Structural Health Monitoring Using Ultrasonic Guided Waves And Piezoelectric Wafer Active Sensors

Faisal Haider Mohammad

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STRUCTURAL HEALTH MONITORING USING ULTRASONIC GUIDED WAVES 
AND PIEZOELECTRIC WAFER ACTIVE SENSORS 

by 

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DEDICATION

This work is dedicated to my lovely wife Farzana Yasmeen and adorable daughter Faarisah Mubashshira, who have been a constant source of support and encouragement during the challenges of my life. This work is also dedicated to my parents, and brothers, who have always loved me unconditionally and whose good examples have taught me to work hard for the things that I aspire to achieve.
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This dissertation is organized in three major parts covering ardent and challenging research topics in structural health monitoring (SHM). Part I focuses on the theoretical and numerical analysis of acoustic emission (AE) guided waves in both Cartesian coordinates (2D) and cylindrical coordinates (3D) system using energy approach, which, to the best of our knowledge, has not been yet reported elsewhere. AE guided waves appear due to sudden energy release during incremental crack propagation in a plate. The Helmholtz decomposition approach is applied to the inhomogeneous elastodynamic Navier-Lame equation for both the displacement field and body forces. For the displacement field, we use the usual decomposition in terms of unknown scalar and vector potentials. For the body forces, we hypothesize that they can be also expressed in terms of known scalar and vector excitation potentials. It is shown that these excitation potentials can be traced to the energy released during an incremental crack growth. Thus, the inhomogeneous Navier-Lame equation has been transformed into a system of inhomogeneous wave equations in terms of known excitation potentials and unknown solution potentials. The solution is readily obtained through integral transforms and application of the residue theorem. The resulting solution is a series expansion containing the superposition of all the Lamb waves modes existing for the particular frequency-thickness combination under consideration as well as the bulk waves. A numerical study of the AE guided wave propagation is conducted for a 6 mm thick 304-steel plate. A Gaussian pulse is used to model the growth of the excitation potentials during the AE event; as a result, the actual excitation potential follows the error
function in the time domain. The effect of plate thickness, source depth, rise time, higher propagation modes and propagating distance on guided waves are investigated.

Part II focuses on, an efficient analytical predictive simulation of scattered wave field based on physics of Lamb wave propagation for detecting a crack in a plate with stiffener. In this method, the scattered wave field is expanded in terms of complex Lamb wave modes with unknown amplitudes. These unknown amplitudes are obtained from the boundary conditions using a vector projection utilizing the power expression. An analytical tool (analytical global-local software) is developed to understand the effect of a cracked stiffener on Lamb wave propagation. The simulation results are verified with the finite element modeling and experimental results.

Part III focuses an experimental and analytical study of irreversible changes in the piezoelectric wafer active sensor (PWAS) E/M impedance and admittance signature under high temperature and radiation exposure. For temperature dependent study, circular PWAS transducers were exposed to temperatures between 50°C and 250°C at 50°C intervals. The material properties of PWAS transducer were measured from experimental data taken at room temperature before and after high-temperature exposure. Change in material properties of PWAS transducers may be explained by depinning of domains or by domain wall motion without affecting the microstructure of PWAS transducer material. The degraded PWAS material properties were also determined by matching impedance and admittance spectrums from experimental results with a closed form circular PWAS transducer analytical model. For irradiation test, PWAS were exposed to gamma radiation (a) slow radiation test 100 Gy/hr rate for 20 hours and (b) accelerated irradiation test 1200Gy/hr for 192 hours. Electro-mechanical (E/M) impedance-admittance signatures and
electrical capacitance were measured to evaluate the PWAS performance before and after gamma radiation exposure. The piezoelectric material was investigated microstructurally and crystallographically by using a scanning electron microscope, energy-dispersive X-ray spectroscopy, and X-ray diffraction methods. No noticeable changes in microstructure, crystal structure, unit cell dimension, or symmetry could be observed. This proposed dissertation ends with summary, conclusions, and suggestions for future work.
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CHAPTER 1
INTRODUCTION

1.1 BACKGROUND AND MOTIVATION

Structural health monitoring (SHM) or non-destructive evaluation (NDE) are the terms increasingly used in the last decade to diagnose the “state” of the constituent materials, of the different parts, and of the full assembly of these parts constituting the structure as a whole. The main purposes are to assist and inform operators about continued ‘fitness for purpose’ of structures under gradual or sudden changes to their state, to learn about either or both of the load and response mechanisms. Knowing the integrity of in-service structures on a continuous real-time basis is a paramount objective for manufacturers, end-users and maintenance teams. In effect, SHM allows: (a) optimal use of the structure; (b) a minimized downtime; (c) the avoidance of catastrophic failures gives the constructor an improvement in his products (d) drastically changes the work organization of maintenance services. It helps to prolong the service life of a structure by aiming to replace scheduled and periodic maintenance inspection with performance-based (or condition-based) maintenance (long-term) or at least (short term) by reducing the present maintenance labor.

Recently the concept of smart materials/structures (SMS) has been adopted as a step in the general evolution of smart sensing methods. There is a continuous trend from simple to complex in human production, starting from the use of homogeneous materials, supplied by nature and accepted with their natural properties, followed by multi-materials
in particular, composite materials (Haider et al. 2017d, 2016a, 2015a, 2013) allowing us to use
Introduction to SHM to create structures with properties adapted to specific uses.

To mitigate the demand of SHM or NDE techniques, this dissertation is focused in three major parts. Part I focuses theoretical and numerical analysis of acoustic emission (AE) guided waves in both Cartesian coordinates (2D) and cylindrical coordinates (3D) system using energy approach, for the first time according to the literature survey. Part II focuses on an efficient analytical predictive simulation of scattered wavefield based on physics of Lamb wave propagation for detecting a crack in a plate with stiffener for aerospace applications. Part III focuses on the experimental and analytical study of irreversible change in the piezoelectric wafer active sensor (PWAS) E/M impedance and admittance signature under high temperature and radiation exposure.

Acoustic emission (AE) has been used widely in structural health monitoring (SHM) and nondestructive testing (NDT) for the detection of crack propagation and to prevent ultimate failure (Harris et al., 1974; Han et al., 2014; Tandon et al., 1999; Roberts et al., 2003 and Bassim et al. 1994). Acoustic emission (AE) signals are therefore needed to study extensively in order to characterize the crack in an aging structure. This dissertation presents a theoretical formulation of AE guided wave due to AE events such as crack extension in a structure. Lamb (1917) derived Rayleigh-Lamb wave equations from Navier-Lame elastodynamic equations in an elastic plate. The presence of body force makes the homogeneous Navier-Lame elastodynamic equations into inhomogeneous equations. Helmholtz decomposition principle (Helmholtz, 1858) was used to decompose displacement to unknown scalar and vector potentials, and force vectors to known excitation scaler and vector potentials. There are two types of potentials acting in a plate
for Lamb waves: pressure potential and shear potential. Excitation potentials for body forces can be generalized as a localized AE event. Excitation potentials can be traced to the energy released from the tip of the crack during crack propagation. These potentials satisfy the inhomogeneous wave equations. In this research, inhomogeneous wave equations for unknown potentials were solved due to known generalized excitation potentials in a form, suitable for numerical calculation. The theoretical formulation shows that elastic waves generated in a plate using excitation potentials followed the Rayleigh-Lamb equations.

Another motivation of this research is to develop a physical understanding of the effects of damage on Lamb wave propagation. We aim to use this understanding to create an efficient predictive methodology for damage detection and characterization. Therefore, in one part of this research, we focus on developing analytical tools to tackle the complex phenomena like Lamb waves interaction with damage using a physical understanding of Lamb wave propagation.

Piezoelectric wafer active sensor (PWAS) has been used extensively for detecting damage and flaws in structures (Giurgiutiu, 2014; Giurgiutiu et al. 2002; Zagrai et al., 2010). The electro-mechanical (E/M) impedance and admittance method have been used to characterize the state of a host structure using permanently bonded PWAS transducer. The basic principle of this method is to monitor variation in impedance/admittance signature measured by the PWAS transducers (Park et al., 2006; Bhalla et al., 2004). Through E/M coupling between the electrical response of PWAS transducer and the mechanical properties of the host structure, the state of a host structure is reflected in the E/M impedance and admittance spectra. Resonance/anti-resonance frequency and
amplitude are the essential features of an E/M impedance/admittance spectrum. Damage can be detected by observing the change in peak amplitude and its frequency. However, a problem may arise while using E/M impedance/admittance method on host structures exposed to high temperature or radiation. After high temperature/radiation exposure, the E/M impedance/admittance method may lead to flawed damage detection due to a change in material properties of PWAS transducer itself.

The significance of SHM has been emphasized in so many fields such as dry cask storage canister (nuclear-spent fuel storage), pressure vessel and pipe (PVP), turbine blade and so on; where attention is being drawn to the successful implementation of SHM techniques due to temperature variation. In such cases, SHM can be done at room temperature or relatively lower temperature when the structure is not in operation or out of service. However, during service, permanently bonded PWAS transducer may be exposed to high temperature. The fundamental properties of PWAS transducer material are defined by the piezoelectric, dielectric, elastic coefficients etc. They are temperature dependent and become irreversible when the applied temperature exceeds the characteristic limits of the PWAS transducer material. Changes in resonance or anti-resonance amplitude or frequency may result from a change in above-mentioned PWAS transducer material properties. This irreversible behavior may be explained as thermal hysteresis; possible reasons for this irreversibility may be due to the irreversible domain switching, depinning of domains, or domain wall motion in the PWAS transducer material. Piezoelectric materials PZT consist of domains of aligned electric dipoles, separated by domain walls. A domain must possess at least two stable states, and must have the ability to be reversibly switched from one state to another by the application of an electric field. When an electric
field is applied, the various domains can reorient, leading to switching in the net polarization of the bulk material. High-temperature exposure of the piezoelectric materials may lead to change the domain state and causes irreversible domain switching upon applying an electric field. Moreover, some domains may not move to the polarization direction, and the domain wall sits in a local energy minimum at room temperature. This phenomenon is known as pinning of the domain wall. External energy is required for depinning the domain wall from its pinned position; due to heating of the piezoelectric materials may provide such energy and helps the domain to orient to its favorable direction. This research may help develop a temperature compensation technique for impedance and admittance where used in SHM applications. In order to use the PWAS transducer as an SHM transducer, temperature dependence PWAS material properties should be investigated, and the PWAS transducer should be defect free. Proper transducer characterization allows an SHM system to infer the integrity of the transducers and separate transducer flawed signal from structural defects in an extreme environment.

There is considerable demand for structural health monitoring (SHM) at locations where there are substantial radiation fields such as nuclear reactor components, dry cask storage canister, irradiated fuel assemblies, etc. Piezoelectric wafer active sensors (PWAS) have emerged as one of the major SHM sensing technologies. In order to use PWAS to perform SHM in a nuclear environment, radiation influence on the sensor and sensing capability needs to be investigated to assure the reliability of the PWAS based method. Piezoelectric wafer active sensors (PWAS) have emerged as one of the major SHM sensing technologies. In order to use PWAS to perform SHM in a nuclear environment, radiation influence on the sensor and sensing capability needs to be investigated to assure the
reliability of the PWAS based method. Radiation may cause degradation or even complete failure of sensors. Gamma radiation is one of the significant radiation sources near the nuclear source. Therefore, an experimental investigation is needed on the gamma radiation endurance of piezoelectric sensors.

**1.2 SCOPE OF THIS DISSERTATION**

The scope of this research is to develop modeling based guided wave propagation for damage detection and using PWAS as one of the important SHM sensing technologies. In modeling based guided wave propagation, we aim to analyze an AE event by theoretically and numerically. Also, we aim to develop a better analytical tool to understand the effects of damage on Lamb wave propagation.

![Diagram of acoustic emission propagation and detection](image)

Figure 1.1  Acoustic emission propagation and detection by a sensor installed on a structure

In general, damage detection can be divided into two main groups; active and passive detection. As shown in Figure 1.1 in passive detection, sensors are used to sense an acoustic event such as crack propagation. There are numerous publications to characterize the source from a crack growth or damage. Various deformation sources in solids, such as single forces, step force, point force, dipoles and moments can be represented as AE sources (Ohtsu et al., 1986; Michaels et al., 1981; Hsu et al., 1978; Pao,
A moment tensor analysis to acoustic emission (AE) has been studied to elucidate crack types and orientations of AE sources (Ohtsu, 1995). AE source characteristics are unknown, and the detected AE signals depend on the types of AE source, propagation media, and the sensor response. Therefore extracting AE source feature from a recorded AE waveform is always challenging. In this paper, we proposed a new technique to predict the AE waveform using excitation potentials. Excitation potentials can be traced to the energy released during incremental crack propagation. The main contribution of this paper is to simulate AE elastic waves due to energy released during crack propagation by Helmholtz potential approach, for the first time to the authors’ best knowledge. The time profile of available energy as AE source from a crack can be obtained analytically or from a 3D computational model (Cuadra et al., 2016).

Our analytical model is developed based on the integral transform using Helmholtz potentials (Haider et al. 2018a, 2018b, 2018c, 2017b). The solution is obtained through direct and inverse Fourier transforms and application of the residue theorem. The resulting solution is a series expansion containing the superposition of all the Lamb waves modes and bulk waves existing for the particular frequency-thickness combination under consideration. It is worth mentioning that other methods can be used to solve wave equation of a structure, for example, Green’s functions for forced loading, semi-analytical method; normal mode expansion method, etc. Green’s functions for forced loading can be solved through an integral formulation relying on the elastodynamic reciprocity principle or integral transform. However, the Green function for forced loading problem requires the knowledge of the point load source that generated the AE event. Extracting information about the point load source that generated an actual AE event recorded in practice is quite
challenging. Our approach bypassed this difficulty because it only needs an estimation of the energy released from the tip of the crack during a crack propagation increment. Our proposed method novelty lies on calculating AE waveform using the energy released from the tip of the crack. The semi-analytical method requires extensive computational effort and has convergence issue at high frequency. Integral transform provides an exact solution, and easy to solve numerically.

When a crack grows into a solid, a region of material adjacent to the free surfaces is unloaded, and it releases strain energy. The strain energy is the energy that must be supplied to a crack tip for it to grow and it must be balanced by the amount of energy dissipated due to the formation of new surfaces and other dissipative processes such as plasticity. Fracture mechanics allows calculating strain energy released during propagation of a crack. Figure 1.2a shows a plate with an existing crack length of 2a and Figure 1.2b shows the typical energy release rate with crack half-length (Fischer, 2000). The amount of energy released from a crack, is to be calculated as follow (Fischer, 2000)

\[
U_s = \frac{\sigma_f^2 \pi (a^2)t}{E}
\]  

(1-1)

Therefore, the amount of energy release from crack depends on: applied stress level (\(\sigma_f\)), initial crack half-length (a), incremental crack length (da) and modulus of elasticity (E). For a small increment of crack length \(da\) (Figure 1.2a) the incremental strain energy is to be calculated by following formula

\[
dU_s = \frac{2\sigma_f^2 \pi (a)t}{E} da
\]  

(1-2)
This incremental strain energy can be released at different time rate at different time depending on crack growth and types of material. A real AE source releases energy during a finite period. If time rate of energy released is known (Figure 1.2c), and the integration of that with respect to time gives the time profile of energy (Figure 1.2d) released from a crack. Total released energy from a crack can be decomposed to shear excitation potential and pressure excitation potential. The proceeding section presents a theoretical formulation for Lamb wave solution using a time profile of excitation potentials: a Helmholtz potential technique.
In active damage detection, energy is imparted in the structure using transducers to create elastic waves. These incident waves then travel in the structure and they are scattered when they encounter a damage or sudden change in the geometry or material properties (Figure 1.3). The scatter field is sensed using various types of strain or velocity sensors. The scatter fields are then compared with the incident waves to calculate scatter coefficients. In a typical SHM or NDE system, these scatter coefficients are analyzed to detect and characterize the damage. However, to identify damage using scatter coefficients we need to understand the effects of different types of damage on these scatter coefficients. In plate structures, typically, there are three main types of damage; surface breaking cracks, corrosions, and horizontal cracks or disbond. To successfully identify these types of damage we need to understand their unique characteristics (if any) represented by the scattering coefficients. Also, each of these types of damage has specific features such as, their depth, inclination of surface angle, width, surface roughness, etc. Ideally, the goal of an SHM or NDE system is to identify these characteristics too.

![Figure 1.3 Interaction of Lamb waves due to discontinuity of a structure](image)

In order to use the PWAS transducer as an SHM transducer, temperature/radiation dependence PWAS material properties should be investigated, and the PWAS transducer should be defect free. In a previous publication (Giurgiuțiu, 2016), the effect of temperature and radiation exposure on PWAS transducers was studied, and it was found that variation

Scattered waves = Scatter Coefficients × Incident wave
of the resonance and anti-resonance frequencies and amplitudes may appear due to high-temperature exposure. However, characterization PWAS transducer material properties before and after temperature/radiation exposure were not investigated at that time (Giurgiutiu, 2016). Characterization of such material properties may be relevant to explaining the changes observed in the resonance and anti-resonance amplitudes and frequencies. The major unanswered question that remained to be answered was as to whether there is any microstructural or crystal structure change that may produce these frequencies and amplitudes change. Furthermore, if these material changes are microscale damage in the material structure, then this might prevent further use of PWAS transducers in SHM applications. These transducers are susceptible to damage themselves due to high temperature/radiation exposure. Transducer characterization is essential in such cases for identifying faulty transducers. Moreover, any change in the material properties of PWAS transducer itself due to high temperature/radiation that is responsible for an irreversible and non-linear response might create a further problem for SHM applications. After subjected to temperature/radiation exposure the properties of PWAS transducer material may not return to the original state due to domain hysteresis. Further heating/radiation may create additional change in material properties. Indeed, PWAS transducer material is known to show irreversible behavior due to irreversible domain switching, depinning of domain or domain wall motion (Haider et al. 2016, 2017, 2018). Therefore, change in PWAS transducer material properties needs to be evaluated for temperature/radiation exposure for proper damage detection in SHM applications. The aim of this part of the research is to measure irreversible response and to examine the PWAS transducer microstructure, crystal structure after exposure to temperature/radiation.
1.3 Organization of the Dissertation

This dissertation is divided into total seven chapters including the current chapter. Chapter 2 presents a Helmholtz potential approach to the analysis of straight crested wave generation during acoustic emission events. Chapter 3 presents a Helmholtz potential approach to the analysis of circular crested wave generation during acoustic emission events. Chapter 4 presents an analytical model based on the physics of the Lamb wave modes to predict the scattering of them from stiffener. This chapter also presents their validation using finite element models. Chapter 5 presents irreversibility effects in piezoelectric wafer active sensors after exposure to high temperature. Chapter 6 presents gamma radiation endurance of piezoelectric wafer active transducers. Finally, Chapter 7 presents conclusions from our research and recommended future work along with the significant contributions of this research.
PART I PASSIVE SENSING: AE GUIDED WAVE PROPAGATION MODELING

CHAPTER 2
A HELMHOLTZ POTENTIAL APPROACH TO THE ANALYSIS OF STRAIGHT CRESTED WAVE GENERATION DURING ACOUSTIC EMISSION EVENTS

2.1 State of the art

Lamb (1917) derived Rayleigh-Lamb wave equations from Navier-Lame elastodynamic equations in an elastic plate. The presence of body force makes the homogeneous Navier-Lame elastodynamic equations into inhomogeneous equations. The basic concepts and several authors (Lamb, 1917; Achenbach, 2003; Giurgiutiu, 2014; Graff, 1975; Viktorov, 1967; Landau, 1965; Love, 1944; Aki and Richards, 2002) have explained equations of elastodynamics. Elastic waves emission from a damaging process, a common phenomenon in earthquake seismology. As a result, major references can be found from earthquake seismology. Aki and Richards (2002) have presented a comprehensive study on elastic wave fields due to seismic sources in “Quantitative Seismology” book. A generalized theory of solution for the elastodynamic Green function due to point dislocation source in a homogeneous isotropic bounded medium is discussed in that book. But the solution of green function was not obtained in an elastic plate. Vvendensyaya (1956) developed a system of forces, which is equivalent to the rupture accompanied by slipping in the theory.
of dislocation of Nabarro (1951). Another notable contribution is the development of the force equivalent of dynamic elastic dislocation to the earthquake mechanism by Maruyama (1963).

Lamb (1903) presented the propagation of vibrations over the surface of semi-infinite elastic solids. The vibrations were assumed due to an arbitrary application of force at a point. Rice (1980) discussed a theory of elastic wave emission from damage (e.g., slip and micro cracking). A general representation of the displacement field of an AE event was presented in terms of damage process in the source region. The solution is obtained in an unbounded medium.

Miklowitz (1962), Weaver and Pao (1982) presented the response of transient loads of an infinite elastic plate, using double integral transforms. Based on the generalized theory of AE, the elastodynamic solution due to internal crack or fault can be analyzed by a suitable Green function solution (Ono and Ohtsu, 1984). This Green function solution was obtained for either infinite media or half space media. Later, Ohtsu et al. (1986) characterized the source of AE on the basis of that generalized theory. AE waveforms due to instantaneous formation of a dislocation were presented in that paper. In an inverse study, deconvolution analysis was done to characterize the AE source. Johnson (1974) provided a complete solution to the three-dimensional Lamb’s problem, the problem of determining the elastic disturbance resulting from a point source in a half space. Frank Roth (1990) extended the theory of dislocations to model deformations on the surface of a layered half-space to calculate deformation inside the medium. Lamé constants for each layer were chosen independently of each other. However, a theoretical representation of wave propagation in a plate due to body force is not presented.
Wave propagation in an elastic plate is well known from Lamb (1917) classical work. Achenbach (2003), Giurgiutiu (2014), Graff (1975) and Viktorov (1967) considered Lamb waves, which exists in an elastic plate with traction free boundaries. Achenbach (2003) presented Lamb wave in an isotropic elastic layer generated by a time-harmonic internal/surface point load or line load. Displacements are obtained directly as summations over symmetric and antisymmetric modes of wave propagation. Elastodynamic reciprocity is used in order to obtain the coefficients of the wave mode expansion. Bai et al. (2004) presented three-dimensional steady-state Green functions for Lamb wave in a layered isotropic plate. The elastodynamic response of a layered isotropic plate to a source point load having an arbitrary direction was studied in that paper. A semi-analytical finite element (SAFE) method was used to formulate the governing equations. Using this method, in-plane displacements were accommodated by means of an analytical double integral Fourier transform, while, the anti-plane displacement approximated by using finite elements. They have used same modal summation technique of eigenvectors as described by Liu and Achenbach (1995), Achenbach(2003).

Jacobs et al. (1991) presented an analytical methodology by incorporating a time-dependent acoustic emission signal as a source model to represent an actual crack propagation and arrest event. A review paper on the signal analysis used in acoustic emission (AE) for materials research field published by Ono (2011). This paper reviewed recent progress in methods of signal analysis used in acoustic emission such as deformation, fracture, phase transformation, coating, film, friction, wear, corrosion and stress corrosion for materials research. A novel experimental mechanics technique using scanning electron microscopy (SEM) in conjunction with AE monitoring is discussed by
Wisner et al. (2015). The objective is to investigate microstructure sensitive mechanical behavior and damage of metals, in order to validate AE-related information. Cuadra et al. (2016) proposed a 3D computational model to quantify the energy associated with AE source by energy balance and energy flux approach. The time profile of available energy as AE source was obtained for the first increment of crack. Khalifa et al. (2012) proposed a formulation for modeling the AE sources and for the propagation of guided or Rayleigh waves. They used an exact analytical solution from a fracture-mechanics based model to obtain the crack opening displacement. Green’s functions for Rayleigh wave are calculated using reciprocity considerations without the use of integral transform techniques.

There are numerous publications based on finite element analysis for detecting AE signals. For example, Hamsted et al. (1999) reported wave propagation due to buried monopole and dipole sources with finite element technique. Hill et al. (2004) compared waveforms captured by AE transducer for a step force on the plate surface by finite element modeling. Hamstad (2010) presented the frequencies and amplitudes of AE signals on a plate as a function of source rise time. In that paper, an exponential increase in peak amplitude with source rise time was reported. Sause et al. (2013) presented a finite element approach for modeling of acoustic emission sources and signal propagation in hybrid multi-layered plates. They presented various parameter studies using different layup configurations of multi-layered hybrid plates and validated the results to analytically calculated dispersion curve results.

2.2 FORMATION OF PRESSURE AND SHEAR POTENTIALS

Navier-Lame equations in vector form for Cartesian coordinates is given as
\[(\lambda + \mu)\nabla \left( \nabla \cdot \vec{u} \right) + \mu \nabla^2 \vec{u} = \rho \ddot{\vec{u}} \]  
\hspace{1cm} (2-1) 

where, \( \vec{u} = u_i \hat{i} + u_j \hat{j} + u_k \hat{k} \), with \( \hat{i}, \hat{j}, \hat{k} \) being unit vectors in the \( x, y, z \) directions respectively, \( \lambda, \mu \) = Lame constant, \( \rho \) = density.

If the body force is present, then the Navier-Lame equations are to be written as follow

\[(\lambda + \mu)\nabla \left( \nabla \cdot \vec{u} \right) + \mu \nabla^2 \vec{u} + \rho \vec{f} = \rho \ddot{\vec{u}} \]  
\hspace{1cm} (2-2) 

Helmholtz decomposition states that (Helmholtz, 1858), any vector can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal vector field, where an irrotational vector field has a scalar potential, and a solenoidal vector field has a vector potential. Potentials are useful and convenient in several wave theory derivations (Giurgiutiu, 2014; Graff, 1975).

Assume that the displacement \( \vec{u} \) can be expressed in terms of two potential functions, a scalar potential \( \Phi \) and a vector potential \( \vec{H} \),

\[\vec{u} = \nabla \Phi + \nabla \times \vec{H} = \nabla \Phi + \nabla \times \vec{H} \]  
\hspace{1cm} (2-3) 

Where

\[\vec{H} = H_x \hat{i} + H_y \hat{j} + H_z \hat{k} \]  
\hspace{1cm} (2-4) 

Equation (2-3) is known as the Helmholtz equation, and is complemented by the uniqueness condition, i.e.,

\[\nabla \cdot \vec{H} = 0 \]  
\hspace{1cm} (2-5)
Introducing additional scalar and vector potentials $A^*$ and $B^*$ for body force $\vec{f}$ (Graff, 1975; Aki and Richards, 2002)

$$\vec{f} = \text{grad } A^* + \text{curl } B^* = \vec{\nabla} A^* + \vec{\nabla} \times B^*$$

(2-6)

Here,

$$\vec{B}^* = B_x^* \vec{i} + B_y^* \vec{j} + B_z^* \vec{k}$$

(2-7)

The uniqueness condition is

$$\vec{\nabla} \cdot \vec{B}^* = 0$$

(2-8)

Upon substitution of Equations (2-3), (2-6) into Equation (2-2), Equation (2-2) becomes,

$$(\lambda + \mu) \vec{\nabla}(\nabla^2 \Phi) + \mu \left( \vec{\nabla}(\nabla^2 \Phi) + \vec{\nabla} \times \nabla^2 \vec{H} \right) + \rho (\vec{\nabla} A^* + \vec{\nabla} \times \vec{B}^*) = \rho \left( \vec{\nabla} \Phi + \vec{\nabla} \times \vec{H} \right)$$

(2-9)

Upon rearrangement of Equation (2-9)

$$\vec{\nabla}[(\lambda + 2\mu)(\nabla^2 \Phi) + \rho A^* - \rho \Phi] + \vec{\nabla} \times (\mu \nabla^2 \vec{H} + \rho \vec{B}^* - \rho \vec{H}) = 0$$

(2-10)

For Equation (2-10) to hold at any place and any time, the components in parentheses must be independently zero, i.e.,

$$(\lambda + 2\mu)(\nabla^2 \Phi) + \rho A^* - \rho \Phi = 0$$

(2-11)

$$\mu \nabla^2 \vec{H} + \rho \vec{B}^* - \rho \vec{H} = 0$$

(2-12)

Upon division by $\rho$ and rearrangement,

$$c_p^2 \nabla^2 \Phi + A^* = \Phi$$

(2-13)
\[ c_s^2 \nabla^2 \bar{H} + \bar{B}^* = \ddot{H} \]  

(2-14)

Here, \( c_p^2 = \frac{\lambda + 2\mu}{\rho} \); \( c_s^2 = \frac{\mu}{\rho} \).

Equations (2-13) and (2-14) are the wave equations for the scalar potential and the vector potential respectively. Equation (2-13) indicates that the scalar potential, \( \Phi \), propagates with the pressure wave speed, \( c_p \) due to excitation potential \( A' \), whereas Equation (2-14) indicates that the vector potential, \( \bar{H} \), propagates with the shear wave speed, \( c_s \) due to excitation potential \( \bar{B}^* \). The excitation potentials are indeed a true description of energy released from a crack. The unit of the excitation potentials are \( \mu J/kg \). Inhomogeneous wave equations for potentials are well known in electrodynamics and electromagnetic fields. Wolski (2011), Jackson (1999) and Uman et al. (1975) treated the inhomogeneous wave equation for scalar and vector potential with the presence of current and charge density. The source can be treated as a localized event and Green function can be used as the solution of inhomogeneous wave equations. In this paper, wave equations for scalar potential \( \Phi \) and vector potential \( \bar{H} \) need to be solved with the presence of known source potentials \( A' \) and \( \bar{B}^* \).

The derivation Lamb wave equation will be provided by solving wave Equations (2-13) and (2-14) for the potentials. In this derivation, we will consider the generation of straight crested lamb wave due to time-harmonic excitation potentials. The assumption of straight-crested waves makes the problem \( z \)-invariant.

Upon expansion of Equation (2-2), i.e.,
\[
\rho \frac{\partial^2 u_x}{\partial t^2} - (\lambda + \mu) \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} + \frac{\partial^2 u_z}{\partial x \partial z} \right) + \mu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) = \rho f_x
\]

\[
\rho \frac{\partial^2 u_y}{\partial t^2} - (\lambda + \mu) \left( \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial y \partial z} \right) + \mu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) = \rho f_y
\]  \hspace{1cm} (2-15)

\[
\rho \frac{\partial^2 u_z}{\partial t^2} - (\lambda + \mu) \left( \frac{\partial^2 u_x}{\partial x \partial z} + \frac{\partial^2 u_y}{\partial y \partial z} + \frac{\partial^2 u_z}{\partial z^2} \right) + \mu \left( \frac{\partial^2 u_z}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} + \frac{\partial^2 u_z}{\partial z^2} \right) = \rho f_z
\]

Under z invariant condition \( \left( \frac{\partial}{\partial z} = 0 \right) \), Equation (2-15) becomes

\[
\rho \frac{\partial^2 u_x}{\partial t^2} - (\lambda + \mu) \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial x \partial y} \right) + \mu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) = \rho f_x
\]

\[
\rho \frac{\partial^2 u_y}{\partial t^2} - (\lambda + \mu) \left( \frac{\partial^2 u_x}{\partial x \partial y} + \frac{\partial^2 u_y}{\partial y^2} \right) + \mu \left( \frac{\partial^2 u_y}{\partial x^2} + \frac{\partial^2 u_y}{\partial y^2} \right) = \rho f_y \]  \hspace{1cm} (2-16)

\[
\rho \frac{\partial^2 u_z}{\partial t^2} - \mu \left( \frac{\partial^2 u_x}{\partial x^2} + \frac{\partial^2 u_z}{\partial y^2} \right) = \rho f_z
\]

Corresponding displacement and force equations from Equations (2-3) and (2-6), under z invariant condition can be written as

\[
\begin{align*}
  u_x &= \frac{\partial \Phi}{\partial x} + \frac{\partial H_z}{\partial y} \\
  u_y &= \frac{\partial \Phi}{\partial y} - \frac{\partial H_z}{\partial x} \\
  u_z &= \frac{\partial H_y}{\partial x} - \frac{\partial H_x}{\partial y}
\end{align*}
\]  \hspace{1cm} (2-17)
\[
\begin{align*}
 f_x &= \frac{\partial A^*}{\partial x} + \frac{\partial B_z^*}{\partial y} \\
 f_y &= \frac{\partial A^*}{\partial y} - \frac{\partial B_z^*}{\partial x} \\
 f_z &= \frac{\partial B_y^*}{\partial x} - \frac{\partial B_x^*}{\partial y}
\end{align*}
\]

(2-18)

Corresponding potentials Equations (2-13) and (2-14) become

\[
\begin{align*}
 c_s^2 \nabla^2 \Phi + A^* &= \Phi \\
 c_s^2 \nabla^2 H_x + B_x^* &= \tilde{H}_x \\
 c_s^2 \nabla^2 H_y + B_y^* &= \tilde{H}_y \\
 c_s^2 \nabla^2 H_z + B_z^* &= \tilde{H}_z
\end{align*}
\]

(2-19)

(2-20)

Equation (2-16) implies that, it is possible to separate the solution into two parts:

1. Solution for \( u_x \) and \( u_y \), with excitation forces \( f_x \) and \( f_y \), which depend on the potentials \( \Phi, H_z, A^*, B_z^* \). The solution will be the combination of pressure wave \( P \) represented by potential \( \Phi \) and a shear vertical wave \( SV \) represented by potential \( H_z \) due to excitation potentials \( A^*, B_z^* \) respectively.

2. Solution for \( u_z \) due to excitation force \( f_z \) which depend on the potentials \( H_x, H_y, B_x^*, B_y^* \). The solution for \( u_z \) displacement will give shear horizontal wave \( SH \) represented by potential \( H_x, H_y \) due to excitation potentials \( B_x^*, B_y^* \) respectively.

For \( P+SV \) waves in a plate,
\[ u_x = \frac{\partial \Phi}{\partial x} + \frac{\partial H_z}{\partial y} \]
\[ u_y = \frac{\partial \Phi}{\partial y} - \frac{\partial H_x}{\partial x} \]  
\( (2-21) \)

\[ f_x = \frac{\partial A^*}{\partial x} + \frac{\partial B^*_z}{\partial y} \]
\[ f_y = \frac{\partial A^*}{\partial y} - \frac{\partial B^*_z}{\partial x} \]  
\( (2-22) \)

\[ u_x, u_y = f(\Phi, H_z) \]  
\( (2-23) \)

\[ f_x, f_y = f(A^*, B^*_z) \]  
\( (2-24) \)

Therefore, for P+SV waves, the relevant potentials are \( \Phi, H_z, A, B_z \). Equations \( (2-19) \) and \( (2-20) \) are condensed into two equations, i.e.,

\[ c_p^2 \nabla^2 \Phi + A^* = \ddot{\Phi} \]  
\( (2-25) \)

\[ c_s^2 \nabla^2 H_z + B^*_z = \ddot{H}_z \]  
\( (2-26) \)

Upon rearranging

\[ \nabla^2 \Phi + \frac{1}{c_p^2} A^* = \frac{1}{c_p^2} \ddot{\Phi} \]  
\( (2-27) \)

\[ \nabla^2 H_z + \frac{1}{c_s^2} B^*_z = \frac{1}{c_s^2} \ddot{H}_z \]  
\( (2-28) \)

The P waves and SV waves give rise to the Lamb waves, which consist of a pattern of standing waves in the thickness y-direction (Lamb wave modes) behaving like traveling waves in the x-direction in a plate of thickness \( h=2d \) (Figure 2.1). Excitation potentials are
concentrated (z invariant) at the origin. If the source is vibrating harmonically, then equations for excitation potentials are

\[ A^* = c_p^2 A \delta(x) \delta(y)e^{-i\omega t} \]  \hspace{1cm} (2-29)

\[ B^*_z = c_s^2 B_z \delta(x) \delta(y)e^{-i\omega t} \]  \hspace{1cm} (2-30)

Here \( A \) and \( B_z \) are the source amplitude.

Figure 2.1 Plate of thickness 2d in which straight crested Lamb waves (P+SV) propagate in the x-direction due to concentrated potentials at the origin.

If the potentials \( \Phi, H_z \), have the harmonic response, then

\[ \Phi(x, y, t) = \Phi(x, y)e^{-i\omega t} \]
\[ H_z(x, y, t) = H_z(x, y)e^{-i\omega t} \]  \hspace{1cm} (2-31)

Then, Equations (2-27) and (2-28) become

\[ \frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} + \frac{\omega^2}{c_p^2} \Phi = -A \delta(x) \delta(y) \]  \hspace{1cm} (2-32)

\[ \frac{\partial^2 H_z}{\partial x^2} + \frac{\partial^2 H_z}{\partial y^2} + \frac{\omega^2}{c_s^2} H_z = -B_z \delta(x) \delta(y) \]  \hspace{1cm} (2-33)

Equations (2-32) and (2-33) must be solved subject to zero-stress boundary conditions at the free top
and bottom surfaces of the plate, i.e.,

\[
\sigma_{yy} \bigg|_{y=\pm d} = 0, \quad \sigma_{xy} \bigg|_{y=\pm d} = 0
\]  

(2-34)

2.3 Solution in Terms of Displacement

Taking Fourier transform of Equations (2-32) and (2-33), in x-direction, \( \Phi(x, y) \rightarrow \Phi(\xi, y) \) and \( H(x, y) \rightarrow H(\xi, y) \)

\[
-\xi^2 \Phi(\xi, y) + \Phi''(\xi, y) + \frac{\omega^2}{c_p^2} \Phi(\xi, y) = -A\delta(y)
\]

(2-35)

\[
-\xi^2 H_\xi(\xi, y) + H_\xi''(\xi, y) + \frac{\omega^2}{c_s^2} H_\xi(\xi, y) = -B_\xi\delta(y)
\]

(2-36)

Here, \( A\delta(y) = \int_{-\infty}^{\infty} A\delta(x)[\delta(x)e^{-i\xi x} dx] \) and \( B_\xi\delta(y) = \int_{-\infty}^{\infty} B_\xi\delta(y)[\delta(x)e^{-i\xi x} dx] \)

Upon rearranging

\[
\Phi''(\xi, y) + \eta_p^2 \Phi(\xi, y) = -A\delta(y)
\]

(2-37)

\[
H_\xi''(\xi, y) + \eta_s^2 H_\xi(\xi, y) = -B_\xi\delta(y)
\]

(2-38)

Here,

\[
\begin{pmatrix}
\frac{\omega^2}{c_p^2} - \xi^2 \\
\frac{\omega^2}{c_s^2} - \xi^2
\end{pmatrix} = \eta_p^2
\]

(2-39)
Equations (2-37) and (2-38) are the second order ordinary differential equation (ODE) in \( y \) direction. The total solution of Equations (2-37) and (2-38) consists of a representation of two solutions (Jensen et al., 1975):

(a) The complementary solution for the homogeneous equation

(b) A particular solution of that satisfies source effect from right-hand side

Solution can be assumed as

\[
\overline{\Phi}(\xi, y) = \overline{\Phi}_0(\xi, y) + \overline{\Phi}_1(\xi, y) \tag{2-40}
\]

\[
\overline{H}_c(\xi, y) = i \left( \overline{H}_{c0}(\xi, y) + \overline{H}_{c1}(\xi, y) \right) \tag{2-41}
\]

The functions \( \overline{\Phi}_0 \) and \( \overline{H}_{c0} \) satisfy the corresponding homogeneous differential equations

\[
\overline{\Phi}_0^*(\xi, y) + \eta^2 \overline{\Phi}_0(\xi, y) = 0 \tag{2-42}
\]

\[
\overline{H}_{c0}^*(\xi, y) + \eta^2 \overline{H}_{c0}(\xi, y) = 0 \tag{2-43}
\]

Equations (2-42) and (2-43) give the complementary solutions, i.e.,

\[
\overline{\Phi}_0(\xi, y) = C_1 \sin \eta_p y + C_2 \cos \eta_p y \tag{2-44}
\]

\[
\overline{H}_{c0}(\xi, \eta) = D_1 \sin \eta_s y + D_2 \cos \eta_s y \tag{2-45}
\]

The functions \( \overline{\Phi}_1(\xi, y) \) and \( \overline{H}_{c1}(\xi, y) \) from Equation (2-41) satisfy the corresponding inhomogeneous equations

\[
\overline{\Phi}_1^*(\xi, y) + \eta^2 \overline{\Phi}_1(\xi, y) = -A(y)\delta(y) \tag{2-46}
\]

\[
\overline{H}_{c1}^*(\xi, y) + \eta^2 \overline{H}_{c1}(\xi, y) = -B_c(y)\delta(y) \tag{2-47}
\]
Here, $\Phi_1(\xi, y)$ and $H_{z1}(\xi, y)$ are the Fourier transform of the free field depth dependent function.

The solution of Equation (2-46) is obtained using Fourier transform in $y$ direction

$$-\eta^2\Phi_1(\xi, \eta) + \eta_p^2\Phi_1(\xi, \eta) = -A \quad (2-48)$$

Here, $A = \int_{-\infty}^{\infty} A\delta(y)e^{-\imath \eta y}dy$

$$\Phi(\xi, \eta) = \frac{A}{\eta^2 - \eta_p^2} \quad (2-49)$$

The solution of Equation (2-47) is obtained in a similar way

$$-\eta^2H_{z1}(\xi, \eta) + \eta_p^2H_{z1}(\xi, \eta) = -B_z$$

$$H_{z1}(\xi, \eta) = \frac{B_z}{\eta^2 - \eta_p^2} \quad (2-50)$$

In order to get the solution in $y$ domain, taking inverse Fourier transform of Equations (2-49) and (2-50), i.e.,

$$\Phi_1(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A \frac{e^{\imath \eta y}}{\eta^2 - \eta_p^2} d\eta \quad (2-51)$$

$$H_{z1}(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B_z \frac{e^{\imath \eta y}}{\eta^2 - \eta_p^2} d\eta \quad (2-52)$$

The evaluation of the integral in Equations (2-51) and (2-52) can be done by the residue theorem using a close contour (Remmert, 2012; Cohen, 2010; Krantz, 2007; Watanabe, 2014).
The evaluation of the integral in Equation (2-51) is to be done by the residue theorem using a close contour. Since the poles are on the real axis, the integral is to be evaluated as

\[
\int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{\eta y} d\eta = \pi i \sum_{\text{real axis}} \text{Res} \left( f(\eta); \eta_j \right) \tag{2-53}
\]

\[
\text{Res} \left( f(\eta) \right) = (\eta - \eta_j) f(\eta_j) \tag{2-54}
\]

Here \( \eta_j \) are the poles of \( \eta \) on the real axis. Poles of \( \eta \) for Equation (2-53) are

\[
\eta = \pm \eta_p \tag{2-55}
\]

To evaluate the integration of Equation (2-51) two cases are considered as follow,

**Case 1:** \( y > 0 \)

\[
\Phi_1(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{\eta y} d\eta \tag{2-56}
\]

In order to evaluate the contour integration, an arc at the infinity on the upper half plane (Figure 2.2) is added because the function converges in upper half circle.

