2018

Special Fiber Rings of Certain Height Four Gorenstein Ideals

Jaree Hudson
University of South Carolina

Special Fiber Rings of Certain Height Four Gorenstein Ideals

by

Jaree Hudson

Bachelor of Science
Armstrong Atlantic State University 2008

Master of Science
University of Oregon 2011

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2018

Accepted by:

Andrew Kustin, Major Professor

Adela Vraciu, Committee Member

Matthew Ballard, Committee Member

George Androulakis, Committee Member

Paramita Chakraborty, Committee Member

Cheryl L. Addy, Vice Provost and Dean of the Graduate School

Acknowledgments

I would like to thank my advisor, Andrew Kustin, whose office I always left more encouraged than when I entered. Thank you also to my mentor through the AWM, Emily Witt, whose advice and experience helped me to keep working through tough spots. Finally, I could never have completed this disseratation without the love and support of Rade Musulin, my best friend and partner of the last four years.

Abstract

Let S be a set of four variables, $\boldsymbol{k}$ a field of characteristic not equal to two such that $\boldsymbol{k}$ contains all square roots, and I a height four Gorenstein ideal of $\boldsymbol{k}[S]$ generated by nine quadratics so that I has a Gorenstein-linear resolution. We define a complex $X_\bullet$ so that each module of $X_\bullet$ is the tensor product of a certain polynomial ring Q in nine variables and a direct sum of indecomposable $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules and the differential maps are Q- and $\boldsymbol{k}[\mathrm{Sym}(S)]$-module homomorphisms. Work with the Macaulay2 software suggests that $H_0(X_\bullet)$ is the special fiber ring of I and $H_1(X_\bullet) = 0$.

Table of Contents

Chapter 1

Introduction

1.1 Special Fiber Rings and Blowup Algebras

Definition 1.1. Let $\boldsymbol{k}$ be a field. Given homogeneous polynomials $f_1, f_2, \ldots, f_m$ in $P = \boldsymbol{k}[x_1, x_2, \ldots, x_d]$, all of the same degree, the subring $\boldsymbol{k}[f_1, f_2, \ldots, f_m]$ of P is called the *special fiber ring* of the ideal $(f_1, f_2, \ldots, f_m)$.

Given $f_1, f_2, \ldots, f_m$ as in the definition above, there is a well-defined map of projective spaces $\Psi : \mathbb{P}^{d-1} \to \mathbb{P}^{m-1}$ given by

$$\Psi(X) = [f_1(X) : f_2(X) : \cdots : f_m(X)].$$

In algebraic geometry terms, the special fiber ring is the homogeneous coordinate ring of $\overline{\mathrm{Im}(\Psi)}$, the closure of $\mathrm{Im}(\Psi)$ in the Zariski topology.

Definition 1.2. Given the same setup as above, the *Rees algebra* of I, denoted $\mathcal{R}(I)$, is the graded subalgebra $P[It]$ of the polynomial ring $P[t]$. That is,

$$\mathcal{R}(I) = P \oplus It \oplus I^2t^2 \oplus \cdots = \bigoplus_{j=0}^{\infty} I^j t^j.$$

Letting $\mathfrak{m}$ be the maximal homogeneous ideal of P, the special fiber ring is a quotient of the Rees algebra:

$$\mathcal{R}(I)/\mathfrak{m}\mathcal{R}(I) \cong \boldsymbol{k}[f_1, f_2, \ldots, f_m].$$

To algebraic geometers, the Rees algebra is the bi-homogeneous coordinate ring of the graph of Ψ. Special fiber rings and Rees algebras are so-called *blowup algebras* that occur when one "blows up" a variety along a subvariety. Information about the geometry of $\overline{\mathrm{Im}(\Psi)}$ can be gleaned from these algebras. The multiplicity of the special fiber ring gives the degree of the parametrization [21] and the degrees of the defining equations of the Rees algebra give information about singularities [6].

The other important algebra to consider in this context is, using the same notation as before, the symmetric algebra of I over the polynomial ring P:

$$\mathrm{Sym}_\bullet^P(I) = P \oplus I \oplus \mathrm{Sym}_2^P(I) \oplus \mathrm{Sym}_3^P(I) \oplus \cdots .$$

There is a natural epimorphism

$$\mathrm{Sym}_\bullet^P(I) \to \mathcal{R}(I)$$

from the universal mapping property of $\mathrm{Sym}_\bullet^P(I)$. Blowup algebras arising from ideals so that the aforementioned epimorphism $\mathrm{Sym}_\bullet^P(I) \to \mathcal{R}(I)$ is an isomorphism has been understood for decades [15, 17].

The situtation in which I is primary to the maximal homogenous ideal is of interest currently. In this setting, Bruns, Conca, and Varbaro [3] have studied the case that I is a determinantal ideal; results concerning the case of perfect ideals of height two can be found in [29, 30, 16, 6, 20, 1, 27, 2, 22, 31], among other sources. Since height three Gorenstein ideals have been characterized by Buchsbaum and Eisenbud [5], this case is also of interest, for instance to Johnson and Morey [18, 29]. Kustin, Polini, and Ulrich [23] found the defining equations of blowup algebras when the corresponding ideal is height three Gorenstein, primary to the maximal ideal, and has a homogeneous presentation matrix consisting of linear forms. We would like to say what we can in the case where I is a height four Gorenstein ideal.

Since a characterization of height four Gorenstein ideals analogous to that of height three Gorenstein ideals is not known, we approach this case from another angle. As before, let $\boldsymbol{k}$ be a field and P be a polynomial ring in d variables, $P = \boldsymbol{k}[x_1, x_2, \dots, x_d]$. Let n be a positive integer. When I is a homogeneous ideal in P

generated by forms of degree n so that P/I is an Artinian Gorenstein algebra that has a Gorenstein linear resolution, El Khoury and Kustin [8, 9, 10] used Macaulay Inverse Systems to give a minimal homogeneous resolution of P/I by free P-modules. A pleasant fact about this resolution is that is it constructed in a polynomial manner from the coefficients of the generator of the Macaulay Inverse System.

1.2 Divided Power Algebras

The notion of a divided power algebra is used to define Macaulay Inverse Systems and we will use divided power algebras in our eventual complexes, so we present the pertinent ideas now.

Definition 1.3. Let R be a commutative ring and A a graded, commutative R-algebra with $A_0 = R$. A **system of divided powers** in A is a collection of functions

$$\bigcup_{i>0} A_i \to A,\; x \mapsto x^{(d)},$$

one for each nonnegative integer d, satisfying each of the following for any $x, y \in \bigcup\limits_{i>0} A_i$ and $d, e \in \mathbb{Z}_{\geq 0}$:

(1) $x^{(0)} = 1$ and $x^{(1)} = x$

(2) $\deg\left(x^{(d)}\right) = d \cdot \deg(x)$

(3) $x^{(d)}x^{(e)} = \dfrac{(d+e)!}{d!e!}x^{(d+e)}$

(4) $\left(x^{(d)}\right)^{(e)} = \dfrac{(de)!}{e!(d!)^e}x^{(de)}$

(5) $(xy)^{(d)} = d!x^{(d)}y^{(d)} = x^d y^{(d)} = x^{(d)}y^d$

(6) $(ax)^{(d)} = a^d x^{(d)}$ for $a \in A_0$

(7) $(x+y)^{(d)} = \sum\limits_{e=0}^{d} x^{(e)}y^{(d-e)}$

Let n be a positive integer and $x \in \bigcup_{i>0} A_i$. It is easily proven by (1) and (3) above that $x^n = n! x^{(n)}$. Hence, if $\mathrm{char}(R) = 0$, then $x^{(n)} = x^n/n!$. All of the properties above follow from treating $x^{(n)}$ like $x^n/n!$ if this were defined.

A slightly more general definition of divided powers can be found in [7, Appendix 2]. We will use one particular example of a divided power algebra, which we now present.

Definition 1.4. Let $\boldsymbol{k}$ be a field. If U is a d-dimensional vector space over $\boldsymbol{k}$ then the divided power algebra $D^{\boldsymbol{k}}_{\bullet}(U^*)$ is defined as a vector space by

$$D^{\boldsymbol{k}}_{\bullet}(U^*) = \bigoplus_{j \geq 0} D^{\boldsymbol{k}}_{j}(U^*),$$

where

$$D^{\boldsymbol{k}}_{j}(U^*) = \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{Sym}^{\boldsymbol{k}}_{j}(U), \boldsymbol{k}).$$

To get the algebra structure on $D^{\boldsymbol{k}}_{\bullet}(U^*)$, first recall that there is a coalgebra structure on $\mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U)$. The comultiplication map $\Delta : \mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U) \to \mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U \oplus U) = \mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U) \otimes_{\boldsymbol{k}} \mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U)$ is the $\boldsymbol{k}$-algebra homomorphism defined by $\Delta(u) = u \otimes 1 + 1 \otimes u$ for $u \in \mathrm{Sym}^{\boldsymbol{k}}_{1}(U)$. The counit is the projection $\varepsilon : \mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U) \to \mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U)/U\,\mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U) = \mathrm{Sym}^{\boldsymbol{k}}_{0}(U) = \boldsymbol{k}$. There are also, for any nonnegative integers i and j, $\boldsymbol{k}$-linear maps $\tau_{i,j} : D^{\boldsymbol{k}}_{i}(U^*) \otimes_{\boldsymbol{k}} D^{\boldsymbol{k}}_{j}(U^*) \to \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{Sym}^{\boldsymbol{k}}_{i}(U) \otimes_{\boldsymbol{k}} \mathrm{Sym}^{\boldsymbol{k}}_{j}(U), \boldsymbol{k}) \subseteq \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U) \otimes_{\boldsymbol{k}} \mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(U), \boldsymbol{k})$ defined by

$$(\tau_{i,j}(\phi \otimes \psi))(u_i \otimes u_j) = \phi(u_i)\psi(u_j).$$

Taking the $\boldsymbol{k}$-dual of Δ and composing with a $\tau_{i,j}$ gives a $\boldsymbol{k}$-linear map

$$\Delta^* \circ \tau_{i,j} : D^{\boldsymbol{k}}_{i}(U^*) \otimes_{\boldsymbol{k}} D^{\boldsymbol{k}}_{j}(U^*) \to D^{\boldsymbol{k}}_{\bullet}(U^*).$$

The $\Delta^* \circ \tau_{i,j}$ maps, for all nonnegative integers i and j, define the multiplication on homogeneous elements of $D^{\boldsymbol{k}}_{\bullet}(U^*)$. The unit of $D^{\boldsymbol{k}}_{\bullet}(U^*)$ is $\varepsilon^*(1)$, the image of $1 \in \boldsymbol{k}$

under the $\boldsymbol{k}$-dual of ε. For proof that these do indeed define $D_\bullet^{\boldsymbol{k}}(U^*)$ as a $\boldsymbol{k}$-algebra, see [7, Appendix 2].

It is true that $D_\bullet^{\boldsymbol{k}}(U^*)$ has a system of divided powers. It is easiest to describe the divided powers explicitly if we choose a basis for U. Let $\{x_1, x_2, \ldots, x_d\}$ is a basis for $U = \mathrm{Sym}_1^{\boldsymbol{k}}(U)$. Then there is a dual basis $\{x_1^*, x_2^*, \ldots, x_d^*\}$ for $U^* = D_1^{\boldsymbol{k}}(U^*)$ and, for each positive integer i, a basis for $\mathrm{Sym}_i^{\boldsymbol{k}}(U)$ consisting of the degree i monomials in the variables $x_1, x_2, \ldots, x_d$. There is a divided power structure on $D_\bullet^{\boldsymbol{k}}(U^*)$ satisfying, for nonnegative integers $a_1, a_2, \ldots, a_d$,

$$(x_1^*)^{(a_1)}(x_2^*)^{(a_2)} \cdots (x_d^*)^{(a_d)} = (x_1^{a_1} x_2^{a_2} \cdots x_d^{a_d})^*.$$

The properties of divided powers then give the definition of a divided power of any element of $D_\bullet^{\boldsymbol{k}}(U^*)$. Again, all relevant proofs are found in [7, Appendix 2].

It is also the case that $D_\bullet^{\boldsymbol{k}}(U^*)$ has a comultiplication map $\Delta_D : D_\bullet^{\boldsymbol{k}}(U^*) \to D_\bullet^{\boldsymbol{k}}(U^*) \otimes_{\boldsymbol{k}} D_\bullet^{\boldsymbol{k}}(U^*)$ defined by $\Delta_D(\psi) = \psi \otimes 1 + 1 \otimes \psi$ for $\psi \in D_1^{\boldsymbol{k}}(U^*)$. We will only use this in the next remark.

Remark 1.5. *If $A : U^* \to W_1$ and $B : U^* \to W_2$ are $\boldsymbol{k}$-module maps, then there is a unique $\boldsymbol{k}$-module map $\varphi : D_2^{\boldsymbol{k}}(U^*) \to W_1 \otimes_{\boldsymbol{k}} W_2$ such that $\varphi(x^{(2)}) = A(x) \otimes B(x)$ for all $x \in U^* = D_1^{\boldsymbol{k}}(U^*)$. This map is the composition*

$$D_2^{\boldsymbol{k}}(U^*) \to D_1^{\boldsymbol{k}}(U^*) \otimes_{\boldsymbol{k}} D_1^{\boldsymbol{k}}(U^*) \to W_1 \otimes_{\boldsymbol{k}} W_2$$

$$(A \otimes B) \circ (\pi_{D_1(U^*) \otimes D_1(U^*)} \circ \Delta_D).$$

An unsophisticated way to see this is to see that by (7) of Definition 1.3, $(x+y)^{(2)} = x^{(2)} + xy + y^{(2)}$ for $x, y \in U^*$. Applying φ and using the fact that we have a $\boldsymbol{k}$-module map produces

$$\varphi(xy) = A(x) \otimes B(y) + A(y) \otimes B(x)$$

for any $x, y \in U^*$.

1.2.1 The algebra $D_\bullet^{\boldsymbol{k}}(U^*)$ is a $\mathrm{Sym}_\bullet^{\boldsymbol{k}}(U)$-module

Given an element $u_i \in \mathrm{Sym}_i^{\boldsymbol{k}}(U)$ and a $w_j \in D_j^{\boldsymbol{k}}(U^*)$, define $u_i \cdot w_j \in D_{j-i}^{\boldsymbol{k}}(U^*) = \mathrm{Hom}_{\boldsymbol{k}}\left(\mathrm{Sym}_{j-i}^{\boldsymbol{k}}(U), \boldsymbol{k}\right)$ to be the map

$$(u_i \cdot w_j)(s_{j-i}) = w_j(u_i s_{j-i})$$

for $s_{j-i} \in \mathrm{Sym}_{j-i}^{\boldsymbol{k}}(U)$. One can check that this does define a $\mathrm{Sym}_\bullet^{\boldsymbol{k}}(U)$-module structure on $D_\bullet^{\boldsymbol{k}}(U^*)$.

If we have a basis $\{x_1, x_2, \ldots, x_d\}$ for U with dual basis $\{x_1^*, x_2^*, \ldots, x_d^*\}$, then we can also express the $\mathrm{Sym}_\bullet^{\boldsymbol{k}}(U)$-module structure on $D_\bullet^{\boldsymbol{k}}(U^*)$ in terms of the basis. Letting $i \in \{1, 2, \ldots, d\}$ and $a_1, a_2, \ldots, a_d$ be nonnegative integers with a_i positive, $x_i \in \mathrm{Sym}_1^{\boldsymbol{k}}(U)$ acts on monomials in $D_\bullet^{\boldsymbol{k}}(U^*)$ as

$$x_i \cdot \prod_{j=1}^{d} (x_j^*)^{(a_j)} = \prod_{j=1}^{i-1} (x_j^*)^{(a_j)} \cdot (x_i^*)^{(a_i - 1)} \cdot \prod_{j=i+1}^{d} (x_j^*)^{(a_j)}.$$

Definition 1.6. Under the given module structure, we can define the annihilator of a subset $S \subseteq \mathrm{Sym}_\bullet^{\boldsymbol{k}}(U)$ to be

$$\mathrm{Ann}_{D(U^*)}(S) = \{w \in D_\bullet^{\boldsymbol{k}}(U^*) \mid s \cdot w = 0 \text{ for all } s \in S\}$$

and the annihilator of a subset $T \subseteq D_\bullet^{\boldsymbol{k}}(U^*)$ to be

$$\mathrm{Ann}_{\mathrm{Sym}_\bullet^{\boldsymbol{k}}(U)}(T) = \{p \in \mathrm{Sym}_\bullet^{\boldsymbol{k}}(U) \mid p \cdot t = 0 \text{ for all } t \in T\}.$$

It is also true that $\mathrm{Sym}_\bullet^{\boldsymbol{k}}(U)$ is a $D_\bullet^{\boldsymbol{k}}(U^*)$-module, but we will not need this structure.

1.3 Indecomposable Representations of S_4

The special fiber ring we will be interested in is a subring of a polynomial ring in four variables. Since none of these four variables will be treated differently from the others, it is reasonable to consider the action by the group of permutations of the variables. This action will induce an action on the modules of the complex we will define. First, we recall a few relevant definitions and facts from representation theory:

Let $\boldsymbol{k}$ be a field. A *representation* of a group G is a group homomorphism $\rho : G \to \mathrm{GL}(U)$ for some $\boldsymbol{k}$-vector space U. Given a representation $\rho : G \to \mathrm{GL}(U)$ of a group G, one can give U the structure of a module over the group algebra $\boldsymbol{k}[G]$ by defining, for $g \in G$ and $u \in U$,

$$g \cdot u = \rho(g)(u)$$

and then extending this definition by linearity to all of $\boldsymbol{k}[G]$. Conversely, if U is a $\boldsymbol{k}[G]$-module, then the definition

$$\rho(g)(u) = g \cdot u$$

is a group homomorphism $\rho : G \to \mathrm{GL}(U)$. It is clear that the two operations above are inverse to each other, so that we can interchangeably talk about $\boldsymbol{k}[G]$-modules or representations of G.

A module is *decomposable* if can be written as the direct sum of two nonzero proper submodules and is *indecomposable* otherwise. A main result of from representation theory of finite groups is the following.

Theorem 1.7. *If $\boldsymbol{k}$ is algebraically closed with $\mathrm{char}(\boldsymbol{k}) = 0$ and G is a finite group, then any $\boldsymbol{k}[G]$-module is a direct sum of indecomposable $\boldsymbol{k}[G]$-modules and, up to isomorphism, there are only finitely many indecomposable $\boldsymbol{k}[G]$-modules. In fact, the number of non-isomorphic indecomposable $\boldsymbol{k}[G]$-modules is equal to the number of conjugacy classes in G.*

Definition 1.8. Given a representation $\rho : G \to \mathrm{GL}(U)$ of a finite group G over a field $\boldsymbol{k}$, where U is finite-dimensional, the *character* of ρ is the function $\chi : G \to \boldsymbol{k}$, $\chi(g) = \mathrm{tr}(\rho(g))$, the trace of the linear map $\rho(g)$.

Facts 1.9. *The following facts about characters over algebraically closed fields of zero characteristic are useful to recall. The specifics can be found in [11, Lecture 2].*

1. *Up to isomorphism, a representation is defined uniquely by its character.*

2. *There is an inner product defined on characters.*

3. *A representation is indecomposable if and only if the inner product of its character with itself is 1.*

4. *There are orthogonality relations on the characters of the indecomposable representations of a finite group.*

5. *The character of a direct sum of representations is the sum of the characters.*

6. *The character of a tensor product of representations is the pointwise multiplication of the characters.*

We now consider the setup pertaining to our eventual complex. Let $\boldsymbol{k}$ be a field, S a set of four variables, and V the $\boldsymbol{k}$-vector space spanned by S. We consider $\mathrm{Sym}(S)$, the group of permutations of S. The complex to be defined in Chapter 3 will be made of $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules and $\mathrm{Sym}(S)$-equivariant maps.

Since there are exactly five conjugacy classes in $\mathrm{Sym}(S)$, there are exactly five indecomposable $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules up to isomorphism. For a general discussion of these representations (i.e., representations of the group S_4), see [11, Section 2.3]. We will give these representations in contexts in which they will appear in Chapter 3.

There are two one-dimensional $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules. The first is the trivial one, for which every $\sigma \in \mathrm{Sym}(S)$ acts as the identity; this is isomorphic to $\bigwedge_{\boldsymbol{k}}^0 V$. The other is isomorphic to $\bigwedge_{\boldsymbol{k}}^4 V$, where $\sigma \in \mathrm{Sym}(S)$ acts as the sign of σ; this is referred to as the "alternating" representation. It is clear that these do define representations of $\mathrm{Sym}(S)$. Since each is one-dimensional, they are indecomposable.

There are two three-dimensional indecomposable $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules:

$$\frac{V}{\left\langle \sum\limits_{s\in S} s \right\rangle}$$

and

$$\frac{V}{\left\langle \sum\limits_{s\in S} s \right\rangle} \otimes_{\boldsymbol{k}} \textstyle\bigwedge_{\boldsymbol{k}}^4 V.$$

It is easy to see that these are indeed representations of $\mathrm{Sym}(S)$, as $\sigma \cdot \sum\limits_{s\in S} s = \sum\limits_{s\in S} s$ and the tensor product of representations is a representation. The fact that these are indecomposable if $\boldsymbol{k}$ is characteristic zero and algebraically closed can be shown via their characters.

Lastly, there is one two-dimensional indecomposable $\boldsymbol{k}[\mathrm{Sym}(S)]$-module, and it is isomorphic to the subvector space of $\mathrm{Sym}_2(V) = \boldsymbol{k}[S]_2$ spanned by the set

$$\left\{(s_1 - s_2)(s_3 - s_4) \mid S = \{s_1, s_2, s_3, s_4\}\right\}.$$

To see that this is indeed a $\boldsymbol{k}[\mathrm{Sym}(S)]$-module, note that every element of the form $(s_1 - s_2)(s_3 - s_4)$ for $S = \{s_1, s_2, s_3, s_4\}$ acted on by a permutation of S will be sent to either another such element or the negative of another such element. Also, if we let $S = \{s_1, s_2, s_3, s_4\}$, then this subvector space is generated by the set

$$\{(s_1 - s_2)(s_3 - s_4), (s_1 - s_3)(s_2 - s_4), (s_1 - s_4)(s_2 - s_3)\}$$

and, since

$$0 = (s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3)$$

in $\boldsymbol{k}[S]_2$, the given module can be generated as a vector space by two elements. It is not difficult to see that it cannot be generated by one element and is, hence, two-dimensional. The fact that it is indecomposable if $\boldsymbol{k}$ is characteristic zero and algebraically closed is again most easily done by computing the character.

Thus, if $\mathrm{char}(\boldsymbol{k}) = 0$ and $\boldsymbol{k}$ is algebraically closed, then the five indecomposable $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules are have been found. Computation of the characters of these modules gives the following character table:

	id	τ_2	τ_3	τ_4	$\tau_{2,2}$
$\bigwedge_{\boldsymbol{k}}^0 V$	1	1	1	1	1
$\bigwedge_{\boldsymbol{k}}^4 V$	1	-1	1	-1	1
$V/\langle \sum s \rangle$	3	1	0	-1	-1
$V/\langle \sum s \rangle \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^4 V$	3	-1	0	1	-1
$\langle (s_1 - s_2)(s_3 - s_4) \mid S = \{s_1, s_2, s_3, s_4\} \rangle$	2	0	-1	0	2,

where id is the identity of $\mathrm{Sym}(S)$, τ_i is a representative of the conjugacy class of i-cycles for $i \in \{1, 2, 3\}$, and $\tau_{2,2}$ is a representative of the conjugacy class of consisting of permutations that can be written as two disjoint 2-cycles.

