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Local Rings and Golod Homomorphisms

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by

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Abstract

The Poincaré series of a local ring is the generating function of the Betti numbers for the residue field. The question of when this series represents a rational function is a classical problem in commutative algebra. Golod rings were introduced by Golod in 1962 and are one example of a class of rings that have rational Poincaré series. The idea was generalized to Golod homomorphisms by Levin in 1975.

In this paper we prove two homomorphisms are Golod. The first is a class of ideals such that the natural projection to the quotient ring is a Golod homomorphism. The second deals with Golod homomorphisms between certain fiber products. To prove the second we give a construction for a resolution of a module over a fiber product.

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Chapter 1

Introduction

Let R be a local ring, $\mathfrak{m}$ its maximal ideal, and $\boldsymbol{k} = R/\mathfrak{m}$ its residue field. Then we can define the Poincaré series of R

$$P_{\boldsymbol{k}}^R(t) = \sum_{n=0}^{\infty} \dim_{\boldsymbol{k}} \operatorname{Tor}_n^R(\boldsymbol{k}, \boldsymbol{k}) t^n.$$

The study of this series is a classical problem in commutative algebra. Since R is local the residue field can be resolved by free R-modules. The minimal resolution of $\boldsymbol{k}$ has the form

$$0 \longleftarrow R^{\beta_0} \longleftarrow R^{\beta_1} \longleftarrow \dots \longleftarrow R^{\beta_n} \longleftarrow \dots$$

where β_n is the nth Betti number. It follows that $\dim_{\boldsymbol{k}} \operatorname{Tor}_n^R(\boldsymbol{k}, \boldsymbol{k}) = \beta_n$. So the Poincaré series of R is the generating function of the sequence of Betti numbers.

Of particular interest is the question of when this series equals a rational function in t.

For example, if $R = \boldsymbol{k}[x]/(x^2)$, then $\operatorname{Tor}_n^R(\boldsymbol{k}, \boldsymbol{k}) = 1$ for all n. So

$$P_{\boldsymbol{k}}^R(t) = \sum_{n=0}^{\infty} t^n = \frac{1}{1-t},$$

a rational function.

Both Kaplansky and Serre asked whether the Poincaré series of a local ring is always a rational function. Kaplansky did not publish the question. Serre posed the question in [15, pg. 118].

There is some evidence that this may be the case as several familiar rings have rational Poincaré series. For example:

1. Let R is a regular local ring of Krull dimension n. The residue field $\boldsymbol{k}$ of R is resolved by the Koszul Complex. In this case

$$P_{\boldsymbol{k}}^R(t) = (1+t)^{\beta_1}.$$

2. Let R be a regular local ring of Krull dimension n and $f_1, \ldots, f_c \in R$ a regular sequence. Then Tate shows in [16] that the complete intersection $S = R/(f_1, \ldots, f_c)$ has a rational Poincaré series and that

$$P_{\boldsymbol{k}}^S(t) = \frac{(1+t)^n}{(1-t^2)^c}.$$

However the answer to the question of Kaplansky and Serre was shown to be no by David Anick [1] in 1982. Ancik's counterexample was the ring

$$\frac{\boldsymbol{k}[x_1, \ldots, x_5]}{(x_1^2, x_2^2, x_4^2, x_5^2, x_1x_2, x_4x_5, x_1x_3 + x_3x_4 + x_2x_5)}$$

where $\boldsymbol{k}$ is a field of characteristic not equal to two.

The examples of rings with rational and irrational Poincaré series do not give much insight into which of the following should be expected:

1. Most rings have rational Poincaré series. Rings with irrational Poincaré series are rare.

2. While some special classes of rings have rational Poincaré series, most rings have irrational Poincaré series.

3. Rational and irrational Poincaré series both commonly occur.

This is difficult to determine since the Poincaré series of a ring is not easily computable unless additional hypothesis about the ring are made.

It was shown by Serre that if S is a regular local ring with the same residue field as R and $S \to R$ is a surjective ring homomorphism then the Poincaré series of R is term-wise bounded above by the series

$$\frac{P^S_{\boldsymbol{k}}(t)}{1 - t\,(P^S_R(t) - 1)}. \tag{1.1}$$

See [2, 3.3.2] for a proof.

A ring that meets this upper-bound is called a Golod ring, named for Evgenii Golod who first examined such rings in [4]. This upper-bound is a rational function in t since both $P^S_{\boldsymbol{k}}(t)$ and $P^S_R(t)$ are polynomials. So Golod rings are another example of rings with rational Poincaré series. Such rings exhibit maximal growth of Betti numbers.

The Golod property can be generalized in two ways:

1. The Poincaré series can be generalized to any R-module M. When S is a regular local ring with the same residue field as R and $S \to R$ is a surjective ring homomorphism an upper-bound similar to (1.1) for $P^R_M(t)$ exists. This allows us to define a Golod module.

2. The upper-bound in (1.1) still holds when the regular condition is removed from S.

In Chapter 2 notation is established and key definitions are given. These include algebra retracts, fiber products, Golod homomorphisms, and large homomorphisms. Several preliminary results are also given.

In Chapter 3 a question asked by Gupta in [5] is answered. Here the ideal

$$I(s,t,u,v) = (x_1,\ldots,x_m)^s + (x_1,\ldots,x_m)^u(y_1,\ldots,y_n)^v + (y_1,\ldots,y_n)^t$$

of $\boldsymbol{k}[x_1,\ldots,x_m,y_1,\ldots,y_n]$ is considered. Using the results of [9] we show that the ring

$$\frac{\boldsymbol{k}[x_1,\ldots,x_m,y_1,\ldots,y_n]}{I(s,t,u,v)}$$

is Golod when $1 \le u < s$ and $1 \le v < t$.

In Chapter 4 we turn our attention to the fiber product, $S \times_R T$, of S and T over R that fits the commutative diagram

$$\begin{tikzcd} S \times_R T \arrow[r, "\pi_T"] \arrow[d, "\pi_S"] & T \arrow[d, "p_T"] \\ S \arrow[r, "p_S"] & R. \end{tikzcd}$$

In this chapter we produce a minimal free resolution for an S-module M over $S \times_R T$ using the minimal free resolutions of R and M as S-modules and R as a T-module. The definition of the resolution was inspired by the work of Moore in [13], who produced a resolution in the case that $R = \boldsymbol{k}$, the shared residue field of S and T. To produce a resolution in the case that R is more general the hypothesis that R is an algebra retract of S and of T is added.

In Chapter 5 we use the structure of the resolution given in Chapter 4 to find the following equality of the Poincaré series of an S-module M over the fiber product $S \times_R T$

$$\frac{1}{P_M^A(t)} = \frac{P_R^S(t)}{P_M^S(t)} \left(\frac{1}{P_R^S(t)} + \frac{1}{P_R^T(t)} - 1 \right).$$

This equality is not original to this paper. It was first shown by Dress and Krämer in [3] in the case that $R = M = \boldsymbol{k}$. Herzog showed the equality in [6] using the assumption that the maps p_S and p_T are equivalent representations of R. That is, there exists maps $\phi_S : S \to T$ and $\phi_T : T \to S$ with the property that the diagram

$$\begin{tikzcd} & T \arrow[dl, shift left, "\phi_T"'] \arrow[d, "p_T"] \\ S \arrow[ur, shift left, "\phi_S"'] \arrow[r, "p_S"'] & R \end{tikzcd}$$

commutes.

Neither of these previous works produced a resolution that reflected the equality.

Chapter 2

Notation, Definitions and Preliminaries

In this chapter we establish notation and definitions that will be used throughout the remainder of the paper. We also record a few results that will be used later.

We use $(R, \mathfrak{m}, \boldsymbol{k})$ for a local ring R with unique maximal ideal $\mathfrak{m}$ and residue field $\boldsymbol{k} = R/\mathfrak{m}$.

All rings are assumed to be Noetherian and all modules are assumed to be finitely generated.

If $F(t) = \sum a_n t^n$ and $G(t) = \sum b_n t^n$ are power series in t with $b_n \leq a_n$ for all n, then we say that $F(t)$ is term-wise bounded above by $F(t)$ and write

$$G(t) \preceq F(t).$$

Observation 2.0.1. *Let $F(t)$ and $G(t)$ be power series in t with positive coefficients and $G(t) \preceq F(t)$. If $1/F(t)$ and $1/G(t)$ exist, then $1/F(t) \preceq 1/G(t)$.*

Proof. Let

$$F(t) = \sum_{n=0}^{\infty} a_n t^n, \qquad G(t) = \sum_{n=0}^{\infty} b_n t^n,$$

$$\frac{1}{F(t)} = \sum_{n=0}^{\infty} c_n t^n, \quad \text{and} \quad \frac{1}{G(t)} = \sum_{n=0}^{\infty} d_n t^n.$$

We will show that $c_n \leq d_n$ for all n. Note that since $1/F(t)$ and $1/G(t)$ exist a_0 and b_0 are invertible and hence non-zero.

We proceed by induction on n. For $n = 0$ note that

$$c_0 = \frac{1}{a_0} \quad \text{and} \quad d_0 = \frac{1}{b_0}.$$

Since $G(t) \preceq F(t)$, $b_0 \leq a_0$. So $c_0 \leq d_0$.

Assume that $c_k \leq d_k$ for $k < n$. Now note that

$$c_n = \frac{1}{a_0} \sum_{k=1}^{\infty} (-a_k) c_{n-k}$$

and

$$d_n = \frac{1}{b_0} \sum_{k=1}^{\infty} (-b_k) d_{n-k}.$$

Since $a_k \geq b_k$, $-a_k \leq -b_l$ and $1/a_0 \leq 1/b_0$. By hypothesis $c_{n-k} \leq d_{n-k}$ for $k = 1, \ldots, n$. It follows that $c_n \leq d_n$. $\square$

2.1 Algebra Retracts

Definition 2.1.1. *Let A and T be commutative rings and $p : A \to T$ a ring homomorphism. The ring T is an algebra retract of A if there is a ring homomorphism $i : T \to A$ with $pi = id_T$. The ring homomorphism i is call a section of p.*

The homomorphism i in the above definition need not be unique. We will call any homomorphism i that satisfies the condition $pi = \mathrm{id}_T$ a section of p.

The datum of an algebra retract are: The rings T and A, a surjective homomorphism $p : A \to T$, and at least one section $i : T \to A$. We write

$$T \xrightarrow{i} A \xrightarrow{p} T$$

when T is an algebra retract of A and i is a section of p.

Note that the condition that $pi = \mathrm{id}_T$ forces i to be injective and p to be surjective. We can consider A to be an T module via the map i. So $t \cdot a = i(t)a$. Note that the maps p and i are $T-$module homomorphisms:

$$p(t \cdot a) = p(i(t)a) = p(i(t))p(a) = tp(a) = t \cdot p(a)$$

and

$$i(t't) = i(t')i(t) = t' \cdot i(t).$$

Also p is an $A-$module homomorphism:

$$p(a' \cdot a) = p(a'a) = p(a')p(a) = a' \cdot p(a).$$

However i is not an $A-$module homomorphism:

$$i(a \cdot t) = i(p(a)t) = i(p(a))i(t) = i(p(a)) \cdot i(t).$$

In general $i(p(a))i(t) \neq ai(t)$. For example if $t = 1_T$ and $a \in \ker p$ non-zero. Then $i(1_T) = 1_A$ since i is a ring homomorphism and $i(p(a)) = 0_A$. Then $i(p(a))i(1_T) = 0$ but $ai(1_T) = a \neq 0_A$. There is the exact sequence of T or $A-$modules:

$$0 \longleftarrow T \xleftarrow[p]{} A \longleftarrow \ker \pi_T \longleftarrow 0.$$

So $T \cong A/\ker \pi_T$ as T or $A-$modules.

2.2 Fiber Products

Definition 2.2.1. *Suppose R, S and T are rings and $p_S : S \to R$ and $p_T : T \to R$ are ring homomorphisms. The fiber product over R of S and T, denoted $S \times_R T$, is the ring*

$$S \times_R T = \{(s,t) \in S \times T : p_S(s) = p_T(t) \in R\}.$$

There is a ring homomorphism $\pi_S : S \times_R T \to S$ given by $\pi_S((s,t)) = s$ and a ring homomorphism $\pi_T : S \times_R T \to R$ given by $\pi_T((s,t)) = t$. We will call the maps π_S and π_T the projection homomorphisms.

Note that the diagram

$$\begin{tikzcd} S \times_R T \arrow[r, "\pi_T"] \arrow[d, "\pi_S"] & T \arrow[d, "p_T"] \\ S \arrow[r, "p_S"] & R \end{tikzcd} \tag{2.1}$$

commutes.

Example 2.2.1. *Let* $(R, \mathfrak{m}, \boldsymbol{k})$ *be a local ring. Define* $S = R[x_1, \ldots, x_m]$ *and* $T = R[y_1, \ldots, y_n]$*. Let* $I \subseteq (x_1, \ldots, x_m)$ *be an ideal of* S *and* $J \subseteq (y_1, \ldots, y_n)$ *be an ideal of* T*. Then*

$$\frac{S}{I} \times_R \frac{T}{J} \cong \frac{R[x_1, \ldots, x_m, y_1, \ldots, y_n]}{I + J + (x_1, \ldots, x_m)(y_1, \ldots, y_n)}$$

We record a few important details about $S \times_R T, S, T$ and R.