According to the residue theorem, the integrant is

\[
\int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{\eta y} d\eta = \pi i \sum_{\text{real axis}} \text{Res} \left( f(\eta); \eta_j \right) \tag{2-57}
\]

Here \( \eta_j \) are the poles of \( f(\eta) = \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{\eta y} \)
Figure 2.2 Close contour for evaluating the inverse Fourier transform by residue theorem for $y > 0$

Using the poles, the residue theorem gives,

$$\int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{iy} d\eta = \pi i \left[ (\eta - \eta_p) \frac{A}{(\eta + \eta_p)(\eta - \eta_p)} e^{iy} \right]_{\eta = \eta_p} + \pi i \left[ (\eta + \eta_p) \frac{A}{(\eta + \eta_p)(\eta - \eta_p)} e^{iy} \right]_{\eta = -\eta_p}$$ (2-58)

Upon rearrangement

$$\int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{iy} d\eta = \frac{-A\pi}{\eta_p} \left( \frac{e^{i\eta_p y}}{2i} - \frac{e^{-i\eta_p y}}{2i} \right)$$ (2-59)

Using Euler formula in Equation (2-59), i.e.,

$$\int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{iy} d\eta = -\frac{A\pi}{\eta_p} \sin \eta_p y$$ (2-60)

**Case 2:** $y < 0$

For $y < 0$; $y = -|y|$
\[ \Phi_1(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{-iy|\eta|} d\eta \] (2-61)

This time an arc at the infinity on the upper half plane cannot be added because the function does not converge in upper half circle. To evaluate the contour integration of above function, an arc at the infinity on the lower half plane (Figure 2.3) can be added because the function converges in a lower half circle.

Figure 2.3 Close contour for evaluating the inverse Fourier transform by residue theorem for \( y < 0 \)

\[ \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{-iy|\eta|} d\eta = -\pi i \sum_{\text{real axis}} \text{Res} (f(\eta); \eta_j) \] (2-62)

Using the poles, residue theorem gives

\[
\int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{-iy|\eta|} d\eta = -\pi i \left[ \frac{(\eta - \eta_p)}{(\eta + \eta_p)(\eta - \eta_p)} \right]_{\eta = \eta_p} A e^{-iy|\eta|} \\
-\pi i \left[ \frac{(\eta + \eta_p)}{(\eta + \eta_p)(\eta - \eta_p)} \right]_{\eta = -\eta_p} A e^{-iy|\eta|} \]

(2-63)

Upon rearrangement
Using Euler formula in Equation (2-64), i.e.,

\[ \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{-i\eta|y|} d\eta = -A\pi \left( \frac{e^{i\eta_p|y|}}{2i} - \frac{e^{-i\eta_p|y|}}{2i} \right) \]  

(2-64)

By observing both cases the following summary can be made:

For \( y > 0 \)

\[ \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{i\eta y} d\eta = -A\pi \frac{\sin \eta_p}{\eta_p} |y| \]  

(2-66)

For \( y < 0 \)

\[ \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{-i\eta y} d\eta = -A\pi \frac{\sin \eta_p}{\eta_p} |y| \]  

(2-67)

Therefore, whether, \( y \) is positive or negative, the contour integral result is same.

Integration formula for Equation (2-51) becomes

\[ \overline{\Phi}_1(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{i\eta y} d\eta = -\frac{A}{2\eta_p} \sin \eta_p |y| \]  

(2-68)

Similarly, from Equation (2-52), the evaluation of the integral is as follow

\[ \overline{H}_{z\ell}(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{B_z}{\eta^2 - \eta_p^2} e^{i\eta y} d\eta = -\frac{B_z}{2\eta_p} \sin \eta_p |y| \]  

(2-69)

Integration formula for Equations (2-51) and (2-52) are
\[
\Phi_1(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{i\eta y} d\eta = -\frac{A}{2\eta_p} \sin \eta_p |y|
\]

(2-70)

\[
\bar{H}_z(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{B_z}{\eta^2 - \eta_s^2} e^{i\eta y} d\eta = -\frac{B_z}{2\eta_s} \sin \eta_s |y|
\]

(2-71)

Equations (2-70), (2-71) represent the particular solution. The total solution is a superposition of the complimentary solution Equations (2-44), (2-45) and particular solution Equations (2-70), (2-71), i.e.,

\[
\Phi(\xi, y) = C_1 \sin \eta_p y + C_2 \cos \eta_p y - \frac{A}{2\eta_p} \sin \eta_p |y|
\]

(2-72)

\[
\bar{H}_z(\xi, y) = i \left( D_1 \sin \eta_s y + D_2 \cos \eta_s y - \frac{B_z}{2\eta_s} \sin \eta_s |y| \right)
\]

(2-73)

The coefficients \( C_1, C_2, D_1, D_2 \) of the complimentary solutions Equations (2-44), (2-45) are to be determined by using the Fourier transformed boundary conditions i.e.,

\[
\bar{\sigma}_{yy} \bigg|_{y=\pm d} = 0, \quad \bar{\sigma}_{xy} \bigg|_{y=\pm d} = 0
\]

(2-74)

Stress equations in terms of displacements for \( z \)-invariant motions are

\[
\sigma_{yy} = \lambda \frac{\partial u_x}{\partial x} + (\lambda + 2\mu) \frac{\partial u_y}{\partial y}; \\
\sigma_{xy} = \mu \left( \frac{\partial u_x}{\partial y} + \frac{\partial u_y}{\partial x} \right)
\]

(2-75)

Using Equation (2-21) into Equation (2-75) gives the stress expression in terms of potentials, i.e.,
\[
\sigma_{yy} = \lambda \frac{\partial^2 \Phi}{\partial x^2} + (\lambda + 2\mu) \frac{\partial^2 \Phi}{\partial y^2} - 2\mu \frac{\partial^2 H}{\partial x \partial y};
\]
\[
\sigma_{xy} = \mu \left( 2 \frac{\partial^2 \Phi}{\partial x \partial y} - \frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} \right) \tag{2-76}
\]

Apply the x-domain Fourier transform to Equation (2-76) and obtain
\[
\bar{\sigma}_{xy} = -\lambda \bar{\epsilon}^2 \Phi + (\lambda + 2\mu) \frac{\partial^2 \bar{\Phi}}{\partial y^2} - 2i\bar{\epsilon} \mu \frac{\partial \bar{H}}{\partial y};
\]
\[
\bar{\sigma}_{xy} = \mu \left( 2i\bar{\epsilon} \frac{\partial \bar{\Phi}}{\partial y} + \bar{\epsilon}^2 \bar{H} + \frac{\partial^2 \bar{H}}{\partial y^2} \right) \tag{2-77}
\]

The differentiation properties of Equations (2-72), (2-73) are
\[
\frac{\partial \Phi(\xi, y)}{\partial y} = C_1\eta_p \cos \eta_{p} y - C_2\eta_p \sin \eta_{p} y - \eta_p \frac{A}{2\eta_p} \cos \eta_p |y|
\]
\[
\frac{\partial^2 \Phi(\xi, y)}{\partial y^2} = -C_1\eta_p^2 \sin \eta_{p} y - C_2\eta_p^2 \cos \eta_{p} y + (\eta_p^2) \frac{A}{2\eta_p} \sin \eta_p |y| = -\eta_p^2 \Phi(\xi, y) \tag{2-78}
\]
\[
\frac{\partial H}{\partial y}(\xi, y) = i \left( D_1\eta_i \cos \eta_i y - D_2\eta_i \sin \eta_i y - \eta_i \frac{B}{2\eta_i} \cos \eta_i |y| \right)
\]
\[
\frac{\partial^2 H}{\partial y^2}(\xi, y) = i \left( -D_1\eta_i^2 \sin \eta_i y - D_2\eta_i^2 \cos \eta_i y + (\eta_i^2) \frac{B}{2\eta_i} \sin \eta_i |y| \right) = -\eta_i^2 \bar{H}_z(\xi, y) \tag{2-79}
\]

Using Equations (2-72), (2-73) and differentiation property Equations (2-78), (2-79) into Equation (2-77) i.e.,
\[
\sigma_{y y} = \left( \xi^2 - \eta_s^2 \right) \left( C_1 \sin \eta_p y + C_2 \cos \eta_p y - \frac{A}{2\eta_p} \sin \eta_p y \right) \\
+ 2\eta_s \left( D_1 \eta_s \cos \eta_s y - D_2 \eta_s \sin \eta_s y - \frac{B}{2\eta_s} \cos \eta_s y \right) \\
- \left( \xi^2 - \eta_s^2 \right) \left( D_1 \sin \eta_s y + D_2 \cos \eta_s y - \frac{B}{2\eta_s} \sin \eta_s y \right)
\] 

(2-80)

Using Equation (2-80) into the boundary conditions, Equation (2-74) yields

\[
\sigma_{y y} \bigg|_{y=d} = 0
\]

\[
\left( \xi^2 - \eta_s^2 \right) \left( C_1 \sin \eta_p d + C_2 \cos \eta_p d - \frac{A}{2\eta_p} \sin \eta_p d \right) \\
+ 2\eta_s \left( D_1 \eta_s \cos \eta_s d - D_2 \eta_s \sin \eta_s d - \frac{B}{2\eta_s} \cos \eta_s d \right) = 0
\]

(2-81)

\[
\sigma_{y y} \bigg|_{y=-d} = 0
\]

\[
\left( \xi^2 - \eta_s^2 \right) \left( -C_1 \sin \eta_p d + C_2 \cos \eta_p d - \frac{A}{2\eta_p} \sin \eta_p d \right) \\
+ 2\eta_s \left( D_1 \eta_s \cos \eta_s d + D_2 \eta_s \sin \eta_s d - \frac{B}{2\eta_s} \cos \eta_s d \right) = 0
\]

(2-82)
\[ \sigma_{xy} \bigg|_{y=d} = 0 \]

\[ 2\xi \left( C_s \eta_p \cos \eta_p d - C_z \eta_p \sin \eta_p d - \eta_p \frac{A}{2\eta_p} \cos \eta_p d \right) + \left( \xi^2 - \eta_p^2 \right) \left( D_1 \sin \eta_s d + D_2 \cos \eta_s d - \frac{B_2}{2\eta_s} \sin \eta_s d \right) = 0 \] (2-83)

\[ \sigma_{xy} \bigg|_{y=-d} = 0 \]

\[ 2\xi \left( C_s \eta_p \cos \eta_p d + C_z \eta_p \sin \eta_p d - \eta_p \frac{A}{2\eta_p} \cos \eta_p d \right) + \left( \xi^2 - \eta_p^2 \right) \left( -D_1 \sin \eta_s d + D_2 \cos \eta_s d - \frac{B_2}{2\eta_s} \sin \eta_s d \right) = 0 \] (2-84)

Equations (2-81)-(2-84) are a set of four equations with four unknowns. The equations can be separated into a couple of two equations with two unknowns, one for symmetric motion and one for anti-symmetric motion.

2.4 **Symmetric Lamb wave solution**

Addition of the Equations (2-81) and (2-82), and subtraction of the Equations (2-83) and (2-84), yields

\[
\begin{bmatrix}
(\xi^2 - \eta_p^2) \cos \eta_p d & 2\xi \eta_s \cos \eta_s d \\
-2\xi \eta_p \sin \eta_p d & (\xi^2 - \eta_p^2) \sin \eta_s d
\end{bmatrix}
\begin{bmatrix}
C_2 \\
D_1
\end{bmatrix}
= \begin{bmatrix}
(\xi^2 - \eta_p^2) \frac{A}{2\eta_p} \sin \eta_p d + \xi B_2 \cos \eta_s d \\
0
\end{bmatrix}
\] (2-85)
Equation (2-85) represents an algebraic system that can be solved for $C_2, D_1$ provided the system determinant does not vanish

$$D_s = \begin{vmatrix} (\xi^2 - \eta_s^2) \cos \eta_s d & 2 \xi \eta_s \cos \eta_s d \\ -2 \xi \eta_p \sin \eta_p & (\xi^2 - \eta_s^2) \sin \eta_s d \end{vmatrix} \neq 0$$

Let,

$$P_s = \left( (\xi^2 - \eta_s^2) \frac{A}{2 \eta_p} \sin \eta_p d + \xi B_z \cos \eta_s d \right)$$ \hspace{1cm} (2-86)

$P_s$ is the source term for a symmetric solution which contains source potentials $A$ and $B_z$.

Then,

$$\begin{bmatrix} C_2 \\ D_1 \end{bmatrix} = \begin{bmatrix} (\xi^2 - \eta_s^2) \sin \eta_s d \\ 2 \xi \eta_p \sin \eta_p d \\ (\xi^2 - \eta_s^2) \cos \eta_s d \end{bmatrix} \begin{bmatrix} P_s \\ 0 \end{bmatrix}$$ \hspace{1cm} (2-87)

$$\begin{bmatrix} C_2 \\ D_1 \end{bmatrix} = \frac{1}{D_s} \begin{bmatrix} P_s (\xi^2 - \eta_s^2) \sin \eta_s d \\ P_s 2 \xi \eta_p \sin \eta_p d \end{bmatrix}$$ \hspace{1cm} (2-88)

Here,

$$D_s = \begin{vmatrix} (\xi^2 - \eta_s^2) \cos \eta_p d & 2 \xi \eta_s \cos \eta_s d \\ -2 \xi \eta_p \sin \eta_p & (\xi^2 - \eta_s^2) \sin \eta_s d \end{vmatrix} = (\xi^2 - \eta_s^2)^2 \cos \eta_p d \sin \eta_s d + 4 \xi^2 \eta_s \eta_p \cos \eta_s d \sin \eta_p d$$ \hspace{1cm} (2-89)

By equating to zero the $D_s(\xi)$ term of Equation (2-89), one gets the symmetric Rayleigh-Lamb equation.
2.5 Anti-symmetric Lamb wave solution

Subtraction of the Equations (2-81) from (2-82), and addition of the Equations (2-83) and (2-84), yields

\[
\begin{bmatrix}
(\xi^2 - \eta_s^2) \sin \eta_d d & -2\xi \eta_s \sin \eta_d d \\
2\xi \eta_p \cos \eta_p d & (\xi^2 - \eta_s^2) \cos \eta_d d
\end{bmatrix}
\begin{bmatrix}
C_1 \\
D_2
\end{bmatrix}
= 0
\]

\[
\begin{bmatrix}
(\xi^2 - \eta_s^2) \sin \eta_d d + (\xi^2 - \eta_s^2) \frac{B_z}{2\eta_s} \sin \eta_d d
\end{bmatrix}
\]

Equation (2-90) represents an algebraic system that can be solved for \(C_1, D_2\) provided the system determinant does not vanish

\[
D_A = \begin{vmatrix}
(\xi^2 - \eta_s^2) \sin \eta_d d & -2\xi \eta_s \sin \eta_d d \\
2\xi \eta_p \cos \eta_p d & (\xi^2 - \eta_s^2) \cos \eta_d d
\end{vmatrix} \neq 0
\]

Let,

\[
P_A = \xi A \cos \eta_p d + (\xi^2 - \eta_s^2) \frac{B_z}{2\eta_s} \sin \eta_d d
\]  

\[P_A\] is the source term for an anti-symmetric solution which contains source potentials \(A\) and \(B_z\).

Then,

\[
\begin{bmatrix}
C_1 \\
D_2
\end{bmatrix}
= \begin{bmatrix}
(\xi^2 - \eta_s^2) \cos \eta_d d & 2\xi \eta_s \sin \eta_d d \\
-2\xi \eta_p \cos \eta_p d & (\xi^2 - \eta_s^2) \sin \eta_d d
\end{bmatrix}
\begin{bmatrix}
P_A \\
D_A
\end{bmatrix}
\]
$$\begin{bmatrix} C_1 \\ D_2 \end{bmatrix} = \frac{1}{D_A} \begin{bmatrix} P_A 2 \xi \eta, \sin \eta_d \\ P_A (\xi^2 - \eta^2) \sin \eta_p \eta_d \end{bmatrix}$$

(2-93)

Here,

$$D_A = \begin{vmatrix} (\xi^2 - \eta^2) \sin \eta_p \eta_d & -2 \xi \eta, \sin \eta_d \\ 2 \xi \eta_p \cos \eta_p & (\xi^2 - \eta^2) \cos \eta_p \eta_d \end{vmatrix}$$

(2-94)

$$= (\xi^2 - \eta^2)^2 \sin \eta_p \eta_d \cos \eta_d + 4 \xi^2 \eta, \eta_p \sin \eta_p \eta_d \cos \eta_p \eta_d$$

By equating to zero the $D_A(\xi)$ term of Equation (2-94), one gets the anti-symmetric Rayleigh-Lamb equation.

### 2.6 Displacement Solution in the Wavenumber Domain

Equation (2-17) gives the out-of-plane displacement $u_y$, in terms of potentials, i.e.,

$$u_y = \frac{\partial \Phi}{\partial y} - \frac{\partial H_z}{\partial x}$$

(2-95)

In x direction wavenumber domain the displacement $u_y$ becomes

$$\tilde{u}_y = \frac{\partial \Phi}{\partial y} - i \xi \tilde{H}_z$$

(2-96)

By substituting of Equations (2-72), (2-73) into Equation (2-96) yields the expressions for out-of-plane displacement in the wavenumber domain in terms of coefficients $C_1, C_2, D_1, D_2$ which are functions of the wavenumber $\xi$, i.e.,

$$\tilde{u}_y = \left[ (C_1 \eta_p \cos \eta_p, y - \eta_p C_2 \sin \eta_p, y) + \xi (D_1 \sin \eta_p, y + D_2 \cos \eta_p, y) \right]
- \frac{A}{2} \cos \eta_p \eta_d - \xi \frac{B}{2 \eta} \sin \eta_i \eta_d$$

(2-97)
Evaluation of Equation (2-97) at the plate top surface, \( \bar{u}_y \mid_{y=d} : \)

\[
\bar{u}_y = \left[ \left( C_1 \eta_p \cos \eta_p d - \eta_p C_2 \sin \eta_p d \right) + \xi \left( D_1 \sin \eta_s d + D_2 \cos \eta_s d \right) \right] - \frac{A}{2} \cos \eta_p d - \frac{B}{2 \eta_s} \sin \eta_s d
\]

(2-98)

The complete solution of displacement is the superposition of the symmetric, antisymmetric and bulk wave solution. By using Equations (2-88) and (2-93) into Equation (2-98) and obtain

\[
\bar{u}_y = \left( P_S \frac{N_s}{D_S} + P_A \frac{N_A}{D_A} \right) - \frac{A}{2} \cos \eta_p d - \frac{B}{2 \eta_s} \sin \eta_s d
\]

(2-99)

Where,

\[
N_s = 2 \xi \eta_p \sin \eta_p d \sin \eta_s d - (\xi^2 - \eta_s^2) \eta_p \sin \eta_p d \sin \eta_s d
\]

\[
N_A = 2 \xi \eta_s \eta_p \sin \eta_s d \cos \eta_p d + \xi (\xi^2 - \eta_s^2) \sin \eta_p d \cos \eta_s d
\]

(2-100)

2.7 Displacement Solution in the Physical Domain

Displacement solution in the physical domain is obtained by applying the inverse Fourier transform to Equation (2-99) into x-direction, i.e.,

\[
u_y = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( P_S \frac{N_s}{D_S} + P_A \frac{N_A}{D_A} - \frac{A}{2} \cos \eta_p d - \frac{B}{2 \eta_s} \sin \eta_s d \right) e^{i\xi \xi} d\xi
\]

(2-101)

Let,

\[
u_y = I_1 - I_2 - I_3
\]

(2-102)

Where,
\[ I_1 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( P_S \frac{N_S}{D_S} + P_A \frac{N_A}{D_A} \right) e^{i\xi \sigma} d\xi \]  
\[ (2-103) \]

\[ I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{A}{2} \cos \eta \sigma d \right) e^{i\xi \sigma} d\xi \]  
\[ (2-104) \]

\[ I_3 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \frac{B}{2\eta} \sin \eta \sigma d \right) e^{i\xi \sigma} d\xi \]  
\[ (2-105) \]

The integrant in Equation (2-103) is singular at the roots of \( D_S \) and \( D_A \), i.e.,

\[ D_S = 0 \]  
\[ (2-106) \]

\[ D_A = 0 \]  
\[ (2-107) \]

Equations (2-106) and (2-107) are the Rayleigh-Lamb equations for symmetric and anti-symmetric modes.

The evaluation of integral in Equation (2-103) can be done by residue theorem using a close contour as shown in Figure 2.4. The positive roots correspond to forward propagating waves. To satisfy the radiation boundary condition at \( x = \infty \), that is no incoming waves from infinity; the negative real poles are avoided in the contour \( (x > 0) \).

The integration of Equation (2-101) can be written as the sum of the residues,

\[ I_s = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left[ P_s(\xi) \frac{N_s(\xi)}{D_s(\xi)} e^{i\xi \sigma} d\xi + P_a(\xi) \frac{N_a(\xi)}{D_a(\xi)} e^{i\xi \sigma} d\xi \right] \]

\[ = \frac{1}{2\pi} \sum_{\xi_k} \text{Res}(\xi_k) \]  
\[ (2-108) \]

For the poles \( \xi_k \), Equation (2-108) becomes
\[ I_1 = i \sum_{\xi} \text{Res}(\xi) \] (2-109)

Figure 2.4 Close contour for evaluating the inverse Fourier transform by residue theorem for (a) \( x > 0 \) (b) \( x < 0 \)

From residue theorem, if \( f(z) = \frac{N(z)}{D(z)} \) and \( z = a \) is a simple pole then

\[ \text{Res}(a) = \frac{N(a)}{D'(a)}, \] where \( D'(a) = \frac{dD}{dz} \bigg|_{z=a}. \) Applying residue theorem to obtain the integration of Equation (2-108) we get

\[ I_1 = i \left( \sum_{j=0}^{j_s} P_S(\xi^S_j) \frac{N_S(\xi^S_j)}{D'(\xi^S_j)} e^{i\xi^S_j x} + \sum_{j=0}^{j_A} P_A(\xi^A_j) \frac{N_A(\xi^A_j)}{D'(\xi^A_j)} e^{i\xi^A_j x} \right) \] (2-110)

The summation of Equation (2-110) is taken over the symmetric and anti-symmetric positive real wavenumbers \( \xi^S_j \) and \( \xi^A_j \). At given frequency \( \omega \), there are \( j = 0,1,2,\ldots, j_s \) symmetric Lamb wave modes and \( j = 0,1,2,\ldots, j_A \) anti-symmetric Lamb wave modes.

Using Euler formula in Equation (2-104) and obtain
\[ I_2 = \frac{1}{2\pi} \int_{-\infty}^{\infty} A \cos(\eta \xi) \left( \cos \xi x + i \sin \xi x \right) d\xi \]  

(2-111)

By using even and odd function property of the integrand, Equation (2-111) becomes

\[ I_2 = \frac{A}{2\pi} \int_{0}^{\infty} \cos(\eta \xi) \cos \xi x d\xi \]  

(2-112)

\[ I_2 = \frac{A}{2\pi} \int_{0}^{\infty} \cos \left( \sqrt{\left(\omega/c_p\right)^2 - \xi^2} \right) \cos \xi x d\xi \]  

(2-113)

Apply integration by parts to Equation (2-113)

\[ u = \cos \left( \sqrt{\left(\omega/c_p\right)^2 - \xi^2} \right); \quad v = \cos \xi x; \quad \int u v d\xi = u \int v d\xi - \int \left( \frac{du}{d\xi} \right) \int v d\xi \]  

(2-114)

After integrating by parts using Equation (2-114), Equation (2-114) becomes

\[ I_2 = \frac{A}{2\pi} \int_{0}^{\infty} \frac{d}{\xi} \sin \left( \sqrt{\left(\omega/c_p\right)^2 - \xi^2} \right) \sin \xi x d\xi \]  

(2-115)

The integration of Equation (2-115) is to be evaluated with the aid of ref Erdélyi, (19540, pp 35 (23) for \( \nu = 1/2 \) and using \( J_{1/2} = \sqrt{\pi/2} \sin(x) \) i.e.,

\[ I_2 = \frac{A}{4} \left( \omega/c_p \right) d \left( x^2 + d^2 \right)^{1/2} Y_1 \left( \left( \omega/c_p \right) \left[ x^2 + d^2 \right]^{1/2} \right) \]  

(2-116)

Here \( Y_1 \) is the Bessel function of second kind of order 1.

Upon rearrangement of Equation (2-105)
\[ I_3 = \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \xi \frac{B_z}{2\sqrt{\left(\omega/c_s\right)^2 - \xi^2}} \sin \sqrt{\left(\omega/c_s\right)^2 - \xi^2} d \xi \right) e^{i\xi x} d\xi \quad (2-117) \]

The integration of Equation (2-117) is to be evaluated by contour integration with branch cut. The poles are \( \xi = \pm \omega/c_s \). The closed path for contour integral is shown in Figure 2.5 (Watanabe, 2014). The corresponding arguments are listed in Table 2.1.

Table 2.1  Arguments for inverse Fourier transform

<table>
<thead>
<tr>
<th>Line</th>
<th>( \xi )</th>
<th>( \sqrt{\left(\omega/c_s\right)^2 - \xi^2} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \rightarrow DE )</td>
<td>( +i\eta )</td>
<td>( \sqrt{\left(\omega/c_s\right)^2 + \eta^2} )</td>
</tr>
<tr>
<td>( \rightarrow EF )</td>
<td>( +\eta )</td>
<td>( \sqrt{\left(\omega/c_s\right)^2 - \eta^2} )</td>
</tr>
<tr>
<td>( \rightarrow GH )</td>
<td>( +\eta )</td>
<td>( -\sqrt{\left(\omega/c_s\right)^2 - \eta^2} )</td>
</tr>
<tr>
<td>( \rightarrow HI )</td>
<td>( +i\eta )</td>
<td>( -\sqrt{\left(\omega/c_s\right)^2 + \eta^2} )</td>
</tr>
</tbody>
</table>

After applying the arguments as shown in Table 1 the Equation (2-117) becomes


Figure 2.5  Close contour for evaluating the inverse Fourier transform for \( x > 0 \)
Equation (2-118) turns into three integrals. Let assume that

\[ I_3 = I_{31} + I_{32} + I_{33} \]

(2-119)

Where,

\[ I_{31} = \frac{-B_z}{\pi} \int_0^\infty \eta \left( \frac{\sin \sqrt{(\omega/c_z)^2 + \eta^2}}{\sqrt{(\omega/c_z)^2 + \eta^2}} \right) e^{-\eta x} d\eta \]

(2-120)

\[ I_{32} = \frac{B_z}{\pi} \int_0^{\pi} \eta \left( \frac{\sin \sqrt{(\omega/c_z)^2 - \eta^2}}{\sqrt{(\omega/c_z)^2 - \eta^2}} \right) \cos \eta x d\eta \]

(2-121)

\[ I_{33} = \frac{iB_z}{\pi} \int_0^{\pi} \eta \left( \frac{\sin \sqrt{(\omega/c_z)^2 - \eta^2}}{\sqrt{(\omega/c_z)^2 - \eta^2}} \right) \sin \eta x d\eta \]

(2-122)

Equation (2-120) accepts imaginary wave number. The imaginary wavenumber correspond to non-propagating evanescent waves and will not be considered here.

The integrant in Equation (2-121) can be evaluated by with the aid of ref (Erdélyi, 1954; Bailey, 1938), i.e.,
\[ I_{32} = \frac{B}{2} \zeta c (x^2 + d^2)^{-\frac{1}{2}} J_0 \left[ \left( \omega / c_i \right) \left( x^2 + d^2 \right)^{\frac{1}{2}} \right] \]  

(2-123)

Here \( J_0 \) is the Bessel function of the first kind of order 0.

Equation (2-122) is to be determined from ref (Erdélyi, 1954), pp. 35 (23), by putting \( \nu = \zeta c \) and using the identity of Bessel function, \( J_{\zeta c}(x) = \sqrt{2/\pi x} \sin x \) i.e.,

\[ I_{33} = \frac{iB}{2} (\omega / c_i) x(x^2 + d^2)^{-\frac{1}{2}} Y_1 \left[ (\omega / c_i) (x^2 + d^2)^{\frac{1}{2}} \right] \]  

(2-124)

Using Equations (2-102), (2-110), (2-116), (2-123), (2-124) the total displacement becomes

\[ u_y = i \left( \sum_{j=0}^j P_S(\zeta_j^S) N_S \left( \zeta_j^S \right) D_j \left( \zeta_j^S \right) \right) e^{i \xi j x} + \sum_{j=0}^j P_A(\zeta_j^A) N_A \left( \zeta_j^A \right) D_A \left( \zeta_j^A \right) e^{i \xi j x} \]

\[ -\frac{A}{4} \left( \zeta c / c_i \right) d \left( x^2 + d^2 \right)^{-\frac{1}{2}} Y_1 \left( \zeta c / c_i \left( x^2 + d^2 \right)^{\frac{1}{2}} \right) \]

\[ -\frac{B}{2} \left( \zeta c / c_i \right) x(x^2 + d^2)^{-\frac{1}{2}} J_0 \left( \zeta c / c_i \left( x^2 + d^2 \right)^{\frac{1}{2}} \right) \]

\[ -\frac{iB}{2} \left( \zeta c / c_i \right) x(x^2 + d^2)^{-\frac{1}{2}} Y_1 \left( \zeta c / c_i \left( x^2 + d^2 \right)^{\frac{1}{2}} \right) \]

(2-125)

After rearranging Equation (2-125) and reinstating explicitly the harmonic time dependency, we get,
\[ u_y = i \left( \sum_{j=0}^{j_1} \left[ P_s \left( \xi_j^S \right) \frac{N_s \left( \xi_j^S \right)}{D_s' \left( \xi_j^S \right)} \right] e^{i \left( \xi_j^S x - \omega t \right)} + \sum_{j=0}^{j_A} \left[ P_A \left( \xi_j^A \right) \frac{N_A \left( \xi_j^A \right)}{D'_A \left( \xi_j^A \right)} \right] e^{i \left( \xi_j^A x - \omega t \right)} \right) \]

\[- \frac{A}{4} \left( \frac{\omega}{c_p} \right) d \left( x^2 + d^2 \right)^{1/2} Y_1 \left( \frac{\omega}{c_p} \left[ \frac{d^2 + x^2}{2} \right]^1 \right) e^{-i \omega t} \]

\[- \frac{B}{2} \left( \frac{\omega}{c_p} \right) x \left( x^2 + d^2 \right)^{1/2} J_0 \left[ \frac{\omega}{c_p} \left( x^2 + d^2 \right)^{1/2} \right] e^{-i \omega t} \]

Assume that,

\[ u_y = u_{y_L} + u_{y_A} + u_{y_B} \]

Where,

\[ u_{y_L} = i \left( \sum_{j=0}^{j_1} \left[ P_s \left( \xi_j^S \right) \frac{N_s \left( \xi_j^S \right)}{D_s' \left( \xi_j^S \right)} \right] e^{i \left( \xi_j^S x - \omega t \right)} + \sum_{j=0}^{j_A} \left[ P_A \left( \xi_j^A \right) \frac{N_A \left( \xi_j^A \right)}{D'_A \left( \xi_j^A \right)} \right] e^{i \left( \xi_j^A x - \omega t \right)} \right) \] (2-126)

\[ u_{y_A} = - \frac{A}{4} \left( \frac{\omega}{c_p} \right) x^{-\gamma/2} \left( d^2 + x^2 \right)^{1/2} d \left( \frac{\omega}{c_p} \left[ d^2 + x^2 \right]^{1/2} \right) e^{-i \omega t} \] (2-127)

\[ u_{y_B} = - \frac{B}{2} \left( \frac{\omega}{c_p} \right) x \left( x^2 + d^2 \right)^{1/2} J_0 \left[ \frac{\omega}{c_p} \left( x^2 + d^2 \right)^{1/2} \right] e^{-i \omega t} \] (2-128)

\[- \frac{iB}{2} \left( \frac{\omega}{c_p} \right) x \left( x^2 + d^2 \right)^{1/2} \bigg[ \left( \frac{\omega}{c_p} \right) \left( x^2 + d^2 \right)^{1/2} \bigg] e^{-i \omega t} \] (2-129)

\[ u_{y_L} \] is the out-of-plane displacement containing Lamb wave mode and \( u_{y_A}, u_{y_B} \) are the out-of-plane displacements of bulk wave for excitation potentials \( A \) and \( B \), respectively.
Theoretical derivation reveals that, bulk wave is present in addition to Lamb wave. The Lamb waves, which consist of a pattern of standing waves in the thickness direction (Lamb wave modes) behaving like traveling waves in the x-direction in a plate. Guided waves can propagate long distances, yields an easy inspection of a wide variety of structures. Whereas, bulk waves generate from a source and directly reach to the AE sensor without interfering with the boundaries. Bulk waves are categorized into longitudinal (pressure) and transverse (shear) waves. The displacements $u_{yA}$ and $u_{yB}$ are the longitudinal and transverse bulk waves respectively.

### 2.8 Effect of depth of source

If the source is present at a different location other than mid-plane, i.e., $y = y_0$ (Figure 2.6), then equations for excitation potentials are

\[
A = A(x, y)\delta(x)\delta(y - y_0)e^{-i\omega t} 
\]

\[
B_z = B_z(x, y)\delta(x)\delta(y - y_0)e^{-i\omega t} 
\]

Recall Equations (2-37) and (2-38) for new location of excitation potentials

\[
\bar{\Phi}''(\xi, y) + \eta^2_r \bar{\Phi}(\xi, y) = -A(y)\delta(y - y_0) 
\]

\[
\bar{H}''(\xi, y) + \eta^2_r \bar{H}(\xi, y) = -B_z(y)\delta(y - y_0) 
\]

A complementary solution of Equation (2-132) and (2-133) will be the same as before. The only particular solution will change due to the source in the right hand side.
Figure 2.6 Plate of thickness 2d in which straight crested Lamb waves (P+SV) propagate in the x-direction due to concentrated potentials at \( y = y_0 \)

Fourier shift transform states that,

\[
F\{ f(y - y_0) \} = e^{-i\eta y_0} F\{ f(y) \}
\]  

Equation 2-134)

By using Fourier shift transform theorem Equation (2-134), the particular solutions of Equations (2-132), (2-133) become

\[
\Phi_1(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A e^{-i\eta y_0} e^{i\eta y} \frac{e^{-i\eta_2}}{\eta^2 - \eta_p^2} d\eta
\]  

Equation 2-135)

\[
H_{z1}(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B e^{-i\eta y_0} e^{i\eta y} \frac{e^{-i\eta_2}}{\eta^2 - \eta_p^2} d\eta
\]  

Equation 2-136)

Upon rearranging,

\[
\Phi_1(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} A e^{i\eta(y - y_0)} d\eta
\]  

Equation 2-137)

\[
H_{z1}(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} B e^{i\eta(y - y_0)} d\eta
\]  

Equation 2-138)
Integration of Equations (2-137) and (2-138) can be done by using residue theorem. The details solution is already presented before from Equations (2-51) to (2-71). For brevity, only the final results of Equations (2-137), (2-138) are presented here. From the residue theorem the integrands become,

$$\Phi_1(\xi, y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{\eta^2 - \eta_p^2} e^{i(x-y)} d\eta = -\frac{A}{2\eta_p} \sin \eta_p \left| y - y_0 \right|$$  \hfill (2-139)

Similarly,

$$\Phi_1(\xi, \eta) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{B}{\eta^2 - \eta_s^2} e^{i(x-y)} d\eta = -\frac{B s}{2\eta_s} \sin \eta_s \left| y - y_0 \right|$$  \hfill (2-140)

The corresponding source terms Equations (2-86) and (2-91) for a symmetric and antisymmetric solution will be

$$P_S = (\xi^2 - \eta_s^2) \frac{A}{2\eta_p} \sin \eta_p d_1 + \xi B_c \cos \eta_s d_1$$  \hfill (2-141)

$$P_A = \xi A \cos \eta_p d_1 + (\xi^2 - \eta_s^2) \frac{B_s}{2\eta_s} \sin \eta_s d_1$$  \hfill (2-142)

Here, \(d_1 = d - y_0\).

The corresponding displacement Equations (2-127), (2-128), (2-129) become

$$u_{s,x} = i \left( \sum_{j=0}^{b} P_s(\xi_j) \left( \frac{N_s(\xi_j)}{D_j(\xi_j)} \right) e^{i(\xi_j x - \alpha x)} + \sum_{j=0}^{a} P_A(\xi_j) \left( \frac{N_A(\xi_j)}{D_A(\xi_j)} \right) e^{i(\xi_j x - \alpha x)} \right)$$  \hfill (2-143)

$$u_{s,y} = -\frac{A}{4} \left( \frac{\alpha}{\alpha_p} \right) \left( x^2 + d_1^2 \right)^{\frac{1}{2}} Y_1 \left( \frac{\alpha}{\alpha_p} \left[ x^2 + d_1^2 \right]^{\frac{1}{2}} \right) e^{-i\alpha x}$$  \hfill (2-144)
AE guided waves will be generated by the AE event. AE guided waves will propagate through the structure according to the structural transfer function. The out-of-plane displacement of the guided waves can be captured by conventional AE transducer installed on the surface of the structure as shown in Figure 2.7.

![Diagram of AE guided wave propagation](image)

**Figure 2.7** Acoustic emission propagation and detection by a sensor installed on a structure

For numerical analysis, 304-stainless steel material with 6 mm thickness was chosen as a case study. The signal was received at 500 mm distance from the source. The excitation source was located at mid-plane. Very short peak time of 3 μs was used for a case study. Later, the effects of source of depth and propagation distance were included in the study. The details methodology of numerical analysis is described in Figure 2.8. In numerical analysis, direct Fourier transforms of the time domain excitation signal was performed to get the frequency contents of the signal and an inverse Fourier transform was
done to get the time domain output signal from the frequency response of out-of-plane displacement.

Figure 2.8   Methodology for guided wave analysis
2.10 Numerical Studies

This section includes guided wave simulation in a plate due to excitation potentials.

2.10.1 Time-dependent excitation potentials

The time-dependent excitation potentials depend on time-dependent energy released from a crack. A real AE source releases energy during a finite period. In the beginning, the rate of energy released from a crack increases sharply with time and reaches a maximum peak value within a very short time; then decreases asymptotically toward the steady-state value, usually zero. The time rate of energy released from a crack can be modeled as a Gaussian pulse. Figure 2.9a and Figure 2.10a show the time rate of pressure potential and shear potential. The corresponding equations are,

\[
\frac{\partial A^*}{\partial t} = A_0 t^2 e^{-\frac{t^2}{\tau^2}} \\
\frac{\partial B_\tau^*}{\partial t} = B_{z0} t^2 e^{-\frac{t^2}{\tau^2}}
\]

(2-146)

Here \( A_0 \) and \( B_{z0} \) are the scaling factors. The time profile of the potentials are to be evaluated by integrating Equation (2-146), i.e.,

\[
A^* = A_0 \left( \sqrt{\pi} \text{erf} \left( \frac{1}{\tau} \right) - 2te^{-\frac{t^2}{\tau^2}} \right)
\]

(2-147)

\[
B_\tau^* = B_{z0} \left( \sqrt{\pi} \text{erf} \left( \frac{1}{\tau} \right) - 2te^{-\frac{t^2}{\tau^2}} \right)
\]
Figure 2.9  (a) Time rate of pressure potential \( \left( \frac{\partial A^*}{\partial t} \right) \) and (b) Time profile of pressure potential \( (A^*) \)

Figure 2.10  (a) Time rate of shear potential \( \left( \frac{\partial B^*}{\partial t} \right) \) and (b) Time profile of shear potential \( (B^*) \)

The time profile of potential is shown in Figure 2.9b and Figure 2.10b. The time profile of AE source may follow a cosine bell function (Hamstad, 2010; Sause et al., 2013) or error function. In this research, a Gaussian pulse is used to model the growth of the
excitation potentials during the AE event; as a result, the actual excitation potential follows
the error function variation in the time domain. Cuadra et al. (2016) obtained similar time
profile of AE energy from a crack by using 3D computation method.

The key characteristics of the excitation potentials are:

- peak time: time required to reach the time rate of potential to a maximum value
- rise time: time required to reach the potential to 98% of maximum value or steady
  state value
- peak value: the maximum value of time rate of potential
- maximum potential: the maximum value of time profile of potential

The amplitude of the source \( A \) and \( B_z \) for unit width is,

\[
A = \frac{h}{c_p} A_0 \left( \sqrt{\pi} \text{erf} \left( \frac{t}{\tau} \right) - 2te^{-\frac{t^2}{\tau^2}} \right)
\]

\[
B_z = \frac{h}{c_x} B_{z0} \left( \sqrt{\pi} \text{erf} \left( \frac{t}{\tau} \right) - 2te^{-\frac{t^2}{\tau^2}} \right)
\]

(2-148)

Here, plate thickness \( h = 2d \).

2.10.2 Phase velocity and group velocity dispersion curves

A solution of the Rayleigh-Lamb equation for symmetric and antisymmetric modes
(Equations (2-89) and (2-94)) yields the wave number and hence the phase velocity for
each given frequency. Multiple solutions exist, thus multiple Lamb modes also exist.
Differentiation of phase velocity with respect to frequency yields the group velocity. The
phase velocity dispersion curves and group velocity dispersion curves are shown in Figure
2.11a and Figure 2.11b respectively. The existence of certain Lamb mode depends on the plate thickness and frequency. The fundamental S0 and A0 modes will always exist. The phase velocity (Figure 2.11a) is associated with the phase difference between the vibrations observed at two different points during the passage of the wave. The phase velocity is used to calculate the wavelength of each mode.

![Lamb wave dispersion curves](image)

**Figure 2.11** Lamb wave dispersion curves: (a) phase velocity and (b) group velocity of symmetric and anti-symmetric Lamb wave modes

The fundamental wave mode of symmetric (S0) and anti-symmetric (A0) is considered in this analysis. The signal is received at some distance from the point of excitation potentials, only the modes with real-valued wavenumbers are included in the simulation. The corresponding modes for imaginary and complex-valued wavenumbers are ignored because they propagate with decaying amplitude; at sufficiently far from the excitation point their amplitudes are negligible. Since different Lamb wave modes traveling with different wave speeds exist simultaneously, the excitation potentials will generate S0 and A0 wave packets. The group velocity (Figure 2.11b) of Lamb waves is important when examining the traveling of Lamb wave packets. These wave packets will travel independently through the plate and will arrive at different times. Due to the multi-mode character of guided Lamb wave propagation, the received signal has at least two
separate wave packets, S0 and A0. All wave modes propagate independently in the structure. The final waveform will be the superposition of all the propagating waves and will have the contribution from each Lamb wave mode.