For the complex we define, we will not necessarily assume that our field has characteristic zero or is algebraically closed. Thus, it does not have to be the case that every $\boldsymbol{k}[\mathrm{Sym}(S)]$-module is a direct sum of the modules above or that the above are indecomposable over our field. However, the $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules used in the complex to be defined will be direct sums of the modules given above.

Chapter 2

The Idea

Let V be a vector space of dimension 4 over a field $\boldsymbol{k}$ and I be a height 4 Gorenstein ideal in $\mathrm{Sym}_{\bullet}^{\boldsymbol{k}}(V)$ which is generated by quadratics and for which $\mathrm{Sym}_{\bullet}^{\boldsymbol{k}}(V)/I$ has a Gorenstein-linear resolution. We will also assume that $\mathrm{char}(\boldsymbol{k}) \neq 2$ to be able to take advantage of results about nondegenerate symmetric bilinear forms over such fields, as well as assuming that $\boldsymbol{k}$ is closed under square roots in order to simplify maps.

We now give definitions of the concepts needed. The following come from [8]. Gorenstein rings and ideals can be defined more generally (see, for instance, [4]), but we present the definitions in the context we use.

Definition 2.1. If $\boldsymbol{k}$ is a field and $A = \bigoplus_{i \geq 0} A_i$ is a graded Artinian $\boldsymbol{k}$-algebra with $A_0 = \boldsymbol{k}$ and maximal ideal $\mathfrak{m} = \bigoplus_{i>0} A_i$, then

(a) The *socle* of A is $0 :_A \mathfrak{m} = \{a \in A \mid a\mathfrak{m} = 0\}$.

(b) The algebra A is *Gorenstein* if $\dim_{\boldsymbol{k}} (0 :_A \mathfrak{m}) = 1$. In this case, the *socle degree* of A is the degree of a generator of $0 :_A \mathfrak{m}$.

Definition 2.2. Let $\boldsymbol{k}$ be a field and a I a homogeneous ideal of a polynomial ring $P = \boldsymbol{k}[x_1, x_2, \ldots, x_d]$.

(a) The ideal I is a *height d Gorenstein ideal* if P/I is an Artinian Gorenstein $\boldsymbol{k}$-algebra.

(b) If I is a height d Gorenstein ideal, then P/I has a *Gorenstein-linear* resolution if there is a positive integer n so that the minimal homogeneous resolution of P/I

by free P-modules has the form

$$0 \to P(-2n-d+2) \to P(-n-d+2)^{\beta_{d-1}} \to P(-n-d+3)^{\beta_{d-2}} \to \dots$$
$$\to \dots \to P(-n-2)^{\beta_3} \to P(-n-1)^{\beta_2} \to P(-n)^{\beta_1} \to P.$$

The β_i for $i \in \{1, 2, \dots, d-1\}$ can be found from the Herzog-Kühl formulas [14].

The remaining required definition is that of a Macaulay Inverse System, the existence of which comes from the following theorem.

Theorem (**Macaulay**, [28])**.** *Let U be a dimension d vector space over a field $\boldsymbol{k}$. The annihilators with respect to the $\mathrm{Sym}_\bullet^{\boldsymbol{k}}(U)$-module structure on $D_\bullet^{\boldsymbol{k}}(U^*)$ given in Section 1.2.1 give a one-to-one correspondence*

$$\left\{\begin{array}{c} \textit{nonzero homogeneous} \\ \textit{grade } d \textit{ Gorenstein} \\ \textit{ideals of } \mathrm{Sym}_\bullet^{\boldsymbol{k}} U \end{array}\right\} \leftrightarrow \left\{\begin{array}{c} \textit{nonzero homogeneous} \\ \textit{cyclic submodules} \\ \textit{of } D_\bullet^{\boldsymbol{k}}(U^*) \end{array}\right\}$$
$$I \mapsto \mathrm{Ann}_{D(U^*)}(I)$$
$$\mathrm{Ann}_{\mathrm{Sym}(U)}(M) \leftarrow M$$

Furthermore, for a nonzero homogeneous height d Gorenstein ideal I of $\mathrm{Sym}_\bullet^{\boldsymbol{k}} U$, the socle degree of $\mathrm{Sym}_\bullet^{\boldsymbol{k}}(U)/I$ is equal to the degree of a homogeneous generator of $\mathrm{Ann}(I)$.

Definition 2.3. In the setting of Macaulay's Theorem, the homogeneous cyclic submodule $\mathrm{Ann}(I)$ of $D_\bullet^{\boldsymbol{k}}(U^*)$ is the *Macaulay Inverse System* of the ideal I.

In [8, Proposition 1.8], El Khoury and Kustin give various properties equivalent to a homogeneous Gorenstein ideal in a polynomial ring having a Gorenstein-linear resolution. For our specific situation, the proposition is as follows.

Proposition 2.4 (**El Khoury and Kustin**, [8])**.** *Let $\boldsymbol{k}$ be a field and I a height 4 Gorenstein ideal of $P = \boldsymbol{k}[x_1, x_2, x_3, x_4]$ which is generated by quadratics. Let U*

be the vector space P_1 and $\phi \in D^{\boldsymbol{k}}_{\bullet}(U^*)$ a homogeneous generator of the Macaulay Inverse System of I. The following are equivalent.

(a) The ideal I has a Gorenstein-linear resolution.

(b) The minimal homogeneous resolution of P/I by free P-modules has the form

$$0 \to P(-6) \to P(-4)^9 \to P(-3)^{16} \to P(-2)^9 \to P.$$

(c) All minimal generators of I have degree 2 and the socle of P/I has degree 2.

(d) Both $I_1 = 0$ and $[P/I]_3 = 0$.

(e) $\phi \in D^{\boldsymbol{k}}_2(U^*)$ and the homomorphism $U \to U^*$ given by $u \mapsto u \cdot \phi \in U^* = D^{\boldsymbol{k}}_1(U^*)$, where $u \cdot \phi$ represents the result of $u \in U = \mathrm{Sym}^{\boldsymbol{k}}_1(U)$ acting on $\phi \in D^{\boldsymbol{k}}_2(U^*)$ as defined in subsection 1.2.1, is an isomorphism.

(f) $\phi \in D^{\boldsymbol{k}}_2(U^*)$ and the matrix $(\phi(x_i x_j))_{i,j=1,2,3,4}$ is invertible.

2.1 The Precise Project

Let $\boldsymbol{k}$ be a field with $\mathrm{char}(\boldsymbol{k}) \neq 2$ that is closed under square roots and let V be a dimension 4 $\boldsymbol{k}$-vector space. Let I be a height 4 Gorenstein ideal of $\boldsymbol{k}[V]$ generated by quadratics which has a Gorenstein-linear resolution. Since I is generated by quadratics, the special fiber ring is $\boldsymbol{k}[I_2]$.

Macaulay's Theorem gives a $\phi \in D^{\boldsymbol{k}}_2(V^*)$ so that $I = \mathrm{Ann}(\phi)$. Suppose that $\mathrm{Sym}^{\boldsymbol{k}}_{\bullet}(V) \cong \boldsymbol{k}[x_1, x_2, x_3, x_4]$. By Proposition 2.4, the matrix $(\phi(x_i x_j))_{i,j=1,2,3,4}$ is invertible. Notice that $(\phi(x_i x_j))_{i,j=1,2,3,4}$ is also symmetric. By properties of nondegenerate symmetric bilinear forms and the fact that $\mathrm{char}(\boldsymbol{k}) \neq 2$, there is a basis for V

with respect to which the matrix is diagonal. So we may assume that

$$(\phi(x_ix_j))_{i,j=1,2,3,4} = \begin{pmatrix} u_1 & 0 & 0 & 0 \\ 0 & u_2 & 0 & 0 \\ 0 & 0 & u_3 & 0 \\ 0 & 0 & 0 & u_4 \end{pmatrix}$$

for some $u_1, u_2, u_3, u_4 \in \boldsymbol{k}$. Since the matrix is invertible, u_1, u_2, u_3, and u_4 are units. Since $\boldsymbol{k}$ is closed under square roots, we may alter the basis elements again so that we can assume that $u_1 = u_2 = u_3 = u_4 = 1$. Then we have that

$$I = \operatorname{Ann}(\phi) = (x_1x_2, x_1x_3, x_1x_4, x_2x_3, x_2x_4, x_3x_4, x_2^2 - x_1^2, x_3^2 - x_1^2, x_4^2 - x_1^2).$$

In the next chapter, we will define a complex $X_\bullet$. Computer experimentation with Macaulay2 [12] indicates that $H_1(X_\bullet) = 0$ and that $H_0(X_\bullet)$ is isomorphic to the special fiber ring $\boldsymbol{k}[I_2]$.

Chapter 3

The Complex

Let $\boldsymbol{k}$ be a field. Our goal is to define a complex of free modules with zeroth homology equal to the subring

$$\boldsymbol{k}[xy, xz, xw, yz, yw, y^2 - x^2, z^2 - x^2, w^2 - x^2]$$

of $\boldsymbol{k}[x, y, z, w]$. As noted in Section 2.1, if $\mathrm{char}(\boldsymbol{k}) \neq 2$ and $\boldsymbol{k}$ is closed under square roots, then if I is a height four Gorenstein ideal in $\boldsymbol{k}[x, y, z, w]$ that is generated by quadratics and has a Gorenstein-linear resolution, then we may assume that I is the ideal $(xy, xz, xw, yz, yw, y^2 - x^2, z^2 - x^2, w^2 - x^2)$ after a change of variables.

Before starting the specific complex we want here, we present a nice, formal complex arising from a small complex. The complex we will define will have a subcomplex of this form. The ideas are derived from the work of Tate and can be found in [13].

Lemma 3.1. *Let $\boldsymbol{k}$ be a field, R a commutative $\boldsymbol{k}$-algebra, and N_1, N_2 be free R-modules. If $N_2 \xrightarrow{\delta_2} N_1 \xrightarrow{\delta_1} R$ is a complex, then so is*

$$D_2^{\boldsymbol{k}}(N_2) \xrightarrow{\Delta_3} N_1 \otimes_{\boldsymbol{k}} N_2 \xrightarrow{\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)} N_2 \oplus {\textstyle\bigwedge_{\boldsymbol{k}}^2} N_1 \xrightarrow{\delta_2 \oplus \mathrm{Kos}^{\delta_1}} N_1 \xrightarrow{\delta_1} R, \tag{3.1}$$

where $\mathrm{Kos}^{\delta_1} : \bigwedge_{\boldsymbol{k}}^2 N_1 \to N_1$ is the Koszul map associated to δ_1, $q : N_1 \otimes_{\boldsymbol{k}} N_1 \to \bigwedge_{\boldsymbol{k}}^2 N_1$ is the natural quotient map, and $\Delta_3(\theta^{(2)}) = \delta_2(\theta) \otimes \theta$ for $\theta \in N_2$.

By Remark 1.5, the above definition of Δ_3 does define Δ_3 on all of $D_2^{\boldsymbol{k}}(N_2)$. The Koszul map $\mathrm{Kos}^{\delta_1} : \bigwedge_{\boldsymbol{k}}^2 N_1 \to N_1$ is the map defined by $\mathrm{Kos}^{\delta_1}(m \wedge n) = \delta_1(m) \cdot n - \delta_1(n) \cdot m$ for $m, n \in N_1$; that is, the action of $\bigwedge_{\boldsymbol{k}}^\bullet N_1^*$ on $\bigwedge_{\boldsymbol{k}}^\bullet N_1$ arising from the homomorphism $\delta_1 : N_1 \to R$.

Proof of Lemma 3.1. The Tate technique of killing cycles shows that the complex $N_2 \xrightarrow{\delta_2} N_1 \xrightarrow{\delta_1} R$ induces a Differential Graded R-Algebra $\bigwedge_R^\bullet N_1 \otimes D_\bullet^R N_2$. We give a hands-on argument for the relevant part of this technique.

The fact that $\delta_1 \circ (\delta_2 \oplus \mathrm{Kos}^{\delta_1}) = 0$ is immediate by assumption and the definition of Koszul maps. Let $n_1 \in N_1$ and $n_2 \in N_2$. Then

$$\begin{aligned}
&\Big((\delta_2 \oplus \mathrm{Kos}^{\delta_1}) \circ (\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2))\Big)(n_1 \otimes n_2) \\
&= (\delta_2 \oplus \mathrm{Kos}^{\delta_1})\,(\delta_1(n_1) \cdot n_2 - n_1 \wedge \delta_2(n_2)) \\
&= \delta_2\,(\delta_1(n_1) \cdot n_2) - \mathrm{Kos}^{\delta_1}\,(n_1 \wedge \delta_2(n_2)) \\
&= \delta_1(n_1)\delta_2(n_2) - \delta_1(n_1)\delta_2(n_2) + \delta_1(\delta_2(n_2)) \cdot n_1 \\
&= 0
\end{aligned}$$

since $\delta_1 \circ \delta_2 = 0$ by assumption.

Finally, since all the modules in (3.1) are free R-modules, Lemma 1.3 from [19] implies that we need only show that $(\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)) \circ \Delta_3$ is zero on elements of the form $\theta^{(2)} \in D_2^{\boldsymbol{k}}(N_2)$ to show that $(\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)) \circ \Delta_3$ is identically zero. So, let $\theta \in N_2$. Then, since $\delta_1 \circ \delta_2 = 0$,

$$\begin{aligned}
((\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)) \circ \Delta_3)\,(\theta^{(2)}) &= (\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2))\,(\delta_2(\theta) \otimes \theta) \\
&= \delta_1(\delta_2(\theta)) \cdot \theta - \delta_2(\theta) \wedge \delta_2(\theta) \\
&= 0.
\end{aligned}$$

Thus, (3.1) is, in fact, a complex. $\square$

3.1 Set-Up

The following definitions give the basic structure we will use to define our complex, as well as the ring over which the complex will be defined.

(a) Let $\boldsymbol{k}$ be a field, S be a set of 4 variables, V be the vector space spanned by S, sf be the set of square free quadratic monomials in the variables S, ps be the set of perfect square quadratic monomials in the variables S, SF be the vector space spanned by sf, PS be the vector space spanned by ps. Observe that

$$\mathrm{Sym}_2^{\boldsymbol{k}}(V) = \mathrm{SF} \oplus \mathrm{PS}\,.$$

(b) Of course, $S = \{s|s \in S\}$ and $S^* = \{s^*|s \in S\}$ are dual bases for V and V^*.

(c) Let one : $V \to \boldsymbol{k}$ be defined by $\mathrm{one}(s) = 1$ for all $s \in S$. So, one $\in V^*$ and

$$\mathrm{one} = \sum_{s \in S} s^*.$$

(d) Observe that $\mathrm{one}^{(2)}$ is a well-defined element in $D_2^{\boldsymbol{k}}(V^*)$. Furthermore, $\mathrm{one}^{(2)}(m) = 1$ for all monomials m of degree two in S. Furthermore,

$$\mathrm{one}^{(2)} = \sum_{s_1 \neq s_2 \in S} s_1^* s_2^* + \sum_{s \in S} s^{*(2)}.$$

(e) The modules $D_2^{\boldsymbol{k}}(V^*)$ and $\mathrm{Sym}_2^{\boldsymbol{k}}(V)$ are dual to one another; hence, $\bigwedge_{\boldsymbol{k}}^{\bullet} D_2^{\boldsymbol{k}}(V^*)$ acts on $\bigwedge_{\boldsymbol{k}}^{\bullet} \mathrm{Sym}_2^{\boldsymbol{k}}(V)$. In particular, $\mathrm{one}^{(2)} : \bigwedge_{\boldsymbol{k}}^3 \mathrm{PS} \to \bigwedge_{\boldsymbol{k}}^2 \mathrm{PS}$ behaves as follows:

$$\mathrm{one}^{(2)}(s_1^2 \wedge s_2^2 \wedge s_3^2) = s_2^2 \wedge s_3^2 - s_1^2 \wedge s_3^2 + s_1^2 \wedge s_2^2,$$

for $s_1, s_2, s_3 \in S$.

(f) Let $Q = \mathrm{Sym}_{\bullet}^{\boldsymbol{k}}\left(\mathrm{SF} \oplus \frac{\bigwedge_{\boldsymbol{k}}^2 \mathrm{PS}}{(\mathrm{one}^{(2)}(\bigwedge_{\boldsymbol{k}}^3 \mathrm{PS}))}\right)$.

The complex that we will define in this chapter and the following be will a Q-module complex. Notice that Q is isomorphic to a polynomial ring in 9 variables.

(g) Let $\Omega \in \mathrm{Sym}_4^{\boldsymbol{k}}(V)$ be the product of the four elements of S.

(h) If $m \in \mathrm{sf}$, then let $\overline{m}$ be the unique element of sf with $m\overline{m} = \Omega$.

(i) Recall the definition of $\overline{m}$ for $m \in \mathrm{sf}$ as given in (h). Define

$$\overline{\mathrm{SF}} = \frac{\mathrm{SF}}{\langle\{m - \overline{m} \mid m \in \mathrm{sf}\}\rangle}.$$

If $m \in \mathrm{sf}$, then let $\langle m \rangle$ be the image of m in $\overline{\mathrm{SF}}$; in particular, for $m \in \mathrm{sf}$, $\langle m \rangle = \langle \overline{m} \rangle$. Furthermore, $\langle m \rangle$ is a coset which consists of the two elements m and $\overline{m}$ of sf. Observe that $\overline{\mathrm{SF}}$ is a vector space of dimension 3.

(j) Recall the element $\mathrm{one}^{(2)} \in D_2^{\boldsymbol{k}}(V^*)$. That is, $\mathrm{one}^{(2)}$ is a homomorphism

$$\mathrm{one}^{(2)} : \mathrm{Sym}_2^{\boldsymbol{k}}(V) \to \boldsymbol{k}.$$

Let $\mathrm{one}^{(2)}|_{\mathrm{SF}}$ be the restriction of $\mathrm{one}^{(2)}$ to SF. Observe that $\mathrm{one}^{(2)}|_{\mathrm{SF}} : \mathrm{SF} \to \boldsymbol{k}$ factors through $\mathrm{SF} \to \overline{\mathrm{SF}}$. Let $\overline{\mathrm{one}^{(2)}|_{\mathrm{SF}}} : \overline{\mathrm{SF}} \to \boldsymbol{k}$ be the induced map. Of course,

$$\overline{\mathrm{one}^{(2)}|_{\mathrm{SF}}} \in \overline{\mathrm{SF}}^*.$$

(k) Observe that $\langle xy \rangle$, $\langle xz \rangle$, $\langle xw \rangle$ is a basis for $\overline{\mathrm{SF}}$ if $S = \{x, y, z, w\}$. (One gets the same basis for any choice of names for the elements of S.) So, $\langle xy \rangle^*$, $\langle xz \rangle^*$, $\langle xw \rangle^*$ is the basis for $\overline{\mathrm{SF}}^*$ which is dual to the basis $\langle xy \rangle$, $\langle xz \rangle$, $\langle xw \rangle$ for $\overline{\mathrm{SF}}$. When $\overline{\mathrm{one}^{(2)}|_{\mathrm{SF}}}$ is written in terms of the basis $\langle xy \rangle^*$, $\langle xz \rangle^*$, $\langle xw \rangle^*$, one obtains

$$\overline{\mathrm{one}^{(2)}|_{\mathrm{SF}}} = \langle xy \rangle^* + \langle xz \rangle^* + \langle xw \rangle^* \in \overline{\mathrm{SF}}^*.$$

3.1.1 The $\boldsymbol{k}[\mathrm{Sym}(S)]$-module structure of the defined spaces

As stated previously, our complex will be a Q-module complex. In addition, the complex will also have the structure of $\boldsymbol{k}[\mathrm{Sym}(S)]$-module complex.

We can observe that the $\boldsymbol{k}$-vector spaces we have so far defined are $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules. The fact that V, SF, and PS are is clear. To see that $\overline{\mathrm{SF}}$ is as well, recall the defintion of $\overline{\mathrm{SF}}$: $\overline{\mathrm{SF}} = \dfrac{\mathrm{SF}}{\langle \{m - \overline{m} \mid m \in \mathrm{sf}\} \rangle}$. Let $S = \{s_1, s_2, s_3, s_4\}$. Then $\mathrm{sf} = \{s_1 s_2, s_3 s_4, s_1 s_3, s_2 s_4, s_1 s_4, s_2 s_3\}$. It is easy to check that each of the elements $(s_1 s_2)$ and $(s_1 s_2 s_3 s_4)$ of $\mathrm{Sym}(S)$ send every element of $\{m - \overline{m} \mid m \in \mathrm{sf}\}$ into $\{m - \overline{m} \mid m \in \mathrm{sf}\}$. Since $(s_1 s_2)$ and $(s_1 s_2 s_3 s_4)$ generate $\mathrm{Sym}(S)$, we have that

$$\sigma \cdot (m_1 - \overline{m_1}) \in \{m - \overline{m} \mid m \in \mathrm{sf}\}$$

for any $m_1 \in \mathrm{sf}$ and any $\sigma \in \mathrm{Sym}(S)$. It follows that $\overline{\mathrm{SF}}$ is a $\boldsymbol{k}[\mathrm{Sym}(S)]$-module. Since V, PS, and $\overline{\mathrm{SF}}$ are now $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules, so are $\bigwedge_{\boldsymbol{k}}^i V$, $\bigwedge_{\boldsymbol{k}}^i \overline{\mathrm{SF}}$, $\bigwedge_{\boldsymbol{k}}^i \mathrm{PS}$, $\mathrm{Sym}_i^{\boldsymbol{k}}(V)$, and $D_i^{\boldsymbol{k}}(V^*)$ for each $i > 0$.

For the remainder of the section, we consider the map one : $V \to \boldsymbol{k}$ and associated maps and modules. Both one $\in D_1^{\boldsymbol{k}}(V^*) = V^*$ and $\mathrm{one}^{(2)} \in D_2^{\boldsymbol{k}}(V^*)$ are $\mathrm{Sym}(S)$-equivariant: this follows from the facts that one is constant on S, $\mathrm{one}^{(2)}$ is constant on $\mathrm{ps} \cup \mathrm{sf}$, and $\mathrm{Sym}(S)$ acts trivially on elements of $\boldsymbol{k}$. Alternatively, writing one $= \sum\limits_{s \in S} s^*$ and $\mathrm{one}^{(2)} = \sum\limits_{s_1 \neq s_2 \in S} s_1^* s_2^* + \sum\limits_{s \in S} s^{*(2)}$ as in (c) and (d) of the previous section makes the $\mathrm{Sym}(S)$-equivariance clear.