Observation 2.2.2. *The kernel of* π_T *is* $\{(s, 0) : s \in \ker p_S\}$ *and the kernel of* π_S *is* $\{(0, t) : t \in \ker p_T\}$.

Proof. Clearly if $(s, t) \in \{(s, 0) : s \in \ker p_S\}$ then $(s, t) \in \ker \pi_T$.

On the other hand suppose that $(s, t) \in \ker \pi_T$. Then $\pi_T((s, t)) = t = 0$. So $(s, t) = (s, 0)$. Also, since $(s, t) \in \{(s, 0) : s \in \ker p_S\}$. So $\ker \pi_T = \{(s, 0) : s \in \ker p_S\}$.

Similarly $\ker \pi_S = \{(0, t) : t \in \ker p_T\}$ □

Observation 2.2.3. *The ideals* $\ker \pi_T$ *and* $\ker \pi_S$ *of* A *satisfy* $(\ker \pi_T)(\ker \pi_S) = \ker \pi_T \cap \ker \pi_S = 0$.

Proof. Since $(\ker \pi_T)(\ker \pi_S) \subseteq \ker \pi_T \cap \ker \pi_S$ it suffices to show that $\ker \pi_T \cap \ker \pi_S = 0$. Let $(s, t) \in \ker \pi_T \cap \ker \pi_S$. Then $(s, t) \in \ker \pi_S$. So, by (2.2.2), $(s, t) = (0, t)$ for $t \in \ker p_T$. Similarly, since $(s, t) \in \ker \pi_T$, $(s, t) = (s, 0)$ for $s \in \ker p_S$. It follows that $(s, t) = (0, 0)$. □

Observation 2.2.4. *If* p_S *is surjective, then so is* π_T.

Proof. Let $t \in T$. Since p_S is surjective, there exists an $s \in S$ with $p_S(s) = p_T(t)$. So $(s, t) \in S \times_R T$ and $\pi_T((s, t)) = t$. □

Similarly, if p_T is surjective then so is π_S.

From this point forward we will assume that both p_S and p_T are surjective.

Observation 2.2.5. *If S, T, and R are local rings with maximal ideals $\mathfrak{m}_S, \mathfrak{m}_T$, and $\mathfrak{m}_R$ respectively, then $S \times_R T$ is local with maximal ideal*

$$\mathfrak{m}_{S\times_R T} = \{(s,t) \in \mathfrak{m}_S \times \mathfrak{m}_T : p_S(s) = p_T(t) \in R\}.$$

Proof. First note that

$$\mathfrak{m}_{S\times_R T} = \pi_S^{-1}(\mathfrak{m}_S) = \pi_T^{-1}(\mathfrak{m}_T).$$

Since $\mathfrak{m}_S$ is a maximal ideal, $\mathfrak{m}_{S\times_R T}$ is also.

Now we will show that $\mathfrak{m}_{S\times_R T}$ is the unique maximal ideal. Suppose that $(s,t) \in S \times_R T$ and $(s,t) \notin \mathfrak{m}_{S\times_R T}$. We will show that (s,t) is a unit. Since $(s,t) \notin \mathfrak{m}_{S\times_R T}$, either $s \notin \mathfrak{m}_S$ or $t \notin \mathfrak{m}_T$. Assume with out loss of generality that $s \notin \mathfrak{m}_S$. Then $p_S(s) \notin \mathfrak{m}_R$. Since $p_T(t) = p_S(s)$, $t \notin \mathfrak{m}_T$. Since both S and T are local rings s and t are invertible. It follows that (s,t) is invertible.

Thus $S \times_R T$ is a local ring with $\mathfrak{m}_{S\times_R T}$ its maximal ideal. □

Definition 2.2.6. *A fiber product of the form*

$$A \times_R R$$

is called a trivial fiber product.

Note that the trivial fiber product $A \times_R R$ is isomorphic to A.

Observation 2.2.7. *Suppose that $A = S \times_R T$ is a fiber product and either of the projection maps is an isomorphism. Then the fiber product is trivial.*

Proof. Without loss of generality we assume that $\pi_S : A \to S$ is an isomorphism. Then, by (2.2.2), $0 = \ker \pi_S = \{(0,t) : t \in \ker p_T\}$. So $\ker p_T = 0$. Since p_T is surjective by assumption, $T \cong R$. Then

$$A = S \times_R T \cong A \times_T T$$

a trivial fiber product. □

The next theorem is useful for determining which rings can be realized as a non-trivial fiber product.

Theorem 2.2.8. *Let A be a local ring. One can realize A as a fiber product if and only if there exists two non-zero ideals I and J of A with $I \cap J = 0$. In this case*

$$A \cong \frac{A}{I} \times_{\frac{A}{I+J}} \frac{A}{J}.$$

Proof. Suppose that A is a non-trivial fiber product fiber product. So

$$A \cong S \times_R T$$

and there are ring homomorphisms $\pi_S : A \to S$ and $\pi_T : A \to T$. Then $\ker \pi_S$ and $\ker \pi_T$ are ideals of A. Neither of these ideals is zero, since otherwise $S \times_R T$ would be a trivial fiber product by (2.2.7). By (2.2.3), $\ker \pi_S \cap \ker \pi_T = 0$.

On the other hand, suppose that I and J are two non-zero ideals with $I \cap J = 0$. Let

$$S = \frac{A}{I}, \quad T = \frac{A}{J}, \quad \text{and} \quad R = \frac{A}{I+J}.$$

We will show that $A \cong S \times_R T$. For $a \in A$, let $\bar{a}$, $\hat{a}$, and $\tilde{a}$ denote the image of a in S, T, and R, respectively. Then we define the map $\theta : A \to S \times_R T$ by

$$\theta(a) = (\bar{a}, \hat{a}).$$

This map is well-defined since the projection from A to R can be factored through the projection to S or T.

If $a \in \ker \theta$, then $\bar{a} = 0$. So $a \in I$. Similarly, $a \in J$. Then $a \in I \cap J = 0$. So θ is injective.

To see that θ is surjective let $(\bar{a_1}, \bar{a_2}) \in S \times_R T$. Then $\tilde{a_1} = \tilde{a_2}$ in R. So $a_1 - a_2 \in I + J$. Let $a_1 - a_2 = i + j$ for $i \in I$ and $j \in J$. Let $a = a_1 - i = a_2 + j$. Then

$$\theta(a) = (\overline{a_1 - i}, \widehat{a_2 + j}) = (\bar{a}_1, \hat{a}_2).$$

Thus θ is surjective and $A \cong S \times_R T$. □

2.2.9 (Fiber product of modules). *Let M be an S-module, N a T-module, and P an R-module. We can consider P as an S or T-module via p_S or p_T, respectively. Suppose that $\mu : M \to P$ and $\nu : N \to P$ are surjective S and T-module homomorphisms, respectively. Then we define the $A-$module*

$$M \times_P N = \{(m, n) \in M \times N : \mu(m) = \nu(n) \in P\}.$$

The A action on $M \times_P N$ is given by

$$a \cdot (m, n) = (\pi_S(a) \cdot m, \pi_T(a) \cdot n).$$

The above is an element of $M \times_P N$ since

$$\mu(\pi_S(a) \cdot m) = p_S(\pi_S(a)) \cdot \mu(m).$$

and

$$\nu(\pi_T(a) \cdot n) = p_T(\pi_T(a)) \cdot \nu(n).$$

Further $p_T(\pi_T(a)) = p_S(\pi_S(a))$, since diagram (2.1) commutes, and $\mu(m) = \nu(n)$, since $(m, n) \in M \times_P N$, so that

$$\mu(\pi_S(a) \cdot m) = \nu(\pi_T(a) \cdot n).$$

2.2.10 (Fiber Product of Retracts). *Suppose that $R \xrightarrow{i_T} T \xrightarrow{p_T} R$ and $R \xrightarrow{i_S} S \xrightarrow{p_S} R$ are algebra retracts.*

Then $S \times_R T$ is a T-algebra via the map $\mu_T : T \to S \times_R T$ given by $\mu_T(t) = (i_S(p_T(t)), t)$. This is a well-defined map to $S \times_R T$ since $p_S \circ i_S = id_R$ so that $p_S(i_S(p_T(t))) = p_T(t)$. Further, T is an algebra retract of $S \times_R T$ and $\pi_T \mu_T = id_T$ where $\pi_T : S \times_R T \to T$ is the natural projection to T.

Similarly $S \times_R T$ is a S-algebra and S is an algebra retract of $S \times_R T$.

2.3 Poincaré Series and Golod Modules.

Definition 2.3.1. *If* $(R, \mathfrak{m}, \boldsymbol{k})$ *is a local ring and* M *is an* R*-module, then the Poincaré series of* M *is*

$$P_M^R(t) = \sum_{n=0}^{\infty} \dim_{\boldsymbol{k}} \operatorname{Tor}_n^R(M, \boldsymbol{k}) t^n.$$

Proposition 2.3.2. *Let* $\varphi : T \to R$ *be a surjective ring homomorphism and* M *an* R*-module. Then*

$$P_M^R(t) P_{\boldsymbol{k}}^T(t) \preceq P_{\boldsymbol{k}}^R(t) P_M^T(t).$$

See [2, 3.3.3] for proof.

Observation 2.3.3. *Suppose that* $(R, \mathfrak{m}, \boldsymbol{k})$ *is a local ring and*

$$0 \longleftarrow M' \longleftarrow M \longleftarrow M'' \longleftarrow 0$$

is a split exact sequence. Then

$$P_M^R(t) = P_{M'}^R(t) + P_{M''}^R(t).$$

Proof. Since the sequence is split exact $M \cong M' \bigoplus M''$. Then claim follows from the definition of the Poincaré series and the fact that Tor commutes with direct sums. □

If $(S, \mathfrak{m}_S, \boldsymbol{k})$ is a regular local ring and $(R, \mathfrak{m}_R, \boldsymbol{k})$ is a local ring with $\varphi : S \to R$ a surjective ring homomorphisms, then Serre showed that

$$P_{\boldsymbol{k}}^R(t) \preceq \frac{P_{\boldsymbol{k}}^S(t)}{1 - t\left(P_R^S(t) - 1\right)}. \tag{2.2}$$

Definition 2.3.4. *The ring* R *is Golod if equality holds in (2.2).*

Golod rings have maximal growth of Betti numbers and their Poincaré series is a rational function. Golod characterized rings that meet this upper bound in [4] by introducing higher homology operations.

Definition 2.3.5 (Trivial Massey Operation)**.** *Let A be a differentially graded algebra with $H_0(A) \cong \boldsymbol{k}$. We say that A admits a trivial Massey operation if for some $\boldsymbol{k}$-basis $\mathbf{b} = \{h_\lambda\}_{\lambda\in\Lambda}$ of $H_{\geq 1}(A)$ there exists a function $\mu : \sqcup_{i=1}^{\infty}\mathbf{b}^i \to A$ such that*

$$\mu\left(h_\lambda\right) = z_\lambda \in Z(A) \quad \textit{with} \quad \textit{cls}(z_\lambda) = h_\lambda$$

and

$$\partial\mu\left(h_{\lambda_1}, \ldots, h_{\lambda_p}\right) = \sum_{j=1}^{p-1} \overline{\mu\left(h_{\lambda_1}, \ldots, h_{\lambda_j}\right)}\mu\left(h_{\lambda_{j+1}}, \ldots, h_{\lambda_p}\right)$$

where $\bar{a} = (-1)^{\deg a+1}a$ for $a \in A$.

Golod showed that if the Koszul complex of a local ring $(R, \mathfrak{m}_R, \boldsymbol{k})$ admits a trivial Massey operation then the ring R is Golod.

The upper bound given in (2.2) still holds when the condition that S be regular is removed. Levin generalized the Golod definition in [11]. In this paper Levin noted that the Golod property really depends on the homomorphisms between S and R, not on S and R themselves.

Definition 2.3.6 (Golod Homomorphisms)**.** *Suppose that $\varphi : S \to R$ is a surjective ring homomorphism of local rings. The homomorphism φ is called Golod it*

$$P_{\boldsymbol{k}}^R(t) = \frac{P_{\boldsymbol{k}}^S(t)}{1 - t(P_R^S(t) - 1)}.$$

This property can be further generalized by replacing the residue field with any finitely generated $R-$module.

