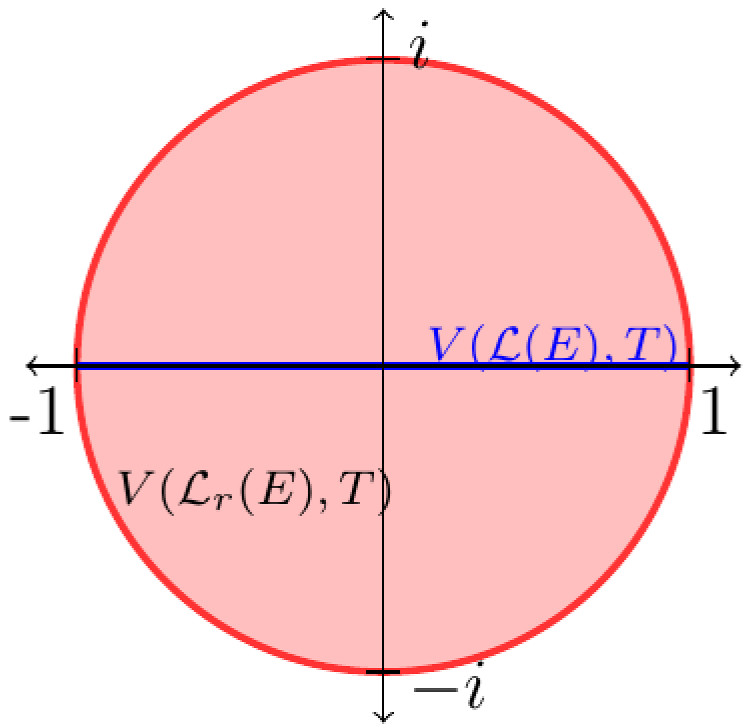

Figure 2.1 A comparison of the classical and regular numerical ranges

Now that we know how an operator disjoint with the identity behaves, we will look at another special type of operator, operators in the center. First we will need the following lemma.

Lemma 2.6. *Let μ be a complex regular Borel measure on a compact Hausdorff space K such that $\mu(K) = 1 = |\mu|(K)$, where $|\mu|$ is the total variation of μ. Then μ is a positive measure.*

Proof. Assume that there exists some measurable set $E \subset K$ such that $\mu(E) = c$ for some $c \in \mathbb{C} \backslash \mathbb{R}$. By the additivity of a complex measure $\mu(E^c) = 1 - c$. However,

$$|\mu|(K) \geq |\mu(E)| + |\mu(E^c)| > 1.$$

This yields a contradiction so μ must be a real measure.

Assume that there exists some measureable set $E \subset K$ such that $\mu(E) = -a$ for some $a > 0$. By the additivity of the measure $\mu(E^c) = 1 + a$. However,

$$|\mu|(K) \geq |\mu(E)| + |\mu(E^c)| > 1.$$

This yields a contradiction so μ must be a positive measure. $\square$

Lemma 2.7. *Consider* $\Phi \in \mathcal{L}_r(E)^*$ *such that* $\Phi(I) = 1 = ||\Phi||$. *Then* $\Phi\mid_{\mathcal{Z}(E)} \geq 0$..

Proof. By the Kakutani theorem 1.18, there exists a compact Hausdorff space K such that $\mathcal{Z}(E)$ is lattice isomorphic to $C(K)$. Hence $\Phi\mid_{\mathcal{Z}(E)}$ can be identified with some $\Psi \in C(K)^*$. By the Riesz representation theorem every functional on $C(K)$ can be represented by a regular complex Borel measure μ on K such that $\Psi(f) = \int_K f d\mu$ and $||\mu|| = |\mu|(K) = ||\Psi||$. Thus we have that $\mu(K) = 1 = |\mu|(K)$ and by lemma 2.6 this means that μ must be a postive measure, which implies that Ψ must be a positive functional, and by the lattice isomorphism implies that $\Phi\mid_{\mathcal{Z}(E)} \geq 0$. $\square$

Remark 2.8. As mentioned in example 1.22 the operators in the center of $\ell_p(n)$ for $1 \leq p \leq \infty$ are the diagonal matrices. The above lemma says that all regular states on $\mathcal{L}_r(\ell_p(n))$ must be non-negative on the diagonal.

Lemma 2.9. *Let* E *be a Banach lattice and* $T \in \mathcal{Z}(E)$. *Then* $V(\mathcal{L}_r(E), T) = V(\mathcal{Z}(E), T)$

Proof. By the Hahn-Banach theorem, the mapping $\Phi \to \Phi|_{\mathcal{Z}(E)}$ maps $\{\Phi \in \mathcal{L}_r(E)^* : ||\Phi|| = 1 = \Phi(I)\}$ onto $\{\Phi \in \mathcal{Z}(E)^* : ||\Phi|| = 1 = \Phi(I)\}$. Therefore we have that

$$\{\Phi(T) : \Phi \in \mathcal{L}_r(E)^*, ||\Phi|| = 1 = \Phi(I)\} = \{\Phi(T) : \Phi \in \mathcal{Z}(E)^* : ||\Phi|| = 1 = \Phi(I)\}$$

which proves the result. $\square$

Theorem 2.10. *Let E be a Banach lattice and $T \in \mathcal{Z}(E)$. Then $V(\mathcal{L}_r(E), T) = co(\sigma(T))$.*

Proof. By lemma 2.9 we have that $V(\mathcal{L}_r(E), T) = V(\mathcal{Z}(E), T)$. By the Kakutani Theorem 1.18 we have that $\mathcal{Z}(E)$ is lattice isomorphic to $C(K)$ for some compact Hausdorff space K. We can identity T with some $f \in C(K)$ such that $V(\mathcal{Z}(E), T) = V(C(K), f)$. By example 1.36 we have that $V(C(K), f) = \text{co}(f(K))$. It is also known that $\sigma(f) = f(K)$. By the lattice isomorphism given by Kakutani we thus have $V(\mathcal{L}_r(E), T) = \text{co}(\sigma(T))$. $\square$

Throughout the study of numerical ranges people have studied Hermitian operators, those operators whose numerical range is a subset of the real line. For this reason it is worthwhile to consider the operators whose regular algebra numerical range is a subset of the real line.

Theorem 2.11. *Let E be a Banach lattice and $T \in \mathcal{L}_r(E)$. Then $V(\mathcal{L}_r(E), T) \subset \mathbb{R}_+$ if and only if $T \geq 0$ and $T \in \mathcal{Z}(E)$*

Proof. Consider $0 \leq T \in \mathcal{Z}(E)$. By lemma 2.9, we have $V(\mathcal{L}_r(E), T) = V(\mathcal{Z}(E), T)$. Let $\Phi \in \mathcal{L}_r(E)^*$ such that $\Phi(I) = 1 = ||\Phi||$. By lemma 2.7 we have $\Phi\mid_{\mathcal{Z}(E)} \geq 0$, so we have $\Phi\mid_{\mathcal{Z}(E)}(T) \geq 0$ and so $V(\mathcal{L}_r(E), T) \subset \mathbb{R}_+$.
Now assume that $V(\mathcal{L}_r(E), T) \subseteq \mathbb{R}_+$. Since $T \in \mathcal{L}_r(E)$, $T = T_1 + T_2$ where $T_1 \in \mathcal{Z}(E)$ and $T_2 \perp I$. For any regular state, Φ, we have

$$\Phi(T) = \Phi(T_1) + \Phi(T_2).$$

Conside any regular state $\Phi \in \mathcal{L}_r(E)^*$. Since the restriction to the center is a contraction and $I \in \mathcal{Z}(E)$ we have that $\Phi\mid_{\mathcal{Z}(E)}$ is also a state. Thus for any Φ, $\Phi(T_1) = \Phi\mid_{\mathcal{Z}(E)}(T_1) = \Phi\mid_{\mathcal{Z}(E)}(T) \geq 0$. Thus $\Phi(T_1) \geq 0$ for all regular states Φ and thus $T_1 \geq 0$. Since $\Phi(T_1) \in \mathbb{R}$ for all regular states, we must have that $\Phi(T_2) \in \mathbb{R}$ for all regular states. If $T_2 \neq 0$ then by theorem 2.4 there exists a regular state Ψ such that $\Psi(T_2) \in \mathbb{C}\backslash\mathbb{R}$ which is a contradiction of $V(\mathcal{L}_r(E), T) \subseteq \mathbb{R}_+$. Hence $T_2 = 0$ and thus $0 \leq T = T_1 \in \mathcal{Z}(E)$. $\square$

Corollary 2.12. *Let E be a Banach lattice and $T \in \mathcal{L}_r(E)$. Then $V(\mathcal{L}_r(E), T) \subset \mathbb{R}$ if and only if $T \in \mathcal{Z}_{\mathbb{R}}(E)$.*

Thus the hermitian operators for the regular algebra numerical range are the real central operators.

2.2 Positive Numerical Ranges

Before continuing the study of the regular algebra numerical range we will discuss other possible numerical ranges that can be considered once you have the lattice structure, namely positivity. A similar question was studied in [15]. In that paper, they showed results for positive operators in the classical numerical ranges. We instead continue to use the regular algebra numerical range. For the following, let E be a complex Dedekind complete Banach lattice.

Definition 2.13. The *positive spatial numerical range* of a linear operator $T \in \mathcal{L}_r(E)$ is defined to be

$$V_+(T) := \{f(Tx) : 0 \leq x \in E, 0 \leq f \in E^*, ||x|| = ||f|| = 1 = f(x)\}$$

Definition 2.14. The *positive algebra numerical range* of a linear operator $T \in \mathcal{L}_r(E)$ is defined to be

$$V_+(\mathcal{L}_r(E), T) := \{\Phi(T) : 0 \leq \Phi \in \mathcal{L}_r(E)^*, ||\Phi|| = 1 = \Phi(I)\}$$

Remark 2.15. Although the above definitions are defined for all $T \in \mathcal{L}_r(E)$, the main focus will be on operators that are also positive.