Note that the decomposition $\mathrm{Sym}_2^{\boldsymbol{k}}(V) = \mathrm{PS} \oplus \mathrm{SF}$ is a decomposition not just of vector spaces, but also of $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules and that

$$D_2^{\boldsymbol{k}}(V^*) = \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{Sym}_2^{\boldsymbol{k}}(V), \boldsymbol{k}) = \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{PS} \oplus \mathrm{SF}, \boldsymbol{k}) = \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{PS}, \boldsymbol{k}) \oplus \mathrm{Hom}_{\boldsymbol{k}}(\mathrm{SF}, \boldsymbol{k}).$$

As a consequence, both $\mathrm{one}^{(2)}|_{\mathrm{PS}}$ and $\mathrm{one}^{(2)}|_{\mathrm{SF}}$ are $\mathrm{Sym}(S)$-equivariant. Additionally, $\overline{\mathrm{one}^{(2)}|_{\mathrm{SF}}} : \overline{\mathrm{SF}} \to \boldsymbol{k}$ is $\mathrm{Sym}(S)$-equivariant, which again follows easily from either of the facts that $\mathrm{one}^{(2)}|_{\mathrm{SF}}$ is constant on sf or from writing the map in the form of part (k) of the previous section.

Now, for any positive integer n, the Koszul maps $\bigwedge_{\boldsymbol{k}}^n \mathrm{PS} \to \bigwedge_{\boldsymbol{k}}^{n-1} \mathrm{PS}$, $\bigwedge_{\boldsymbol{k}}^n \mathrm{SF} \to \bigwedge_{\boldsymbol{k}}^{n-1} \mathrm{SF}$, and $\bigwedge_{\boldsymbol{k}}^n \overline{\mathrm{SF}} \to \bigwedge_{\boldsymbol{k}}^{n-1} \overline{\mathrm{SF}}$ induced by $\mathrm{one}^{(2)}|_{\mathrm{PS}}$, $\mathrm{one}^{(2)}|_{\mathrm{SF}}$, and $\overline{\mathrm{one}^{(2)}|_{\mathrm{SF}}}$, respectively, are $\mathrm{Sym}(S)$-equivariant. To verify this assertion, $m_1 \wedge m_2 \wedge \cdots \wedge m_n \in \bigwedge_{\boldsymbol{k}}^n \mathrm{SF}$ and $\sigma \in \mathrm{Sym}(S)$. Then

$$\begin{aligned}
&\mathrm{one}^{(2)}|_{\mathrm{SF}}(\sigma \cdot (m_1 \wedge m_2 \wedge \cdots \wedge m_n)) \\
&= \mathrm{one}^{(2)}|_{\mathrm{SF}}(\sigma \cdot m_1 \wedge \sigma \cdot m_2 \wedge \cdots \sigma \cdot m_n) \\
&= \sum_{i=1}^n (-1)^{i+1} \mathrm{one}^{(2)}(\sigma \cdot m_i) \sigma \cdot m_1 \wedge \cdots \wedge \sigma m_{i-1} \wedge \sigma \cdot m_{i+1} \wedge \cdots \wedge \sigma \cdot m_n \\
&= \sum_{i=1}^n (-1)^{i+1} \sigma \cdot \mathrm{one}^{(2)}(m_i) \sigma \cdot m_1 \wedge \cdots \wedge \sigma m_{i-1} \wedge \sigma \cdot m_{i+1} \wedge \cdots \wedge \sigma \cdot m_n \\
&= \sum_{i=1}^n (-1)^{i+1} \mathrm{one}^{(2)}(m_i) \sigma \cdot m_1 \wedge \cdots \wedge \sigma m_{i-1} \wedge \sigma \cdot m_{i+1} \wedge \cdots \wedge \sigma \cdot m_n
\end{aligned}$$

$$= \sigma \cdot \sum_{i=1}^{n} (-1)^{i+1} \operatorname{one}^{(2)}(m_i) m_1 \wedge \cdots m_{i-1} \wedge m_{i+1} \wedge \cdots \wedge m_n$$

$$= \sigma \cdot \operatorname{one}^{(2)}|_{\mathrm{SF}}(m_1 \wedge m_2 \wedge \cdots \wedge m_n).$$

The fourth equality above is due to the fact that $\mathrm{Sym}(S)$ acts trivially on $\boldsymbol{k}$ and $\operatorname{one}^{(2)}(m_i) \in \boldsymbol{k}$ for any $m_i \in \mathrm{SF}$. The argument for $\bigwedge^n \mathrm{PS}$ or $\bigwedge^n \overline{\mathrm{SF}}$ is the exactly the same.

Knowing now that SF, $\bigwedge_{\boldsymbol{k}}^2 \mathrm{PS}$ and $\bigwedge_{\boldsymbol{k}}^3 \mathrm{PS}$ are $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules and that the map the induced map $\operatorname{one}^{(2)} : \bigwedge_{\boldsymbol{k}}^3 \mathrm{PS} \to \bigwedge_{\boldsymbol{k}}^2 \mathrm{PS}$ is $\mathrm{Sym}(S)$-equivariant, we can conclude that $Q = \mathrm{Sym}_{\bullet}^{\boldsymbol{k}}\left(\mathrm{SF} \oplus \frac{\bigwedge_{\boldsymbol{k}}^2 \mathrm{PS}}{\left(\operatorname{one}^{(2)}(\bigwedge_{\boldsymbol{k}}^3 \mathrm{PS})\right)}\right)$ is $\boldsymbol{k}[\mathrm{Sym}(S)]$-module as well.

Although we have not yet mentioned the module $\dfrac{\bigwedge_{\boldsymbol{k}}^2 \overline{\mathrm{SF}}}{\left(\operatorname{one}^{(2)}|_{\mathrm{SF}}\left(\bigwedge_{\boldsymbol{k}}^3 \overline{\mathrm{SF}}\right)\right)}$, it will be used later, so we note now that the above shows that it is a module over $\boldsymbol{k}[\mathrm{Sym}(S)]$.

3.1.2 Preliminary Map Definitions

As the final bit of set-up before we define the first complex, we define a few homomorphisms that will be relevant.

(a) Recall that

$$\mathrm{Sym}_2^{\boldsymbol{k}}(V) = \mathrm{SF} \oplus \mathrm{PS}$$

and

$$\textstyle\bigwedge_{\boldsymbol{k}}^2 \mathrm{Sym}_2^{\boldsymbol{k}}(V) = \bigwedge_{\boldsymbol{k}}^2(\mathrm{SF} \oplus \mathrm{PS}) = \bigwedge_{\boldsymbol{k}}^2 \mathrm{SF} \oplus (\mathrm{SF} \otimes_{\boldsymbol{k}} \mathrm{PS}) \oplus \bigwedge_{\boldsymbol{k}}^2 \mathrm{PS}\,.$$

Let $T : \mathrm{Sym}_2^{\boldsymbol{k}} V \to Q$ be defined by

$$T|_{\mathrm{SF}} \text{ is the natural injection}$$

and

$$T|_{\mathrm{PS}} \text{ is the zero map.}$$

Define $U : \bigwedge_{\boldsymbol{k}}^2 \mathrm{Sym}_2^{\boldsymbol{k}} V \to Q$ by

$$U|_{\bigwedge_{\boldsymbol{k}}^2 \mathrm{SF} \oplus (\mathrm{SF} \otimes_{\boldsymbol{k}} \mathrm{PS})} \text{ is the zero map}$$

and

$$U|_{\bigwedge^2_{\boldsymbol{k}} \mathrm{PS}} \text{ is the natural quotient map.}$$

Observation 3.2. *Recall that* $Q = \mathrm{Sym}^{\boldsymbol{k}}_{\bullet}\left(\mathrm{SF} \oplus \frac{\bigwedge^2_{\boldsymbol{k}} \mathrm{PS}}{\left(\mathrm{one}^{(2)}(\bigwedge^3_{\boldsymbol{k}} \mathrm{PS})\right)}\right)$. *With the above definitions of* T *and* U*, we may now think of* Q *as the ring of polynomials in the variables* $\{T(s_i s_j)\}$ *and* $\{U(s_i^2 \wedge s_j^2)\}$ *for* s_i, s_j *distinct elements of* S *subject to: for any distinct* $s_i, s_j, s_k \in S$*,*

$$U(s_i^2 \wedge s_j^2) - U(s_i^2 \wedge s_k^2) + U(s_j^2 \wedge s_k^2) = 0. \tag{3.2}$$

(b) Define the Q-module augmentation map $\varepsilon : Q \to \mathrm{Sym}^{\boldsymbol{k}}_{\bullet} V$ by

$$\varepsilon(T(m)) = m \quad \text{for } m \in \mathrm{sf} \text{ and}$$
$$\varepsilon(U(x^2 \wedge y^2)) = \mathrm{one}^{(2)}(x^2 \wedge y^2) = y^2 - x^2 \quad \text{for } x, y \in S.$$

(c) Define $\alpha : \bigwedge^4_{\boldsymbol{k}} V \to \bigwedge^3_{\boldsymbol{k}} \overline{\mathrm{SF}}$ to be the $\boldsymbol{k}$-isomorphism such that

$$\alpha(s_1 \wedge s_2 \wedge s_3 \wedge s_4) = \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle.$$

We need to see that α makes sense. From subsection 3.1.1, we know that both $\bigwedge^4_{\boldsymbol{k}} V$ and $\bigwedge^3_{\boldsymbol{k}} \overline{\mathrm{SF}}$ are $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules.

Consider the action of the cycles $(s_1 s_2)$ and $(s_1 s_2 s_3 s_4)$:

$$\begin{aligned}
(s_1 s_2) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle &= \langle s_2 s_1 \rangle \wedge \langle s_2 s_3 \rangle \wedge \langle s_2 s_4 \rangle \\
&= \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \wedge \langle s_1 s_3 \rangle \\
&= -\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle
\end{aligned}$$

$$\begin{aligned}
(s_1 s_2 s_3 s_4) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle &= \langle s_2 s_3 \rangle \wedge \langle s_2 s_4 \rangle \wedge \langle s_2 s_1 \rangle \\
&= \langle s_1 s_4 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_2 \rangle \\
&= -\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle
\end{aligned}$$

Since $\{(s_1s_2), (s_1s_2s_3s_4)\}$ is a generating set for $\mathrm{Sym}(S)$, we conclude that, for any $\sigma \in \mathrm{Sym}(S)$,

$$\sigma \cdot \langle s_1s_2\rangle \wedge \langle s_1s_3\rangle \wedge \langle s_1s_4\rangle = \mathrm{sign}(\sigma)\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle \wedge \langle s_1s_4\rangle,$$

where $\mathrm{sign} : \mathrm{Sym}(S) \to \{\pm 1\}$ is the sign map. Hence, α is well-defined and $\mathrm{Sym}(S)$-equivariant.

(d) If M is a module with submodules A and B so that $M = A \oplus B$, let $\pi_A : M \to A$ be the projection of M onto A.

3.2 Module Definitions

Now we are ready to define the modules of our complex. Call the complex $X_\bullet$, which will be a Q-module complex

$$X_\bullet : Q \otimes_{\boldsymbol{k}} C_4 \to Q \otimes_{\boldsymbol{k}} C_3 \to Q \otimes_{\boldsymbol{k}} C_2 \to Q \otimes_{\boldsymbol{k}} C_1 \to Q.$$

(a) Define $C_1 = C_{1,1} \oplus C_{1,2} \oplus C_{1,3}$, where:

(i) $C_{1,1} = \dfrac{\bigwedge^2_{\boldsymbol{k}} \overline{\mathrm{SF}}}{\left(\overline{\mathrm{one}^{(2)}\left(\bigwedge^3_{\boldsymbol{k}} \overline{\mathrm{SF}}\right)}\right)}.$

We note the following for future reference:

$$\langle s_is_k\rangle \wedge \langle s_is_l\rangle - \langle s_is_j\rangle \wedge \langle s_is_l\rangle + \langle s_is_j\rangle \wedge \langle s_is_k\rangle = 0 \tag{3.3}$$

in $C_{1,1}$ for any labeling $S = \{s_i, s_j, s_k, s_l\}$.

(ii) $C_{1,2} = \bigwedge^2_{\boldsymbol{k}} V$.

(iii) $C_{1,3}$ is the subvector space of $\mathrm{Sym}_2^{\boldsymbol{k}}(V) = \boldsymbol{k}[S]_2$ spanned by

$$\{(s_1 - s_2)(s_3 - s_4) \mid \{s_1, s_2, s_3, s_4\} = S\}.$$

Notice that if $S = \{s_1, s_2, s_3, s_4\}$, then

$$(s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3) = 0. \tag{3.4}$$

Hence, $C_{1,3}$ can be generated by $\{(s_1 - s_2)(s_3 - s_4), (s_1 - s_3)(s_2 - s_4)\}$.

(b) Define $C_2 = C_{2,1} \oplus C_{2,2} \oplus \bigwedge_{\boldsymbol{k}}^2 C_1$, where

(i) $C_{2,1} = \dfrac{V}{\left\langle \sum\limits_{s \in S} s \right\rangle} \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^4 V$ and

(ii) $C_{2,2}$ is the subvector space of $\bigwedge_{\boldsymbol{k}}^2 V \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^2 V$ spanned by

$$\{s_1 \wedge s_2 \otimes s_3 \wedge s_4 \mid \{s_1, s_2, s_3, s_4\} = S\}\,.$$

(c) Define $C_3 = C_{3,1} \oplus (C_1 \otimes_{\boldsymbol{k}} (C_{2,1} \oplus C_{2,2}))$, where $C_{3,1} = \bigwedge_{\boldsymbol{k}}^2 V$.

(d) Define $C_4 = C_{4,1} \oplus C_{4,2} \oplus D_2^{\boldsymbol{k}}(C_{2,1} \oplus C_{2,2})$, where

(i) $C_{4,1} = \dfrac{V}{\left\langle \sum\limits_{s \in S} s \right\rangle}$ and

(ii) $C_{4,2} = \mathrm{SF}\,$.

Thus, $X_\bullet$ will look like Q tensored over $\boldsymbol{k}$ with

$$\begin{pmatrix} C_{4,1} \oplus C_{4,2} \\ \oplus \\ D_2^{\boldsymbol{k}}(C_{2,1} \oplus C_{2,2}) \end{pmatrix} \to \begin{pmatrix} C_{3,1} \\ \oplus \\ C_1 \otimes_{\boldsymbol{k}} (C_{2,1} \oplus C_{2,2}) \end{pmatrix} \to \begin{pmatrix} C_{2,1} \oplus C_{2,2} \\ \oplus \\ \bigwedge_{\boldsymbol{k}}^2 C_1 \end{pmatrix} \to C_1 \to \boldsymbol{k}$$

Notice that, if we ignore $C_{4,1} \oplus C_{4,2}$ and $C_{3,1}$, the above looks to be of the form (3.1).

Observation 3.3. *Since $C_1 = C_{1,1} \oplus C_{1,2} \oplus C_{1,3}$, there is an isomorphism*

$$\bigwedge_{\boldsymbol{k}}^2 C_1 \cong \left\{ \begin{matrix} \left(\bigwedge_{\boldsymbol{k}}^2 C_{1,1}\right) \oplus (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,2}) \oplus (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,3}) \\ \oplus \\ \left(\bigwedge_{\boldsymbol{k}}^2 C_{1,2}\right) \oplus (C_{1,2} \otimes_{\boldsymbol{k}} C_{1,3}) \oplus \left(\bigwedge_{\boldsymbol{k}}^2 C_{1,3}\right) \end{matrix} \right\}$$

given by

$$\begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \wedge \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \mapsto \begin{array}{c} \begin{bmatrix} a_1 \wedge b_1 \\ a_1 \otimes_{\boldsymbol{k}} b_2 - b_1 \otimes_{\boldsymbol{k}} a_2 \\ a_1 \otimes_{\boldsymbol{k}} b_3 - b_1 \otimes_{\boldsymbol{k}} a_3 \\ a_2 \wedge b_2 \\ a_2 \otimes_{\boldsymbol{k}} b_3 - b_2 \otimes_{\boldsymbol{k}} a_3 \\ a_3 \wedge b_3 \end{bmatrix} \end{array} \begin{array}{l} \in \wedge_{\boldsymbol{k}}^2 C_{1,1} \\ \in C_{1,1} \otimes_{\boldsymbol{k}} C_{1,2} \\ \in C_{1,1} \otimes_{\boldsymbol{k}} C_{1,3} \\ \in \wedge_{\boldsymbol{k}}^2 C_{1,2} \\ \in C_{1,2} \otimes_{\boldsymbol{k}} C_{1,3} \\ \in \wedge_{\boldsymbol{k}}^2 C_{1,3} \end{array} \tag{3.5}$$

We will use this isomorphism, writing elements of $\bigwedge_{\boldsymbol{k}}^2 C_1$ *as elements of* $\bigwedge_{\boldsymbol{k}}^2 C_{1,1} \oplus (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,2}) \oplus (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,3}) \oplus \bigwedge_{\boldsymbol{k}}^2 C_{1,2} \oplus (C_{1,2} \otimes_{\boldsymbol{k}} C_{1,3}) \oplus \bigwedge_{\boldsymbol{k}}^2 C_{1,3}$ *instead.*

3.2.1 The C_i modules are $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules

From section 1.3 and subsection 3.1.1, we know that $C_{1,1}$, $C_{1,2}$, $C_{1,3}$, $C_{2,1}$, $C_{3,1}$, $C_{4,1}$, and $C_{4,2}$ are $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules. Since $C_1 = C_{1,1} \oplus C_{1,2} \oplus C_{1,3}$, $C_2 = C_{2,1} \oplus C_{2,2} \oplus \bigwedge_{\boldsymbol{k}}^2 C_{1,1}$, $C_3 = C_{3,1} \oplus (C_1 \otimes_{\boldsymbol{k}} (C_{2,1} \oplus C_{2,2}))$, and $C_4 = C_{4,1} \oplus C_{4,1} \oplus D_2^{\boldsymbol{k}}(C_{2,1} \oplus C_{2,2})$, to show that C_1, C_2, C_3, and C_4 are modules over $\boldsymbol{k}[\mathrm{Sym}(S)]$, it suffices to show that $C_{2,2}$ is such.

Let $S = \{s_1, s_2, s_3, s_4\}$. Remember that $C_{2,2}$ is the subvector space of $\bigwedge_{\boldsymbol{k}}^2 V \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^2 V$ spanned by

$$\{s_i \wedge s_j \otimes s_k \wedge s_l \mid \{i, j, k, l\} = \{1, 2, 3, 4\}\}.$$

Any permutation of S will clearly send an element of this set to either an element of the set or the negative of an element of the set, so $C_{2,2}$ is a $\boldsymbol{k}[\mathrm{Sym}(S)]$-module.

Now we want to investigate how the modules we have are related to indecomposable $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules. Using the same notation for τ_i and $\tau_{2,2}$ as in subsection 1.3, computation of the characters gives:

	id	τ_2	τ_3	τ_4	$\tau_{2,2}$
$\bigwedge_{\boldsymbol{k}}^0 V$	1	1	1	1	1
$\bigwedge_{\boldsymbol{k}}^4 V$	1	-1	1	-1	1
$C_{4,1} = V/\langle\sum s\rangle$	3	1	0	-1	-1
$C_{2,1} = V/\langle\sum s\rangle \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^4 V$	3	-1	0	1	-1
$C_{1,3} = \langle (s_1 - s_2)(s_3 - s_4) \mid S = \{s_1, s_2, s_3, s_4\}\rangle$	2	0	-1	0	2
$C_{1,1}$	2	0	-1	0	2
$C_{1,2}$	6	0	0	0	-2
$C_{2,2}$	6	-2	0	0	2
$C_{3,1}$	6	0	0	0	-2
$C_{4,2}$	6	2	0	0	2
$\bigwedge_{\boldsymbol{k}}^2 C_{1,2}$	15	-3	0	1	-1.

Using the above and Facts 1.9 from subsection 1.3, we can compute the following:

	id	τ_2	τ_3	τ_4	$\tau_{2,2}$
$C_1 \otimes_{\boldsymbol{k}} C_{2,1}$	30	0	0	0	-2
$C_1 \otimes_{\boldsymbol{k}} C_{2,2}$	60	0	0	0	4.

Since there is an isomorphism $D_2(C_{2,1} \oplus C_{2,2}) \cong D_2(C_{2,1}) \oplus (C_{2,1} \otimes_{\boldsymbol{k}} C_{2,2}) \oplus D_2(C_{2,2})$, we find the characters for the summands:

	id	τ_2	τ_3	τ_4	$\tau_{2,2}$
$D_2(C_{2,1})$	6	2	0	0	2
$C_{2,1} \otimes_{\boldsymbol{k}} C_{2,2}$	18	2	0	0	-2
$D_2(C_{2,2})$	21	5	0	1	5

Therefore, we can write each of the the C_i modules as direct sums of indecomposable $\boldsymbol{k}[\mathrm{Sym}(S)]$-modules:

$$C_1 \cong \frac{V}{\langle \sum_{s\in S} s\rangle} \oplus \left[\frac{V}{\langle \sum_{s\in S} s\rangle} \otimes_{\boldsymbol{k}} \bigwedge\nolimits_{\boldsymbol{k}}^4 V \right] \oplus C_{1,3}^2$$

$$C_2 \cong \left[\bigwedge\nolimits_{\boldsymbol{k}}^{4} V\right]^2 \oplus \left[\frac{V}{\langle \sum\limits_{s\in S} s\rangle} \otimes_{\boldsymbol{k}} \bigwedge\nolimits_{\boldsymbol{k}}^{4} V\right]^2 \oplus C_{1,3}$$

$$C_3 \cong \left[\bigwedge\nolimits_{\boldsymbol{k}}^{0} V\right]^4 \oplus \left[\bigwedge\nolimits_{\boldsymbol{k}}^{4} V\right]^4 \oplus \left[\frac{V}{\langle \sum\limits_{s\in S} s\rangle}\right]^{12} \oplus \left[\frac{V}{\langle \sum\limits_{s\in S} s\rangle} \otimes_{\boldsymbol{k}} \bigwedge\nolimits_{\boldsymbol{k}}^{4} V\right]^{12} \oplus C_{1,3}^{8}$$

$$C_4 \cong \left[\bigwedge\nolimits_{\boldsymbol{k}}^{0} V\right]^6 \oplus \left[\frac{V}{\langle \sum\limits_{s\in S} s\rangle}\right]^{9} \oplus \left[\frac{V}{\langle \sum\limits_{s\in S} s\rangle} \otimes_{\boldsymbol{k}} \bigwedge\nolimits_{\boldsymbol{k}}^{4} V\right]^{3} \oplus C_{1,3}^{6}.$$

3.3 Definitions of the Differential Maps

We now will define d_1, d_2, d_3, d_4 for $X_\bullet$:

$$X_\bullet : Q \otimes_{\boldsymbol{k}} C_4 \xrightarrow{d_4} Q \otimes_{\boldsymbol{k}} C_3 \xrightarrow{d_3} Q \otimes_{\boldsymbol{k}} C_2 \xrightarrow{d_2} Q \otimes_{\boldsymbol{k}} C_1 \xrightarrow{d_1} Q.$$

We noted before that we can consider Q the polynomial ring in the variables $T(s_i s_j)$ and $U(s_i^2 \wedge s_j^2)$ for s_i, s_j distinct elements of S. In the modules $Q \otimes_{\boldsymbol{k}} C_i$ for $i \in \{1, 2, 3, 4\}$, we will think of elements of Q as being coefficients of elements of C_i , so we will use muliplicative "$\cdot$" or juxtaposition notation between elements of Q and elements of $C_\bullet$, as well as identifying $\gamma = 1 \cdot \gamma = 1 \otimes \gamma \in Q \otimes C_i$ for an element $\gamma \in C_i$, $i \in \{1, 2, 3, 4\}$.

