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Special Fiber Rings of Certain Height Four Gorenstein Ideals

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Special Fiber Rings of Certain Height Four Gorenstein Ideals

by

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Abstract

Let $S$ be a set of four variables, $k$ a field of characteristic not equal to two such that $k$ contains all square roots, and $I$ a height four Gorenstein ideal of $k[S]$ generated by nine quadratics so that $I$ has a Gorenstein-linear resolution. We define a complex $X_\bullet$ so that each module of $X_\bullet$ is the tensor product of a certain polynomial ring $Q$ in nine variables and a direct sum of indecomposable $k[\text{Sym}(S)]$-modules and the differential maps are $Q$- and $k[\text{Sym}(S)]$-module homomorphisms. Work with the Macaulay2 software suggests that $H_0(X_\bullet)$ is the special fiber ring of $I$ and $H_1(X_\bullet) = 0$. 
TABLE OF CONTENTS

ACKNOWLEDGMENTS ................................................................. iii

ABSTRACT ................................................................................ iv

CHAPTER 1 INTRODUCTION ........................................................ 1
  1.1 Special Fiber Rings and Blowup Algebras ............................... 1
  1.2 Divided Power Algebras .................................................... 3
  1.3 Indecomposable Representations of $S_4$ ............................... 6

CHAPTER 2 THE IDEA ............................................................... 11
  2.1 The Precise Project ........................................................... 13

CHAPTER 3 THE COMPLEX ........................................................ 15
  3.1 Set-Up ............................................................................... 16
  3.2 Module Definitions .......................................................... 22
  3.3 Definitions of the Differential Maps .................................... 26
  3.4 The maps and modules of $X_\bullet$ form a complex .................. 38
  3.5 The Homology of $X_\bullet$ by Macaulay2 ............................... 63

BIBLIOGRAPHY ......................................................................... 65
Chapter 1

Introduction

1.1 Special Fiber Rings and Blowup Algebras

Definition 1.1. Let $k$ be a field. Given homogeneous polynomials $f_1, f_2, \ldots, f_m$ in $P = k[x_1, x_2, \ldots, x_d]$, all of the same degree, the subring $k[f_1, f_2, \ldots, f_m]$ of $P$ is called the special fiber ring of the ideal $(f_1, f_2, \ldots, f_m)$.

Given $f_1, f_2, \ldots, f_m$ as in the definition above, there is a well-defined map of projective spaces $\Psi : \mathbb{P}^{d-1} \to \mathbb{P}^{m-1}$ given by

$$\Psi(X) = [f_1(X) : f_2(X) : \cdots : f_m(X)].$$

In algebraic geometry terms, the special fiber ring is the homogeneous coordinate ring of $\overline{\text{Im}(\Psi)}$, the closure of $\text{Im}(\Psi)$ in the Zariski topology.

Definition 1.2. Given the same setup as above, the Rees algebra of $I$, denoted $\mathcal{R}(I)$, is the graded subalgebra $P[It]$ of the polynomial ring $P[t]$. That is,

$$\mathcal{R}(I) = P \oplus It \oplus I^2t^2 \oplus \cdots = \bigoplus_{j=0}^{\infty} I^jt^j.$$ 

Letting $m$ be the maximal homogeneous ideal of $P$, the special fiber ring is a quotient of the Rees algebra:

$$\mathcal{R}(I)/m\mathcal{R}(I) \cong k[f_1, f_2, \ldots, f_m].$$

To algebraic geometers, the Rees algebra is the bi-homogeneous coordinate ring of the graph of $\Psi$. Special fiber rings and Rees algebras are so-called blowup algebras.
that occur when one “blows up” a variety along a subvariety. Information about the geometry of $\text{Im}(\Psi)$ can be gleaned from these algebras. The multiplicity of the special fiber ring gives the degree of the parametrization [21] and the degrees of the defining equations of the Rees algebra give information about singularities [6].

The other important algebra to consider in this context is, using the same notation as before, the symmetric algebra of $I$ over the polynomial ring $P$:

$$\text{Sym}^P(I) = P \oplus I \oplus \text{Sym}^2_P(I) \oplus \text{Sym}^3_P(I) \oplus \cdots.$$ 

There is a natural epimorphism

$$\text{Sym}^P(I) \to R(I)$$

from the universal mapping property of $\text{Sym}^P(I)$. Blowup algebras arising from ideals so that the aforementioned epimorphism $\text{Sym}^P(I) \to R(I)$ is an isomorphism has been understood for decades [15, 17].