Lemma 2.16. *For each $T \in \mathcal{L}_r(E)$, $V_+(\mathcal{L}_r(E), T)$ is a compact convex set.*

Proof. Define $D(\mathcal{L}_r(E), 1) := \{0 \leq \Phi \in \mathcal{L}_r(E)^* : ||\Phi|| \leq 1 \text{ and } \Phi(I) = 1\}$. Then we have $D(\mathcal{L}_r(E), 1)$ is a convex weak* compact subset of $\mathcal{L}_r(E)^*$. The set $V_+(\mathcal{L}_r(E), T)$ is the image of $D(\mathcal{L}_r(E), 1)$ under the weak* continuous linear mapping $\Phi \to \Phi(T)$ and thus $V_+(\mathcal{L}_r(E), T)$ is a compact convex set. □

Corollary 2.17. *Let $0 \leq T \in \mathcal{L}_r(E)$. We have that $V_+(\mathcal{L}_r(E), T)$ is an interval in $[0, \infty)$.*

Proof. Let $0 \leq \Phi \in \mathcal{L}_r(E)^*, ||\Phi|| = 1 = \Phi(I)$. Since $T \geq 0$ we have that $\Phi(T) \geq 0$ and thus $V_+(\mathcal{L}_r(E), T) \subseteq [0, \infty)$. However by lemma 2.16 we know the set is convex, we must have that $V_+(\mathcal{L}_r(E), T)$ is an interval. □

Proposition 2.18. *Let $T \in \mathcal{L}_r(E)$. Then we have $V_+(T) \subset V_+(\mathcal{L}_r(E), T)$.*

Proof. Given $0 \leq x \in E, 0 \leq f \in E^*, ||x|| = ||f|| = 1 = f(x)$ define

$$\Phi(S) = f(Sx) \, \forall \, S \in \mathcal{L}_r(E)$$

Then we have that $0 \leq \Phi$ and $||\Phi|| = 1 = \Phi(I)$. Thus $f(Tx) = \Phi(T) \in V_+(\mathcal{L}_r(E), T)$. □

Recall the definition of the numerical radius for a linear operator T, $v(T) := \sup\{|\lambda| : \lambda \in V(T)\}$. In order to form a comparison we also define a similar supremum for the positive numerical range.

Definition 2.19. The *positive numerical radius* of a linear operator T on a Banach lattice E is defined to be

$$v_+(T) := \sup\{|\lambda| : \lambda \in V_+(\mathcal{L}_r(E), T)\}.$$

Proposition 2.20. *Let $0 \leq T \in \mathcal{L}_r(E)$. Then $v(T) = v_+(T)$.*

Proof. Clearly

$$v_+(T) \leq v(T) \tag{2.1}$$

. Consider $||f|| = ||x|| = 1 = f(x)$. We have

$$1 = f(x) = |f(x)| \leq |f|(|x|) \leq || \, |f| \, || \cdot || \, |x| \, || = 1.$$

Thus we have that $|f|, |x|$ are a set of postive norm attainers. Furthermore, $|f(Tx)| \leq |f|(T|x|)$ so we must have that $v(T) \leq v_+(T)$. This inequality along with (2.1) gives the desired result. □

Lemma 2.21. *Let $0 \leq T \in \mathcal{L}_r(E)$ with $T \perp I$. Then we have $0 \in \overline{V_+(T)}$.*

Proof. Let $0 < \varepsilon < 1$ be given. Let $0 < x \in E$. Since $T \geq 0$ and by proposition 1.23 we have that there exists a component of x, a, such that $P_a(Ta) \leq \varepsilon a$ where P_a is the band projection of E onto the band generated by a. By normalizing, we can assume that $||a|| = 1$. Now consider a norm-attainer $0 \leq g \in E^*$ such that $||g|| = ||a|| = 1 = g(a)$. Now note that $||g|_{P_a}|| = 1 = g|_{P_a}(a)$ so $g|_{P_a}$ is also a norm-attainer. We have

$$|g|_{P_a}(Ta)| = |g|_{P_a}(P_a(Ta))| \leq |g(\varepsilon a)| = \varepsilon.$$

Thus every ε-neighborhood of 0 contains an element of $V_+(T)$, which implies that $0 \in \overline{V_+(T)}$. □

Lemma 2.22. *Let $T \in \mathcal{L}_r(E)$. Then we have $V_+(T - \delta I) = \{\lambda - \delta : \lambda \in V_+(T)\}$ for all $\delta \in \mathbb{R}$.*

Proof. Let $0 \leq x \in E, 0 \leq f \in E^*$ with $||x|| = ||f|| = 1 = f(x)$. Then

$$f((T - \delta I)x) = f(Tx - \delta Ix) = f(Tx) - f(\delta x) = f(Tx) - \delta.$$

Hence for each pair of norm-attainers, the resulting element of $V_+(T - \delta I)$ is a shift by δ from corresponding element of $V_+(T)$, giving the result. □

Similarly to $v_+(T)$, which is the supremum of the positive spatial numerical range, we also want to define the infimum of the positive spatial numerial range.

Definition 2.23. Let $T \in \mathcal{L}_r(E)$. Then $\mu_+(T) := \inf\{|\lambda| : \lambda \in V_+(T)\}$.

Using the following theorem we can relate this infimum to how the operator dominates the identity.

Theorem 2.24. *Let $0 \leq T \in \mathcal{L}_r(E)$. Then we have that $\mu_+(T) = \sup\{c : T \geq cI\}$.*

Proof. Let $\delta := \sup\{c : T \geq cI\}$. By definition, $T - \delta I \geq 0$ however $T - (\delta + \varepsilon)I \ngeq 0$ for all $\varepsilon > 0$. Let $\mathcal{P}(Q)$ be the projection of $Q \in \mathcal{L}_r(E)$ onto the center, $\mathcal{Z}(E)$. This projection is a positive map so we have

$$\mathcal{P}(T - \delta I) = \mathcal{P}(T) - \delta I \geq 0.$$

Also,

$$\mathcal{P}(T - (\delta + \varepsilon)I) = \mathcal{P}(T) - (\delta + \varepsilon)I \ngeq 0$$

for all $\varepsilon > 0$.

We may assume that $\delta = 0$, for if not, we can consider $T' = T - \delta I \geq 0$. Thus we have, $(\mathcal{P}(T) - \varepsilon I)^- \gneq 0$. Hence there must exist some $0 \neq y \in E$ such that $\mathcal{P}(T)y < \varepsilon y$. Now consider P_y the band projection onto B_y the band generated by y. For each $x \in B_y$ we must also have $\mathcal{P}(T)x \leq \varepsilon x$ for if not then $0 \leq (\mathcal{P}(T) - \varepsilon I)x \leq (\mathcal{P}(T) - \varepsilon I)\lambda y$ where $0 < \lambda \in \mathbb{R}$. This is a contradiction as it implies that $\mathcal{P}(T)y \geq \varepsilon y$.

Now consider $T = \mathcal{P}(T) + T_1$ where importantly $T_1 \perp I$. Thus there exists a non-zero component of y, y_0 such that $P_{y_0}(T_1 y_0) \leq \varepsilon y_0$. We can normalize y_0 to ensure that $||y_0|| = 1$. Consider a norm-attainer, g_0, such that $||g_0|| = ||y_0|| = 1 = g_0(y_0)$. Now considering the original T we have

$$g_0(Ty_0) = g_0((\mathcal{P}(T) + T_1)y_0) = g_0(\mathcal{P}(T)y_0) + g_0(T_1 y_0) < g_0(\varepsilon y_0) + g_0(T_1 y_0) < 2\varepsilon.$$

Thus every ε-neighborhood of 0 contains an element of $V_+(T)$. Since T was assumed positive, which implies that $V_+(T) \subset [0, \infty)$, we must have that $\mu_+(T) = 0$. $\square$

The final theorem of this section relates the two positive numerical ranges. As with the traditional numerical ranges these two numerical ranges are related by a closure, however since they are intervals, the convex hull is unnecessary.

Theorem 2.25. *Let $T \in \mathcal{L}_r(E)$. If $T \geq 0$, then $\overline{V_+(T)} = V_+(\mathcal{L}_r(E), T)$.*

Proof. Based on corollary 2.17, proposition 2.20, and proposition 2.18, we already have that $v_+(T) = \sup\{\lambda : \lambda \in V_+(\mathcal{L}_r(E), T)\} = \sup\{\gamma : \gamma \in V_+(T)\}$, that both sets are intervals on the positive real axis, and $V_+(T) \subseteq V_+(\mathcal{L}_r(E), T)$. Now assume that $\inf\{\lambda : \lambda \in V_+(\mathcal{L}_r(E), T)\} < \mu^+(T)$. Consider $T - \mu^+(T)$. By theorem 2.24 we have that $(T - \mu^+(T)) \geq 0$ and that $0 \in V_+(T - \mu^+(T))$. However since $\inf\{\lambda : \lambda \in V_+(\mathcal{L}_r(E), T)\} < \mu^+(T)$ we must have that $V_+(\mathcal{L}_r(E), T - \mu^+(T)) \not\subseteq [0, \infty) \rightarrow\leftarrow$ This is a contradiction because of the positivity of $T - \mu^+(T)$. Hence both sets are intervals on the postive real axis with the same endpoints, so their closures must be equal. $\square$

2.3 A Note on Duality

To end this chapter we take a brief look at adjoint maps in the numerical ranges already defined. To begin we give a known result related to this topic. Proofs and further reading can be found in [6].

Proposition 2.26. *Let $T \in \mathcal{L}(E)$ for some Banach space E. Then*

(i) $V(T) \subset V(T^*)$

(ii) $\overline{co}(V(T)) = \overline{co}(V(T^*))$

(iii) $v(T) = v(T^*)$

We begin our study of adjoints by considering the positive spatial numerical range of the adjoint T^* of a linear operator $T \in \mathcal{L}_r(E)$.

Lemma 2.27. *Let $T \in \mathcal{L}_r(E)$ with $T \geq 0$. Then we have $V_+(T) \subseteq V_+(T^*)$*

Proof. Consider $\lambda \in V_+(T)$. Then there exists $0 \leq f \in E^*, 0 \leq x \in E$ such that $||f|| = ||x|| = 1 = f(x)$ and $f(Tx) = \lambda$. Then we have $(T^*f)(x) = x^{**}(T^*f)$ where $||x^{**}|| = ||f|| = 1$ and $x^{**}(f) = f(x) = 1$. Thus $\lambda \in V_+(T^*)$. □

Theorem 2.28. *Let $T \in \mathcal{L}_r(E)$ with $T \geq 0$. Then we have $\overline{V_+(T)} = \overline{V_+(T^*)}$*

Proof. Based on proposition 2.26, we have that $v(T) = v(T^*)$. From proposition 2.20 we also know that $v(T) = v_+(T)$. Combining these facts gives us $v_+(T) = v(T) = v(T^*) = v_+(T*)$. Hence both intervals have the same maximums. Now assume $\mu^+(T^*) < \mu^+(T)$ and consider $(T - \mu^+(T)) \geq 0$. $(T - \mu^+(T))^* = T^* - \mu^+(T)$. As proved, in theorem 2.24, $\mu^+(T - \mu^+(T)) = 0$ which based on our assumption would imply that $\mu^+(T^* - \mu^+(T)) < 0 \rightarrow\leftarrow$ This is a contradiction because we must have that $T^* - \mu^+(T) \geq 0 \Rightarrow \mu^+(T^* - \mu^+(T)) \geq 0$.
Thus both sets have the same maximum and minimum, and both sets are intervals on the real line, so we must have that $\overline{V_+(T)} = \overline{V_+(T^*)}$. □

As with the classical numerical ranges there is a closure representation between the positive spatial numerical range of an operator and its adjoint. Since both sets are intervals the convex hull condition is not needed. As a corollary to the previous theorem we also get a relationship between the positive algebra numerical range of an operator and its adjoint.