2.10.3 Example of AE guided wave propagation in a 6mm plate

A test case example is presented in this section to show how AE guided waves propagate over a certain distance using excitation potentials. A 304-stainless steel plate with 6 mm thickness was chosen for this purpose. The signal was received at 500 mm distance from the source. Excitation sources were located at mid-plane of the plate ($d = 3$ mm). The time profile excitation potentials (Figure 2.9b and Figure 2.10b) were used to simulate AE waves in the plate. Excitation potential can be calculated from the time rate of potential released during crack propagation. Peak time, rise time, peak value and maximum potential of excitation potentials are $3 \mu$s, $6.6 \mu$s, $0.28 \mu$W/kg and $1 \mu$J/kg. Based on this information a numerical study on AE Lamb wave propagation is conducted. Peak time or rise time is one of the major characteristics of AE source. Several researchers investigate the potential effect of rise time on AE signal by varying the source rise time in-between $0.1$ $\mu$s to $15$ $\mu$s (Hamstad, 2010; Sause et al., 2013; Michaels et al., 1981; Haider et al., 2017e).
Figure 2.12  Lamb wave (S0 and A0 mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 $\mu$s) located at mid plane.

Each excitation potential was considered separately to simulate the Lamb waves, and then the effect of both potentials was analyzed. Figure 2.12a shows the Lamb wave (S0 and A0 mode) and bulk wave propagation using pressure excitation potential. Signals are normalized by their peak amplitudes, i.e., amplitude/peak amplitude. It can be inferred from the figures that both A0 and S0 are dispersive.
S0 contains a high frequency component at which it is dispersive, whereas, A0 contains low-frequency component at which it is also dispersive. Bulk wave shows nondispersive behavior, but the peak amplitude is not significant compared to peak S0 and A0 amplitude. The notable characteristic is that the peak amplitude of A0 is higher than peak S0 amplitude. Figure 2.12b shows the Lamb wave (S0 and A0 mode) and bulk wave propagation using shear potential only. The peak amplitude of A0 is higher than peak S0 amplitude while using shear potential only (Figure 2.12b). By comparing Figure 2.12a and Figure 2.12b the important observation can be made as, pressure excitation potential has more contribution to the peak S0 amplitude over shear excitation potentials whereas, shear excitation potential has more contribution to the peak A0 amplitude. It should be noted that the amplitude of the bulk wave is much smaller than the peak A0 and S0 amplitudes for both pressure and shear excitation potentials. But their contribution to the S0 and A0 peak amplitude might change due to change in depth of the source and propagating distance.

The proceeding section discusses the effect of depth of the source and propagating distance in AE guided wave propagation using excitation potentials.

2.10.4 Effect of plate thickness

The numerical analysis of AE Lamb wave propagation was performed using excitation potentials in 2mm, 6mm and 12 mm thick 304-steel plate. Effect of pressure and shear excitation potentials acting alone and in combination was considered for each plate thickness.
Figure 2.13  Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 2 mm 304-steel plate for different excitation potentials (peak time = 3 $\mu$s) located at midplane
Figure 2.14  Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 6 mm 304-steel plate for different excitation potentials (peak time = 3 μs) located at midplane
Figure 2.15  Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 12 mm 304-steel plate for different excitation potentials (peak time = 3 µs) located at midplane

Figure 2.13, Figure 2.14 and Figure 2.15 show the Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 2 mm, 6mm and 12 mm thick 304-steel plate for (a)
pressure potential only (b) shear potential only and (c) both shear and pressure potential. Excitation potentials are located at the mid-plane and the peak time of the time rate of excitation potentials are $3 \mu s$. Figure 2.13a and Figure 2.13b show the individual contribution of pressure potential and shear potential in a 2mm plate. Shear potential shows more contribution to the peak amplitude of the S0 wave packets, and both pressure and shear potential have an almost equal contribution to the peak A0 amplitude. The contribution of pressure potential to the peak amplitude of S0 increases with increasing plate thickness (Figure 2.13a, Figure 2.14a, and Figure 2.15a) but contribution to the peak A0 amplitude initially decreases and then increases with increasing plate thickness. The contribution of shear potential to the peak amplitude of S0 increases with increasing plate thickness (Figure 2.13b, Figure 2.14b and Figure 2.15b) but contribution to the peak A0 amplitude decreases with increasing plate thickness. By comparing Figure 2.13, Figure 2.14 and Figure 2.15, it can be inferred that the peak amplitude of S0 increases with increasing plate thickness whereas, the peak amplitude of A0 initially decreases and then increases with increasing plate thickness. The peak amplitude of A0 is dominant over S0 in a 2 mm thick plate while using both excitation potentials. As a result, the S0 wave packet may not be significant in real AE signal at this plate thickness. However, for 12 mm thick steel plate S0 and A0 have equal amplitude. At this thickness, S0 and A0 Lamb wave modes in a plate can be viewed as the propagation of a Rayleigh surface wave with equal amplitude. It should be noted that S0 is non-dispersive in 2mm thick plate due to the low thickness-frequency product at which S0 is non-dispersive (Figure 2.13). However, S0 becomes more dispersive (Figure 2.14) for 6 mm thick plate due to the high thickness-
frequency product. In 12 mm thick plate, Rayleigh surface waves (Figure 2.15) show nondispersive behavior.

2.10.5 Effect of depth of the source

To study the effect of depth of source, a 6 mm thick plate was chosen. The signal was received at 500 mm distance from the source. The excitation sources were located at the top surface, 1.5 mm and 3 mm deep (mid plane) from the top surface. Figure 2.16, Figure 2.17, Figure 2.18 show the out-of-plane displacement (S0 and A0 mode, bulk wave) vs. time at 500 mm propagation distance in 6mm thick plate for different source location. Signals are normalized by their peak amplitudes, i.e., amplitude/ peak amplitude. Figure 2.16 shows that A0 mode appears only while using pressure potential on the top surface whereas, S0 mode appears only while using shear potential. Therefore, pressure potential contributes to the peak A0 amplitude and shear potential contributes to the peak S0 amplitude in the final waveform. Another notable characteristic is that pressure potential has a contribution to the high amplitude of the trailing edge (low-frequency component) of the A0 wave packet for the top-surface source. Only, pressure potential has contribution to the A0 amplitude (Figure 2.16a); as a result, the peak amplitude can be seen at the trailing edge of the A0 waveform. With increasing depth of source the effect of the high amplitude of low-frequency decreases (Figure 2.17a and Figure 2.18a) in A0 signal. By comparing Figure 2.17, and Figure 2.18, clearly amplitude scales show that the increase in peak A0 amplitude and a decrease in peak S0 amplitude with increasing depth of source while using pressure excitation potential only.
Figure 2.16 Lamb wave (S0 and A0 mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 μs) located on the top surface.

Figure 2.16b, Figure 2.17b and Figure 2.18b show some qualitative and quantitative changes in spectrum while using shear excitation potentials only. The amplitude of S0 signal decreases and A0 signal increases with increasing depth of source while using shear excitation potential only (Figure 2.17b and Figure 2.18b). Qualitative change in S0 signal refers to the change in the frequency content of the signal.
Figure 2.17  Lamb wave (S0 and A0 mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 μS) located at 1.5 mm depth from the top surface.

It should be noted that the peak amplitude of S0 and A0 signal due to shear excitation potential located at 1.5 mm depth from the top surface is more significant than pressure potential (Figure 2.17). However, with increasing depth of source S0 signals become more significant due to pressure excitation potential over shear excitation potential (Figure 2.17 and Figure 2.18). For all the AE source location, the shear potential part of the AE source has more contribution to the peak A0 amplitude than pressure potential. The
peak amplitude of bulk waves increases with increasing depth of source (Figure 2.17 and Figure 2.18). However, the amplitude of the bulk wave is much smaller than the peak A0 and S0 amplitudes. Therefore, the peak amplitude of the bulk wave may not appear in real AE signal.

(a) Pressure potential excitation  
(b) Shear potential excitation

Figure 2.18  Lamb wave (S0 and A0 mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 $\mu$s) located at mid plane
2.10.6 Effect of peak time

Figure 2.19, Figure 2.20 and Figure 2.21 show the out-of-plane displacement in time domains for propagating distance of 500 mm from the excitation source in a 6mm plate. The numerical study was done for a selection of different peak times of 3 μs, 6 μs, and 9 μs. These figures are for a source located at midplane of the plate. Clearly, the amplitude scales show that the rise time affects low-frequency A0 amplitude.

The important observation is that the low-frequency amplitude of A0 increases with increasing peak time. S0 has a high-frequency component at this plate thickness, and rise time has barely effect on high-frequency amplitude. As a result, peak S0 amplitude is not responsive to peak time. Another important observation is that peak A0 amplitude decreases with increasing peak time for both excitation potential (Figure 2.19c, Figure 2.20c, and Figure 2.21c). These figures also show some other qualitative changes with changing in peak time. The shapes in the time domain of the waveforms change significantly with peak time. Peak time might have more influence on amplitude and shape if the source located near the top surface rather than at midplane.

A future study on the synergistic effect of rising time and depth of source is recommended added in the future study section. If the peak amplitude and shape of the waveform are compared with peak time and thickness of the plate, it might be possible to distinguish between different source types and source location.
Figure 2.19  Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 6 mm 304-steel plate for different excitation potentials (peak time = 3 $\mu$s) located at mid-plane.
Figure 2.20  Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 6 mm 304-steel plate for different excitation potentials (peak time = 6 $\mu$s) located at mid-plane
Figure 2.21  Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 6 mm 304-steel plate for different excitation potentials (peak time = 9 $\mu$s) located at mid-plane
2.10.7 Effect of higher order Lamb wave mode

Since AE signals have a wideband response; higher order modes should appear in addition to the fundamental S0 and A0 Lamb wave modes. This section discusses the effect of higher-order modes on AE signal due to shear potential excitation and pressure potential excitation.

![Graphs showing the effect of higher order Lamb waves](image)

(a) Pressure potential excitation  
(b) Shear potential excitation

Figure 2.22 Higher-order Lamb waves (S1 and A1) modes at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 $\mu$s) located at mid-plane

A real AE sensor does not have ultimate high-frequency response; therefore, the signals were filtered at 10 kHz and 700 kHz frequency. Figure 2.22 shows the higher order Lamb wave (S1 and A1) modes at 500 mm distance in 6 mm 304-steel plate. The A1 signal is more dispersive for the shear excitation potential than the pressure excitation potential.
The peak amplitude of the A1 mode is higher than the peak amplitude of the S1 mode for both shear and pressure potentials. The peak S1 and A1 amplitudes for pressure potential are higher than the shear potential. By comparing Figure 2.12 and Figure 2.22, it can be inferred that the peak amplitude of fundamental Lamb wave mode (S0 and A0) is higher than peak S1 and A1 amplitude. Therefore, the peak amplitude of higher order mode may not be significant in real AE signal.

2.10.8 Effect of propagation distance

The attenuation of the peak amplitude of the signal as a function of propagating distance was also determined. Figure 2.23, Figure 2.24 shows the normalized peak amplitude of out-of-displacement (S0, A0, bulk wave) against propagation distance from 100 mm to 500 mm for pressure potential, shear potential respectively. The amplitudes are normalized by their peak amplitudes. A 6 mm thick 304-steel plate with both excitation potentials are considered. Potentials are located at midplane with a peak time of 3 $\mu$s.

Figure 2.23, Figure 2.24 show significant attenuation of S0, A0 and bulk wave signal with propagating distance 100 mm to 50 mm. However, larger attenuation of peak bulk wave amplitude is observed compared to peak S0 and A0 amplitudes. Therefore, far from the source-bulk wave becomes less significant and may not be captured by the AE transducer. Another important observation is that peak A0 amplitude attenuates more than peak S0 amplitude while using pressure potential only whereas, peak S0 amplitude attenuates more than peak A0 amplitude while using shear potential only. The attenuation of the peak amplitude is expected due to the dispersion of the signal.
Figure 2.23  Effect of pressure excitation potential: variation of out-of-plane displacement (S0, A0 and bulk wave) with propagation distance in 6 mm 304-steel plate for source (peak time = 3 $\mu$s) located at the midplane

Figure 2.24  Effect of shear excitation potential: variation of out-of-plane displacement with propagation distance in a 6-mm 304-steel plate for the source (peak time = 3 $\mu$s) located at the midplane
2.10.9 Time-frequency analysis

Time-frequency analysis provides an effective tool to study a signal in both the time and frequency domains simultaneously. The main objectives of the time-frequency distributions are to obtain time-frequency localized amplitude distribution with high resolution.

![Time-frequency spectrum](image1)

![Time domain signal](image2)

(a) Time-frequency spectrum (b) time domain signal of S0 and A0 waves for propagation distance of 500 mm in 2 mm 304-steel plate for both excitation potentials (peak time = 3 μs) located at the midplane.

For brevity, the following extreme cases are considered for time-frequency analysis:

(i) Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 2 mm 304-steel plate for both excitation potentials (peak time = 3 μs) located at mid-plane.
(ii) Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 6 mm 304-steel plate for both excitation potentials (peak time = 3 $\mu$s) located at mid-plane.

(iii) Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 6 mm 304-steel plate for both potentials (peak time = 3 $\mu$s) located on top surface.

Lamb wave (S0 and A0 mode) propagation at 500 mm distance in 6 mm 304-steel plate for both potentials (peak time = 9 $\mu$s) located at mid-plane. Figure 2.25a shows the time-frequency spectrum of S0 and A0 waves in a 2 mm plate. The source was kept at mid-plane and peak time of 3 $\mu$s was considered in this case. Figure 2.25a shows that the peak amplitude of S0 has a frequency component of 200-400 kHz at which it is non-dispersive. Therefore a sharp S0 signal is observed in the time-domain signal (Figure 2.25b). The peak amplitude of A0 has the frequency component of 50-100 kHz at which it is dispersive. The dispersive A0 signal can be observed in the time domain signal (Figure 2.25b). The A0 signal also contains low amplitude of high-frequency components that arrive earlier than high amplitude low-frequency components.

Figure 2.26a shows the time-frequency spectrum of S0 and A0 waves in a 6 mm plate. The peak time of the excitation potential and source location is kept same as 2mm plate to compare the effect of thickness. This figure shows the peak amplitude of S0 can be seen around 500 kHz at which it is very dispersive. The dispersive behavior of the S0 signal can be seen from time domain S0 signal (Figure 2.26b). The peak amplitude of S0 reaches in-between 200-300 $\mu$s. A0 has a high amplitude component at around 100-300 kHz. Frequency components of peak amplitude for 6 mm plate are higher than 2 mm plate.
Therefore, the peak amplitude of A_0 reaches 6 mm plate (~200 \mu s) earlier than 2 mm plate (~400 \mu s). The A_0 signal also contains low amplitude of low-frequency component.

![Time-frequency spectrum](image1)

**Figure 2.26** (a) Time-frequency spectrum (b) time domain signal of S_0 and A_0 waves for propagation distance of 500 mm in 6 mm 304-steel plate for both excitation potentials (peak time = 3 \mu s) located at the mid plane

Figure 2.27a shows the time-frequency spectrum of S_0 and A_0 waves in a 6mm plate. The peak time of the excitation potential is three as before, and source location has changed from midplane to top surface to study the effect of source of depth. These figures show some qualitative and quantitative changes in the spectrum. The peak amplitude of S_0 is observed at 400 kHz at which it is very dispersive. The frequency content in S_0 signal for the top-surface source is smaller than frequency content in S_0 signal for the mid-plane source (Figure 2.26a and Figure 2.27a). There is a leading edge of low amplitude S_0 signal
containing frequency in between 300-400 kHz. S0 signal is dispersive at this frequency range. The dispersive behavior can be observed in the time domain signal of S0 (Figure 2.27b). There are some qualitative and quantitative changes also observed in A0 time-frequency spectrum. The peak amplitude of A0 is observed below 100 kHz. The peak amplitude of A0 reaches at around 300-400 $\mu$s as shown in Figure 2.27b. Another major observation is that A0 contains low frequency with high amplitude signal spreading in between 200-600 $\mu$s.

![Figure 2.27](image)

(a) Time-frequency spectrum (b) time domain signal of S0 and A0 waves for propagation distance of 500 mm in 6 mm 304- steel plate for both excitation potentials (peak time = 3 $\mu$s) located on the top surface
Figure 2.28 (a) Time-frequency spectrum (b) time domain signal of S0 and A0 waves for propagation distance of 500 mm in 6 mm 304 steel plate for both excitation potentials (peak time = 9 $\mu s$) located at the midplane.

Figure 2.28a shows the time-frequency spectrum of S0 and A0 waves in the same 6mm plate. The peak time of the excitation potential is changed to 9 $\mu s$ from 3 $\mu s$ and but the source location is kept at mid plane to compare the effect of rise time with case (ii). Figure 2.28a shows the similar trend for S0 as a case (ii), containing peak amplitude at around 500 kHz at which it is dispersive. However, A0 shows some qualitative changes in the signal. It contains peak amplitude around 200 kHz, but it also contains high amplitude low frequency (50-100 kHz) component that reaches at 400 $\mu s$, as can be seen in the time domain signal (Figure 2.28b). The significant difference between the 3 $\mu s$ peak time source and nine $\mu s$ peak time source is that nine $\mu s$ peak time source creates long trailing edge extended up to 800 $\mu s$ which contains very low-frequency components.
CHAPTER 3
A HELMHOLTZ POTENTIAL APPROACH TO THE ANALYSIS OF
CIRCULAR CRESTED WAVE GENERATION DURING ACOUSTIC EMISSION
EVENTS

3.1 FORMATION OF PRESSURE AND SHEAR POTENTIALS

Navier-Lame equations for displacement in the absence of body force in vector form given as

\[(\lambda + \mu) \nabla \text{div} \ u - \mu \nabla \times \nabla \times \ u = \rho \ddot{u} \tag{3-1}\]

where, \(\ddot{u}\) is the displacement vector, \(\lambda, \mu\) = Lame constant, \(\rho\) = density.

If the body force is present, then the Navier-Lame equations can be written as follow

\[(\lambda + \mu) \nabla \text{div} \ u - \mu \nabla \times \nabla \times \ u + \rho \ddot{f} = \rho \ddot{u} \tag{3-2}\]

Helmholtz decomposition principle (Helmholtz, 1858) states that, any vector can be resolved into the sum of an irrotational (curl-free) vector field and a solenoidal vector field, where an irrotational vector field has a scalar potential and a solenoidal vector field has a vector potential. Potentials are useful and convenient in several wave theory derivations (Giurgiuitiu, 2014; Graff, 1975; Aki and Richards, 2002, Bhuiyan et al., 2016).

Assume that the displacement \(\ddot{u}\) can be expressed in terms of two potential functions, a scalar potential \(\Phi\) and a vector potential \(\vec{H}\)
\[ u = \text{grad } \Phi + \text{curl } \vec{H} \]  

(3-3)

where

\[ \vec{H} = H_r \hat{e}_r + H_\theta \hat{e}_\theta + H_z \hat{e}_z \]  

(3-4)

Equation (3-3) is known as the Helmholtz equation, and is complemented by the uniqueness condition, i.e.,

\[ \vec{\nabla} \cdot \vec{H} = f(r, \theta) \]  

(3-5)

Introducing additional scalar and vector potentials \( A^* \) and \( B^* \) for body force \( \vec{f} \) (Aki and Richards, 2002; Graff, 1991)

\[ \vec{f} = \text{grad } A^* + \text{curl } B^* \]  

(3-6)

Here,

\[ \vec{B}^* = B_r^* \hat{e}_r + B_\theta^* \hat{e}_\theta + B_z^* \hat{e}_z \]  

(3-7)

The uniqueness condition is,

\[ \vec{\nabla} \cdot \vec{B}^* = f(r, \theta) \]  

(3-8)

Using the Equation (3-3) and (3-6), into Equation (3-2),

\[ (\lambda + \mu) \text{grad div (grad } \Phi) + (\lambda + \mu) \text{grad div (curl } \vec{H}) - \mu \text{ curl curl (grad } \Phi) - \mu \text{ curl curl (curl } \vec{H}) + \rho(\text{grad } A^* + \text{curl } B^*) = \rho(\text{grad } \Phi + \text{curl } \vec{H}) \]  

(3-9)

Here,

\[ \text{grad div (curl } \vec{H}) = 0 \text{ and curl curl (grad } \Phi) = 0 \]
Therefore, Equation (3-9) becomes,

\[(\lambda + \mu) \operatorname{grad} \operatorname{div} (\operatorname{grad} \Phi) - \mu \operatorname{curl} \operatorname{curl} (\operatorname{curl} \vec{H}) \]

\[+ \rho(\operatorname{grad} A^* + \operatorname{curl} \vec{B}^*) = \rho(\operatorname{grad} \Phi + \operatorname{curl} \vec{H}) \]

(3-10)

Upon rearranging of Equation (3-10)

\[\operatorname{grad}[((\lambda + \mu) \operatorname{div} \operatorname{grad} \Phi + \rho A^* - \rho \Phi)] \]

\[- \operatorname{curl}[\mu \operatorname{curl} \operatorname{curl} \vec{H} - \rho \vec{B}^* + \rho \vec{H}] = 0 \]

(3-11)

Equation (3-11) to hold at any place and any time, the components in parentheses must be independently zero, i.e.,

\[(\lambda + \mu) \operatorname{div} \operatorname{grad} \Phi + \rho A^* - \rho \Phi = 0 \]

(3-12)

\[\mu \operatorname{curl} \operatorname{curl} \vec{H} - \rho \vec{B}^* + \rho \vec{H} = 0 \]

(3-13)

Upon division by \(\rho\) and rearrangement,

\[c_p^2 \operatorname{div} \operatorname{grad} \Phi + A^* - \Phi = 0 \]

(3-14)

\[c_s^2 \operatorname{curl} \operatorname{curl} \vec{H} - \vec{B}^* + \vec{H} = 0 \]

(3-15)

Here, \(c_p^2 = \frac{\lambda + 2\mu}{\rho} \); \(c_s^2 = \frac{\mu}{\rho}\)

Equations (3-14) and (3-15) are the wave equations for the scalar potential and the vector potential respectively. Equation (3-14) indicates that the scalar potential, \(\Phi\), propagates with the pressure wave speed, \(c_p\) due to excitation potential \(A^*\), whereas Equation (3-15) indicates that the vector potential, \(\vec{H}\), propagates with the shear wave speed, \(c_s\) due to excitation potential \(\vec{B}^*\). The excitation potentials are indeed a true description of energy...
released from a crack. The unit of the excitation potentials are J/kg. The source can be treated as localized event and Green function can be used as the solution of inhomogeneous wave equations. In this paper, wave equations for scalar potential $\Phi$ and vector potential $\vec{H}$ need to be solved with the presence of source potentials $A^*$ and $B^*$. The derivation of Lamb wave equation will be provided by solving wave Equations (3-14) and (3-15) for the potentials. In this derivation we will consider generation of axis symmetric circular crested lamb wave due to time harmonic excitation potentials. The assumption of axis-symmetric waves makes the problem $\theta$-invariant i.e., $\frac{\partial}{\partial \theta} = 0$.

Upon expansion of Equation (3-2) in cylindrical coordinate, i.e.,

$$(\lambda + \mu) \frac{\partial}{\partial r} \left( 1 \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right) + \mu \left( \Delta^2 u_r - \frac{u_r}{r^2} - \frac{2}{r^2} \frac{\partial u_\theta}{\partial \theta} \right) + \rho f_r = \rho \frac{\partial^2 u_r}{\partial t^2}$$

$$(\lambda + \mu) \frac{1}{r} \frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right) + \mu \left( \Delta^2 u_\theta - \frac{u_\theta}{r^2} + \frac{2}{r^2} \frac{\partial u_r}{\partial \theta} \right) + \rho f_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2}$$

$$(\lambda + \mu) \frac{\partial}{\partial z} \left( \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{1}{r} \frac{\partial u_\theta}{\partial \theta} + \frac{\partial u_z}{\partial z} \right) + \mu \Delta^2 u_z + \rho f_z = \rho \frac{\partial^2 u_z}{\partial t^2}$$

For axis symmetric problem $\left( \frac{\partial}{\partial \theta} = 0 \right)$, Equation (3-16) becomes

$$(\lambda + \mu) \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} \right) + \mu \left( \Delta^2 u_r - \frac{u_r}{r^2} \right) + \rho f_r = \rho \frac{\partial^2 u_r}{\partial t^2}$$

$$\mu \left( \Delta^2 u_\theta - \frac{u_\theta}{r^2} \right) + \rho f_\theta = \rho \frac{\partial^2 u_\theta}{\partial t^2}$$

(3-17)

$$(\lambda + \mu) \frac{\partial}{\partial z} \left( \frac{1}{r} \frac{\partial (ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} \right) + \mu \Delta^2 u_z + \rho f_z = \rho \frac{\partial^2 u_z}{\partial t^2}$$

Corresponding displacement and force equations from Equations (3-3) and (3-6), for axis symmetric condition,
\[ u_r = \frac{\partial \Phi}{\partial r} - \frac{\partial H_\theta}{\partial z} \]
\[ u_\theta = \frac{\partial H_r}{\partial z} - \frac{\partial H_z}{\partial r} \]
\[ u_z = \frac{\partial \Phi}{\partial z} + \frac{1}{r} \frac{\partial (r H_\theta)}{\partial r} \]  \hspace{2cm} (3-18)

\[ f_r = \frac{\partial A^*}{\partial r} - \frac{\partial B_\theta^*}{\partial z} \]
\[ f_\theta = \frac{\partial B_r^*}{\partial z} - \frac{\partial B_z^*}{\partial r} \]
\[ f_z = \frac{\partial A^*}{\partial z} + \frac{1}{r} \frac{\partial (r B_\theta^*)}{\partial r} \]  \hspace{2cm} (3-19)

Equation (3-16) implies that, it is possible to separate the solution into two parts:

(3) Solution for \( u_r \) and \( u_z \), with excitation force \( f_r \) and \( f_z \), which depend on the potentials \( \Phi, H_\theta, A^*, B_\theta^* \). The solution will be the combination of pressure wave \( P \) represented by potential \( \Phi \) and a shear vertical wave \( SV \) represented by potential \( H_\theta \) due to excitation potentials \( A^*, B_\theta^* \).

(4) Solution for \( u_\theta \) due to excitation force \( f_\theta \) which depend on the potentials \( H_r, H_z, B_r^*, B_z^* \). The solution for \( u_\theta \) displacement will give shear horizontal wave \( SH \) represented by potential \( H_r, H_z \) due to excitation potentials \( B_r^*, B_z^* \).

Corresponding potentials Equations (3-14) and (3-15) become

\[ c_p^2 \text{ div grad } \Phi + A^* = \Phi \]

\[ c_z^2 \text{ curl curl } H_r - B_r^* + \hat{H}_r = 0 \]
\[ c_z^2 \text{ curl curl } H_\theta - B_\theta^* + \hat{H}_\theta = 0 \]
\[ c_z^2 \text{ curl curl } H_z - B_z^* + \hat{H}_z = 0 \]  \hspace{2cm} (3-21)
Now for P+SV waves in a plate

\[ u_r = \frac{\partial \Phi}{\partial r} - \frac{\partial H_\theta}{\partial z} \]
\[ u_z = \frac{\partial \Phi}{\partial z} + \frac{1}{r} \frac{\partial (rH_\theta)}{\partial r} \]  

\( \text{(3-22)} \)

\[ f_r = \frac{\partial A^*}{\partial r} - \frac{\partial B_\theta^*}{\partial z} \]
\[ f_z = \frac{\partial A^*}{\partial z} + \frac{1}{r} \frac{\partial (rB_\theta^*)}{\partial r} \]  

\( \text{(3-23)} \)

\[ u_r, u_z = f(\Phi, H_\theta) \]  

\( \text{(3-24)} \)

\[ f_r, f_z = f(A^*, B_\theta^*) \]  

\( \text{(3-25)} \)

Therefore, for P+SV waves, the relevant potentials are \( \Phi, H_\theta, A, B_\theta \). Equations (3-20) and (3-21) condensed to two equations, i.e.,

\[ c_p^2 \text{ div grad } \Phi + A^* - \ddot{\Phi} = 0 \]  

\( \text{(3-26)} \)

\[ c_s^2 \text{ curl curl } H_\theta - B_\theta^* + \ddot{H}_\theta = 0 \]  

\( \text{(3-27)} \)

Upon rearranging,

\[ \text{ div grad } \Phi + \frac{A^*}{c_p^2} - \frac{\ddot{\Phi}}{c_p^2} = 0 \]  

\( \text{(3-28)} \)

\[ \text{ curl curl } H_\theta - \frac{B_\theta^*}{c_s^2} + \frac{\ddot{H}_\theta}{c_s^2} = 0 \]  

\( \text{(3-29)} \)

The P waves and SV waves give rise to the Lamb waves, which consist of a pattern of standing waves in the thickness \( Z \) direction (Lamb wave modes) behaving like traveling waves in the \( r \) direction in a plate of thickness \( h=2d \).
3.2 Source Localization:

Pressure and shear excitation potentials are assumed to be located at \( z_0 \) distance (Figure 3.1) from the mid-plane (origin).

![Source location](image)

Figure 3.1  Pressure and shear excitation potentials location in thickness wise direction in a plate

Excitation pressure potentials can be assumed as an axis-symmetric point source (Figure 3.2) emitting from location \( O(0, z_0) \) in a volume

\[
A^* = c_p^2 \frac{A}{2\pi} \frac{\delta(r)}{r} \delta(z - z_0)
\]  \quad (3-30)

Here \( A \) is the amplitude of the source.

Excitation shear potential is a point source by limiting the ring source (Figure 3.3) as

\[
B^*_{\theta} = c_s^2 \frac{B_{\theta}}{2\pi} \left[ \lim_{r_0 \to 0} \frac{\delta(r - r_0)}{r_0 r} \delta(z - z_0) \right]
\]  \quad (3-31)

Here \( B_{\theta} \) is the amplitude of the source.
Figure 3.2 Concentrated pressure excitation potentials located at \( O (r = 0, z = z_0) \) in a plate of thickness \( 2d \)

Figure 3.3 Shear excitation potentials from a point source at \( O (r = 0, z = z_0) \) by limiting the ring source of radius \( r_o \) in a plate of thickness \( 2d \)

3.3 Solutions in Terms of Displacement

If the source is vibrating harmonically, then equations for pressure and shear excitation potentials are

\[
A^* = c_p^2 \frac{A}{2\pi} \frac{\delta(r)}{r} \delta(z-z_0)e^{-i\omega t} \tag{3-32}
\]

\[
B_{\theta}^* = c_s^2 \frac{B_{\theta}}{2\pi} \lim_{r_o \to 0} \frac{\delta(r-r_o)}{r} \delta(z-z_0)e^{-i\omega t} \tag{3-33}
\]

If the potentials \( \Phi, H_{\theta} \), has the harmonic response then
\[ \Phi(r, z, t) = \Phi(x, y)e^{-i\omega t} \]
\[ H_\theta(r, z, t) = H_\theta(x, y)e^{-i\omega t} \]  

Equations (3-28) and (3-29) become

\[
\text{div grad } \Phi + \frac{\omega^2}{c_p^2} \Phi = -\frac{A}{2\pi} \frac{\delta(r)}{r} \delta(z - z_0) \tag{3-35}
\]

\[
\text{curl curl } H_\theta - \frac{\omega^2}{c_s^2} H_\theta = \frac{B_\theta}{2\pi} \left[ \lim_{r_o \to 0} \frac{\delta(r - r_o)}{r_o r} \delta(z - z_0) \right] \tag{3-36}
\]

In expanded form of Equations (3-35) and (3-36) are

\[
\frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial z^2} + \frac{\omega^2}{c_p^2} \frac{\partial \Phi}{\partial t} = -\frac{A}{2\pi} \frac{\delta(r)}{r} \delta(z - z_0) \tag{3-37}
\]

\[
\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rH_\theta)}{\partial r} \right) + \frac{\partial^2 H_\theta}{\partial z^2} + \frac{\omega^2}{c_s^2} H_\theta = -\frac{B_\theta}{2\pi} \left[ \lim_{r_o \to 0} \frac{\delta(r - r_o)}{r_o r} \delta(z - z_0) \right] \tag{3-38}
\]

Here,

\[
\text{div grad } \Phi = \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial z^2} \tag{3-39}
\]

\[
\text{curl curl } H_\theta = -\frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rH_\theta)}{\partial r} \right) + \frac{\partial^2 H_\theta}{\partial z^2} \tag{3-40}
\]

Equations (3-37) and (3-38) must be solved to the boundary conditions

\[
\sigma_{zz} \bigg|_{z=\pm d} = 0 ; \quad \sigma_{rz} \bigg|_{z=\pm d} = 0 \tag{3-41}
\]

Recall the Hankel transform of order \( \nu \)

\[
(\tilde{f}(\xi))_{\nu} = \int_{0}^{\infty} f(r) r J_\nu(\xi r)dr \tag{3-42}
\]
\[
f(r) = \int_{0}^{\infty} (\tilde{f}(\xi))_{r_{2}} \tilde{\xi}^{\nu} J_{\nu}(\tilde{\xi} r) d\tilde{\xi}
\]

(3-43)

Here, \( f(r) \) is a generic function and \( J_{\nu} \) is Bessel function of order \( \nu \).

\[
J_{\nu}(x) = \left( \frac{1}{2} \right)^{\nu} \sum_{k=0}^{\infty} \frac{(-1)^{k} x^{2k}}{k! \Gamma(\nu + k + 1)}
\]

(3-44)

Apply the Hankel transform of order 0 to Equation (3-37)

\[
\begin{align*}
\int_{0}^{\infty} & \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) r J_{0}(\xi r) dr + \int_{0}^{\infty} \frac{\partial^{2} \Phi}{\partial \xi^{2}} r J_{0}(\xi r) dr - \int_{0}^{\infty} \frac{\omega^{2}}{c^{2}_{p}} r J_{0}(\xi r) dr \\
& = -\int_{0}^{\infty} \frac{A}{2\pi} \frac{\delta(r)}{r} \delta(z - z_{0}) r J_{0}(\xi r) dr
\end{align*}
\]

(3-45)

Using Hankel transform identity from ref. (Giurgiutiu, 2014; pp. 608,609) and Dirac-delta identity, the integrals from Equation (3-45) are

\[
\begin{align*}
\int_{0}^{\infty} & \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) r J_{0}(\xi r) dr = -\xi^{2} \Phi_{J_{0}} \\
\int_{0}^{\infty} \Phi r J_{0}(\xi r) dr & = \Phi_{J_{0}} \\
\int_{0}^{\infty} \frac{\partial^{2} \Phi}{\partial \xi^{2}} r J_{0}(\xi r) dr & = \frac{\partial^{2} \Phi_{J_{0}}}{\partial \xi^{2}} \\
\int_{0}^{\infty} \frac{\omega^{2}}{c^{2}_{p}} \Phi r J_{0}(\xi r) dr & = \frac{\omega^{2}}{c^{2}_{p}} \Phi_{J_{0}} \\
\int_{0}^{\infty} \frac{A}{2\pi} \frac{\delta(r)}{r} \delta(z - z_{0}) r J_{0}(\xi r) dr & = \frac{A}{2\pi} \delta(z - z_{0}) J_{0}(0) = \frac{A}{2\pi} \delta(z - z_{0})
\end{align*}
\]

(3-46) (3-47) (3-48) (3-49) (3-50)
Here, \( J_0(0) = 1 \)

Therefore, Equation (3-45) becomes

\[
-\xi^2 \Phi_{10} + \frac{\partial^2 \Phi_{10}}{\partial z^2} + \frac{\omega^2}{c_p^2} \Phi_{10} = -\frac{A}{2\pi} \delta(z - z_0)
\]  

(3-51)

Upon rearrangement

\[
\frac{\partial^2 \Phi_{10}}{\partial z^2} + \left(\frac{\omega^2}{c_p^2} - \xi^2\right) \Phi_{10} = -\frac{A}{2\pi} \delta(z - z_0)
\]  

(3-52)

Apply the Hankel transform of order 1 to Equation (3-38)

\[
\int_0^\infty \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \left(r \Phi_{10}(\xi)\right)}{\partial r} \right) r J_1(\xi r) dr + \int_0^\infty \frac{\partial^2 H_\theta}{\partial z} r J_1(\xi r) dr - \int_0^\infty \frac{\omega^2}{c_p^2} r J_1(\xi r) dr = -\int_0^\infty \frac{B_\theta}{2\pi} \lim_{\xi \to 0} \frac{\delta(r - r_0)}{r} \delta(z - z_0) \ r J_1(\xi r) dr
\]  

(3-53)

Using Hankel transform identity from ref. (Giurgiutiu, 2014; pp. 609,610) and Dirac-delta identity, the integrals from Equation (3-53) are

\[
\int_0^\infty \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial \left(r H_{\theta1}(\xi)\right)}{\partial r} \right) r J_1(\xi r) dr = -\xi^2 \tilde{H}_{\theta1}
\]  

(3-54)

\[
\int_0^\infty \frac{\partial^2 H_\theta}{\partial z^2} r J_1(\xi r) dr = \frac{\partial^2 \tilde{H}_{\theta1}}{\partial z^2}
\]  

(3-55)

\[
\int_0^\infty \frac{\partial^2 H_\theta}{\partial z^2} r J_1(\xi r) dr = \frac{\partial^2 \tilde{H}_{\theta1}}{\partial z^2}
\]  

(3-56)

\[
\int_0^\infty \frac{\omega^2}{c_s^2} H_\theta r J_1(\xi r) dr = \frac{\omega^2}{c_s^2} \tilde{H}_{\theta1}
\]  

(3-57)
\[
\int_0^\infty \frac{B_\theta}{2\pi} \left[ \lim_{r_0 \to 0} \frac{\delta(r-r_0)}{r_0 r} \delta(z-z_0) \right] r J_1(\xi r) dr
\]
\[
= \frac{B_\theta}{2\pi} \delta(z-z_0) \lim_{r_0 \to 0} J_1(\xi r_0) = \frac{B_\theta}{2\pi} \delta(z-z_0) \frac{\xi}{2}
\]

Here,
\[
\lim_{r_0 \to 0} J_1(\xi r_0) = \frac{\xi}{2}
\]

Hence, Equation (3-53) becomes
\[
-\xi^2 \ddot{H}_{\theta_1} + \frac{\partial^2 \ddot{H}_{\theta_1}}{\partial z^2} + \frac{\omega^2}{c_s^2} \ddot{H}_{\theta_1} = -\frac{B_\theta}{4\pi} \xi \delta(z-z_0)
\]

Upon rearrangement,
\[
\frac{\partial^2 \ddot{H}_{\theta_1}}{\partial z^2} + \left( \frac{\omega^2}{c_s^2} - \xi^2 \right) \ddot{H}_{\theta_1} = -\frac{B_\theta}{4\pi} \xi \delta(z-z_0)
\]

Equations (3-52) and (3-60) can be simplified as,
\[
\ddot{\Phi}_{p_0}(\xi, y) + \xi^2 \ddot{\Phi}_{p_0}(\xi, y) = -\frac{A}{2\pi} \delta(z-z_0)
\]
\[
\ddot{\Phi}_{p_1}(\xi, y) + \xi^2 \ddot{\Phi}_{p_1}(\xi, y) = -\frac{B_\theta}{4\pi} \xi \delta(z-z_0)
\]

Here,
\[
\begin{pmatrix}
\frac{\omega^2}{c_p^2} - \xi^2 \\
\frac{\omega^2}{c_s^2} - \xi^2
\end{pmatrix}
= \begin{pmatrix}
\xi_p^2 \\
\xi_s^2
\end{pmatrix}
\]
Equations (3-61) and (3-62) are the second order ordinary differential equation (ODE) in \( z \) direction. The total solution of Equations (3-52) and (3-60) consists of representation of two solutions (Jensen, 2011):

(c) The complimentary solution for the homogenous equation

(d) A particular solution of that satisfies source effect from right hand side

Solution can be assumed as

\[
\tilde{\Phi}_{J_0} (\xi, z) = \tilde{\Phi}_0 (\xi, z) + \tilde{\Phi}_1 (\xi, z) \quad (3-64)
\]

\[
\tilde{H}_{\theta_1} (\xi, z) = i \left( \tilde{H}_{\theta_0} (\xi, z) + \tilde{H}_{\theta_1} (\xi, z) \right) \quad (3-65)
\]

For simplicity, subscript \( J_0 \) and \( J_1 \) are omitted from \( \tilde{\Phi}_0, \tilde{\Phi}_1 \) and \( \tilde{H}_{\theta_0}, \tilde{H}_{\theta_1} \).