The situation in which $I$ is primary to the maximal homogenous ideal is of interest currently. In this setting, Bruns, Conca, and Varbaro [3] have studied the case that $I$ is a determinantal ideal; results concerning the case of perfect ideals of height two can be found in [29, 30, 16, 6, 20, 1, 27, 2, 22, 31], among other sources. Since height three Gorenstein ideals have been characterized by Buchsbaum and Eisenbud [5], this case is also of interest, for instance to Johnson and Morey [18, 29]. Kustin, Polini, and Ulrich [23] found the defining equations of blowup algebras when the corresponding ideal is height three Gorenstein, primary to the maximal ideal, and has a homogeneous presentation matrix consisting of linear forms. We would like to say what we can in the case where $I$ is a height four Gorenstein ideal.

Since a characterization of height four Gorenstein ideals analogous to that of height three Gorenstein ideals is not known, we approach this case from another angle. As before, let $\mathbf{k}$ be a field and $P$ be a polynomial ring in $d$ variables, $P = \mathbf{k}[x_1, x_2, \ldots, x_d]$. Let $n$ be a positive integer. When $I$ is a homogeneous ideal in $P$
generated by forms of degree \( n \) so that \( P/I \) is an Artinian Gorenstein algebra that has a Gorenstein linear resolution, El Khoury and Kustin [8, 9, 10] used Macaulay Inverse Systems to give a minimal homogeneous resolution of \( P/I \) by free \( P \)-modules. A pleasant fact about this resolution is that it is constructed in a polynomial manner from the coefficients of the generator of the Macaulay Inverse System.

1.2 Divided Power Algebras

The notion of a divided power algebra is used to define Macaulay Inverse Systems and we will use divided power algebras in our eventual complexes, so we present the pertinent ideas now.

**Definition 1.3.** Let \( R \) be a commutative ring and \( A \) a graded, commutative \( R \)-algebra with \( A_0 = R \). A system of divided powers in \( A \) is a collection of functions

\[
\bigcup_{i>0} A_i \to A, \ x \mapsto x^{(d)},
\]

one for each nonnegative integer \( d \), satisfying each of the following for any \( x, y \in \bigcup_{i>0} A_i \) and \( d, e \in \mathbb{Z}_{\geq 0} \):

1. \( x^{(0)} = 1 \) and \( x^{(1)} = x \)
2. \( \text{deg} \left( x^{(d)} \right) = d \cdot \text{deg}(x) \)
3. \( x^{(d)} x^{(e)} = \frac{(d + e)!}{d!e!} x^{(d+e)} \)
4. \( \left( x^{(d)} \right)^{(e)} = \frac{(de)!}{e!(d!)^e} x^{(de)} \)
5. \( (xy)^{(d)} = d! x^{(d)} y^{(d)} = x^d y^{(d)} = x^{(d)} y^d \)
6. \( (ax)^{(d)} = a^d x^{(d)} \) for \( a \in A_0 \)
7. \( (x + y)^{(d)} = \sum_{e=0}^{d} x^{(e)} y^{(d-e)} \)
Let \( n \) be a positive integer and \( x \in \bigcup_{i > 0} A_i \). It is easily proven by (1) and (3) above that \( x^n = n!x^{(n)} \). Hence, if \( \text{char}(R) = 0 \), then \( x^{(n)} = x^n / n! \). All of the properties above follow from treating \( x^{(n)} \) like \( x^n / n! \) if this were defined.

A slightly more general definition of divided powers can be found in [7, Appendix 2]. We will use one particular example of a divided power algebra, which we now present.

**Definition 1.4.** Let \( k \) be a field. If \( U \) is a \( d \)-dimensional vector space over \( k \) then the divided power algebra \( D_k^*(U^*) \) is defined as a vector space by

\[
D_k^*(U^*) = \bigoplus_{j \geq 0} D_k^j(U^*),
\]

where

\[
D_k^j(U^*) = \text{Hom}_k(\text{Sym}_k^j(U), k).
\]