Corollary 2.29. *Let $T \in \mathcal{L}_r(E)$ with $T \geq 0$. Then we have $V_+(\mathcal{L}_r(E), T) = V_+(\mathcal{L}_r(E^*), T^*)$.*

Proof. Recall from Theorem 2.25 we have that $\overline{V_+(T)} = V_+(\mathcal{L}_r(E), T)$. Thus we have

$$V_+(\mathcal{L}_r(E), T) = \overline{V_+(T)} = \overline{V_+(T^*)} = V_+(\mathcal{L}_r(E^*), T^*)$$

where the second equality is due to Theorem 2.28 □

Finally we wish to discuss the duality for the regular algebra numerical range. First note that the fact $V(\mathcal{L}(E), T) = V(\mathcal{L}(E^*), T^*)$ is almost trivial due to the characterization given in Theorem 1.41 and the fact that $||T|| = ||T^*||$ for all $T \in \mathcal{L}(E)$. Following a similar path we have a similar proposition for the regular algebra numerical range.

Proposition 2.30. *Let $T \in \mathcal{L}_r(E)$ such that $||T||_r = ||T^*||_r$. Then $V(\mathcal{L}_r(E), T) = V(\mathcal{L}_r(E^*), T^*)$.*

Proof. Since $||T||_r = ||T^*||_r$ we have $||T - zI||_r = ||T^* - zI||_r$ for all $z \in \mathbb{C}$. Hence by the characterization in Theorem 1.41 we have,

$$\begin{aligned} V(\mathcal{L}_r(E), T) &= \cap_{z\in\mathbb{C}}\{\lambda : |z - \lambda| \leq ||T - zI||_r\} \\ &= \cap_{z\in\mathbb{C}}\{\lambda : |z - \lambda| \leq ||T^* - zI||_r\} \\ &= V(\mathcal{L}_r(E^*), T^*). \end{aligned}$$

□

Note that in the above proposition we had to assume that $||T||_r = ||T^*||_r$ because this property is not true for every $T \in \mathcal{L}_r(E)$ as it is in the bounded operator case. There are characterizations of a Banach lattice such that this property will hold for every $T \in \mathcal{L}_r(E)$. The following theorem due to Altin [3] gives such a characterization.

Theorem 2.31. *Let E and F be two Banach lattices with F having a Levi norm, i.e. every norm bounded upward directed set of positive elements has a supremum. Then a continous operator $T : E \to F$ is order bounded if and only if its adjoint $T^* : F^* \to E^*$ is order bounded. In particular, if F also has a Fatou norm (for every increasing net $(x_i)_{i\in\Gamma} \in F$ with the supremum $x \in F$ it follows that $||x|| = \sup\{||x_i|| : i \in \Gamma\}$), then T satisfies $||\,|T|\,|| = ||\,|T^*|\,||$.*

Other conditions to impose on the Banach lattice E are that E has an order continuous norm or that E is reflexive. Both of these properties also have the property that $||T||_r = ||T^*||_r$ for all $T \in \mathcal{L}_r(E)$.

At the time of this dissertation it remains an open question if Proposition 2.30 is true without the assumption $||T||_r = ||T^*||_r$.

Chapter 3

Numerical Range Preserving Maps on $\mathcal{L}_r(\ell_p(n))$

The goal of this chapter is to describe the linear maps on $\mathcal{L}_r(\ell_p(n))$ that preserve the regular algebra numerical range. Although few references are made throughout the chapter, these results were heavily motivated by results of Li and Sourour [11]. They considered a similar problem for bounded numerical ranges. In the case where $p = 1$ and $p = \infty$ our results necessarily coincide with those of Li and Sourour due to Remark 1.16. However, in the case $1 < p < \infty$ our results differ.

3.1 Regular States

Before we tackle the main goal of this chapter we will discuss the regular states $\mathcal{L}_r(\ell_p(n))$. Let $S(A)$ be the set of states on a Banach space A and $S_r(B)$ be the set of regular states on a Banach lattice B. The motivation for this is due to the following theorem by Pelligrini [13].

Theorem 3.1. *Let $\mathcal{A}$ be a unital Banach algebra, $a \in A$ and $\mathcal{F}$ a bounded linear operator on A. Then the following are equivalent:*

(i) $V(\mathcal{A}, \mathcal{F}(a)) \subset V(\mathcal{A}, a)$

(ii) $\mathcal{F}^*(S(\mathcal{A})) \subset S(\mathcal{A})$

In this dissertation we will be concerned with the equality case of the above theorem. In other words, a map is numerical range preserving if and only if its adjoint map preserves the states. In order to describe the linear operators that preserve the regular algebra numerical range it will be important to first understand the regular states. First we prove a simple lemma that will allow us to consider positive states when given any regular state.

Lemma 3.2. *Let E be a Dedekind complete complex Banach lattice. If $\Phi \in S_r(\mathcal{L}_r(E))$ then $|\Phi| \in S_r(\mathcal{L}_r(E))$.*

Proof. Since $\mathcal{L}_r(E)$ has a lattice norm we have that $||\,|\Phi|\,|| = ||\Phi|| = 1$. By Lemma 2.7, $\Phi \mid_{\mathcal{Z}(E)} \geq 0$ and so,

$$1 = \Phi(I) = \Phi \mid_{\mathcal{Z}(E)} (I) = |\Phi| \mid_{\mathcal{Z}(E)} (I) = |\Phi|(I).$$

Thus $||\,|\Phi|\,|| = 1 = |\Phi|(I)$ and $|\Phi| \in S_r(\mathcal{L}_r(E))$. □

We now are able to discuss the regular states on $\mathcal{L}_r(\ell_p(n))$. To this end we have the following two lemmas which describe a way to factor a regular state on $\mathcal{L}_r(\ell_p(n))$ into factors from $\ell_1(n)$ and $\ell_\infty(n)$.

Lemma 3.3. *Let $\Phi_1 = [a_{ij}] \in \mathcal{S}_r(\mathcal{L}_r(\ell_1(n)))$ and $\Phi_2 = [b_{ij}] \in \mathcal{S}_r(\mathcal{L}_r(\ell_\infty(n)))$. Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. If for all $1 \leq i \leq n$ we have $a_{ii} = b_{ii}$, then $\Phi := \Phi_1^{1/p}\Phi_2^{1/p'}$ is in $\mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$, where the exponent and multiplication are done entrywise.*

Proof. Let $\Phi = [\phi_{ij}]$

$$\Phi(I) = \sum_{i=1}^{n} \phi_{ii} = \sum_{i=1}^{n} a_{ii}^{1/p}\, b_{ii}^{1/p'} = \sum_{i=1}^{n} a_{ii}^{1/p}\, a_{ii}^{1/p'} = \sum_{i=1}^{n} a_{ii} = 1.$$

Thus we have that $\Phi(I) = 1$.

$$||\Phi||_r \leq ||\Phi_1||_r^{1/p} ||\Phi_2||_r^{1/p'} = 1^{1/p}1^{1/p'} = 1.$$

Hence we have that $||\Phi||_r \leq 1$. However we already know that $\Phi(I) = 1$ so we must have that $||\Phi||_r = 1$.

This shows that Φ is a state on $\mathcal{L}_r(\ell_p(n))$. □

To prove the converse of the above lemma we will need a theorem proven by Schep in [16].

Theorem 3.4. *Let $1 < p < \infty$. Then the order continuous dual of $\mathcal{L}_r(L^p)$ is equal to $(L_{1,\infty})^{1/p'}(L^t_{1,\infty})^{1/p} = L_{p',\infty}L^t_{p,\infty}$.*

Using this theorem we can factor regular states on $\mathcal{L}_r(\ell_p(n))$ into a regular state on $\mathcal{L}_r(\ell_1(n))$ and a regular state on $\mathcal{L}_r(\ell_\infty(n))$.

Lemma 3.5. *Let $1 < p < \infty$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $0 \leq \Phi \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$, i.e. $||\Phi||_r = 1 = \Phi(I)$. Let Φ have matrix representation $\Phi = [\phi_{ij}]$. There exists $0 \leq \Phi_1 \in \mathcal{S}_r(\mathcal{L}_r(\ell_1(n)))$ and $0 \leq \Phi_2 \in \mathcal{S}_r(\mathcal{L}_r(\ell_\infty(n)))$ with $\Phi_1 = [a_{ij}]$, $\Phi_2 = [b_{ij}]$ and for all $1 \leq i \leq n, a_{ii} = b_{ii}$ such that $\Phi = \Phi_1^{1/p}\Phi_2^{1/p'}$ where the exponent and multiplication are done entry-wise.*

Proof. By Theorem 3.4 there exists $0 \leq \Phi_1 \in \mathcal{L}_r(\ell_1(n))^*$ and $0 \leq \Phi_2 \in \mathcal{L}_r(\ell_\infty(n))^*$ with $||\Phi_1|| = ||\Phi_2|| = 1$ such that $\Phi = \Phi_1^{1/p}\Phi_2^{1/p'}$. Thus all that needs to be shown is that the diagonals of the matrices are equal and that they are in fact states.

First let us show that the diagonal of the matrices Φ_1 and Φ_2 must be equal.

$$1 = \Phi(I) = \sum_{i=1}^{n} \phi_{ii} = \sum_{i=1}^{n} a_{ii}^{1/p} b_{ii}^{1/p'} \leq \left(\sum_{i=1}^{n} a_{ii}\right)^{1/p} \left(\sum_{i=1}^{n} b_{ii}\right)^{1/p'} \leq ||\Phi_1||^{1/p}||\Phi_2||^{1/p'} = 1.$$

Hence we must have equality throughout and by a corollary of the Holder's inequality the a_{ii} and the b_{ii} must be linearly dependent. Thus we must have that $a_{ii} = \mu b_{ii}$.

$$1 = \sum_{i=1}^{n} \phi_{ii} = \sum_{i=1}^{n} (\mu b_{ii})^{1/p} b_{ii}^{1/p'} = \mu^{1/p} \sum_{i=1}^{n} b_{ii}.$$

Since $||\Phi_2|| = 1$ we must have that $\Phi_2(I) = \sum_{i=1}^{n} b_{ii} \leq 1$. This in turn means that $\mu^{1/p} \geq 1 \Rightarrow \mu \geq 1$.