The functions \( \tilde{\Phi}_0 \) and \( \tilde{H}_{\theta_0} \) satisfy the corresponding homogeneous differential equations

\[
\tilde{\Phi}_0'' (\xi, z) + \xi^2 \tilde{\Phi}_0 (\xi, z) = 0 \quad (3-66)
\]

\[
\tilde{H}_{\theta_0}'' (\xi, z) + \xi^2 \tilde{H}_{\theta_0} (\xi, z) = 0 \quad (3-67)
\]

Equations (3-66) and (3-67) gives the complementary solutions, i.e.,

\[
\tilde{\Phi}_0 (\xi, \zeta) = C_1 \sin \zeta_\theta z + C_2 \cos \zeta_\theta z \quad (3-68)
\]

\[
\tilde{H}_{\theta_0} (\xi, \zeta) = D_1 \sin \zeta_\theta z + D_2 \cos \zeta_\theta z \quad (3-69)
\]

The function \( \tilde{\Phi}_1 (\xi, y) \) and \( \tilde{H}_{\theta_1} (\xi, y) \) from Equations (3-64) and (3-65) satisfy the corresponding inhomogeneous equations

\[
\tilde{\Phi}_1'' (\xi, z) + \xi^2 \tilde{\Phi}_1 (\xi, z) = -\frac{A}{2\pi} \delta (z - z_0) \quad (3-70)
\]
\[ \dot{H}_0^*(\xi, z) + \xi^2 \ddot{H}_0^*(\xi, z) = -\frac{B_0}{4\pi} \xi \delta(z - z_0) \quad (3-71) \]

Here, \( \Phi_1(\xi, \eta) \) and \( \ddot{H}_{el}(\xi, \eta) \) are the depth dependent function. The solution of Equation (3-70) is obtained using Fourier transform in \( z \) direction, i.e.,

\[ -\xi^2 \ddot{\Phi}_1(\xi, \zeta) + \zeta^2 \ddot{\Phi}_1(\xi, \zeta) = -\frac{A e^{-i\xi z_0}}{2\pi} \quad (3-72) \]

Here, \( A e^{-i\xi z_0} = \int_{-\infty}^{\infty} A(z) \delta(z - z_0) e^{-i\xi z} \, dz \)

\[ \ddot{\Phi}(\xi, \eta) = \frac{A}{2\pi} \frac{1}{\xi^2 - \zeta_p^2} e^{-i\xi z_0} \quad (3-73) \]

The solution of Equation (3-71) is obtained in a similar way

\[ \ddot{H}_{el}(\xi, \zeta) = \frac{B_0}{4\pi} \frac{1}{\xi^2 - \zeta_s^2} e^{-i\xi z_0} \quad (3-74) \]

In order to get the solution in \( z \) domain, taking inverse Fourier transform of Equations (3-73) and (3-74), i.e.,

\[ \ddot{\Phi}_1(\xi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{A}{2\pi(\xi^2 - \zeta_p^2)} e^{-i\xi z_0} e^{i\zeta \xi} \, d\zeta \quad (3-75) \]

\[ \ddot{H}_{el}(\xi, z) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\xi B_0}{4\pi(\xi^2 - \zeta_s^2)} e^{-i\xi z_0} e^{i\zeta \xi} \, d\zeta \quad (3-76) \]

Upon rearrangement

\[ \ddot{\Phi}_1(\xi, z) = \frac{A}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{\xi^2 - \zeta_p^2} e^{i\xi(z - z_0)} \, d\zeta \quad (3-77) \]
\[ \tilde{H}_{\theta_1}(\xi, z) = \frac{\xi B_0}{8\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \xi^2} e^{i\xi(z-z_0)} d\xi \]  

(3.78)

The evaluation of the integral in Equations (3.77) and (3.78) can be done by the residue theorem using a close contour (Watanabe, 2014; Haider and Giurgiuțiu, 2017e; Remmert, 2012; Cohen, 2010; Krantz, 2007). Since the poles are on the real axis the integral equal to

\[ \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \xi^2} e^{i\xi(z-z_0)} d\xi = \pi i \sum_{\text{real axis}} \text{Res} (f(\xi); \xi_j) \]  

(3.79)

\[ \text{Res} (f(\xi)) = (\xi - \xi_j) f(\xi_j) \]  

(3.80)

Here \( \xi_j \) are the poles of \( \xi \) on real axis. Poles of \( \xi \) for Equation (3.77) are

\[ \xi = \pm \xi_p \]  

(3.81)

To evaluate the integration of Equation (3.77) two cases are considered as follow,

Case 1: \( z - z_0 > 0 \)

\[ \Phi_1(\xi, y) = \frac{A}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \xi^2} e^{i\xi(z-z_0)} d\xi \]  

(3.82)
To evaluate the contour integration an arc at the infinity on the upper half plane (Figure 3.4) is added because the function converges in upper half circle.

According to residue theorem the integrant is,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{i\zeta(z-z_o)} d\zeta = \pi i \sum_{\text{real axis}} \text{Res} \left( f(\zeta); \zeta_j \right)$$  \hspace{1cm} (3-83)

Here $\zeta_j$ are the poles of $f(\zeta) = \frac{1}{\zeta^2 - \zeta_p^2} e^{-i\zeta_0} e^{i\zeta}$

Using the poles, residue theorem gives,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{i\zeta(z-z_o)} d\zeta = \pi i \left[ \frac{1}{(\zeta - \zeta_p)(\zeta - \zeta_p)} e^{i\zeta(z-z_o)} \right]_{\zeta = \zeta_p} + \pi i \left[ \frac{1}{(\zeta - \zeta_p)(\zeta - \zeta_p)} e^{i\zeta(z-z_o)} \right]_{\zeta = -\zeta_p}$$  \hspace{1cm} (3-84)

Upon rearrangement

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{i\zeta(z-z_o)} d\zeta = \frac{-\pi}{\eta_p} \left( \frac{e^{i\zeta_p(z-z_o)}}{2i} - \frac{e^{-i\zeta_p(z-z_o)}}{2i} \right)$$  \hspace{1cm} (3-85)

Using Euler formula in Equation (3-85), i.e.,

$$\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{i\zeta(z-z_o)} d\zeta = -\frac{\pi}{\eta_p} \sin \zeta_p (z - z_o)$$  \hspace{1cm} (3-86)

**Case 2:** $z - z_o < 0$

For $z - z_o < 0 \ ; \ z - z_o = -|z - z_o|$

$$\Phi_j(\zeta, y) = \frac{A}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{-i|\zeta - z_o|} d\zeta$$  \hspace{1cm} (3-87)
This time an arc at the infinity on the upper half plane cannot be added because the function does not converge in upper half circle. To evaluate the contour integration of above function an arc at the infinity on the lower half plane (Figure 3.5) can be added because the function converges in lower half circle.

![Diagram](image)

Figure 3.5 Close contour for evaluating the inverse Fourier transform by residue theorem for $y < 0$

\[
\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{-i|\zeta|} d\zeta = -\pi i \sum_{\text{real axis}} \text{Res} \left( f(\zeta); \zeta_p \right) \tag{3-88}
\]

Using the poles, residue theorem gives,

\[
\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{-i|\zeta|} d\zeta = -\pi i \left[ \left( \zeta - \zeta_p \right) \frac{1}{(\zeta + \zeta_p)(\zeta - \zeta_p)} e^{-i|\zeta|} \right]_{\zeta = \zeta_p}

-\pi i \left[ \frac{1}{(\zeta + \zeta_p)(\zeta - \zeta_p)} e^{-i|\zeta|} \right]_{\zeta = -\zeta_p} \tag{3-89}
\]

Upon rearrangement,

\[
\int_{-\infty}^{\infty} \frac{1}{\zeta_p^2 - \zeta^2} e^{-i|\zeta|} d\zeta = -\pi \frac{e^{i\zeta_p|\zeta_0|}}{2i} \left( \frac{1}{2i} \right) \tag{3-90}
\]

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Using Euler formula in Equation (3-90), i.e.,

\[
\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{-i\zeta(z-z_o)} d\zeta = -\frac{\pi}{\zeta_p} \sin \zeta_p |z-z_o| \tag{3-91}
\]

By observing both cases the following summary can be made:

For \( y > 0 \)

\[
\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{i\zeta(z-z_o)} d\zeta = -\frac{\pi}{\zeta_p} \sin \zeta_p |z-z_o| \tag{3-92}
\]

For \( y < 0 \)

\[
\int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{-i\zeta(z-z_o)} d\zeta = -\frac{\pi}{\zeta_p} \sin \zeta_p |z-z_o| \tag{3-93}
\]

Therefore, whether, \( z-z_o \) is positive or negative, the contour integral result is same.

Integration formula for Equation (3-77) can be written as

\[
\Phi_1(\xi, z) = \frac{A}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{i\zeta(z-z_o)} d\zeta = -\frac{A}{4\pi\zeta_p} \sin \zeta_p |z-z_o| \tag{3-94}
\]

Similarly from Equation (3-78), the evaluation of the integral is as follow,

\[
\tilde{H}_{\theta_1}(\xi, z) = \frac{\xi B_\theta}{8\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_s^2} e^{i\zeta(z-z_o)} d\zeta = -\frac{\xi B_\theta}{8\pi\zeta_s} \sin \zeta_s |z-z_o| \tag{3-95}
\]

Therefore, integration formula for Equations (3-77) and (3-78) are

\[
\Phi_1(\xi, z) = \frac{A}{4\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_p^2} e^{i\zeta(z-z_o)} d\zeta = -\frac{A}{4\pi\zeta_p} \sin \zeta_p |z-z_0| \tag{3-96}
\]

\[
\tilde{H}_{\theta_1}(\xi, z) = \frac{\xi B_\theta}{8\pi^2} \int_{-\infty}^{\infty} \frac{1}{\zeta^2 - \zeta_s^2} e^{i\zeta(z-z_o)} d\zeta = -\frac{\xi B_\theta}{8\pi\zeta_s} \sin \zeta_s |z-z_0| \tag{3-97}
\]
Equations (3-96), (3-97) represent the particular solution. The total solution is superposition of the complimentary solution Equations (3-68), (3-69) and particular solution Equations (3-96), (3-97), i.e.,

\[ \tilde{\Phi}_j (\xi, z) = C_1 \sin \zeta_p z + C_2 \cos \zeta_p z - \frac{A}{4\pi \xi_p} \sin \zeta_p |z - z_0| \]  
(3-98)

\[ \tilde{H}_{\theta j} (\xi, z) = D_1 \sin \zeta_s z + D_2 \cos \zeta_s z - \frac{\xi B_\theta}{8\pi \zeta_s} \sin \zeta_s |z - z_0| \]  
(3-99)

The coefficients \( C_1, C_2, D_1, D_2 \) from Equations (3-98) and (3-99) are to be found from the Hankel transforms of the boundary conditions Equation (3-41), i.e.,

\[ (\tilde{\sigma}_{zz})_j |_{z=\pm d} = 0; \quad (\tilde{\sigma}_{rz})_j |_{z=\pm d} = 0 \]  
(3-100)

Here \((\tilde{\sigma}_{zz})_j\) is the Hankel transform of order 0 of \( \tilde{\sigma}_{zz} \) and \((\tilde{\sigma}_{rz})_j\) is the Hankel transform of order 1 of \( \tilde{\sigma}_{rz} \). It should be noted that, the total field Equations (3-98) and (3-99) must be used when satisfying the boundary conditions.

The differentiation properties of Equations (3-98), (3-99) are,

\[ \frac{\partial (\tilde{\Phi}_j (\xi, \zeta))}{\partial z} = C_1 \zeta_p \cos \zeta_p z - C_2 \zeta_p \sin \zeta_p z - \frac{A}{4\pi \xi_p} \cos \zeta_p |z - z_0| \]  
(3-101)

\[ \frac{\partial^2 (\tilde{\Phi}_j (\xi, \zeta))}{\partial z^2} = - \zeta_p^2 C_1 \sin \zeta_p z - \zeta_p^2 C_2 \cos \zeta_p z + \frac{A}{4\pi \xi_p} \sin \zeta_p |z - z_0| \]  
(3-102)

\[ \frac{\partial (\tilde{H}_{\theta j} (\xi, \zeta))}{\partial z} = \zeta_s D_1 \cos \zeta_s z - \zeta_s D_2 \sin \zeta_s z - \frac{\xi B_\theta}{8\pi \eta_s} \cos \zeta_s |z - z_0| \]  
(3-103)
\[
\frac{\partial^2 (\vec{H}_{\theta_{11}}(\xi, \zeta))}{\partial \zeta^2} = -\zeta^2 D_1 \sin \zeta z - \zeta^2 D_2 \cos \zeta z + \frac{\xi B_\theta}{8\pi \eta_z} \sin \zeta |z - z_0| \quad (3-104)
\]

\[
= -\zeta^2 \vec{H}_{\theta_{11}}(\xi, \zeta)
\]

Axis-symmetric stress equations in cylindrical coordinates are

\[
\sigma_{zz} = \lambda \text{div} \, \bar{u} + 2\mu \frac{\partial u_r}{\partial r} \quad (3-105)
\]

\[
\sigma_{rz} = \mu \left( \frac{\partial u_z}{\partial r} + \frac{\partial u_r}{\partial z} \right) \quad (3-106)
\]

Here,

\[
div \, \bar{u} = div \left( \text{grad} \, \Phi + curl \, \vec{H} \right) = div \, \text{grad} \, \Phi \quad (3-107)
\]

Using Equations (3-22), (3-107), (3-39) into Equations (3-105) and (3-106) yield

\[
\sigma_{zz} = \lambda \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) + \frac{\partial^2 \Phi}{\partial z^2} \right] + 2\mu \frac{\partial^2 \Phi}{\partial z^2} + 2\mu \frac{1}{r} \frac{\partial^2 (rH_{\theta})}{\partial z \partial r} \quad (3-108)
\]

\[
\sigma_{rz} = \mu \left( \frac{2}{r} \frac{\partial^2 \Phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rH_{\theta})}{\partial r} \right) - \frac{\partial^2 H_{\theta}}{\partial z^2} \right) \quad (3-109)
\]

Applying Hankel transform of order 0 to \( \sigma_{zz} \), Equation (3-108), yields \( \left( \tilde{\sigma}_{zz} \right)_0 \), i.e.,

\[
\left( \tilde{\sigma}_{zz} \right)_0 = \lambda \int_0^\infty \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) r J_0(\xi r) dr \\
+ (\lambda + 2\mu) \int_0^\infty \frac{\partial^2 \Phi}{\partial z^2} r J_0(\xi r) dr + 2\mu \int_0^\infty \frac{1}{r} \frac{\partial^2 (rH_{\theta})}{\partial z \partial r} r J_0(\xi r) dr \quad (3-110)
\]

Calculating the individual terms of Equation (3-110) using ref. (Giurgiutiu, 2014 pp. 611-612)
\[
\int_{0}^{\infty} \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \Phi}{\partial r} \right) r J_0(\xi r) dr = -\xi^2 \Phi J_0
\]  \hspace{1cm} (3-111)

\[
\int_{0}^{\infty} \frac{\partial^2 \Phi}{\partial z^2} r J_0(\xi r) dr = \frac{\partial^2 \Phi J_0}{\partial z^2}
\]  \hspace{1cm} (3-112)

\[
\int_{0}^{\infty} \frac{1}{r} \frac{\partial^2 (rH_{\theta})}{\partial \xi \partial r} r J_0(\xi r) dr = \xi \frac{\partial H_{\theta}}{\partial \xi}
\]  \hspace{1cm} (3-113)

Substituting of Equations (3-111), (3-112), (3-113) into Equation (3-110) yields

\[
(\tilde{\sigma}_{zz})_{1_0} = -\lambda \xi^2 \Phi J_0 + (\lambda + 2\mu) \frac{\partial^2 \Phi J_0}{\partial z^2} + 2\mu \xi \frac{\partial H_{\theta}}{\partial z}
\]  \hspace{1cm} (3-114)

Applying Hankel transform of order 1 to \( \sigma_{rz} \), Equation (3-109), yields \( (\tilde{\sigma}_{rz})_{1_1} \), i.e.,

\[
\sigma_{rz} = \mu \left( 2 \frac{\partial^2 \Phi}{\partial r \partial z} + \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rH_{\theta})}{\partial r} \right) - \frac{\partial^2 H_{\theta}}{\partial z^2} \right)
\]  \hspace{1cm} (3-115)

Calculating the individual terms of Equation (3-115) using from ref. (Giurgiuțiu, 2014; pp. 611-612)

\[
(\tilde{\sigma}_{rz})_{1_1} = \mu \left\{ 2 \int_{0}^{\infty} \frac{\partial^2 \Phi}{\partial r \partial z} r J_1(\xi r) dr + \int_{0}^{\infty} \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial (rH_{\theta})}{\partial r} \right) r J_1(\xi r) dr \right\}
\]  \hspace{1cm} (3-116)

\[
- \int_{0}^{\infty} \frac{\partial^2 H_{\theta}}{\partial z^2} r J_1(\xi r) dr
\]

\[
\int_{0}^{\infty} \frac{\partial^2 \Phi}{\partial r \partial z} r J_1(\xi r) dr = -\xi \frac{\partial \Phi J_0}{\partial z}
\]  \hspace{1cm} (3-117)
Substituting of Equations (3-117), (3-118), (3-119) into Equation (3-116) yields

\[
(\hat{\sigma}_{zz})_1 = -2\mu\hat{\varepsilon}\frac{\partial J_0}{\partial z} - \mu\hat{\varepsilon}^2\hat{H}_{\theta 1} - \mu\frac{\partial^2 \hat{H}_{\theta 1}}{\partial^2 z}
\]  

(3-120)

Using Equations (3-98), (3-102), (3-103) into Equation (3-114) gives,

\[
(\hat{\sigma}_{zz})_0 = -\lambda\hat{\varepsilon}^2 \left( C_1 \sin \zeta_p z + C_2 \cos \zeta_p z - \frac{A}{4\pi\hat{\varepsilon}_p} \sin \zeta_p |z - z_0| \right)
\]

\[-\hat{\varepsilon}_p^2(\lambda + 2\mu) \left( C_1 \sin \zeta_p z + C_2 \cos \zeta_p z - \frac{A}{4\pi\hat{\varepsilon}_p} \sin \zeta_p |z - z_0| \right)
\]

\[+2\mu\hat{\varepsilon} \left( \zeta_s D_1 \cos \zeta_s z - \zeta_s D_2 \sin \zeta_s z - \frac{\hat{\xi} B_\theta}{8\pi\hat{\varepsilon}_s} \cos \zeta_s |z - z_0| \right)
\]

(3-121)

Upon rearrangement,

\[
(\hat{\sigma}_{zz})_0 = -\left( \lambda\hat{\varepsilon}^2 + \hat{\varepsilon}_p^2(\lambda + 2\mu) \right) \left( C_1 \sin \zeta_p z + C_2 \cos \zeta_p z - \frac{A}{4\pi\hat{\varepsilon}_p} \sin \zeta_p |z - z_0| \right)
\]

\[+2\mu\hat{\varepsilon} \left( \zeta_s D_1 \cos \zeta_s z - \zeta_s D_2 \sin \zeta_s z - \frac{\hat{\xi} B_\theta}{8\pi\hat{\varepsilon}_s} \cos \zeta_s |z - z_0| \right)
\]

(3-122)

Here,

\[
\left\{ \lambda\hat{\varepsilon}^2 + \hat{\varepsilon}_p^2(\lambda + 2\mu) \right\} = -\mu(\hat{\varepsilon}_z^2 - \hat{\varepsilon}_s^2)
\]

(3-123)

Hence, Equation (3-122) becomes
\[
\frac{1}{\mu} \left( \frac{\partial\sigma_{zz}}{\partial z} \right)_0 = \left( \xi^2 - \xi_s^2 \right) \left( C_1 \sin \zeta_p z + C_2 \cos \zeta_p z - \frac{A}{4\pi\zeta_p} \sin \zeta_p |z - z_0| \right)
\]
(3-124)

\[
+ 2\xi \left( \xi_s D_1 \cos \zeta_s z - \xi_s D_2 \sin \zeta_s z - \frac{\xi B_\theta}{8\pi} \cos \zeta_s |z - z_0| \right)
\]

\[
\left( \sigma_{rz} \right)_1 = -2\mu \xi \frac{\partial \Phi_{f0}}{\partial z} - \mu \xi^2 \tilde{H}_{\theta1} - \mu \frac{\partial^2 \tilde{H}_{\theta1}}{\partial^2 z}
\]
(3-125)

Using Equations (3-99), (3-101), (3-104) into Equations (3-120) gives,

\[
\left( \sigma_{rz} \right)_1 = -2\mu \xi \left( C_1 \xi_p \cos \zeta_p z - C_2 \xi_p \sin \zeta_p z - \frac{A}{4\pi\zeta_p} \cos \zeta_p |z - z_0| \right)
\]

\[- \mu \xi^2 \left( D_1 \sin \zeta_s z + D_2 \cos \zeta_s z - \frac{\xi B_\theta}{8\pi\zeta_s} \sin \zeta_s |z - z_0| \right) \]

\[+ \mu \xi \frac{\partial \Phi_{f0}}{\partial z} - \mu \xi^2 \tilde{H}_{\theta1} - \mu \frac{\partial^2 \tilde{H}_{\theta1}}{\partial^2 z}
\]
(3-126)

Upon rearrangement

\[
\frac{1}{\mu} \left( \frac{\partial\sigma_{zz}}{\partial z} \right)_1 = -2\xi \left( C_1 \xi_p \cos \zeta_p z - C_2 \xi_p \sin \zeta_p z - \frac{A}{4\pi\zeta_p} \cos \zeta_p |z - z_0| \right)
\]

\[- \xi^2 \left( D_1 \sin \zeta_s z + D_2 \cos \zeta_s z - \frac{\xi B_\theta}{8\pi\zeta_s} \sin \zeta_s |z - z_0| \right) \]
(3-127)

Applying boundary conditions Equation (3-100) to Equations (3-126) and (3-127)

\[
\left( \sigma_{zz} \right)_0 \bigg|_{z=d} = 0
\]

\[
\left( \xi^2 - \xi_s^2 \right) \left( C_1 \sin \zeta_p z + C_2 \cos \zeta_p z - \frac{A}{4\pi\zeta_p} \sin \zeta_p |z - z_0| \right)
\]

\[+ 2\xi \left( \xi_s D_1 \cos \zeta_s z - \xi_s D_2 \sin \zeta_s z - \frac{\xi B_\theta}{8\pi} \cos \zeta_s |z - z_0| \right) = 0
\]
(3-128)
\[
(\overline{\sigma}_{zz})_0 \bigg|_{z=-d} = 0
\]

\[
\left(\xi^2 - \zeta_s^2\right) \left(-C_1 \sin \zeta_p d + C_2 \cos \zeta_p d - \frac{A}{4\pi \zeta_p} \sin \zeta_p |d - z_0|\right) + 2\xi \left(\zeta_s D_1 \cos \zeta_s d + \zeta_s D_2 \sin \zeta_s d - \frac{\xi B_0}{8\pi} \cos \zeta_s |d - z_0|\right) = 0
\]

\[
(\overline{\sigma}_{rz})_1 \bigg|_{z=-d} = 0
\]

\[
-2\xi \left(C_1 \zeta_p \cos \zeta_p d - C_2 \zeta_p \sin \zeta_p d - \frac{A}{4\pi} \cos \zeta_p |d - z_0|\right) - (\xi^2 - \zeta_s^2) \left(D_1 \sin \zeta_s d + D_2 \cos \zeta_s d - \frac{\xi B_0}{8\pi \zeta_s} \sin \zeta_s |d - z_0|\right) = 0
\]

\[
(\overline{\sigma}_{rz})_1 \bigg|_{z=-d} = 0
\]

\[
-2\xi \left(C_1 \zeta_p \cos \zeta_p d + C_2 \zeta_p \sin \zeta_p d - \frac{A}{4\pi} \cos \zeta_p |d - z_0|\right) - (\xi^2 - \zeta_s^2) \left(-D_1 \sin \zeta_s d + D_2 \cos \zeta_s d - \frac{\xi B_0}{8\pi \zeta_s} \sin \zeta_s |d - z_0|\right) = 0
\]

Equations (3-128)-(3-131) are a set of four equations with four unknowns. The equations can be separated into a couple of two equations with two unknowns, one for symmetric motion and one for anti-symmetric motion.

**3.4 Symmetric Lamb wave solution**

Addition of the Equations (3-128) and (3-129), and subtraction of the Equation (3-131) from Equation (3-130), yield
\[
\begin{bmatrix}
(\xi^2 - \eta^2) \cos \zeta_p d + 2\xi \zeta_s \cos \zeta_s d \\
-2\xi \zeta_p \sin \zeta_p d + (\xi^2 - \eta^2) \sin \zeta_s d
\end{bmatrix}
\begin{bmatrix}
C_2 \\
D_1
\end{bmatrix}
= \begin{bmatrix}
(\xi^2 - \eta^2) \frac{A}{4\pi \zeta_p} \sin \zeta_p |d - z_0| + \\
\xi^2 \left( \frac{B_0}{4\pi} \right) \cos \zeta_s |d - z_0|
\end{bmatrix}
\]

Equation (3-132) represents an algebraic system that can be solved for \( C_2, D_1 \) provided the system determinant does not vanish

\[
D_1 = \begin{vmatrix}
(\xi^2 - \xi_s^2) \cos \zeta_p d + 2\xi \zeta_s \cos \zeta_s d \\
-2\xi \zeta_p \sin \zeta_p d + (\xi^2 - \xi_s^2) \sin \zeta_s d
\end{vmatrix} \neq 0
\]

Let,

\[
P_S = (\xi^2 - \xi_s^2) \frac{A}{4\pi \zeta_p} \sin \zeta_p |d - z_0| + \xi^2 \left( \frac{B_0}{4\pi} \right) \cos \zeta_s |d - z_0|
\]

\( P_S \) is the source term for symmetric solution which contains source potentials \( A \) and \( B_0 \).

Upon substitution of \( d_1 = d - z_0 \) into Equation (3-133)

\[
P_S = (\xi^2 - \xi_s^2) \frac{A}{4\pi \zeta_p} \sin \zeta_p d_1 + \xi^2 \left( \frac{B_0}{4\pi} \right) \cos \zeta_s d_1
\]

Here \( d_1 \) is the depth of source.

Then,

\[
\begin{bmatrix}
C_2 \\
D_1
\end{bmatrix}
= \begin{bmatrix}
(\xi^2 - \xi_s^2) \sin \zeta_s d + -2\xi \eta_s \cos \zeta_s d \\
2\xi \zeta_p \sin \zeta_p d + (\xi^2 - \xi_s^2) \cos \zeta_s d
\end{bmatrix}
\begin{bmatrix}
P_S \\
0
\end{bmatrix}
\]

Upon rearrangement
\[
\begin{bmatrix}
C_2 \\
D_1 \\
\end{bmatrix}
= \frac{1}{D_i}
\begin{bmatrix}
P_s (\xi^2 - \zeta_i^2) \sin \zeta_s d \\
P_s 2 \xi \zeta_p \sin \zeta_p d \\
\end{bmatrix}
\] (3-136)

Here,
\[
D_s = \begin{vmatrix}
(\xi^2 - \zeta_i^2) \cos \zeta_p d & 2 \xi \eta_i \cos \zeta_i d \\
-2 \xi \zeta_p \sin \zeta_p & (\xi^2 - \zeta_i^2) \sin \zeta_i d \\
\end{vmatrix}
\] (3-137)

By equating to zero the \( D_s (\xi) \) term of Equation (3-137), one gets the symmetric Rayleigh-Lamb wave equation.

3.5 **Anti-symmetric Lamb wave solution**

Subtraction of the Equation (3-128) from (3-129), and addition of the Equations (3-130) and (3-131) yield
\[
\begin{bmatrix}
(\xi^2 - \zeta_i^2) \sin \zeta_p d & -2 \xi \zeta_p \sin \zeta_i d \\
2 \xi \zeta_p \cos \zeta_p d & (\xi^2 - \zeta_i^2) \cos \zeta_i d \\
\end{bmatrix}
\begin{bmatrix}
C_1 \\
D_2 \\
\end{bmatrix}
= \begin{bmatrix}
0 \\
\xi A / 2 \pi \cos \zeta_p |d - z_o| \\
+ (\xi^2 - \zeta_i^2) \xi B_\theta / 8 \pi \zeta_s |d - z_o| \\
\end{bmatrix}
\] (3-138)

Equation (3-138) represents an algebraic system that can be solved for \( C_1, D_2 \) provided the system determinant does not vanish
\[
D_A = \begin{vmatrix}
(\xi^2 - \eta_i^2) \sin \zeta_p d & -2 \xi \zeta_p \sin \zeta_i d \\
2 \xi \eta_p \cos \zeta_p & (\xi^2 - \zeta_i^2) \cos \zeta_i d \\
\end{vmatrix} \neq 0
\]

Let,
\[ P_A = \xi \left( \frac{A}{2\pi} \right) \cos \zeta_p |d - z_o| + \xi (\xi^2 - \zeta_s^2) \frac{B_0}{8\pi \zeta_s} \sin \zeta_s |d - z_0| \]  

(3-139)

\( P_A \) is the source term for anti-symmetric solution which contains source potentials \( A \) and \( B_0 \).

Upon substitution of \( d_i = d - z_o \) into Equation (3-139)

\[ P_A = \xi \left( \frac{A}{2\pi} \right) \cos \zeta_p d_i + \xi (\xi^2 - \zeta_s^2) \frac{B_0}{8\pi \zeta_s} \sin \zeta_s d_i \]  

(3-140)

Then,

\[
\begin{bmatrix}
C_1 \\
D_2
\end{bmatrix}
= \begin{bmatrix}
(\xi^2 - \zeta_s^2) \cos \zeta_s d & 2 \xi \zeta_s \sin \zeta_s d \\
-2 \xi \zeta_p \cos \zeta_p d & (\xi^2 - \zeta_s^2) \sin \zeta_p d
\end{bmatrix}
\begin{bmatrix}
0 \\
P_A
\end{bmatrix}
\]

(3-141)

Upon rearrangement

\[
\begin{bmatrix}
C_1 \\
D_2
\end{bmatrix}
= \frac{1}{D_A} \begin{bmatrix}
P_A 2 \xi \zeta_s \sin \zeta_s d \\
\frac{P_A(\xi^2 - \zeta_s^2) \sin \zeta_p d}{D_A}
\end{bmatrix}
\]

(3-142)

Here,

\[ D_A = \begin{bmatrix}
(\xi^2 - \zeta_s^2) \sin \zeta_s d & -2 \xi \zeta_s \sin \zeta_s d \\
2 \xi \zeta_p \cos \zeta_p d & (\xi^2 - \zeta_s^2) \cos \zeta_s d
\end{bmatrix}
\]

(3-143)

By equating to zero the \( D_A(\xi) \) term of Equation (3-143), one gets the anti-symmetric Rayleigh-Lamb wave equation.
3.6 Complete solution in the wavenumber domain

Equation (3-18) gives the displacement \( u_z \), in terms of potentials, i.e.,

\[
 u_z = \frac{\partial \Phi}{\partial z} + \frac{1}{r} \frac{\partial (r H_\theta)}{\partial r} \quad (3-144)
\]

After applying Hankel transform of zero order to Equation (3-144), displacement \( u_z \) become,

\[
 \tilde{u}_z = \frac{\partial \tilde{\Phi}_0}{\partial z} + \xi \tilde{H}_\theta \quad (3-145)
\]

By substituting of Equations (3-99) and (3-101) into Equation (3-145) yields the expressions for out-of-plane velocity in the wavenumber domain in terms of coefficients \( C_1, C_2, D_1, D_2 \) which are functions of the wavenumber \( \xi \)

\[
 \tilde{u}_z = \left[ (C_1 \xi_p \cos \xi_p z - C_2 \xi_p \sin \xi_p z - \frac{A}{4\pi} \cos \xi_p |z - z_0|) \right] \\
\quad + \xi \left( D_1 \sin \xi_p z + D_2 \cos \xi_p z \right) - \xi \frac{B_p}{8\pi \xi_s} \sin \xi_s |z - z_0| \quad (3-146)
\]

Evaluation of Equation (3-146) at the plate top surface \( \tilde{u}_z |_{z=d} \):

\[
 \tilde{u}_z = \left[ (C_1 \xi_p \cos \xi_p d - C_2 \xi_p \sin \xi_p d - \frac{A}{4\pi} \cos \xi_p |d - z_0|) \right] \\
\quad + \xi \left( D_1 \sin \xi_p d + D_2 \cos \xi_p d \right) - \xi \frac{B_p}{8\pi \xi_s} \sin \xi_s |d - z_0| \quad (3-147)
\]

The complete solution is the superposition of the symmetric and antisymmetric solutions.

By using Equations (3-136) and (3-142) into Equation (3-147) and obtain,
\[ \tilde{u}_z = \frac{1}{\xi} \left\{ \frac{P_s}{D_s} N_s + P_A \frac{N_A}{D_A} - \xi \left( \frac{A}{4\pi} \right) \cos \zeta \sin \zeta d - \xi^3 \frac{B_0}{8\pi \zeta'} \sin \zeta' \right\} \]  

Here

\[ N_s = 2\xi^3 \zeta' \sin \zeta d \cos \zeta' \sin \zeta d - \xi \left( \xi^2 - \zeta' \right) \zeta' \sin \zeta d \sin \zeta', d \]

\[ N_A = 2\xi^3 \zeta^2 \sin \zeta d \cos \zeta d + \xi^2 \left( \xi^2 - \zeta^2 \right) \sin \zeta d \cos \zeta d \]  

Upon substitution of \( d_1 = d - z_0 \) into Equation (3-149)

\[ \tilde{u}_z = \frac{1}{\xi} \left\{ \frac{P_s}{D_s} N_s + P_A \frac{N_A}{D_A} - \xi \left( \frac{A}{4\pi} \right) \cos \zeta \sin \zeta d_1 - \xi^3 \frac{B_0}{8\pi \zeta'} \sin \zeta' d_1 \right\} \]

3.7 COMPLETE SOLUTION IN THE PHYSICAL DOMAIN

Lamb wave solution in the physical domain by applying inverse Hankel transforms to Equation (3-150) and obtain,

\[ u_z = \int_0^\infty \left\{ \frac{P_s}{D_s} N_s + P_A \frac{N_A}{D_A} - \xi \left( \frac{A}{4\pi} \right) \cos \left( \frac{\xi}{\zeta} d_1 \right) \right\} \xi J_0(\xi r) d\xi \]  

Upon rearrangement of Equation (3-151)

\[ u_z = \int_0^\infty \left\{ \frac{P_s}{D_s} N_s + P_A \frac{N_A}{D_A} - \xi \left( \frac{A}{4\pi} \right) \cos \left( \frac{\xi}{\zeta} d_1 \right) \right\} J_0(\xi r) d\xi \]

Let,
\[ u_c = I_1 - I_2 - I_3 \] (3-153)

Where,

\[ I_1 = \int_0^\infty \left( \frac{P_s N_s}{D_s} + P_A \frac{N_A}{D_A} \right) J_0(\xi r) \, d\xi \] (3-154)

\[ I_2 = \int_0^\infty \xi \left( \frac{A}{4\pi} \right) \cos(\xi^3 d_1) J_0(\xi r) \, d\xi \] (3-155)

\[ I_3 = \int_0^\infty \xi^3 \left( \frac{B_0}{8\pi \xi_s} \right) \sin(\xi^3 d_1) J_0(\xi r) \, d\xi \] (3-156)

The integrant in Equation (3-154) is singular at the roots \( D_s \) and \( D_A \), i.e.,

\[ D_s = 0 \] (3-157)

\[ D_A = 0 \] (3-158)

The Equations (3-157) and (3-158) are the Rayleigh-Lamb equations for symmetric and anti-symmetric modes. The evaluation of integral can be done by residue theorem. In order to use residue theorem by close contour limit of the integral should be \(-\infty \) to \( \infty \). This aspect is obtained through the contour unfolding method described by ref (Giutgiutiu, 2016) pp 613-614.

Let,

\[ f(\xi) = \left( \frac{P_s N_s}{D_s} + P_A \frac{N_A}{D_A} \right) \] (3-159)

The function \( f(\xi) \) of Equation (3-159) is an odd function i.e., \( f(-\xi) = -f(\xi) \); hence equation (3-159) become
\[
I_1 = \frac{1}{2} \int_{-\infty}^{\infty} f(\xi) H_0^{(1)}(\xi r) \, d\xi
\]  
(3-160)

Here, \( H_0^{(1)} \) is the Hankel transform of the first kind and order 0. The evaluation of integral in Equation (3-160) can be done by residue theorem by using a close contour as shown in Figure 3.6. The positive roots correspond to forward propagating waves. To satisfy the radiation boundary condition at \( x = \infty \), that is no incoming waves from infinity, the negative real poles are avoided in the contour \( (x > 0) \). The integration of Equation (3-154) can be written as the sum of the residues,

\[
I_1 = \frac{1}{2} \int_{-\infty}^{\infty} \left[ P_s(\xi) \frac{N_s(\xi)}{D_s(\xi)} \right] H_0^{(1)}(\xi r) \, d\xi + \left[ P_A(\xi) \frac{N_A(\xi)}{D_A(\xi)} \right] H_0^{(1)}(\xi r) \, d\xi
\]  
(3-161)

For the poles \( \xi_k \), Equation (3-161) becomes

\[
I_1 = \pi i \sum_{\xi_k} \text{Res}(\xi_k)
\]  
(3-162)

Figure 3.6  Close contour for evaluating the inverse Fourier transform by residue theorem for (a) \( x > 0 \) (b) \( x < 0 \)
From residue theorem, if \( f(z) = \frac{N(z)}{D(z)} \) and \( z = a \) is a simple pole then

\[
\text{Res}(a) = \frac{N(a)}{D'(a)}, \quad \text{where } D'(a) = \frac{dD}{dz} \bigg|_{z=a}.
\]

Applying residue theorem to obtain displacement equation

\[
u_y = \pi i \left( \sum_{j=0}^{j_S} \left[ P_S(\xi_j^S) \frac{N_S(\xi_j^S)}{D'_S(\xi_j^S)} \right] H_0^i(\xi_j^S r) + \sum_{j=0}^{j_A} \left[ P_A(\xi_j^A) \frac{N_A(\xi_j^A)}{D'_A(\xi_j^A)} \right] H_0^i(\xi_j^A r) \right)
\]

Here

\[
D'_S(\xi_j^S) = \frac{4\xi_j^S}{\zeta_p^S\zeta_s^S} \left[ \xi_j^S (\zeta^2_p + \zeta^2_s) - 2 (\zeta_p^S \zeta_s^S) \right] \cos(\zeta_p^S d) \sin(\zeta_p^S d)
-
d \frac{\xi_j^S}{\zeta_p^S} \left[ (\zeta_p^S - \zeta_s^S)^2 + 4(\xi_j^S)^2 \right] \sin(\zeta_p^S d) \sin(\zeta_p^S d)
+ d \frac{\zeta_j^S}{\zeta_s^S} (\zeta^2_p + \zeta^2_s)^2 \cos(\zeta_p^S d) \cos(\zeta_p^S d)
- 8\zeta_j^S (\zeta_p^S - \zeta_s^S)^2 \sin(\zeta_p^S d) \cos(\zeta_p^S d)
\]

\[
D'_A(\xi_j^A) = 4\xi_j^A \left( \frac{\zeta_j^A}{\zeta_p^A} \xi_j^A + \frac{\zeta_j^A}{\zeta_s^A} \right) \sin(\zeta_p^A d) \cos(\zeta_p^A d)
+ 4d \zeta_j^A \xi_j^A \cos(\zeta_p^A d) \cos(\zeta_p^A d) - 4d \zeta_j^A \xi_j^A \sin(\zeta_p^A d) \sin(\zeta_p^A d)
- 8\zeta_j^A (\zeta_p^A - \zeta_s^A)^2 \sin(\zeta_p^A d) \cos(\zeta_p^A d) - d (\zeta_p^A - \zeta_s^A)^2 \frac{\xi_j^A}{\zeta_s^A} \sin(\zeta_p^A d) \sin(\zeta_p^A d)
\]

The summation of Equation (3-163) is taken over the symmetric and anti-symmetric positive real wavenumbers \( \xi_j^S \) and \( \xi_j^A \). At given frequency \( \omega \), there are \( j = 0, 1, 2, \ldots, j_S \) symmetric Lamb wave modes and \( j = 0, 1, 2, \ldots, j_A \) anti-symmetric Lamb wave modes.
Upon substitution of Equation (3-63) into Equation (2-104), we get

\[
I_2 = \int_0^\infty \frac{\xi A}{4\pi} \cos d_1 \left( \sqrt{\left(\frac{\omega}{c_p}\right)^2 - \xi^2} \right) J_0(\xi r) d\xi
\]  

(3-166)

Apply integration by parts to Equation (3-166)

\[
u = \cos \left( d_1 \sqrt{\left(\frac{\omega}{c_p}\right)^2 - \xi^2} \right); \quad v = \xi J_0(\xi r)
\]

\[
\int uv d\xi = u \int vd\xi - \int \frac{du}{d\xi} \int vd\xi d\xi
\]

After integrating by parts using Equation (3-167), Equation (3-166) becomes

\[
I_2 = \frac{A}{4\pi} \int_0^\infty \frac{\xi^2}{r} \sin \left( \sqrt{\left(\frac{\omega}{c_p}\right)^2 - \xi^2} \right) \frac{d_1}{\sqrt{\left(\frac{\omega}{c_p}\right)^2 - \xi^2}} J_1(\xi r) d\xi
\]  

(3-168)

Only the positive wavenumbers correspond to forward propagating waves will be considered here. For positive real valued wave number the integration of Equation (3-168) can be determined using inverse transform of ref (Erdélyi, 1954), pp 35 (23) for \( \nu = 1 \), i.e.,

\[
I_2 = \frac{A}{4\sqrt{2\pi}} d_1 \left( \frac{\omega}{c_p} \right)^{3/2} (r^2 + d_1^2)^{-3/4} Y_{3/2} \left[ \frac{\omega}{c_p} (r^2 + d_1^2)^{3/2} \right]
\]  

(3-169)

Here \( Y \) is the Bessel function of second kind.

Recall Equation (3-156)

\[
I_3 = \frac{B_o}{8\pi} \int_0^\infty \frac{\xi^3}{2\zeta s} \sin \left( \zeta s d_1 \right) J_0(\xi r) d\xi
\]  

(3-170)

Using Equation (3-63) into Equation (3-170) yields

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\[ I_3 = \frac{B_{\theta}}{8\pi} \int_0^\infty \xi^3 \sin \left( d_1 \sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2} \right) \frac{J_0(\xi r)}{\sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2}} d\xi \]  
\hspace{1cm} (3-171)

Upon rearrangement of Equation (3-171) in the following way

\[ I_3 = \frac{B_{\theta}}{8\pi} r^{-3/2} \int_0^\infty \xi^{3/2} \sin \left( d_1 \sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2} \right) \frac{J_0(\xi r)(\xi r)^{1/2}}{\sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2}} d\xi \]  
\hspace{1cm} (3-172)

Using general formula presented by reference (Erdélyi, 1954), pp 5 (3) into Equation (3-173), for \( m = 1 \) and \( \nu = 0; \)

\[ I_3 = \frac{B_{\theta}}{8\pi} \frac{1}{r} \frac{d}{dr} \left[ r^{3/2} \int_0^\infty \xi^{3/2} \sin \left( d_1 \sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2} \right) \frac{J_0(\xi r)(\xi r)^{1/2}}{\sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2}} d\xi \right] \]  
\hspace{1cm} (3-173)

Considering the integral part of Equation (3-174), i.e.,

\[ \int_0^\infty \xi^{3/2} \sin \left( d_1 \sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2} \right) \frac{J_0(\xi r)d\xi}{\sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2}} \]  
\hspace{1cm} (3-174)

For positive real valued wave number the integration of Equation (3-174) can be determined by using formula presented by ref. (Erdélyi, 1954) pp 35 (23) for \( \nu = 1, \) i.e.,

\[ \int_0^\infty \xi^{3/2} \sin \left( d_1 \sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2} \right) \frac{J_0(\xi r)d\xi}{\sqrt{\left( \frac{\omega}{c_s} \right)^2 - \xi^2}} = \sqrt{\pi/2} \left( \frac{\omega}{c_s} \right)^{3/4} (d_1^2 + r^2)^{-3/4} r Y_{3/2} \left( \frac{\omega}{c_s} (d_1^2 + r^2)^{1/2} \right) \]  
\hspace{1cm} (3-175)

Here \( Y \) is the Bessel function of second kind.