Definition 2.3.7 (φ-Golod Modules)**.** *Suppose that $\varphi : S \to R$ is a surjective ring homomorphism of local rings. If M is an R-module, then it can be considered as a S-module via φ. M is called φ-Golod R-module when*

$$P_M^R(t) = \frac{P_M^S(t)}{1 - t(P_R^S(t) - 1)}.$$

Proposition 2.3.8. *Let $\varphi : T \to R$ be a surjective ring homomorphism and M an R-module. If M is φ-Golod, then φ is a Golod homomorphism.*

Proof. M is φ-Golod. So by definition the equality

$$P_M^R(t) = \frac{P_M^T(t)}{1 - t(P_R^T(t) - 1)}$$

holds. Also

$$P_M^R(t) P_{\boldsymbol{k}}^T(t) \preceq P_{\boldsymbol{k}}^R(t) P_M^T(t)$$

by (2.3.2). Combining the above and the upper-bound for $P_{\boldsymbol{k}}^R(t)$ given by Serre we get

$$\frac{P_M^T(t)}{1 - t(P_R^T(t) - 1)} P_{\boldsymbol{k}}^T(t) = P_M^R(t) P_{\boldsymbol{k}}^T(t) \preceq P_{\boldsymbol{k}}^R(t) P_M^T(t) \preceq P_M^T(t) \frac{P_{\boldsymbol{k}}^T(t)}{1 - t(P_R^T(t) - 1)}.$$

Since the outside terms are equal, it follows that

$$P_{\boldsymbol{k}}^R(t) P_M^T(t) = P_M^T(t) \frac{P_{\boldsymbol{k}}^T(t)}{1 - t(P_R^T(t) - 1)}.$$

Dividing both sides by $P_M^T(t)$ shows that φ is a Golod homomorphism. □

Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a local ring and M an R-module. Let $\hat{R}$ denote the completion of R. Since $\hat{R}$ is a flat R-module

$$P_M^R(t) = P_M^{\hat{R}}(t).$$

By the Cohen Structure Theorem there is a regular local ring T and an ideal $I \subset T$ with $\hat{R} \cong T/I$. We call the ring R Golod if π is Golod, where π is the quotient map from T to $\hat{R}$. An R-module M is called Golod if M is π-Golod.

Example 2.3.1. *The following are all examples of Golod homomorphisms.*

1. *If $(R, \mathfrak{m}, \boldsymbol{k})$ is a regular local ring and $f \in \mathfrak{m}^2$ is a regular element, then the natural quotient map*

$$\pi : R \to \frac{R}{(f)}$$

is a Golod homomorphism. See [2, 5.1].

2. *If $(R, \mathfrak{m}, \boldsymbol{k})$ is a regular local ring and I an ideal, then it is shown in [8] that the natural quotient map*

$$\pi : R \to \frac{R}{I^s}$$

is a Golod homomorphisms for $s \gg 0$.

3. *The previous example can be improved when R is a polynomial ring over a field and I is a monomial ideal. In this case π is a Golod homomorphisms for $s \geq 2$ [9].*

4. *Let $(S, \mathfrak{m}_S, \boldsymbol{k}), (S', \mathfrak{m}_{S'}, \boldsymbol{k}), (T, \mathfrak{m}_T, \boldsymbol{k})$, and $(T', \mathfrak{m}_{T'}, \boldsymbol{k})$ be local rings. If $\varphi : S \to S'$ and $\psi : T \to T'$ are surjective homomorphisms, then there is a surjective homomorphism*

$$\theta : S \times_{\boldsymbol{k}} T \to S' \times_{\boldsymbol{k}} T'$$

induced by the universal property for fiber products. If φ and ψ are Golod homomorphisms, then θ is as well. See [13, 4.2.2]

The following example illustrates an example of a non-Golod homomorphism.

Example 2.3.2. *Let $(R, \mathfrak{m}, \boldsymbol{k})$ be a regular local ring of dimension n, $f_1, \ldots, f_c$ a regular sequence in R and*

$$S = \frac{R}{(f_1, \ldots, f_c)}.$$

If $c > 1$, then the natural quotient map $\pi : R \to S$ is not a Golod homomorphism

Proof. The Poincaré series of S is

$$P_{\boldsymbol{k}}^S(t) = \frac{(1+t)^n}{(1-t^2)^c}.$$

The Poincaré series for S is given in [16]. On the other hand the upper bound given by Serre is

$$\frac{P_{\boldsymbol{k}}^R(t)}{1 - t\,(P_S^R(t) - 1)} = \frac{(1+t)^n}{1 - t\,((1+t)^c - 1)}.$$

Since $c > 1$ the denominators are different. So π is not a Golod homomorphism. $\square$

The last example leads to the following observation about Golod homomorphisms.

Observation 2.3.9. *The composition of Golod homomorphisms is not necessarily Golod.*

Proof. Let $\boldsymbol{k}$ be a field and $R = \boldsymbol{k}[x, y]$. Then

$$\pi_1 : R \to \frac{R}{(x^2)} \quad \text{and} \quad \pi_2 : \frac{R}{(x^2)} \to \frac{R}{(x^2, y^2)}$$

are Golod homomorphisms by (2.3.1). However their composition

$$\pi_2 \circ \pi_1 : R \to \frac{R}{(x^2, y^2)}$$

is not a Golod homomorphisms by (2.3.2). □

2.4 Large Homomorphisms

Levin introduced the idea of large homomorphisms in [12].

Definition 2.4.1. *Let $(R, \mathfrak{m}_R, \boldsymbol{k})$ and $(S, \mathfrak{m}_S, \boldsymbol{k})$ be local rings. The surjective ring homomorphism $\varphi : R \to S$ is large if the induced map $\varphi_* : \mathrm{Tor}^R(\boldsymbol{k}, \boldsymbol{k}) \to \mathrm{Tor}^S(\boldsymbol{k}, \boldsymbol{k})$ is surjective.*

The following theorem gives several equivalent conditions for a surjective ring homomorphism to be large. It was proven by Levin in [12].

Theorem ([12, 1.1])**.** *Let $(R, \mathfrak{m}_R, \boldsymbol{k})$ and $(S, \mathfrak{m}_S, \boldsymbol{k})$ be local rings and $\varphi : R \to S$ a local surjective ring homomorphism. Then the following are equivalent*

1. *The homomorphism φ is large.*

2. *For any finitely generated S-module M, considered as an R-module via φ,*

$$P_M^R(t) = P_M^S(t) P_S^R(t).$$

3. *The homomorphism* $\varphi_* : \operatorname{Tor}^R(S, \boldsymbol{k}) \to \operatorname{Tor}^R(\boldsymbol{k}, \boldsymbol{k})$ *induced by the canonical projection* $p : S \to \boldsymbol{k}$ *is injective.*

4. *For any finitely generated* S*-module* M*, regarded as an* R*-module via* φ*, the induced homomorphisms* $\operatorname{Tor}^R(M, \boldsymbol{k}) \to \operatorname{Tor}^S(M, \boldsymbol{k})$ *are surjective.*

5. *There is an exact sequence of algebras*

$$\boldsymbol{k} \longleftarrow \operatorname{Tor}^S(\boldsymbol{k}, \boldsymbol{k}) \longleftarrow \operatorname{Tor}^R(\boldsymbol{k}, \boldsymbol{k}) \longleftarrow \operatorname{Tor}^R(S, \boldsymbol{k}) \longleftarrow \boldsymbol{k}.$$

Example 2.4.1. *The following are large homomorphisms*

1. *Let* $(R, \mathfrak{m}, \boldsymbol{k})$ *be a local ring. The natural projection* $R \to \boldsymbol{k}$ *is a large homomorphism.*

2. *Let* $(R, \mathfrak{m}, \boldsymbol{k})$ *be a local ring. If* $x \in \mathfrak{m}/\mathfrak{m}^2$*, then* $R \to R/(x)$ *is large when*

   a) x *is a non zero-divisor,*

   b) $x \in (0 : \mathfrak{m})$.

3. *Let* $(S, \mathfrak{m}_S, \boldsymbol{k})$ *and* $(T, \mathfrak{m}_T, \boldsymbol{k})$ *be local rings. The projections* $S \times_{\boldsymbol{k}} T \to S$ *and* $S \times_{\boldsymbol{k}} \to T$ *are large homomorphisms [3].*

The following was proven by Herzog in [6, Theorem 1].

Proposition 2.4.2. *Suppose that* $T \xrightarrow{i} A \xrightarrow{p} T$ *is an algebra retract. The homomorphism* p *is large.*

This proposition will play an important role in chapter 5 where we consider the Poincaré series of modules over the fiber product $S \times_R T$ and R is an algebra retract of both S and T.

2.5 Module Homomorphisms

Here we record a few results about module homomorphisms.

Theorem 2.5.1. *Let R be a commutative Noetherian ring and M be a finitely generated $R-$module. If $f : M \to M$ is a surjective $R-$module homomorphism, then it is an isomorphism.*

Proof. Since R is Noetherian and M is finitely generated, M is a Noetherian module. Then we have the ascending chain of submodules of M

$$\ker f \subseteq \ker f^2 \subseteq \ker f^3 \subseteq \dots$$

which must stabilize. So there is some n with $\ker f^n = \ker f^N$ for $n \leq N$. Now let, $x \in \ker f$. Since f is surjective and the composition of surjective maps is surjective, f^n is surjective. So there exists some y with $x = f^n(y)$. Then $0 = f(x) = f^{n+1}(y)$. So $y \in \ker f^{n+1} = \ker f^n$. Then $x = f^n(y) = 0$. This shows that f is injective. Since f is surjective by assumption, f is an isomorphism. □

Corollary 2.5.2. *Let R be a commutative Noetherian ring and M and N be a finitely generated $R-$modules. If $f : M \to N$ and $g : N \to M$ are surjective $R-$module homomorphisms, then both are isomorphisms.*

Proof. The maps $gf : M \to M$ is a surjection. So, by (2.5.1) gf is an isomorphism. If $x \in \ker f$, then $gf(x) = g(f(x)) = 0$. So $x \in \ker gf = 0$, since gf is an isomorphism. Then $\ker f = 0$, showing that f is injective. So f is an isomorphism. Similarly, g is an isomorphism. □

Chapter 3

An Answer to a Question of Gupta

Here we let k be a field of any characteristic and $R = k[x_1, \dots, x_m, y_1, \dots, y_n]$.

In [5] the author poses the following question:

Let

$$I(s,t,u,v) = (x_1, \dots, x_m)^s + (x_1, \dots, x_m)^u (y_1, \dots, y_n)^v + (y_1, \dots, y_n)^t \subset R.$$

[5, Remark 5.2] For which values of $(s,t,u,v) \in \mathbb{N}^4$ is $R/I(s,t,u,v)$ Golod?

In [5] the author showed that $R/I(s,t,u,v)$ is Golod when $t = 2, v = 1$ and $u < s$.

Theorem 3.0.1. *The ring $R/I(s,t,u,v)$ is Golod if and only if $s = 1, t = 1$ or $1 \leq u < s$ and $1 \leq v < t$.*

The proof of (3.0.1) uses results established in [9]. The following definitions will be needed.

The authors in [9] define for the polynomial ring $k[x_1, \dots, x_n]$

$$d^r(f) = \frac{f(0, \dots, 0, x_r, \dots, x_n) - f(0, \dots, 0, x_{r+1}, \dots, x_n)}{x_r}.$$

Note that $d^r(f+g) = d^r(f) + d^r(g)$ for $f, g \in k[x_1, \dots, x_n]$. Also note that for a monomial u we have

$$d^r(u) = \begin{cases} \frac{u}{x_r}, & \text{if } r \text{ is the smallest integer such that } x_r \text{ divides } u \text{, and} \\ 0, & \text{otherwise.} \end{cases}$$

If $I = (f_1, \dots f_m) \subset k[x_1, \dots, x_n]$ is an ideal, then $d(I)$ is the ideal generated by the elements $d^r(f_i)$ for $r = 1, \dots, n$ and $i = 1, \dots, m$. The operator d^r depends on the ordering of the variables. If σ is a permutation, then define

$$d_\sigma^r(f) = \sigma(d^r(f(x_{\sigma^{-1}(1)}, \dots x_{\sigma^{-1}(n)}))).$$

If $I = (f_1, \dots f_m) \subset k[x_1, \dots, x_n]$ is an ideal, then $d_\sigma(I)$ is the ideal generated by the elements $d_\sigma^r(f_i)$ for $r = 1, \dots, n$ and $i = 1, \dots, m$.

We make use of the following theorem:

Theorem ([9, 2.3])**.** *Let $I \subset k[x_1, \dots, x_n]$ be a proper ideal with $(d_\sigma(I))^2 \subset I$ for some permutation σ. Then $k[x_1, \dots, x_n]/I$ is a Golod ring.*

Proof of 3.0.1. First consider the case that $1 < s, t$ and $s \leq u$ or $t \leq v$ then

$$I(s,t,u,v) = (x_1, \dots, x_m)^s + (y_1, \dots, y_n)^t.$$

The ring $k[x_1, \dots, x_m, y_1, \dots, y_n]/I(s,t,u,v)$ is a retract of $k[x_1, y_1]/(x_1^s, y_1^t)$. The later ring is not Golod (see (2.3.2)). Then $I(s,t,u,v)$ is not Golod by [5, 4.2].

Henceforth assume that $u < s$ and $v < t$.

In the case that $s = 1$

$$\frac{R}{I(s,t,u,v)} \cong \begin{cases} \frac{k[y_1,\dots,y_m]}{(y_1,\dots,y_m)^t} & t > 1 \\ \boldsymbol{k} & t = 1 \end{cases}$$

A field is a regular ring and the identity map is obviously Golod. Then ring

$$\frac{k[y_1, \dots, y_m]}{(y_1, \dots, y_m)^t}$$

is Golod by [9, 3.1].

The case that $t = 1$ is similar.