Now consider $b_{ii} = \frac{1}{\mu} a_{ii}$.

$$1 = \sum_{i=1}^{n} \phi_{ii} = \sum_{i=1}^{n} a_{ii}^{1/p} \left(\frac{1}{\mu} a_{ii}\right)^{1/p'} = \left(\frac{1}{\mu}\right)^{1/p'} \sum_{i=1}^{n} a_{ii}.$$

As above $\Phi_1(I) = \sum_{i=1}^{n} a_{ii} \leq 1$ so we must have that $\left(\frac{1}{\mu}\right)^{1/p'} \geq 1 \Rightarrow \mu^{1/p'} \leq 1 \Rightarrow \mu \leq 1$.

Combining the inequalities involving μ gives us that $\mu = 1$ and thus $a_{ii} = b_{ii}$ and the diagonals of Φ_1 and Φ_2 are equal. Since the diagonals are equal we must now have that

$$\phi_{ii} = a_{ii}^{1/p} b_{ii}^{1/p'} = a_{ii}^{1/p} a_{ii}^{1/p'} = a_{ii}.$$

Thus,

$$\Phi_1(I) = \sum_{i=1}^{n} a_{ii} = \sum_{i=1}^{n} \phi_{ii} = 1.$$

The same argument can be used to show that $\Phi_2(I) = 1$. Since we already have that $||\Phi_1|| = ||\Phi_2|| = 1$ we have that they are both states. $\square$

We now know that there is a factorization for any regular state on $\mathcal{L}_r(\ell_p(n))$. The following proposition states that we can determine the maximum of each coordinate in the state.

Proposition 3.6. *Let $1 < p < \infty$ and $n < \infty$. Let $\lambda_1 + \lambda_2 + \cdots + \lambda_n = 1$ with $\lambda_i \geq 0$ for all $1 \leq i \leq n$. Let $0 \leq \Phi \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$ with $\Phi = [\phi_{ij}]$ and $\phi_{ii} = \lambda_i$ for all $0 \leq i \leq n$. The maximum (entry-wise) that Φ can be is*

$$\phi_{ij} = \lambda_j^{1/p} \lambda_i^{1/p'}.$$

Proof. As stated in proposition 3.5 , Φ can be factorized into a Φ_1 and Φ_2. To maximize Φ you can maximize both Φ_1 and Φ_2. The maximum (entry-wise) Φ_1 can be is

$$\Phi_1 = \begin{bmatrix} \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1 & \lambda_2 & \cdots & \lambda_n \end{bmatrix}.$$

The maximum Φ_2 can be is

$$\Phi_2 = \begin{bmatrix} \lambda_1 & \lambda_1 & \cdots & \lambda_1 \\ \lambda_2 & \lambda_2 & \cdots & \lambda_2 \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_n & \lambda_n & \cdots & \lambda_n \end{bmatrix}.$$

Multiplying as in the previous proposition gives the desired form for Φ. $\square$

We are now able to describe the regular states on $\mathcal{L}_r(\ell_p(n))$ for $1 \leq p \leq \infty$. Next we would like to determine what the extreme points of the regular states, $\mathcal{E}(\mathcal{S}_r(\mathcal{L}_r(\ell_p(n))))$, are. The following theorem uses the factorization above to describe the extreme points of the regular states as vector states.

Theorem 3.7. *Let $1 < p < \infty$ and $\frac{1}{p} + \frac{1}{p'} = 1$. Then, $\mathcal{E}(\mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))) =$*

$$\{\Phi \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n))) : |\Phi| = f \otimes x, f, x \geq 0, ||f||_{p'} = ||x||_p = 1 = f(x)\}.$$

Proof. **[First Direction: If $|\Phi| = f \otimes x \Rightarrow \Phi$ is extreme]**
First we will show that if $|\Phi| = f \otimes x$ then $|\Phi|$ is extreme. To that end, assume in order to reach a contradiction, that $|\Phi|$ is not extreme. Then $|\Phi|$ can be written as a linear combination of two states in $\mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$. Hence,

$$|\Phi| = \lambda\Psi_1 + (1-\lambda)\Psi_2$$

where $0 < \lambda < 1$ and $\Psi_1, \Psi_2 \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$. We have that if $\Psi \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$ then $|\Psi| \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$ so there exists a state Φ' such that

$$\Phi' = \lambda|\Psi_1| + (1-\lambda)|\Psi_2|.$$

We clearly have that $|\Phi| \leq \Phi'$. However since $|\Phi| = f \otimes x$ and by the factorization of states on $\mathcal{L}_r(\ell_p(n))$ we must have that $|\Phi| = \Phi'$ and thus

$$|\Phi| = \lambda|\Psi_1| + (1-\lambda)|\Psi_2|$$

where $0 < \lambda < 1$ and $|\Psi_1|, |\Psi_2| \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))_+$.

Now consider $|\Phi| = [|\phi_{ij}|]$. Since the linear combination is done coordinate-wise we can say

$$|\phi_{ij}| = \lambda|\psi_{1_{ij}}| + (1-\lambda)|\psi_{2_{ij}}|.$$

There exists some i', j' such that $|\phi_{i'j'}| \neq |\psi_{1_{i'j'}}|$. Since everything is positive either $|\psi_{1_{i'j'}}| \geq |\phi_{i'j'}|$ or $|\psi_{2_{i'j'}}| \geq |\phi_{i'j'}|$. By the factorization of regular states on $\ell_p(n)$ mentioned in proposition 3.6, $|\Phi|$ contains the largest values in each coordinate in order to remain a state. This yields a contradiction of the fact that $\Psi_1, \Psi_2 \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$. Hence we must have had that $|\Phi|$ is extreme.

Now we will show that if $|\Phi| = f \otimes x$ then Φ is extreme. Again, in order to reach a contradiction assume that Φ is not extreme. Since Φ is not extreme

$$\Phi = \lambda\Psi_1 + (1-\lambda)\Psi_2$$

where $0 < \lambda < 1$ and $\Psi_1, \Psi_2 \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$. This gives us that

$$|\Phi| \leq \lambda|\Psi_1| + (1-\lambda)|\Psi_2|.$$

However from above we must have that $|\Phi| = |\Psi_1| = |\Psi_2|$. Now if we consider the coordinates of Φ we must have

$$\phi_{ij} = \lambda\psi_{1_{ij}} + (1-\lambda)\psi_{2_{ij}}$$

where $|\phi_{ij}| = |\psi_{1_{ij}}| = |\psi_{2_{ij}}|$. We can normalize the above equality to get

$$\frac{\phi_{ij}}{|\phi_{ij}|} = \lambda\frac{\psi_{1_{ij}}}{|\psi_{1_{ij}}|} + (1-\lambda)\frac{\psi_{2_{ij}}}{|\psi_{2_{ij}}|}.$$

In other words we have a complex number of modulus 1 that is a linear combination of two complex numbers of modulus 1. The only way this is possible is if all three complex numbers are equal. Hence for all i, j we have

$$\phi_{ij} = \psi_{1_{ij}} = \psi_{2_{ij}},$$

and thus $\Phi = \Psi_1 = \Psi_2$. This is a contradiction that Φ is a strict linear combination of Ψ_1 and Ψ_2 so we must have that $\Phi \in \mathcal{E}(\mathcal{S}_r(\mathcal{L}_r(\ell_p(n))))$.

[Second Direction: If Φ is extreme, then $|\Phi| = f \otimes x$]

First assume that $0 \leq \Phi \in \mathcal{E}(\mathcal{S}_r(\mathcal{L}_r(\ell_p(n))))$. Let $\Phi = [\phi_{ij}]$. Assume, in order to reach a contradiction, that $\Phi = |\Phi| \neq f \otimes x$. As shown in proposition 3.6, a state is maximal in each coordinate if it has the form

$$\Phi = \left[\lambda_j^{1/p}\lambda_i^{1/p'}\right] = (\lambda_1^{1/p'}, ..., \lambda_n^{1/p'}) \otimes (\lambda_1^{1/p}, ..., \lambda_n^{1/p}).$$

We must have that there is some coordinate $\phi_{i'j'}$ such that Φ is not maximal in that coordinate. In other words there exists some $\varepsilon > 0$ such that

$$\Psi_1 = \begin{cases} [\phi_{ij}] & i \neq i' \text{ or } j \neq j' \\ [\phi_{i'j'} + \varepsilon] & i = i', j = i' \end{cases} \quad \text{and } \Psi_2 = \begin{cases} [\phi_{ij}] & i \neq i' \text{ or } j \neq j' \\ [\phi_{i'j'} - \varepsilon] & i = i', j = i' \end{cases}$$

are in $\mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$. However now we can write

$$\Phi = \frac{1}{2}\Psi_1 + \frac{1}{2}\Psi_2.$$

This is a contradiction of the fact that $\Phi \in \mathcal{E}(\mathcal{S}_r(\mathcal{L}_r(\ell_p(n))))$ and thus we must have that $\Phi = f \otimes x$.

Now consider $\Phi \in \mathcal{E}(\mathcal{S}_r(\mathcal{L}_r(\ell_p(n))))$.
By Lemma 3.2 we have that $|\Phi| \in \mathcal{E}(\mathcal{S}_r(\mathcal{L}_r(\ell_p(n))))$. As seen above this implies that $|\Phi| = f \otimes x$, which completes the proof. □

To finish this section we discuss a known map that will preserve the regular states on $\ell_p(n)$.

Proposition 3.8. *Let P be a permutation matrix on $\ell_p(n)$. Given any state $\Phi \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$, $P^*\Phi P \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n)))$.*

Proof. First consider $P^*\Phi P$ as a matrix

$$P^*\Phi P = \left[\phi_{\sigma^{-1}(i)\sigma^{-1}(j)}\right]$$

Since P is a permutation it must be onto, so we have the following:

$$P^*\Phi P(I) = \sum_{i=1}^{n} \phi_{\sigma(i)\sigma(i)} = \sum_{i=1}^{n} \phi_{ii} = 1.$$

For the r-norm of $P^*\Phi P$, we can use the fact that P is interval perserving and P^* is a lattice homomorphism to equate $|P^*\Phi P| = P^*|\Phi|P$. Thus we can simplify as follows:

$$||P^*\Phi P||_r = ||\,|P^*\Phi P|\,|| = ||P^*|\Phi|P|| = \max\{P^*|\Phi|P(T) : ||T|| \leq 1\} =$$

$$= \max\{|\Phi|(PTP^*) : ||T|| \leq 1\} = \max|\Phi|(T) : ||T|| \leq 1 = ||\,|\Phi|\,|| = ||\Phi||_r = 1.$$

Thus we have that $P^*\Phi P$ is a regular state. □

Remark 3.9. Proposition 3.8 is also true for more general symmetric norms, but was stated in this way to emphasize the main result in the ensuing section.