Using Equation (3-175) into Equation (3-173), we get

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\[ I_s = \frac{B_s}{8 \sqrt{2\pi}} (\omega/c_s)^{3/2} (d_i^2 + r^2)^{-3/4} r^{1/2} \]

\[
\begin{align*}
&= \frac{3}{2} \omega/c_s (d_i^2 + r^2)^{1/2} Y_{3/2} \left[ \omega/c_s (d_i^2 + r^2)^{1/2} \right] \\
&+ \frac{3}{2} Y_{3/2} \left[ \omega/c_s (d_i^2 + r^2)^{1/2} \right] \\
&- \frac{3}{4} Y_{3/2} \left[ \omega/c_s (d_i^2 + r^2)^{1/2} \right] 
\end{align*}
\]

(3-176)

The displacement Equation (3-176) becomes

\[
u_i = \pi i \left( \sum_{j=0} \left[ P_i (\xi_j) N_s (\xi_j) \right] H_i (\xi_j^*) r + \sum_{j=0} \left[ P_A (\xi_j^A) N_A (\xi_j^A) \right] H_A (\xi_j^* r) \right) \\
- \frac{A}{4 \sqrt{2\pi}} d_i (\omega/c_p)^{3/2} (r^2 + d_i^2)^{-3/4} Y_{3/2} \left[ \omega/c_p (r^2 + d_i^2)^{1/2} \right] \\
- \frac{B_s}{8 \sqrt{2\pi}} \rho^2 (\omega/c_s)^{3/2} (d_i^2 + r^2)^{-3/4} r^{1/2} \]

(3-177)

After rearranging Equation (3-177) we get,

\[
u_i = \pi i \left( \sum_{j=0} \left[ P_i (\xi_j) N_s (\xi_j) \right] H_i (\xi_j^*) r e^{-i\omega t} + \sum_{j=0} \left[ P_A (\xi_j^A) N_A (\xi_j^A) \right] H_A (\xi_j^* r) e^{-i\omega t} \right) \\
- \frac{A}{4 \sqrt{2\pi}} d_i (\omega/c_p)^{3/2} (r^2 + d_i^2)^{-3/4} Y_{3/2} \left[ \omega/c_p (r^2 + d_i^2)^{1/2} \right] e^{-i\omega t} \\
- \frac{B_s}{8 \sqrt{2\pi}} \rho^2 (\omega/c_s)^{3/2} (d_i^2 + r^2)^{-3/4} r^{1/2} \]

(3-178)
Assume that,

\[ u_z = u_{zL} + u_{zA} + u_{zB} \quad (3-179) \]

Where,

\[ u_{zL} = \pi i \left\{ \sum_{j=0}^{j_A} \left[ P_s(\xi_j^S) \frac{N_s(\xi_j^S)}{D_s(\xi_j^S)} \right] H_0(\xi_j^S r)e^{-i\omega t} \right\} \]

\[ + \sum_{j=0}^{j_A} \left[ P_A(\xi_j^A) \frac{N_A(\xi_j^A)}{D_A(\xi_j^A)} \right] H_0(\xi_j^A r)e^{-i\omega t} \quad (3-180) \]

\[ u_{zA} = -\frac{A}{4\sqrt{2\pi}} d_1 \left( \frac{\omega}{c_p} \right)^{3/2} (r^2 + d_1^2)^{-3/4} Y_{3/2} \left[ \omega/c_p (\pi^2 + d_1^2)^{1/2} \right] e^{-i\omega t} \quad (3-181) \]

\[ u_{zB} = -\frac{B_0}{8\sqrt{2\pi}} \left( \frac{\omega}{c_s} \right)^{3/2} (d_1^2 + r^2)^{-3/4} r^{1/2} \]

\[ \left[ \frac{3}{2} Y_{3/2} \left[ \frac{\omega}{c_s} (d_1^2 + r^2)^{1/2} \right] \right] e^{-i\omega t} \quad (3-182) \]

\[ u_{zL} \] is the out-of-plane displacement containing Lamb wave mode and \( u_{zA}, u_{zB} \) are the out-of-plane displacements of bulk wave due to excitation potentials \( A \) and \( B_0 \) respectively.

### 3.8 AE ELASTIC WAVE PROPAGATION

AE elastic waves will be generated by the AE event. AE elastic waves will propagate through the structure according to structural transfer function. The out-of-plane displacement of the elastic waves can be captured by conventional AE transducer installed on the surface of the structure as shown in Figure 3.7.
Figure 3.7  Acoustic emission propagation and detection by a sensor installed on a structure

![Diagram of acoustic emission propagation](image)

**Excitation Potentials:**
- Time rate of excitation potential $\frac{\partial A^t(t)}{\partial t}$ and $\frac{\partial B^s_2(t)}{\partial t}$
- Integrate: Time profile of released energy $A^t(t)$ and $B^s_2(t)$
- Calculate: Amplitude of the source
  
  $A(t) = \frac{1}{c^2_2} A^t(t)$; $B_2(t) = \frac{c_2^2}{c^2_s} B^s_2(t)$

\[\downarrow\]

**Fourier transform of the signal**

$A(t) \rightarrow A(\omega)$; $B_2(t) \rightarrow B_2(\omega)$

\[\downarrow\]

**Calculate source term** $P_2(x^2)$ and $P_4(x^4)$

\[\downarrow\]

**Calculate structural transfer function**

$\frac{N_2(x^2)}{D_2(x^2)} H_2^1(x^2, x_2)$ and $\frac{N_4(x^4)}{D_4(x^4)} H_4^1(x^4, x_4)$

\[\downarrow\]

**Product of structural transfer functions and source term**

$u_2(\omega) = P_2(x^2, \omega) \frac{N_2(x^2)}{D_2(x^2)} H_2^1(x^2, x_2) + P_4(x^4, \omega) \frac{N_4(x^4)}{D_4(x^4)} H_4^1(x^4, x_4)$

**Calculate bulk wave displacement in frequency domain**

$u_2(\omega); u_B(\omega)$

\[\downarrow\]

**Perform inverse Fourier transform**

$u_2(\omega); u_B(\omega); u_B(\omega) \rightarrow u_2(t); u_2(t); u_B(t)$

Figure 3.8  Methodology for elastic wave analysis
For numerical analysis 304-stainless steel material with 6 mm thickness was chosen as a case study. The signal was received at 500 mm distance from the source. Excitation source was located at mid-plane. A very short peak time of 3 μs was used for case study. Later, the effects of source of depth and propagation distance were included into the study. The details methodology of numerical analysis is described in Figure 3.8.

3.9 Numerical Studies

This section includes elastic wave simulation in a plate due to excitation potentials.

3.9.1 Time dependent excitation potentials

The time-dependent excitation potentials depend on time-dependent energy released from a crack. A real AE source releases energy during a finite time period. At the beginning, the rate of energy released from a crack increases sharply with time and reaches a maximum peak value within very short time; then decreases asymptotically toward the steady state value, usually zero. In this research, a Gaussian pulse is used to model the growth of the excitation potentials during the AE event; as a result, the actual excitation potential follows the error function variation in the time domain. Figure 3.9a and Figure 3.10a show the time rate of pressure potential and shear potential. The corresponding equations are,

\[
\frac{\partial A^*_p}{\partial t} = A_0 t^2 e^{\frac{t^2}{\tau^2}} \\
\frac{\partial B^*_p}{\partial t} = B_{00} t^2 e^{\frac{t^2}{\tau^2}}
\]

(3-183)

Here \( A_0 \) and \( B_{00} \) are the scaling factors. Time profile of the potentials are to be evaluated by integrating Equation (3-183), i.e.,
Time profile of potential is shown in Figure 3.9b and Figure 3.10b. The key characteristics of the excitation potentials are:

- peak time: time required to reach the time rate of potential to maximum value
- rise time: time required to reach the potential to 98% of maximum value or steady state value
- peak value: maximum value of time rate of potential
- maximum potential: maximum value of time profile of potential

Amplitude of the source $A$ and $B_\theta$ for unit volume is,

$$A = \frac{1}{c_p^2} A^*$$

$$B_\theta = \frac{d}{c_s^2} B_\theta^*$$

Here, $d$ is the plate half thickness.
Figure 3.9  (a) Time rate of pressure potential \( \left( \frac{\partial A^*}{\partial t} \right) \) and (b) Time profile of pressure potential \( (A^*) \)

Figure 3.10  (a) Time rate of shear potential \( \left( \frac{\partial B^*_\theta}{\partial t} \right) \) and (b) Time profile of shear potential \( (B^*_\theta) \)

3.9.2 Phase velocity and group velocity dispersion curves

Solution of the Rayleigh-Lamb equation for symmetric and antisymmetric modes (Equations (3-137) and (3-143)) yields the wave number and hence the phase velocity for each given frequency. Multiple solutions exist, hence multiple Lamb modes also exist. Differentiation of phase velocity with respect to frequency yields the group velocity. The
phase velocity dispersion curves and group velocity dispersion curves are shown in Figure 3.11a and Figure 3.11b respectively. The existence of certain Lamb mode depends on the plate thickness and frequency. The fundamental S0 and A0 modes will always exist. The phase velocity (Figure 3.11a) is associated with the phase difference between the vibrations observed at two different points during the passage of the wave. The phase velocity is used to calculate the wave length of each mode.

Figure 3.11 Lamb wave dispersion curves: (a) phase velocity and (b) group velocity of symmetric and anti-symmetric Lamb wave modes

The fundamental wave mode of symmetric (S0) and anti-symmetric (A0) is considered in this analysis. Signal is received at some distance from the point of excitation potentials, only the modes with real-valued wavenumbers are included in the simulation. The corresponding modes for imaginary and complex-valued wavenumbers are ignored because they propagate with decaying amplitude; at sufficiently far from the excitation point their amplitudes are negligible. Since different Lamb wave modes traveling with different wave speeds exist simultaneously, the excitation potentials will generate S0 and A0 wave packets. The group velocity (Figure 3.11b) of Lamb waves is important when examining the traveling of Lamb wave packets. These wave packets will travel independently through the plate and will arrive at different times. Due to the multi-mode
character of elastic Lamb wave propagation, the received signal has at least two separate wave packets, S0 and A0. All wave modes propagate independently in the structure. The final waveform will be the superposition of all the propagating waves and will have the contribution from each Lamb wave mode.

3.9.3 Example of AE elastic wave propagation in a 6mm plate

A test case example is presented in this section to show how AE elastic waves propagate over a certain distance using excitation potentials. A 304-stainless steel plate with 6 mm thickness was chosen for this purpose. The signal was received at 500 mm distance from the source. Excitation sources were located at mid-plane of the plate ($d_t = 3$mm). The time profile excitation potentials (Figure 3.9b and Figure 3.10b) were used to simulate AE waves in the plate. Excitation potential can be calculated from time rate of potential released during crack propagation. Peak time, rise time, peak value and maximum potential of excitation potentials are $3 \mu$s, $6.6 \mu$s, $0.28 \mu$W/kg and $1 \mu$J/kg respectively. Based on this information, a numerical study on AE Lamb wave propagation is conducted.

In this thesis, each excitation potential was considered separately to simulate the Lamb waves. Total released energy from a crack can be decomposed to pressure and shear excitation potentials. This article presents the individual effect of Figure 3.12a shows the Lamb wave (S0 and A0 mode) and bulk wave propagation using pressure excitation potential. Signals are normalized by their individual peak amplitudes, i.e., amplitude/ peak amplitude.
It can be inferred from the figures that both A0 and S0 are dispersive. S0 contains high frequency component at which it is dispersive, whereas, A0 contains low frequency component at which it is also dispersive. Bulk wave shows non dispersive behavior, but the peak amplitude is not significant compared to peak S0 and A0 amplitude. The notable characteristic is that, peak amplitude of A0 is higher than peak S0 amplitude. Figure 3.12b
shows the Lamb wave (S0 and A0 mode) and bulk wave propagation using shear potential only. Peak amplitude of A0 is higher than peak S0 amplitude while using shear potential only (Figure 3.12b).

By comparing Figure 3.12a and Figure 3.12b the important observation can be made as, shear excitation potential has more contribution to the peak S0 amplitude over pressure excitation potentials whereas, pressure and shear excitation potentials have almost equal contribution to the peak A0 amplitude. However, their contribution to the S0 and A0 peak amplitude might change due to change in depth of source and propagating distance. The proceeding section discusses the effect of depth of source and propagating distance in AE elastic wave propagation using excitation potentials.

3.9.4 Effect of depth of source

To study the effect of depth of source, a 6 mm thick plate was chosen. The signal was received at 500 mm distance from the source. The excitation sources were located at top surface, 1.5 mm and 3 mm deep (mid plane) from the top surface. Figure 3.13, Figure 3.14, Figure 3.15 show the out-of-plane displacement (S0 and A0 mode, bulk wave) vs time at 500 mm propagation distance in 6mm thick plate for different source location. Signals are normalized by their individual peak amplitudes, i.e., amplitude/ peak amplitude. Figure 3.13 shows that, A0 mode appears only while using pressure potential on the top surface whereas, S0 mode and bulk wave appear while using shear potential.
Therefore, pressure potential contributes to the peak $A_0$ amplitude and shear potential contributes to the peak $S_0$ and bulk wave amplitudes in the final waveform. Other notable characteristic is that, pressure potential has contribution to the trailing edge (low frequency component) of $A_0$ wave packet for top-surface source. With increasing depth of source the effect of high amplitude of low frequency decreases (Figure 3.14a and Figure 3.15a) in $A_0$ signal. By comparing Figure 3.14 and Figure 3.15, clearly amplitude scales
show that, the increase in peak $A_0$ amplitude and decrease in peak $S_0$ amplitude with increasing depth of source while using pressure excitation potential only.

(c) Pressure potential excitation

(d) Shear potential excitation

Figure 3.14  Lamb wave ($S_0$ and $A_0$ mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = $3 \mu s$) located at 1.5 mm depth from top surface
Figure 3.15  Lamb wave (S0 and A0 mode) and bulk wave propagation at 500 mm distance in 6 mm 304-steel plate for (a) pressure potential excitation (b) shear potential excitation (peak time = 3 \( \mu s \)) located at 3.0 mm depth from top surface.

Figure 3.13b, Figure 3.14b and Figure 3.15b show some qualitative and quantitative changes in spectrum while using shear excitation potentials only. Amplitude of S0 signal decreases and A0 signal increases with increasing depth of source while using shear excitation potential only (Figure 3.14b and Figure 3.15b). Qualitative change in S0 signal refers to the change in frequency content of the signal. It should be noted that, peak
amplitude of S0 due to shear excitation potential located at 1.5 mm depth from the top surface is more significant than pressure potential (Figure 3.14).

For all AE source location, the shear potential part of the AE source has more contribution to the peak S0 amplitude than pressure potential. However, pressure and shear excitation potentials have almost equal contribution to the peak A0 amplitude for all plate thickness. Peak amplitude of bulk waves increases with increasing depth of source (Figure 3.14 and Figure 3.15). However, the amplitude of bulk wave is much smaller than the peak A0 and S0 amplitudes. Therefore, peak amplitude of bulk wave may not appear in real AE signal.

3.9.5 Effect of propagation distance

The attenuation of the peak amplitude of the signal as a function of propagating distance was also determined. Figure 3.16 and Figure 3.17 show the normalized peak amplitude of out-of-displacement (S0, A0, bulk wave) against propagation distance from 100 mm to 500 mm for pressure potential, shear potential respectively. The amplitudes are normalized by their individual peak amplitudes. A 6 mm thick 304-steel plate with both excitation potentials are considered. Potentials are located at mid plane with peak time of 3 $\mu$s. Figure 3.16 and Figure 3.17 show significant attenuation of S0, A0 and bulk wave signal with propagating distance 100 mm to 50 mm. However, larger attenuation of peak bulk wave amplitude is observed compared to peak S0 and A0 amplitudes. Therefore, far from the source bulk wave becomes less significant and may not be captured by the AE transducer. Another important observation is that, peak S0 amplitude attenuates more than peak A0 amplitude for both excitation potentials. The attenuation of the peak amplitude is expected due to dispersion of the signal.
Figure 3.16 Effect of pressure excitation potential: variation of out-of-plane displacement (S0, A0 and bulk wave) with propagation distance in 6 mm 304-steel plate for source (peak time = 3 $\mu$s) located at the mid plane.

Figure 3.17 Effect of shear excitation potential: variation of out-of-plane displacement (S0, A0 and bulk wave) with propagation distance in a 6-mm 304-steel plate for source (peak time = 3 $\mu$s) located at the mid plane.
CHAPTER 4

HYBRID GLOBAL ANALYTICAL AND LOCAL CMEP ANALYSIS FOR DETECTING CRACK IN A STIFFENER PLATE

4.1 STATE OF THE ART

In the field of non-destructive evaluation (NDE) and structural health monitoring (SHM), methods using ultrasonic plate guided waves are popular for their potential use in the inspection of large structures. Due to this popularity, the prediction of the scattering of Lamb waves from damage has been a major focus for researchers in NDE and SHM. Damage characterization in particular, as an inverse problem, requires fast and accurate prediction of scattered waves. But the solution of the scattering problem is highly challenging because of the existence of multiple dispersive modes of Lamb waves at any frequency along with mode conversion at the damage location (Haider et al. 2018e, Bhuiyan et al. 2018a, 2018b). Therefore, well-developed numerical methods such as the finite element method (FEM) and the boundary element method (BEM) have been popular (Cho and Rose 2000; Galán and Abascal 2005; Mackerle 2004; Moser, Jacobs, and Qu 1999). However commercial FEM codes are time consuming and they do not provide much insight of the wave field in the structure, especially near the damage location. Therefore, for efficiency of simulation, researchers developed hybrid methods using FEM, BEM, and
normal mode expansion (Cho and Rose 2000; Galán and Abascal 2005; Gunawan and Hirose 2004; Terrien et al. 2007).

One of the main challenges of the scattering problem is to satisfy the thickness dependent boundary conditions at the discontinuity (Castaings, Le Clezio, and Hosten 2002; Terrien et al. 2007). Gregory et al. (Gregory and Gladwell 1982, 1983) developed the ‘projection method’ to satisfy these thickness dependent boundary conditions. This method was developed to predict the singularity in stresses in the case of geometric discontinuities (Flores-López and Douglas Gregory 2006). A scalar form of the projection method was also used by Grahn (Grahn 2003); though simple, this approach proved to be not very stable and to have slow convergence due to the use of simple sine and cosine functions (Moreau et al. 2012). Moreau et al. (Moreau et al. 2011) used the displacement components of the complex Lamb wave modes shapes instead of simple sine and cosine functions and attained a faster convergence in the projection method.

4.2 **CMEP for Stiffener**

4.2.1 **Problem Setup for a Stiffener**

To begin our analysis, let us consider a plate with a cross section as shown in Figure 4.1. We assume that there is an incident straight-crested Lamb wave mode travelling from the left towards a stiffener. Upon interacting with the stiffener, it will result in reflected wave modes, transmitted wave modes, and wave modes trapped in the stiffener. As shown in Figure 4.1, the stiffener is located at a distance \(x = x_0\) with the thickness of the plate being \(h_1\). The height of the stiffener is \(h_2\) with the width of \(L = 2b\). At the stiffener, we define height ration as \(R_h = (h_2 - h_1)/h_1\) and width ratio as \(R_w = 2b/h_1\). Also, let us imagine that
the incident wave field is represented by \((\Phi_0, H_0)\), travelling in +ve \(x\) direction in the Region 1. We define the reflected wave field as \((\Phi_1, H_1)\) and the transmitted wave field in Region 3 as \((\Phi_3, H_3)\). We also define the trapped wave field inside the stiffener in Region 2 as \((\Phi_2, H_2)\).

The incident and scattered wave fields satisfy the generic wave equations

\[
\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{1}{C_p^2} \Phi \\
\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{1}{C_s^2} H
\]  

where, \(C_p\) and \(C_s\) are the wave speeds of pressure wave and shear waves, respectively.

These must satisfy the zero-stress boundary condition at the top and bottom of the plate,

\[
\sigma_{yy} \bigg|_{x < x_0 - b; \; y = \pm \frac{h_1}{2}} = \sigma_{yy} \bigg|_{x > x_0 + b; \; y = \pm \frac{h_1}{2}} = 0 \\
\tau_{xy} \bigg|_{x < x_0 - b; \; y = \pm \frac{h_1}{2}} = \tau_{xy} \bigg|_{x > x_0 + b; \; y = \pm \frac{h_1}{2}} = 0
\]

The stress and displacement fields associated with the wave fields are expressed as

\[
\sigma = \begin{bmatrix} \sigma_{xx} & \tau_{xx} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}; \quad \tilde{\sigma} = \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}; \quad \tilde{u} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}
\]

where, \(\sigma\) is stress tensor, \(\tilde{\sigma}\) is stress vector, and \(\tilde{u}\) is displacement vector. For the displacements, subscripts \(x\) and \(y\) indicate the directions of the displacement. For stresses, the subscripts \(xx\), \(yy\) stand for the normal stress in \(x\), \(y\) directions, respectively and the subscript \(xy\) stands for the shear stress. The boundary conditions at the stiffener are illustrated in Figure 4.2.
Figure 4.1 Schematic of Lamb waves interacting with a stiffener

Figure 4.2 Boundary conditions at the stiffener

The boundary conditions at the interface at $x = x_0 - b$ are,
\[\sigma_2 \cdot \hat{n}_x = \begin{cases} 0, & x = x_0 - b, \quad h_1/2 \leq y \leq h_2 - h_1/2 \\ (\sigma_0 + \sigma_1) \cdot \hat{n}_x, & x = x_0 - b, \quad -h_1/2 \leq y \leq h_1/2 \end{cases} \]  
\hspace{1cm} (4-4)

\[\tilde{u}_0 + \tilde{u}_1 = \tilde{u}_2, \quad x = x_0 - b, \quad -h_1/2 \leq y \leq h_1/2 \]  
\hspace{1cm} (4-5)

The boundary conditions at the interface at \( x = x_0 + b \) are,

\[\sigma_2 \cdot \hat{n}_x = \begin{cases} 0, & x = x_0 - b, \quad h_1/2 \leq y \leq h_2 - h_1/2 \\ \sigma_3 \cdot \hat{n}_{x_3}, & x = x_0 + b, \quad -h_1/2 \leq y \leq h_1/2 \end{cases} \]  
\hspace{1cm} (4-6)

\[\tilde{u}_3 = \tilde{u}_2, \quad x = x_0 + b, \quad -h_1/2 \leq y \leq h_1/2 \]  
\hspace{1cm} (4-7)

where \( \hat{n}_{x_1}, \hat{n}_{x_3} \) are the unit surface normal vectors of the interface surfaces represented by \( x_1(y), x_3(y) \) and \( \hat{n}_x \) is the unit vector in +ve \( x \) direction as shown in Figure 4.2. Also, note that for a vertical stiffener, \( \hat{n}_{x_1} = \hat{n}_x \) and \( \hat{n}_{x_3} = -\hat{n}_x \). The subscript 0 stands for incident waves in Region 1, subscript 1 stands for reflected waves in Region 1, subscript 2 stands for the trapped waves in Region 2 and subscript 3 stands for transmitted waves in Region 3. Let us assume the incident wave field to be a harmonic wave field of the form

\[\Phi_0 = f_0(y)e^{i(\xi_0 x - \omega t)} \quad H_0 = i\xi_0(y)e^{i(\xi_0 x - \omega t)} \]  
\hspace{1cm} (4-8)

Assuming \( \xi_0 \) to be one of the roots of Rayleigh-Lamb equation for the plate in Region 1, Eq. (4-1)and (4-2) are satisfied by definition of Lamb waves and the incident wave becomes one of the modes of Lamb waves.

**4.2.2 Complex Modes Expansion of the Scattered Wave Field for a Stiffener**

We expand the transmitted, reflected, and trapped wave fields in terms of all possible complex Lamb wave modes corresponding to the complex roots of Rayleigh-Lamb
frequency equation (Haider et al. 2018h). Therefore, the scattered wave fields are expressed as

\[
\Phi_1 = \sum_{n_1=1}^{\infty} C_{1B,n_1} \Phi_{1B,n_1} = \sum_{n_1=1}^{\infty} C_{1B,n_1} f_{n_1} \left( y_1 \right) e^{i \xi_{1B,n_1} x - \omega t} \\
H_1 = \sum_{n_1=1}^{\infty} C_{1B,n_1} H_{1B,n_1} = \sum_{n_1=1}^{\infty} C_{1B,n_1} i h_{n_1} \left( y_1 \right) e^{i \xi_{1B,n_1} x - \omega t} \\
\Phi_2 = \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} \Phi_{2F,n_2} + C_{2B,n_2} \Phi_{2B,n_2} \right) \\
= \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} f_{n_2} \left( y_2 \right) e^{i \xi_{2F,n_2} x - \omega t} + C_{2B,n_2} f_{n_2} \left( y_2 \right) e^{i \xi_{2B,n_2} x - \omega t} \right) \\
H_2 = \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} H_{2F,n_2} + C_{2B,n_2} H_{2B,n_2} \right) \\
= \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} i h_{n_2} \left( y_2 \right) e^{i \xi_{2F,n_2} x - \omega t} + C_{2B,n_2} i h_{n_2} \left( y_2 \right) e^{i \xi_{2B,n_2} x - \omega t} \right) \\
\Phi_3 = \sum_{n_3=1}^{\infty} C_{3F,n_3} \Phi_{3F,n_3} = \sum_{n_3=1}^{\infty} C_{3F,n_3} f_{n_3} \left( y_3 \right) e^{i \xi_{3F,n_3} x - \omega t} \\
H_3 = \sum_{n_3=1}^{\infty} C_{3F,n_3} H_{3F,n_3} = \sum_{n_3=1}^{\infty} C_{3F,n_3} i h_{n_3} \left( y_3 \right) e^{i \xi_{3F,n_3} x - \omega t} \\
(4-9)
\]

where, \( y_1 \) and \( y_2 \) are connected by the expression \( y_2 = y_1 + a \) with \( a = (h_1 - h_2) / 2 \) being the eccentricity between Region 1 and Region 2 and \( y_1 = y_3 \). The wavenumber \( \xi_{1B,n_1} \) is the \( n_1 \)th complex root of the Rayleigh-Lamb equation corresponding to backward propagating Lamb waves in Region 1 and the wavenumbers \( \xi_{2F,n_2} \) and \( \xi_{2B,n_2} \) are \( n_2 \)th complex root of the Rayleigh-Lamb equation corresponding to forward and backward propagating waves in Region 2, respectively. Similarly, wavenumber \( \xi_{3F,n_3} \) is the \( n_3 \)th complex root of the Rayleigh-Lamb equation corresponding to forward propagating waves in Region 3. The coefficient \( C_{1B,n_1} \) is the unknown amplitude of the \( n_1 \)th mode of backward propagating...
Lamb waves in Region 1 whereas $C_{2F,n_2}$ and $C_{2B,n_2}$ are the unknown amplitudes of the $n_2$-th mode of forward and backward propagating Lamb waves in Region 2, respectively. Similarly, the coefficient $C_{3F,n_3}$ is the unknown amplitude of the $n_3$-th mode of forward propagating Lamb waves in Region 3. The amplitudes $C_{1B,n_1}$, $C_{2F,n_2}$, $C_{2B,n_2}$ and $C_{3F,n_3}$ of these modes have to be determined through the boundary matching process.

We express the stress and displacement fields of Eqs. (4-4), (4-5), (4-6) and (4-7) using the complex Lamb wave mode expansion of Eq. (4-9), i.e.,

$$
\bar{u}_1 = \sum_{n_1=1}^{\infty} C_{1B,n_1} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{1B,n_1} ; \sigma_1 = \sum_{n_1=1}^{\infty} C_{1B,n_1} \begin{bmatrix} \sigma_{xx} & \tau_{xx} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}_{1B,n_1}
$$

$$
\bar{u}_2 = \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F,n_2} + C_{2B,n_2} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2B,n_2} \right)
$$

$$
\sigma_2 = \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} \begin{bmatrix} \sigma_{xx} & \tau_{xx} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}_{2F,n_2} + C_{2B,n_2} \begin{bmatrix} \sigma_{xx} & \tau_{xx} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}_{2B,n_2} \right)
$$

$$
\bar{u}_3 = \sum_{n_3=1}^{\infty} C_{3F,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3F,n_3} ; \sigma_3 = \sum_{n_3=1}^{\infty} C_{3F,n_3} \begin{bmatrix} \sigma_{xx} & \tau_{xx} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}_{3F,n_3}
$$

In the same vein, the incident wave field uses subscript 0, i.e.,

$$
\bar{u}_0 = \begin{bmatrix} u_x \\ u_y \end{bmatrix}_0 ; \sigma_0 = \begin{bmatrix} \sigma_{xx} & \tau_{xy} \\ \tau_{xy} & \sigma_{yy} \end{bmatrix}_0
$$

Therefore, for a vertical stiffener, using Eqs.(4-10), (4-11) into Eqs. (4-4), (4-5), (4-6) and (4-7) yield

$$
\bar{\sigma}_2 = \begin{cases} 0, & x = x_0 - b, \quad h_1 / 2 \leq y \leq h_2 - h_1 / 2 \\ \bar{\sigma}_0 + \bar{\sigma}_1, & x = x_0 - b, \quad -h_1 / 2 \leq y \leq h_1 / 2 \end{cases}
$$

$$
\bar{u}_0 + \bar{u}_1 = \bar{u}_2, \quad x = x_0 - b, \quad -h_1 / 2 \leq y \leq h_1 / 2
$$
\[ \bar{\sigma}_2 = \begin{cases} 0, & x = x_0 + b, \quad h_2 - h_1 / 2 \leq y \leq h_1 / 2 \\ \bar{\sigma}_3, & x = x_0 + b, \quad -h_1 / 2 \leq y \leq h_2 - h_1 / 2 \end{cases} \] (4-14)

\[ \bar{u}_3 = \bar{u}_2, \quad x = x_0 + b, \quad -h_1 / 2 \leq y \leq h_1 / 2 \] (4-15)

Therefore, Eqs. (4-12), (4-13), (4-14) and (4-15) represent the thickness dependent boundary conditions at the stiffener.

**4.2.3 Vector Projection of the Boundary Conditions for a Stiffener**

CMEP formulation incorporates the average power flow associated with the reflected, transmitted, and trapped wave fields. CMEP uses the time averaged power flow expression, which uses stress-velocity product. Thus, in Region 1, we project the displacement boundary conditions onto the conjugate stress vector space of the complex Lamb wave modes; in Region 2, we project the stress boundary conditions onto the conjugate displacement vector space of the complex Lamb wave modes. Similarly, in Region 3, we project the displacement boundary conditions onto the conjugate stress vector space of the complex Lamb wave modes.

The projection vector space for Eq. (4-12) is

\[ \bar{u}_{2B} = \text{conj} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2B,n_2} = \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix}_{2B,n_2}, \quad n_2 = 1, 2, 3, ... \] (4-16)

After projecting Eq. (4-12) onto Eq. (4-16), the stress boundary conditions in Eq. (4-12) takes the form,
\[ 
\int_{-h/2}^{h/2} \left( \sigma_0 + \sigma_1 \right) \cdot \vec{u}_{2B} \, dy = \int_{-h/2}^{h/2} \left( \sigma_2 \cdot \vec{u}_{2B} \right) \, dy
\]

\[ 
= \sum_{n_1=1}^{\infty} \left( C_{2F,n_1} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2F,n_1}, \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}_{2F,n_1} \right)_{h/2} + \sum_{n_2=1}^{\infty} \left( C_{2B,n_2} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2B,n_2}, \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}_{2B,n_2} \right)_{-h/2} \]

\[ 
= \sum_{n_1=1}^{\infty} \left( C_{2F,n_1} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2F,n_1}, \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}_{2F,n_1} \right)_{h/2} + \sum_{n_2=1}^{\infty} \left( C_{2B,n_2} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2B,n_2}, \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}_{2B,n_2} \right)_{-h/2} \]

\[ 
= \sum_{n_1=1}^{\infty} \left( C_{2F,n_1} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2F,n_1}, \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}_{2F,n_1} \right)_{h/2} + \sum_{n_2=1}^{\infty} \left( C_{2B,n_2} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2B,n_2}, \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}_{2B,n_2} \right)_{-h/2} \]

\[ 
\; n_1, n_2 = 1, 2, 3, \ldots
\]

where, \( \int_a^b P \cdot Q \, dy = \langle P, Q \rangle^b_a \) represents the inner product. Similarly, the projection vector space for Eq. (4-14) is

\[ 
\vec{u}_{2F} = \text{conj} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F,n_2} = \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}_{2F,n_2}, \vec{n}_2 = 1, 2, 3, \ldots
\]  

(4-18)

Upon projecting Eq. (4-14) onto Eq. (4-18), the stress boundary conditions in Eq. (4-14) takes the form,

\[ 
\int_{-h/2}^{h/2} \vec{\sigma}_3 \cdot \vec{u}_{2F} \, dy = \int_{-h/2}^{h/2} \vec{\sigma}_2 \cdot \vec{u}_{2F} \, dy
\]

\[ 
= \sum_{n_1=1}^{\infty} \left( C_{3F,n_1} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{3F,n_1}, \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}_{3F,n_1} \right)_{h/2} \]

(4-19)

\[ 
\; n_2, n_3 = 1, 2, 3, \ldots
\]

The projection vector space for Eq. (4-13) is,
\[ \bar{\sigma}_{1B} = \text{conj} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{1B, n_i} = \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\sigma}_{xy} \end{bmatrix}_{1B, n_i}, \quad n_i = 1, 2, 3,... \] (4-20)

After projecting Eq. (4-13) onto Eq. (4-20), the displacement boundary conditions in Eq. (4-13), take the form,

\[ \int_{-h/2}^{h/2} (\bar{u}_0 + \bar{u}_1) \cdot \bar{\sigma}_{1B} \, dy = \int_{-h/2}^{h/2} \bar{u}_2 \cdot \bar{\sigma}_{1B} \, dy \]

\[ \Rightarrow \sum_{n_z=1}^{\infty} C_{2F, n_z} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F, n_z} , \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{1B, n_i} \Bigg|_{-h/2}^{h/2} + C_{2B, n_z} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2B, n_z} , \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\sigma}_{xy} \end{bmatrix}_{1B, n_i} \Bigg|_{-h/2}^{h/2} \]

\[ -\sum_{n_z=1}^{\infty} C_{1B, n_i} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{1B, n_z} , \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\sigma}_{xy} \end{bmatrix}_{1B, n_i} \Bigg|_{-h/2}^{h/2} = \begin{bmatrix} u_x \\ u_y \end{bmatrix}_0 , \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\sigma}_{xy} \end{bmatrix}_{1B, n_i} \Bigg|_{-h/2}^{h/2} \]

\[ \quad n_i, n_z = 1, 2, 3,... \] (4-21)

The projection vector space for Eq. (4-15) is,

\[ \bar{\sigma}_{3F} = \text{conj} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix}_{3F, n_i} = \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\sigma}_{xy} \end{bmatrix}_{3F, n_i}, \quad n_i = 1, 2, 3,... \] (4-22)

Upon projecting Eq. (4-15) onto Eq. (4-22), the displacement boundary conditions in Eq. (4-15) takes the form,

\[ \int_{-h/2}^{h/2} \bar{u}_3 \cdot \bar{\sigma}_{3F} \, dy = \int_{-h/2}^{h/2} \bar{u}_2 \cdot \bar{\sigma}_{3F} \, dy \]

\[ \Rightarrow \sum_{n_z=1}^{\infty} \left[ C_{2F, n_z} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F, n_z} , \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\sigma}_{xy} \end{bmatrix}_{3F, n_i} \Bigg|_{-h/2}^{h/2} + C_{2B, n_z} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2B, n_z} , \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\sigma}_{xy} \end{bmatrix}_{3F, n_i} \Bigg|_{-h/2}^{h/2} \right] \]

\[ = \sum_{n_z=1}^{\infty} C_{3F, n_z} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3F, n_z} , \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\sigma}_{xy} \end{bmatrix}_{3F, n_i} \Bigg|_{-h/2}^{h/2} \]

\[ \quad n_z, n_i = 1, 2, 3,... \] (4-23)
4.2.4 Numerical Solution for Cracked Stiffener

For numerical calculation we consider finite values for the indices \( n_1, n_2, n_3, \bar{n}_1, \bar{n}_2, \bar{n}_3 \).

We assume, \( n_1 = 1, 2, 3, \ldots, N_1 \), \( n_2 = 1, 2, 3, \ldots, N_2 \), \( n_3 = 1, 2, 3, \ldots, N_3 \), \( \bar{n}_1 = 1, 2, 3, \ldots, \bar{N}_1 \), \( \bar{n}_2 = 1, 2, 3, \ldots, \bar{N}_2 \), \( \bar{n}_3 = 1, 2, 3, \ldots, \bar{N}_3 \). Then, Eq. (4-17) contains \( \bar{N}_1 \) linear equations with \( (N_1 + 2N_2) \) unknowns and Eq. (4-19) contains \( \bar{N}_3 \) linear equations with \( (N_3 + 2N_2) \). Also Eq. (4-21) contains \( \bar{N}_2 \) linear equations with \( (N_1 + 2N_2) \) and Eq. (4-23) contains \( \bar{N}_2 \) linear equations with \( (N_3 + 2N_2) \) unknowns. Recall that the \( (N_1 + 2N_2 + N_3) \) unknowns are the complex Lamb wave mode amplitudes \( \left( C_{1B,n_1}, C_{2F,n_2}, C_{2B,n_3}, C_{3F,n_3} \right) \). Thus, Eqs. (4-17), (4-19), (4-21), (4-23) combined are a set of \( (\bar{N}_1 + 2\bar{N}_2 + \bar{N}_3) \) linear algebraic equations in \( (N_1 + 2N_2 + N_3) \) unknowns \( C_{1B,n_1}, C_{2F,n_2}, C_{2B,n_3}, C_{3F,n_3} \). By assuming \( N_1 = N_2 = N_3 = \bar{N}_1 = \bar{N}_2 = \bar{N}_3 = N \), we get \( 4N \) equations in \( 4N \) unknowns. Then Eqs. (4-17) and (4-19) can be written as

\[
\begin{bmatrix}
C_{2F,1} \\
\vdots \\
C_{2F,N}
\end{bmatrix}_{N \times 1} + \begin{bmatrix}
C_{2B,1} \\
\vdots \\
C_{2B,N}
\end{bmatrix}_{N \times 1} - \begin{bmatrix}
C_{1B,1} \\
\vdots \\
C_{1B,N}
\end{bmatrix}_{N \times 1} + \begin{bmatrix}
C_{3F,1} \\
\vdots \\
C_{3F,N}
\end{bmatrix}_{N \times 1} = \{E\}_{N \times 1}
\]

\[
\begin{bmatrix}
C_{2F,1} \\
\vdots \\
C_{2F,N}
\end{bmatrix}_{N \times 1} + \begin{bmatrix}
C_{2B,1} \\
\vdots \\
C_{2B,N}
\end{bmatrix}_{N \times 1} + \begin{bmatrix}
C_{1B,1} \\
\vdots \\
C_{1B,N}
\end{bmatrix}_{N \times 1} - \begin{bmatrix}
C_{3F,1} \\
\vdots \\
C_{3F,N}
\end{bmatrix}_{N \times 1} = \{0\}_{N \times 1}
\]

Similarly from Eqs. (4-21) and (4-23) can be written as

\[
\left( N_1 + 2N_2 \right) + \left( N_3 + 2N_2 \right) = \left( N_1 + 2N_2 + N_3 \right)
\]
\[
[J]_{N \times N} \begin{bmatrix} C_{2F,1} \\ \vdots \\ C_{2F,N} \end{bmatrix} + [K]_{N \times N} \begin{bmatrix} C_{2B,1} \\ \vdots \\ C_{2B,N} \end{bmatrix} - [L]_{N \times N} \begin{bmatrix} C_{1B,1} \\ \vdots \\ C_{1B,N} \end{bmatrix} + [0]_{N \times 1} \begin{bmatrix} C_{3F,1} \\ \vdots \\ C_{3F,N} \end{bmatrix} = \{M\}_{N \times 1}
\]
(4-26)

\[
[N]_{N \times N} \begin{bmatrix} C_{2F,1} \\ \vdots \\ C_{2F,N} \end{bmatrix} + [O]_{N \times N} \begin{bmatrix} C_{2B,1} \\ \vdots \\ C_{2B,N} \end{bmatrix} + [0]_{N \times N} \begin{bmatrix} C_{1B,1} \\ \vdots \\ C_{1B,N} \end{bmatrix} - [P]_{N \times 1} \begin{bmatrix} C_{3F,1} \\ \vdots \\ C_{3F,N} \end{bmatrix} = \{0\}_{N \times 1}
\]
(4-27)

In Eqs. (4-24), (4-25), (4-26) and (4-27) the coefficient matrices \([A], [B], [D], [E], [F], [G], [H], [J], [K], [L], [M], [N], [O]\) and \([P]\) are known matrices containing the vector projected boundary conditions. Combining them we get

\[
\begin{bmatrix} A & B & -D & 0 \\ F & G & 0 & -H \\ J & K & -L & 0 \\ N & O & 0 & -P \end{bmatrix}_{4N \times 4N} \begin{bmatrix} C_{2F,1} \\ \vdots \\ C_{2B,N} \\ C_{1B,1} \\ \vdots \\ C_{1B,N} \\ C_{3F,1} \\ \vdots \\ C_{3F,N} \end{bmatrix}_{4N \times 1} = \begin{bmatrix} E \\ 0 \\ 0 \\ 0 \end{bmatrix}_{4N \times 1}
\]
(4-28)

\[
\Rightarrow [Q]_{4N \times 4N} \{C\}_{4N \times 1} = \{R\}_{4N \times 1}
\]

Eq. (4-28) can be solved for the unknown amplitudes of the reflected and transmitted Lamb wave modes as

\[
\{C\}_{4N \times 1} = [Q]^{-1}_{4N \times 4N} [R]_{4N \times 1}
\]
(4-29)
4.2.5 Results and discussion

As a test case we consider a vertical stiffener in an aluminum plate with $E = 70$ GPa, $\rho = 2780$ kg/m$^3$, $\nu = 0.33$, $h_1 = 1/6''$, $h_2 = 1/3''$ mm, and $2b = 1/3''$ mm. This results in depth ratio $R_d = (h_2 - h_1)/h_1 = 2$, and width ratio $R_w = 2b/h_1 = 2$. We also consider both S0 and A0 as the incident Lamb wave modes. We use a frequency-thickness range of up to 350 kHz for its relevance to practical applications. We perform convergence studies to determine the maximum number of complex roots of the Rayleigh-Lamb equation needed to calculate the first two scattered Lamb wave modes S0 and A0 with high accuracy. Figure 4.3 shows the convergence study for the amplitudes of the first two modes of Lamb waves, S0 and A0. Considering 27 modes in the expansion ensured convergence to less than 0.5% error. Another important verification of convergence is the power flow balance. Figure 4.4 shows that a 27-mode expansion gave a balanced average power flow though the vertical stiffener with different widths over the whole frequency range up to 750 kHz.