Now consider the case that $1 \leq u < s$ and $1 \leq v < t$. Order the variables by $x_1, \dots, x_m, y_1, \dots y_n$. Then the following hold:

1. $$d((x_1, \dots, x_m)^s) = (x_1, \dots, x_m)^{s-1},$$

2.

$$d((y_1,\dots,y_n)^t) = (y_1,\dots,y_n)^{t-1}, \text{ and}$$

3.

$$d((x_1,\dots,x_m)^u(y_1,\dots,y_n)^v) = (x_1,\dots,x_m)^{u-1}(y_1,\dots,y_n)^v.$$

Proof of (1). If $x_{i_1}^{e_1}\dots x_{i_k}^{e_k}$ is a generator of $(x_1,\dots,x_m)^s$ with $i_1 < i_2 < \dots < i_k$ and e_1 positive, then

$$d^j\left(x_{i_1}^{e_1}\dots x_{i_k}^{e_k}\right) = \begin{cases} x_{i_1}^{e_1-1}\dots x_{i_k}^{e_k}, & \text{if } j = i_1, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

In either case $d^j\left(x_{i_1}^{e_1}\dots x_{i_k}^{e_k}\right) \in (x_1,\dots,x_m)^{s-1}$.

On the other hand if $x_{i_1}^{e_1}\dots x_{i_k}^{e_k}$ is a generator of $(x_1,\dots,x_m)^{s-1}$ with $i_1 < \dots < i_k$ then

$$d^{i_1}\left(x_{i_1}^{e_1+1}\dots x_{i_k}^{e_k}\right) = x_{i_1}^{e_1}\dots x_{i_k}^{e_k}.$$

Proof of (2). The equality is established in a similar way as (1).

Proof of (3). If $x_{i_1}^{e_1}\dots x_{i_k}^{e_k}y_{j_1}^{f_1}\dots y_{j_\ell}^{f_\ell}$ is a generator of $(x_1,\dots,x_m)^u(y_1,\dots,y_n)^v$ with $i_1 < \dots < i_k$, $j_1 < \dots < j_\ell$ and e_1 positive, then

$$d^j\left(x_{i_1}^{e_1}\dots x_{i_k}^{e_k}y_{j_1}^{f_1}\dots y_{j_\ell}^{f_\ell}\right) = \begin{cases} x_{i_1}^{e_1-1}\dots x_{i_k}^{e_k}y_{j_1}^{f_1}\dots y_{j_\ell}^{f_\ell}, & \text{if } j = i_1, \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

In either case $d^j\left(x_{i_1}^{e_1}\dots x_{i_k}^{e_k}y_{j_1}^{f_1}\dots y_{j_\ell}^{f_\ell}\right) \in (x_1,\dots,x_m)^{u-1}(y_1,\dots,y_n)^v$.

On the other hand if $x_{i_1}^{e_1}\dots x_{i_k}^{e_k}y_{j_1}^{f_1}\dots y_{j_\ell}^{f_\ell}$ is a generator of $(x_1,\dots,x_m)^{u-1}(y_1,\dots y_n)^v$ with $i_1 < \dots < i_k$ and $j_1 < \dots < j_\ell$ then

$$d^{i_1}\left(x_{i_1}^{e_1+1}\dots x_{i_k}^{e_k}y_{j_1}^{f_1}\dots y_{j_\ell}^{f_\ell}\right) = x_{i_1}^{e_1}\dots x_{i_k}^{e_k}y_{j_1}^{f_1}\dots y_{j_\ell}^{f_\ell}.$$

The three assertions have been established; it follows that

$$d\left(I(s,t,u,v)\right) = (x_1,\ldots,x_m)^{s-1} + (x_1,\ldots,x_m)^{u-1}(y_1,\ldots,y_n)^{v} + (y_1,\ldots,y_n)^{t-1}$$

and

$$\begin{aligned}\left(d\left(I(s,t,u,v)\right)\right)^2 = {} & (x_1,\ldots,x_m)^{2s-2} + (x_1,\ldots,x_m)^{s+u-2}(y_1,\ldots,y_n)^{v} \\ & + (x_1,\ldots,x_m)^{s-1}(y_1,\ldots,y_n)^{t-1} + (x_1,\ldots,x_m)^{2u-2}(y_1,\ldots,y_n)^{2v} \\ & + (x_1,\ldots,x_m)^{u-1}(y_1,\ldots,y_n)^{t+v-1} + (y_1,\ldots,y_n)^{2t-2}.\end{aligned}$$

So $d\left(I(s,t,u,v)\right)^2 \subset I$ when $1 < u$. Thus, by [9, 2.3], $I(s,t,u,v)$ is Golod.

If $u = 1$ and $1 < v$, then $d_\sigma\left(I(s,t,u,v)\right)^2 \subset I(s,t,u,v)$ where σ is the permutation that gives the variables the order $y_1,\ldots,y_n,x_1,\ldots,x_m$. Thus, by [9, 2.3], $R/I(s,t,u,v)$ is Golod.

Finally when $u = v = 1$, then we note that the quotient ring is isomorphic to the fiber product over k of Golod Rings.

$$\frac{k[x_1,\ldots,x_m,y_1,\ldots,y_n]}{I(s,t,1,1)} \cong \frac{k[x_1,\ldots,x_m]}{(x_1,\ldots,x_m)^s} \times_k \frac{k[y_1,\ldots,y_n]}{(y_1,\ldots,y_n)^t}.$$

Then $I(s,t,1,1)$ is Golod by [10, 4.1].

□

Chapter 4

Resolutions over Fiber Products

4.1 Construction of Resolution

4.1.1. *Let $(R, \mathfrak{m}_R, \boldsymbol{k}), (S, \mathfrak{m}_S, \boldsymbol{k})$ and $(T, \mathfrak{m}_T, \boldsymbol{k})$ be local commutative rings with surjective ring homomorphisms $p_S : S \to R$ and $p_T : T \to R$. Define $A = S \times_R T$ and let $\pi_S : A \to S$ and $\pi_T : A \to T$ be the projection homomorphisms. Suppose further that $R \xrightarrow{i_S} S \xrightarrow{p_S} R$ and $R \xrightarrow{i_T} T \xrightarrow{p_T} R$ are algebra retracts. Then, by (2.2.10), $S \xrightarrow{\mu_S} A \xrightarrow{\pi_S} S$ and $T \xrightarrow{\mu_T} A \xrightarrow{\pi_T} T$ are algebra retracts. The diagram*

$$\begin{tikzcd}
A \arrow[d, shift right, "\pi_S"'] \arrow[r, shift right, "\pi_T"'] & T \arrow[l, shift right, "\mu_T"'] \arrow[d, shift right, "p_T"'] \\
S \arrow[u, shift right, "\mu_S"'] \arrow[r, shift right, "p_S"'] & R \arrow[l, shift right, "i_S"'] \arrow[u, shift right, "i_T"']
\end{tikzcd} \tag{4.1}$$

commutes.

Let M be an S-module. We consider M to be an A-module via $\pi_S : A \to S$.

Our goal is to resolve M as an A-module.

Let $\mathcal{B}'$ be a minimal resolution of R as a T-module. Let p_T be the augmentation from $\mathcal{B}'$ to R. Also let $\mathcal{C}'$ and $\mathcal{D}'$ be minimal resolutions of R and M as S-modules, respectively. Let p_S be the augmentation from $\mathcal{C}'$ to R. Let $\mathcal{B} = \mathcal{B}' \otimes_T A$, $\mathcal{C} = \mathcal{C}' \otimes_S A$ and $\mathcal{D} = \mathcal{D}' \otimes_A S$. Since $\_\_ \otimes_T A$ and $\_\_ \otimes_S A$ are functors, $\mathcal{B}, \mathcal{C}$ and $\mathcal{D}$ are complexes of free A-modules.

Since $\mathcal{B}'$ is minimal, $\operatorname{Im} \partial_d^{\mathcal{B}'} \subset \mathfrak{m}_T B'_{d-1}$. It follows that $\operatorname{Im} \partial_d^{\mathcal{B}} \subset \mathfrak{m}_A B_{d-1}$. Similarly, $\operatorname{Im} \partial_d^{\mathcal{C}} \subset \mathfrak{m}_A C_{d-1}$ and $\operatorname{Im} \partial_d^{\mathcal{D}} \subset \mathfrak{m}_A D_{d-1}$.

Observation 4.1.2. *There are isomorphisms of A-modules:*

1. $\operatorname{Im} \partial_1^{\mathcal{B}} \cong \ker \pi_S$.

2. $\operatorname{Im} \partial_1^{\mathcal{C}} \cong \ker \pi_T$.

Proof. First note that $\operatorname{Im} \partial_1^{\mathcal{B}'} = \ker p_T$. Then

$$\operatorname{Im} \partial_1^{\mathcal{B}} = \partial_1^{\mathcal{B}}(B_1' \otimes_T A) = \partial_1^{\mathcal{B}'}(B_1') \otimes_T A = \ker p_T \otimes_T A \cong (\ker p_T)A.$$

The last isomorphisms is as A-modules given by $k \otimes a \mapsto \mu_T(k)a$.

We will show that $(\ker p_T)A = \ker \pi_S = \{(0,t) | t \in \ker p_T\}$. If $k \in \ker p_T$ and $(s,t) \in A$, then

$$k(s,t) = (\pi_S(\mu_T(k))s, \pi_T(\mu_T(k))t) = (i_S(p_T(k))s, kt) = (0, kt) \in \ker \pi_S.$$

On the other hand, if $(0,k) \in \ker \pi_S$ then $k \in \ker p_T$ and

$$(0,k) = k(1,1) \in (\ker p_T)A.$$

So $\operatorname{Im} \partial_1^{\mathcal{B}} \cong (\ker p_T)A = \ker \pi_S$. Similarly $\operatorname{Im} \partial_1^{\mathcal{C}} \cong \ker \pi_T$. $\square$

We now define the double complex $\mathcal{G}$. We will show in (4.2.1) that the total complex of $\mathcal{G}$ is a minimal resolution of M as an A-module.

Let $G_{d,1} = C_d$. For ℓ even define

$$G_{d,\ell} = \bigoplus_{d'=1}^{d-\ell+2} B_{d'} \otimes_A G_{d-d',\ell-1}$$

and for $\ell > 1$ odd define

$$G_{d,\ell} = \bigoplus_{d'=1}^{d-\ell+2} C_{d'} \otimes_A G_{d-d',\ell-1}.$$

Consider the picture

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial^v_{3,4}"] & \vdots \arrow[d, "\partial^v_{4,4}"] & \vdots \arrow[d, "\partial^v_{5,4}"] & \vdots \arrow[d, "\partial^v_{6,4}"] & \\
0 & G_{2,3} \arrow[l] \arrow[d, "\partial^v_{2,3}"] & G_{3,3} \arrow[l, "\partial^h_{3,3}"'] \arrow[d, "\partial^v_{3,3}"] & G_{4,3} \arrow[l, "\partial^h_{4,3}"'] \arrow[d, "\partial^v_{4,3}"] & G_{5,3} \arrow[l, "\partial^h_{5,3}"'] \arrow[d, "\partial^v_{5,3}"] & \dots \arrow[l] \\
\mathcal{G} :0 & G_{1,2} \arrow[l] \arrow[d, "\partial^v_{1,2}"] & G_{2,2} \arrow[l, "\partial^h_{2,2}"'] \arrow[d, "\partial^v_{2,2}"] & G_{3,2} \arrow[l, "\partial^h_{3,2}"'] \arrow[d, "\partial^v_{3,2}"] & G_{4,2} \arrow[l, "\partial^h_{4,2}"'] \arrow[d, "\partial^v_{4,2}"] & \dots \arrow[l] \\
0 & G_{0,1} \arrow[l] \arrow[d] & G_{1,1} \arrow[l, "\partial^h_{1,1}"'] \arrow[d] & G_{2,1} \arrow[l, "\partial^h_{2,1}"'] \arrow[d] & G_{3,1} \arrow[l, "\partial^h_{3,1}"'] \arrow[d] & \dots \arrow[l] \\
 & 0 & 0 & 0 & 0. &
\end{tikzcd}
\tag{4.2}
$$

We describe the horizontal and vertical maps of $\mathcal{G}$.