The complex $X_\bullet$ will have a subcomplex of the form (3.1) if we remove the $C_{4,1} \oplus C_{4,2}$ summands of C_4 and the $C_{3,1}$ summand of C_3. We will refer to the subsequence of maps and modules of the form (3.1) as the "Tate portion" of our sequence of maps and modules. In order to make the homomorphisms below more understandable, we define many of the maps in pieces, giving the definition on and to various summands of the C_i modules.

(a) Define $d_1 : Q \otimes_{\boldsymbol{k}} C_1 \to Q$ to be the Q-module map such that:

1.) For $m_1, m_2 \in \mathrm{sf}$, $d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}} : Q \otimes_{\boldsymbol{k}} C_{1,1} \to Q$ is

$$d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}(\langle m_1\rangle \wedge \langle m_2\rangle) = T(m_1)T(\overline{m_1}) - T(m_2)T(\overline{m_2}).$$

The facts that the map above is antisymmetric in m_1 and m_2 and well-defined for $\langle m_1 \rangle = \langle \overline{m_1} \rangle$ and $\langle m_2 \rangle = \langle \overline{m_2} \rangle$ are immediate. Applying the map to an expression of the form $\langle s_i s_k \rangle \wedge \langle s_i s_j \rangle - \langle s_i s_j \rangle \wedge \langle s_i s_l \rangle + \langle s_i s_j \rangle \wedge \langle s_i s_k \rangle$ for $\{s_i, s_j, s_k, s_l\} = S$ (see formula (3.3)) does give zero, so the above is a well-defined map on $Q \otimes_{\boldsymbol{k}} C_{1,1} = Q \otimes_{\boldsymbol{k}} \dfrac{\bigwedge^2_{\boldsymbol{k}} \overline{\mathrm{SF}}}{\left(\overline{\mathrm{one}^{(2)}}\left(\bigwedge^3_{\boldsymbol{k}} \overline{\mathrm{SF}}\right)\right)}$. See Proposition 3.4 for the proof that the above is $\boldsymbol{k}[\mathrm{Sym}(S)]$-equivariant.

2.) For $\{s_1, s_2, s_3, s_4\} = S$, $d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,2}} : Q \otimes_{\boldsymbol{k}} C_{1,1} \to Q$ is

$$d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,2}}(s_1 \wedge s_2) = T(s_3 s_4) U(s_1^2 \wedge s_2^2) + T(s_1 s_3) T(s_1 s_4) - T(s_2 s_3) T(s_2 s_4).$$

The definition is antisymmetric in s_1 and s_2 and symmetric in s_3 and s_4, and is therefore well-defined on $Q \otimes_{\boldsymbol{k}} C_{1,2} = Q \otimes_{\boldsymbol{k}} \bigwedge^2_{\boldsymbol{k}} V$. The $\mathrm{Sym}(S)$-equivariance is clear.

3.) For $\{s_1, s_2, s_3, s_4\} = S$, $d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,3}} : Q \otimes_{\boldsymbol{k}} C_{1,1} \to Q$ is

$$d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,3}}((s_1 - s_2)(s_3 - s_4)) = \begin{cases} +U(s_1^2 \wedge s_2^2) U(s_3^2 \wedge s_4^2) \\ -T(s_1 s_3)^2 + T(s_1 s_4)^2 \\ +T(s_2 s_3)^2 - T(s_2 s_4)^2. \end{cases}$$

The definition above satisfies

$$d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,3}}((s_1 - s_2)(s_3 - s_4)) = -d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,3}}((s_2 - s_1)(s_3 - s_4)),$$

$$d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,3}}((s_1 - s_2)(s_3 - s_4)) = -d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,3}}((s_1 - s_2)(s_4 - s_3)),$$

and

$$d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,3}}((s_1 - s_2)(s_3 - s_4)) = d_1|_{Q \otimes_{\boldsymbol{k}} C_{1,3}}((s_3 - s_4)(s_1 - s_2)).$$

The fact that the map is zero on the expression

$$(s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3)$$

is proven in Proposition 3.4. Therefore, $d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,3}}$ is well-defined. The $\mathrm{Sym}(S)$-equivariance is easy to check.

(b) Define $d_2 : Q \otimes_{\boldsymbol{k}} C_2 \to Q \otimes_{\boldsymbol{k}} C_1$ to be the Q-module homomorphism described below.

1.) For $s \in S$ and $\omega \in \bigwedge^4 V$, there is a basis $\{t \otimes w \mid t \in S\}$ of $C_{2,1}$. Define $d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,1}}$ on this basis by the following and then extend the definitions linearly to get a homomorphism.

$$\pi_{Q\otimes_{\boldsymbol{k}} C_{1,1}} \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,1}}(s \otimes \omega) = \sum_{t\in S} U(t^2 \wedge s^2) \cdot \langle st\rangle^*(\alpha(\omega)),$$

$$\pi_{Q\otimes_{\boldsymbol{k}} C_{1,2}} \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,1}}(s \otimes \omega) = \sum_{t\in S} T(\overline{ts}) \cdot (t^* \wedge s^*)(\omega),$$

and

$$\pi_{Q\otimes_{\boldsymbol{k}} C_{1,3}} \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,1}} = 0.$$

Notice that the above do make sense, as if $t = s$, then $U(t^2 \wedge s^2) = 0$ and $t^* \wedge s^* = 0$, so that we do not use any $\langle st\rangle$ or $\overline{st}$ with $st \in \mathrm{ps}$, which we have not defined. The fact that

$$\begin{aligned}
&\sum_{s\in S} \pi_{Q\otimes_{\boldsymbol{k}} C_{1,1}} \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,1}}(s \otimes \omega) \\
&= \sum_{s\in S}\sum_{t\in S} U(t^2 \wedge s^2) \cdot \langle st\rangle^*(\alpha(\omega)) \\
&= \sum_{\{s,t\}\subset S} \Big(U(t^2 \wedge s^2) \cdot \langle st\rangle^*(\alpha(\omega)) + U(s^2 \wedge t^2) \cdot \langle ts\rangle^*(\alpha(\omega))\Big) \\
&= 0
\end{aligned}$$

shows that $\pi_{Q\otimes_{\boldsymbol{k}} C_{1,1}} \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,1}}$ is well-defined. The argument for $\pi_{Q\otimes_{\boldsymbol{k}} C_{1,2}} \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,1}}$ is similar. It is again quickly seen that the definition is $\mathrm{Sym}(S)$-equivariant.

2.) For $\{s_1, s_2, s_3, s_4\} = S$, $d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,2}} : Q \otimes_{\boldsymbol{k}} C_{2,2} \to Q \otimes_{\boldsymbol{k}} C_{1,1}$ is defined on the basis $\{s_i \wedge s_j \otimes s_k \wedge s_l | \{i,j,k,l\} = \{1,2,3,4\}, i < j, k < l\}$ of $C_{2,2}$ by

$$\pi_{Q\otimes_{\boldsymbol{k}} C_{1,1}} \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,2}}(s_1 \wedge s_2 \otimes s_3 \wedge s_4) = -2T(s_3 s_4) \cdot \langle s_1 s_4 \rangle \wedge \langle s_1 s_3 \rangle.$$

The antisymmetry in s_1 and s_2 and in s_3 and s_4 follows from the fact that $\langle m \rangle = \langle \overline{m} \rangle$ in $\overline{\mathrm{SF}}$. Otherwise, well-definedness and $\mathrm{Sym}(S)$-equivariance is easily seen.

3.) For $\{s_1, s_2, s_3, s_4\} = S$, define $d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,2}} : Q \otimes_{\boldsymbol{k}} C_{2,2} \to Q \otimes_{\boldsymbol{k}} C_{1,2}$ by

$$\pi_{Q\otimes_{\boldsymbol{k}} C_{1,2}} \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,2}}(s_1 \wedge s_2 \otimes s_3 \wedge s_4) = \begin{cases} -U(s_1^2 \wedge s_2^2) \cdot s_3 \wedge s_4 \\ -T(s_1 s_4) \cdot s_1 \wedge s_3 + T(s_1 s_3) \cdot s_1 \wedge s_4 \\ +T(s_2 s_4) \cdot s_2 \wedge s_3 - T(s_2 s_3) \cdot s_2 \wedge s_4. \end{cases}$$

Again, the required antisymmetry and $\mathrm{Sym}(S)$-equivariance is clear and the definition above is the definition on the basis $\{s_i \wedge s_j \otimes s_k \wedge s_l | \{i,j,k,l\} = \{1,2,3,4\}, i < j, k < l\}$ of $C_{2,2}$ and then extended linearly.

4.) For $\{s_1, s_2, s_3, s_4\} = S$, $d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,2}} : Q \otimes_{\boldsymbol{k}} C_{2,2} \to Q \otimes_{\boldsymbol{k}} C_{1,3}$ is

$$\pi_{Q\otimes_{\boldsymbol{k}} C_{1,3}} \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,2}}(s_1 \wedge s_2 \otimes s_3 \wedge s_4) = T(s_1 s_2) \cdot (s_1 - s_2)(s_3 - s_4).$$

The definition above is well-defined and equivariant with respect to the $\mathrm{Sym}(S)$ action.

5.) On $Q \otimes_{\boldsymbol{k}} \bigwedge^2 C_1$, define

$$d_2|_{Q\otimes_{\boldsymbol{k}} \bigwedge^2 C_1} = \mathrm{Kos}^{d_1},$$

the Koszul map $\bigwedge^2 C_1 \to C_1$ induced by d_1. This is part of the Tate portion of our sequence of maps and modules. Since d_1 is $\mathrm{Sym}(S)$-equivariant, so is Kos^{d_1}.

(c) Define $d_3 : Q \otimes_{\boldsymbol{k}} C_3 \to Q \otimes_{\boldsymbol{k}} C_2$ to be the Q-module homomorphism that is given below. For parts c1) through c5), the $\mathrm{Sym}(S)$-equivariance, the antisymmetry in s_1 and s_2, and the symmetry in s_3 and s_4, are clear, so those maps are $\mathrm{Sym}(S)$-equivariant and well-defined on $Q \otimes_{\boldsymbol{k}} C_3 = Q \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^2 V$.

1.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}} : Q \otimes_{\boldsymbol{k}} C_{3,1} \to Q \otimes_{\boldsymbol{k}} C_{2,1}$ is defined by

$$\pi_{Q \otimes_{\boldsymbol{k}} C_{2,1}} \circ d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}}(s_1 \wedge s_2) = 2T(s_3 s_4) \cdot (s_3 - s_4) \otimes s_1 \wedge s_2 \wedge s_3 \wedge s_4.$$

2.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}} : Q \otimes_{\boldsymbol{k}} C_{3,1} \to Q \otimes_{\boldsymbol{k}} C_{2,2}$ is defined by

$$\pi_{Q \otimes_{\boldsymbol{k}} C_{2,2}} \circ d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}}(s_1 \wedge s_2) = \begin{cases} +U(s_3^2 \wedge s_4^2) \cdot (s_1 \wedge s_2 \otimes s_3 \wedge s_4) \\ +T(s_2 s_3) \cdot (s_1 \wedge s_3 \otimes s_2 \wedge s_4) \\ +T(s_2 s_4) \cdot (s_1 \wedge s_4 \otimes s_2 \wedge s_3) \\ -T(s_1 s_3) \cdot (s_2 \wedge s_3 \otimes s_1 \wedge s_4) \\ -T(s_1 s_4) \cdot (s_2 \wedge s_4 \otimes s_1 \wedge s_3). \end{cases}$$

3.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}} : Q \otimes_{\boldsymbol{k}} C_{3,1} \to Q \otimes_{\boldsymbol{k}} C_{1,1} \otimes_{\boldsymbol{k}} C_{1,2}$ is defined by

$$\pi_{Q \otimes_{\boldsymbol{k}} (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,2})} \circ d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}}(s_1 \wedge s_2) = \begin{cases} -2\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \otimes s_1 \wedge s_2 \\ -2\langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \otimes s_1 \wedge s_2. \end{cases}$$

4.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}} : Q \otimes_{\boldsymbol{k}} C_{3,1} \to Q \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^2 C_{1,2}$ is defined by

$$\pi_{Q \otimes_{\boldsymbol{k}} \bigwedge^2 C_{1,2}} \circ d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}}(s_1 \wedge s_2) = \begin{cases} +(s_1 \wedge s_3) \wedge (s_2 \wedge s_3) \\ +(s_1 \wedge s_4) \wedge (s_2 \wedge s_4). \end{cases}$$

5.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}} : Q \otimes_{\boldsymbol{k}} C_{3,1} \to Q \otimes_{\boldsymbol{k}} C_{1,2} \otimes_{\boldsymbol{k}} C_{1,3}$ is defined by

$$\pi_{Q \otimes_{\boldsymbol{k}} (C_{1,2} \otimes_{\boldsymbol{k}} C_{1,3})} \circ d_3|_{Q \otimes_{\boldsymbol{k}} C_{3,1}}(s_1 \wedge s_2) = -(s_3 \wedge s_4) \cdot (s_1 - s_2)(s_3 - s_4).$$

6.) The images of $d_3|_{Q\otimes_{\boldsymbol{k}} C_{3,1}}$ in the $Q\otimes_{\boldsymbol{k}}\bigwedge_{\boldsymbol{k}}^2 C_{1,1}$, $Q\otimes_{\boldsymbol{k}} C_{1,1}\otimes_{\boldsymbol{k}} C_{1,3}$, and $Q\otimes_{\boldsymbol{k}}\bigwedge_{\boldsymbol{k}}^2 C_{1,3}$ summands of $Q \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^2 C_1$ are all zero.

7.) Finally, $d_3|_{Q\otimes_{\boldsymbol{k}}(C_1\otimes(C_{2,1}\oplus C_{2,2}))} : Q \otimes_{\boldsymbol{k}} (C_1 \otimes (C_{2,1} \oplus C_{2,2})) \to Q \otimes_{\boldsymbol{k}} C_2$ is the map

$$d_1 \otimes 1 - \operatorname{quot} \circ (1 \otimes d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,1}\oplus C_{2,2}}),$$

where $\operatorname{quot} : C_1 \otimes_{\boldsymbol{k}} C_1 \to \bigwedge_{\boldsymbol{k}}^2 C_1$ is the natural quotient map. This piece of d_3 is part of the Tate portion. It is $\operatorname{Sym}(S)$-equivariant by the $\operatorname{Sym}(S)$-equivariance of d_1, d_2, and quot.

(d) Define $d_4 : Q \otimes_{\boldsymbol{k}} C_4 \to Q \otimes_{\boldsymbol{k}} C_3$ by:

1.) For $s \in S$ define $d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}} : Q \otimes_{\boldsymbol{k}} C_{4,1} \to Q \otimes_{\boldsymbol{k}} C_{3,1}$ by

$$\pi_{Q\otimes_{\boldsymbol{k}} C_{3,1}} \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}(s) = \sum_{t\in S} T(st) \cdot t \wedge s.$$

Extending the definition linearly gives a homomorphism on $Q \otimes_{\boldsymbol{k}} V$. It gives a map on $Q \otimes_{\boldsymbol{k}} C_4 = Q \otimes_{\boldsymbol{k}} V/\langle \sum_{t\in S} t\rangle$ since

$$\begin{aligned}
\pi_{Q\otimes_{\boldsymbol{k}} C_{3,1}} \left(\sum_{s\in S} d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}(s) \right) &= \sum_{s\in S}\sum_{t\in S} T(st) \cdot t \wedge s \\
&= \sum_{\{s,t\}\subseteq S} \Big(T(st) \cdot t \wedge s + T(st) \cdot s \wedge t \Big) \\
&= 0.
\end{aligned}$$

The equivariance with respect to $\operatorname{Sym}(S)$ is easy to check. The fact that $d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}$ is independent of a particular enumeration of $S \setminus \{s\}$, see Proposition 3.4.

2.) For $s \in S$ and $S = \{s, s_1, s_2, s_3\}$, $d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}} : Q \otimes_{\boldsymbol{k}} C_{4,1} \to Q \otimes_{\boldsymbol{k}} C_{1,1} \otimes_{\boldsymbol{k}} C_{2,1}$ is

$$\pi_{Q\otimes_{\boldsymbol{k}} C_{1,1}\otimes C_{2,1}} \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}(s) = \begin{cases} -2(\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle) \otimes (s_1 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s) \\ +2(\langle s_1s_2\rangle \wedge \langle s_2s_3\rangle) \otimes (s_2 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s) \\ -2(\langle s_1s_3\rangle \wedge \langle s_2s_3\rangle) \otimes (s_3 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s). \end{cases}$$

The above is $\mathrm{Sym}(S)$-equivariant by inspection. See Proposition 3.4 to see that it is well-defined on $Q \otimes_{\boldsymbol{k}} C_4 = Q \otimes_{\boldsymbol{k}} V/\langle \sum_{t \in S} t \rangle$.

3.) For $s \in S$ and $S = \{s, s_1, s_2, s_3\}$, define $d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,1}} : Q \otimes_{\boldsymbol{k}} C_{4,1} \to Q \otimes_{\boldsymbol{k}} C_{1,2} \otimes_{\boldsymbol{k}} C_{2,2}$ by

$$\pi_{Q \otimes_{\boldsymbol{k}} C_{1,2} \otimes C_{2,2}} \circ d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,1}}(s) = \begin{cases} +(s_1 \wedge s_2) \otimes (s \wedge s_3 \otimes s_1 \wedge s_2) \\ +(s_1 \wedge s_3) \otimes (s \wedge s_2 \otimes s_1 \wedge s_3) \\ +(s_2 \wedge s_3) \otimes (s \wedge s_1 \otimes s_2 \wedge s_3). \end{cases}$$

Again, we leave the proof that the map is well-defined on $Q \otimes_{\boldsymbol{k}} C_4 = Q \otimes_{\boldsymbol{k}} V/\langle \sum_{t \in S} t \rangle$ to Proposition 3.4 and claim that $\mathrm{Sym}(S)$-equivariance is clear.

4.) The images of $d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,1}}$ in the summands $Q \otimes_{\boldsymbol{k}} C_{1,2} \otimes_{\boldsymbol{k}} C_{2,1}$, $Q \otimes_{\boldsymbol{k}} C_{1,3} \otimes_{\boldsymbol{k}} C_{2,1}$, $Q \otimes_{\boldsymbol{k}} C_{1,1} \otimes_{\boldsymbol{k}} C_{2,2}$, and $Q \otimes_{\boldsymbol{k}} C_{1,3} \otimes_{\boldsymbol{k}} C_{2,2}$ are all zero.

5.) For $s_1 s_2 \in \mathrm{sf}$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,2}} : Q \otimes_{\boldsymbol{k}} C_{4,2} \to Q \otimes_{\boldsymbol{k}} C_{3,1}$ is defined by

$$\pi_{Q \otimes_{\boldsymbol{k}} C_{3,1}} \circ d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,2}}(s_1 s_2) = \begin{cases} +T(s_1 s_3) \cdot s_1 \wedge s_4 + T(s_1 s_4) \otimes s_1 \wedge s_3 \\ +T(s_2 s_3) \cdot s_2 \wedge s_4 + T(s_2 s_4) \otimes s_2 \wedge s_3 \\ -U(s_3^2 \wedge s_4^2) \cdot s_3 \wedge s_4. \end{cases}$$

Then extend this definition from the basis sf of SF to all of $Q \otimes_{\boldsymbol{k}} C_{4,2} = Q \otimes_{\boldsymbol{k}} \mathrm{SF}$. The definition is symmetric in s_1 and s_2, as well as in s_3 and s_4, so it is well-defined on sf and is independent of enumeration of $S \setminus \{s_1, s_2\}$, so it is well-defined. The fact that the map is $\mathrm{Sym}(S)$-equivariant is obvious.

6.) For $s_1 s_2 \in \mathrm{sf}$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,2}} : Q \otimes_{\boldsymbol{k}} C_{4,2} \to Q \otimes_{\boldsymbol{k}} C_{1,1} \otimes_{\boldsymbol{k}} C_{2,2}$ is defined to be

$$\pi_{Q \otimes_{\boldsymbol{k}} C_{1,1} \otimes_{\boldsymbol{k}} C_{2,2}} \circ d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,2}}(s_1 s_2) = 2(\langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \otimes (s_1 \wedge s_2 \otimes s_3 \wedge s_4).$$

Again, the map is well-defined and $\mathrm{Sym}(S)$-equivariant, as long as one recalls that $\langle m\rangle = \langle\overline{m}\rangle$ in $\overline{\mathrm{SF}}$. We claim that the $\mathrm{Sym}(S)$-equivariance and well-definedness is also obvious for parts d7) through d9) below.

7.) For $s_1s_2 \in \mathrm{sf}$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}} : Q\otimes_{\boldsymbol{k}}C_{4,2} \to Q\otimes_{\boldsymbol{k}}C_{1,2}\otimes_{\boldsymbol{k}}C_{2,1}$ is defined to be

$$\pi_{Q\otimes_{\boldsymbol{k}}C_{1,2}\otimes_{\boldsymbol{k}}C_{2,1}} \circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}}(s_1s_2) = \begin{cases} +2(s_3\wedge s_4)\otimes s_1\otimes(s_1\wedge s_2\wedge s_3\wedge s_4) \\ -2(s_3\wedge s_4)\otimes s_2\otimes(s_1\wedge s_2\wedge s_3\wedge s_4). \end{cases}$$

8.) For $s_1s_2 \in \mathrm{sf}$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}} : Q\otimes_{\boldsymbol{k}}C_{4,2} \to Q\otimes_{\boldsymbol{k}}C_{1,2}\otimes_{\boldsymbol{k}}C_{2,2}$ is defined to be

$$\pi_{Q\otimes_{\boldsymbol{k}}C_{1,2}\otimes_{\boldsymbol{k}}C_{2,2}} \circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}}(s_1s_2) = \begin{cases} -(s_1\wedge s_3)\otimes(s_2\wedge s_3\otimes s_1\wedge s_4) \\ -(s_2\wedge s_3)\otimes(s_1\wedge s_3\otimes s_2\wedge s_4) \\ -(s_1\wedge s_4)\otimes(s_2\wedge s_4\otimes s_1\wedge s_3) \\ -(s_2\wedge s_4)\otimes(s_1\wedge s_4\otimes s_2\wedge s_3). \end{cases}$$

9.) For $s_1s_2 \in \mathrm{sf}$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}} : Q\otimes_{\boldsymbol{k}}C_{4,2} \to Q\otimes_{\boldsymbol{k}}C_{1,3}\otimes_{\boldsymbol{k}}C_{2,2}$ is defined to be

$$\pi_{Q\otimes_{\boldsymbol{k}}C_{1,3}\otimes_{\boldsymbol{k}}C_{2,2}} \circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}}(s_1s_2) = (s_1-s_2)(s_3-s_4)\otimes(s_3\wedge s_4\otimes s_1\wedge s_2).$$

10.) The images of $d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}}$ in the summands $Q\otimes_{\boldsymbol{k}}C_{1,1}\otimes_{\boldsymbol{k}}C_{2,1}$ and $Q\otimes_{\boldsymbol{k}}C_{1,3}\otimes_{\boldsymbol{k}}C_{2,1}$ are both zero.

11.) For $\theta \in C_{2,1}\oplus C_{2,2}$,

$$d_4|_{Q\otimes_{\boldsymbol{k}}D_2(C_{2,1}\oplus C_{2,2})}(\theta^{(2)}) = d_2|_{Q\otimes_{\boldsymbol{k}}C_{2,1}\oplus C_{2,2}}(\theta)\otimes\theta.$$

This map is part of the Tate portion and, by Remark 1.5, this is sufficient to understand the map on all of $Q\otimes_{\boldsymbol{k}} D_2(C_{2,1}\oplus C_{2,2})$.