As shown in Figure 4.5 - Figure 4.8 the scattered wave amplitudes and phase angle for symmetric and antisymmetric modes are calculated using CMEP method for a vertical stiffener on an aluminum plate. We can also see that the power flow across the stiffener is conserved, as the incident wave power and the scattered wave power are identical.
Figure 4.3 Convergence of (a) amplitudes, (b) phases of scattered Lamb wave modes for S0 mode incident on a vertical stiffener with $R_d=0.5$ and $R_w=0.5$ over a wide frequency range of 50 kHz.mm to 750 kHz.

Figure 4.4 Variation of time average power flow $P_{av}$ through the vertical stiffener as calculated by CMEP with a 27-mode expansion for S0 mode incident
Figure 4.5  Normalized amplitude of $u_x$ displacement of transmitted and reflected modes for incident S0 mode

Figure 4.6  Phase of $u_x$ displacement of transmitted and reflected modes for incident S0 mode
Figure 4.7  Normalized amplitude of $u_x$ displacement of transmitted and reflected modes for incident A0 mode

Figure 4.8  Phase of $u_x$ displacement of transmitted and reflected modes for incident A0 mode
4.3 CMEP FOR A CRACKED STIFFENER

4.3.1 Problem Setup for cracked Stiffener

To begin our analysis, let us consider a plate with a cross section as shown in Figure 4.9. We assume that there is an incident straight-crested Lamb wave mode travelling from the left towards a cracked stiffener. Upon interacting with the cracked stiffener, it will result in reflected wave modes, transmitted wave modes, and wave modes trapped in the stiffener. As shown in Figure 4.9, the stiffener is located at a distance $x = x_0$ with the thickness of the plate being $h_1$. The height of the stiffener is $h_2$ with the width of $L = 2b$. Crack appears at the left end of the stiffener with half of the stiffener width. At the stiffener, we define height ratio as $R_h = (h_2 - h_1)/h_1$ and width ratio as $R_w = 2b/h_1$. Also, let us imagine that the incident wave field is represented by $(\Phi_0, H_0)$, travelling in $+ve$ $x$ direction in the Region 1. We define the reflected wave field as $(\Phi_1, H_1)$ and the transmitted wave field in Region 4 as $(\Phi_4, H_4)$. We also define the trapped wave field inside the stiffener in Region 2 as $(\Phi_2, H_2)$ and region 3 as $(\Phi_3, H_3)$.

The incident and scattered wave fields satisfy the generic wave equations

$$\frac{\partial^2 \Phi}{\partial x^2} + \frac{\partial^2 \Phi}{\partial y^2} = \frac{1}{C_p^2} \Phi$$

$$\frac{\partial^2 H}{\partial x^2} + \frac{\partial^2 H}{\partial y^2} = \frac{1}{C_s^2} \ddot{H}$$

(4-30)

where, $C_p$ and $C_s$ are the wave speeds of pressure wave and shear waves, respectively.

These must satisfy the zero-stress boundary condition at the top and bottom of the plate,
\begin{align}
\sigma_{yy} \left\{ \begin{array}{l}
\text{for } x < x_0 - b; \quad y = \pm \frac{h}{2} \\
\text{Region 1}
\end{array} \right. &= 0 \\
\tau_{xy} \left\{ \begin{array}{l}
\text{for } x < x_0 - b; \quad y = \pm \frac{h}{2} \\
\text{Region 1}
\end{array} \right. &= 0 \\
\sigma_{yy} \left\{ \begin{array}{l}
\text{for } 0 > x > x_0 - b; \quad y = h_2 - h_1 \frac{h}{2} \\
\text{Region 2}
\end{array} \right. &= 0 \\
\tau_{xy} \left\{ \begin{array}{l}
\text{for } 0 > x > x_0 - b; \quad y = h_2 - h_1 \frac{h}{2} \\
\text{Region 2}
\end{array} \right. &= 0 \\
\sigma_{yy} \left\{ \begin{array}{l}
\text{for } x_0 + b > x > 0; \quad y = \frac{h}{2} - h_1 \frac{h}{2} \\
\text{Region 3}
\end{array} \right. &= 0 \\
\tau_{xy} \left\{ \begin{array}{l}
\text{for } x_0 + b > x > 0; \quad y = \frac{h}{2} - h_1 \frac{h}{2} \\
\text{Region 3}
\end{array} \right. &= 0 \\
\sigma_{yy} \left\{ \begin{array}{l}
\text{for } x > x_0 + b; \quad y = \frac{h}{2} \\
\text{Region 4}
\end{array} \right. &= 0 \\
\tau_{xy} \left\{ \begin{array}{l}
\text{for } x > x_0 + b; \quad y = \frac{h}{2} \\
\text{Region 4}
\end{array} \right. &= 0
\end{align}

Figure 4.9 Schematic of Lamb waves interacting with a stiffener
The stress and displacement fields associated with the wave fields are expressed as

\[
\sigma = \begin{bmatrix}
\sigma_{xx} & \tau_{xx} \\
\tau_{xy} & \sigma_{yy}
\end{bmatrix}; \quad \bar{\sigma} = \begin{bmatrix}
\sigma_{xx} \\
\tau_{xy}
\end{bmatrix}; \quad \bar{u} = \begin{bmatrix}
u_x \\
u_y
\end{bmatrix}
\]

(4-32)

where, \(\sigma\) is stress tensor, \(\bar{\sigma}\) is stress vector, and \(\bar{u}\) is displacement vector. For the displacements, subscripts \(x\) and \(y\) indicate the directions of the displacement. For stresses, the subscripts \(xx\), \(yy\) stand for the normal stress in \(x\), \(y\) directions, respectively and the subscript \(xy\) stands for the shear stress. The boundary conditions at the stiffener are illustrated in Figure 4.10.

The boundary conditions at the interface at \(x = x_0\) are,

\[
\bar{\sigma}_3 = \begin{cases}
\bar{\sigma}_2, & x = 0, \quad h_0/2 \leq y \leq h_2 - h_0/2 \\
(\bar{\sigma}_0 + \bar{\sigma}_1), & x = 0, \quad -h_0/2 \leq y \leq h_0/2
\end{cases}
\]

(4-33)

\[
\bar{u}_3 = \begin{cases}
\bar{u}_2, & x = 0, \quad h_0/2 \leq y \leq h_2 - h_0/2 \\
\bar{u} + \bar{u}_1, & x = 0, \quad -h_0/2 \leq y \leq h_0/2
\end{cases}
\]

(4-34)

The boundary conditions at the interface at \(x = x_0 + b\) are,
\[ \sigma_3 = \begin{cases} 0, & x = x_0 + b, \quad h_i/2 \leq y \leq h_2 - h_i/2 \\ -\sigma_4, & x = x_0 + b, \quad -h_i/2 \leq y \leq h_i/2 \end{cases} \quad (4-35) \]

\[ \tilde{u}_3 = \{ \tilde{u}_4, \quad x = 0, \quad -h_i/2 \leq y \leq h_i/2 \} \quad (4-36) \]

The boundary conditions at the interface at \( x = x_0 - b \) are,

\[ -\sigma_2 = \begin{cases} 0, & x = x_0 - b, \quad h_i/2 \leq y \leq h_2 - h_i/2 \end{cases} \quad (4-37) \]

Therefore, Eqs.(4-33), (4-34), (4-35) (4-36) and (4-6) represent the thickness dependent boundary conditions at the stiffener.

The subscript 0 stands for incident waves in Region 1, subscript 1 stands for reflected waves in Region 1, subscripts 2 and 3 stand for the trapped waves in Regions 2 and 3, respectively. The subscript 4 stands for transmitted waves in Region 4. Let us assume the incident wave field in Region 1to be harmonic of the form

\[ \Phi_0 = f_0(y)e^{i(\xi_0 x - \omega t)} \quad H_0 = i f_0(y)e^{i(\xi_0 x - \omega t)} \quad (4-38) \]

Assuming \( \xi_0 \) to be one of the roots of Rayleigh-Lamb equation for the plate in Region 1, Eq. (4-1)and (4-38) are satisfied by definition of Lamb waves and the incident wave becomes one of the modes of Lamb waves.

**4.3.2 Complex Modes Expansion of the Scattered Wave Field for a Stiffener**

We expand the transmitted, reflected, and trapped wave fields in terms of all possible complex Lamb wave modes corresponding to the complex roots of Rayleigh-Lamb frequency equation. Therefore, the scattered wave fields are expressed as
\[
\Phi_1 = \sum_{n_1}^\infty C_{1B,n_1}^{1B} \Phi_{1B,n_1} = \sum_{n_1}^\infty C_{1B,n_1} f_{n_1}(y_1)e^{i(z_{1B,n_1}y - \alpha_1)}
\]

\[
H_1 = \sum_{n_1}^\infty C_{1B,n_1} H_{1B,n_1} = \sum_{n_1}^\infty C_{1B,n_1} h_{n_1}(y_1)e^{i(z_{1B,n_1}y - \alpha_1)}
\]

\[
\Phi_2 = \sum_{n_2}^\infty \left( C_{2F,n_2} \Phi_{2F,n_2} + C_{2B,n_2} \Phi_{2B,n_2} \right)
= \sum_{n_2}^\infty \left( C_{2F,n_2} f_{n_2}(y_2)e^{i(z_{2F,n_2}y - \alpha_2)} + C_{2B,n_2} f_{n_2}(y_2)e^{i(z_{2B,n_2}y - \alpha_2)} \right)
\]

\[
H_2 = \sum_{n_2}^\infty \left( C_{2F,n_2} H_{2F,n_2} + C_{2B,n_2} H_{2B,n_2} \right)
= \sum_{n_2}^\infty \left( C_{2F,n_2} h_{n_2}(y_2)e^{i(z_{2F,n_2}y - \alpha_2)} + C_{2B,n_2} h_{n_2}(y_2)e^{i(z_{2B,n_2}y - \alpha_2)} \right)
\]

\[
\Phi_3 = \sum_{n_3}^\infty \left( C_{3F,n_3} \Phi_{3F,n_3} + C_{3B,n_3} \Phi_{3B,n_3} \right)
= \sum_{n_3}^\infty \left( C_{3F,n_3} f_{n_3}(y_3)e^{i(z_{3F,n_3}y - \alpha_3)} + C_{3B,n_3} f_{n_3}(y_3)e^{i(z_{3B,n_3}y - \alpha_3)} \right)
\]

\[
H_3 = \sum_{n_3}^\infty \left( C_{3F,n_3} H_{3F,n_3} + C_{3B,n_3} H_{3B,n_3} \right)
= \sum_{n_3}^\infty \left( C_{3F,n_3} h_{n_3}(y_3)e^{i(z_{3F,n_3}y - \alpha_3)} + C_{3B,n_3} h_{n_3}(y_3)e^{i(z_{3B,n_3}y - \alpha_3)} \right)
\]

\[
\Phi_4 = \sum_{n_4}^\infty C_{4F,n_4} \Phi_{4F,n_4} = \sum_{n_4}^\infty C_{4F,n_4} f_{n_4}(y_4)e^{i(z_{4F,n_4}y - \alpha_4)}
\]

\[
H_4 = \sum_{n_4}^\infty C_{3F,n_4} H_{3F,n_4} = \sum_{n_4}^\infty C_{3F,n_4} h_{n_4}(y_4)e^{i(z_{3F,n_4}y - \alpha_4)}
\]

where, subscripts \( B \) and \( F \) stand for backward and forward propagating wave, \( y_1, y_2, y_3 \) are connected by the expressions \( y_2 = y_1 + a_2 \) and \( y_3 = y_1 + a_3 \) with \( a_2 = (h_1 - h_2) / 2 \) and \( a_3 = -h_2 / 2 \) being the eccentricities of Region 2 and Region 3 from Region 1 and \( y_1 = y_4 \). The wavenumber \( z_{1B,n_1} \) is the \( n_1 \)th complex root of the Rayleigh-Lamb equation corresponding to backward propagating Lamb waves in Region 1 and the wavenumbers
$\xi_{2F,n_2}$ and $\xi_{2B,n_2}$ are $n_2$th complex root of the Rayleigh-Lamb equation corresponding to forward and backward propagating waves in Region 2, respectively. Similarly, wavenumber $\xi_{4F,n_4}$ is the $n_4$th complex root of the Rayleigh-Lamb equation corresponding to forward propagating waves in Region 4 and the wavenumbers $\xi_{3F,n_3}$ and $\xi_{3B,n_3}$ are $n_3$th complex root of the Rayleigh-Lamb equation corresponding to forward and backward propagating waves in Region 3, respectively. The coefficient $C_{1B,n_1}$ is the unknown amplitude of the $n_1$th mode of backward propagating Lamb waves in Region 1 whereas $C_{2F,n_2}$ and $C_{2B,n_2}$ are the unknown amplitudes of the $n_2$th mode of forward and backward propagating Lamb waves in Region 2, respectively. Similarly, the coefficient $C_{4F,n_4}$ is the unknown amplitude of the $n_4$th mode of forward propagating Lamb waves in Region 4 whereas $C_{3F,n_3}$ and $C_{3B,n_3}$ are the unknown amplitudes of the $n_3$th mode of forward and backward propagating Lamb waves in Region 3, respectively. The amplitudes $C_{1B,n_1}$, $C_{2F,n_2}$, $C_{2B,n_2}$, $C_{3F,n_3}$, $C_{3B,n_3}$ and $C_{4F,n_4}$ of these modes have to be determined through the boundary matching process. Recall the boundary conditions at the interfaces at $x = x_0 - b$; $x = 0$ and $x = x_0 + b$ are given by equations (4-33), (4-34), (4-35), (4-36) and (4-37).

We express the stress and displacement fields of Eqs. (4-33), (4-34), (4-35), (4-36) and (4-37) using the complex Lamb wave mode expansion of Eq.(4-9), i.e.,
\[\vec{u}_r = \sum_{n_1=1}^{\infty} C_{1B,n_1} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{1B,n_1}; \quad \vec{\sigma}_r = \sum_{n_1=1}^{\infty} C_{1B,n_1} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{1B,n_1}\]

\[\vec{u}_2 = \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F,n_2} + C_{2B,n_2} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{2F,n_2} \right)\]

\[\vec{\sigma}_2 = \sum_{n_2=1}^{\infty} \left( C_{2F,n_2} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2F,n_2} + C_{2B,n_2} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2F,n_2} \right)\]

\[\vec{u}_3 = \sum_{n_3=1}^{\infty} \left( C_{3F,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3F,n_3} + C_{3B,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3F,n_3} \right)\]

\[\vec{\sigma}_3 = \sum_{n_3=1}^{\infty} \left( C_{3F,n_3} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{3F,n_3} + C_{3B,n_3} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{3F,n_3} \right)\]

\[\vec{u}_4 = \sum_{n_4=1}^{\infty} C_{4F,n_4} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{4F,n_4}; \quad \vec{\sigma}_4 = \sum_{n_4=1}^{\infty} C_{4F,n_4} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{4F,n_4}\]

(4-40)

In the same vein, the incident wave field uses subscript 0, i.e.,

\[\vec{u}_0 = \begin{bmatrix} u_x \\ u_y \end{bmatrix}_0; \quad \vec{\sigma}_0 = \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_0\]

(4-41)

Using equations (4-38) and (4-9) into equations (4-4), (4-5), (4-35) (4-36) and (4-6) yield

\[\vec{\sigma}_3 = \begin{cases} \vec{\sigma}_2, & x = 0, \ h_1/2 \leq y \leq h_2 - h_1/2 \\ \vec{\sigma}_0 + \vec{\sigma}_1, & x = 0, \ -h_1/2 \leq y \leq h_1/2 \end{cases}\]

(4-42)

\[\vec{u}_0 + \vec{u}_1 = \vec{u}_3, \quad x = 0, \ -h_1/2 \leq y \leq h_1/2\]

(4-43)

\[\vec{u}_2 = \vec{u}_3, \quad x = 0, \ h_1/2 \leq y \leq h_2 - h_1/2\]

(4-44)

\[\vec{\sigma}_3 = \begin{cases} 0, & x = x_0 + b, \ h_1/2 \leq y \leq h_2 - h_1/2 \\ \vec{\sigma}_4, & x = x_0 + b, \ -h_1/2 \leq y \leq h_1/2 \end{cases}\]

(4-45)
\[ \bar{u}_3 = \bar{u}_4, \quad x = 0, \quad -h_1/2 \leq y \leq h_1/2 \]  \hfill (4-46)

\[ \bar{\sigma}_2 = \{0, \quad x = x_0 - b, \quad h_1/2 \leq y \leq h_2 - h_1/2 \} \]  \hfill (4-47)

Therefore, equations (4-42), (4-43), (4-44), (4-45), (4-46) and (4-47) represent the thickness dependent boundary conditions at the cracked stiffener.

### 4.3.3 Vector Projection of the Boundary Conditions for cracked Stiffener

CMEP formulation incorporates the average power flow associated with the reflected, transmitted, and trapped wave fields. CMEP uses the time averaged power flow expression, which uses stress-velocity product. Thus, in Region 1, we project the displacement boundary conditions onto the conjugate stress vector space of the complex Lamb wave modes; in Region 2, we project the stress boundary conditions onto the conjugate displacement vector space of the complex Lamb wave modes. Similarly, in Region 3, we project the displacement boundary conditions onto the conjugate stress vector space of the complex Lamb wave modes.

The projection vector space for Eq. (4-42) is

\[ \bar{u}_{3B} = \text{conj} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3B,n_i} = \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix}_{3B,n_i}, \quad n_i = 1, 2, 3, \ldots \]  \hfill (4-48)

After projecting Eq. (4-42) onto Eq. (4-48), the stress boundary conditions in Eq. (4-42) takes the form,
\[
\begin{split}
\int_{-\frac{h}{2}}^{\frac{h}{2}} (\bar{\sigma}_0 + \bar{\sigma}_1) \cdot \vec{u}_3 \, dy + \int_{\frac{h}{2}}^{h - \frac{h}{2}} \bar{\sigma}_2 \cdot \vec{u}_3 \, dy = \int_{-\frac{h}{2}}^{h - \frac{h}{2}} \bar{\sigma}_3 \cdot \vec{u}_3 \, dy
\end{split}
\]

\[
\Rightarrow - \sum_{n_1=1}^{\infty} \left[ C_{1B,n_1} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix} \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix} \right]_{-\frac{h}{2}}^{\frac{h}{2}}
\]

\[
- \sum_{n_2=1}^{\infty} \left[ C_{2F,n_2} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix} \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix} \right]_{-\frac{h}{2}}^{h - \frac{h}{2}}
\]

\[
+ \sum_{n_3=1}^{\infty} \left[ C_{3B,n_3} \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix} \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix} \right]_{h/2}^{h - h/2}
\]

\[
= \begin{bmatrix} \sigma_{xx} \\ \sigma_{xy} \end{bmatrix} \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix} \left. \right|_{-\frac{h}{2}}^{\frac{h}{2}}
\]

\[; n_1, n_2, n_3 = 1, 2, 3, \ldots\]

where, \( \int_a^b P \cdot Q \, dy = \langle P, Q \rangle_a^b \) represents the inner product. Similarly, the projection vector space for Eq. (4-45) is

\[
\vec{u}_{3F} = \text{conj} \begin{bmatrix} u_x \\ u_y \end{bmatrix} = \begin{bmatrix} \vec{u}_x \\ \vec{u}_y \end{bmatrix}_{3F,n_3}, \ n_3 = 1, 2, 3, \ldots
\]

Upon projecting Eq. (4-46) onto Eq. (4-18), the stress boundary conditions in Eq. (4-46) takes the form,
The projection vector space for Eq. (4-47) is

\[
\tilde{\mathbf{u}}_{3B} = \text{conj} \begin{bmatrix} u_x \\ u_y \end{bmatrix}_{3B, n_3} = \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix}_{3B, n_3}, \quad n_3 = 1, 2, 3, \ldots
\]  

Upon projecting Eq. (4-47) onto Eq. (4-52), the stress boundary conditions in Eq. (4-47) takes the form,

\[
\int_{h_1/2}^{h_1-h/2} \tilde{\sigma}_2 \cdot \tilde{\mathbf{u}}_{3B} \, dy = 0;
\]

\[
\Rightarrow \sum_{n_3=1}^{\infty} C_{2F, n_3} \left< \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2B, n_3}, \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix}_{3B, n_3} \right>_{h_1/2}^{h_1-h/2} + C_{2B, n_3} \left< \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{2B, n_3}, \begin{bmatrix} \bar{u}_x \\ \bar{u}_y \end{bmatrix}_{3B, n_3} \right>_{h_1/2}^{h_1-h/2} = 0 \quad (4-53)
\]

\( n_3 = 1, 2, 3, \ldots \)

The projection vector space for Eq. (4-43) is,

\[
\tilde{\sigma}_{1B} = \text{conj} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}_{1B, n_1} = \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\tau}_{xy} \end{bmatrix}_{1B, n_1}, \quad n_1 = 1, 2, 3, \ldots
\]  

After projecting Eq. (4-43) onto Eq. (4-20), the displacement boundary conditions in Eq. (4-43), take the form,
\[ \int_{-h/2}^{h/2} \left( \tilde{u}_0 + \tilde{u}_1 \right) \cdot \tilde{\sigma}_{1B} \, dy = \int_{-h/2}^{h/2} \tilde{u}_3 \cdot \tilde{\sigma}_{1B} \, dy \]

\[ \Rightarrow \sum_{n_3=1}^{\infty} \left( C_{3F,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} \tilde{\sigma}_{xx} \\ \tilde{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2} + C_{3B,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_{xx} \\ \tilde{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2} \]

\[ \label{eq:4-55} \]

\[ -\sum_{n_3=1}^{\infty} \left( C_{3F,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} \tilde{\sigma}_{xx} \\ \tilde{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2} = \begin{bmatrix} u_x \\ u_y \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_{xx} \\ \tilde{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2} \]

\[ n_1, n_3 = 1, 2, 3, \ldots \]

The projection vector space for Eq. (4.46) is,

\[ \tilde{\sigma}_{3B} = \text{conj} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}, \quad \tilde{n}_3 = 1, 2, 3, \ldots \]

Upon projecting Eq. (4.46) onto Eq. (4.22), the displacement boundary conditions in Eq. (4.46) takes the form,

\[ \int_{h/2}^{h/2} \tilde{u}_2 \cdot \tilde{\sigma}_{3F} \, dy = \int_{h/2}^{h/2} \tilde{u}_3 \cdot \tilde{\sigma}_{3F} \, dy \]

\[ \Rightarrow \sum_{n_3=1}^{\infty} \left( C_{2F,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} \tilde{\sigma}_{xx} \\ \tilde{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2} + C_{2B,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_{xx} \\ \tilde{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2} \]

\[ \label{eq:4-57} \]

\[ \sum_{n_3=1}^{\infty} \left( C_{3F,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \cdot \begin{bmatrix} \tilde{\sigma}_{xx} \\ \tilde{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2} + C_{3B,n_3} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \begin{bmatrix} \tilde{\sigma}_{xx} \\ \tilde{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2} \]

\[ n_2, n_3 = 1, 2, 3, \ldots \]

The projection vector space for Eq. (4.46) is,

\[ \tilde{\sigma}_{4F} = \text{conj} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix} \begin{bmatrix} \sigma_{xx} \\ \tau_{xy} \end{bmatrix}, \quad \tilde{n}_3 = 1, 2, 3, \ldots \]

Upon projecting Eq. (4.58) onto Eq. (4.46), the displacement boundary conditions in Eq. (4.46) takes the form,
\[
\int_{-h/2}^{h/2} \bar{u}_3 \cdot \bar{\sigma}_{4F} dy = \int_{-h/2}^{h/2} \bar{u}_4 \cdot \bar{\sigma}_{4F} dy
\]

\[
\Rightarrow \sum_{n=1}^{\infty} \left( C_{3F,n_1} \begin{bmatrix} u_x \\ u_y \end{bmatrix}, \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2} + C_{3B,n_1} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2}
\]

\[
= \sum_{n=1}^{\infty} C_{4F,n_4} \begin{bmatrix} u_x \\ u_y \end{bmatrix} \begin{bmatrix} \bar{\sigma}_{xx} \\ \bar{\tau}_{xy} \end{bmatrix} \right)_{-h/2}^{h/2}
\]

\[
n_2, n_4 = 1, 2, 3, ...
\]

### 4.3.4 Numerical Solution for Cracked Stiffener

For numerical calculation we consider finite values for the indices

\[
n_1, n_2, n_3, n_4, \tilde{n}_1, \tilde{n}_2, \tilde{n}_3, \tilde{n}_4.
\]

We assume, \(n_1 = 1, 2, 3, N_1\), \(n_2 = 1, 2, 3, N_2\), \(n_3 = 1, 2, 3, N_3\), \(n_4 = 1, 2, 3, N_4\), \(\tilde{n}_1 = 1, 2, 3, \tilde{N}_1\), \(\tilde{n}_2 = 1, 2, 3, \tilde{N}_2\), \(\tilde{n}_3 = 1, 2, 3, \tilde{N}_3\), \(\tilde{n}_4 = 1, 2, 3, \tilde{N}_4\). Equations (4-49), (4-51), (4-53), (4-55), (4-57), (4-59) combine to form a set of \( (\tilde{N}_1 + 2\tilde{N}_2 + 2\tilde{N}_3 + \tilde{N}_4) \) linear algebraic equations in \( (N_1 + 2N_2 + 2N_3 + N_4) \) unknowns \( C_{1B,n_1}, C_{2F,n_2}, C_{2B,n_2}, C_{3F,n_1}, C_{3B,n_1}, C_{4F,n_4} \). By assuming

\[
N_1 = N_2 = N_3 = N_4 = \tilde{N}_1 = \tilde{N}_2 = \tilde{N}_3 = \tilde{N}_4 = N,
\]

we get \(6N\) equations in \(6N\) unknowns.

Then, equations, (4-49), (4-51), (4-53), can be written as

\[
- [A]_{N\times N} \{C_{1B}\}_{N\times 1} - [B]_{N\times N} \{C_{2F}\}_{N\times 1} - [D]_{N\times N} \{C_{2B}\}_{N\times 1} + [F]_{N\times N} \{C_{3F}\}_{N\times 1} + [E]_{N\times N} \{C_{3B}\}_{N\times 1} = 0_{N\times 1}
\]

(4-60)

\[
- [G]_{N\times N} \{C_{3B}\}_{N\times 1} - [0]_{N\times N} \{C_{4F}\}_{N\times 1} = 0_{N\times 1}
\]

(4-61)

\[
- [H]_{N\times N} \{C_{3F}\}_{N\times 1} + [I]_{N\times N} \{C_{4F}\}_{N\times 1} = 0_{N\times 1}
\]

(4-62)
Similarly equations (4-21), (4-23) and (4-59) can be written as

\[
\begin{align*}
-\begin{bmatrix} M_N & & & & \end{bmatrix}_{N \times N} & \{ C_{1B} \}_{N \times 1} + \begin{bmatrix} 0 & & & \end{bmatrix}_{N \times N} \{ C_{2F} \}_{N \times 1} + \begin{bmatrix} 0 & & \end{bmatrix}_{N \times N} \{ C_{2B} \}_{N \times 1} + \begin{bmatrix} N & & \end{bmatrix}_{N \times N} \{ C_{3F} \}_{N \times 1} \\
+ \begin{bmatrix} O_N & & & & \end{bmatrix}_{N \times N} \{ C_{3B} \}_{N \times 1} - \begin{bmatrix} 0 & & & \end{bmatrix}_{N \times N} \{ C_{4F} \}_{N \times 1} &= \{ P \}_{N \times 1}  \\
\end{align*}
\]  

(4-63)

\[
\begin{align*}
-\begin{bmatrix} 0 & & & \end{bmatrix}_{N \times N} \{ C_{1B} \}_{N \times 1} + \begin{bmatrix} 0 & & & \end{bmatrix}_{N \times N} \{ C_{2F} \}_{N \times 1} + \begin{bmatrix} 0 & & \end{bmatrix}_{N \times N} \{ C_{2B} \}_{N \times 1} + \begin{bmatrix} Q \end{bmatrix}_{N \times N} \{ C_{3F} \}_{N \times 1} \\
+ \begin{bmatrix} R_N & & & & \end{bmatrix}_{N \times N} \{ C_{3B} \}_{N \times 1} - \begin{bmatrix} S \end{bmatrix}_{N \times N} \{ C_{4F} \}_{N \times 1} &= \{ 0 \}_{N \times 1}  \\
\end{align*}
\]  

(4-64)

\[
\begin{align*}
-\begin{bmatrix} 0 & & & \end{bmatrix}_{N \times N} \{ C_{1B} \}_{N \times 1} - \begin{bmatrix} U \end{bmatrix}_{N \times N} \{ C_{2F} \}_{N \times 1} - \begin{bmatrix} V \end{bmatrix}_{N \times N} \{ C_{2B} \}_{N \times 1} + \begin{bmatrix} W \end{bmatrix}_{N \times N} \{ C_{3F} \}_{N \times 1} \\
+ \begin{bmatrix} X \end{bmatrix}_{N \times N} \{ C_{3B} \}_{N \times 1} - \begin{bmatrix} 0 \end{bmatrix}_{N \times N} \{ C_{4F} \}_{N \times 1} &= \{ 0 \}_{N \times 1}  \\
\end{align*}
\]  

(4-65)

In equations (4-60), (4-61), (4-62), (4-63), (4-64) and (4-65) the coefficient matrices \([A], [B], [D], [E], [F], [G], [H], [I], [J], [K], [L], [M], [N], [O], [P], [Q], [R], [S], [U], [V], [W], [X]\) are known matrices containing the vector-projected boundary conditions; the vectors \{C_{1B}\}, \{C_{2F}\}, \{C_{2B}\}, \{C_{3F}\}, \{C_{3B}\} and \{C_{4F}\} contain the unknown coefficients. Combining equations (4-60) through (4-65) we get

\[
\begin{bmatrix}
-A & -B & -D & F & G & 0 \\
0 & 0 & 0 & H & I & -J \\
0 & 0 & 0 & K & L & 0 \\
-M & 0 & 0 & N & O & 0 \\
0 & 0 & 0 & Q & R & -S \\
0 & -U & -V & W & X & 0
\end{bmatrix}_{6N \times 6N} \begin{bmatrix}
C_{1B} \\
C_{2F} \\
C_{2B} \\
C_{3F} \\
C_{3B} \\
C_{4F}
\end{bmatrix}_{6N \times 1} = \begin{bmatrix}
E \\
0 \\
0 \\
C_{3F} \\
0 \\
0
\end{bmatrix}_{6N \times 1}  
\]  

(4-66)

\[
\Rightarrow \begin{bmatrix} Y \end{bmatrix}_{6N \times 6N} \begin{bmatrix} C \end{bmatrix}_{6N \times 1} = \begin{bmatrix} \Lambda \end{bmatrix}_{6N \times 1}
\]

Equation (4-66) can be solved for the unknown amplitudes of the reflected and transmitted Lamb wave modes as

\[
\begin{bmatrix} C \end{bmatrix}_{6N \times 1} = \begin{bmatrix} Y \end{bmatrix}_{6N \times 6N}^{-1} \begin{bmatrix} \Lambda \end{bmatrix}_{6N \times 1}  
\]  

(4-67)
Figure 4.11 Convergence of (a) amplitudes, (b) phases of scattered Lamb wave modes for S0 mode incident on a vertical stiffener with $R_d=0.5$ and $R_w=0.5$ over a wide frequency range of 50 kHz to 750 kHz.

As a test case we consider a vertical stiffener in an aluminum plate with $E = 70$ GPa, $\rho = 2780$ kg/m$^3$, $\nu = 0.33$, $h_1 = 1/6''$, $h_2 = 1/3''$ mm, and $2b = 1/3''$ mm. This results in depth ratio $R_d = (h_2 - h_1)/h_1 = 2$, and width ratio $R_w = 2b/h_1 = 2$.

We use a frequency-thickness range of up to 350 kHz for its relevance to practical applications. We perform convergence studies to determine the maximum number of
complex roots of the Rayleigh-Lamb equation needed to calculate the first two scattered Lamb wave modes S0 and A0 with high accuracy. Figure 4.11 shows the convergence study for the amplitudes of the first two modes of Lamb waves, S0 and A0. Considering 27 modes in the expansion ensured convergence to less than 0.5% error. Another important verification of convergence is the power flow balance. Figure 4.12 shows that a 27-mode expansion gave a balanced average power flow though the vertical stiffeners with different widths over the whole frequency range up to 350 kHz.

Figure 4.13  Normalized amplitude of $u_x$, displacement of transmitted and reflected modes for incident S0 mode
Figure 4.14  Phase of $u_x$ displacement of transmitted and reflected modes for incident S0 mode

Figure 4.15  Normalized amplitude of $u_x$ displacement of transmitted and reflected modes for incident A0 mode
Figure 4.16  Phase of \( u_x \) displacement of transmitted and reflected modes for incident A0 mode

As shown in Figure 4.13-Figure 4.16 the scattered wave amplitudes and phase angle for symmetric and antisymmetric modes are calculated using CMEP method for a vertical stiffener on an aluminum plate. We can also see that the power flow across the stiffener is conserved, as the incident wave power and the scattered wave power are identical.

4.4 **Verification of CMEP for a Stiffener Using FEM**

For the validation of the results from CMEP, we made finite element models (FEM) using the commercial software ANSYS. The dimensions and the material properties were chosen as described in previous sections. To ensure accuracy of the FEM results we follow Moser et al. (Moser, Jacobs, and Qu 1999). We used PLANE182 elements in ANSYS multiphysics. We ensured the value of \( \lambda/l_{FEM} \) to be at least 30 (Poddar et al., 2016) which is higher than the recommended value for an accurate result. We also performed convergence study with progressively smaller element sizes with the value of \( \lambda/l_{FEM} \) to be up to 40.
4.4.1 Finite element modeling

As a test case we consider a vertical stiffener in an aluminum plate with $E = 70$ GPa, $\rho = 2780$ kg/m$^3$, $\nu = 0.33$, $h_1 = 1/6''$, $h_2 = 1/3''$ mm, and $2b = 1/3''$ mm. This results in height ratio $R_h = (h_2 - h_1)/h_1 = 2$, and width ratio $R_w = 2b/h_1 = 2$. The frequency range for comparison was chosen to be 50 kHz to 350 kHz to avoid exciting the propagating A1 mode (Poddar et al., 2016). Figure 4.17 and Figure 4.18 shows the schematics of the finite element model with nonreflecting boundary for harmonic analysis for stiffener and cracked stiffener respectively. The S0 Lamb wave mode was excited by applying force at the top and bottom node of the plate in phase at the transmission location. The A0 Lamb wave mode was excited by applying force at the top and bottom node of the plate in opposite phase at the transmission location. However, at the specified frequencies, this type of excitation will create the S0 Lamb wave mode along with large number of non-propagating evanescent modes. Therefore, for successful simulation of the S0 and A0 modes incident on the stiffener, the distance between the transmission location and the step should be sufficiently large for the evanescent modes to die out.

![Figure 4.17](image)

Figure 4.17 Schematics of the finite element model with nonreflecting boundary for harmonic analysis
Figure 4.18  Schematics of the finite element model with nonreflecting boundary for harmonic analysis

Figure 4.19  Normalized amplitude of $u_x$ displacement (CMEP and FEM) of transmitted and reflected modes from stiffener for incident S0 mode
Figure 4.20  Normalized amplitude of $u_x$ displacement (CMEP and FEM) of transmitted and reflected modes from stiffener for incident A0 mode

Figure 4.21  Normalized amplitude of $u_x$ displacement (CMEP and FEM) of transmitted and reflected modes from cracked stiffener for incident S0 mode
We performed a harmonic analysis in ANSYS to verify the CMEP results in the frequency domain. To achieve a transient response from a finite dimensional model from the harmonic analysis, we introduced non-reflective boundaries (NRB) at both the ends of the model to eliminate the boundary reflections and thereby eliminating standing waves. The NRBs were created using the COMBIN14 spring damper element (Shen and Giurgiutiu 2015). These elements were arranged at the top and the bottom surfaces of the NRBs and also at both the ends. The damping coefficients of the elements were varied gradually in a sinusoidal pattern starting from zero. This eliminated any reflection from the edge of the NRB itself. A similar model was created without the step to capture the incident wave field only. We subtracted the incident wave field from the wave field in the Region 1 obtained from the step model to get the reflected wave fields. The transmitted wave field was obtained directly from the Region 2. The symmetric and the antisymmetric modes
were separated by averaging the summation and by subtraction of displacements at the top and the bottom nodes.

We can see that the convergence of the FEM model was quite expensive in terms of computational time and it took 200 times more computational time than the CMEP code to obtain the same results. The disparity in computational time was because the FEM model required a very high level of discretization with more than 30 elements per wavelength to obtain convergence of the phase (Poddar, 2016). This confirms that the CMEP method is a reliable and accurate method for predicting the scattering of Lamb waves. It is very important to obtain correct scatter coefficients quickly for NDE and SHM. This can be done easily by using the specialized analytical model, CMEP which can predict the scatter coefficients in seconds instead of hours.

4.5 Analytical Global-Local (AGL) Analysis

In order to understand the effect of Lamb wave propagation due to damage an analytical tool (analytical global-local analysis) is developed. In analytical global local analysis, a global solution is valid outside the damage region and at the damage a local analysis by using CMEP can be done to incorporate the damage coefficients (Figure 4.23). Analytical global-local analysis provides much better computational time over entire FEM analysis (Haider 2018h). Figure 4.24 shows the hybrid global-local analysis GUI layout.
Figure 4.23 Interaction of Lamb waves due to discontinuity of a structure

Figure 4.24 Hybrid global-local analysis software layout
4.5.1 Local analysis using CMEP

Calculating scattering coefficients using local analysis CMEP have been already discussed in the previous section, these scattering coefficients will be then fed into the global analytical method. The material properties and dimension of the stiffener are listed in the Table 4.1 and Table 4.2. Figure 4.25 shows the Geometry and dimensions (a) pristine stiffener (b) cracked stiffener.

Table 4.1 Material properties of aluminum

<table>
<thead>
<tr>
<th>Elastic modulus</th>
<th>Density</th>
<th>Poisson ratio</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E$ (GPa)</td>
<td>$\rho$ (kg/m$^3$)</td>
<td>$\nu$</td>
</tr>
<tr>
<td>70.4</td>
<td>2780</td>
<td>0.33</td>
</tr>
</tbody>
</table>

Table 4.2 Material and dimensions of the pristine stiffener and cracked stiffener

<table>
<thead>
<tr>
<th>Stiffener types</th>
<th>Material</th>
<th>Plate thickness</th>
<th>Stiffener height</th>
<th>Stiffener width</th>
<th>Crack length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pristine stiffener</td>
<td>Aluminum</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>0</td>
</tr>
<tr>
<td>Cracked stiffener</td>
<td>Aluminum</td>
<td>4</td>
<td>8</td>
<td>8</td>
<td>4</td>
</tr>
</tbody>
</table>

For theoretical simulations incident, S0 Lamb wave mode is considered as incident wave. Recommended by practical application, frequency range from 50 kHz to 350 kHz is considered for excitation. Figure 4.26 (a), (b) represents the S0, and A0 scattered wave amplitudes and phase angle for pristine stiffener on an aluminum body. Figure 4.26 (c), (d) is the S0 and A0 scattered wave amplitudes and phase angle for pristine stiffener on an aluminum plate.
The peak of scattering coefficients changes due to the presence of crack as it can be interpreted from Figure 4.26 (a) and (c). Evidently, the minimum peak of S0 transmitted has shifted from 160 kHz to 120 kHz due to the presence of a crack. Also, the amplitude of S0 transmitted at resonance increases due to the energy redistribution as a result of the crack.

The changes for A0 transmitted, S0 reflect and A0 reflected modes due to crack can also be observed from Figure 4.26 (a) and (c). Not only amplitude change, but phase change also occurs due to the influence of crack (Figure 4.26 (b), (d)). Therefore, the scattered wave modes can be utilized for predicting the presence of a crack in the stiffener. Scatter coefficient plots can predict the appropriate frequency range of excitation to excite Lamb wave to get the damage information from the time domain signal. From Figure 4.26 (a) and (c), for the present material and geometry, 120 kHz to 160 kHz would be an adequate choice of frequency for both S0 and A0 incident wave to detect the crack.
4.5.2 Global Solution

AGL approach is analytical. The advantage of this approach is that this approach is much faster than conventional global analytical local FEM or entire FEM analysis. The AGL method takes nearly 40 seconds to complete the whole analysis whereas the global analytical local FEM approach or complete FEM analysis takes several hours to complete the same analysis. A schematic of the AGL analysis is shown in Figure 4.27. The algorithm
of the steps of the global analytical solution to incorporate the local scattering coefficients is presented by using a flowchart as shown in Figure 4.28. The global analytical solution performs the wave generation by a transmitter sensor, wave propagation through the structure, incorporation of the scattering coefficients in particular to damage (pristine stiffener and cracked stiffener, in this case), and detection by a receiver sensor. The whole process utilizes the structural transfer function, the time domain signal generation, Fast Fourier transform (FFT) and inverse FFT. Therefore, the combination of global and local CMEP method provides a complete analytical solution for Lamb wave scattering and propagation. The following steps are followed for a global analytical solution.

![Flowchart of global analytical solution](image)

**Figure 4.27** Transmitter and sensor locations for predicting Lamb waves scatter due to discontinuity of a structure

1. First, consider a time-domain excitation signal from transmitter PWAS transducer. For tone burst excitation signal the following equation can be used

\[
V_T(t) = V_0 \left( 1 - \cos(\omega_c t) \right) \sin(\omega_c t) \tag{4-68}
\]

Here, \(V_0\) is the amplitude, \(\omega_c\) is the center frequency and \(t\) is the time.