To get the algebra structure on \( D_k^*(U^*) \), first recall that there is a coalgebra structure on \( \text{Sym}_k^*(U) \). The comultiplication map \( \Delta : \text{Sym}_k^*(U) \to \text{Sym}_k^*(U \oplus U) = \text{Sym}_k^1(U) \otimes_k \text{Sym}_k^1(U) \) is the \( k \)-algebra homomorphism defined by \( \Delta(u) = u \otimes 1 + 1 \otimes u \) for \( u \in \text{Sym}_k^1(U) \). The counit is the projection \( \varepsilon : \text{Sym}_k^*(U) \to \text{Sym}_k^1(U) / U \text{Sym}_k^0(U) = \text{Sym}_k^0(U) = k \). There are also, for any nonnegative integers \( i \) and \( j \), \( k \)-linear maps \( \tau_{i,j} : D_k^i(U^*) \otimes_k D_k^j(U^*) \to \text{Hom}_k(\text{Sym}_k^i(U) \otimes_k \text{Sym}_k^j(U), k) \subseteq \text{Hom}_k(\text{Sym}_k^i(U) \otimes_k \text{Sym}_k^j(U), k) \) defined by

\[
(\tau_{i,j}(\phi \otimes \psi))(u_i \otimes u_j) = \phi(u_i)\psi(u_j).
\]

Taking the \( k \)-dual of \( \Delta \) and composing with a \( \tau_{i,j} \) gives a \( k \)-linear map

\[
\Delta^* \circ \tau_{i,j} : D_k^i(U^*) \otimes_k D_k^j(U^*) \to D_k^*(U^*).
\]

The \( \Delta^* \circ \tau_{i,j} \) maps, for all nonnegative integers \( i \) and \( j \), define the multiplication on homogeneous elements of \( D_k^*(U^*) \). The unit of \( D_k^*(U^*) \) is \( \varepsilon^*(1) \), the image of 1 ∈ \( k \).
under the $k$-dual of $\varepsilon$. For proof that these do indeed define $D^k_\bullet(U^*)$ as a $k$-algebra, see [7, Appendix 2].

It is true that $D^k_\bullet(U^*)$ has a system of divided powers. It is easiest to describe the divided powers explicitly if we choose a basis for $U$. Let $\{x_1, x_2, \ldots, x_d\}$ is a basis for $U = \text{Sym}^1_1(U)$. Then there is a dual basis $\{x_1^*, x_2^*, \ldots, x_d^*\}$ for $U^* = D^k_\bullet(U^*)$ and, for each positive integer $i$, a basis for $\text{Sym}^i_1(U)$ consisting of the degree $i$ monomials in the variables $x_1, x_2, \ldots, x_d$. There is a divided power structure on $D^k_\bullet(U^*)$ satisfying,

$$ (x_1^*)^{(a_1)}(x_2^*)^{(a_2)} \cdots (x_d^*)^{(a_d)} = (x_1^{a_1}x_2^{a_2}\cdots x_d^{a_d})^* .$$

The properties of divided powers then give the definition of a divided power of any element of $D^k_\bullet(U^*)$. Again, all relevant proofs are found in [7, Appendix 2].

It is also the case that $D^k_\bullet(U^*)$ has a comultiplication map $\Delta_D : D^k_\bullet(U^*) \to D^k_\bullet(U^*) \otimes_k D^k_\bullet(U^*)$ defined by $\Delta_D(\psi) = \psi \otimes 1 + 1 \otimes \psi$ for $\psi \in D^k_1(U^*)$. We will only use this in the next remark.

**Remark 1.5.** If $A : U^* \to W_1$ and $B : U^* \to W_2$ are $k$-module maps, then there is a unique $k$-module map $\varphi : D^k_2(U^*) \to W_1 \otimes_k W_2$ such that $\varphi(x^{(2)}) = A(x) \otimes B(x)$ for all $x \in U^* = D^k_1(U^*)$. This map is the composition

$$ D^k_2(U^*) \to D^k_1(U^*) \otimes_k D^k_1(U^*) \to W_1 \otimes_k W_2 $$

$$ (A \otimes B) \circ (\pi_{D^k_1(U^*)} \otimes \pi_{D^k_1(U^*)}) \circ \Delta_D ) .$$

An unsophisticated way to see this is to see that by (7) of Definition 1.3, $(x+y)^{(2)} = x^{(2)} + xy + y^{(2)}$ for $x, y \in U^*$. Applying $\varphi$ and using the fact that we have a $k$-module map produces

$$ \varphi(xy) = A(x) \otimes B(y) + A(y) \otimes B(x) $$

for any $x, y \in U^*$.
1.2.1 The algebra $D^k_1(U^*)$ is a $\text{Sym}^k_1(U)$-module