3.2 Regular Algebra Numerical Range Preserving Maps

To begin this section we define a regular algebra numerical range preserving map.

Definition 3.10. Let $\mathcal{L} : \mathcal{L}_r(E) \to \mathcal{L}_r(E)$. We say $\mathcal{L}$ is *regular algebra numerical range preserving* if

$$V(\mathcal{L}_r(E), \mathcal{L}(T)) = V(\mathcal{L}_r(E), T) \text{ for all } T \in \mathcal{L}_r(E).$$

The final goal of this section is to prove the following theorem describing all numerical range preserving maps on $\mathcal{L}_r(\ell_p(n))$.

Theorem 3.11. *Let $\mathcal{L}$ be a regular numerical range preserving map on $\mathcal{L}_r(\ell_p(n))$. Then*

$$\mathcal{L}(T) = U \star \left(P^t \mathcal{Q} T P\right),$$

where P is a permutation matrix, $U = [u_{ij}]$ is a matrix such that $|u_{ij}| = 1$ and $u_{ii} = 1$ for all $i = \{1, ..., n\}$, $(\star)$ represents Hadamard multiplication, and $\mathcal{Q}$ is a map that permutes the off-diagonal entries of T. Moreover given the value of p we have *the following necessary and sufficient conditions on $\mathcal{Q}$ for the map $\mathcal{L}$ to be regular numerical range preserving.*

(a) *If $p = 1$ then $\mathcal{Q}$ permutes the column vectors.*

(b) *If $p = \infty$ then $\mathcal{Q}$ permutes the row vectors.*

(c) *If $p = 2$ then $\mathcal{Q}$ is a partial transpose, i.e. there exists $A \subset \{(i,j) : 1 \leq i < j \leq n\}$ such that $\mathcal{Q}(t_{i,j}) = t_{j,i}$ for all $(i,j) \in A$ and $(j,i) \in A$ and $\mathcal{Q}(t_{i,j}) = t_{i,j}$ otherwise.*

(d) *If $p \neq \{1, 2, \infty\}$, then $\mathcal{Q}$ is the identity.*

To that end we first determine how a numerical range preserving map will behave on the center of the Banach lattice. For this section let $1 \leq n < \infty$ and $1 \leq p \leq \infty$ (unless otherwise stated).

First we require a theorem proven by Phelps in [14]. As stated this is a special case of the theorem proven by Phelps.

Theorem 3.12. *Suppose T is a linear operator from $C(X)$ to $C(X)$ for some compact Hausdorff space X. Then T is an isometry and $T(I) = 1$ if and only if $\overline{co}(Tf)(X) = \overline{co}\, f(X)$ for each $f \in C(X)$.*

We now use this theorem to prove the following lemma describing how a linear regular numerical range preserving map will act on the center of a Banach lattice.

Lemma 3.13. *Let $\mathcal{L} : \mathcal{L}_r(\ell_p(n)) \to \mathcal{L}_r(\ell_p(n))$ be a regular numerical range preserving map. The restriction of $\mathcal{L}$ to the center of $\mathcal{L}_r(\ell_p(n))$, $\mathcal{L}|_{\mathcal{Z}(\ell_p(n))}$, is a permutation.*

Proof. For simplicity of notation let $\mathcal{L}|_{\mathcal{Z}(\ell_p(n))} = \mathcal{L}_0$. Note that we have $\mathcal{L}_0(\mathcal{Z}(\ell_p(n))) = \mathcal{Z}(\ell_p(n))$. By the Kakutani theorem 1.18, $\mathcal{Z}(\ell_p(n))$ is lattice isomorphic to $C(K)$ for some compact Hausdorff space K. In fact, for $\ell_p(n)$, $K =$

$\{1, 2, ..., n\}$. By example 1.36, $\mathcal{L}_0$ must be a range preserving map. By Phelps 3.12, $\mathcal{L}_0$ must be an isometry and thus must be given by a permutation. $\square$

Hence there is a permutation matrix P such that $\mathcal{L}(T) = P^*TP$ for all $T \in \mathcal{Z}(\ell_p(n))$. We can create a new linear operator $\mathcal{L}_1 = P^*\mathcal{L}P$ such that $\mathcal{L}_1(T) = T$ for all $T \in \mathcal{Z}(\ell_p(n))$, i.e. $\mathcal{L}_1$ will fix the diagonal, and $\mathcal{L}_1$ is still regular algebra numerical range preserving.

Now that we know how the regular numerical range preserving map will behave on the diagonal, we wish to know how it will behave on the off-diagonal.

Lemma 3.14. *Let $\mathcal{L} : \mathcal{L}_r(\ell_p(n)) \to \mathcal{L}_r(\ell_p(n))$ be a regular numerical range preserving map such that $\mathcal{L}$ fixes the diagonal. Let $\mathcal{O} := \{I\}^d$, i.e. the set of matrices with 0 diagonal. Then $\mathcal{L}(\mathcal{O}) \subseteq \mathcal{O}$.*

Proof. Let $T : \ell_p(n) \to \ell_p(n)$. Assume $\mathcal{L}(T) = S$ for some $S : \ell_p(n) \to \ell_p(n)$. Let P be the projection onto the center so that $T = P(T) + T_1$ and $S = P(S) + S_1$. We have

$$\begin{aligned}\mathcal{L}(T) &= S \\ &= P(S) + S_1 \\ &= P(T) + S_1\end{aligned}$$

where the third equality is because the diagonal is fixed. By subtracting $P(T)$ from both sides of the above equation we have

$$S_1 = \mathcal{L}(T) - P(T) = \mathcal{L}(T) - \mathcal{L}(P(T)) = \mathcal{L}(T - P(T)) = \mathcal{L}(T_1)$$

and thus $\mathcal{L}$ preserves the 0 diagonal. $\square$

Lemma 3.15. *Let $\mathcal{L} : \mathcal{L}_r(\ell_p(n)) \to \mathcal{L}_r(\ell_p(n))$ be a regular numerical range preserving map such that $\mathcal{L}$ fixes the diagonal. Then $\mathcal{L}^*$ also preserves the diagonal and preserves the 0 diagonal.*

Proof. By Lemma 3.14 we have that $\mathcal{L}$ will preserve the 0-diagonal as well. Let P be the projection of $\mathcal{L}_r(E)$ onto $\mathcal{Z}(E)$. Since $\mathcal{L}$ preserves the center

$$\mathcal{L}P = P\mathcal{L}P.$$

By taking adjoints of both sides we get

$$P^*\mathcal{L}^* = P^*\mathcal{L}^*P^*.$$

Now consider $\Phi \in \mathcal{O}^*$, i.e. an off-diagonal matrix in the dual.

$$P^*\mathcal{L}^*P^*\Phi = P^*\mathcal{L}^*(0) = P^*(0) = 0.$$

Therefore, $P^*\mathcal{L}^*\Phi = 0$ which implies that $\mathcal{L}^*(\Phi) \in \mathcal{O}^*$ and $\mathcal{L}^*$ preserves the 0-diagonal.

Now consider $(I - P)$ which will be the projection of $\mathcal{L}_r(E)$ onto the off-diagonal. By similar arguments as before we have the following,

$$\mathcal{L}(I - P) = (I - P)\mathcal{L}(I - P)$$

$$(I - P)^*\mathcal{L}^* = (I - P)^*\mathcal{L}^*(I - P)^*.$$

Now let $\Phi \in \mathcal{Z}(E^*)$. We have

$$(I - P)^*\mathcal{L}^*(I - P)^*\Phi = 0.$$

Thus $(I - P)^*\mathcal{L}^*\Phi = 0$ which implies $\mathcal{L}^*\Phi \in \mathcal{Z}(E^*)$. □

Lemma 3.16. *Let $\mathcal{L}$ be a linear regular numerical range preserving map on $\mathcal{L}_r(\ell_p(n))$ such that $\mathcal{L}$ fixes the diagonal. Let Φ be a state on $\mathcal{L}_r(\ell_p(n))$ such that $\Phi = [c_{ij}]$. Then $\mathcal{L}^*(\Phi) = [u_{ij}c_{\sigma(ij)}]$ for $i \neq j$ where $|u_{ij}| = 1$ and σ is a permutation on the off-diagonal elements.*

Proof. By Lemma 3.15, $\mathcal{L}^*$ also fixes the diagonal and preserves the 0 diagonal. Fix a diagonal $L := (\lambda_1, ..., \lambda_n)$ such that $\lambda_i \geq 0$ and $\lambda_1 + \cdots + \lambda_n = 1$. Let

$\Lambda := \{\Phi \in \mathcal{S}_r(\mathcal{L}_r(\ell_p(n))) : \mathrm{diag}(\Phi) = L\}$. Clearly $\mathcal{L}^*(\Lambda) = \Lambda$. We can define a map $\mathcal{S}_L : \ell_\infty(n^2 - n) \to \ell_\infty(n^2 - n)$ with $\mathcal{S}_L((c_{ij})) = (d_{ij})$ for each $\Phi \in \Lambda$. Since $\mathcal{L}^*$ is bijective, so is $\mathcal{S}_L$ and thus $\mathcal{S}_L(\mathrm{Ext}(\ell_\infty(n^2 - n))) = \mathrm{Ext}(\ell_\infty(n^2 - n))$. By the linearity of $\mathcal{L}^*$ and thus $\mathcal{S}_L$ we have $\mathcal{S}_L(\mathrm{co}(\mathrm{Ext}(\ell_\infty(n^2-n)))) = \mathrm{co}(\mathrm{Ext}(\ell_\infty(n^2-n)))$. In $\ell_\infty(n^2 - n)$ we have that $\mathrm{co}(\mathrm{Ext}(\ell_\infty(n^2 - n))) = B(\ell_\infty(n^2 - n))$. Thus $\mathcal{S}_L$ is a bijective map that preserves the unit ball and thus $\mathcal{S}_L$ is an isometry. Hence $\mathcal{S}_L$ must have a representation as

$$\mathcal{S}_L((c_{ij})) = (u_{ij}c_{\sigma(ij)}) \tag{3.1}$$

for all $i \neq j$ and where $|u_{ij}| = 1$ and σ permutes the off-diagonal entries.

Now consider a $\Psi \notin \Lambda$. There exists a $\Phi \in \Lambda$ and a Ψ_1 with $\mathrm{diag}(\Psi_1) = \mathrm{diag}(\Psi)$ and off-$\mathrm{diag}(\Psi_1)$ = off-$\mathrm{diag}(\Phi)$. This is true because if you have a regular state you can decrease the modulus of the off-diagonal elements as much as you want and remain a regular state. By linearity the map must behave the same on Ψ_1 as it does on Φ. This shows that the representation for $\mathcal{S}_L$ is independent of your choice for L. Since Ψ was chosen arbitrarily, the representation for $\mathcal{S}_L$ must be true for all regular states. □

The following four lemmas give conditions for the permuation of the off-diagonal entries given in (3.1).