2. Perform Fourier transform of the time-domain excitation signal to obtain the frequency domain excitation signal

3. Calculate the frequency-domain structural transfer function up to the damage location
\[
G(x_R, \omega) = e_x(x_R, t) = -i \frac{\alpha \tau_0}{\mu} \left\{ \sum_{\xi_S} \left( \frac{N_S(\xi_S)}{D_S(\xi_S)} \right) \frac{\sin(\xi_S a)}{\xi_S^2} e^{-i(\xi_S x_R - \omega t)} \right\} 
+ \sum_{\xi_A} \left( \frac{N_A(\xi_A)}{D_A(\xi_A)} \right) \frac{\sin(\xi_A a)}{\xi_A^2} e^{-i(\xi_A x_R - \omega t)} \right\}
\]

(4-69)

where \( \xi \) is the frequency dependent wave number of each Lamb wave mode and the superscripts S and A refer to symmetric and antisymmetric Lamb wave modes.

\[
N_S(\xi) = \xi \beta (\xi^2 + \beta^2) \cos \alpha d \cos \beta d;
\]
\[
D_S = (\xi^2 - \beta^2)^2 \cos \alpha d \sin \beta d + 4 \xi^2 \alpha \beta \sin \alpha d \cos \beta d
\]
\[
N_A(\xi) = -\xi \beta (\xi^2 + \beta^2) \sin \alpha d \sin \beta d;
\]
\[
D_A = (\xi^2 - \beta^2)^2 \sin \alpha d \cos \beta d + 4 \xi^2 \alpha \beta \cos \alpha d \sin \beta d
\]

\[
\alpha^2 = \frac{\omega^2}{c_p^2} - \xi^2; \quad \beta^2 = \frac{\omega^2}{c_s^2} - \xi^2; \quad c_p = \sqrt{\frac{\lambda + 2 \mu}{\rho}}; \quad c_s = \sqrt{\frac{\mu}{\rho}}; \quad k_{\text{PWAS}} = -\frac{i \alpha \tau_0}{\mu}
\]

\( a \) is the half-length of PWAS size; \( d \) is plate half thickness. The modal participation functions \( S(\omega) \) and \( A(\omega) \) determine the amplitudes of the S0 and A0 wave modes. The terms \( \sin(\xi_S a) \) and \( \sin(\xi_A a) \) control the tuning between the PWAS transducer and the Lamb waves. \( \lambda \) and \( \mu \) are Lame’s constants of the structural material; \( \rho \) is the material density. The wavenumber \( \xi \) of a specific mode for certain frequency \( \omega \) is calculated from Rayleigh-Lamb equation:

4. Multiply the structural transfer function by the frequency-domain excitation signal to obtain the frequency domain signal up to the discontinuity

\[
\tilde{V}_{sc}(x_R, \omega) = G(x_R, \omega) \cdot \tilde{V}_{T}(\omega)
\]

(4-71)
5. At this stage, frequency-dependent scattering coefficient (transmission, reflection and
mode conversion) from discontinuity need to insert in the wave signal. Scattering
coefficients can be determined by using local CMEP approach. The following Table
4.3 illustrated all the scattered coefficients need to be inserted into the wave signal.
Scattering coefficients $SR_{S0}$ and $\phi^{SR_{S0}}$ are the amplitude and phase of the reflected $S_0$
mode for incident $S_0$ mode respectively. The same terminology will be applied for
other scatter coefficients.

Table 4.3  List of scattering coefficients from a discontinuity for AGL analysis

<table>
<thead>
<tr>
<th>Incident wave mode</th>
<th>Direct scattered wave</th>
<th>Mode Conversion</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Reflection</td>
<td>Transmission</td>
</tr>
<tr>
<td>$S0$</td>
<td>$SR_{S0}$, $\phi^{SR_{S0}}$</td>
<td>$ST_{S0}$, $\phi^{ST_{S0}}$</td>
</tr>
<tr>
<td>$A0$</td>
<td>$AR_{A0}$, $\phi^{AR_{A0}}$</td>
<td>$AT_{A0}$, $\phi^{AT_{A0}}$</td>
</tr>
</tbody>
</table>

6. The guided waves from the new wave sources (at damage location) propagate through
the structure and arrive at the receiver location.

For reflected wave:

$$\vec{V}_{r}(x_r, \omega) = \left[ SR_{S0} e^{-i\omega SR_{S0}} \cdot \vec{V}_{Sc}^S(x_r, \omega) + SR_{A0} e^{-i\omega SR_{A0}} \cdot \vec{V}_{Sc}^A(x_r, \omega) \right] e^{-i\omega(-x_k)}$$

$$+ \left[ AR_{S0} e^{-i\omega AR_{S0}} \cdot \vec{V}_{Sc}^A(x_r, \omega) + AR_{A0} e^{-i\omega AR_{A0}} \cdot \vec{V}_{Sc}^S(x_r, \omega) \right] e^{-i\omega(-x_k)}$$

\[ (4-72) \]

For transmitted wave:

$$\vec{V}_{t}(x_r, x_t, \omega) = \left[ ST_{S0} e^{-i\omega ST_{S0}} \cdot \vec{V}_{Sc}^S(x_r, \omega) + ST_{A0} e^{-i\omega ST_{A0}} \cdot \vec{V}_{Sc}^A(x_r, \omega) \right] e^{-i\omega(x_t-x_k)}$$

$$+ \left[ AT_{S0} e^{-i\omega AT_{S0}} \cdot \vec{V}_{Sc}^A(x_r, \omega) + AT_{A0} e^{-i\omega AT_{A0}} \cdot \vec{V}_{Sc}^S(x_r, \omega) \right] e^{-i\omega(x_t-x_k)}$$

\[ (4-73) \]
7. Perform inverse Fourier transform to obtain the time-domain receiver signal.

\[ V_{Rf/Tr}(x_d, x_r, t) = IFFT\{\tilde{V}_{Rf/Tr}(x_d, x_r, \omega)\} \] (4-74)
4.5.3 Finite element modeling

FEM analysis was performed in ANSYS to verify the CMEP predictions. From the scattered wave amplitude plots obtained in section 3.3, the excitation frequency is chosen as 120 kHz. Analytical prediction of guided wave propagation done as explained in section 4 is compared with FEM results. Figure 4.29 and Figure 4.30 shows the schematic diagram of the FEM model for both pristine stiffener and cracked stiffener. S0 Lamb wave mode is generated using simultaneous excitation of two transmitter PWAS attached to the plate 200 mm from the stiffener. Transmitted Lamb wave modes are received using the receiver PWAS and reflected using the transmitter PWAS. Three PWAS of dimensions 7 mm width and 0.2 mm thickness are modeled using coupled field multi-physics analysis in ANSYS. Plane 82 elements are used for modeling of PWAS. APC-850 material properties assigned to the PWAS are as follows.

\[
\left[ C_p \right] = \begin{bmatrix}
97 & 49 & 49 & 0 & 0 & 0 \\
49 & 84 & 49 & 0 & 0 & 0 \\
49 & 49 & 97 & 0 & 0 & 0 \\
0 & 0 & 0 & 24 & 0 & 0 \\
0 & 0 & 0 & 22 & 0 \\
0 & 0 & 0 & 0 & 22 \\
\end{bmatrix} \text{GPa} \quad (4-75)
\]

\[
\left[ \varepsilon_p \right] = \begin{bmatrix}
947 & 0 & 0 \\
0 & 605 & 0 \\
0 & 0 & 947 \\
\end{bmatrix} \times 10^{-8} \text{F/m} \quad (4-76)
\]

\[
\left[ e_p \right] = \begin{bmatrix}
0 & 0 & 0 & 12.84 & 0 & 0 \\
-8.02 & 18.31 & -8.02 & 0 & 0 & 0 \\
0 & 0 & 0 & 12.84 & 0 \\
\end{bmatrix} \text{C/m}^2 \quad (4-77)
\]
where $[C_p]$ is the stiffness matrix, $[\varepsilon_p]$ is the dielectric matrix, and $[e_p]$ is the piezoelectric matrix. The density of the material considered is $\rho = 7600 \text{ kg/m}^3$. Node merge command in ANSYS is used to merge PWAS to the aluminum plate.

![Figure 4.29 Schematic of the plate with pristine stiffener for finite element analysis](image)

Non-reflective boundaries (NRB) was applied at both ends of the model to eliminate the boundary reflections and thereby eliminating standing waves. The NRBs were created using the spring damper element COMBIN14. These elements were arranged at the top and bottom surfaces as well as at both ends of the NRBs. The damping coefficients of the elements were varied gradually in a sinusoidal pattern starting from zero. This eliminated any reflection from the edge of the NRB itself. PLANE182 elements were used in ANSYS multi-physics.
4.5.4 Results and Discussion

The results obtained from the global-local analytical approach are summarized in Figure 4.31 and Figure 4.32. S0 mode with 3-count tone burst at 120 kHz is considered as incident wave. The comparison between analytical and FEM of the corresponding scattered waves of the pristine stiffener is presented in Figure 4.31. The time domain signals are in good agreement. The reflected and transmitted Lamb waves show a similar pattern in the time-domain signals. Also, the reflected and transmitted Lamb waves show a similar pattern in the time domain signals for cracked stiffener case. A little discrepancy between AGL method and FEM method can be observed, and it is expected due to the receiver and transducer design in AGL and FEM model.
Comparing Figure 4.31 and Figure 4.32, we see the influence of the presence of a crack in the signals received using PWAS. The relative S0 to A0 amplitude is changed due to the presence of a crack in case of the transmitted signal. We see distortions in the reflected signals also due to the presence of a crack. Clearly the signature of crack presence is included in the reflected and transmitted signals, which can be observed from the AGL predictions. Also, AGL predictions are verified using FEM analysis in this section, showing very good agreement. The results shows that AGL is a reliable tool to predict the scattering of the Lamb waves from geometric discontinuities. In comparison with other NDE or SHM technique, this analytical method capitalizing on Lamb waves scattering that can offer faster, more accurate and cost-effective evaluation of different types of damage. This method is useful for different types of damage to which ultrasonic Lamb waves are
particularly sensitive including discontinuity, corrosion patch, debonding, cracking, etc. Damage identification technique using AGL is envisioned to be a promising method in addition to traditional NDE/SHM approaches.

![Graphs of AGL vs. FEM results for different scattered Lamb waves from cracked stiffener: (a) analytical reflected waves (b) analytical transmitted waves (c) FEM reflected waves (d) FEM transmitted waves.](image)

**Figure 4.32** AGL vs. FEM results for different scattered Lamb waves from cracked stiffener: (a) analytical reflected waves (b) analytical transmitted waves (c) FEM reflected waves (d) FEM transmitted waves.

### 4.6 Experiments and comparison with analytical results

This section describes an experimental validation of the global-local analytical results. The analysis of the scattered Lamb wave has been performed to detect the crack in the stiffener on the plate.

#### 4.6.1 Experimental procedure

Two aluminum plates are manufactured for conducting the experiments: (a) plate with the pristine stiffener (Figure 4.33), and (b) plate with the cracked stiffener (Figure 4.34) with
same geometric dimension as described in Section 4.2, 4.3 and 4.4. Electrical discharge machining (EDM) method was used to create a crack along the entire length of the stiffener (Figure 4.34a).

![Image of plate with PWAS transducers and dimensions](image)

**Figure 4.33** Aluminum plate with pristine stiffener (a) plate with dimension (b) after applying wave-absorbing clay

In both plates, the PWAS transducers were bonded 200 mm away from the stiffener. Two PWAS were excited simultaneously with a 180-degree phase difference to generate A0 Lamb waves, selectively. A 3.5 count tone burst at 150 kHz was applied as excitation signal by using a Tektronics AFG3052C dual channel function generator. The wave-absorbing clay boundary around the plate was used to avoid reflection from the plate edges.
Two 60 mm × 5 mm × 0.2 mm piezoelectric wafer active sensors (PWAS) are bonded in a straight line on top and bottom surfaces of the plate to create a line source.

![Image](image1.png)

(a) Cracked stiffener on the plate

![Image](image2.png)

(b) Top view of the plate

![Image](image3.png)

Zoomed in view of cracked stiffener

Crack along the entire length of stiffener

Figure 4.34  (a) Aluminum plate with cracked stiffener; 60-mm long PWAS strips are bonded on top and bottom of the plate; zoomed in view of the crack along the entire length of the stiffener (b) top and bottom view of the plate after applying wave-absorbing clay boundary.

The wave fields were measured using scanning LDV. Power amplifiers were used to improve the signal to noise ratio to achieve better out-of-plane velocity signal by LDV. The experimental setup is illustrated in Figure 4.35. The LDV scanning results are illustrated in Figure 4.36. It shows that this arrangement of PWAS transducers successfully generated the straight crested A0 Lamb wave modes. It also shows a minimal reflection
from the plate edges due to the use of absorbing clay boundaries. Figure 4.36 (b) and (c) show the scattered wave field from pristine stiffener and Figure 4.36 (d) and (e) show the scattered wave field from cracked stiffener. These figures also show that the scattered wave fields are also straight crest waves after interacting with a discontinuity, as expected.

Figure 4.35  Experimental setup for LDV scanning to measure the out-of-plane velocity of the scatter wave field

To compare the scattering coefficients of CMEP and global analytical results with experimental results, the out of plane velocity \( (v_y) \) was measured at a point on the plate surface using the LDV. Single point measurement locations are shown in Figure 4.37. The reflected, and transmitted wave fields are measured at 170 mm and 200 mm, respectively, from the stiffener.
Experimentally measured scattered wave field. The scattered wave straight crested Lamb wave modes using scanning LDV.
4.6.2 Experiment vs. analytical analysis

In this section, a comparison of the experimental results and the proposed global-local analytical results are compared. At first, we give the detail steps of the global-local analytical approach. As scattering coefficients have already been determined as discussed in the previous section, these scattering coefficients are then fed into the global analytical method. Therefore, the total approach is analytical. The advantage of this approach is that it is much faster than conventional global analytical local FEM or entire FEM analysis. The current analytical method takes nearly 40 seconds to complete the entire analysis, whereas the global analytical local FEM approach takes several hours to complete the same analysis. global analytical solution performed the wave generation by a transmitter, wave propagation through the structure, incorporation of the scattering coefficients in particular to damage (pristine stiffener and cracked stiffener, in this case), and detection by a receiver sensor.
The results obtained from the global-local analytical approach are summarized in Figure 4.38 and Figure 4.39. In the global-local analysis, the signals are calculated at the same location where the signals were obtained experimentally (Figure 4.37). The comparison between analytical and experimental scattered waveform, as well as FFT of the corresponding scattered waves of the pristine stiffener is illustrated in Figure 4.38. Although it is difficult to have exact matching between two time-domain signals, the FFT of the two signals are in good agreement as depicted in Figure 4.38(c), (f). Also, the reflected and transmitted Lamb waves show a similar pattern in the time-domain signals and their FFTs.

Figure 4.39 shows the comparison of analytical and experimental scattered waveform, as well as FFT of the corresponding scattered waves for the cracked stiffener. The FFT of the two signals are in good agreement as depicted in Figure 4.39 (c), (f). The reflected and transmitted Lamb waves show a similar pattern in the time-domain signals and their FFTs.

Also, the shifting in frequency response due to the presence of the crack is an important phenomenon to note. The transmitted A0 Lamb wave has clear anti-resonance at 150 kHz that can be detected by both analytical and experimental results. Such information may be useful for crack detection in the complex geometry. This shows that global-local analytical could be a reliable tool to predict the scattering of the Lamb waves from geometric discontinuities.
Figure 4.38  Experiment vs. global-local analytical results for different scattered Lamb waves from pristine stiffener: (a) analytical reflected waves (b) analytical transmitted waves (c) FFT of the analytical scattered waves (d) experimental reflected waves (e) experimental transmitted waves (f) FFT of the experimental scattered waves
Figure 4.39  Experiment vs. global-local analytical results for different scattered Lamb waves from cracked stiffener: (a) analytical reflected waves (b) analytical transmitted waves (c) FFT of the analytical scattered waves (d) experimental reflected waves (e) experimental transmitted waves (f) FFT of the experimental scattered waves.
4.6.3 Comparison between CMEP Scattered Coefficients vs. Experimentally Measured Scattered Coefficients

Figure 4.40 shows the comparison between CMEP results and experimental results for pristine stiffener and Figure 4.41 shows the comparison between CMEP results and experimental results for a pristine stiffener.

![Graphs showing comparison between CMEP and experimental results for different Lamb wave modes](image)

(a) (b) (c) (d)

Figure 4.40  Experiment vs. CMEP result for variation of scattering coefficients of different Lamb wave modes for incident A0 mode in the plate with pristine stiffener (a) reflected S0 (b) transmitted S0 (c) reflected A0 (d) transmitted A0

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Figure 4.40 and Figure 4.41 show that the experimental results are in agreement with the CMEP results. The experimentally obtained scatter coefficients of reflected and transmitted A0 modes are lower than the predicted coefficients. No material damping was considered in CMEP methodology. However, material damping and its effect are always present in the structure.

![Graphs showing experimental vs. CMEP results for scattering coefficients of different modes](image)

**Figure 4.41** Experiment vs. CMEP result for variation of scattering coefficients of different Lamb wave modes for incident A0 mode in the plate with cracked stiffener (a) reflected S0 (b) transmitted S0 (c) reflected A0 (d) transmitted A0
However, material damping has a more significant effect on A0 mode, rather than S0 mode. Therefore, experimental results show a lower amplitude of transmitted and reflected A0 modes than predicted ones. Antisymmetric modes have much higher out of plane velocity amplitude than the symmetric mode. Therefore, LDV cannot measure the out-of-plane velocity of symmetric mode accurately. As a result, scattering coefficients of symmetric mode differ slightly from the predicted scatter coefficients.

By comparing all of the plots, it can be inferred that the scattering coefficients predicted by CMEP are close to experimentally obtained results. This proves that CMEP is a reliable tool to predict the scattering coefficients of the Lamb waves from stiffener. FFT of the scattered signal shows a clear change in amplitude with frequency due to the presence of a crack. For example, transmitted A0 has a clear anti-resonance frequency at \( \sim 150 \) kHz when the crack is present. Such anti-resonance frequency cannot be observed for the pristine stiffener case.
PART III: HIGH TEMPERATURE AND RADIATION ENDURANCE OF PWAS

CHAPTER 5

IRREVERSIBILITY EFFECTS IN PIEZOELECTRIC WAFER ACTIVE SENSORS AFTER EXPOSURE TO HIGH TEMPERATURE

5.1 STATE OF THE ART REVIEW OF SMART SENSING SYSTEM

Piezoelectric wafer active sensors (PWAS) have emerged as one of the major SHM technologies; the same sensor installation can be used with a variety of damage detection methods: propagating ultrasonic guided waves, standing waves (E/M impedance) and phased arrays (Giurgiutiu, 2014, 2002a, 2002b, 2002c, 2000, 1999 Zagrai et al., 2002) PWAS transducers are very small, lightweight and inexpensive transducers. PWAS transducers can be bonded on a host structure or between layers of a structure easily. PWAS transducers require low power, which enables it to be feasible for onsite inspection in SHM and NDE applications. PWAS transducers are made of piezoelectric material with electrodes deposited on the upper and lower surfaces as grounding and supplying electrodes to polarize the electric field through the thickness. The traditional PWAS transducers use piezoelectric material Lead Zirconate Titanate (PZT). PWAS transducer material has been attracted by researchers due to its enhanced sensing, actuation or both capabilities. PZT with large coupling coefficient, high permittivity and quick time to respond make it to be an excellent candidate as piezoelectric transducer in SHM and NDE
applications (Newnham et al., 1990). Property enhancement was made at the expense of temperature, electric field, and stress stability. Property enhancement is achieved by chemical composition to allow domain wall motion to extensively contribute to the piezoelectric effect. This compound class shows much better piezo-electrical and piezo-mechanical efficiency than naturally occurring piezoelectric materials (Verissimo et al., 2003). Commercially available APC 850 PWAS transducer was used in this research (APC 850 datasheet). The wafers disk is 7 mm in diameter and 0.2 mm in thickness. The wafer has a PZT thin film with silver (Ag) electrode on upper and lower surfaces. The Curie point of APC 850 PWAS transducer is 350°C (Kamas et al., 2015). The Curie transition temperature was kept well above the maximum operating temperature (250°C) in this study.

For SHM applications at elevated temperatures using E/M impedance/admittance method by PWAS transducer, only a few tentative trials have so far been reported but material characterization has not been done yet (Kamas et al., 2015; Park et al., 1999). Bastani et al. (2012) investigated the effect of temperature variation on piezoelectric transducers used in SHM applications. In that paper the structure was selected approximately for an operational range of 29°C–59°C. The experimental results showed that the peak magnitude of real part of impedance decrease monotonically with increasing temperature for both pristine structure and damage structure. The variations in both the amplitude and the frequency were analyzed experimentally by Baptista et al. (2014) for temperatures ranging from 25°C to 102°C. Experimental results showed that the variation in the amplitude is due to the dielectric properties of piezoelectric transducers. The frequency shifts of the resonance peaks were observed with increasing temperature. Baba
et al. (2010) developed a high temperature transducer using Lithium Niobate (LiNbO$_3$) and that worked up to 1000°C without failure. Patel et al. (1990) used AlN thin film for high frequency and high temperature ultrasonic transducer. No damage was observed for AlN after exposure to 1220°C. Stubbs et al. (1996) developed and tested a sensor that is capable of emitting and receiving ultrasonic energy at temperatures exceeding 900°C and pressures above 150 MPa. Radiation, temperature and vacuum effects on PWAS transducers were studied by Giurgiutiu et al. (2016). The change in resonance and anti-resonance frequencies and amplitudes were obtained experimentally in that paper.

The PZT material used in PWAS transducer is a ferroelectric material and, for most ferroelectric materials the existence of domain structure or domain wall make a significant influence on the material properties. In PZT solid solution system the material properties may be changed due to change in the domain size and domain wall motion (Kamel et al., 2008). The material properties of PWAS transducer depend on both intrinsic and extrinsic properties. The material properties from a single domain are denoted as the intrinsic properties of the material, while the contributions from extrinsic parts of the material mainly from domain wall motion. It is expected that both intrinsic and extrinsic contributions are influenced by domain size and domain wall motion. So the dielectric constant, piezoelectric constant and elastic compliance depend on both extrinsic and intrinsic contribution of PWAS transducer material.

The thermal anisotropy may contribute to the internal stress development during cooling of PWAS transducer material and that is released by changing the domain size as a result of multi domain formation, resulting in easy domain switching without changing in microstructure (Herbiet et al., 1989; Xu et al., 2001; Zhang et al., 1994; Sabat et al.,
Another possible reason is depinning of domains, this process helps previously restricted domain to reorient in favorable direction during heating. Upon cooling of PWAS, piezoelectric, elastic and dielectric constant show irreversible response due to irreversible domain dynamics (Sabat et al., 2007). Therefore after subjected to high temperature, the PWAS transducer material properties become non-reversible and eventually affect the E/M impedance/admittance signature.

5.2 Theory

The PWAS transducer can be modeled as a simple electro-mechanical system as shown in Figure 5.1. The mechanical aspects of the transducers can be described by its mechanical impedance. Driving parameters for the mechanical impedance is mass, stiffness and damping. The electro-mechanics of PWAS transducer is typically represented by its transduction relations, while such transduction relations for a PWAS transducer material are usually referred to as the constitutive relations. The dynamics of the PWAS transducer can generally be represented by its structural impedance. The electro-mechanical system can be represented by electrical admittance or impedance which is affected by the PWAS transducer dynamics and mechanical impedance.

![Figure 5.1 Schematic of a circular PWAS under electric excitation](image)
The coupled relation between the mechanical impedance and complex electrical impedance/admittance of PWAS transducer can be shown as follow (Zagrai, 2002).

**Admittance:**

\[
\tilde{Y}(\omega) = i\omega C \left[ 1 - \tilde{k}_p^2 (1 - \frac{(1 + \nu) J_0(\tilde{\zeta})}{\tilde{\zeta} J_0(\tilde{\zeta}) - (1 - \nu) J_1(\tilde{\zeta})} \right]^{-1}
\]

**Impedance:**

\[
\tilde{Z}(\omega) = \frac{1}{i\omega C} \left[ 1 - \tilde{k}_p^2 (1 - \frac{(1 + \nu) J_0(\tilde{\zeta})}{\tilde{\zeta} J_0(\tilde{\zeta}) - (1 - \nu) J_1(\tilde{\zeta})} \right]^{-1}
\]

where, \(\tilde{k}_p^2 = \frac{2 \frac{d_{31}^2}{1 - \nu} \varepsilon_{33} \pi a^2}{s_{44}^p t_a} ; \quad \tilde{\zeta} = \varepsilon_{33}^p \frac{\pi a^2}{t_a} ; \quad \tilde{\zeta} = \sqrt{\frac{1}{\rho s_{44}^p (1 - \nu^2)}} ; \quad \tilde{\zeta} = \frac{\omega a}{\tilde{c}_p}.

\(a\) = radius of the PWAS transducer, \(t_a\) = thickness of the PWAS transducer and \(\rho\) = density of the PWAS transducer, \(d_{31}\) = in-plane piezoelectric coefficient, \(\varepsilon_{33}^p\) = complex compliance coefficient, \(\nu\) = Poisson ratio, \(J_0(\tilde{\zeta})\) and \(J_1(\tilde{\zeta})\) are the Bessel functions of order 0 and 1, \(\omega\) is the angular frequency.

Complex compliance and dielectric constant can be expressed as

\[
\varepsilon_{11}^E = s_{11}^E (1 - i\eta)
\]

\[
\varepsilon_{11}^E = s_{11}^E (1 - i\delta)
\]

where \(\eta\) & \(\delta\) are dielectric and mechanical loss factor respectively.

### 5.3 Experiments of PWAS Transducer

E/M impedance/admittance and different material properties of a set of six nominally identical free PWAS transducers were measured at room temperature after exposure to various high temperature values. The following experimental procedure was followed.
1. E/M impedance/admittance and different material properties of free PWAS transducers were measured at baseline room temperature.

2. PWAS transducers were heated to various temperature levels (50°C to 250°C with 50°C steps) at 1-2°C/min heating rate in an oven; then PWAS transducers were kept at that temperature for couple of hours. After exposure to that temperature, PWAS transducers were extracted from the oven and allowed to cool in air at room temperature.

3. E/M impedance/admittance and other material properties of PWAS transducers were measured again at room temperature to investigate material response of PWAS transducers.

4. Microstructural and crystallographic investigation of PWAS transducer was performed after exposure to 250°C temperature.

5.3.1 E/M Impedance/Admittance of Free PWAS Transducer

Experimental set up for E/M impedance/admittance measurement is shown in Figure 5.2. A commercial HP 4194A impedance analyzer was used for E/M impedance/admittance measurement. PWAS transducers were loaded in a test stand and connected to impedance analyzer by wires. PWAS transducers were measured in a stress-free state by using pogo pins that only apply small spring forces to the PWAS surface (Figure 5.2). An oven with PID temperature controller was used to elevate the PWAS to high temperature. A data acquisition system was used to collect the E/M impedance/admittance. The E/M impedance/admittance values were collected from 250 kHz to 350 kHz with a step size of 50 Hz. The real part of the impedance/admittance is used for E/M impedance/admittance method as it has been used for damage detection in SHM applications (Kamas et al., 2015;
Park et al., 2003). The frequency range is determined by trial and error method and only first PWAS anti-resonance (≈ 330 kHz) and resonance (≈ 290 kHz) was considered in this article. In earlier publication by Lin et al. (Sun et al., 20015) showed that, 1st resonance and anti-resonance spectrum shows more stable response than higher resonance or anti-resonance spectrum. In order to detect damage properly it would be recommended to follow change in frequencies of 1st resonance and anti-resonance. Peaks in the PWAS transducer E/M impedance/admittance were seen based on electromechanical coupling with mechanical impedance/admittance of the PWAS transducer. Eqs. (5-1) & (5-2) show the coupled relation between the PWAS transducer mechanical impedance and complex electrical impedance/admittance. E/M impedance/admittance spectra depend on PWAS transducer material properties such as stiffness coefficient, piezoelectric constant, dielectric constant, density and different losses in PWAS transducer material. Any kind of change in PWAS transducer material state can be noticed as peak shifts in E/M impedance/admittance. The spectral peaks observed in the real part of the E/M impedance spectrum follows the PWAS transducer anti-resonances and E/M admittance spectrum follows the resonance. Figure 5.3(a) shows the E/M impedance spectra of a free PWAS transducer after exposure to different temperature and Figure 5.3(b) shows the admittance spectra. In these figures, RT denotes room temperature data, RT_50 denotes as data was taken at room temperature after exposure to 50°C, and so on. It can be seen that, for impedance spectra, both the impedance amplitude and the frequency of the peak impedance decreases with increasing temperature. In case of admittance spectra, admittance amplitude and the frequency of the peak increase with increasing temperature. So, frequency and
The amplitude of the spectral peaks vary with temperature, which may indicate a change in the PWAS transducer material properties.

Figure 5.2 Experimental setup for E/M impedance/admittance measurement

Figure 5.3 (a) E/M impedance and (b) E/M admittance of a PWAS transducer after exposure to different temperature
The E/M impedance/admittance spectral peak can be characterized by the quality factor $Q$. The quality factor $Q$ can be defined as the ratio of the energy stored in an oscillating resonator to the energy dissipated due to loss factors, i.e.,

$$Q = 2\pi \frac{\text{Energy stored}}{\text{Energy dissipated per cycle}}$$

The quality factor $Q$ can also be expressed as

$$Q = \frac{f_c}{f_1 - f_2}$$

where $f_c$ is the center frequency and $f_1 - f_2$ is the 3dB bandwidth.

Figure 5.4 shows the $Q$ values based on impedance and admittance at different temperature. Error bars were obtained from the data of six identical PWAS transducers. The $Q$ values decrease with increasing temperature for both impedance and admittance.

The resonance quality factor $Q_R$ values are lower than the anti-resonance $Q_A$ and it is desired due to more mechanical friction during resonance. Decreased values of the quality factor relates to the increased dissipative losses in PWAS transducer material. Losses in PWAS transducer material are phenomenologically considered to have three coupled mechanisms: dielectric, elastic, and piezoelectric. It should be noted that, R squared values of the regression lines are found to be 0.978 and 0.9777 for impedance and admittance respectively. High R squared value implies that, the regression lines fit the data very well.
5.3.2 Characterization of PWAS Transducer

For characterizing PWAS transducer, the dielectric constant, dielectric loss factor, and piezoelectric constant were measured. Later, PWAS transducer microstructural, crystallographic investigation was done to facilitate understanding of PWAS transducer material behavior in extreme environment for SHM applications.

5.3.3 Dielectric constant and dielectric loss

Low field measurement of dielectric constant and dielectric loss of PWAS transducers were measured by E4980A Precision LCR Meter test system at 1 kHz [Appendix]. Only a small potential was applied (20 mV) to measure the dielectric properties. Dielectric constant relates to a material’s ability to resist an electric field. Dielectric constant is directly related to electric susceptibility, which is a measure of how easily PWAS material domains...
polarize in response to an electric field. So the dielectric constant is influenced by the polarization of domains and domain wall motion of PWAS material.

In order to measure dielectric constant and dielectric loss by Precision LCR meter, a voltage $U_o$ with a fixed frequency $\frac{\omega}{2\pi}$ is applied to the sample. Voltage $U_o$ causes a current $I_o$ at the same frequency in the sample. In addition, there will generally be a phase shift between current and voltage described by the phase angle $\phi$. The ratio between $U_o$ and $I_o$ and the phase angle $\phi$ are determined by the sample material electrical properties and by the sample geometry. So the appropriate relations in complex notation can be expressed as

$$U(t) = U_o \cos(\omega t) = Re \left( U^* \exp(i\omega t) \right)$$

$$I(t) = I_o \cos(\omega t + \phi) = Re \left( I^* \exp(i\omega t) \right)$$

With

$$U^* = U_o$$

And

$$I^* = I' + iI''$$

$$I_o = \sqrt{I'^2 + I''^2}$$

$$\tan(\phi) = \frac{I''}{I'}$$

The measured impedance of the sample is

$$Z^* = Z' + iZ'' = \frac{U^*}{I^*}$$
There are several polarization mechanisms contributing to the dielectric response (Kamel et al., 2008): (i) electric polarization: the relative displacement of the negatively charged electron shell with respect to the positively charged core; (ii) ionic polarization: as observed in ionic crystals and describes the displacement of the positive and negative sublattices under an applied electric field; (iii) orientation polarization: the alignment of permanent dipoles via rotational movement; (iv) space charge polarization: polarization due to spatial inhomogeneities of charge carrier densities; (v) domain wall motion: movement of high energy domain wall due to reorientation of dipole. Domain wall motion plays a decisive role in ferroelectric materials and contributes significantly to the overall dielectric response (Zhang et al., 1994; Sabat et al., 2007; Taylor et al., 1997; Wolf et al., 2004, Haider et al. 2017a). The change in dielectric properties may arise from extrinsic response originating from depinning of domains or domain wall motion. Any change in domain size ultimately
affect domain configuration and domain wall mobility which contribute to an irreversible change in dielectric properties.

Figure 5.5 shows the change in dielectric constant after exposure to different elevated temperatures. All the data were taken at room temperature and all the temperature values here are referred to the temperature at which PWAS transducers were heated. Permittivity jumped from 50°C to 100°C and then increased continuously from 100°C to 250°C. Such scenario implies a definite change in domain size and domain wall mobility of PWAS transducer material. R squared value of the regression lines is found to be 0.978. Therefore the variance of its errors is 97.8% less than the variance of the dependent variable.

Figure 5.6 Dielectric loss factor of PWAS after exposure to elevated temperatures

Figure 5.6 shows that the dielectric loss factor increases after exposure to different elevated temperatures. Dielectric loss quantifies a dielectric material's inherent dissipation.
of electromagnetic energy (e.g. heat). It can be parameterized in terms of the corresponding loss tangent tan \( \delta \). Movement of domain walls also contribute to the dielectric loss of ferroelectric materials. Dielectric loss increases with temperature but remains within 0.2 \% of initial room temperature value. R squared value of 0.8151 implies a relative larger variation in the measured data. It is expected due to very low values of dielectric loss tangent. Therefore, energy loss of the system may not be linear with the temperature.

5.3.4 **In plane piezoelectric coefficient**

In-plane piezoelectric coefficient \( d_{31} \) were measured using an optical-fiber strain transducer system. In plane piezoelectric coefficient is the ratio of mechanical in-plane strain to applied electric field (Units: m/V). At zero stress, in-plane piezoelectric coefficient \( d_{31} \) couples between applied transverse electric field \( E_3 \) and in plane strain \( \varepsilon_1 \), i.e.,

\[
\varepsilon_1 = d_{31} E_3
\]

(5-14)

Fiber Bragg Grating (FBG) optical strain sensors were used to measure the in-plane strain. When mechanical strain \( \varepsilon \) is present, the change in wavelength of FBG sensor is related by (Lin et al., 2015; Haider et al., 2016b)

\[
\Delta \lambda_B = \lambda_B (1 - \rho_a) \varepsilon
\]

(5-15)

where \( \rho_a \) is the effective strain-optic coefficient. After the wavelength shift \( \Delta \lambda_B \), Eq. (5-15) is used to deduce the strain \( \varepsilon \) and hence Eq. (5-14) is used to calculate \( d_{31} \). The experimental setup is shown in Figure 5.7. In this research two-component high temperature M-bond 600 strain gage adhesive was used to attach the FBG sensor on PWAS
surface. The highest operating temperature of M-bond is 260° C (M-bond 600 data sheet)) which is higher than the maximum temperature (250° C) used in this research.

Figure 5.7 Experimental setup for measuring piezoelectric coefficient $d_{31}$ of PWAS transducer

Figure 5.8 shows the change in piezoelectric coefficient $d_{31}$ at room temperature after exposure to higher temperatures. The variance of its errors is 83.6% ($R^2 = 0.836$) less than the variance of the dependent variable. In order to investigate the irreversible behavior of piezoelectric coefficient, data were taken at room temperature after exposure to different elevated temperature ($50^0\text{C} - 250^0\text{C}$). Dipole moment of PWAS transducer material may be changed due to change in the domain configuration after cooling it down to room temperature from high temperature. This change affects the piezoelectric coefficient. It was found that the piezoelectric coefficient increases after exposure to elevated temperature. The piezoelectric effect is a linear coupling between the polarization and the applied stress field. High temperature exposure of the piezoelectric materials may help to switch the domain in favorable position upon applying an electric field, hence leading to an improved
net polarization. Moreover, due to heating of the piezoelectric materials may help to depinning the domains from its pinned position. Therefore, an improved piezoelectric coefficient can be observed.

It should be noted that during $d_{31}$ measurement FBG strain sensors were bonded permanently to the PWAS transducer. So, after different temperature exposure, different PWAS transducers and different FBG strain sensors were used. Due to complex experimental procedure, only single data point was taken at each temperature. Moreover, it is very difficult to get exact values of in-plane strain experimentally for small PWAS transducer (in the order of micro-strain). The manufacturer value of $d_{31}$ for a typical PWAS transducer material is $-175 \times 10^{-12} \text{ m/V}$ at room temperature (APC-850); however, our measured experimental value was found to be $-125 \times 10^{-12} \text{ m/V}$ at room temperature.

![Figure 5.8](image_url)  
**Figure 5.8** In-plane piezoelectric coefficient $d_{31}$ of PWAS transducer after exposure to elevated temperatures
5.3.5 Microstructural and crystallographic investigation

Imaging technique is widely popular for identifying material state change (Haider et al, 2017d; 2015a, 2015b, 2013; Majumdar et al. 2015, 2013a, 2013b, 2013c, 2013d). To further facilitate understanding of PWAS transducer material state, microstructural and crystallographic investigation were done. A scanning Electron Microscopy (SEM) system was used to visualize the PWAS transducer (Ag/PZT/Ag) cross section (Figure 5.9). PWAS transducers have a dense structure with PZT grains composed of 2-3 \( \mu m \). No variation was found in the microstructure or PZT grains in the PWAS transducer due to elevated temperature exposure (Figure 5.9a and Figure 5.9b).

![SEM micrograph of cross section of Ag/PZT/Ag PWAS transducer](image)

**Figure 5.9** SEM micrograph of cross section of Ag/PZT/Ag PWAS transducer (a) at room temperature (b) at room temperature after exposure to 250°C for 2 hours
Figure 5.10 shows the X-ray powder diffraction (XRD) spectrum for PWAS transducer material at room temperature (Figure 5.10a) and at room temperature after exposure to 250°C (Figure 5.10b). XRD is a rapid analytical technique primarily used for phase identification of a crystalline material and can provide information on unit cell dimensions. XRD measures the X-ray diffraction peak with diffraction angle (2θ). Position of the peaks with diffraction angle (2θ) is the important characteristic of the XRD pattern, which act as a unique characteristic of the crystallographic unit cell. By comparing measured peak positions, change in unit cell dimension and symmetry can be obtained.
Figure 5.10 shows no noticeable change in X-ray diffraction peak position. That means, there is no significant change in crystal structure, unit cell dimension, and symmetry after exposure to elevated temperature. Microstructural and crystallographic studies confirm that PWAS transducer can be used as a SHM transducer without any damage after heating to high temperature.

5.4 E/M impedance/admittance modeling of free PWAS Transducer

This subsection presents numerical results of E/M impedance/admittance of circular PWAS transducer influenced by increasing temperature. Driving parameters for E/M impedance and admittance are density, Poisson’s ratio, elastic compliance, mechanical loss factor, dielectric constant, dielectric loss and in plane piezoelectric constant. It is very difficult to measure all the PWAS transducer material properties (e.g. elastic coefficient) experimentally. A proper numerical model is essential to understand effect of PWAS transducer material properties on E/M impedance/admittance with temperature. For E/M impedance/admittance modeling based on Eqs. (5-1) and (5-2), the material properties were taken from the manufacturer data sheet (Table 5.1) and experimental results (Table 5.2). Figure 5.11(a) and Figure 5.11(b) show the experimental impedance and admittance spectra. Figure 5.11 (c) and Figure 5.11 (d) show the numerical results for E/M impedance and admittance. The corresponding modified material properties for E/M impedance and admittance modeling to match experimental results are listed in Table 5.3 and Table 5.4. For numerical model density, Poisson’s ratio, dielectric constant, dielectric loss, in plane piezoelectric constant and elastic compliance were initially taken from manufacturer data sheet (Table 5.1) and experimental results (Table 5.2); later the values were adjusted with temperature to fit the model with experimental E/M impedance/admittance values. One
important observation is that during numerical model the mechanical loss factor for admittance model is higher than for the impedance model. During the resonance the transducer goes through mechanical friction and contributed to more dissipation loss. During anti-resonance the transducer moves hardly resulting in very little mechanical friction and thus it showed very low mechanical loss. Mechanical loss factors are related with the quality factor (Figure 5.4). Mechanical loss factor is the inverse of quality factor during resonance (Mezheritsky, 2002a; 2002b).

\[ \eta^R = \frac{1}{Q_R} \]  

(5-16)

Here, \( \eta^R \) = mechanical loss factor during resonance and \( Q_R \) = quality factor during resonance. Mechanical loss factor is not directly the inverse of quality factor during anti-resonance. Mechanical loss factor during anti-resonance can be expressed as (Mezheritsky, 2002a)

\[ \eta^A = (2\varphi - \delta) + \frac{1}{2} \left( \frac{1}{Q_A} - \frac{1}{Q_R} \right) \left[ 1 + \left( \frac{1}{k_{31}} - k_{31} \right)^2 \Omega^2 \right] \]  

(5-17)

Here, \( \eta^A \) = mechanical loss factor during anti-resonance; \( \delta \) = dielectric loss factor; \( \varphi \) = piezoelectric loss factor; \( Q_A \) = quality factor during anti-resonance; \( k_{31} \) = piezoelectric coupling coefficient; \( \Omega = \omega d / 2v \); \( \omega \) = anti-resonance frequency, \( v \) = wave speed; \( d \) = diameter of the PWAS transducer.

Therefore, mechanical loss factor of anti-resonance depends on piezoelectric coupling coefficient, dielectric loss, and piezoelectric loss; where mechanical loss factor during resonance depends on \( \sqrt{Q_R} \) only.
The major observations from numerical results are:

1. The degraded mechanical, electrical, and piezoelectric properties of PWAS transducer were used to simulate the temperature effects on E/M admittance and impedance peaks.

2. Density, Poisson’s ratio and compliance coefficient is similar for both impedance and admittance model. Density, Poisson’s ratio decreases with increasing temperature and compliance coefficient increases with temperature.

3. Mechanical loss factor for admittance model is higher than for the impedance model. Mechanical loss factor is directly proportional to the inverse of quality.
factor during resonance. But, Mechanical loss factor is not directly proportional to the inverse of quality factor rather, it depends on other losses.

4. There are very slight variation in dielectric constant and dielectric loss between experimental results and analytical results. Dielectric constant and dielectric loss could be different during anti-resonance and resonance condition due to domain wall motion.

5. Modified in-plane piezoelectric coefficient for analytical model differs significantly from experimental results. A better experimental procedure is required for determining in plane piezoelectric coefficient experimentally.