Given an element $u_i \in \text{Sym}^k_1(U)$ and a $w_j \in D^k_1(U^*)$, define $u_i \cdot w_j \in D^k_{j-i}(U^*) = \text{Hom}_k\left(\text{Sym}^k_{j-i}(U), k\right)$ to be the map

$$(u_i \cdot w_j)(s_{j-i}) = w_j(u_is_{j-i})$$

for $s_{j-i} \in \text{Sym}^k_{j-i}(U)$. One can check that this does define a $\text{Sym}^k_1(U)$-module structure on $D^k_1(U^*)$.

If we have a basis $\{x_1, x_2, \ldots, x_d\}$ for $U$ with dual basis $\{x_1^*, x_2^*, \ldots, x_d^*\}$, then we can also express the $\text{Sym}^k_1(U)$-module structure on $D^k_1(U^*)$ in terms of the basis. Letting $i \in \{1, 2, \ldots, d\}$ and $a_1, a_2, \ldots, a_d$ be nonnegative integers with $a_i$ positive, $x_i \in \text{Sym}^k_1(U)$ acts on monomials in $D^k_1(U^*)$ as

$$x_i \cdot \prod_{j=1}^{d} (x_j^*)^{(a_j)} = \prod_{j=1}^{i-1} (x_j^*)^{(a_j)} \cdot (x_i^*)^{(a_i-1)} \cdot \prod_{j=i+1}^{d} (x_j^*)^{(a_j)}.$$  

**Definition 1.6.** Under the given module structure, we can define the annihilator of a subset $S \subseteq \text{Sym}^k_1(U)$ to be

$$\text{Ann}_{D(U^*)}(S) = \{w \in D^k_1(U^*) \mid s \cdot w = 0 \text{ for all } s \in S\}$$

and the annihilator of a subset $T \subseteq D^k_1(U^*)$ to be

$$\text{Ann}_{\text{Sym}^k_1(U)}(T) = \{p \in \text{Sym}^k_1(U) \mid p \cdot t = 0 \text{ for all } t \in T\}.$$  

It is also true that $\text{Sym}^k_1(U)$ is a $D^k_1(U^*)$-module, but we will not need this structure.

1.3 Indecomposable Representations of $S_4$

The special fiber ring we will be interested in is a subring of a polynomial ring in four variables. Since none of these four variables will be treated differently from the others, it is reasonable to consider the action by the group of permutations of the
variables. This action will induce an action on the modules of the complex we will define. First, we recall a few relevant definitions and facts from representation theory:

Let $k$ be a field. A representation of a group $G$ is a group homomorphism $\rho : G \to \text{GL}(U)$ for some $k$-vector space $U$. Given a representation $\rho : G \to \text{GL}(U)$ of a group $G$, one can give $U$ the structure of a module over the group algebra $k[G]$ by defining, for $g \in G$ and $u \in U$,

$$g \cdot u = \rho(g)(u)$$

and then extending this definition by linearity to all of $k[G]$. Conversely, if $U$ is a $k[G]$-module, then the definition

$$\rho(g)(u) = g \cdot u$$

is a group homomorphism $\rho : G \to \text{GL}(U)$. It is clear that the two operations above are inverse to each other, so that we can interchangeably talk about $k[G]$-modules or representations of $G$.

A module is decomposable if can be written as the direct sum of two nonzero proper submodules and is indecomposable otherwise. A main result of from representation theory of finite groups is the following.

**Theorem 1.7.** If $k$ is algebraically closed with $\text{char}(k) = 0$ and $G$ is a finite group, then any $k[G]$-module is a direct sum of indecomposable $k[G]$-modules and, up to isomorphism, there are only finitely many indecomposable $k[G]$-modules. In fact, the number of non-isomorphic indecomposable $k[G]$-modules is equal to the number of conjugacy classes in $G$.

**Definition 1.8.** Given a representation $\rho : G \to \text{GL}(U)$ of a finite group $G$ over a field $k$, where $U$ is finite-dimensional, the character of $\rho$ is the function $\chi : G \to k$, $\chi(g) = \text{tr}(\rho(g))$, the trace of the linear map $\rho(g)$.
**Facts 1.9.** The following facts about characters over algebraically closed fields of zero characteristic are useful to recall. The specifics can be found in [11, Lecture 2].