Now, we fill in the remaining details of showing that the homomorphisms d_1, d_2, d_3, and d_4 are well-defined and $\mathrm{Sym}(S)$-equivariant.

Proposition 3.4. *With maps as previously defined, we have the following three facts.*

1. *The map $d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}$ is $\mathrm{Sym}(S)$-equivariant.*
2. *The element $(s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3)$ is in the kernel of $d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,3}}$.*
3. *The map $d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}$ is zero on the $\boldsymbol{k}$-subvector space $\left\langle \sum_{t\in S} t \right\rangle$ of V and the definition of $d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}(s)$ for $s \in S$ does not depend on a particular enumeration of $S \setminus \{s\}$.*

Proof. First, we show that $d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}} : Q \otimes_{\boldsymbol{k}} C_{1,1} \to Q$ is $\mathrm{Sym}(S)$-equivariant. Let $S = \{s_1, s_2, s_3, s_4\}$. The elements $\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle$ and $\langle s_1s_2\rangle \wedge \langle s_1s_4\rangle$ generate $C_{1,1}$ and (s_1s_2) and $(s_1s_2s_3s_4)$ generate $\mathrm{Sym}(S)$. It is sufficient to check that the action of (s_1s_2) and $(s_1s_2s_3s_4)$ commute with applying $d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}$ to each of $\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle$ and $\langle s_1s_2\rangle \wedge \langle s_1s_4\rangle$:

$$\begin{aligned}
&d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}((s_1s_2)\cdot(\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle))\\
&= d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}(\langle s_1s_2\rangle \wedge \langle s_2s_3\rangle)\\
&= T(s_1s_2)T(s_3s_4) - T(s_2s_3)T(s_1s_4)\\
&= (s_1s_2)\cdot(T(s_1s_2)T(s_3s_4) - T(s_1s_2)T(s_2s_4))\\
&= (s_1s_2)\cdot d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}(\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle).
\end{aligned}$$

$$\begin{aligned}
&d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}((s_1s_2s_3s_4)\cdot(\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle))\\
&= d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}(\langle s_2s_3\rangle \wedge \langle s_2s_4\rangle)\\
&= T(s_2s_3)T(s_1s_4) - T(s_2s_4)T(s_1s_3))\\
&= (s_1s_2s_3s_4)\cdot(T(s_1s_2)T(s_3s_4) - T(s_1s_3)T(s_2s_4))
\end{aligned}$$

$$= (s_1s_2s_3s_4) \cdot d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}(\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle).$$

$$\begin{aligned}
&d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}((s_1s_2) \cdot (\langle s_1s_2\rangle \wedge \langle s_1s_4\rangle)) \\
&= d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}(\langle s_1s_2\rangle \wedge \langle s_2s_4\rangle) \\
&= T(s_1s_2)T(s_3s_4) - T(s_2s_4)T(s_1s_3) \\
&= (s_1s_2) \cdot (T(s_1s_2)T(s_3s_4) - T(s_1s_4)T(s_2s_3)) \\
&= (s_1s_2) \cdot d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}(\langle s_1s_2\rangle \wedge \langle s_1s_4\rangle).
\end{aligned}$$

$$\begin{aligned}
&d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}((s_1s_2s_3s_4) \cdot (\langle s_1s_2\rangle \wedge \langle s_1s_4\rangle)) \\
&= d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}(\langle s_2s_3\rangle \wedge \langle s_2s_1\rangle) \\
&= T(s_2s_3)T(s_1s_4) - T(s_1s_2)T(s_3s_4) \\
&= (s_1s_2s_3s_4) \cdot (T(s_1s_2)T(s_3s_4) - T(s_1s_4)T(s_2s_3)) \\
&= (s_1s_2s_3s_4) \cdot d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,1}}(\langle s_1s_2\rangle \wedge \langle s_1s_4\rangle).
\end{aligned}$$

Now, still letting $S = \{s_1, s_2, s_3, s_4\}$ we see that $d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,3}}$ is zero on

$$(s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3).$$

Indeed,

$$\begin{cases}
+d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,3}}((s_1 - s_2)(s_3 - s_4)) \\
-d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,3}}((s_1 - s_3)(s_2 - s_4)) \\
+d_1|_{Q\otimes_{\boldsymbol{k}} C_{1,3}}((s_1 - s_4)(s_2 - s_3))
\end{cases}$$

$$= \begin{cases}
+U(s_1^2 \wedge s_2^2)U(s_3^2 \wedge s_4^2) - T(s_1s_3)^2 + T(s_1s_4)^2 + T(s_2s_3)^2 - T(s_2s_4)^2 \\
-(U(s_1^2 \wedge s_3^2)U(s_2^2 \wedge s_4^2) - T(s_1s_2)^2 + T(s_1s_4)^2 + T(s_2s_3)^2 - T(s_3s_4)^2) \\
+U(s_1^2 \wedge s_4^2)U(s_2^2 \wedge s_3^2) - T(s_1s_2)^2 + T(s_1s_3)^2 + T(s_2s_4)^2 - T(s_3s_4)^2
\end{cases}$$

$$= \begin{cases} +U(s_1^2 \wedge s_2^2)U(s_3^2 \wedge s_4^2) \\ -U(s_1^2 \wedge s_3^2)U(s_2^2 \wedge s_4^2) \\ +U(s_1^2 \wedge s_4^2)U(s_2^2 \wedge s_3^2) \end{cases}$$

$$= \begin{cases} +U(s_1^2 \wedge s_2^2)U(s_3^2 \wedge s_4^2) \\ -(U(s_1^2 \wedge s_2^2) + U(s_2^2 \wedge s_3^2))U(s_2^2 \wedge s_4^2) \\ +U(s_1^2 \wedge s_4^2)U(s_2^2 \wedge s_3^2) \end{cases}$$

where the last equality comes from the formula (3.2) from Observation 3.2. By combining terms and then using formula (3.2) again, we get that the above is equal to

$$\begin{cases} +U(s_1^2 \wedge s_2^2)(U(s_3^2 \wedge s_4^2) - U(s_2^2 \wedge s_4^2)) \\ +U(s_2^2 \wedge s_3^2)(U(s_1^2 \wedge s_4^2) - U(s_2^2 \wedge s_4^2)) \end{cases} = \begin{cases} +U(s_1^2 \wedge s_2^2)U(s_3^2 \wedge s_2^2) \\ +U(s_2^2 \wedge s_3^2)U(s_1^2 \wedge s_2^2) \end{cases} = 0.$$

Next, given $s \in S$, we prove that the definition of $d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,1}}(s)$ does not depend on a particular enumeration of $S \setminus \{s\}$. Since

$$d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}(s) = \begin{cases} +T(ss_1) \cdot s_1 \wedge s + T(ss_2) \cdot s_2 \wedge s + T(ss_3) \cdot s_3 \wedge s \\ \hline -2(\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle) \otimes (s_1 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s) \\ +2(\langle s_1 s_2 \rangle \wedge \langle s_2 s_3 \rangle) \otimes (s_2 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s) \\ -2(\langle s_1 s_3 \rangle \wedge \langle s_2 s_3 \rangle) \otimes (s_3 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s) \\ \hline +(s_1 \wedge s_2) \otimes (s \wedge s_3 \otimes s_1 \wedge s_2) \\ +(s_1 \wedge s_3) \otimes (s \wedge s_2 \otimes s_1 \wedge s_3) \\ +(s_2 \wedge s_3) \otimes (s \wedge s_1 \otimes s_2 \wedge s_3) \end{cases}$$

one can see that the elements $(s_1 s_2)$, $(s_1 s_3)$, and $(s_2 s_3) = (s_1 s_2)(s_1 s_3)(s_1 s_2)$ of $\mathrm{Sym}(S)$ will leave the formula invariant by properties of exterior powers and the fact that $\langle m \rangle = \langle \overline{m} \rangle$ in $C_{1,1}$ for all $m \in \mathrm{sf}$. Thus, the formula does not depend on a particular choice of names $\{s_1, s_2, s_3\} = S \setminus \{s\}$.

For $d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}}$, it remains to show that

$$\sum_{s\in S}\pi_{Q\otimes_{\boldsymbol{k}}C_1\otimes_{\boldsymbol{k}}C_{2,1}}\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}}(s)=0$$

and

$$\sum_{s\in S}\pi_{Q\otimes_{\boldsymbol{k}}C_1\otimes_{\boldsymbol{k}}C_{2,2}}\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}}(s)=0.$$

Now, for $\sum\limits_{s\in S}\pi_{Q\otimes_{\boldsymbol{k}}C_1\otimes_{\boldsymbol{k}}C_{2,1}}\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}}(s)$, recall that in $\overline{\mathrm{SF}}$, $\langle m\rangle=\langle\overline{m}\rangle$ for any $m\in\mathrm{sf}$. Thus,

$$\begin{aligned}
&\sum_{s\in S}\pi_{Q\otimes_{\boldsymbol{k}}C_1\otimes_{\boldsymbol{k}}C_{2,1}}\circ\sum_{s\in S}d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}}(s)\\
&=\left\{\begin{array}{l}
-2(\langle s_2s_3\rangle\wedge\langle s_2s_4\rangle)\otimes(s_2\otimes s_2\wedge s_3\wedge s_4\wedge s_1)\\
+2(\langle s_2s_3\rangle\wedge\langle s_3s_4\rangle)\otimes(s_3\otimes s_2\wedge s_3\wedge s_4\wedge s_1)\\
-2(\langle s_2s_4\rangle\wedge\langle s_3s_4\rangle)\otimes(s_4\otimes s_2\wedge s_3\wedge s_4\wedge s_1)\\ \hline
-2(\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes(s_1\otimes s_1\wedge s_3\wedge s_4\wedge s_2)\\
+2(\langle s_1s_3\rangle\wedge\langle s_3s_4\rangle)\otimes(s_3\otimes s_1\wedge s_3\wedge s_4\wedge s_2)\\
-2(\langle s_1s_4\rangle\wedge\langle s_3s_4\rangle)\otimes(s_4\otimes s_1\wedge s_3\wedge s_4\wedge s_2)\\ \hline
-2(\langle s_1s_2\rangle\wedge\langle s_1s_4\rangle)\otimes(s_1\otimes s_1\wedge s_2\wedge s_4\wedge s_3)\\
+2(\langle s_1s_2\rangle\wedge\langle s_2s_4\rangle)\otimes(s_2\otimes s_1\wedge s_2\wedge s_4\wedge s_3)\\
-2(\langle s_1s_4\rangle\wedge\langle s_2s_4\rangle)\otimes(s_4\otimes s_1\wedge s_2\wedge s_4\wedge s_3)\\ \hline
-2(\langle s_1s_2\rangle\wedge\langle s_1s_3\rangle)\otimes(s_1\otimes s_1\wedge s_2\wedge s_3\wedge s_4)\\
+2(\langle s_1s_2\rangle\wedge\langle s_2s_3\rangle)\otimes(s_2\otimes s_1\wedge s_2\wedge s_3\wedge s_4)\\
-2(\langle s_1s_3\rangle\wedge\langle s_2s_3\rangle)\otimes(s_3\otimes s_1\wedge s_2\wedge s_3\wedge s_4)
\end{array}\right.\\
&=\left\{\begin{array}{l}
-2\langle s_1s_2\rangle\wedge\langle s_1s_3\rangle\\
+2\langle s_1s_2\rangle\wedge\langle s_1s_4\rangle\\
-2\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle
\end{array}\right\}\otimes\sum_{t\in S}(t\otimes s_1\wedge s_2\wedge s_3\wedge s_4)\\
&=0
\end{aligned}$$

by formula (3.3).

Lastly,

$$
\begin{aligned}
&\sum_{s \in S} \pi_{Q\otimes_{\boldsymbol{k}} C_1 \otimes_{\boldsymbol{k}} C_{2,2}} \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}(s) \\
&= \begin{cases}
+(s_2 \wedge s_3) \otimes (s_1 \wedge s_4 \otimes s_2 \wedge s_3) \\
+(s_2 \wedge s_4) \otimes (s_1 \wedge s_3 \otimes s_2 \wedge s_4) \\
+(s_3 \wedge s_4) \otimes (s_1 \wedge s_2 \otimes s_3 \wedge s_4) \\
\hline
+(s_1 \wedge s_3) \otimes (s_2 \wedge s_4 \otimes s_1 \wedge s_3) \\
+(s_1 \wedge s_4) \otimes (s_2 \wedge s_3 \otimes s_1 \wedge s_4) \\
+(s_3 \wedge s_4) \otimes (s_2 \wedge s_1 \otimes s_3 \wedge s_4) \\
\hline
+(s_1 \wedge s_2) \otimes (s_3 \wedge s_4 \otimes s_1 \wedge s_2) \\
+(s_1 \wedge s_4) \otimes (s_3 \wedge s_2 \otimes s_1 \wedge s_4) \\
+(s_2 \wedge s_4) \otimes (s_3 \wedge s_1 \otimes s_2 \wedge s_4) \\
\hline
+(s_1 \wedge s_2) \otimes (s_4 \wedge s_3 \otimes s_1 \wedge s_2) \\
+(s_1 \wedge s_3) \otimes (s_4 \wedge s_2 \otimes s_1 \wedge s_3) \\
+(s_2 \wedge s_3) \otimes (s_4 \wedge s_1 \otimes s_2 \wedge s_3)
\end{cases} \\
&= 0,
\end{aligned}
$$

which is easy to see. □

3.4 The maps and modules of $X_\bullet$ form a complex

Proposition 3.5. *The sequence of maps and modules*

$$Q \otimes_{\boldsymbol{k}} C_4 \xrightarrow{d_4} Q \otimes_{\boldsymbol{k}} C_3 \xrightarrow{d_3} Q \otimes_{\boldsymbol{k}} C_2 \xrightarrow{d_2} Q \otimes_{\boldsymbol{k}} C_1 \xrightarrow{d_1} Q \xrightarrow{\varepsilon} \mathrm{Sym}^{\boldsymbol{k}}_\bullet(V)$$

is a complex.

Proof. If $m_1, m_2 \in \mathrm{sf}$, then

$$\begin{aligned}(\varepsilon \circ d_1)(\langle m_1 \rangle \wedge \langle m_2 \rangle) &= \varepsilon(T(m_1)T(\overline{m_1}) - T(m_2)T(\overline{m_2})) \\ &= m_1\overline{m_1} - m_2\overline{m_2} \\ &= \Omega - \Omega = 0.\end{aligned}$$

Now assume that $S = \{s_1, s_2, s_3, s_4\}$. Then

$$\begin{aligned}(\varepsilon \circ d_1)(s_1 \wedge s_2) &= \varepsilon(T(s_3s_4)U(s_1^2 \wedge s_2^2) + T(s_1s_3)T(s_1s_4) - T(s_2s_3)T(s_2s_4)) \\ &= s_3s_4 \cdot (s_2^2 - s_1^2) + s_1s_3 \cdot s_1s_4 - s_2s_3 \cdot s_2s_4 \in \mathrm{Sym}_\bullet^{\boldsymbol{k}}(V) \\ &= 0\end{aligned}$$

and

$$\begin{aligned}&(\varepsilon \circ d_1)((s_1 - s_2)(s_3 - s_4)) \\ &= \varepsilon(U(s_1^2 \wedge s_2^2)U(s_3^2 \wedge s_4^2) - T(s_1s_3)^2 + T(s_1s_4)^2 + T(s_2s_3)^2 - T(s_2s_4)^2) \\ &= (s_2^2 - s_1^2)(s_4^2 - s_3^2) - s_1s_3 \cdot s_1s_3 + s_1s_4 \cdot s_1s_4 + s_2s_3 \cdot s_2s_3 - s_2s_4 \cdot s_2s_4 \\ &= 0.\end{aligned}$$

Thus,

$$\varepsilon \circ d_1 = 0.$$

Now consider $d_1 \circ d_2$. Since d_2 on $Q \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^2 C_1$ is the Koszul map induced by d_1, it is immediate that $d_1 \circ d_2|_{Q \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^2 C_1} = 0$, so we only need show that $d_1 \circ d_2|_{Q \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^2 C_{2,1}}$ and $d_1 \circ d_2|_{Q \otimes_{\boldsymbol{k}} C_{2,2}}$ are zero.

For $d_1 \circ d_2|_{Q \otimes_{\boldsymbol{k}} C_{2,1}}$, let $s \in S$ and $S = \{s, t_1, t_2, t_3\}$. Then

$$(d_1 \circ d_2|_{Q \otimes_{\boldsymbol{k}} C_{2,1}})(s \otimes (s \wedge t_1 \wedge t_2 \wedge t_3))$$

$$= \begin{cases} +d_1\Big(U(t_1^2 \wedge s^2) \cdot \langle st_1\rangle^*(\langle st_1\rangle \wedge \langle st_2\rangle \wedge \langle st_3\rangle)\Big) \\ +d_1\Big(U(t_2^2 \wedge s^2) \cdot \langle st_2\rangle^*(\langle st_1\rangle \wedge \langle st_2\rangle \wedge \langle st_3\rangle)\Big) \\ +d_1\Big(U(t_3^2 \wedge s^2) \cdot \langle st_3\rangle^*(\langle st_1\rangle \wedge \langle st_2\rangle \wedge \langle st_3\rangle)\Big) \\ +d_1\Big(T(\overline{t_1 s}) \cdot (t_1^* \wedge s^*)(s \wedge t_1 \wedge t_2 \wedge t_3)\Big) \\ +d_1\Big(T(\overline{t_2 s}) \cdot (t_2^* \wedge s^*)(s \wedge t_1 \wedge t_2 \wedge t_3)\Big) \\ +d_1\Big(T(\overline{t_3 s}) \cdot (t_3^* \wedge s^*)(s \wedge t_1 \wedge t_2 \wedge t_3)\Big) \end{cases}$$

$$= \begin{cases} +U(t_1^2 \wedge s^2)d_1(\langle st_2\rangle \wedge \langle st_3\rangle) \\ -U(t_2^2 \wedge s^2)d_1(\langle st_1\rangle \wedge \langle st_3\rangle) \\ +U(t_3^2 \wedge s^2)d_1(\langle st_1\rangle \wedge \langle st_2\rangle) \\ +T(t_2 t_3)d_1(t_2 \wedge t_3) \\ -T(t_1 t_3)d_1(t_1 \wedge t_3) \\ +T(t_1 t_2)d_1(t_1 \wedge t_2). \end{cases}$$

Expanding all of the d_1 terms, the above is seen to be zero, as long as one recalls formula (3.2): $U(s_i^2 \wedge s_j^2) - U(s_i^2 \wedge s_k^2) + U(s_j^2 \wedge s_k^2) = 0$ for any three distinct $s_i, s_j, s_k \in S$. This is sufficient to show that $d_1 \circ d_2|_{Q \otimes_{\boldsymbol{k}} C_{2,1}} = 0$ since any element of $Q \otimes_{\boldsymbol{k}} C_{2,1}$ is a multiple of an element of the form $s \otimes (s \wedge t_1 \wedge t_2 \wedge t_3)$ for $s, t_1, t_2, t_3 \in S$.

For the remaining summand of $Q \otimes_{\boldsymbol{k}} C_2$, let $S = \{s_1, s_2, s_3, s_4\}$. Then

$$(d_1 \circ d_2|_{Q\otimes_{\boldsymbol{k}} C_{2,2}})(s_1 \wedge s_2 \otimes s_3 \wedge s_4) = \begin{cases} -2d_1(T(s_3 s_4) \cdot \langle s_1 s_4 \rangle \wedge \langle s_1 s_3 \rangle) \\ -d_1(U(s_1^2 \wedge s_2^2) \cdot s_3 \wedge s_4) \\ -d_1(T(s_1 s_4) \cdot s_1 \wedge s_3) \\ +d_1(T(s_1 s_3) \cdot s_1 \wedge s_4) \\ +d_1(T(s_2 s_4) \cdot s_2 \wedge s_3) \\ -d_1(T(s_2 s_3) \cdot s_2 \wedge s_4) \\ +d_1(T(s_1 s_2) \cdot (s_1 - s_2)(s_3 - s_4)). \end{cases}$$

Again to see that the above is zero requires only the use of the definition of d_1 and formula (3.2).

Hence,

$$d_1 \circ d_2 = 0.$$

Since we now know $d_1 \circ d_2 = 0$, we can invoke Lemma 3.1 to conclude that the Tate portion of the defined sequence of maps and modules is a complex. That is,

$$d_2 \circ d_3|_{Q\otimes_{\boldsymbol{k}} C_1 \otimes_{\boldsymbol{k}} (C_{2,1} \oplus C_{2,2})} = 0$$

and

$$d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} D_2^{\boldsymbol{k}}(C_{2,1} \oplus C_{2,2})} = 0,$$

so, to complete the proof, it remains to show that $d_2 \circ d_3|_{Q\otimes_{\boldsymbol{k}} C_{3,1}}$, $d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}$, and $d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}}$ are all zero.