Table 5.1 Material properties of APC 850 [8]

<table>
<thead>
<tr>
<th>Property</th>
<th>Symbol</th>
<th>Value</th>
</tr>
</thead>
<tbody>
<tr>
<td>Compliance, in plane</td>
<td>$\varepsilon_{11}^E$</td>
<td>$17.2 \times 10^{-12}$ Pa$^{-1}$</td>
</tr>
<tr>
<td>Dielectric constant</td>
<td>$\varepsilon_{33}^T$</td>
<td>1900</td>
</tr>
<tr>
<td>In plane piezoelectric</td>
<td>$d_{31}$</td>
<td>$-175 \times 10^{-12}$ m/V</td>
</tr>
<tr>
<td>Coupling factor</td>
<td>$k_{31}$</td>
<td>0.36</td>
</tr>
<tr>
<td>Poisson ratio</td>
<td>$\nu$</td>
<td>0.35</td>
</tr>
<tr>
<td>Density</td>
<td>$\rho$</td>
<td>7600 kg/m$^3$</td>
</tr>
<tr>
<td>Curie Temperature</td>
<td>$T_c$</td>
<td>360°C</td>
</tr>
</tbody>
</table>

Table 5.2 Material properties deduced from experimental measurements

<table>
<thead>
<tr>
<th>Temperature ($^\circ$C)</th>
<th>$\left(\frac{1}{Q_A}\right)$ %</th>
<th>$\left(\frac{1}{Q_R}\right)$ %</th>
<th>Dielectric constant</th>
<th>Dielectric loss ($\delta$) %</th>
<th>Piezoelectric coefficient</th>
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<tr>
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<tr>
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<tr>
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<td>1.88</td>
<td>-160</td>
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Table 5.3 Modified material properties for E/M impedance modeling to match experimental results

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Density (kg/m³)</th>
<th>Poisson’s ratio ν</th>
<th>$s_{11}$ (x10⁻¹² m²/N)</th>
<th>Mechanical loss factor</th>
<th>Dielectric constant</th>
<th>Dielectric loss (%)</th>
<th>Piezoelectric coefficient $(d_{31})$ (x10⁻¹² m/V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RT</td>
<td>7600</td>
<td>0.350</td>
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<td>(f₀)</td>
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<td>7580</td>
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<td>0.01</td>
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<td>17.25</td>
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<tr>
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Table 5.4 Modified material properties for E/M admittance modeling to match experimental results

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<tr>
<th>Temperature (°C)</th>
<th>Density (kg/m³)</th>
<th>Poisson’s ratio ν</th>
<th>$s_{11}$ (x10⁻¹² m²/N)</th>
<th>Mechanical loss factor $(\eta^R)$ (%)</th>
<th>Dielectric constant</th>
<th>Dielectric loss $(\delta^R)$ %</th>
<th>Piezoelectric coefficient $(d_{31})$ (x10⁻¹² m/V)</th>
</tr>
</thead>
<tbody>
<tr>
<td>RT</td>
<td>7600</td>
<td>0.350</td>
<td>17.2</td>
<td>1.2</td>
<td>1950</td>
<td>1.7</td>
<td>-160</td>
</tr>
<tr>
<td>50</td>
<td>7580</td>
<td>0.351</td>
<td>17.22</td>
<td>1.25</td>
<td>2000</td>
<td>1.8</td>
<td>-165</td>
</tr>
<tr>
<td>100</td>
<td>7550</td>
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<td>17.25</td>
<td>1.28</td>
<td>2200</td>
<td>1.82</td>
<td>-170</td>
</tr>
<tr>
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<td>7520</td>
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<td>17.28</td>
<td>1.30</td>
<td>2300</td>
<td>1.84</td>
<td>-175</td>
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<td>1.33</td>
<td>2500</td>
<td>1.88</td>
<td>-185</td>
</tr>
</tbody>
</table>

5.5 TEMPERATURE CORRECTION OF E/M IMPEDANCE AND ADMITTANCE OF PWAS TRANSDUCER FOR SHM APPLICATION

Reliable high temperature operation of PWAS transducers are desired for SHM applications. To evaluate the temperature effect from the E/M impedance or admittance spectra, a tentative statistical analysis was done. For statistical purpose, a set of six nominally identical PWAS transducers were exposed to different temperatures. Mean value, standard deviation (STD), and % change of mean value of resonance frequency, anti-resonance frequency and amplitude are listed in Table 5.5 and Table 5.6. It can be inferred from these table that the maximum percentage changes in anti-resonance and...
resonance frequencies with temperature are 0.41% and 1.01% respectively. The changes in anti-resonance and resonance amplitudes are 18.1% and 11.6%, respectively. For SHM application by using E/M impedance and admittance method, it is important to notice that a free PWAS transducer does not show significance change in anti-resonance or resonance frequency.

Table 5.5 Temperature effect on anti-resonance and resonance frequency of PWAS transducer

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>Anti-resonance frequency (kHz)</th>
<th>Resonance frequency (kHz)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Average</td>
<td>STD</td>
</tr>
<tr>
<td>25</td>
<td>331.3</td>
<td>1.8</td>
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<tr>
<td>50</td>
<td>330.2</td>
<td>2.2</td>
</tr>
<tr>
<td>100</td>
<td>329.4</td>
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<td>328.1</td>
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<tr>
<td>200</td>
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<td>3.1</td>
</tr>
<tr>
<td>250</td>
<td>325.6</td>
<td>3.4</td>
</tr>
</tbody>
</table>

Table 5.6 Temperature effect on anti-resonance and resonance amplitude of PWAS transducer

<table>
<thead>
<tr>
<th>Temperature (°C)</th>
<th>ReZ (ohm)</th>
<th>ReY (S)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
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<td>STD</td>
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<tr>
<td>25</td>
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<tr>
<td>100</td>
<td>8827.6</td>
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<td>200</td>
<td>6538.6</td>
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</tr>
<tr>
<td>250</td>
<td>5804.2</td>
<td>1130.7</td>
</tr>
</tbody>
</table>
Figure 5.12 Temperature effect on PWAS transducer anti-resonance and resonance frequency (experimental results)

Plots of these irreversible changes in anti-resonance and resonance frequency as a function of temperature are shown in Figure 5.12. Both the anti-resonance frequencies and the resonance frequencies have a linear relationship with temperature. Values of anti-resonance frequency decrease gradually as temperature increases whereas values of resonance frequency increase as temperature increases. That means that in both cases irreversibility seems to depend linearly on temperature. Hence, we suggest using the following temperature correction formula

\[ f(T) = f_0 + \frac{\partial f}{\partial T} T \]  \hspace{1cm} (5-18)

In our particular situation, the experimental values of Eq. (11) are:

\[ f_{\text{AR}}^* (kHz) = 331.750 - 0.0246 T \]  \hspace{1cm} (Impedance) \hspace{1cm} (5-19)

\[ f_{\text{R}}^* (kHz) = 275.118 + 0.0327 T \]  \hspace{1cm} (Admittance) \hspace{1cm} (5-20)
By using Eqs. (12) and (13), temperature corrections can be made to compensate for the irreversible temperature effects. The irreversible temperature sensitivity of anti-resonance and resonance frequency are -0.0246 kHz/°C and 0.0327 kHz/°C respectively. The regression lines fit the data very well due to high R squared values of 0.994 and 0.9754 for impedance and admittance respectively.
CHAPTER 6

GAMMA RADIATION ENDURANCE OF PIEZOELECTRIC WAFER ACTIVE TRANSUDCERS

6.1 STATE OF THE ART

Piezoelectric wafer active sensors (PWAS) have emerged as one of the major SHM technologies; the same sensor installation can be used with a variety of damage detection methods: propagating ultrasonic guided waves, standing waves (E/M impedance) and phased arrays (Zagrai et al., 2010; Park et al., 2006) PWAS transducers are very small, lightweight and inexpensive transducers. PWAS transducers can be bonded on a host structure or between layers of a structure easily. PWAS transducers require low power that makes it feasible for onsite inspection in SHM and NDE applications. PWAS transducers are made of piezoelectric material with electrodes on both sides. To use PWAS transducer as an SHM transducer, electrodes must be connected to the piezoelectric material and the piezoelectric materials must be polarized. Out-of-plane electric field is applied in order to polarize the PWAS transducer in the same direction. Piezoelectric materials convert mechanical energy into electrical energy or vice versa. The traditional PWAS transducers use piezoelectric material Lead Zirconate Titanate (PZT). PWAS transducer material has been attracted by researchers due to its enhanced sensing, actuation or both capabilities. PZT with large coupling coefficient, high permittivity and quick time to respond make it to be an excellent candidate as piezoelectric transducer in SHM and NDE applications.
(Newnham, 1990). Property enhancement was made at the expense of temperature, electric field, and stress stability. Property enhancement is achieved by chemical composition to allow domain wall motion to extrinsically contribute to the piezoelectric effect. This compound class shows much better piezo-electrical and piezo-mechanical efficiency than naturally occurring piezoelectric materials (Veríssimo, 2003). Commercially available APC 850 PWAS transducer was used in this research (APC 850 materials). The wafers disk is 7 mm in diameter and 0.2 mm in thickness. The wafer has a PZT thin film with silver (Ag) electrode on both sides. The Curie point of APC 850 PWAS transducer is 350°C (APC 850 materials).

GaPO4 material is quartz type (α-quartz) belongs to the class of compounds M-X-O4. Single crystal GaPO4 as many researchers (Giurgiutiu et al., 2010; Haines et al., 2006) have investigated piezoelectric materials. GaPO4 possess a low piezoelectric constant and low dielectric constant compared to PZT. But compared with quartz crystal it possesses nearly all the advantages of quartz with higher electromechanical coupling and has thermally stable physical properties up to 950°C. Furthermore, it displays no pyroelectric effect. This article also presents the potential impact on EMIS signature and material properties of GaPO4 as a piezoelectric material after exposure to elevated temperature. Commercially available high quality GaPO4-PWAS (Piezocryst GMBH) was used in this research. The wafers were x-cut GaPO4 single crystal disks of 7 mm diameter and 0.2 mm thickness. The wafer had a GaPO4 single crystal thin film with electrode on both sides.

In physics, radiation is the emission or transmission of energy in the form of waves or particles through space or through a material medium. There are two types of radiation: ionizing (more than 10 eV) and non-ionizing. A common source of ionizing radiation is
radioactive materials that emit α, β, or γ radiation. Gamma radiation consist gamma ray (γ), which is extremely high-frequency electromagnetic radiation and therefore consists of high-energy photons (Augereau, 2008). Gamma radiation is the main source of radiation near DCSS. The measurement unit for gamma radiation dose is the Gray, equal to 1 Joule of absorbed energy per kg of material. The damage caused by gamma rays to PWAS sensors is dependent on the total accumulated dose. The major mechanisms for piezo material performance degradation via gamma ray interaction are: The primary degradation mechanism is depoling, accumulated exposure to ionizing radiation can cause internal defect and radiation-induced charges can be trapped near the electrodes. Such concentration of charge could potentially affect polarizability (Kundzins et al., 2001; Fawzy et al., 2008; Yang et al, 2014; Lin et al., 2012). A comprehensive literature study was already conducted by Sinclair et al in a review paper (Sinclair et al., 2015). In this paper, effects of gamma radiation on piezoelectric properties for several candidate materials were presented. Reliable operation was found of PZT after doses of 1.5 MGy. Ionization damage threshold is 400 MGy of gamma, but only if temperature and neutron fluence are kept low. Signal amplitude decrease of 13% after dose of 22.7 MGy in a customized transducer assembly; drop is believed to be due to a change in piezoelectric efficiency. Tittmann et al. (2014) reported Radiation tolerance of piezoelectric bulk single-crystal aluminum nitride. Theoretical study of ferroelectric properties degradation in perovskite ferroelectrics and anti-ferroelectrics under neutron irradiation was conducted by Kulikov et al.(2004). Radiation, temperature and vacuum effects on piezoelectric wafer active sensors were studied by Giurgiutiu et al. (2016). The change in resonance and anti-resonance frequencies and amplitudes were obtained experimentally in that paper.
6.2 EXPERIMENTS OF PWAS TRANSDUCER

This section presents details experimental procedure for evaluating PWAS performance after exposure to (a) slow irradiation test (b) accelerated irradiation test.

6.2.1 Slow irradiation test

For slow radiation experimental study, radiation dose was set to 0.1 kGy/hr for 20 hours (Cumulative dose: 2 kGy). PWAS were exposed to the radiation for 5 times with 4 hours for each time. A set of ten (numbered from 1 to 10) nominally identical free PWAS (PZT and GaPO4) were tested (Figure 6.1 (a)). The irradiation test was done in a Co-60 gamma irradiator facility (by JL Shepherd and Associate) as shown in Figure 6.1(b). The experimental procedure is shown in Figure 6.2 (Haider et al. 2017 c, 2018g).

Figure 6.1 (a) PWAS for radiation exposure (b) JL Shepherd and associate irradiator facility
Figure 6.2  Experimental procedure flow chart for slow radiation test

Figure 6.3  Irradiator facility
6.2.2 Accelerated irradiation test

For accelerated radiation experiment study, radiation dose was set to 1.233 kGy/hr for 192 hours (Cumulative dose: ~ 223 kGy). The irradiation test was done in a Co-60 Gamma Irradiator facility as shown in Figure 6.3. Radiation dose analysis is shown in Table 6.1. Last 80 hours radiation test was done only to a subset (4 samples from each group). In order to determine the stability of PWAS transducers under gamma radiation, the following experimental procedure was followed (Haider et al. 2018 f, 2018g).

**Step 1:** Transducers were delivered to the testing laboratory for first run with the following steps

- Transducers were placed in petri dish
- Transducers were spread out well in order to be contact free from each other
- Transducers were placed at the center of the irradiator facility (dose rate: 1.233 kGy/hour)
- Transducers were exposed for 16 hours continuously
- Temperature of the sample/chamber was recorded periodically
- Transducers were extracted after the irradiation test and were kept at room temperature

**Step 2:** Transducers were returned back to USC-LAMSS for evaluation

- Electro-mechanical impedance and admittance values were evaluated
- Electrical capacitance was measured

**Step 3:** Repeat Step 1 and Step 2 for additional 16 hours.
**Step 4: Intermediate evaluation**

- SEM system was used to visualize the cross section of PWAS Transducer
- XRD spectrum was used to identify change in crystal structure, unit cell dimension and symmetry.

**Step 5:** Repeat Step 1 and Step 2 for additional 160 hours with 80 hours interval.

**Step 6:** Final evaluation

- SEM system was used to visualize the cross section of PWAS Transducer
- XRD spectrum was used to identify change in crystal structure, unit cell dimension and symmetry.

Table 6.1 Radiation dose analysis for DCSS applications

<table>
<thead>
<tr>
<th>Radiation rate (kGy/hr)</th>
<th>Radiation exposure time (hours)</th>
<th>Radiation exposure (kGy)</th>
<th>Accumulated exposure time (hours)</th>
<th>Accumulated radiation (kGy)</th>
<th>Equivalent to application dose (years)</th>
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<tbody>
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</table>
6.2.3 EMIA measurement

EMIA and electrical capacitance of a set of nominally identical free PWAS transducers were measured in each step. Figure 6.4 shows the experimental set up for EMIA measurement. For PZT, the EMIA were collected from 250 kHz to 350 kHz with a step size of 50 Hz during in plane mode and 10 MHz to 13 MHz with a step size of 250 Hz during thickness mode measurement. For GaPO4, the EMIA were collected from 270 kHz to 300 kHz with a step size of 1 Hz during in plane mode and 8 MHz to 11 MHz with a step size of 250 Hz during thickness mode measurement. The real part of the EMIA is used in this research (Haider et al. 2017). First PWAS anti-resonance and resonance was considered in this article as it shows more stable response than higher resonance or anti-resonance spectrum (Haider et al. 2017).

![Experimental setup for EMIA measurement](image)

Figure 6.4 Experimental setup for EMIA measurement

Capacitance of the transducer can be measured by using following formula,

\[
\bar{C} = \bar{\varepsilon} \frac{A}{d}
\]  

(6-1)
Here, $\bar{\varepsilon} = \varepsilon' - i\varepsilon'' = \frac{-i}{\omega Z'(\omega)C_0}$ and $C_0 = \frac{\varepsilon_0 A}{d}$; $\varepsilon_0 = 8.85 \times 10^{-12}$ F/m; $A$ is the area of the sample and $d$ is the thickness of the sample.

The measured impedance of the transducer is $Z' = Z' + iZ''$. Here, $Z'$ is the real part and $Z''$ is the imaginary part of the impedance.

6.3 RESULTS AND DISCUSSION

6.3.1 EMIA of PZT-PWAS transducers

Figure 6.5 shows the typical EMIA spectra of a circular PWAS. Anti-resonance frequency is the frequency at which the impedance spectral peak is observed and the corresponding value of the peak is the anti-resonance amplitude. Similarly, resonance frequency is the frequency at which the admittance spectral peak is observed and the corresponding value of the peak is the anti-resonance amplitude.

![Graph showing typical EMIA spectra of a circular PWAS](a) impedance spectra (b) admittance spectra)

Change in the resonance/anti-resonance frequency and amplitude are the key factors for evaluating PWAS transducer piezoelectric performance. The major mechanisms
for piezoelectric material properties degradation via gamma ray interaction are considered as follows:

1. The primary degradation mechanism is pinning of domain
2. Accumulated exposure to ionizing radiation can cause internal defect
3. Radiation-induced charges can be trapped near the electrodes. Such concentration of charge could potentially affect polarizability.

Figure 6.6 (a)-(d) show the change of resonance frequency and amplitude for both radially and thickness vibrating PZT-PWAS transducers with accumulated gamma radiation. To facilitate the understanding the plot of PWAS EMIA with exposed radiation, slow radiation part and accelerated radiation part of the total dose is shown in Figure 6.6 (a). Rest of the figures carry the same information as Figure 6.6 (a). In order to respond to skewness towards large radiation dose, the cumulative radiation dose values are plotted in log scale (x-axis). A small radiation exposure (~30 Gy) was assigned to the before radiation test data in order to plot in the log scale.

Figure 6.6 (a) and (c) shows the change in resonance frequency with accumulated radiation dose for in-place and thickness vibrating PZT-PWAS respectively. Whereas Figure 6.6 (b) and (d) shows the change in resonance amplitude with accumulated radiation dose for in-place and thickness vibrating PZT-PWAS respectively. The mean resonance frequency before radiation of radially and thickness vibrating PWAS are 287 kHz and 1134 kHz respectively. The resonance frequency (Figure 6.6 (a) and (c)) increases at higher rate initially but at higher cumulative radiation dose the frequency increases at slower rate. Therefore, the resonance frequency follows a logarithmic increment with radiation dose. The resonance amplitude shows the opposite phenomenon i.e., resonance amplitude decreases logarithmically with radiation dose.
Figure 6.6 Change in (a) –(b) Resonance frequency and amplitude for radially vibrating PZT-PWAS; (c)-(d) Resonance frequency and amplitude for thickness vibrating PZT-PWAS; with cumulative gamma radiation dose
However, the change in resonance frequency for thickness vibrating PZT-PWAS is significant compared to radially vibrating PZT-PWAS. Eqs. (1) and (5) shows that amplitude of impedance and admittance value depend on capacitance and piezoelectric coupling factor. Admittance of a piezoelectric sensor is proportional to capacitance, concluding that capacitance of the piezoelectric sensor decreases with increasing radiation dose. Figure 6.7 shows the change in electrical capacitance after exposure to different radiation dose. Electrical capacitance decreases logarithmically as radiation absorbed dose increases. Many researchers have reported decreasing capacitance value with increasing radiation dose. Several polarization mechanisms are responsible for changing the dielectric response. Among them, pinning of domain is the key factor to the change the dielectric response of PZT-PWAS transducer. Pinning of domains may be defined as restricted domain configuration in unfavorable position. Dipole moment of PWAS transducer material may be changed due to change in the domain configuration after radiation exposure. This change affects the piezoelectric coefficient as well as piezoelectric coupling. The piezoelectric effect is a linear coupling between the polarization and the applied stress field. Radiation exposure of the piezoelectric materials may restrict the
domain in unfavorable position, hence leading to degraded net polarization. Therefore, piezoelectric coupling factor decreases as radiation exposure increases.

![Graphs showing relationship between antiresonance frequency/amplitude and cumulative radiation dose](image)

Figure 6.8 Change in (a) –(b) Anti-resonance frequency and amplitude for radially vibrating PZT-PWAS; (c)-(d) Anti-resonance frequency and amplitude for thickness vibrating PZT-PWAS; with cumulative gamma radiation dose

Temperature may have also effect on dielectric properties of the PWAS transducers. Therefore, temperature was monitored periodically during the radiation exposure test. Only 1-2°C temperature fluctuation from room temperature was observed.
Figure 6.8 (a)-(d) show the change of anti-resonance frequency and amplitude for both radially and thickness vibrating PZT-PWAS transducers with accumulated gamma radiation. Figure 6.8 shows that both anti-resonance frequency and amplitude increases logarithmically with increasing radiation dose. The anti-resonance (Eqs. (1) and (5)) amplitude is inversely proportion the capacitance. Anti-resonance value is also depend on piezoelectric coupling. Therefore decreasing capacitance and piezoelectric coupling value resulting increasing of anti-resonance amplitude. Anti-resonance frequency shows larger variation than resonance frequency with applied radiation dose. Resonance frequency depends on poison’s ratio, density, stiffness, and mechanical loss factor. Extrinsic contribution of domain may contribute to change stiffness and mechanical loss factor resulting a change in resonance frequency. However, anti-resonance frequency depends on poison’s ratio, density, stiffness, piezoelectric coupling factor, electrical capacitance, mechanical loss factor, and dielectric loss. The change in piezoelectric coupling factor, electrical capacitance, mechanical loss factor, and dielectric loss contributes a larger change in anti-resonance frequency.

6.3.2 EMIA of GaPO₄-PWAS transducers

Figure 6.9 (a) and (c) show the change of resonance frequency with radiation for in-plane and thickness modes respectively. The resonance frequency increases slightly with increasing exposed radiation dose whereas, the resonance amplitude decreases with increasing exposed radiation dose. However due to large scatter of resonance amplitude between the samples resulting a large standard error (error bar). Due to high electrical resistivity of GaPO₄-PWAS, it shows very low resonance amplitude (Figure 6.9 (b)). The
change in resonance frequency and amplitude can be explained by the role of Ga–O–P interactions.

Figure 6.9 Change in (a)–(b) Resonance frequency and amplitude for radially vibrating GaPO₄ PWAS; (c)–(d) Resonance frequency and amplitude for thickness vibrating GaPO₄ PWAS; with cumulative gamma radiation dose.
The Ga–O and P–O distances may contract due to radiation exposure, based on the ionic radii it may change Ga–O–P angle. The change in Ga-O-P angle may contribute to change in capacitance and piezoelectric performance as well as mechanical and electrical loss. Figure 6.10 shows the change in electrical capacitance of GaPO₄-PWAS with cumulative radiation dose. The capacitance value decreases with increasing radiation dose. The resonance amplitude is proportional to capacitance therefore; decreasing resonance amplitude with increasing radiation dose is expected. Figure 6.11 shows the change in anti-resonance frequency and amplitude with exposed radiation dose. For both radially vibrating and thickness vibrating GaPO₄-PWAS the resonance frequency increases logarithmically (Figure 6.11 (a) and Figure 6.11 (c)) with increasing radiation dose. Decreasing electrical capacitance value and piezoelectric coupling resulting an increase of anti-resonance amplitude (Figure 6.11 (c) and Figure 6.11 (d)). The anti-resonance frequency shows larger variation than resonance frequency. Dielectric and mechanical loss due to change in Ga-O-P angle may contribute to the larger variation in anti-resonance frequency.
Figure 6.11  Change in (a) –(b) Anti-resonance frequency and amplitude for radially vibrating GaPO₄ PWAS; (c)-(d) Anti-resonance frequency and amplitude for thickness vibrating GaPO₄ PWAS; with cumulative gamma radiation dose

6.4 MICROSTRUCTURAL AND CRYSTALLOGRAPHIC INVESTIGATION

The microstructure of the PWAS was examined using scanning electron microscopy (SEM) where, crystal structure was investigated by X-ray diffraction (XRD) method. SEM
and XRD examinations were performed on four cases (a) before exposure to radiation (b) after 2 kGy radiation exposure and (c) after 40.7 kGy radiation exposure (d) after 225 kGy radiation exposure. SEM images of PWAS transducers (Ag/PZT/Ag and Pt/GaPO₄/Pt) cross section are shown in Figure 6.12 and Figure 6.13. PZT-PWAS transducer has two Ag electrodes on top and bottom with multi domain PZT in between them (Figure 6.12). GaPO₄ has also two Pt electrodes on top and bottom with single crystal GaPO₄ in between them (Figure 6.13). No significant variation was found in the microstructure (PZT grains and GaPO₄ single crystal) PWAS transducers due to radiation exposure (Figure 6.12 and Figure 6.13). Figure 6.14 and Figure 6.15 show the X-ray powder diffraction (XRD) spectrum for PWAS transducers material. The important feature of the XRD measurement is X-ray diffraction peak with diffraction angle (2θ). Any change in the crystal structure shifts the peak with diffraction angle (2θ). Figure 6.14 show no significant change in X-ray diffraction peak position of PZT-PWAS transducer material. Figure 6.15 shows that the overall XRD patterns of GaPO₄-PWAS transducer material remain the same before and after radiation exposure. A very small shift in position of X-ray diffraction peak toward the higher angle side (first large X-ray diffraction peak appears at 26.25°, 26.35°, 26.39° and 26.45° for unexposed, 2 kGy, 40.7 kGy and 225 kGy radiated GaPO₄-PWAS respectively) indicates a slight contraction in the lattice due to radiation induced effect. SEM and XRD studies confirm that, there was no significant variation in PWAS transducer microstructure and crystal structure.
(a) Before irradiation

(b) After 2 kGy

(c) After 40.7 kGy

(d) After 225 kGy

Figure 6.12 SEM micrograph of cross section of Ag/PZT/Ag PWAS transducer (a) before irradiation (b) after slow irradiation (2 kGy) (c) after accelerated irradiation of 40.7 kGy (d) after accelerated radiation of 225 kGy
Figure 6.13  SEM micrograph of cross section of Pt/GaPO$_4$/Pt PWAS transducer (a) before irradiation (b) after slow irradiation (2 kGy) (c) after accelerated irradiation of 40.7 kGy (d) after accelerated radiation of 225 kGy
Figure 6.14  XRD of PZT-PWAS transducer (a) before irradiation (b) after 2 kGy radiation (c) after 40.7 kGy radiation (d) after 225 kGy radiation
Figure 6.15 XRD of GaPO₄-PWAS transducer (a) before irradiation (b) after 2 kGy radiation (c) after 40.7 kGy radiation (d) after 225 kGy radiation
6.5 Radiation Dependent Sensitivity of EMIA

PWAS transducers shows some changes in EMIA spectra with radiation. In order to use the PWAS transducers in SHM applications radiation dependent sensitivity need to be determined and the radiation effect on EMIA reading need to be evaluated. Mean values and standard errors from a set of identical PWAS transducers are plotted in Section 6.3. The changes in anti-resonance and resonance frequency and amplitude as a function of radiation dose are shown in their respective plots. From these figures, it can be inferred that anti-resonance amplitude decrease logarithmically with exposed radiation whereas values of resonance amplitude increase as radiation dose increases. Values of anti-resonance and resonance frequencies decrease logarithmically as radiation dose increases. Hence, the following radiation compensation formula can be proposed

\[
f_{R/AR}^{R/AR}(R) = f_{o/AR}^{R/AR} + m_{f}^{R/AR} \log_{e} R_{d}
\]

Here \( f_{R/AR} \) is resonance/anti-resonance frequency; \( R_{d} \) is accumulated radiation dose; \( m_{f}^{R/AR} \) is the slope of the resonance/anti-resonance curve; \( f_{o/AR} \) is the frequency axis intercept.

Slope of the resonance/anti-resonance curve can be written as

\[
\frac{\partial (f_{R/AR})}{\partial (\log_{e} R_{d})} = m_{f}^{R/AR}
\]

Here \( m_{f}^{R/AR} \) is defined as logarithmic sensitivity of equation (11). Therefore, change of frequency at particular radiation dose depend on logarithmic value of radiation dose.
Similarly, for resonance/anti-resonance amplitude and electrical capacitance sensitivity can be defined as

$$\frac{\partial(A^{R/AR})}{\partial(\log_r R_d)} = m^{R/AR}_A$$

(6-4)

$$\frac{\partial(C)}{\partial(\log_r R_d)} = m_C$$

(6-5)

Here $A^{R/AR}$ is resonance/anti-resonance amplitude; $C$ is the electrical capacitance; $R_d$ is accumulated radiation dose; $m^{R/AR}_A$ and $m_C$ is the slope of the amplitude and electrical capacitance curve.

Table 6.2 Gamma radiation effect on EMIA of PWAS transducers.

<table>
<thead>
<tr>
<th>Direction</th>
<th>PWAS types</th>
<th>Sensitivity with radiation $\left( \frac{\partial(f^{\pm\omega}, A^{\pm\omega})}{\partial(\log_r R_d)} \right)$</th>
<th>Capacitance $\left( \frac{\partial(C)}{\partial(\log_r R_d)} \right)$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Resonance frequency</td>
<td></td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{\partial(f^+)}{\partial(\log_r R_d)}$</td>
<td></td>
</tr>
<tr>
<td>In-plane</td>
<td>PZT</td>
<td>0.244</td>
<td>-3e-4</td>
</tr>
<tr>
<td>Thickness</td>
<td>PZT</td>
<td>7.44</td>
<td>-0.002</td>
</tr>
<tr>
<td>In-plane</td>
<td>GaPO_4</td>
<td>0.0629</td>
<td>-0.033</td>
</tr>
<tr>
<td>Thickness</td>
<td>GaPO_4</td>
<td>2.454</td>
<td>-0.001</td>
</tr>
</tbody>
</table>

Table 6.2 shows the instantaneous sensitivity of PWAS resonance/anti-resonance frequency, amplitude and electrical capacitance. Radiation dependent logarithmic sensitivity of PZT-PWAS in-plane and thickness resonance frequency was estimated as 0.244 kHz and 7.44 kHz respectively whereas; logarithmic sensitivity of PZT-PWAS in-
plane and thickness anti-resonance frequency is higher 0.674 KHz and 11.7 kHz respectively. The similarly, the logarithmic sensitivity of GaPO₄-PWAS in-plane and thickness resonance frequency was estimated as 0.0629 kHz and 2.454 kHz respectively whereas; logarithmic sensitivity of GaPO₄-PWAS in-plane and thickness anti-resonance frequency is higher 0.18 KHz and 3.42 kHz respectively. Therefore, GaPO₄-PWAS EMIA spectra shows more gamma radiation endurance than PZT-PWAS. By observing Table 6.2 it can be inferred that the resonance frequency shows less radiation sensitivity than anti-resonance frequency. The same conclusion can be drawn for resonance amplitude over anti-resonance amplitude.
CHAPTER 7
SUMMARY, CONCLUSIONS AND FUTURE WORK

7.1 SUMMARY

The guided waves generated by an acoustic emission (AE) event were analyzed through a Helmholtz potential approach. The inhomogeneous elastodynamic Navier-Lame equation was expressed as a system of wave equations in terms of unknown scalar and vector solution potentials, \( \Phi \), \( \vec{H} \), and known scalar and vector excitation potentials, \( A \), \( \vec{B} \). The excitation potentials \( A \), \( \vec{B} \) were traced to the energy released during an incremental crack propagation.

The solution was readily obtained through direct and inverse integral transforms and application of the residue theorem. The resulting solution took the form of a series expansion containing the superposition of all the Lamb waves modes existing for the particular frequency-thickness combination under consideration.

A numerical study of the AE guided wave propagation in a 6 mm thick 304-steel plate was conducted in order to predict the out-of-plane displacement that would be recorded by an AE sensor placed on the plate surface at some distance away from the source. Parameter studies were performed to evaluate: (a) the effect of the pressure and shear potentials; (b) the effect of the thickness-wise location of the excitation potential
sources varying from mid-plane to the top surface (depth of source effect); (c) the effect of propagating distance away from the source.

This dissertation also illustrated how the Lamb wave complex eigen space can be used efficiently to project the thickness dependent boundary conditions encountered in the Lamb wave scatter problem. The convergence and accuracy of the CMEP method was verified over a wide range of frequency. It was found that the CMEP method is more than two orders of magnitude faster than FEM for the same accuracy. As a byproduct, the CMEP method also yields the local vibration field near the damage which is dominated by the evanescent and complex wave modes. guides and vertical free-ends to implement the orthogonality relations.

E/M impedance/admittance response of free PWAS transducer after exposure to various high temperature environments (50°C to 250°C) was investigated. The PWAS transducers were made of APC-850 PZT material. Both frequency shift and amplitude change in E/M impedance/admittance were observed due to change in PWAS transducer material state. Slight variation of 0.41% and 1.01% in resonance and anti-resonance frequency were observed respectively. The observed changes in anti- resonance and resonance amplitudes were larger, 18.1% and 11.6%, respectively.

Peaks in the impedance/admittance signature depend on material properties such as stiffness coefficient, piezoelectric constant, dielectric constant and density. The change in dielectric properties, piezoelectric constant and elastic coefficients may be due to extrinsic response originating from domain wall motion without changes in microstructure or crystal structure. Scanning Electron Microscopy (SEM) was used to examine the microstructure of the PWAS transducer material and no change in microstructure was observed. Due to
irreversible domain dynamics, piezoelectric, elastic, and dielectric constants show irreversible response upon cooling from heating. X-ray powder diffraction (XRD) spectrum was examined to see the change in crystal structure, unit cell dimension, and symmetry. No significant change in crystal structure, unit cell dimension, and symmetry was observed.

Numerical simulation was also performed; Numerical results show that the impedance and admittance strongly depends on elastic coefficient, dielectric constant, dielectric loss tangent, mechanical loss and in-plane piezoelectric constant that can change significantly after high temperature exposure. The change in material properties of PWAS transducer are responsible for changing in amplitude and frequency of the peak amplitude of impedance or admittance. A tentative statistical analysis was conducted to find sensitivity of resonance and anti-resonance frequencies and amplitudes with temperature. Resonance frequency and anti-resonance frequency shows less sensitive to temperature than the corresponding amplitudes.

To develop proper SHM techniques for DCSS, EMIA response of free PWAS transducers were evaluated after gamma radiation exposure. The PWAS transducers were made of PZT material and GaPO$_4$ crystal. Piezoelectric material degradation of PWAS transducers were observed due to gamma radiation exposure. Values of anti-resonance and resonance frequencies decrease logarithmically as radiation dose increases. The tentative statistical analysis was conducted to find the logarithmic sensitivity of resonance and anti-resonance frequencies and amplitudes with radiation. Radiation dependent logarithmic sensitivity of PZT-PWAS in-plane and thickness resonance frequency $\left( \frac{\partial (f^*)}{\partial (\log R_y)} \right)$ was estimated as 0.244 kHz and 7.44 kHz respectively.
whereas; the logarithmic sensitivity $\frac{\partial (f^*)}{\partial (\log_e R)}$ of GaPO$_4$-PWAS in-plane and thickness resonance frequency was estimated as 0.0629 kHz and 2.454 kHz respectively. Scanning Electron Microscopy (SEM) and X-ray diffraction method (XRD) was used to investigate the microstructure and crystal structure of the PWAS transducer material receptively. No significant change in microstructure morphology, crystal structure was observed.

The change in material properties of PZT-PWAS may be explained by the pinning of domain walls by some radiation-induced effect, whereas, material properties degradation behavior of GaPO$_4$-PWAS may be explained by the change in Ga-O; P-O distances and Ga-O-P angle.

7.2 CONCLUSIONS

This article has shown that pressure and shear source potentials can be used to model the guided wave generation and propagation due to an AE event associated with incremental crack growth. The advantage of this approach is to decouple the inhomogeneous Navier-Lame equations.

The numerical studies performed over a range of parameters have shown that:

i. Both A0 and S0 modes generated by an AE excitation in a 6-mm steel plate are dispersive. The high frequency components of the AE excitation are responsible for the dispersive part of the S0 mode, whereas the low frequency components of the AE excitation are responsible for the dispersive part of the A0 mode.

ii. The peak amplitude of A0 mode is higher than the peak amplitude of the S0 mode for all cases.
iii. The amplitude of bulk wave is much smaller than peak A0 and S0 amplitude. Therefore, peak amplitude of bulk waves may not be significant in real AE signal.

iv. For mid-plane AE source location, the shear potential part of the AE source has more contribution to the peak A0 amplitude whereas the pressure potential part of the AE source has more contribution to the peak S0 amplitude.

v. An increase in the depth of source increases the peak A0 and S0 amplitude while using pressure potential only, whereas, an increase in the depth of source increases the peak A0 and decreases the peak S0 amplitude while using shear potential only.

vi. If the AE source is located at top surface, the effect of the excitation pressure and shear potentials on the S0 and A0 modes seems to be decoupled: pressure potential does not contribute to S0 and bulk wave amplitude whereas shear potential does not contribute to the A0 and bulk wave amplitude.

vii. For top-surface AE source, pressure excitation potential has contribution to the high amplitude low-frequency component of the A0 wave packet. This contribution decreases as the depth of source increases.

viii. Pressure potential has contribution to the high amplitude of low frequency component of A0 wave packet for top-surface source.

ix. Peak amplitude of A0 shifts to lower frequency components with decreasing depth of source.

x. Frequency components of S0 changes with changing depth of source.

xi. Peak time has notable contribution to the low frequency component of A0.
xii. Peak S0 amplitude is not sensitive to peak time.

xiii. Low frequency A0 amplitude increases with increasing peak time, however maximum peak was observed relatively at higher frequency.

For Lamb wave scattering problem, an efficient analytical global-local method was presented in this paper to determine Lamb wave scattering from a discontinuity. A local analytical method called CMEP was employed to determine the scattering coefficients. The scattered wave fields from a discontinuity were expanded in terms of complex Lamb wave modes with unknown scatter coefficients. These unknown coefficients were obtained from the boundary conditions using a vector projection utilizing the power expression. A convergence study was performed; it was found that 35 modes in the expansion ensured convergence to less than 2% error. Two cases were considered in this research: (a) a plate with a pristine stiffener and (b) a plate with a cracked stiffener. Complex-valued scattering coefficients are calculated from 50 kHz to 350 kHz for both S0 and A0 incident waves.

The frequency dependent complex valued scattering coefficients are then inserted into the global analytical model. The global analytical solution provides time domain scattered signal from the discontinuity. An experiment was conducted for both pristine stiffener and cracked stiffener to validate the CMEP and global analytical results. A long strip PWAS was used to create straight crested Lamb wave modes in the plate. Antisymmetric Lamb wave mode selectively excited by using two PWAS in out of phase on opposite sides of the plate. Single point laser Doppler vibrometer (LDV) measurement was done on the plate to obtain the out-of-plane velocity of scattered Lamb wave fields. The obtained experimental results agree well with the CMEP and global analytical predictions.
The combined analytical global-local method is a reliable tool for predicting the scattering of Lamb waves even in a complex structure. The scattered signals from the global-local method can be used to detect a crack in complex geometry. FEM, BEM or other numerical techniques require extensive computation time to attain computational accuracy. The analytical global-local method can predict the scattering signals in seconds instead of hours. Therefore, the combined analytical global-local method shows remarkable performance regarding computational efficiency. It is also important to obtain scattering coefficients with sufficient computational accuracy to simulate the scatter signal. CMEP can predict the scattering coefficients with high computational efficiency.

The experiment showed that PWAS transducers might have some sensitivity to high temperature and gamma radiation. A detailed experiment was conducted to evaluate PWAS performance after high temperature and gamma radiation exposure. A compensation technique is proposed in this dissertation base on the fact that, irreversible changes in anti-resonance and resonance frequencies have a linear relationship with temperature. This relation could provide temperature compensation in high temperature environment and could be useful for proper damage detection. Obviously, temperature measurements will need to be included with the acquisition of each impedance/admittance signature. However, these PWAS transducers are susceptible to damage themselves after heating to elevated temperature. Therefore, for proper structural health monitoring system it is important to characterize the transducer systematically before installing on the host structure. PWAS transducer characterization allows a SHM system to infer the integrity of the transducers and separate flawed signal from structural defects. Changes in resonance/anti-resonance frequencies and amplitudes in the PWAS transducers were also
evaluated after exposure to gamma radiation. For GaPO₄-PWAS, slight variation in resonance and anti-resonance frequencies of both in-plane and thickness modes were observed. Whereas, for PZT-PWAS larger variation in resonance and anti-resonance frequencies of both in-plane and thickness modes were observed. Therefore, it can be concluded that GaPO₄-PWAS transducers show more radiation endurance than PZT-PWAS. It was found that the changes in resonance and anti-resonance frequencies have a logarithmic relationship with radiation dose. Hence, a compensation technique has been proposed to take care of this aspect. This compensation technique could be useful to distinguish irradiation effect from EMIA signal for structural damage detection in nuclear-spent fuel storage facilities. This research could provide a number of future benefits: (a) radiation compensation for proper damage detection (b) developing proper damage detection method in SHM applications for nuclear environment (c) a proper SHM technique in nuclear environment with limited number of transducers (d) developing a method for transducer characterization to separate defective transducers for impedance and admittance based SHM technique.

7.3 Future work

Substantial future work is still needed to verify the hypotheses and substantiate the calculation of the AE source potentials that produce the guided wave excitation. The extensive experimental AE monitoring data existing in the literature should be explored to find actual physical signals that could be compared with numerical predictions in order to extract factual data about the amplitude and time-evolution of the AE source potentials. A frequency analysis of time domain signal should be done to analyze the frequency content of the captured AE signals. Frequency content may help to distinguish different source
types and source location. If necessary, additional experiments with wider band AE sensors should be conducted. An inverse algorithm could then be developed to characterize the AE source during crack propagation. The source characterization can provide information about amount of energy released from the crack. Therefore, it may help to generate a qualitative as well quantitative description of the crack propagation phenomenon. A further extensive study on the effect of plate thickness, AE rise time, and AE depth of source would be recommended. A non-axisymmetric circular crested of AE elastic wave generation due to excitation potentials need to be investigated.

The CMEP method was demonstrated for straight crested Lamb waves which exist in $z$ invariant condition. Similar approach can be taken to analyze interactions of non-straight crested Lamb waves with damages. However, in addition to Lamb waves, we need to consider shear horizontal waves for such analysis in non $z$ invariant elastodynamic field. Further research would be recommended in order to conduct a statistical distribution for in-plane piezoelectric coefficient and better experimental procedure can be used to measure the in-plane strain (e.g. digital image correlation). A future study would also be recommended to examine the domain configuration of PZT by transmission electron microscopy (TEM). A special arrangement would be necessary to heat up the sample and then cool down to room temperature for in situ inspection of piezoelectric domain under TEM.
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