1. Up to isomorphism, a representation is defined uniquely by its character.

2. There is an inner product defined on characters.

3. A representation is indecomposable if and only if the inner product of its character with itself is 1.

4. There are orthogonality relations on the characters of the indecomposable representations of a finite group.

5. The character of a direct sum of representations is the sum of the characters.

6. The character of a tensor product of representations is the pointwise multiplication of the characters.

We now consider the setup pertaining to our eventual complex. Let $k$ be a field, $S$ a set of four variables, and $V$ the $k$-vector space spanned by $S$. We consider $\text{Sym}(S)$, the group of permutations of $S$. The complex to be defined in Chapter 3 will be made of $k[\text{Sym}(S)]$-modules and $\text{Sym}(S)$-equivariant maps.

Since there are exactly five conjugacy classes in $\text{Sym}(S)$, there are exactly five indecomposable $k[\text{Sym}(S)]$-modules up to isomorphism. For a general discussion of these representations (i.e., representations of the group $S_4$), see [11, Section 2.3]. We will give these representations in contexts in which they will appear in Chapter 3.

There are two one-dimensional $k[\text{Sym}(S)]$-modules. The first is the trivial one, for which every $\sigma \in \text{Sym}(S)$ acts as the identity; this is isomorphic to $\bigwedge^0_k V$. The other is isomorphic to $\bigwedge^2_k V$, where $\sigma \in \text{Sym}(S)$ acts as the sign of $\sigma$; this is referred to as the “alternating” representation. It is clear that these do define representations of $\text{Sym}(S)$. Since each is one-dimensional, they are indecomposable.
There are two three-dimensional indecomposable $k[\text{Sym}(S)]$-modules:

$$V \langle \sum_{s \in S} s \rangle$$

and

$$V \langle \sum_{s \in S} s \rangle \otimes_k \wedge_4^k V.$$

It is easy to see that these are indeed representations of $\text{Sym}(S)$, as $\sigma \cdot \sum_{s \in S} s = \sum_{s \in S} s$ and the tensor product of representations is a representation. The fact that these are indecomposable if $k$ is characteristic zero and algebraically closed can be shown via their characters.

Lastly, there is one two-dimensional indecomposable $k[\text{Sym}(S)]$-module, and it is isomorphic to the subvector space of $\text{Sym}_2(V) = k[S]_2$ spanned by the set

$$\{(s_1 - s_2)(s_3 - s_4) \mid S = \{s_1, s_2, s_3, s_4\}\}.$$

To see that this is indeed a $k[\text{Sym}(S)]$-module, note that every element of the form $(s_1 - s_2)(s_3 - s_4)$ for $S = \{s_1, s_2, s_3, s_4\}$ acted on by a permutation of $S$ will be sent to either another such element or the negative of another such element. Also, if we let $S = \{s_1, s_2, s_3, s_4\}$, then this subvector space is generated by the set

$$\{(s_1 - s_2)(s_3 - s_4), (s_1 - s_3)(s_2 - s_4), (s_1 - s_4)(s_2 - s_3)\}$$

and, since

$$0 = (s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3)$$

in $k[S]_2$, the given module can be generated as a vector space by two elements. It is not difficult to see that it cannot be generated by one element and is, hence, two-dimensional. The fact that it is indecomposable if $k$ is characteristic zero and algebraically closed is again most easily done by computing the character.
Thus, if $\text{char}(k) = 0$ and $k$ is algebraically closed, then the five indecomposable $k[\text{Sym}(S)]$-modules are have been found. Computation of the characters of these modules gives the following character table:

<table>
<thead>
<tr>
<th></th>
<th>id</th>
<th>$\tau_2$</th>
<th>$\tau_3$</th>
<th>$\tau_4$</th>
<th>$\tau_{2,2}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\bigwedge^0_k V$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>$\bigwedge^4_k V$</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>$V/\langle \sum s \rangle$</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>$V/\langle \sum s \rangle \otimes_k \bigwedge^4_k V$</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>$\langle (s_1 - s_2)(s_3 - s_4) \mid S = {s_1, s_2, s_3, s_4} \rangle$</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
</tbody>
</table>

where id is the identity of $\text{Sym}(S)$, $\tau_i$ is a representative of the conjugacy class of $i$-cycles for $i \in \{1, 2, 3\}$, and $\tau_{2,2}$ is a representative of the conjugacy class of consisting of permutations that can be written as two disjoint 2-cycles.