Lemma 3.17. *Let $p = 1$. Let $\mathcal{S}((c_{ij})) = (u_{ij}c_{\sigma(ij)})$ as in Lemma 3.16. Then $\sigma = \sigma_1 \cdots \sigma_n$ where $\sigma_j : \{(i,j) : i \in \{1, ..., n\}, i \neq j\} \to \{(i,j) : i \in \{1, ..., n\}, i \neq j\}$ is a permutation of the j^{th} column vector.*

Proof. First we show that such a representation will preserve the regular states. To that end let $\lambda_1 + \cdots + \lambda_n = 1$ with $\lambda_i \geq 0$ for every $1 \leq i \leq n$. A matrix Φ with $\mathrm{diag}(\Phi) = (\lambda_1, ..., \lambda_n)$ is a regular state for $\ell_1(n)$ if and only if $|\phi_{ij}| \leq \lambda_j$ for every $i \neq j$. Clearly, permuting the column entries will preserve this condition.

Now assume there is (i,j) such that $\sigma((i,j)) = (k,l)$ for $j \neq l$. There exists a state $\Psi = [\psi_{ij}]$ such that $\mathrm{diag}(\Psi) = (\kappa_1, ..., \kappa_n)$ and $\kappa_j > \kappa_l$. Furthermore, let $\psi_{ij} = \kappa_j$ for all $1 \leq i,j \leq n$. Clearly, $\mathcal{L}^*(\Psi)$ is no longer a state, which yields a contradiction. Hence we must have that σ only permutes within each column vector. $\square$

Lemma 3.18. *Let $p = \infty$. Let $\mathcal{S}((c_{ij})) = (u_{ij}c_{\sigma(ij)})$ as in Lemma 3.16. Then $\sigma = \sigma_1 \cdots \sigma_n$ where $\sigma_i : \{(i,j) : j \in \{1,...,n\}, i \neq j\} \to \{(i,j) : j \in \{1,...,n\}, i \neq j\}$ is a permutation of the i^{th} row vector.*

Proof. The proof is the same as lemma 3.17 with columns replaced by rows. $\square$

Lemma 3.19. *Let $p = 2$. Let $\mathcal{S}((c_{ij})) = (u_{ij}c_{\sigma(ij)})$ as in Lemma 3.16. There exists a $A \subset \{(i,j) : i < j\}$ such that $\sigma((i,j)) = (j,i)$ for each $(i,j) \in A$ and $\sigma((i,j)) = (i,j)$ otherwise. In other words σ is a partial transpose.*

Proof. First we show that such a representation will preserve the regular states. To that end let $\lambda_1 + \cdots + \lambda_n = 1$ with $\lambda_i \geq 0$ for every $1 \leq i \leq n$. Consider an extremal state $\Phi = [\phi_{ij}]$ where $|\phi_{ij}| = \lambda_i^{1/2}\lambda_j^{1/2}$ as in proposition 3.6. Since $\lambda_i^{1/2}\lambda_j^{1/2} = \lambda_j^{1/2}\lambda_i^{1/2}$ partial transposes will preserve the extremal regular states and thus all regular states. Now assume there is (i,j) such that $\sigma((i,j)) = (j,l)$ with $i \neq l$. There exists a state $\Psi = [\psi_{ij}]$ such that $\operatorname{diag}(\Psi) = (\kappa_1, ..., \kappa_n)$ and $\kappa_i > \kappa_l$. Furthermore, let $\psi_{ij} = \kappa_i^{1/2}\kappa_j^{1/2}$ for all $1 \leq i,j \leq n$. Clearly $\kappa_i^{1/2}\kappa_j^{1/2} > \kappa_j^{1/2}\kappa_l^{1/2}$ and thus $\mathcal{L}(\Psi)$ is no longer a regular state, which yields a contradiction.

The same argument as above will show that there cannot be an (i,j) such that $\sigma((i,j)) = (k,i)$ where $k \neq j$. Together these arguments show that σ must be a partial transpose. $\square$

Lemma 3.20. *Let $p \neq \{1,2,\infty\}$. Let $\mathcal{S}((c_{ij})) = (u_{ij}c_{\sigma(ij)})$ as in Lemma 3.16. Then $c_{\sigma(ij)} = c_{ij}$ i.e. the map fixes the modulus of each element.*

Proof. It is obvious in this case that such a representation will preserve the regular states. Now assume there exists an (i,j) such that $\sigma((i,j)) = (k,j)$ where $i \neq k$. There exists a state $\Psi = [\psi_{ij}]$ such that $\operatorname{diag}(\Psi) = (\kappa_1, ..., \kappa_n)$ and $\kappa_i > \kappa_k$.

Furthermore, let $\psi_{ij} = \kappa_i^{1/p'}\kappa_j^{1/p}$ for all $1 \leq i, j \leq n$. Clearly $\kappa_i^{1/p'}\kappa_j^{1/p} > \kappa_k^{1/p'}\kappa_j^{1/p}$ and thus $\mathcal{L}(\Psi)$ is no longer a regular state, which yields a contradiction.
The same argument as above will show that there cannot be an (i, j) such that $\sigma((i, j)) = (i, l)$ where $j \neq l$. Together these arguments show that σ must be the identity. $\square$

We now know how the adjoint of a regular numerical range preserving map acts on the diagonal and off-diagonal of the regular states for all $1 \leq p \leq \infty$. This allows us to prove the following lemma describing the structure of this adjoint map.

Lemma 3.21. *Let $\mathcal{L}$ be a regular numerical range preserving map on $\mathcal{L}_r(\ell_p(n))$. Then*

$$\mathcal{L}^*(\Phi) = U \star (\mathcal{Q}(P^*\Phi P)),$$

where P is a permutation matrix, $U = [u_{ij}]$ is a matrix such that $|u_{ij}| = 1$ and $u_{ii} = 1$ for all $i = \{1, ..., n\}$, $(\star)$ represents Hadamard multiplication, and $\mathcal{Q}$ is a map that permutes the off-diagonal entries of T. Moreover we have the following necessary and sufficient conditions on $\mathcal{Q}$ given the value of p.

(a) If $p = 1$ then $\mathcal{Q}$ permutes the column vectors.

(b) If $p = \infty$ then $\mathcal{Q}$ permutes the row vectors.

(c) If $p = 2$ then $\mathcal{Q}$ is a partial transpose, i.e. there exists $A \subset \{(i, j) : 1 \leq i < j \leq n\}$ such that $\mathcal{Q}(t_{i,j}) = t_{j,i}$ for all $(i, j) \in A$ and $(j, i) \in A$ and $\mathcal{Q}(t_{i,j}) = t_{i,j}$ otherwise.

(d) If $p \neq \{1, 2, \infty\}$, then $\mathcal{Q}$ is the identity.

Proof. By Lemma 3.13, $\mathcal{L}$ is a permutation on the center and we can undo this permutation so that the map fixes the diagonal. By Lemma 3.15, $\mathcal{L}^*$ with this permutation also preserves the center and off-diagonal. By Lemma 3.16 each off-diagonal entry $t_{ij} \mapsto u_{ij}t_{\sigma(ij)}$. By noting that $u_{ii} = 1$ for all $1 \leq i \leq n$ there is a matrix U such that the Hadamard multiplication by U sends $t_{ij} \mapsto u_{ij}t_{ij}$. Finally, there is some off-diagonal operator $\mathcal{Q}$ which maps $t_{ij} \mapsto t_{\sigma(ij)}$ for $i \neq j$. By composing the previous maps in order we get the desired representation for the adjoint of a regular numerical range preserving map on $\ell_p(n)$. Lemmas 3.17, 3.18, 3.19, and 3.20 give the extra conditions on $\mathcal{Q}$ described in this lemma. □

Remark 3.22. In the above representation $\mathcal{Q}$ is applied to the state before the Hadamard multiplication by U. By adjusting the permutation and the entries in U it is possible to define a U_1 and $\mathcal{Q}_1$ such that the Hadamard multiplication by U_1 is done prior to the off-diagonal permutations of $\mathcal{Q}_1$.

Now that we know the representation for the adjoint of the regular algebra numerical range preserving map we can now prove the main result of this section, theorem 3.11.

Proof of Theorem 3.11. By Lemma 3.21 we have that $\mathcal{L}^*(\Phi) = U\star(\mathcal{Q}\,(P^*\Phi P))$. Now consider some $T \in \mathcal{L}_r(E)$. Then

$$\Phi(\mathcal{L}(T)) = \mathcal{L}^*(\Phi)(T) = U \star (\mathcal{Q}\,(P^*\Phi P))\,(T).$$

Let σ represent the permutation of P on the coordinates of a matrix and let τ represent the action of $\mathcal{Q}$ on the matrix, where $\tau(ii) = ii$ for all i. Let $U = [u_{ij}], \Phi = [\phi_{ij}]$ and $T = [t_{ij}]$. Then our representation gives us

$$\begin{aligned} U \star (\mathcal{Q}\,(P^*\Phi P))\,(T) &= [u_{ij}\phi_{\tau(\sigma(i),\sigma(j))}t_{ij}] \\ &= [u_{\tau^{-1}(ij)}\phi_{\sigma(i),\sigma(j)}t_{\tau^{-1}(ij)}] \\ &= [\phi_{ij}u_{\sigma^{-1}(\tau^{-1}(ij))}t_{\sigma^{-1}(\tau^{-1}(ij))}]. \end{aligned}$$

Thus there exists some U_1 of elements of modulus 1 with $u_{ii} = 1$ for all i, some $\mathcal{Q}_1$ that acts on the off-diagonal (note that if $\mathcal{Q}$ permutes columns vector or permutes row vectors or is a partial transpose then so will $\mathcal{Q}_1$) and some permuation P_1 such that

$$\Phi(\mathcal{L}(T)) = \Phi(U_1 \star (P_1^*\mathcal{Q}_1 T P_1)\,.$$

Thus given the representation for $\mathcal{L}^*$ in Lemma 3.21 we have the desired representation for $\mathcal{L}$ in Theorem 3.11. $\square$

We now have a categorization of all regular numerical range preserving maps on $\mathcal{L}_r(\ell_p(n))$. These maps are similar to the maps in [11], but have an additional Hadamard multiplication by a unimodular matrix. Some of the techniques used throughout this chapter are unique to the $\ell_p(n)$ case. In the next section we give a framework for continuing into more general norms as well as looking briefly at the infinite dimensional case.