First define the horizontal maps. Note that $G_{d,1} = D_d$. So define

$$\partial^h_{d,1} = \partial^{\mathcal{D}}_d.$$

If ℓ is even

$$\partial^h_{d,\ell}|_{B_{d'} \otimes G_{d-d',\ell}} = \begin{cases} \partial^{\mathcal{B}}_{d'} \otimes 1 + (-1)^{d'} 1 \otimes \partial^h_{d-d',\ell-1} & \text{if } d' \geq 2 \\ (-1)^{d'} 1 \otimes \partial^h_{d-d',\ell-1} & \text{if } d' = 1. \end{cases}$$

If $\ell > 1$ is odd

$$\partial^h_{d,\ell}|_{C_{d'} \otimes G_{d-d',\ell}} = \begin{cases} \partial^{\mathcal{C}}_{d'} \otimes 1 + (-1)^{d'} 1 \otimes \partial^h_{d-d',\ell-1} & \text{if } d' \geq 2 \\ (-1)^{d'} 1 \otimes \partial^h_{d-d',\ell-1} & \text{if } d' = 1. \end{cases}$$

Now we define the vertical maps. If ℓ is even

$$\partial^v_{d,\ell}|_{B_{d'} \otimes G_{d-d',\ell}} = \begin{cases} \partial^{\mathcal{B}}_1 \cdot 1 & \text{if } d' = 1 \\ 0 & \text{if } d' > 1. \end{cases}$$

If $\ell > 1$ is odd

$$\partial^v_{d,\ell}|_{C_{d'} \otimes G_{d-d',\ell}} = \begin{cases} \partial^{\mathcal{C}}_1 \cdot 1 & \text{if } d' = 1 \\ 0 & \text{if } d' > 1. \end{cases}$$

Proposition 4.1.3. *The picture (4.2) is a double complex.*

Proof. The rows are complexes. We proceed by induction on ℓ. When $\ell = 1$ we show that

$$G_{*,1} : 0 \longleftarrow G_{0,1} \xleftarrow[\partial^h_{1,1}]{} G_{1,1} \xleftarrow[\partial^h_{2,1}]{} G_{2,1} \xleftarrow[\partial^h_{3,1}]{} G_{3,1} \xleftarrow[\partial^h_{4,1}]{} \cdots .$$

is a complex. That is

$$0 \longleftarrow D_0 \xleftarrow[\partial^{\mathcal{D}}_1]{} D_1 \xleftarrow[\partial^{\mathcal{D}}_2]{} D_2 \xleftarrow[\partial^{\mathcal{D}}_3]{} D_3 \xleftarrow[\partial^{\mathcal{D}}_4]{} \cdots .$$

which is a complex. So $G_{*,1}$ is a complex.

Assume that $G_{*,k}$ is a complex for $k < \ell$. The modules in $G_{*,\ell}$ depend on whether ℓ is even or odd. We show the case that ℓ is even. Let $b \otimes g \in G_{d,\ell}$ with $b \in \mathsf{B}_{d'}$ and $g \in G_{d-d',\ell-1}$. If $d' \geq 3$, then

$$\begin{aligned}
\partial^h_{d-1,\ell}\left(\partial^h_{d,\ell}(b \otimes g)\right) &= \partial^h_{d-1,\ell-1}\left(\partial^{\mathcal{B}}_{d'}(b) \otimes g + (-1)^{d'} b \otimes \partial^h_{d-d',\ell-1}(g)\right) \\
&= \partial^{\mathcal{B}}_{d'-1}\left(\partial^{\mathcal{B}}_{d'}(b)\right) \otimes g + (-1)^{d'-1}\partial^{\mathcal{B}}_{d'}(b) \otimes \partial^h_{d-d',\ell-1}(g) \\
&+ (-1)^{d'}\partial^{\mathcal{B}}_{d'}(b) \otimes \partial^h_{d-d',\ell-1}(g) + (-1)^{2d'-1} b \otimes \partial^h_{d-d'-1,\ell-1}\left(\partial^h_{d-d',\ell-1}(g)\right) \\
&= 0.
\end{aligned}$$

The first summand is zero because $\mathcal{B}$ is a complex. The last summand is zero because $G_{*,\ell-1}$ is a complex by the inductive hypothesis. The middle terms are the same except with opposite signs. The cases that $d' = 1, 2$ are similar. The case that ℓ is odd is similar.

The columns are complexes. We show that $\partial^v_{d+1,\ell+1} \circ \partial^v_{d,\ell} = 0$. The module $G_{d+1,\ell+1}$ depends on whether ℓ is even or odd. We show the case that ℓ is odd. Let $b \otimes g \in G_{d+1,\ell+1}$ with $b \in \mathsf{B}_{d'}$ and $g \in G_{d,\ell}$. Then

$$g = \sum_{d'=1}^{d-\ell+2} \sum_{c \in \mathsf{C}_{d'}} a_c c \otimes g_c$$

for $a_c \in A$ and $g_c \in G_{d-d',\ell-1}$. If $d' > 1$, then $\partial^v_{d+1,\ell+1}(b \otimes g) = 0$ by definition. Consider $d' = 1$. Then

$$\begin{aligned}
\partial^v_{d,\ell}\left(\partial^v_{d-1,\ell-1}(b \otimes g)\right) &= \partial^v_{d,\ell}\left(\partial^{\mathcal{B}}_1(b)g\right) \\
&= \partial^{\mathcal{B}}_1(b)\partial^v_{d,\ell}\left(\sum_{d'=1}^{d-\ell+2}\sum_{c \in \mathsf{C}_{d'}} a_c c \otimes g_c\right) \\
&= \partial^{\mathcal{B}}_1(b)\sum_{d'=1}^{d-\ell+2}\sum_{c \in \mathsf{C}_{d'}} \partial^v_{d,\ell}(a_c c \otimes g_c) \\
&= \partial^{\mathcal{B}}_1(b)\sum_{c \in \mathsf{C}_1} a_c \partial^{\mathcal{C}}_1(c) g_c \\
&\subset (\ker \pi_T)(\ker \pi_S) \\
&= 0.
\end{aligned}$$

The reduction on the fourth line occurs because of the definition of $\partial^v_{d,\ell}$. The containment on the next to last line is due (4.1.2).

The case that ℓ is even is similar.

The squares commute. We will show that the square

$$\begin{tikzcd}
G_{d,\ell+1} \arrow[d, "\partial^v_{d,\ell+1}"] & G_{d+1,\ell+1} \arrow[l, "\partial^h_{d+1,\ell+1}"'] \arrow[d, "\partial^v_{d+1,\ell+1}"] \\
G_{d-1,\ell} & G_{d,\ell} \arrow[l, "\partial^h_{d,\ell}"]
\end{tikzcd}$$

commutes. The module $G_{d+1,\ell+1}$ depends on whether ℓ is even or odd. We show the case that ℓ is odd. Let $b \otimes g \in G_{d+1,\ell+1}$ with $b \in \mathsf{B}_{d'}$ and $g \in G_{d+1-d',\ell}$. If $d' \geq 2$, then

$$\partial^h_{d,\ell}(\partial^v_{d+1,\ell+1}(b_d \otimes g)) = \partial^h_{d,\ell}(0) = 0.$$

On the other hand

$$\partial^v_{d,\ell+1}\left(\partial^h_{d+1,\ell+1}(b \otimes g)\right) = \partial^v_{d,\ell+1}\left(\partial^{\mathcal{B}}_{d'}(b) \otimes g + (-1)^{d'} b \otimes \partial^h_{d+1-d',\ell}(g)\right).$$

Note that $\partial^{\mathcal{B}}_{d'}(b) \otimes g \in B_{d'-1} \otimes G_{d+1-d',\ell}$ and $b \otimes \partial^h_{d+1-d',\ell}(g) \in B_{d'} \otimes G_{d-d',\ell}$. If $d' \geq 3$, then the vertical map sends both to zero since $d', d'-1 \geq 2$. If $d' = 2$, then

$$\partial^v_{d,\ell+1}\left(\partial^{\mathcal{B}}_{d'}(b) \otimes g + (-1)^{d'} b \otimes \partial^h_{d+1-d',\ell}(g)\right) = \partial^{\mathcal{B}}_1\left(\partial^{\mathcal{B}}_2(b)\right) g = 0$$

since $\left(\partial^{\mathcal{B}}\right)^2 = 0$.

If $d' = 1$, then

$$\partial^h_{d,\ell}(\partial^v_{d+1,\ell+1}(b_d \otimes g)) = \partial^h_{d,\ell}(\partial^{\mathcal{B}}_1(b)g) = \partial^{\mathcal{B}}_1(b)\partial^h_{d,\ell}(g).$$

On the other hand

$$\partial^v_{d,\ell+1}\left(\partial^h_{d+1,\ell+1}(b \otimes g)\right) = \partial^v_{d,\ell+1}\left(-b \otimes \partial^h_{d,\ell}(g)\right) = -\partial^{\mathcal{B}}_1(b)\partial^h_{d,\ell}(g).$$

So the squares commute. The case that ℓ is even is similar.

□

4.2 The total complex of $\mathcal{G}$ is a minimal Resolution

Theorem 4.2.1. *Given the hypothesis of (4.1.1), the total complex of the double complex $\mathcal{G}$ is a minimal resolution of M as an A-module.*

Before proving (4.2.1) we consider the complex

$$0 \longleftarrow \frac{\ker \partial^{\mathcal{B}}_1}{(\ker \pi_T)B_1} \underset{\bar{\partial}^{\mathcal{B}}_2}{\longleftarrow} \frac{B_2}{(\ker \pi_T)B_2} \underset{\bar{\partial}^{\mathcal{B}}_3}{\longleftarrow} \frac{B_3}{(\ker \pi_T)B_3} \longleftarrow \ldots . \tag{4.3}$$

First note that $\partial^{\mathcal{B}}_1((\ker \pi_T)B_1) = \ker \pi_T \partial^{\mathcal{B}}_1(B_1) = (\ker \pi_T)(\ker \pi_S) = 0$. So $(\ker \pi_T)B_1 \subset \ker \partial^{\mathcal{B}}_1$. Since $\mathcal{B}$ is a complex, $\operatorname{Im} \partial^{\mathcal{B}}_2 \subseteq \ker \partial^{\mathcal{B}}_1$. So (4.3) is induced on quotients by the complex

$$0 \longleftarrow \ker \partial^{\mathcal{B}}_1 \underset{\partial^{\mathcal{B}}_2}{\longleftarrow} B_2 \underset{\partial^{\mathcal{B}}_3}{\longleftarrow} B_3 \longleftarrow \ldots .$$

Lemma 4.2.2. *The complex (4.3) is exact.*

Proof. First note that we have $(\ker \pi_T)B_1 \subseteq \ker \partial^{\mathcal{B}}_1 \subseteq B_1$. Then we have the inclusion of modules

$$\frac{\ker \partial^{\mathcal{B}}_1}{(\ker \pi_T)B_1} \subseteq \frac{B_1}{(\ker \pi_T)B_1}.$$

Note that

$$\frac{\frac{B_1}{(\ker \pi_T)B_1}}{\frac{\ker \partial^{\mathcal{B}}_1}{(\ker \pi_T)B_1}} \cong \frac{B_1}{\ker \partial^{\mathcal{B}}_1}.$$

So we have a short exact sequence.

$$0 \longleftarrow \frac{B_1}{\ker \partial_1^{\mathcal{B}}} \xleftarrow[\pi]{} \frac{B_1}{(\ker \pi_T)B_1} \xleftarrow[i]{} \frac{\ker \partial_1^{\mathcal{B}}}{(\ker \pi_T)B_1} \longleftarrow 0.$$

Now consider the short exact sequence of complexes

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\mathcal{E} : 0 & \frac{\ker \partial_1^{\mathcal{B}}}{(\ker \pi_T)B_1} \arrow[l] \arrow[d, "i"] & \frac{B_2}{(\ker \pi_T)B_2} \arrow[l, "\bar{\partial}_2^{\mathcal{B}}"'] \arrow[d, "\mathrm{id}"] & \frac{B_3}{(\ker \pi_T)B_3} \arrow[l, "\bar{\partial}_3^{\mathcal{B}}"'] \arrow[d, "\mathrm{id}"] & \dots \arrow[l] \\
\mathcal{F} : 0 & \frac{B_1}{(\ker \pi_T)B_1} \arrow[l] \arrow[d, "\pi"] & \frac{B_2}{(\ker \pi_T)B_2} \arrow[l, "\bar{\partial}_2^{\mathcal{B}}"'] \arrow[d] & \frac{B_3}{(\ker \pi_T)B_3} \arrow[l, "\bar{\partial}_3^{\mathcal{B}}"'] \arrow[d] & \dots \arrow[l] \\
\mathcal{H} : 0 & \frac{B_1}{\ker \partial_1^{\mathcal{B}}} \arrow[l] \arrow[d] & 0 \arrow[l] & 0 \arrow[l] & \dots \arrow[l] \\
 & 0. & & &
\end{tikzcd}$$

This leads us to the long exact sequence on homology:

$$\dots \longleftarrow \mathrm{H}_{n-1}(\mathcal{E}) \longleftarrow \mathrm{H}_n(\mathcal{H}) \longleftarrow \mathrm{H}_n(\mathcal{F}) \longleftarrow$$

$$\longleftarrow \mathrm{H}_n(\mathcal{E}) \longleftarrow \mathrm{H}_{n+1}(\mathcal{H}) \longleftarrow \dots .$$

Clearly

$$\mathrm{H}_n(\mathcal{H}) = \begin{cases} \frac{B_1}{\ker \partial_1^{\mathcal{B}}} & n = 0 \\ 0 & n \neq 0 \end{cases}$$

Note that

$$\frac{B_n}{(\ker \pi_T)B_n} \cong B_n \otimes_A T \cong B'_n.$$

Then the following diagram commutes:

$$\begin{tikzcd}
\frac{B_{n-1}}{(\ker \pi_T)B_{n-1}} \arrow[d, "\cong"] & \frac{B_n}{(\ker \pi_T)B_n} \arrow[l, "\bar{\partial}_n^{\mathcal{B}}"'] \arrow[d, "\cong"] \\
B'_{n-1} & B'_n. \arrow[l, "\partial_n^{\mathcal{B}'}"']
\end{tikzcd}$$

So $\mathcal{F}$ is isomorphic to

$$0 \longleftarrow B'_1 \xleftarrow[\partial_2^{\mathcal{B}'}]{} B'_2 \xleftarrow[\partial_3^{\mathcal{B}'}]{} B'_3 \longleftarrow \cdots.$$

Then

$$\mathrm{H}_n(\mathcal{F}) = \begin{cases} \frac{B'_1}{\ker \partial_1^{\mathcal{B}'}} & n = 0 \\ 0 & n \neq 0. \end{cases}$$

If $n \neq 0$, then

$$\mathrm{H}_n(\mathcal{F}) \longleftarrow \mathrm{H}_n(\mathcal{E}) \longleftarrow \mathrm{H}_{n+1}(\mathcal{H})$$

is exact and the left and right modules are zero. So $\mathrm{H}_n(\mathcal{E}) = 0$ for $n \neq 0$. For $n = 0$, the sequence

$$0 \longleftarrow \mathrm{H}_0(\mathcal{H}) \xleftarrow[\bar{\pi}]{} \mathrm{H}_0(\mathcal{F}) \longleftarrow \mathrm{H}_0(\mathcal{E}) \longleftarrow 0$$

is exact, where $\bar{\pi}$ is the map induced on homology by π. Note that $\bar{\pi}$ is surjective.