Let $s_1, s_2 \in S$ and $S \setminus \{s_1, s_2\} = \{s_3, s_4\}$. Then

$$(d_2 \circ d_3)|_{Q \otimes_{\boldsymbol{k}} C_{3,1}} (s_1 \wedge s_2) = \begin{cases} +2d_2\Big(T(s_3 s_4) \cdot (s_3 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s_4)\Big) \\ -2d_2\Big(T(s_3 s_4) \cdot (s_4 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s_4)\Big) \\ +d_2\Big(U(s_3^2 \wedge s_4^2) \cdot (s_1 \wedge s_2 \otimes s_3 \wedge s_4)\Big) \\ +d_2\Big(T(s_2 s_3) \cdot (s_1 \wedge s_3 \otimes s_2 \wedge s_4)\Big) \\ +d_2\Big(T(s_2 s_4) \cdot (s_1 \wedge s_4 \otimes s_2 \wedge s_3)\Big) \\ -d_2\Big(T(s_1 s_3) \cdot (s_2 \wedge s_3 \otimes s_1 \wedge s_4)\Big) \\ -d_2\Big(T(s_1 s_4) \cdot (s_2 \wedge s_4 \otimes s_1 \wedge s_3)\Big) \\ -2d_2\Big(\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \otimes s_1 \wedge s_2\Big) \\ -2d_2\Big(\langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \otimes s_1 \wedge s_2\Big) \\ +d_2\Big((s_1 \wedge s_3) \wedge (s_2 \wedge s_3)\Big) \\ +d_2\Big((s_1 \wedge s_4) \wedge (s_2 \wedge s_4)\Big) \\ -d_2\Big((s_3 \wedge s_4) \otimes (s_1 - s_2)(s_3 - s_4)\Big). \end{cases}$$

We consider the projections to the direct summands of $Q \otimes_{\boldsymbol{k}} C_1$. To start, the image of $(d_2 \circ d_3)|_{Q\otimes_{\boldsymbol{k}} C_{3,1}}(s_1 \wedge s_2)$ in $Q \otimes_{\boldsymbol{k}} C_{1,1}$ is

$$\begin{cases}
+2d_1(s_1 \wedge s_2) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \\
+2d_1(s_1 \wedge s_2) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \\
+2T(s_3 s_4) \sum_{t \in S} U(t^2 \wedge s_3^2) \cdot \langle s_3 t \rangle^* (\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \\
-2T(s_3 s_4) \sum_{t \in S} U(t^2 \wedge s_4^2) \cdot \langle s_4 t \rangle^* (\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \\
-2U(s_3^2 \wedge s_4^2) T(s_3 s_4) \cdot \langle s_1 s_4 \rangle \wedge \langle s_1 s_3 \rangle \\
+2T(s_2 s_3) T(s_2 s_4) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \\
-2T(s_1 s_3) T(s_1 s_4) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \\
+2T(s_2 s_3) T(s_2 s_4) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \\
-2T(s_1 s_3) T(s_1 s_4) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle .
\end{cases}$$

Collecting the same $\langle m_1 \rangle \wedge \langle m_2 \rangle$ terms for $m_1, m_2 \in \mathrm{sf}$, we get

$$\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}} C_{1,1}} \circ d_2 \circ d_3)\Big|_{Q\otimes_{\boldsymbol{k}} C_{3,1}} (s_1 \wedge s_2) \\
&= \begin{cases}
+2d_1(s_1 \wedge s_2) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \\
+2(T(s_2 s_3) T(s_2 s_4) - T(s_1 s_3) T(s_1 s_4)) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \\
\hline
+2d_1(s_1 \wedge s_2) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \\
+2(T(s_2 s_3) T(s_2 s_4) - T(s_1 s_3) T(s_1 s_4)) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \\
\hline
+2T(s_3 s_4) \sum_{t \in S} U(t^2 \wedge s_3^2) \cdot \langle s_3 t \rangle^* (\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \\
-2T(s_3 s_4) \sum_{t \in S} U(t^2 \wedge s_4^2) \cdot \langle s_4 t \rangle^* (\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \\
\hline
-2U(s_3^2 \wedge s_4^2) T(s_3 s_4) \cdot \langle s_1 s_4 \rangle \wedge \langle s_1 s_3 \rangle .
\end{cases}
\end{aligned}$$

By definition of d_1, we have

$$(\pi_{Q\otimes_{\boldsymbol{k}} C_{1,1}} \circ d_2 \circ d_3)\Big|_{Q\otimes_{\boldsymbol{k}} C_{3,1}} (s_1 \wedge s_2)$$

$$
= \begin{cases}
+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle \\
+2T(s_3s_4)\sum_{t \in S} U(t^2 \wedge s_3^2) \cdot \langle s_3t \rangle^*(\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \\
-2T(s_3s_4)\sum_{t \in S} U(t^2 \wedge s_4^2) \cdot \langle s_4t \rangle^*(\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \\
-2U(s_3^2 \wedge s_4^2)T(s_3s_4) \cdot \langle s_1s_4 \rangle \wedge \langle s_1s_3 \rangle.
\end{cases}
$$

After expanding the summations and collecting terms, we get

$$
(\pi_{Q\otimes_{\boldsymbol{k}} C_{1,1}} \circ d_2 \circ d_3)\Big|_{Q\otimes_{\boldsymbol{k}} C_{3,1}} (s_1 \wedge s_2) = \left\{\begin{array}{l}
+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
+2T(s_3s_4)U(s_2^2 \wedge s_3^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
-2T(s_3s_4)U(s_1^2 \wedge s_4^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
\hline
+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle \\
-2T(s_3s_4)U(s_1^2 \wedge s_3^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle \\
+2T(s_3s_4)U(s_2^2 \wedge s_4^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle \\
\hline
-2T(s_3s_4)U(s_3^2 \wedge s_4^2) \cdot \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle.
\end{array}\right.
$$

Using (3.2) followed by (3.3), the above becomes

$$
\left\{\begin{array}{l}
-2T(s_3s_4)U(s_3^2 \wedge s_4^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
\hline
+2T(s_3s_4)U(s_3^2 \wedge s_4^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle \\
\hline
-2T(s_3s_4)U(s_3^2 \wedge s_4^2) \cdot \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle
\end{array}\right. = 0.
$$

Moving on, the portion of $\left.(d_2 \circ d_3)\right|_{Q \otimes_{\boldsymbol{k}} C_{3,1}} (s_1 \wedge s_2)$ contained in $Q \otimes_{\boldsymbol{k}} C_{1,2}$ is

$$
\begin{cases}
-2d_1(\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle) \cdot s_1 \wedge s_2 - 2d_1(\langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle) \cdot s_1 \wedge s_2 \\
+d_1((s_1 - s_2)(s_3 - s_4)) \cdot s_3 \wedge s_4 \\
+d_1(s_1 \wedge s_3) \cdot s_2 \wedge s_3 - d_1(s_2 \wedge s_3) \cdot s_1 \wedge s_3 \\
+d_1(s_1 \wedge s_4) \cdot s_2 \wedge s_4 - d_1(s_2 \wedge s_4) \cdot s_1 \wedge s_4 \\
+2T(s_3 s_4) \sum_{t \in S} T(\overline{t s_3}) \cdot (t^* \wedge s_3^*)(s_1 \wedge s_2 \wedge s_3 \wedge s_4) \\
-2T(s_3 s_4) \sum_{t \in S} T(\overline{t s_4}) \cdot (t^* \wedge s_4^*)(s_1 \wedge s_2 \wedge s_3 \wedge s_4) \\
-U(s_3^2 \wedge s_4^2) T(s_1 s_4) \cdot s_1 \wedge s_3 + U(s_2^2 \wedge s_4^2) T(s_1 s_4) \cdot s_1 \wedge s_3 \\
+U(s_3^2 \wedge s_4^2) T(s_1 s_3) \cdot s_1 \wedge s_4 - U(s_3^2 \wedge s_2^2) T(s_1 s_3) \cdot s_1 \wedge s_4 \\
+U(s_3^2 \wedge s_4^2) T(s_2 s_4) \cdot s_2 \wedge s_3 - U(s_1^2 \wedge s_4^2) T(s_2 s_4) \cdot s_2 \wedge s_3 \\
-U(s_3^2 \wedge s_4^2) T(s_2 s_3) \cdot s_2 \wedge s_4 - U(s_1^2 \wedge s_3^2) T(s_2 s_3) \cdot s_2 \wedge s_4 \\
-U(s_1^2 \wedge s_2^2) U(s_3^2 \wedge s_4^2) \cdot s_3 \wedge s_4 \\
-2T(s_1 s_3) T(s_2 s_4) \cdot s_1 \wedge s_2 - 2T(s_1 s_4) T(s_2 s_3) \cdot s_1 \wedge s_2 \\
+T(s_1 s_2) T(s_2 s_4) \cdot s_1 \wedge s_3 + T(s_1 s_3) T(s_3 s_4) \cdot s_1 \wedge s_3 \\
+T(s_1 s_4) T(s_4 s_3) \cdot s_1 \wedge s_4 + T(s_1 s_2) T(s_2 s_3) \cdot s_1 \wedge s_4 \\
-T(s_2 s_3) T(s_3 s_4) \cdot s_2 \wedge s_3 - T(s_1 s_2) T(s_1 s_4) \cdot s_2 \wedge s_3 \\
-T(s_2 s_4) T(s_3 s_4) \cdot s_2 \wedge s_4 - T(s_1 s_2) T(s_1 s_3) \cdot s_2 \wedge s_4 \\
-T(s_2 s_3)^2 \cdot s_3 \wedge s_4 + T(s_2 s_4)^2 \cdot s_3 \wedge s_4 \\
-T(s_1 s_4)^2 \cdot s_3 \wedge s_4 + T(s_1 s_3)^2 \cdot s_3 \wedge s_4.
\end{cases}
$$

Collecting terms containing the same $s_i \wedge s_j$ for pairs $\{i, j\}$, we get that the above expression is equal to

$$
\left\{
\begin{array}{l}
-2d_1(\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle) \cdot s_1 \wedge s_2 - 2d_1(\langle s_1s_2\rangle \wedge \langle s_1s_4\rangle) \cdot s_1 \wedge s_2 \\
-2T(s_1s_3)T(s_2s_4) \cdot s_1 \wedge s_2 - 2T(s_1s_4)T(s_2s_3) \cdot s_1 \wedge s_2 \\
\hline
-d_1(s_2 \wedge s_3) \cdot s_1 \wedge s_3 \\
-U(s_3^2 \wedge s_4^2)T(s_1s_4) \cdot s_1 \wedge s_3 + U(s_2^2 \wedge s_4^2)T(s_1s_4) \cdot s_1 \wedge s_3 \\
+T(s_1s_2)T(s_2s_4) \cdot s_1 \wedge s_3 + T(s_1s_3)T(s_3s_4) \cdot s_1 \wedge s_3 \\
\hline
-d_1(s_2 \wedge s_4) \cdot s_1 \wedge s_4 \\
+U(s_3^2 \wedge s_4^2)T(s_1s_3) \cdot s_1 \wedge s_4 - U(s_3^2 \wedge s_2^2)T(s_1s_3) \cdot s_1 \wedge s_4 \\
+T(s_1s_4)T(s_4s_3) \cdot s_1 \wedge s_4 + T(s_1s_2)T(s_2s_3) \cdot s_1 \wedge s_4 \\
\hline
+d_1(s_1 \wedge s_3) \cdot s_2 \wedge s_3 \\
+U(s_3^2 \wedge s_4^2)T(s_2s_4) \cdot s_2 \wedge s_3 - U(s_1^2 \wedge s_4^2)T(s_2s_4) \cdot s_2 \wedge s_3 \\
-T(s_2s_3)T(s_3s_4) \cdot s_2 \wedge s_3 - T(s_1s_2)T(s_1s_4) \cdot s_2 \wedge s_3 \\
\hline
+d_1(s_1 \wedge s_4) \cdot s_2 \wedge s_4 \\
-U(s_3^2 \wedge s_4^2)T(s_2s_3) \cdot s_2 \wedge s_4 - U(s_1^2 \wedge s_3^2)T(s_2s_3) \cdot s_2 \wedge s_4 \\
-T(s_2s_4)T(s_3s_4) \cdot s_2 \wedge s_4 - T(s_1s_2)T(s_1s_3) \cdot s_2 \wedge s_4 \\
\hline
+d_1((s_1 - s_2)(s_3 - s_4)) \cdot s_3 \wedge s_4 \\
-U(s_1^2 \wedge s_2^2)U(s_3^2 \wedge s_4^2) \cdot s_3 \wedge s_4 \\
-T(s_2s_3)^2 \cdot s_3 \wedge s_4 + T(s_2s_4)^2 \cdot s_3 \wedge s_4 \\
-T(s_1s_4)^2 \cdot s_3 \wedge s_4 + T(s_1s_3)^2 \cdot s_3 \wedge s_4 \\
\hline
+2T(s_3s_4) \sum_{t \in S} T(\overline{ts_3}) \cdot (t^* \wedge s_3^*)(s_1 \wedge s_2 \wedge s_3 \wedge s_4) \\
-2T(s_3s_4) \sum_{t \in S} T(\overline{ts_4}) \cdot (t^* \wedge s_4^*)(s_1 \wedge s_2 \wedge s_3 \wedge s_4)
\end{array}
\right.
$$

$$
= \begin{cases}
-4T(s_1s_2)T(s_3s_4)\cdot s_1\wedge s_2 \\
+2T(s_1s_3)T(s_3s_4)\cdot s_1\wedge s_3 \\
+2T(s_1s_4)T(s_4s_3)\cdot s_1\wedge s_4 \\
-2T(s_2s_3)T(s_3s_4)\cdot s_2\wedge s_3 \\
-2T(s_2s_4)T(s_3s_4)\cdot s_2\wedge s_4 \\
+0\cdot s_3\wedge s_4 \\
\hline
+2T(s_3s_4)\sum_{t\in S} T(\overline{ts_3})\cdot (t^*\wedge s_3^*)(s_1\wedge s_2\wedge s_3\wedge s_4) \\
-2T(s_3s_4)\sum_{t\in S} T(\overline{ts_4})\cdot (t^*\wedge s_4^*)(s_1\wedge s_2\wedge s_3\wedge s_4)
\end{cases}
$$

$$
= 0,
$$

where the second-to-last equality is simply from the definition of d_1. Thus,

$$
\pi_{Q\otimes_{\boldsymbol{k}} C_{1,2}} \circ d_2 \circ d_3|_{Q\otimes_{\boldsymbol{k}} C_{3,1}}(s_1\wedge s_2) = 0.
$$

For the final summand of $Q\otimes_{\boldsymbol{k}} C_{1,3}$,

$$
(\pi_{Q\otimes_{\boldsymbol{k}} C_{1,3}} \circ d_2 \circ d_3)\Big|_{Q\otimes_{\boldsymbol{k}} C_{3,1}}(s_1\wedge s_2) = \begin{cases}
-d_1(s_3\wedge s_4)\cdot(s_1-s_2)(s_3-s_4) \\
+U(s_3^2\wedge s_4^2)T(s_1s_2)\cdot(s_1-s_2)(s_3-s_4) \\
+T(s_1s_3)T(s_2s_3)\cdot(s_1-s_3)(s_2-s_4) \\
-T(s_1s_3)T(s_3s_2)\cdot(s_1-s_4)(s_2-s_3) \\
-T(s_1s_4)T(s_2s_4)\cdot(s_1-s_3)(s_2-s_4) \\
+T(s_1s_4)T(s_2s_4)\cdot(s_1-s_4)(s_2-s_3)
\end{cases}
$$

Recalling (3.4) the previous expression becomes

$$\begin{cases} -d_1(s_3 \wedge s_4) \cdot (s_1 - s_2)(s_3 - s_4) \\ +U(s_3^2 \wedge s_4^2)T(s_1s_2) \cdot (s_1 - s_2)(s_3 - s_4) \\ +T(s_1s_3)T(s_2s_3) \cdot (s_1 - s_2)(s_3 - s_4) \\ -T(s_1s_4)T(s_2s_4) \cdot (s_1 - s_2)(s_3 - s_4), \end{cases}$$

which is zero by expanding $d_1(s_3 \wedge s_4)$. Therefore, we have finished showing

$$d_2 \circ d_3 = 0.$$

Next consider $d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}$. For $s \in S$ and $\{s, t_1, t_2, t_3\} = S$,

$$(d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}})(s) = \begin{cases} +d_3\Big(T(st_1) \cdot t_1 \wedge s\Big) \\ +d_3\Big(T(st_2) \cdot t_2 \wedge s\Big) \\ +d_3\Big(T(st_3) \cdot t_3 \wedge s\Big) \\ -2d_3\Big((\langle t_1t_2\rangle \wedge \langle t_1t_3\rangle) \otimes (t_1 \otimes t_1 \wedge t_2 \wedge t_3 \wedge s)\Big) \\ +2d_3\Big((\langle t_1t_2\rangle \wedge \langle t_1s\rangle) \otimes (t_2 \otimes t_1 \wedge t_2 \wedge t_3 \wedge s)\Big) \\ -2d_3\Big((\langle t_1t_3\rangle \wedge \langle t_1s\rangle) \otimes (t_3 \otimes t_1 \wedge t_2 \wedge t_3 \wedge s)\Big) \\ +d_3\Big((t_1 \wedge t_2) \otimes (s \wedge t_3 \otimes t_1 \wedge t_2)\Big) \\ +d_3\Big((t_1 \wedge t_3) \otimes (s \wedge t_2 \otimes t_1 \wedge t_3)\Big) \\ +d_3\Big((t_2 \wedge t_3) \otimes (s \wedge t_1 \otimes t_2 \wedge t_3)\Big). \end{cases}$$

Again, we find the projections of $(d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}})(s)$ onto the direct summands of $Q \otimes_{\boldsymbol{k}} C_2$ and show these are all zero. Recall that we use the isomorphism (3.5) to express $\bigwedge^2 C_1$ as a direct sum. Specifically,

$$C_2 = C_{2,1} \oplus C_{2,2} \oplus \textstyle\bigwedge_{\boldsymbol{k}}^2 C_1 = \left\{ \begin{array}{c} C_{2,1} \oplus C_{2,2} \\ \oplus \\ \bigwedge_{\boldsymbol{k}}^2 C_{1,1} \oplus (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,2}) \oplus (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,3}) \\ \oplus \\ \bigwedge_{\boldsymbol{k}}^2 C_{1,2} \oplus (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,3}) \oplus \bigwedge_{\boldsymbol{k}}^2 C_{1,3} \end{array} \right\}$$

Two of the projections of $(d_3 \circ d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,1}})(s)$ are immediately zero:

$$(\pi_{Q \otimes_{\boldsymbol{k}} (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,3})} \circ d_3 \circ d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,1}})(s) = 0$$

and

$$(\pi_{Q \otimes_{\boldsymbol{k}} \bigwedge_{\boldsymbol{k}}^2 C_{1,3}} \circ d_3 \circ d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,1}})(s) = 0.$$

We now calculate the remaining six projections.

$$(\pi_{Q \otimes_{\boldsymbol{k}} C_{2,1}} \circ d_3 \circ d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,1}})(s) = \begin{cases} +2T(st_1)T(t_2t_3) \cdot ((t_2 - t_3) \otimes t_1 \wedge s \wedge t_2 \wedge t_3) \\ +2T(st_2)T(t_1t_3) \cdot ((t_1 - t_3) \otimes t_2 \wedge s \wedge t_1 \wedge t_3) \\ +2T(st_3)T(t_1t_2) \cdot ((t_1 - t_2) \otimes t_3 \wedge s \wedge t_1 \wedge t_2) \\ \hline +2d_1(\langle t_1t_2 \rangle \wedge \langle st_2 \rangle) \cdot (t_1 \otimes s \wedge t_1 \wedge t_2 \wedge t_3) \\ -2d_1(\langle t_1t_2 \rangle \wedge \langle st_1 \rangle) \cdot (t_2 \otimes s \wedge t_1 \wedge t_2 \wedge t_3) \\ +2d_1(\langle st_2 \rangle \wedge \langle st_1 \rangle) \cdot (t_3 \otimes s \wedge t_1 \wedge t_2 \wedge t_3). \end{cases}$$

Using the definition of d_1 shows easily that $(\pi_{Q \otimes_{\boldsymbol{k}} C_{2,1}} \circ d_3 \circ d_4|_{Q \otimes_{\boldsymbol{k}} C_{4,1}})(s) = 0$.

For the image of $d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}})(s)$ in the $Q \otimes_{\boldsymbol{k}} C_{2,2}$ summand is

$$\begin{cases} +U(t_2^2 \wedge t_3^2)T(st_1) \cdot (t_1 \wedge s \otimes t_2 \wedge t_3) + U(t_1^2 \wedge t_3^2)T(st_2) \cdot (t_2 \wedge s \otimes t_1 \wedge t_3) \\ +U(t_1^2 \wedge t_2^2)T(st_3) \cdot (t_3 \wedge s \otimes t_1 \wedge t_2) \\ \hline +T(st_1)T(st_2) \cdot (t_1 \wedge t_2 \otimes s \wedge t_3) + T(st_1)T(st_3) \cdot (t_1 \wedge t_3 \otimes s \wedge t_2) \\ -T(st_1)T(t_1t_2) \cdot (s \wedge t_2 \otimes t_1 \wedge t_3) - T(st_1)T(t_1t_3) \cdot (s \wedge t_3 \otimes t_1 \wedge t_2) \\ +T(st_1)T(st_2) \cdot (t_2 \wedge t_1 \otimes s \wedge t_3) + T(st_2)T(st_3) \cdot (t_2 \wedge t_3 \otimes s \wedge t_1) \\ -T(st_2)T(t_2t_1) \cdot (s \wedge t_1 \otimes t_2 \wedge t_3) - T(st_2)T(t_2t_3) \cdot (s \wedge t_3 \otimes t_2 \wedge t_1) \\ +T(st_1)T(st_3) \cdot (t_3 \wedge t_1 \otimes s \wedge t_2) + T(st_2)T(st_3) \cdot (t_3 \wedge t_2 \otimes s \wedge t_1) \\ -T(st_3)T(t_3t_1) \cdot (s \wedge t_1 \otimes t_3 \wedge t_2) - T(st_3)T(t_3t_2) \cdot (s \wedge t_2 \otimes t_3 \wedge t_1) \\ \hline +d_1(t_1 \wedge t_2) \cdot (s \wedge t_3 \otimes t_1 \wedge t_2) + d_1(t_1 \wedge t_3) \cdot (s \wedge t_2 \otimes t_1 \wedge t_3) \\ +d_1(t_2 \wedge t_3) \cdot (s \wedge t_1 \otimes t_2 \wedge t_3). \end{cases}$$

Collecting terms by the generators of $C_{2,2}$ gives

$$(\pi_{Q\otimes_{\boldsymbol{k}} C_{2,2}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}})(s)$$

$$= \begin{cases} +d_1(t_2 \wedge t_3) \cdot (s \wedge t_1 \otimes t_2 \wedge t_3) + U(t_2^2 \wedge t_3^2)T(st_1) \cdot (t_1 \wedge s \otimes t_2 \wedge t_3) \\ -T(st_2)T(t_2t_1) \cdot (s \wedge t_1 \otimes t_2 \wedge t_3) - T(st_3)T(t_3t_1) \cdot (s \wedge t_1 \otimes t_3 \wedge t_2) \\ \hline +d_1(t_1 \wedge t_3) \cdot (s \wedge t_2 \otimes t_1 \wedge t_3) + U(t_1^2 \wedge t_3^2)T(st_2) \cdot (t_2 \wedge s \otimes t_1 \wedge t_3) \\ -T(st_1)T(t_1t_2) \cdot (s \wedge t_2 \otimes t_1 \wedge t_3) - T(st_3)T(t_3t_2) \cdot (s \wedge t_2 \otimes t_3 \wedge t_1) \\ \hline +d_1(t_1 \wedge t_2) \cdot (s \wedge t_3 \otimes t_1 \wedge t_2) + U(t_1^2 \wedge t_2^2)T(st_3) \cdot (t_3 \wedge s \otimes t_1 \wedge t_2) \\ -T(st_1)T(t_1t_3) \cdot (s \wedge t_3 \otimes t_1 \wedge t_2) - T(st_2)T(t_2t_3) \cdot (s \wedge t_3 \otimes t_2 \wedge t_1) \\ \hline +T(st_1)T(st_2) \cdot (t_1 \wedge t_2 \otimes s \wedge t_3) + T(st_1)T(st_2) \cdot (t_2 \wedge t_1 \otimes s \wedge t_3) \\ \hline +T(st_1)T(st_3) \cdot (t_1 \wedge t_3 \otimes s \wedge t_2) + T(st_1)T(st_3) \cdot (t_3 \wedge t_1 \otimes s \wedge t_2) \\ \hline +T(st_2)T(st_3) \cdot (t_2 \wedge t_3 \otimes s \wedge t_1) + T(st_2)T(st_3) \cdot (t_3 \wedge t_2 \otimes s \wedge t_1), \end{cases}$$

which is zero by the definition of d_1.