Chapter 4

Extensions

4.1 General Finite Dimensional Banach Lattices

In this chapter we look at extensions of results obtained in Chapter 3.

First we consider a result proved by Li and Schneider in [10]. This result is useful in the proof of the representation of numerical range preserving maps in [11].

Theorem 4.1. *Let ν be a norm on $\mathcal{F}^n$. Then A is an extreme point of the unit ball of $||\cdot||_v^D$ if and only if $A = xy^*$ such that $x \in \mathcal{E}(B(\nu)^*)$ and $y \in \mathcal{E}(B(\nu))$.*

In notation more similar to that used in Chapter 3 the theorem says

$$\mathcal{E}(B(\mathcal{L}(E)^*)) = \{f \otimes x : f \in \mathcal{E}(B(E^*)), x \in \mathcal{E}(B(E))\}.$$

Furthermore, in [11] it is shown that by intersecting with the conditions to be a state,

$$\mathcal{E}(S(\mathcal{L}(E)^*)) = \{f \otimes x : f \in \mathcal{E}(B(E^*)), x \in \mathcal{E}(B(E)), f(x) = 1\}.$$

Our goal now is to prove a theorem similar to the one above for the regular states, which will allow us to discuss regular algebra numerical range preserving maps in the general finite dimensional case. In order to get there, we must first prove several lemmas. Note that these lemmas are in fact true in the infinite dimensional case as well but are used in a proof that finite dimensions is necessary.

Lemma 4.2. *Let E be a Dedekind complete complex Banach lattice. Let $T \in \mathcal{Z}(E)$ such that $|T| = I$. Then $|T^*| = I$.*

Proof. Via the Stone-Weierstras Theorem there exists $T_k \in \mathcal{Z}(E)$ with $T_k = \sum_1^{n_k} \alpha_{i,k} P_{i,k}$, where $\sum_i^{n_k} P_{i,k} = I$ and $|\alpha_{i,k}| = 1$ for all i such that

$$|T_k - T| \le \varepsilon_k I \text{ with } \varepsilon_k \to 0.$$

By taking adjoints and using the fact that $|T^*| \le |T|^*$ we have,

$$|T_k^* - T^*| \le |T_k - T|^* \le \varepsilon_k I^* = \varepsilon_k I.$$

It follows that

$$|I - |T^*|| \le |T_k^* - T^*| \le \varepsilon_k I.$$

This gives us the desired result that $|T^*| = I$. □

We now use the above lemma in oder to prove a result that a positive operator is related to another operator by a complex rotation.

Lemma 4.3. *Let E be a Dedekind complete complex Banach lattice and let $0 \le f \in E^*$. Then for all $z \in E$ there exists $g \in E^*$ with $|g| = f$ such that $|g(z)| = g(z) = f(|z|)$.*

Proof. There exists an operator T such that $|z| = Tz$ and $|T| = I$. Let $g = T^* f$. Then

$$|g| = |T^* f| = |T^*|(|f|) = |f|,$$

where the final equality is due to Lemma 4.2. We also have

$$g(z) = T^* f(z) = f(Tz) = f(|z|)$$

as desired. □

The following theorem gives a description of the regular norm related to rank one operators.

Theorem 4.4. *Let E be a Dedekind complete complex Banach lattice. Then*

$$||T||_r = \sup\{|\Phi(T)| : \Phi \in \mathcal{L}_r(E)^*, |\Phi| = f \otimes x, ||x|| = ||f|| = 1\}.$$

Proof. By considering rank one operators as functionals similar to 1.37 we get the following equality for the regular norm,

$$\begin{aligned} ||T||_r &= \sup\{f(|T|x) : ||f|| = ||x|| = 1, f, x \geq 0\} \\ &= \sup\{(f \otimes x)(|T|) : ||f|| = ||x|| = 1, f, x \geq 0\} \\ &= \sup\{|\Phi(T)| : \Phi \in \mathcal{L}_r(E)^* : |\Phi| = f \otimes x, ||f|| = ||x|| = 1, f, x \geq 0\}. \end{aligned}$$

□

Before we are able to relate the regular norm of an operator to rank one operators we will need the following theorem due to Milman [17].

Theorem 4.5. *Let A be a compact convex subset of a locally convex space X. Let $B \subset A$ such that $\overline{co}(B) = A$, then $\mathcal{E}(A) \subset \overline{B}$.*

Now we can prove the following corollary

Corollary 4.6. *Let E be a Dedekind complete complex Banach lattice. Then the unit ball of $\mathcal{L}_r(E)^*$ is equal to the weak* closed convex hull of functionals whose modulus is equal to a rank one operator. In particular,*

$$\mathcal{E}(B(\mathcal{L}_r(E)^*)) \subseteq \overline{\{\Phi \in \mathcal{L}_r(E)^* : |\Phi| = f \otimes x, f, x \geq 0, ||f|| = ||x|| = 1\}}. \tag{4.1}$$

Proof. The equality follows from Theorem 4.4 and the Hahn-Banach theorem. The inclusion now follows from Milman's Theorem 4.5 □

Now we restrict ourselves to finite dimensions in order to remove the closure condition from the previous corollary.

Lemma 4.7. *If E is a finite dimensional Dedekind complete complex Banach lattice then,*

$$\mathcal{E}(B(\mathcal{L}_r(E)^*)) \subseteq \{\Phi \in \mathcal{L}_r(E)^* : |\Phi| = f \otimes x, f, x \geq 0, f \in \mathcal{E}(B(E^*)), x \in \mathcal{E}(B(E))\}$$

Proof. Now that we are in finite dimensions the right hand side of equation 4.1 is closed in the norm of $\mathcal{L}_r(E)^*$. Now let $\Phi \in \mathcal{E}(B(\mathcal{L}_r(E)^*))$. It is clear from the description in equation 4.1 that $|\Phi| \in \mathcal{E}(B(\mathcal{L}_r(E)^*))$ and $|\Phi| = f \otimes x$. If f is not extreme then $f = \frac{f_1+f_2}{2}$ and $|\Phi| = f \otimes x = \frac{f_1 \otimes x + f_2 \otimes x}{2}$ which is a contradiction of $|\Phi|$ being extreme. If x is not extreme then $x = \frac{x_1+x_2}{2}$ and $|\Phi| = f \otimes x = \frac{f \otimes x_1 + f \otimes x_2}{2}$ which is again a contradiction of $|\Phi|$ being extreme. Hence both f and x must be extreme as desired. □

The previous theorem gives us one inclusion for the overall desired result similar to Theorem 4.1. Before we prove the reverse inclusion we must reference a result by Grząślewicz and Schaefer [8].

Theorem 4.8. *Let E be a normed vector lattice. Let $0 < x_0 \in \mathcal{E}(B(E))$. Then the following conditions are equivalent:*

(a) $x_0 \in \mathcal{E}(B(E))$

(b) x_0 *is maximal in* $B_+(E)$

(c) *the norm of E is strictly monotone at x_0.*

In other words if $x \in \mathcal{E}(B(E))$ and $x \leq y \in B_+(E)$ with $||x|| \leq ||y||$ then $x = y$.

We now want to prove that we have equality in Lemma 4.7

Theorem 4.9. *Let E be a finite dimensional Dedekind complete complex Banach lattice. Then,*

$$\mathcal{E}(B(\mathcal{L}_r(E)^*)) = \{\Phi \in \mathcal{L}_r(E)^* : |\Phi| = f \otimes x, f, x \geq 0, f \in \mathcal{E}(B(E^*)), x \in \mathcal{E}(B(E))\}$$

Proof. We have one inclusion from Lemma 4.7. To prove the other inclusion let $\Phi \in \mathcal{L}_r(E)^*$ such that $|\Phi| = f \otimes x$ where $f, x \geq 0, f \in \mathcal{E}(B(E^*)), x \in \mathcal{E}(B(E))$. Assume that $\Phi \notin \mathcal{E}(B(\mathcal{L}_r(E)^*))$. Then $\Phi = \lambda\Phi_1 + (1-\lambda)\Phi_2$ with $\Phi_1, \Phi_2 \in B(\mathcal{L}_r(E)^*)$. This implies that $|\Phi| \leq \lambda|\Phi_1| + (1-\lambda)|\Phi_2|$. There exists $\Psi_1, \Psi_2 \in B(\mathcal{L}_r(E)^*)$ such that $|\Phi| = \lambda\Psi_1 + (1-\lambda)\Psi_2$ and $0 \leq \Psi_i \leq |\Phi_i|$ for $i = 1, 2$. However since $||\Phi|| = 1$ and $||\Psi_i|| \leq 1$ we must have that $||\Psi_i|| = 1$ for $i = 1, 2$. In particular we now have that $|\Phi| \notin \mathcal{E}(B(\mathcal{L}_r(E)^*))$.

By assumption $|\Phi| = f \otimes x$. Since $|\Phi|$ is not extreme we have

$$f \otimes x = \sum_1^N \lambda_i \Phi_i$$

for some $N \in \mathbb{N}$ and $\Phi_i \in \mathcal{E}(B(\mathcal{L}_r(E)^*))$. By 4.7 $|\Phi_i| = f_i \otimes x_i$. Hence, $f \otimes x$ is dominated by a convex combination of other f_i and x_i such that

$$f \otimes x \leq \lambda_1 f_1 \otimes x_1 + \cdots \lambda_n f_n \otimes x_n, \quad \lambda_1, ..., \lambda_n > 0, \sum_{i=1}^n \lambda_i = 1.$$

Let $0 \leq u \in B(E)$ such that $f(u) = 1$ and let $f_i(u) = u_i$ for $i = 1, ..., n$. Note that $0 \leq u_j \leq 1$. We now have

$$x = (f \otimes x)(u) \leq \sum_{i=1}^n \lambda_i (f_i \otimes x_i)(u) = \sum_{i=1}^n \lambda_i u_i x_i.$$

Since x is assumed to be an extreme point we have

$$1 \leq ||\sum_{i=1}^n \lambda_i u_i x_i|| \leq ||\sum_{i=1}^n \lambda_i x_i|| = 1.$$

By the theorem of Grząślewicz and Schaefer 4.8, $x = \sum_{i=1}^n \lambda_i u_i x_i$ which implies that $u_i = 1$ and $x = x_i$ for all i.

A similar argument shows that $f = f_i$ for all i. This is a contradiction that $f \otimes x$ was a linear combination of other operators, so we must have the desired inclusion. $\square$

We would now like to go one step further and prove a similar statement regarding the regular states.