For the complex we define, we will not necessarily assume that our field has characteristic zero or is algebraically closed. Thus, it does not have to be the case that every $k[\text{Sym}(S)]$-module is a direct sum of the modules above or that the above are indecomposable over our field. However, the $k[\text{Sym}(S)]$-modules used in the complex to be defined will be direct sums of the modules given above.
CHAPTER 2

THE IDEA

Let $V$ be a vector space of dimension 4 over a field $k$ and $I$ be a height 4 Gorenstein ideal in $\text{Sym}_k^k(V)$ which is generated by quadratics and for which $\text{Sym}_k^k(V)/I$ has a Gorenstein-linear resolution. We will also assume that $\text{char}(k) \neq 2$ to be able to take advantage of results about nondegenerate symmetric bilinear forms over such fields, as well as assuming that $k$ is closed under square roots in order to simplify maps.

We now give definitions of the concepts needed. The following come from [8]. Gorenstein rings and ideals can be defined more generally (see, for instance, [4]), but we present the definitions in the context we use.

**Definition 2.1.** If $k$ is a field and $A = \bigoplus_{i \geq 0} A_i$ is a graded Artinian $k$-algebra with $A_0 = k$ and maximal ideal $m = \bigoplus_{i > 0} A_i$, then

(a) The socle of $A$ is $0 :_A m = \{a \in A \mid am = 0\}$.

(b) The algebra $A$ is Gorenstein if $\dim_k (0 :_A m) = 1$. In this case, the socle degree of $A$ is the degree of a generator of $0 :_A m$.

**Definition 2.2.** Let $k$ be a field and a $I$ a homogeneous ideal of a polynomial ring $P = k[x_1, x_2, \ldots, x_d]$.

(a) The ideal $I$ is a height $d$ Gorenstein ideal if $P/I$ is an Artinian Gorenstein $k$-algebra.

(b) If $I$ is a height $d$ Gorenstein ideal, then $P/I$ has a Gorenstein-linear resolution if there is a positive integer $n$ so that the minimal homogeneous resolution of $P/I$
by free $P$-modules has the form

\[ 0 \to P(-2n - d + 2) \to P(-n - d + 2)^{\beta_{d-1}} \to P(-n - d + 3)^{\beta_{d-2}} \to \ldots \]

\[ \to \ldots \to P(-n - 2)^{\beta_3} \to P(-n - 1)^{\beta_2} \to P(-n)^{\beta_1} \to P. \]

The $\beta_i$ for $i \in \{1, 2, \ldots, d-1\}$ can be found from the Herzog-Kühl formulas [14].

The remaining required definition is that of a Macaulay Inverse System, the existence of which comes from the following theorem.

**Theorem (Macaulay, [28]).** Let $U$ be a dimension $d$ vector space over a field $k$. The annihilators with respect to the $\text{Sym}^k(U)$-module structure on $D^k(U^*)$ given in Section 1.2.1 give a one-to-one correspondence

\[
\begin{align*}
&\quad \begin{cases}
\text{nonzero homogeneous} \\
\text{grade } d \text{ Gorenstein} \\
\text{ideals of } \text{Sym}^k(U)
\end{cases} 
\leftrightarrow \begin{cases}
\text{nonzero homogeneous} \\
\text{cyclic submodules} \\
\text{of } D^k(U^*)
\end{cases}
\end{align*}
\]

\[ I \mapsto \text{Ann}_{D(U^*)}(I) \]

\[ \text{Ann}_{\text{Sym}(U)}(M) \leftrightarrow M \]

Furthermore, for a nonzero homogeneous height $d$ Gorenstein ideal $I$ of $\text{Sym}^k(U)$, the socle degree of $\text{Sym}^k(U)/I$ is equal to the degree of a homogeneous generator of $\text{Ann}(I)$.

**Definition 2.3.** In the setting of Macaulay’s Theorem, the homogeneous cyclic submodule $\text{Ann}(I)$ of $D^k(U^*)$ is the Macaulay Inverse System of the ideal $I$.

In [8, Proposition 1.8], El Khoury and Kustin give various properties equivalent to a homogeneous Gorenstein ideal in a polynomial ring having a Gorenstein-linear resolution. For our specific situation, the proposition is as follows.