For the $Q \otimes_{\boldsymbol{k}} \bigwedge^2_{\boldsymbol{k}} C_{1,1}$ part,

$$
\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}}\bigwedge^2_{\boldsymbol{k}} C_{1,1}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}})(s) \\
&= \begin{cases}
-2\sum\limits_{t\in S} U(t^2 \wedge t_1^2) \cdot \Big(\langle t_1t_2\rangle \wedge \langle st_2\rangle) \wedge (\langle t_1 t\rangle^*(\alpha(s\wedge t_1 \wedge t_2 \wedge t_3))\Big) \\
+2\sum\limits_{t\in S} U(t^2 \wedge t_2^2) \cdot \Big((\langle t_1t_2\rangle \wedge \langle st_1\rangle) \wedge \langle t_2 t\rangle^*(\alpha(s\wedge t_1 \wedge t_2 \wedge t_3))\Big) \\
-2\sum\limits_{t\in S} U(t^2 \wedge t_3^2) \cdot \Big((\langle st_2\rangle \wedge \langle st_1\rangle) \wedge \langle t_3 t\rangle^*(\alpha(s\wedge t_1 \wedge t_2 \wedge t_3))\Big)
\end{cases} \\
&= \begin{cases}
-2U(s^2 \wedge t_1^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_1t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_1t_2\rangle) \\
-2U(t_2^2 \wedge t_1^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_1t_3\rangle) \wedge (\langle t_2t_3\rangle \wedge \langle t_1t_2\rangle) \\
+2U(t_3^2 \wedge t_1^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_1t_3\rangle) \wedge (\langle t_2t_3\rangle \wedge \langle t_1t_2\rangle) \\
-2U(s^2 \wedge t_2^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_2t_3\rangle) \wedge (\langle t_2t_3\rangle \wedge \langle t_1t_2\rangle) \\
+2U(t_1^2 \wedge t_2^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_2t_3\rangle) \wedge (\langle t_2t_3\rangle \wedge \langle t_1t_3\rangle) \\
+2U(t_3^2 \wedge t_2^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_2t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_1t_2\rangle) \\
-2U(s^2 \wedge t_3^2) \cdot (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle) \wedge (\langle t_2t_3\rangle \wedge \langle t_1t_3\rangle) \\
+2U(t_1^2 \wedge t_3^2) \cdot (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle) \wedge (\langle t_2t_3\rangle \wedge \langle t_1t_2\rangle) \\
-2U(t_2^2 \wedge t_3^2) \cdot (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_1t_2\rangle)
\end{cases}
\end{aligned}
$$

due to the fact that $\langle m\rangle = \langle \overline{m}\rangle \in \overline{\mathrm{SF}}$ for any $m \in \mathrm{sf}$. Therefore,

$$
(\pi_{Q\otimes_{\boldsymbol{k}}\bigwedge^2_{\boldsymbol{k}} C_{1,1}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}})(s) = \begin{cases}
-2U(t_2^2 \wedge t_3^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_1t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle) \\
+2U(t_2^2 \wedge t_3^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_2t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle)
\end{cases}
$$

From here, using formula (3.3) gives that

$$
(\pi_{Q\otimes_{\boldsymbol{k}}\bigwedge^2_{\boldsymbol{k}} C_{1,1}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}})(s) = \begin{cases}
-2U(t_2^2 \wedge t_3^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_1t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle) \\
+2U(t_2^2 \wedge t_3^2) \cdot (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle) \\
+2U(t_2^2 \wedge t_3^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_1t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle)
\end{cases}
$$

$$= \begin{cases} -2U(t_2^2 \wedge t_3^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_1t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle) \\ +2U(t_2^2 \wedge t_3^2) \cdot (\langle t_1t_2\rangle \wedge \langle t_1t_3\rangle) \wedge (\langle t_1t_3\rangle \wedge \langle t_2t_3\rangle) \end{cases}$$

$$= 0.$$

For the $Q \otimes_{\boldsymbol{k}} (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,2})$ summand, recall that $\mathrm{quot} : C_1 \otimes_{\boldsymbol{k}} C_1 \to \bigwedge^2_{\boldsymbol{k}} C_1$ was defined (in Section 3.3) as the natural quotient map and that we are using the isomorphism (3.5) from Observation 3.3 to decompose the module $\bigwedge^2_{\boldsymbol{k}} C_1$ as a direct sum. For instance, if $\gamma_1 \in C_{1,1}$ and $\gamma_2 \in C_{1,2}$, then the element $\mathrm{quot}(\gamma_2 \otimes \gamma_1) \in \bigwedge^2 C_1$ is $-\gamma_1 \otimes \gamma_2$ in the $C_{1,1} \otimes_{\boldsymbol{k}} C_{1,2}$ summand of the decomposition of $\bigwedge^2_{\boldsymbol{k}} C_1$ under the isomorphism (3.5). Thus,

$$(\pi_{Q\otimes_{\boldsymbol{k}}(C_{1,1}\otimes_{\boldsymbol{k}}C_{1,2})} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}})(s)$$

$$= \begin{cases} -2T(st_1) \cdot (\langle t_1s\rangle \wedge \langle t_1t_2\rangle + \langle t_1s\rangle \wedge \langle t_1t_3\rangle) \otimes t_1 \wedge s \\ -2T(st_2) \cdot (\langle t_2s\rangle \wedge \langle t_2t_1\rangle + \langle t_2s\rangle \wedge \langle t_2t_3\rangle) \otimes t_2 \wedge s \\ -2T(st_3) \cdot (\langle t_3s\rangle \wedge \langle t_3t_1\rangle + \langle t_3s\rangle \wedge \langle t_3t_2\rangle) \otimes t_3 \wedge s \\ \hline +2T(t_1t_2) \cdot \mathrm{quot}\Big((t_1 \wedge t_2) \otimes \langle st_2\rangle \wedge \langle st_1\rangle\Big) \\ +2T(t_1t_3) \cdot \mathrm{quot}\Big((t_1 \wedge t_3) \otimes \langle st_3\rangle \wedge \langle st_1\rangle\Big) \\ +2T(t_2t_3) \cdot \mathrm{quot}\Big((t_2 \wedge t_3) \otimes \langle st_3\rangle \wedge \langle st_2\rangle\Big) \\ \hline -2\sum_{t\in S} T(\overline{tt_1}) \cdot \mathrm{quot}\Big((\langle t_1t_2\rangle \wedge \langle st_2\rangle) \otimes (t^* \wedge t_1^*)(s \wedge t_1 \wedge t_2 \wedge t_3)\Big) \\ +2\sum_{t\in S} T(\overline{tt_2}) \cdot \mathrm{quot}\Big((\langle t_1t_2\rangle \wedge \langle st_1\rangle) \otimes (t^* \wedge t_2^*)(s \wedge t_1 \wedge t_2 \wedge t_3)\Big) \\ -2\sum_{t\in S} T(\overline{tt_3}) \cdot \mathrm{quot}\Big((\langle st_2\rangle \wedge \langle st_1\rangle) \otimes (t^* \wedge t_3^*)(s \wedge t_1 \wedge t_2 \wedge t_3)\Big) \end{cases}$$

$$
= \begin{cases}
-2T(st_1) \cdot (\langle t_1 s\rangle \wedge \langle t_1 t_2\rangle + \langle t_1 s\rangle \wedge \langle t_1 t_3\rangle) \otimes t_1 \wedge s \\
-2T(st_2) \cdot (\langle t_2 s\rangle \wedge \langle t_2 t_1\rangle + \langle t_2 s\rangle \wedge \langle t_2 t_3\rangle) \otimes t_2 \wedge s \\
-2T(st_3) \cdot (\langle t_3 s\rangle \wedge \langle t_3 t_1\rangle + \langle t_3 s\rangle \wedge \langle t_3 t_2\rangle) \otimes t_3 \wedge s \\
\hline
-2T(t_1 t_2) \cdot \langle st_2\rangle \wedge \langle st_1\rangle \otimes (t_1 \wedge t_2) \\
-2T(t_1 t_3) \cdot \langle st_3\rangle \wedge \langle st_1\rangle \otimes (t_1 \wedge t_3) \\
-2T(t_2 t_3) \cdot \langle st_3\rangle \wedge \langle st_2\rangle \otimes (t_2 \wedge t_3) \\
\hline
-2 \sum_{t\in S} T(\overline{tt_1}) \cdot \langle t_1 t_2\rangle \wedge \langle st_2\rangle \otimes (t^* \wedge t_1^*)(s \wedge t_1 \wedge t_2 \wedge t_3) \\
+2 \sum_{t\in S} T(\overline{tt_2}) \cdot \langle t_1 t_2\rangle \wedge \langle st_1\rangle \otimes (t^* \wedge t_2^*)(s \wedge t_1 \wedge t_2 \wedge t_3) \\
-2 \sum_{t\in S} T(\overline{tt_3}) \cdot \langle st_2\rangle \wedge \langle st_1\rangle \otimes (t^* \wedge t_3^*)(s \wedge t_1 \wedge t_2 \wedge t_3)
\end{cases}
$$

$$= 0,$$

where the last equality comes from noticing that the three summations at the end are exactly what is needed to make the expression zero. It is straightforward to see that the remaining two pieces of $d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}}(s)$ are zero:

$$
(\pi_{Q\otimes_{\boldsymbol{k}}\bigwedge^2_{\boldsymbol{k}} C_{1,2}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,1}})(s) = \left\{ \begin{array}{l}
+T(st_2)\cdot(t_2\wedge t_1)\wedge(s\wedge t_1) \\
+T(st_2)\cdot(t_2\wedge t_3)\wedge(s\wedge t_3) \\
+T(st_3)\cdot(t_3\wedge t_1)\wedge(s\wedge t_1) \\
+T(st_3)\cdot(t_3\wedge t_2)\wedge(s\wedge t_2) \\
+T(st_1)\cdot(t_1\wedge t_2)\wedge(s\wedge t_2) \\
+T(st_1)\cdot(t_1\wedge t_3)\wedge(s\wedge t_3) \\
\hline
+U(s^2\wedge t_3^2)\cdot(t_1\wedge t_2)\wedge(t_1\wedge t_2) \\
+T(st_2)\cdot(t_1\wedge t_2)\wedge(s\wedge t_1) \\
-T(st_1)\cdot(t_1\wedge t_2)\wedge(s\wedge t_2) \\
-T(t_3t_2)\cdot(t_1\wedge t_2)\wedge(t_3\wedge t_1) \\
+T(t_3t_1)\cdot(t_1\wedge t_2)\wedge(t_3\wedge t_2) \\
+U(s^2\wedge t_2^2)\cdot(t_1\wedge t_3)\wedge(t_1\wedge t_3) \\
+T(st_3)\cdot(t_1\wedge t_3)\wedge(s\wedge t_1) \\
-T(st_1)\cdot(t_1\wedge t_3)\wedge(s\wedge t_3) \\
-T(t_2t_3)\cdot(t_1\wedge t_3)\wedge(t_2\wedge t_1) \\
+T(t_2t_1)\cdot(t_1\wedge t_3)\wedge(t_2\wedge t_3) \\
+U(s^2\wedge t_1^2)\cdot(t_2\wedge t_3)\wedge(t_2\wedge t_3) \\
+T(st_3)\cdot(t_2\wedge t_3)\wedge(s\wedge t_2) \\
-T(st_2)\cdot(t_2\wedge t_3)\wedge(s\wedge t_3) \\
-T(t_1t_3)\cdot(t_2\wedge t_3)\wedge(t_1\wedge t_2) \\
+T(t_1t_2)\cdot(t_2\wedge t_3)\wedge(t_1\wedge t_3)
\end{array} \right.
$$

$$
= 0
$$

and

$$(\pi_{Q\otimes_{\boldsymbol{k}}(C_{1,2}\otimes_{\boldsymbol{k}}C_{1,3})}\circ d_3\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}})(s)=\left\{\begin{array}{l}-T(st_1)\cdot(t_2\wedge t_3)\otimes(t_1-s)(t_2-t_3)\\ -T(st_2)\cdot(t_1\wedge t_3)\otimes(t_2-s)(t_1-t_3)\\ -T(st_3)\cdot(t_1\wedge t_2)\otimes(t_3-s)(t_1-t_2)\\ \hline -T(st_3)\cdot(t_1\wedge t_2)\otimes(s-t_3)(t_1-t_2)\\ -T(st_2)\cdot(t_1\wedge t_3)\otimes(s-t_2)(t_1-t_3)\\ -T(st_1)\cdot(t_2\wedge t_3)\otimes(s-t_1)(t_2-t_3)\end{array}\right.$$

$$=0.$$

Hence,

$$d_3\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}}=0.$$

It remains to verify that $d_3\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}}=0$. Let $s_1s_2\in \mathrm{sf}$ and $S=\{s_1,s_2,s_3,s_4\}$. Then

$$(d_3\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}})(s_1s_2)=\left\{\begin{array}{l}+d_3\Big(T(s_1s_3)\cdot s_1\wedge s_4+T(s_1s_4)\cdot s_1\wedge s_3\Big)\\ +d_3\Big(T(s_2s_3)\cdot s_2\wedge s_4+T(s_2s_4)\cdot s_2\wedge s_3\Big)\\ -d_3\Big(U(s_3^2\wedge s_4^2)\cdot s_3\wedge s_4\Big)\\ +2d_3\Big((\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes(s_1\wedge s_2\otimes s_3\wedge s_4)\Big)\\ +2d_3\Big((s_3\wedge s_4)\otimes((s_1-s_2)\otimes(s_1\wedge s_2\wedge s_3\wedge s_4))\Big)\\ -d_3\Big((s_1\wedge s_3)\otimes(s_2\wedge s_3\otimes s_1\wedge s_4)\Big)\\ -d_3\Big((s_2\wedge s_3)\otimes(s_1\wedge s_3\otimes s_2\wedge s_4)\Big)\\ -d_3\Big((s_1\wedge s_4)\otimes(s_2\wedge s_4\otimes s_1\wedge s_3)\Big)\\ -d_3\Big((s_2\wedge s_4)\otimes(s_1\wedge s_4\otimes s_2\wedge s_3)\Big)\\ +d_3\Big((s_1-s_2)(s_3-s_4)\otimes(s_3\wedge s_4\otimes s_1\wedge s_2)\Big)\end{array}\right.$$

and we again look at the projections of the above to the various direct summands of $Q \otimes_{\boldsymbol{k}} C_2$ and show that they are each zero. Four of the projections are short and easily seen to be zero:

$$
\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}} C_{2,1}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}})(s_1 s_2) \\
&= \begin{cases}
+2(T(s_1s_4)T(s_2s_4) - T(s_1s_3)T(s_2s_3)) \cdot ((s_1 - s_2) \otimes s_1 \wedge s_2 \wedge s_3 \wedge s_4) \\
-2T(s_1s_2)U(s_3^2 \wedge s_4^2) \cdot s_1 \otimes s_3 \wedge s_4 \wedge s_1 \wedge s_2 \\
+2T(s_1s_2)U(s_3^2 \wedge s_4^2) \cdot s_2 \otimes s_3 \wedge s_4 \wedge s_1 \wedge s_2 \\
+2d_1(s_3 \wedge s_4) \cdot (s_1 - s_2) \otimes s_1 \wedge s_2 \wedge s_3 \wedge s_4
\end{cases} \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}} \bigwedge^2_{\boldsymbol{k}} C_{1,1}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}})(s_1 s_2) \\
&= 4T(s_3s_4) \cdot (\langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \wedge (\langle s_1s_4\rangle \wedge \langle s_1s_3\rangle) \\
&= 0,
\end{aligned}
$$

$$
\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}} (C_{1,1}\otimes_{\boldsymbol{k}} C_{1,3})} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}})(s_1 s_2) \\
&= \begin{cases}
-2T(s_1s_2) \cdot \mathrm{quot}\Big((\langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \otimes (s_1 - s_2)(s_3 - s_4)\Big) \\
+2T(s_1s_2) \cdot \mathrm{quot}\Big((s_1 - s_2)(s_3 - s_4) \otimes \langle s_3s_2\rangle \wedge \langle s_3s_1\rangle\Big)
\end{cases} \\
&= 0,
\end{aligned}
$$

and

$$
\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}} \bigwedge^2_{\boldsymbol{k}} C_{1,3}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}})(s_1 s_2) \\
&= -T(s_3s_4) \cdot (s_1 - s_2)(s_3 - s_4) \wedge (s_3 - s_4)(s_1 - s_2) \\
&= 0.
\end{aligned}
$$

Now compute the image in $Q \otimes_{\boldsymbol{k}} C_{2,2}$:

$$
\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}} C_{2,2}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}})(s_1 s_2) \\
&= \begin{cases}
+2d_1(\langle s_1 s_4\rangle \wedge \langle s_1 s_3\rangle) \cdot (s_1 \wedge s_2 \otimes s_3 \wedge s_4) \\
+(U(s_2^2 \wedge s_4^2)T(s_1 s_4) + T(s_1 s_2)T(s_2 s_4) - T(s_1 s_3)T(s_3 s_4)) \cdot (s_1 \wedge s_3 \otimes s_2 \wedge s_4) \\
+(U(s_2^2 \wedge s_3^2)T(s_1 s_3) + T(s_1 s_2)T(s_2 s_3) - T(s_1 s_4)T(s_3 s_4)) \cdot (s_1 \wedge s_4 \otimes s_2 \wedge s_3) \\
+(U(s_1^2 \wedge s_4^2)T(s_2 s_4) + T(s_1 s_2)T(s_1 s_4) - T(s_2 s_3)T(s_3 s_4)) \cdot (s_2 \wedge s_3 \otimes s_1 \wedge s_4) \\
+(U(s_1^2 \wedge s_3^2)T(s_2 s_3) + T(s_1 s_2)T(s_1 s_3) - T(s_2 s_4)T(s_3 s_4)) \cdot (s_2 \wedge s_4 \otimes s_1 \wedge s_3) \\
+(T(s_1 s_3)^2 - T(s_1 s_4)^2 + T(s_2 s_4)^2 - T(s_2 s_3)^2) \cdot (s_3 \wedge s_4 \otimes s_1 \wedge s_2) \\
+2d_1(\langle s_1 s_3\rangle \wedge \langle s_1 s_4\rangle) \cdot (s_1 \wedge s_2 \otimes s_3 \wedge s_4) \\
-d_1(s_1 \wedge s_3) \cdot (s_2 \wedge s_3 \otimes s_1 \wedge s_4) \\
-d_1(s_2 \wedge s_3) \cdot (s_1 \wedge s_3 \otimes s_2 \wedge s_4) \\
-d_1(s_1 \wedge s_4) \cdot (s_2 \wedge s_4 \otimes s_1 \wedge s_3) \\
-d_1(s_2 \wedge s_4) \cdot (s_1 \wedge s_4 \otimes s_2 \wedge s_3) \\
+d_1((s_1 - s_2)(s_3 - s_4)) \cdot (s_3 \wedge s_4 \otimes s_1 \wedge s_2) \\
-U(s_1^2 \wedge s_2^2)U(s_3^2 \wedge s_4^2) \cdot (s_3 \wedge s_4 \otimes s_1 \wedge s_2) \\
-T(s_4 s_1)U(s_3^2 \wedge s_4^2) \cdot (s_3 \wedge s_1 \otimes s_4 \wedge s_2) \\
-T(s_4 s_2)U(s_3^2 \wedge s_4^2) \cdot (s_3 \wedge s_2 \otimes s_4 \wedge s_1) \\
+T(s_3 s_1)U(s_3^2 \wedge s_4^2) \cdot (s_4 \wedge s_1 \otimes s_3 \wedge s_2) \\
+T(s_3 s_2)U(s_3^2 \wedge s_4^2) \cdot (s_4 \wedge s_2 \otimes s_3 \wedge s_1).
\end{cases}
\end{aligned}
$$

Use the definition of d_1 and formula (3.2) to get $(\pi_{Q\otimes_{\boldsymbol{k}} C_{2,2}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}})(s_1 s_2) = 0$.

The projection of $d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}})(s_1s_2)$ onto $Q \otimes_{\boldsymbol{k}} (C_{1,1} \otimes_{\boldsymbol{k}} C_{1,2})$ is

$$\begin{cases}
-2T(s_1s_4) \cdot (-\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle + \langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \otimes s_1 \wedge s_3 \\
+2T(s_1s_3) \cdot (\langle s_1s_3\rangle \wedge \langle s_1s_4\rangle + \langle s_1s_2\rangle \wedge \langle s_1s_4\rangle) \otimes s_1 \wedge s_4 \\
+2T(s_2s_4) \cdot (\langle s_1s_2\rangle \wedge \langle s_1s_4\rangle + \langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \otimes s_2 \wedge s_3 \\
-2T(s_2s_3) \cdot (-\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle + \langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \otimes s_2 \wedge s_4 \\
+2T(s_1s_4) \cdot \mathrm{quot}\left((\langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \otimes s_1 \wedge s_3\right) \\
-2T(s_1s_3) \cdot \mathrm{quot}\left((\langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \otimes s_1 \wedge s_4\right) \\
-2T(s_2s_4) \cdot \mathrm{quot}\left((\langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \otimes s_2 \wedge s_3\right) \\
+2T(s_2s_3) \cdot \mathrm{quot}\left((\langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \otimes s_2 \wedge s_4\right) \\
-2T(s_2s_4) \cdot \mathrm{quot}\left((s_2 \wedge s_3) \otimes \langle s_1s_4\rangle \wedge \langle s_1s_2\rangle\right) \\
-2T(s_2s_3) \cdot \mathrm{quot}\left((s_2 \wedge s_4) \otimes \langle s_1s_3\rangle \wedge \langle s_1s_2\rangle\right) \\
-2T(s_1s_4) \cdot \mathrm{quot}\left((s_1 \wedge s_3) \otimes \langle s_2s_4\rangle \wedge \langle s_2s_1\rangle\right) \\
-2T(s_1s_3) \cdot \mathrm{quot}\left((s_1 \wedge s_4) \otimes \langle s_2s_3\rangle \wedge \langle s_2s_1\rangle\right) \\
+2U(s_3^2 \wedge s_4^2) \cdot (\langle s_3s_4\rangle \wedge \langle s_3s_1\rangle) \otimes (s_3 \wedge s_4) \\
+2U(s_3^2 \wedge s_4^2) \cdot (\langle s_3s_4\rangle \wedge \langle s_3s_2\rangle) \otimes (s_3 \wedge s_4) \\
+2U(s_1^2 \wedge s_2^2) \cdot \mathrm{quot}\left((\langle s_1s_3\rangle \wedge \langle s_1s_4\rangle) \otimes s_3 \wedge s_4\right) \\
-2\sum_{t\in S} U(t^2 \wedge s_1^2) \cdot \mathrm{quot}\left((s_3 \wedge s_4) \otimes \langle s_1t\rangle^*(\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle \wedge \langle s_1s_4\rangle)\right) \\
+2\sum_{t\in S} U(t^2 \wedge s_2^2) \cdot \mathrm{quot}\left((s_3 \wedge s_4) \otimes \langle s_2t\rangle^*(\langle s_1s_2\rangle \wedge \langle s_1s_3\rangle \wedge \langle s_1s_4\rangle)\right).
\end{cases}$$