Since B'_n is a free T-module for $n = 0, 1$, the functor $B'_n \otimes_T \_\_$ is exact. Applying this functor to the the exact sequence

$$0 \longleftarrow T \xleftarrow[\pi_T]{} A \xleftarrow[i]{} \ker \pi_T \longleftarrow 0.$$

we get

$$0 \longleftarrow B'_n \xleftarrow[\chi_{B_n}]{} B_n \xleftarrow[B'_n \otimes_T i]{} B'_n \otimes_T \ker \pi_T \longleftarrow 0$$

where $\chi_{B_n}(b \otimes a) = \pi_T(a)b$. If $k \in \ker \partial_1^{\mathcal{B}}$, then $\chi_{B_1}(k) \in \ker \partial_1^{\mathcal{B}'}$ since

$$\begin{tikzcd} B_0 \arrow[d, "\chi_{B_0}"] & B_1 \arrow[l, "\partial_1^{\mathcal{B}}"'] \arrow[d, "\chi_{B_1}"] \\ B'_0 & B'_1 \arrow[l, "\partial_1^{\mathcal{B}'}"] \end{tikzcd}$$

commutes. Let $\chi_{\ker \partial_1^{\mathcal{B}}}$ be the restriction of χ_{B_1} to $\ker \partial_1^{\mathcal{B}}$. Then

$$\begin{tikzcd} 0 & \ker \partial_1^{\mathcal{B}'} \arrow[l] \arrow[d, "i'_1"] & \ker \partial_1^{\mathcal{B}'} \arrow[l, "\chi_{\ker \partial_1^{\mathcal{B}}}"'] \arrow[d, "i_1"] & \ker \chi_{\ker \partial_1^{\mathcal{B}}} \arrow[l] \arrow[d, "a"] & 0 \arrow[l] \\ 0 & B'_1 \arrow[l] & B_1 \arrow[l, "\chi_{B_1}"] & B'_1 \otimes_T \ker \pi_T \arrow[l, "B'_n \otimes_T i"] & 0 \arrow[l] \end{tikzcd}$$

commutes and the rows are exact. The maps i'_1 and i_1 are the natural inclusion maps. So they are injective. Then, by the snake lemma, we have:

$$0 \longleftarrow \mathrm{H}_0(\mathcal{F}) \xleftarrow[\varphi]{} \mathrm{H}_0(\mathcal{H}) \longleftarrow \operatorname{coker} a \longleftarrow 0$$

is exact. In particular φ is surjective. So we have surjections $\bar{\pi} : \mathrm{H}_0(\mathcal{F}) \to \mathrm{H}_0(\mathcal{H})$ and $\varphi : \mathrm{H}_0(\mathcal{H}) \to \mathrm{H}_0(\mathcal{F})$. By (2.5.2) both maps are isomorphisms. Then $\mathrm{H}_0(\mathcal{E}) = \ker \bar{\pi} = 0$. Thus $\mathcal{E}$ is exact.

$\square$

Now we proceed with the proof of (4.2.1)

Proof of (4.2.1). To determine the homology of Tot$\mathcal{G}$ we use the spectral sequence associated to $\mathcal{G}$ filtered by columns. Call this spectral sequence $\mathcal{E}$.

The zero page of $\mathcal{E}$ is $\mathcal{G}$ with only the vertical maps. Then $\mathcal{E}^1_{d,\ell} = H^v_\ell(\mathcal{G}_{d,*})$. So we compute the vertical homology at each spot.

For $\ell = 1$ we compute the homology of

$$\begin{array}{c} G_{d+1,2} \\ \Big\downarrow \partial^v_{d+1,2} \\ G_{d,1} \\ \Big\downarrow \partial^v_{d,1} \\ 0. \end{array}$$

Clearly $\ker \partial^v_{d,1} = G_{d,1}$. We will show that $\operatorname{Im} \partial^v_{d+1,2} = (\ker \pi_S) G_{d,1}$. Let $(0,j)g \in (\ker \pi_T) G_{d,1}$. Note that $\operatorname{Im} \partial^{\mathcal{B}}_1 = \ker \pi_S$. Then there exists $b \in B_1$ with $\partial^{\mathcal{B}}_1(b) = (0,j)$. Then $b \otimes g \in G_{d+1,2}$ and

$$\partial^v_{d+1,2}(b \otimes g) = \partial^{\mathcal{B}}_1(b) g = (0,j)g.$$

On the other hand if $b \otimes g \in G_{d+1,2}$ with $b \in B_{d'}$ and $g \in G_{d+1-d',1}$ then either

$$\partial^v_{d+1,2}(b \otimes g) = 0 \in (\ker \pi_S) G_{d,1}$$

or

$$\partial^v_{d+1,2}(b \otimes g) = \partial^{\mathcal{B}}_1(b)g \in (\ker \pi_S)G_{d,1}.$$

Then

$$H^v_1(\mathcal{G}_{d,*}) = \frac{G_{d,1}}{(\ker \pi_S)G_{d,1}} \cong G_{d,1} \otimes_A S.$$

If ℓ is even, then we compute the homology of

$$\begin{tikzcd} G_{d+1,\ell+1} \arrow[d, "\partial^v_{d+1,\ell+1}"] \\ G_{d,\ell} \arrow[d, "\partial^v_{d,\ell}"] \\ G_{d-1,\ell-1}. \end{tikzcd}$$

First note that

$$\bigoplus_{d'=2}^{d-\ell+2} B_{d'} \otimes_A G_{d-d',\ell-1} \subseteq \ker \partial^v_{d,\ell}$$

by definition.

If $b \otimes g \in \ker^{\mathcal{B}}_1 \otimes_A G_{d-1,\ell-1}$, then

$$\partial^v_{d,\ell}(b \otimes g) = \partial^{\mathcal{B}}_1(b)g = 0.$$

So

$$\ker^{\mathcal{B}}_1 \otimes_A G_{d-1,\ell-1} \oplus \bigoplus_{d'=2}^{d-\ell+2} B_{d'} \otimes_A G_{d-d',\ell-1} \subseteq \ker \partial^v_{d,\ell}$$

Since $G_{d-1,\ell-1}$ is a free A module, it has a basis $\mathsf{G}_{d-1,\ell-1}$. Then $\{b \otimes g : b \in \mathsf{B}_1, g \in \mathsf{G}_{d-1,\ell-1}\}$ is a basis for $B_1 \otimes_A G_{d-1,\ell-1}$. Let

$$\left(\sum_{g \in \mathsf{G}_{d-1,\ell-1}} \sum_{b \in \mathsf{B}_1} a_{b,g} b \otimes g \right) + \sum_{d'=2}^{d-\ell+2} \gamma_{d'} \in \ker \partial^v_{d,\ell}$$

with $\gamma_{d'} \in B_{d'} \otimes_A G_{d-d',\ell-1}$ Then

$$\begin{aligned} 0 &= \partial^v_{d,\ell} \left(\left(\sum_{g \in \mathsf{G}_{d-1,\ell-1}} \sum_{b \in \mathsf{B}_1} a_{b,g} b \otimes g \right) + \sum_{d'=2}^{d-\ell+2} \gamma_{d'} \right) \\ &= \sum_{g \in \mathsf{G}_{d-1,\ell-1}} \sum_{b \in \mathsf{B}_1} a_{b,g} \partial^{\mathcal{B}}_1(b) g + 0 \\ &= \sum_{g \in \mathsf{G}_{d-1,\ell-1}} \partial^{\mathcal{B}}_1 \left(\sum_{b \in \mathsf{B}_1} a_{b,g} b \right) g. \end{aligned}$$

Since $\mathsf{G}_{d-1,\ell-1}$ is a basis,

$$\partial_1^{\mathcal{B}}\left(\sum_{b\in\mathsf{B}_1} a_{b,g}b\right) = 0.$$

So $(\sum_{b\in\mathsf{B}_1} a_{b,g}b) \in \ker \partial_1^{\mathcal{B}}$. Then

$$\begin{aligned}\left(\sum_{g\in\mathsf{G}_{d-1,\ell-1}}\sum_{b\in\mathsf{B}_1} a_{b,g}b\otimes g\right) + \sum_{d'=2}^{d-\ell+2}\gamma_{d'} &= \sum_{g\in\mathsf{G}_{d-1,\ell-1}}\left(\sum_{b\in\mathsf{B}_1} a_{b,g}b\right)\otimes g + \sum_{d'=2}^{d-\ell+2}\gamma_{d'} \\ &\in \ker_1^{\mathcal{B}}\otimes_A G_{d-1,\ell-1}\oplus\bigoplus_{d'=2}^{d-\ell+2} B_{d'}\otimes_A G_{d-d',\ell-1}.\end{aligned}$$

Thus

$$\ker\partial_{d,\ell}^v = \left(\ker\partial_1^{\mathcal{B}}\otimes_A G_{d-1,\ell-1}\right)\oplus\bigoplus_{d'=2}^{d-\ell+2} B_{d'}\otimes_A G_{d-d',\ell-1}.$$

We will show that $\operatorname{Im}\partial_{d+1,\ell+1}^v = (\ker\pi_T)G_{d,\ell}$. Let $(s,0)g\in(\ker\pi_T)G_{d,\ell}$. Note that $\operatorname{Im}\partial_1^{\mathcal{C}} = \ker\pi_T$. Then there exists $c\in C_1$ with $\partial_1^{\mathcal{C}}(c) = (s,0)$. Then $c\otimes g\in G_{d+1,\ell+1}$ and

$$\partial_{d+1,\ell+1}^v(c\otimes g) = \partial_1^{\mathcal{C}}(c)g = (s,0)g.$$

On the other hand if $c\otimes g\in G_{d+1,2}$ with $c\in C_{d'}$ and $g\in G_{d+1-d',1}$ then either

$$\partial_{d+1,2}^v(c\otimes g) = 0\in(\ker\pi_T)G_{d,\ell}$$

or

$$\partial_{d+1,2}^v(c\otimes g) = \partial_1^{\mathcal{C}}(c)g\in(\ker\pi_T)G_{d,\ell}.$$

Then

$$\begin{aligned}H_\ell^v(\mathcal{G}_{d,*}) &= \frac{\left(\ker\partial_1^{\mathcal{B}}\otimes_A G_{d-1,\ell-1}\right)\oplus\bigoplus_{d'=2}^{d-\ell+2} B_{d'}\otimes_A G_{d-d',\ell-1}}{(\ker\pi_T)G_{d,\ell}} \\ &\cong \frac{\ker\partial_1^{\mathcal{B}}\otimes_A G_{d-1,\ell-1}}{\ker\pi_T\left(B_1\otimes_A G_{d-1,\ell-1}\right)}\oplus\left(\bigoplus_{d'=2}^{d-\ell+2} B_{d'}\otimes_A G_{d-d',\ell-1}\right)\otimes_A T.\end{aligned}$$

By a similar argument

$$\begin{aligned}H_\ell^v(\mathcal{G}_{d,*}) &= \frac{\left(\ker\partial_1^{\mathcal{C}}\otimes_A G_{d-1,\ell-1}\right)\oplus\bigoplus_{d'=2}^{d-\ell+2} C_{d'}\otimes_A G_{d-d',\ell-1}}{(\ker\pi_S)G_{d,\ell}} \\ &\cong \frac{\ker\partial_1^{\mathcal{C}}\otimes_A G_{d-1,\ell-1}}{\ker\pi_S\left(C_1\otimes_A G_{d-1,\ell-1}\right)}\oplus\left(\bigoplus_{d'=2}^{d-\ell+2} C_{d'}\otimes_A G_{d-d',\ell-1}\right)\otimes_A S.\end{aligned}$$

for $\ell > 1$ odd.