Theorem 4.10. *Let E be a finite dimensional Dedekind complete complex Banach lattice. Then,*

$$\mathcal{E}(S_r(\mathcal{L}_r(E))) = \mathcal{E}(B(\mathcal{L}_r(E)^*)) \cap \{\Phi : \Phi(I) = 1\}$$

$$= \{\Phi \in \mathcal{L}_r(E)^* : \Phi(I) = 1, |\Phi| = f \otimes x, f \in \mathcal{E}(B(E^*))_+, x \in \mathcal{E}(B(E))_+, f(x) = 1\}.$$

Proof. First note that $S_r(\mathcal{L}_r(E) \subset B(\mathcal{L}_r(E)^*)$ and $S_r(\mathcal{L}_r(E)) \subset \{\Phi : \Phi(I) = 1\}$. Since the extreme points of the larger set are obviously extreme points of the smaller set we have $\mathcal{E}(S_r(\mathcal{L}_r(E))) \supset \mathcal{E}(B(\mathcal{L}_r(E)^*)) \cap \{\Phi : \Phi(I) = 1\}$.

Now let $\Psi \in \mathcal{E}(S_r(\mathcal{L}_r(E)))$ and assume $\Psi = \frac{1}{2}(\Psi_1 + \Psi_2)$ for $\Psi_i \in B(\mathcal{L}_r(E)^*)$. Then

$$1 = \Psi(I) = \frac{1}{2}(\Psi_1(I) + \Psi_2(I)) = \frac{1}{2}(\mathrm{Re}\Psi_1(I) + \mathrm{Re}\Psi_2(I)) \leq \frac{1}{2}(|\Psi_1(I)| + |\Psi_2(I)|) \leq 1$$

This implies that $\Psi_1(I) = \Psi_2(I) = 1$ and thus $\Psi = \Psi_1 = \Psi_2$ and thus $\Psi \in \mathcal{E}(B(\mathcal{L}_r(E)^*))$.

Now consider $\Psi \in \mathcal{E}(B(\mathcal{L}_r(E)^*)) \cap \{\Phi : \Phi(I) = 1\}$. By Theorem 4.9 we have that $|\Psi| = f \otimes x$ with $f \in \mathcal{E}(B(E^*))_+$, and $x \in \mathcal{E}(B(E))_+$. We also have that if $\Psi \in S_r(\mathcal{L}_r(E))$ then $|\Psi| \in S_r(\mathcal{L}_r(E))$. Thus $1 = |\Psi|(I) = f \otimes x(I) = f(Ix) = f(x)$ and we have the inclusion $\mathcal{E}(B(\mathcal{L}_r(E)^*)) \cap \{\Phi : \Phi(I) = 1\} \subset \{\Phi \in \mathcal{L}_r(E)^* : \Phi(I) = 1, |\Phi| = f \otimes x, f \in \mathcal{E}(B(E^*))_+, x \in \mathcal{E}(B(E))_+, f(x) = 1\}$

Finally let $\Psi \in \{\Phi \in \mathcal{L}_r(E)^* : \Phi(I) = 1, |\Phi| = f \otimes x, f \in \mathcal{E}(B(E^*))_+, x \in \mathcal{E}(B(E))_+, f(x) = 1\}$. By Theorem 4.9 we have that $\Psi \in B(\mathcal{L}_r(E)^*)$. It is already assumed that $\Psi(I) = 1$ and thus we have the inclusion $\mathcal{E}(B(\mathcal{L}_r(E)^*)) \cap \{\Phi : \Phi(I) = 1\} \supset \{\Phi \in \mathcal{L}_r(E)^* : \Phi(I) = 1, |\Phi| = f \otimes x, f \in \mathcal{E}(B(E^*))_+, x \in \mathcal{E}(B(E))_+, f(x) = 1\}$. By combining all of the shown inclusion we have the desired statement of the theorem. $\square$

We now have a way to describe the extreme regular states for any finite dimensional Dedekind complete complex Banach lattice. Our goal now is to use this discription to describe the types of linear operators that preserve the regular algebra numerical range for any finite dimensional Banach lattice. To that end, we have several necessary lemmas.

Lemma 4.11. *Let E be a complex Banach lattice and $0 \leq x \in E$. Then $\mathcal{E}(\{z \in E : |z| \leq x\}) = \{z \in E : |z| = x\}$.*

Proof. This follows immediately from the Kakutani Theorem 1.18 applied to the principal ideal generated by x and the well-known characterization of the extreme points of the unit ball of a $C(K)$ space. □

Lemma 4.12. *Let E be a complex Banach space and let $T : E \to E$ be an invertible map such that both T and T^{-1} are contractions. Then T is an isometry.*

Proof. Let $x \in E$. Then

$$||x|| = ||T^{-1}(Tx)|| \le ||T^{-1}|| \, ||Tx|| \le ||Tx|| \le ||x|| \, .$$

□

Corollary 4.13. *Let E be a complex Banach lattice and $0 \le x \in E$. Assume T is an invertible linear map on E such that $T(\{z \in E : |z| \le x\}) \subset \{z \in E : |z| \le x\}$ and $T^{-1}(\{z \in E : |z| \le x\}) \subset \{z \in E : |z| \le x\}$. Then the restriction of T to the principal ideal generated by x is a lattice isometry with respect to the AM-norm.*

Now let $\mathcal{L}^*$ be a regular state preserving map, which fixes all diagonals.

Lemma 4.14. *Let E be a complex finite dimensional Banach lattice. Let $0 \le f \in \mathcal{E}(B(E^*))$ and $0 \le x \in \mathcal{E}(B(E))$ with $f(x) = 1$. Assume $|\Phi| \le f \otimes x$ is a regular state with the same diagonal as $f \otimes x$. Then $|\mathcal{L}^*(\Phi)| \le f \otimes x$.*

Proof. Let $|\Phi| \le f \otimes x \in S_r(\mathcal{L}_r(E))$. Then by Lemma 4.11 there exists Φ_i with $|\Phi_i| = f \otimes x$ and $\lambda_i \ge 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that $\Phi = \sum_{i=1}^m \lambda_i \Phi_i$. Now $|\mathcal{L}^*(\Phi_i)| = f \otimes x$, since $\mathcal{L}^*$ must map extreme points to extreme points. This implies that $|\mathcal{L}^*(\Phi)| \le |\sum_{i=1}^m \lambda_i \mathcal{L}^*(\Phi_i)| \le f \otimes x$. □

We now begin to describe the linear maps that preserve the regular algebra numerical range.

Let $\Lambda = \{\lambda_i \ge 0 : \sum_{i=1}^n \lambda_i = 1\}$ denote a fixed diagonal. Via Lozanovski's factorization theorem (a proof can be found in [7]) we can find $0 \le f \in B(E^*)$ and

$0 \leq x \in B(E)$ with $f(x) = 1$ such that the state $f \otimes x$ has diagonal Λ. Then there exists Φ_i with $|\Phi_i| = f_i \otimes x_i, 0 \leq f_i \in \mathcal{E}(B(E^*))$ and $0 \leq x_i \in \mathcal{E}(B(E))$ with $f_i(x_i) = 1$ and $\lambda_i \geq 0$ with $\sum_{i=1}^m \lambda_i = 1$ such that

$$f \otimes x = \sum_{i=1}^m \lambda_i \Phi_i.$$

Observe that the diagonal of $\sum_{i=1}^m \lambda_i f_i \otimes x_i$ is still Λ. Denote now

$$S_\lambda := \{\Phi : \Phi \in S_r(\mathcal{L}_r(E)) \text{ with diagonal } \Lambda, |\Phi| \leq \sum_{i=1}^m \lambda_i |\Phi_i|\}.$$

Proposition 4.15. *Let $\mathcal{L}^*$ and S_Λ as above. Then $\mathcal{L}^*(S_\Lambda) \subset S_\Lambda$.*

Proof. Using the same ideas of the proof of Lemma 4.14 and the statements of Lemmas 4.11 and 4.14 this follows the same way. □

Theorem 4.16. *Let $\mathcal{L}^*$ and S_Λ as above. Then the restriction of $\mathcal{L}^*$ to the principal ideal generated by $\sum_{i=1}^m \lambda_i |\Phi_I|$ is a lattice isometry with respect to the AM-norm.*

Proof. Applying Proposition 4.15 to $(\mathcal{L}^*)^{-1}$ as well we see that both $\mathcal{L}^*$ and $(\mathcal{L}^*)^{-1}$ are contractions on the principal ideal generated by $\sum_{i=1}^m \lambda_i |\Phi_i|$. The conclusion follows now from Corollary 4.13. □

Now we are in the same position as in $\ell_p(n)$ where we can represent $\mathcal{L}^*$ as a lattice isometry on $\ell_\infty(n^2 - n)$ for each Λ and get a global representation as before, except that in this case we can't say anything about the off-diagonal permutation. In other words, the description of a linear operator that preserves the regular numerical range on a general finite dimensional complex Dedekind complete Banach lattice is the same as stated in Theorem 3.11, but without any concrete conditions on $\mathcal{Q}$.

4.2 Infinite Dimensions

The main results of this dissertation regarding regular numerical range preserving maps have all been under the assumption that we are in a finite dimensional Banach lattice. For this section we give a single result for the infinite dimensional case as well as explain possible difficulties when trying to prove a completely general infinite dimensional version of Theorem 3.11.

First we consider $\ell_p(\mathbb{N})$. By reviewing the proofs of Section 3.1 one will notice that none of the proofs fail when we let the dimension go to infinity. Thus we would have a description of the extreme regular states for $\ell_p(\mathbb{N})$. Hence, a theorem could be formed similar to 3.11 describing the regular algebra numerical range preserving maps. One would run into issue when trying to go between $\mathcal{L}$ and $\mathcal{L}^*$ as was necessary in the proofs given in Chapter 3. For this reason an exact proof is not given in this dissertation.

Now consider any infinite dimensional Banach lattice. When trying to prove results similar to those in Chapter 3 one would run into several issues that would require new techniques. First, the description that relates vector states to the extreme points of the operator ball is no longer true. There is a similar description involving compact operators, but that will not cover every possible state. Another difficulty is that not all infinite Banach lattices are reflexive. There are several times in Chapter 3 where it is necessary that $x^{**} = x$ in order to make sense of what an expression means. When dealing with the infinite dimensional case the second dual will have to be handled with more care. For these reasons an infinite dimensional version of Theorem 3.11 may look very different. By forcing some conditions on the infinite dimensional Banach lattice it is possible to apply the techniques used in Chapter 3 and Section 4.1. However, since these spaces offer little more than the spaces already described in this dissertation, they were not explored further.

In conclusion, the infinite dimensional case of regular algebra numerical range preserving maps will require the development of techniques not used in this dissertation. It is possible to conjecture that a similar result to Theorem 3.11 is true in the infinite dimensional case, but more study would have to be done.

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