Collect terms with the same $s_i \wedge s_j$ for each pair $\{i, j\}$ to obtain

$$(\pi_{Q\otimes_{\boldsymbol{k}}(C_{1,1}\otimes_{\boldsymbol{k}} C_{1,2})} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}})(s_1s_2)$$

$$
= \left\{ \begin{array}{l}
-2T(s_1s_4)\cdot(-\langle s_1s_2\rangle\wedge\langle s_1s_3\rangle+\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes s_1\wedge s_3 \\
+2T(s_1s_4)\cdot\mathrm{quot}\Big((\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes s_1\wedge s_3\Big) \\
-2T(s_1s_4)\cdot\mathrm{quot}\Big((s_1\wedge s_3)\otimes\langle s_2s_4\rangle\wedge\langle s_2s_1\rangle\Big) \\
\hline
+2T(s_1s_3)\cdot(\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle+\langle s_1s_2\rangle\wedge\langle s_1s_4\rangle)\otimes s_1\wedge s_4 \\
-2T(s_1s_3)\cdot\mathrm{quot}\Big((\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes s_1\wedge s_4\Big) \\
-2T(s_1s_3)\cdot\mathrm{quot}\Big((s_1\wedge s_4)\otimes\langle s_2s_3\rangle\wedge\langle s_2s_1\rangle\Big) \\
\hline
+2T(s_2s_4)\cdot(\langle s_1s_2\rangle\wedge\langle s_1s_4\rangle+\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes s_2\wedge s_3 \\
-2T(s_2s_4)\cdot\mathrm{quot}\Big((\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes s_2\wedge s_3\Big) \\
-2T(s_2s_4)\cdot\mathrm{quot}\Big((s_2\wedge s_3)\otimes\langle s_1s_4\rangle\wedge\langle s_1s_2\rangle\Big) \\
\hline
-2T(s_2s_3)\cdot(-\langle s_1s_2\rangle\wedge\langle s_1s_3\rangle+\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes s_2\wedge s_4 \\
+2T(s_2s_3)\cdot\mathrm{quot}\Big((\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes s_2\wedge s_4\Big) \\
-2T(s_2s_3)\cdot\mathrm{quot}\Big((s_2\wedge s_4)\otimes\langle s_1s_3\rangle\wedge\langle s_1s_2\rangle\Big) \\
\hline
+2U(s_3^2\wedge s_4^2)\cdot(\langle s_3s_4\rangle\wedge\langle s_3s_1\rangle)\otimes(s_3\wedge s_4) \\
+2U(s_3^2\wedge s_4^2)\cdot(\langle s_3s_4\rangle\wedge\langle s_3s_2\rangle)\otimes(s_3\wedge s_4) \\
+2U(s_1^2\wedge s_2^2)\cdot\mathrm{quot}\Big((\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\otimes s_3\wedge s_4\Big) \\
-2\sum_{t\in S}U(t^2\wedge s_1^2)\cdot\mathrm{quot}\Big((s_3\wedge s_4)\otimes\langle s_1t\rangle^*(\langle s_1s_2\rangle\wedge\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\Big) \\
+2\sum_{t\in S}U(t^2\wedge s_2^2)\cdot\mathrm{quot}\Big((s_3\wedge s_4)\otimes\langle s_2t\rangle^*(\langle s_1s_2\rangle\wedge\langle s_1s_3\rangle\wedge\langle s_1s_4\rangle)\Big)
\end{array}\right.
$$

As mentioned for the earlier calculation of $\pi_{Q\otimes_{\boldsymbol{k}}(C_{1,1}\otimes_{\boldsymbol{k}}C_{1,2})}\circ d_3\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}}$, quot : $C_1\otimes_{\boldsymbol{k}}C_1\to\bigwedge^2_{\boldsymbol{k}}C_1$ is the natural quotient map and we use the isomorphism (3.5) to write elements of $\bigwedge^2_{\boldsymbol{k}}C_1$. Thus, the terms above the final horizontal line above add to zero. Therefore, the expression becomes

$$\begin{cases}
+2U(s_3^2 \wedge s_4^2) \cdot (\langle s_3 s_4\rangle \wedge \langle s_3 s_1\rangle) \otimes (s_3 \wedge s_4) \\
+2U(s_3^2 \wedge s_4^2) \cdot (\langle s_3 s_4\rangle \wedge \langle s_3 s_2\rangle) \otimes (s_3 \wedge s_4) \\
+2U(s_1^2 \wedge s_2^2) \cdot \operatorname{quot}\Big((\langle s_1 s_3\rangle \wedge \langle s_1 s_4\rangle) \otimes s_3 \wedge s_4\Big) \\
-2 \sum\limits_{t \in S} U(t^2 \wedge s_1^2) \cdot \operatorname{quot}\Big((s_3 \wedge s_4) \otimes \langle s_1 t\rangle^*(\langle s_1 s_2\rangle \wedge \langle s_1 s_3\rangle \wedge \langle s_1 s_4\rangle)\Big) \\
+2 \sum\limits_{t \in S} U(t^2 \wedge s_2^2) \cdot \operatorname{quot}\Big((s_3 \wedge s_4) \otimes \langle s_2 t\rangle^*(\langle s_1 s_2\rangle \wedge \langle s_1 s_3\rangle \wedge \langle s_1 s_4\rangle)\Big).
\end{cases}$$

Expand the two summations to get

$$\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}}(C_{1,1}\otimes_{\boldsymbol{k}}C_{1,2})} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}})(s_1 s_2) \\
&= \left\{\begin{array}{l}
+2U(s_3^2 \wedge s_4^2) \cdot (\langle s_1 s_2\rangle \wedge \langle s_1 s_3\rangle) \otimes (s_3 \wedge s_4) \\
+2U(s_4^2 \wedge s_1^2) \cdot (\langle s_1 s_2\rangle \wedge \langle s_1 s_3\rangle) \otimes (s_3 \wedge s_4) \\
-2U(s_3^2 \wedge s_2^2) \cdot (\langle s_1 s_2\rangle \wedge \langle s_1 s_3\rangle) \otimes (s_3 \wedge s_4) \\ \hline
+2U(s_3^2 \wedge s_4^2) \cdot (\langle s_1 s_2\rangle \wedge \langle s_1 s_4\rangle) \otimes (s_3 \wedge s_4) \\
-2U(s_3^2 \wedge s_1^2) \cdot (\langle s_1 s_2\rangle \wedge \langle s_1 s_4\rangle) \otimes (s_3 \wedge s_4) \\
+2U(s_4^2 \wedge s_2^2) \cdot (\langle s_1 s_2\rangle \wedge \langle s_1 s_4\rangle) \otimes (s_3 \wedge s_4) \\ \hline
+2U(s_1^2 \wedge s_2^2) \cdot (\langle s_1 s_3\rangle \wedge \langle s_1 s_4\rangle) \otimes (s_3 \wedge s_4) \\
+2U(s_2^2 \wedge s_1^2) \cdot (\langle s_1 s_3\rangle \wedge \langle s_1 s_4\rangle) \otimes (s_3 \wedge s_4) \\
-2U(s_1^2 \wedge s_2^2) \cdot (\langle s_1 s_3\rangle \wedge \langle s_1 s_4\rangle) \otimes (s_3 \wedge s_4)
\end{array}\right. \\
&= \left\{\begin{array}{l}
-2U(s_1^2 \wedge s_2^2) \cdot (\langle s_1 s_2\rangle \wedge \langle s_1 s_3\rangle) \otimes (s_3 \wedge s_4) \\ \hline
+2U(s_1^2 \wedge s_2^2) \cdot (\langle s_1 s_2\rangle \wedge \langle s_1 s_4\rangle) \otimes (s_3 \wedge s_4) \\ \hline
+2U(s_2^2 \wedge s_1^2) \cdot (\langle s_1 s_3\rangle \wedge \langle s_1 s_4\rangle) \otimes (s_3 \wedge s_4)
\end{array}\right. \\
&= 0,
\end{aligned}$$

the last equality following from (3.3) and the fact that $U(s_2^2 \wedge s_1^2) = -U(s_1^2 \wedge s_2^2)$. The next part is easy to check.

$$
\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}}\bigwedge^2_{\boldsymbol{k}} C_{1,2}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}} C_{4,2}})(s_1 s_2) \\
&= \begin{cases}
-T(s_1s_3)\cdot(s_1\wedge s_3)\wedge(s_3\wedge s_4) - T(s_1s_3)\cdot(s_1\wedge s_2)\wedge(s_2\wedge s_4) \\
-T(s_1s_4)\cdot(s_1\wedge s_2)\wedge(s_2\wedge s_3) + T(s_1s_4)\cdot(s_1\wedge s_4)\wedge(s_3\wedge s_4) \\
+T(s_2s_4)\cdot(s_1\wedge s_2)\wedge(s_1\wedge s_3) + T(s_2s_4)\cdot(s_2\wedge s_4)\wedge(s_3\wedge s_4) \\
+T(s_2s_3)\cdot(s_1\wedge s_2)\wedge(s_1\wedge s_4) - T(s_2s_3)\cdot(s_2\wedge s_3)\wedge(s_3\wedge s_4) \\
-T(s_1s_4)\cdot(s_2\wedge s_3)\wedge(s_1\wedge s_2) + T(s_1s_2)\cdot(s_2\wedge s_3)\wedge(s_1\wedge s_4) \\
+T(s_3s_4)\cdot(s_2\wedge s_3)\wedge(s_3\wedge s_2) - T(s_3s_2)\cdot(s_2\wedge s_3)\wedge(s_3\wedge s_4) \\
-T(s_1s_3)\cdot(s_2\wedge s_4)\wedge(s_1\wedge s_2) + T(s_1s_2)\cdot(s_2\wedge s_4)\wedge(s_1\wedge s_3) \\
+T(s_4s_3)\cdot(s_2\wedge s_4)\wedge(s_4\wedge s_2) - T(s_4s_2)\cdot(s_2\wedge s_4)\wedge(s_4\wedge s_3) \\
-T(s_2s_4)\cdot(s_1\wedge s_3)\wedge(s_2\wedge s_1) + T(s_2s_1)\cdot(s_1\wedge s_3)\wedge(s_2\wedge s_4) \\
+T(s_3s_4)\cdot(s_1\wedge s_3)\wedge(s_3\wedge s_1) - T(s_3s_1)\cdot(s_1\wedge s_3)\wedge(s_3\wedge s_4) \\
-T(s_2s_3)\cdot(s_1\wedge s_4)\wedge(s_2\wedge s_1) + T(s_2s_1)\cdot(s_1\wedge s_4)\wedge(s_2\wedge s_3) \\
+T(s_4s_3)\cdot(s_1\wedge s_4)\wedge(s_4\wedge s_1) - T(s_4s_1)\cdot(s_1\wedge s_4)\wedge(s_4\wedge s_3) \\
-U(s_3^2\wedge s_4^2)\cdot(s_3\wedge s_1)\wedge(s_4\wedge s_1) \\
-U(s_3^2\wedge s_4^2)\cdot(s_3\wedge s_2)\wedge(s_4\wedge s_2) \\
-U(s_1^2\wedge s_3^2)\cdot(s_2\wedge s_3)\wedge(s_2\wedge s_4) \\
-U(s_1^2\wedge s_4^2)\cdot(s_2\wedge s_4)\wedge(s_2\wedge s_3) \\
-U(s_2^2\wedge s_3^2)\cdot(s_1\wedge s_3)\wedge(s_1\wedge s_4) \\
-U(s_2^2\wedge s_4^2)\cdot(s_1\wedge s_4)\wedge(s_1\wedge s_3) \\
-2\sum_{t\in S} T(\overline{ts_1})\cdot\Big((s_3\wedge s_4)\wedge(t^*\wedge s_1^*)(s_1\wedge s_2\wedge s_3\wedge s_4)\Big) \\
+2\sum_{t\in S} T(\overline{ts_2})\cdot\Big((s_3\wedge s_4)\wedge(t^*\wedge s_2^*)(s_1\wedge s_2\wedge s_3\wedge s_4)\Big)
\end{cases}
\end{aligned}
$$

$$= \begin{cases} -2T(s_1s_3)\cdot(s_1\wedge s_3)\wedge(s_3\wedge s_4)+2T(s_1s_4)\cdot(s_1\wedge s_4)\wedge(s_3\wedge s_4) \\ -2T(s_2s_3)\cdot(s_2\wedge s_3)\wedge(s_3\wedge s_4)+2T(s_2s_4)\cdot(s_2\wedge s_4)\wedge(s_3\wedge s_4) \\ \hline -2\sum_{t\in S} T(\overline{ts_1})\cdot\Big((s_3\wedge s_4)\wedge(t^*\wedge s_1^*)(s_1\wedge s_2\wedge s_3\wedge s_4)\Big) \\ +2\sum_{t\in S} T(\overline{ts_2})\cdot\Big((s_3\wedge s_4)\wedge(t^*\wedge s_2^*)(s_1\wedge s_2\wedge s_3\wedge s_4)\Big) \end{cases}$$

$$= 0.$$

Finally, the one remaining projection of $d_3\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}}(s_1s_2)$ is

$$(\pi_{Q\otimes_{\boldsymbol{k}}(C_{1,2}\otimes_{\boldsymbol{k}}C_{1,3})}\circ d_3\circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}})(s_1s_2)$$

$$= \begin{cases} +U(s_3^2\wedge s_4^2)\cdot(s_1\wedge s_2)\cdot(s_3-s_4)(s_1-s_2) \\ +T(s_1s_3)\cdot\mathrm{quot}\Big((s_2\wedge s_3)\otimes(s_1-s_3)(s_2-s_4)\Big) \\ +T(s_1s_4)\cdot\mathrm{quot}\Big((s_2\wedge s_4)\otimes(s_1-s_4)(s_2-s_3)\Big) \\ +T(s_2s_3)\cdot\mathrm{quot}\Big((s_1\wedge s_3)\otimes(s_2-s_3)(s_1-s_4)\Big) \\ +T(s_2s_4)\cdot\mathrm{quot}\Big((s_1\wedge s_4)\otimes(s_2-s_4)(s_1-s_3)\Big) \\ +T(s_1s_3)\cdot(s_2\wedge s_3)\otimes(s_1-s_4)(s_3-s_2) \\ -T(s_1s_4)\cdot(s_2\wedge s_4)\otimes(s_1-s_3)(s_2-s_4) \\ -T(s_2s_4)\cdot(s_1\wedge s_4)\otimes(s_1-s_4)(s_2-s_3) \\ -T(s_2s_3)\cdot(s_1\wedge s_3)\otimes(s_1-s_3)(s_2-s_4) \\ +U(s_3^2\wedge s_4^2)\cdot\mathrm{quot}\Big((s_1-s_2)(s_3-s_4)\otimes s_1\wedge s_2\Big) \\ +T(s_3s_2)\cdot\mathrm{quot}\Big((s_1-s_2)(s_3-s_4)\otimes s_3\wedge s_1\Big) \\ -T(s_3s_1)\cdot\mathrm{quot}\Big((s_1-s_2)(s_3-s_4)\otimes s_3\wedge s_2\Big) \\ -T(s_4s_2)\cdot\mathrm{quot}\Big((s_1-s_2)(s_3-s_4)\otimes s_4\wedge s_1\Big) \\ +T(s_4s_1)\cdot\mathrm{quot}\Big((s_1-s_2)(s_3-s_4)\otimes s_4\wedge s_2\Big). \end{cases}$$

Collect all terms with the same $s_i \wedge s_j$, we get

$$
\begin{aligned}
&(\pi_{Q\otimes_{\boldsymbol{k}}(C_{1,2}\otimes_{\boldsymbol{k}}C_{1,3})} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}})(s_1s_2) \\
&= \begin{cases}
+T(s_1s_3)\cdot \operatorname{quot}\Big((s_2\wedge s_3)\otimes(s_1-s_3)(s_2-s_4)\Big) \\
-T(s_3s_1)\cdot \operatorname{quot}\Big((s_1-s_2)(s_3-s_4)\otimes s_3\wedge s_2\Big) \\
+T(s_1s_3)\cdot (s_2\wedge s_3)\otimes(s_1-s_4)(s_3-s_2) \\
\hline
+T(s_1s_4)\cdot \operatorname{quot}\Big((s_2\wedge s_4)\otimes(s_1-s_4)(s_2-s_3)\Big) \\
+T(s_4s_1)\cdot \operatorname{quot}\Big((s_1-s_2)(s_3-s_4)\otimes s_4\wedge s_2\Big) \\
-T(s_1s_4)\cdot (s_2\wedge s_4)\otimes(s_1-s_3)(s_2-s_4) \\
\hline
+T(s_2s_3)\cdot \operatorname{quot}\Big((s_1\wedge s_3)\otimes(s_2-s_3)(s_1-s_4)\Big) \\
+T(s_3s_2)\cdot \operatorname{quot}\Big((s_1-s_2)(s_3-s_4)\otimes s_3\wedge s_1\Big) \\
-T(s_2s_3)\cdot (s_1\wedge s_3)\otimes(s_1-s_3)(s_2-s_4) \\
\hline
+T(s_2s_4)\cdot \operatorname{quot}\Big((s_1\wedge s_4)\otimes(s_2-s_4)(s_1-s_3)\Big) \\
-T(s_4s_2)\cdot \operatorname{quot}\Big((s_1-s_2)(s_3-s_4)\otimes s_4\wedge s_1\Big) \\
-T(s_2s_4)\cdot (s_1\wedge s_4)\otimes(s_1-s_4)(s_2-s_3)
\end{cases} \\
&= 0
\end{aligned}
$$

by formula (3.4), as well as the same considerations mentioned for the calcuations of $\pi_{Q\otimes_{\boldsymbol{k}}\bigwedge^2_{\boldsymbol{k}}C_{1,2}} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,1}}$ and $\pi_{Q\otimes_{\boldsymbol{k}}(C_{1,1}\otimes_{\boldsymbol{k}}C_{1,2})} \circ d_3 \circ d_4|_{Q\otimes_{\boldsymbol{k}}C_{4,2}}$.

We can now conclude that $d_3 \circ d_4 = 0$, and we have finished showing that $X_\bullet \to \operatorname{Sym}^{\boldsymbol{k}}_\bullet(V)$ is indeed a complex. $\square$

3.5 The Homology of $X_\bullet$ by Macaulay2

The complex $X_\bullet$ we have defined is a free Q-module complex

$$
X_\bullet : \begin{matrix} Q(-5)^9 \\ \oplus \\ Q(-6)^{45} \end{matrix} \xrightarrow{d_4} \begin{matrix} Q(-4)^6 \\ \oplus \\ Q(-5)^{90} \end{matrix} \xrightarrow{d_3} \begin{matrix} Q(-3)^9 \\ \oplus \\ Q(-4)^{45} \end{matrix} \xrightarrow{d_2} Q(-2)^{10} \xrightarrow{d_1} Q.
$$

Macaulay2 [12] indicates that $H_1(X_\bullet) = 0$ and that, if we refine $X_\bullet$ into a minimal homogeneous resolution of $\boldsymbol{k}[I_2]$, then that resolution will look like:

$$Q(-10) \longrightarrow Q(-9)^9 \longrightarrow Q(-8)^{36} \longrightarrow Q(-7)^{85} \longrightarrow Q(-6)^{125}$$

$$\longrightarrow Q(-5)^{107} \longrightarrow \begin{matrix} Q(-3)^9 \\ \oplus \\ Q(-4)^{39} \end{matrix} \longrightarrow Q(-2)^{10} \longrightarrow Q.$$

Bibliography

[1] T. Cortadellas Benitez and C. D'Andrea, *Minimal generators of the defining ideal of the Rees algebra associated with a rational plane parametrization with* $\mu = 2$, Canad. J. Math. **66** (2014), 1225–1249.

[2] J.A. Boswell and V. Mukundan, *Rees algebras and almost linearly presented ideals*, J. Algebra **460** (2016), 102–127.

[3] W. Bruns, A. Conca, and M. Varbaro, *Maximal minors and linear powers*, J. Reine Angew. Math. **702** (2015), 41–53

[4] W. Bruns and J. Herzog, *Cohen-Macaulay Rings*, Cambridge Studies in Advanced Mathematics **39**, Cambridge University Press, Cambridge, 1998.

[5] D. Buchsbaum and D. Eisenbud, *Algebra structures for finite free resolutions, and some structure theorems for ideals of codimension 3*, Amer. J. Math. **99** (1977), 447–485.

[6] D. Cox, A. Kustin, C. Polini, and B. Ulrich, *A study of singularities on rational curves via syzygies*, Mem. Amer. Math. Soc. **222** (2013), no. 1045.

[7] D. Eisenbud, *Commutative Algebra with a View Toward Algebraic Geometry*, Graduate Texts in Mathematics **150**, Springer-Verlag, New York, 1995.

[8] S. El Khoury and A. Kustin, *Artinian Gorenstein algebras with linear resolutions*, J. Algebra **420** (2014), 402–474.

[9] S. El Khoury and A. Kustin, *The explicit minimal resolution constructed from a Macaulay inverse system*, J. Algebra **440** (2015), 145–186.

[10] S. El Khoury and A. Kustin, *The structure of Gorenstein-linear resolutions of Artinian algebras*, J. Algebra **453** (2016), 492–560.

[11] W. Fulton and J. Harris, *Representation Theory: A First Course*, Graduate Texts in Mathematics **129**, Springer-Verlag, New York, 1991.

[12] D. Grayson and M. Stillman, *Macaulay2, a software system for research in algebraic geometry*. Available at `https://faculty.math.illinois.edu/Macaulay2/`.

[13] T.H. Gulliksen and G. Levin, *Homology of local rings*, Queen's Papers in Pure and Appl. Math. **20**, Kingston, Ontario: Queen's University, 1969.

[14] J. Herzog and M. Kühl, *On the Betti numbers of finite pure and linear resolutions*, Commutative Algebra **12** (1984), 1627–1646.

[15] J. Herzog, A. Simis, and W. Vasconcelos, *Koszul homology and blowing-up rings*, Commutative algebra (Trento, 1981), pp. 79–169, Lecture Notes in Pure and Appl. Math. **84**, Dekker, New York, 1983.

[16] J. Hong, A. Simis, and W.V. Vasconcelos, *On the homology of two-dimensional elimination*, J. Symbolic Comput. **43** (2008), 275–292.

[17] C. Huneke, *On the symmetric and Rees algebra of an ideal generated by a d-sequence*, J. Algebra **62** (1980), 268–275.

[18] M. Johnson, *Second analytic deviation one ideals and their Rees algebras*, J. Pure Appl. Algebra **119** (1997), 171–183.

[19] A. Kustin, *The resolution of the universal ring for modules of rank zero and projective dimension two*, J. Algebra **310** (2007), 261–289.

[20] A. Kustin, C. Polini, and B. Ulrich, *Rational normal scrolls and the defining equations of Rees algebras*, J. Reine Angew. Math. **650** (2011), 23–65.

[21] A. Kustin, C. Polini, and B. Ulrich, *Blowups and fibers of morphisms*, Nagoya Math. J. **224** (2016), 168–201.

[22] A. Kustin, C. Polini, and B. Ulrich, *The bi-graded structure of symmetric algebras with applications to Rees rings*, J. Algebra **469** (2017) 188–250.

[23] A. Kustin, C. Polini, and B. Ulrich, *The equations defining blowup algebras of height three Gorenstein ideals*, Algebra Number Theory **11** (2017), 1489–1525.

[24] A. Kustin and B. Ulrich, *A family of complexes associated to an almost alternating map, with applications to residual intersections*, Mem. Amer. Math. Soc. **95** (1992), no. 461.

[25] A. Iarrobino, *Associated graded algebra of a Gorenstein Artin algebra*, Mem. Amer. Math. Soc. **107** (1994), no. 514.

[26] A. Iarrobino and V. Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics **1721**, Springer-Verlag, Berlin, 1999.

[27] J. Madsen, *Equations of Rees algebras of ideals in two variables*, preprint 2015.

[28] F.S. Macaulay, *The algebraic theory of modular systems*, reissued with an Introduction by P. Roberts in 1994, Cambridge University Press, Cambridge, 1916.

[29] S. Morey, *Equations of blowups of ideals of codimension two and three*, J. Pure Appl. Algebra **109** (1996), 197–211.

[30] S. Morey and B. Ulrich, *Rees algebras of ideals with low codimension*, Proc. Amer. Math. Soc. **124** (1996), 3653–3661.

[31] L. Nguyen, *On Rees algebras of linearly presented ideals*, J. Algebra **420** (2014), 186–200.