Now we will find $\mathcal{E}^2$. We take homology at each spot of the complex

$$\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
0 & \longleftarrow & H_3^v(G_{2,*}) & \xleftarrow[\bar\partial^h_{3,3}]{} & H_3^v(G_{3,*}) & \xleftarrow[\bar\partial^h_{4,3}]{} & H_3^v(G_{4,*}) & \longleftarrow & \dots \\
\mathcal{E}^1 : 0 & \longleftarrow & H_2^v(G_{1,*}) & \xleftarrow[\bar\partial^h_{2,2}]{} & H_2^v(G_{2,*}) & \xleftarrow[\bar\partial^h_{3,2}]{} & H_2^v(G_{3,*}) & \longleftarrow & \dots \\
0 & \longleftarrow & H_1^v(G_{0,*}) & \xleftarrow[\bar\partial^h_{1,1}]{} & H_1^v(G_{1,*}) & \xleftarrow[\bar\partial^h_{2,1}]{} & H_1^v(G_{2,*}) & \longleftarrow & \dots \\
 & & 0 & & 0 & & 0 & &
\end{array}$$

where $\bar\partial^h_{*,*}$ is the map in map induced on homology by $\partial^h_{*,*}$.

For $\ell = 1$ we have the complex

$$0 \longleftarrow G_{0,1} \otimes_A S \xleftarrow[\bar\partial^h_{1,1}]{} G_{1,1} \otimes_A S \xleftarrow[\bar\partial^h_{2,1}]{} G_{2,1} \otimes_A S \longleftarrow \dots . \tag{4.4}$$

The module $G_{d,1} = D_d$. Then $G_{d,1} \otimes_A S = D_d \otimes_A S \cong D'_d$ as S-modules. So (4.4) is isomorphic to $\mathcal{D}'$ as complexes of S-modules. The homology does not change when we treat (4.4) as a complex of A-modules. Thus

$$H_d^h(H_1^v(G_{*,*})) = H_d(\mathcal{D}') = \begin{cases} M & \text{if } d = 0 \\ 0 & \text{if } d > 0. \end{cases}$$

For ℓ even we consider the homology of

$$\begin{aligned}
0 \longleftarrow{} & \frac{\ker \partial_1^{\mathcal{B}} \otimes_A G_{\ell-2,\ell-1}}{\ker \pi_T\left(B_1 \otimes_A G_{\ell-2,\ell-1}\right)} \\
\xleftarrow[\bar\partial^h_{\ell,\ell}]{} & \frac{\ker \partial_1^{\mathcal{B}} \otimes_A G_{\ell-1,\ell-1}}{\ker \pi_T\left(B_1 \otimes_A G_{\ell-1,\ell-1}\right)} \oplus B_2 \otimes_A G_{\ell-2,\ell-1} \otimes_A T \longleftarrow \dots \qquad (4.5) \\
\dots \xleftarrow[\bar\partial^h_{d,\ell}]{} & \frac{\ker \partial_1^{\mathcal{B}} \otimes_A G_{d-1,\ell-1}}{\ker \pi_T\left(B_1 \otimes_A G_{d-1,\ell-1}\right)} \oplus \left(\bigoplus_{d'=2}^{d-\ell+2} B_{d'} \otimes_A G_{d-d',\ell-1}\right) \otimes_A T \longleftarrow \dots .
\end{aligned}$$

Note that (4.5) is isomorphic to the tensor product of the complexes

$$G_{*,\ell-1} : 0 \longleftarrow G_{\ell-2,\ell-1} \longleftarrow G_{\ell-1,\ell-1} \longleftarrow \dots$$

and

$$0 \longleftarrow \frac{\ker \partial_1^{\mathcal{B}}}{\ker \pi_T B_1} \longleftarrow B_2 \otimes_A T \longleftarrow B_3 \otimes_A T \longleftarrow \dots . \tag{4.6}$$

By (4.2.2), (4.6) is exact. The homology does not change when we consider it as a complex of A-modules. Then (4.5) is exact since it is the tensor product of an exact complex and another complex. So

$$H_d^h(H_\ell^v(G_{*,*})) = 0$$

for ℓ even.

By a similar argument

$$H_d^h(H_\ell^v(G_{*,*})) = 0$$

for $\ell > 1$ odd.

Thus the spectral sequence collapses on page 2. Then

$$H_n(\operatorname{Tot} \mathcal{G}) = \bigoplus_{p+q=n} H_{p+q}^h\left(H_{q+1}^v(G_{*,*})\right) = \begin{cases} M & \text{if } p = q = 0 \\ 0 & \text{otherwise.} \end{cases}$$

Thus $\operatorname{Tot} \mathcal{G}$ is a resolution of M by free A-modules. Since $\operatorname{Im} \partial^{\mathcal{B}} \subset \mathfrak{m}_A \mathcal{B}$, $\operatorname{Im} \partial^{\mathcal{C}} \subset \mathfrak{m}_A \mathcal{C}$, and $\operatorname{Im} \partial^{\mathcal{D}} \subset \mathfrak{m}_A \mathcal{D}$, the resolution is minimal. $\square$

Chapter 5

The Poincaré Series of Module over a Fiber Product

The purpose of this chapter is to establish the relationship between Poincaré series over the fiber product $A = S \times_R T$ and Poincaré series over S and T. As stated before this result is already known [3],[6]. Here we show that the structure of the resolution from Chapter 4 reflects this equality. We will then consider the case that S', T', R and S, T, R satisfy the conditions of (4.1.1) with surjective ring homomorphisms $\varphi : S \to S'$ and $\psi : T \to T'$. The universal property for fiber products induces a surjective ring homomorphisms $\theta : S \times_R T \to S' \times_R T'$. We will give conditions for this homomorphism to be Golod.

5.1 Hilbert Series

Definition 5.1.1. *Let M be a graded $\boldsymbol{k}$-vector space with $M_d = 0$ for $d \ll 0$ and $\dim_{\boldsymbol{k}} M_d$ finite for all d. Then the Hilbert series of M is the formal Laurent series*

$$HS_M(t) = \sum_d \dim_{\boldsymbol{k}} M_d t^d.$$

If $\mathrm{HS}_M(t)$ is defined, then we say that M has a Hilbert series. Let

$$\mathbb{T}^\ell_{\boldsymbol{k}}(M) = M^{\otimes_{\boldsymbol{k}} \ell} = \underbrace{M \otimes_{\boldsymbol{k}} M \otimes_{\boldsymbol{k}} \otimes_{\boldsymbol{k}} \ldots \otimes_{\boldsymbol{k}} M}_{\ell \text{ times}}$$

where $\mathbb{T}^0_{\boldsymbol{k}}(M) = \boldsymbol{k}$. Then the tensor algebra of M is

$$\mathbb{T}_{\boldsymbol{k}}(M) = \bigoplus_{\ell=0}^{\infty} \mathbb{T}^\ell_{\boldsymbol{k}}(M).$$

Define

$$\mathbb{T}_{\boldsymbol{k}}^{\ell,d}(M) = \bigoplus_{d_1+d_2+\cdots+d_\ell=d} M_{d_1} \otimes_{\boldsymbol{k}} M_{d_2} \otimes_{\boldsymbol{k}} \ldots \otimes_{\boldsymbol{k}} M_{d_\ell} \subset \mathbb{T}_{\boldsymbol{k}}^{\ell}(M).$$

and note that

$$\mathbb{T}_{\boldsymbol{k}}^{\ell}(M) = \bigoplus_{d=0}^{\infty} \mathbb{T}_{\boldsymbol{k}}^{\ell,d}(M)$$

We record two properties of Hilbert series

Proposition 5.1.2. *Let M and N be graded vector spaces with Hilbert series. Then*

$$HS_{M\otimes_{\boldsymbol{k}} N}(t) = HS_M(t)HS_N(t).$$

Proof. By the definition of the Hilbert series

$$\mathrm{HS}_{M\otimes_{\boldsymbol{k}} N}(t) = \sum_d \dim_{\boldsymbol{k}}(M \otimes_{\boldsymbol{k}} N)_d \, t^d.$$

The $\boldsymbol{k}$-vector space

$$(M \otimes_{\boldsymbol{k}} N)_d = \bigoplus_{i+j=d} M_i \otimes_{\boldsymbol{k}} N_j.$$

Then

$$\dim_{\boldsymbol{k}}(M \otimes_{\boldsymbol{k}} N)_d = \sum_{i+j=d} \dim_{\boldsymbol{k}} M_i \dim_{\boldsymbol{k}} N_j.$$

The right hand side is precisely the coefficient of t^d in the product $\mathrm{HS}_M(t)\mathrm{HS}_N(t)$. □

Proposition 5.1.3. *Let $\mathbb{T}_{\boldsymbol{k}}(M)$ be the tensor algebra of M over $\boldsymbol{k}$. If $M_d = 0$ for $d < 0$, then*

$$HS_{\mathbb{T}_{\boldsymbol{k}}(M)}(t) = \frac{1}{1 - HS_M(t)}.$$

Proof. First we have

$$\mathrm{HS}_{\mathbb{T}_{\boldsymbol{k}}(M)}(t) = \bigoplus_{\ell=0}^{\infty} \mathbb{T}_{\boldsymbol{k}}^{\ell}(M).$$

Since

$$\mathbb{T}_{\boldsymbol{k}}^{\ell}(M) = \underbrace{M \otimes_{\boldsymbol{k}} M \otimes_{\boldsymbol{k}} \otimes_{\boldsymbol{k}} \ldots \otimes_{\boldsymbol{k}} M}_{\ell \text{ times}}$$

the Hilbert series of $\mathbb{T}_{\boldsymbol{k}}^{\ell}(M)$ can be determined by (5.1.2). So

$$\mathrm{HS}_{\mathbb{T}_{\boldsymbol{k}}^{\ell}(M)}(t) = \left(\mathrm{HS}_M(t)\right)^{\ell}.$$

Thus

$$\begin{aligned}\mathrm{HS}_{\mathbb{T}_{\boldsymbol{k}}(M)}(t) &= \sum_{\ell=0}^{\infty} \mathrm{HS}_{\mathbb{T}^{\ell}_{\boldsymbol{k}}(M)}(t) \\ &= \sum_{\ell=0}^{\infty} \left(\mathrm{HS}_M(t)\right)^{\ell} \\ &= \frac{1}{1-\mathrm{HS}_M(t)}.\end{aligned}$$

□

5.2 Fiber Products and Poincaré Series

Definition 5.2.1. *Let $\boldsymbol{k}$ be a field and C be a set. Then let ${}^{\boldsymbol{k}}\mathsf{C}$ be the $\boldsymbol{k}$-vector space with basis C.*

Corollary 5.2.2. *Given the assumptions of 4.2.1 we have*

$$\frac{1}{P^A_M(t)} = \frac{P^S_R(t)}{P^S_M(t)}\left(\frac{1}{P^S_R(t)} + \frac{1}{P^T_R(t)} - 1\right).$$

We prove 5.2.2 using the following lemma

Lemma 5.2.3. *The $\boldsymbol{k}$-vector space $\boldsymbol{k} \otimes_A G_{d,\ell}$ is isomorphic to*

$$\bigoplus_{d_1+d_2=d} \boldsymbol{k} \otimes_{\boldsymbol{k}} \mathbb{T}^{\ell',d_1}_{\boldsymbol{k}}({}^{\boldsymbol{k}}\mathsf{C}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{B}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_2}$$

if $\ell = 2\ell' + 1$ and

$$\bigoplus_{\substack{d_1+d_2+d_3=d \\ d_1 \geq 1}} {}^{\boldsymbol{k}}\mathsf{B}_{d_1} \otimes_{\boldsymbol{k}} \mathbb{T}^{\ell',d_2}_{\boldsymbol{k}}({}^{\boldsymbol{k}}\mathsf{C}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{B}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_3}$$

if $\ell = 2\ell' + 2$.

Proof of 5.2.3. We proceed by induction on ℓ.

For $\ell = 1$ we have $G_{d,\ell} = D_d$, which has basis D_d. The $\boldsymbol{k}$ vector space $\boldsymbol{k} \otimes_{\boldsymbol{k}} \boldsymbol{k} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_d$ has basis $\{1 \otimes 1 \otimes d : d \in \mathsf{D}_d\}$. Then the map $d \mapsto 1 \otimes 1 \otimes d$ fo $d \in \mathsf{D}_d$ gives the isomorphism.

Assume that the claim is true for $\ell' < \ell$. We will show that it holds for ℓ. If $\ell = 2\ell' + 2$, then

$$\begin{aligned}
G_{d,\ell} \otimes_A \boldsymbol{k} &= \left(\bigoplus_{d_1=1}^{d-\ell+2} B_{d_1} \otimes_A G_{d-d_1,\ell-1} \right) \otimes_A \boldsymbol{k} \\
&= \bigoplus_{d_1=1}^{d-\ell+2} (B_{d_1} \otimes_A \boldsymbol{k}) \otimes_{\boldsymbol{k}} (G_{d-d_1,\ell-1} \otimes_A \boldsymbol{k}) \\
&\cong \bigoplus_{d_1=1}^{d-\ell+2} (B_{d_1} \otimes_A \boldsymbol{k}) \otimes_{\boldsymbol{k}} \left(\bigoplus_{d_2+d_3=d-d_1} \boldsymbol{k} \otimes_{\boldsymbol{k}} \mathbb{T}_{\boldsymbol{k}}^{\ell',d_2}({}^{\boldsymbol{k}}\mathsf{C}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{B}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_3} \right) \\
&= \bigoplus_{d_1=1}^{d-\ell+2} \bigoplus_{d_2+d_3=d-d_1} (B_{d_1} \otimes_A \boldsymbol{k}) \otimes_{\boldsymbol{k}} \left(\boldsymbol{k} \otimes_{\boldsymbol{k}} \mathbb{T}_{\boldsymbol{k}}^{\ell',d_2}({}^{\boldsymbol{k}}\mathsf{C}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{B}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_3} \right) \\
&\cong \bigoplus_{d_1=1}^{d-\ell+2} \bigoplus_{d_2+d_3=d-d_1} {}^{\boldsymbol{k}}\mathsf{B}_{d_1} \otimes_{\boldsymbol{k}} \mathbb{T}_{\boldsymbol{k}}^{\ell',d_2}({}^{\boldsymbol{k}}\mathsf{C}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{B}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_3} \\
&= \bigoplus_{\substack{d_1+d_2+d_3=d \\ d_1 \geq 1}} {}^{\boldsymbol{k}}\mathsf{B}_{d_1} \otimes_{\boldsymbol{k}} \mathbb{T}_{\boldsymbol{k}}^{\ell',d_2}({}^{\boldsymbol{k}}\mathsf{C}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{B}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_3}.
\end{aligned}$$

If $\ell = 2\ell' + 1$, then

$$\begin{aligned}
G_{d,\ell} \otimes_A \boldsymbol{k} &= \left(\bigoplus_{d_0=1}^{d-\ell+2} C_{d_0} \otimes_A G_{d-d_0,\ell-1} \right) \otimes_A \boldsymbol{k} \\
&= \bigoplus_{d_0=1}^{d-\ell+2} (C_{d_0} \otimes_A \boldsymbol{k}) \otimes_{\boldsymbol{k}} (G_{d-d_0,\ell-1} \otimes_A \boldsymbol{k}) \\
&\cong \bigoplus_{d_0=1}^{d-\ell+2} (C_{d_0} \otimes_A \boldsymbol{k}) \otimes_{\boldsymbol{k}} \left(\bigoplus_{\substack{d_1+d_2+d_3 \\ =d-d_0}} {}^{\boldsymbol{k}}\mathsf{B}_{d_1} \otimes_{\boldsymbol{k}} \mathbb{T}_{\boldsymbol{k}}^{\ell'-1,d_2}({}^{\boldsymbol{k}}\mathsf{C}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{B}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_3} \right) \\
&= \bigoplus_{d_0=1}^{d-\ell+2} \bigoplus_{\substack{d_1+d_2+d_3 \\ =d-d_0}} {}^{\boldsymbol{k}}\mathsf{C}_{d_0} \otimes_{\boldsymbol{k}} \left({}^{\boldsymbol{k}}\mathsf{B}_{d_1} \otimes_{\boldsymbol{k}} \mathbb{T}_{\boldsymbol{k}}^{\ell'-1,d_2}({}^{\boldsymbol{k}}\mathsf{C}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{B}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_3} \right) \\
&\cong \bigoplus_{d_0=1}^{d-\ell+2} \bigoplus_{\substack{d_1+d_2+d_3 \\ =d-d_0}} {}^{\boldsymbol{k}}\mathsf{C}_{d_0} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{B}_{d_1} \otimes_{\boldsymbol{k}} \mathbb{T}_{\boldsymbol{k}}^{\ell'-1,d_2}({}^{\boldsymbol{k}}\mathsf{B}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{C}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_3} \\
&= \bigoplus_{\substack{d_0+d_1+d_2+d_3=d \\ d_0 \geq 1}} \boldsymbol{k} \otimes_{\boldsymbol{k}} \mathbb{T}_{\boldsymbol{k}}^{\ell',d_0+d_1+d_2}({}^{\boldsymbol{k}}\mathsf{B}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{C}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}_{d_3}.
\end{aligned}$$

□

Proof of 5.2.2. Let $B = {}^{\boldsymbol{k}}\mathsf{B}, C = {}^{\boldsymbol{k}}\mathsf{C}$ and $D = {}^{\boldsymbol{k}}\mathsf{D}$. Then

$$P_R^T(t) = \mathrm{HS}_B(t), P_R^S(t) = \mathrm{HS}_C(t), P_M^S(t) = \mathrm{HS}_D(t)$$

and

$$P^A_M(t) = \mathrm{HS}_{G_*\otimes_A \boldsymbol{k}}(t).$$

Now

$$G_* \otimes_A \boldsymbol{k} \cong \sum_{d,\ell} G_{d,\ell} \otimes_A \boldsymbol{k} \cong {}^{\boldsymbol{k}}\mathsf{B} \otimes_{\boldsymbol{k}} \mathbb{T}_{\boldsymbol{k}}({}^{\boldsymbol{k}}\mathsf{B}_{\geq 1} \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{C}_{\geq 1}) \otimes_{\boldsymbol{k}} {}^{\boldsymbol{k}}\mathsf{D}.$$

The second isomorphism follows from (5.2.3). Then

$$\mathrm{HS}_{G_*\otimes_A \boldsymbol{k}}(t) = \frac{\mathrm{HS}_B(t)\mathrm{HS}_D(t)}{1 - (\mathrm{HS}_B(t) - 1)(\mathrm{HS}_C(t) - 1)}$$

by (5.1.2) and (5.1.3).

That is

$$P^A_M(t) = \frac{P^T_R(t)P^S_M(t)}{1 - (P^T_R(t) - 1)(P^S_R(t) - 1)}. \tag{5.1}$$

Reciprocate both sides to finish the proof.

□

5.2.4. *Let S', T', R and S, T, R both satisfy the hypothesis of (4.1.1). Let $\varphi : S \to S'$ and $\psi : T \to T'$ be surjective ring homomorphisms. Then we have the diagram of rings*

$$\begin{tikzcd}
A := S \times_R T \arrow[rr] \arrow[dd] \arrow[dr, "\theta"] & & T \arrow[d, "\psi"] \\
& A' = S' \times_R T' \arrow[r] \arrow[d] & T' \arrow[d] \\
S \arrow[r, "\varphi"] & S' \arrow[r] & R.
\end{tikzcd} \tag{5.2}$$

Here θ is the map induced from the universal property of fiber products.

Lemma 5.2.5. *Suppose that $\varphi : S \to S'$ is a surjective ring homomorphism, $p_S : S \to R$ and $p_{S'} : S' \to R$ are large homomorphisms. Then the following are equivalent*

1. *The homomorphism φ is Golod.*

2. *When treated as an S'-module via $p_{S'}$, R is φ-Golod.*

3. *Every finitely generated S'-module M is φ-Golod.*

Proof. <u>*1.* $\Rightarrow$ *2.*</u> Suppose that φ if a Golod homomorphisms. Then

$$P_{\boldsymbol{k}}^{S'}(t) = \frac{P_{\boldsymbol{k}}^{S}(t)}{1 - t\,(P_{S'}^{S}(t) - 1)}.$$

Since p_S and $p_{S'}$ are large homomorphisms we have

$$P_{\boldsymbol{k}}^{S}(t) = P_{\boldsymbol{k}}^{R}(t)P_{R}^{S}(t) \quad \text{and} \quad P_{\boldsymbol{k}}^{S'}(t) = P_{\boldsymbol{k}}^{R}(t)P_{R}^{S'}(t)$$

by [12, 1.1]. Then

$$P_{\boldsymbol{k}}^{R}(t)P_{R}^{S'}(t) = \frac{P_{\boldsymbol{k}}^{R}(t)P_{R}^{S}(t)}{1 - t\,(P_{S'}^{S}(t) - 1)}.$$

Divide by $P_{\boldsymbol{k}}^{R}(t)$ to show that R is a φ-Golod S'-module.

<u>*2.* $\Rightarrow$ *3.*</u> Suppose that R is a φ-Golod S'-module and let M be a finitely generated S'-module. Since p_S and p'_S are large homomorphisms we have

$$P_{M}^{S}(t) = P_{M}^{R}(t)P_{R}^{S}(t) \quad \text{and} \quad P_{M}^{S'}(t) = P_{M}^{R}(t)P_{R}^{S'}(t)$$

by [12, 1.1]. Since R is φ-Golod we have

$$P_{R}^{S'}(t) = \frac{P_{R}^{S}(t)}{1 - t\,(P_{S'}^{S}(t) - 1)}.$$

Multiply both sides by $P_{M}^{R}(t)$ to get that M is φ-Golod.

<u>*3.* $\Rightarrow$ *1.*</u> Let $M = \boldsymbol{k}$ to see that φ is a Golod homomorphism.

$\square$

Theorem 5.2.6. *In (5.2), θ is a Golod homomorphism if and only if φ and ψ are Golod homomorphisms.*

Proof. First note that the map $p_S, p_T, p_{S'}$ and $p_{T'}$ are large homomorphisms by (2.4.2). Then by (5.2.5) it suffices to show that R is a θ-Golod A'-module if and only if R is a φ-Golod S'-module and a ψ-Golod T'-module.

Now we have the short exact sequence of R-modules

$$0 \longleftarrow R \xleftarrow[p_{S'} - p_{T'}]{} S' \times T' \xleftarrow[i]{} A' \longleftarrow 0.$$

Also we have an R-module homomorphisms $\chi : R \to S' \times T'$ given by $\chi(r) = (i_{S'}(r), 0)$ which satisfies $(p_{S'} - p_{T'}) \circ \chi = \mathrm{id}_R$. So the sequence is split exact. Then, by (2.3.3)

$$P^R_{A'}(t) = P^R_{S'}(t) + P^R_{T'}(t) - P^R_R(t).$$

Since R is an algebra retract of A we have the following

$$P^A_{A'}(t) = P^R_{A'}(t) P^A_R(t), \quad P^A_{S'}(t) = P^R_{S'}(t) P^A_R(t),$$

$$P^A_{T'}(t) = P^R_{T'}(t) P^A_R(t), \quad P^A_R(t) = P^R_R(t) P^A_R(t).$$

Multiply both sides of

$$P^R_{A'}(t) = P^R_{S'}(t) + P^R_{T'}(t) - P^R_R(t)$$

by $P^A_R(t)$ and using the above equalities of Poincaré series to get

$$P^A_{A'}(t) = P^A_{S'}(t) + P^A_{T'}(t) - P^A_R(t).$$

The ring homomorphisms $\pi_S : A \to S$ and $\pi_T : A \to T$ are large homomorphisms by (2.4.2), since S and T are algebra retracts of A. Combining this and the above we get

$$\begin{aligned}
P^A_{A'}(t) &= P^A_{S'}(t) + P^A_{T'}(t) - P^A_R \\
&= P^S_{S'}(t) P^A_S(t) + P^T_{T'}(t) P^A_T(t) - P^S_R(t) P^A_S(t) \\
&= P^S_{S'}(t) \frac{P^T_R(t)}{1 - (P^T_R(t) - 1)(P^S_R(t) - 1)} + P^T_{T'}(t) \frac{P^S_R(t)}{1 - (P^T_R(t) - 1)(P^S_R(t) - 1)} \\
&- P^S_R(t) \frac{P^T_R(t)}{1 - (P^T_R(t) - 1)(P^S_R(t) - 1)} \\
&= \frac{P^S_{S'}(t) P^T_R(t) + P^T_{T'}(t) P^S_R(t) - P^S_R(t) P^T_R(t)}{P^T_R(t) + P^S_R(t) - P^T_R(t) P^S_R(t)}.
\end{aligned}$$

The third equality follows from (5.1). Hence

$$(P^T_R(t) + P^S_R(t) - P^T_R(t) P^S_R(t))(P^A_{A'}(t) - 1) = P^S_R(t)(P^T_{T'}(t) - 1) + P^T_R(t)(P^S_{S'}(t) - 1). \quad (5.3)$$

Now we have the following sequence of (in)equalities:

$$
\begin{aligned}
&\frac{1 - t(P^A_{A'}(t) - 1)}{P^A_R(t)} \\
&= \frac{\left(1 - t(P^A_{A'}(t) - 1)\right)(P^T_R(t) + P^S_R(t) - P^T_R(t)P^S_R(t))}{P^S_R(t)P^T_R(t)} \\
&= \frac{P^S_R(t) + P^T_R(t) - P^T_R(t)P^S_R(t) - t\left(P^S_R(t)(P^T_{T'}(t) - 1) + P^T_R(t)(P^S_{S'}(t) - 1)\right)}{P^S_R(t)P^T_R(t)} \\
&= \frac{P^S_R(t)(1 - t(P^T_{T'}(t) - 1)) + P^T_R(t)(1 - t(P^S_{S'}(t) - 1)) - P^T_R(t)P^S_R(t)}{P^S_R(t)P^T_R(t)} \\
&= \frac{1 - t(P^T_{T'}(t) - 1)}{P^T_R(t)} + \frac{1 - t(P^S_{S'}(t) - 1)}{P^S_R(t)} - 1 \\
&\preceq \frac{1}{P^{T'}_R(t)} + \frac{1}{P^{S'}_R(t)} - 1 \\
&= \frac{1}{P^{A'}_R(t)}.
\end{aligned}
$$

The first and last equalities follow from (5.2.2). The second equality comes from (5.3). The term-wise inequality follows from Serre's upperbound and (2.0.1). The term-wise inequality is an equality if and only if R is φ-Golod and ψ-Golod.

□

This theorem can be compared to [13, 4.2.2]. Here we have replaced $\boldsymbol{k}$ with R at the cost of adding the hypothesis of (4.1.1).

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