**Proposition 2.4 (El Khoury and Kustin, [8]).** Let $k$ be a field and $I$ a height 4 Gorenstein ideal of $P = k[x_1, x_2, x_3, x_4]$ which is generated by quadratics. Let $U$
be the vector space $P_1$ and $\phi \in D^k(U^*)$ a homogeneous generator of the Macaulay Inverse System of $I$. The following are equivalent.

(a) The ideal $I$ has a Gorenstein-linear resolution.

(b) The minimal homogeneous resolution of $P/I$ by free $P$-modules has the form

$$0 \to P(-6) \to P(-4)^9 \to P(-3)^{16} \to P(-2)^9 \to P.$$

(c) All minimal generators of $I$ have degree 2 and the socle of $P/I$ has degree 2.

(d) Both $I_1 = 0$ and $[P/I]_3 = 0$.

(e) $\phi \in D^k(U^*)$ and the homomorphism $U \to U^*$ given by $u \mapsto u \cdot \phi \in U^* = D^k(U^*)$, where $u \cdot \phi$ represents the result of $u \in U = \text{Sym}^k(U)$ acting on $\phi \in D^k(U^*)$ as defined in subsection 1.2.1, is an isomorphism.

(f) $\phi \in D^k(U^*)$ and the matrix $(\phi(x_i x_j))_{i,j=1,2,3,4}$ is invertible.

2.1 The Precise Project

Let $k$ be a field with $\text{char}(k) \neq 2$ that is closed under square roots and let $V$ be a dimension 4 $k$-vector space. Let $I$ be a height 4 Gorenstein ideal of $k[V]$ generated by quadratics which has a Gorenstein-linear resolution. Since $I$ is generated by quadratics, the special fiber ring is $k[I_2]$.

Macaulay’s Theorem gives a $\phi \in D^k(V^*)$ so that $I = \text{Ann}(\phi)$. Suppose that $\text{Sym}^k(V) \cong k[x_1, x_2, x_3, x_4]$. By Proposition 2.4, the matrix $(\phi(x_i x_j))_{i,j=1,2,3,4}$ is invertible. Notice that $(\phi(x_i x_j))_{i,j=1,2,3,4}$ is also symmetric. By properties of nondegenerate symmetric bilinear forms and the fact that $\text{char}(k) \neq 2$, there is a basis for $V$
with respect to which the matrix is diagonal. So we may assume that

\[
(\phi(x_i x_j))_{i,j=1,2,3,4} = \begin{pmatrix}
    u_1 & 0 & 0 & 0 \\
    0 & u_2 & 0 & 0 \\
    0 & 0 & u_3 & 0 \\
    0 & 0 & 0 & u_4
\end{pmatrix}
\]

for some \( u_1, u_2, u_3, u_4 \in k \). Since the matrix is invertible, \( u_1, u_2, u_3, \) and \( u_4 \) are units. Since \( k \) is closed under square roots, we may alter the basis elements again so that we can assume that \( u_1 = u_2 = u_3 = u_4 = 1 \). Then we have that

\[
I = \text{Ann}(\phi) = (x_1 x_2, x_1 x_3, x_1 x_4, x_2 x_3, x_2 x_4, x_3 x_4, x_2^2 - x_1^2, x_3^2 - x_1^2, x_4^2 - x_1^2).
\]

In the next chapter, we will define a complex \( X_\bullet \). Computer experimentation with Macaulay2 [12] indicates that \( H_1(X_\bullet) = 0 \) and that \( H_0(X_\bullet) \) is isomorphic to the special fiber ring \( k[I_2] \).
Chapter 3

The Complex

Let $k$ be a field. Our goal is to define a complex of free modules with zeroth homology equal to the subring $k[x, y, z, w, y^2 - x^2, z^2 - x^2, w^2 - x^2]$ of $k[x, y, z, w]$. As noted in Section 2.1, if $\text{char}(k) \neq 2$ and $k$ is closed under square roots, then if $I$ is a height four Gorenstein ideal in $k[x, y, z, w]$ that is generated by quadratics and has a Gorenstein-linear resolution, then we may assume that $I$ is the ideal $(xy, xz, xw, yz, yw, y^2 - x^2, z^2 - x^2, w^2 - x^2)$ after a change of variables.

Before starting the specific complex we want here, we present a nice, formal complex arising from a small complex. The complex we will define will have a subcomplex of this form. The ideas are derived from the work of Tate and can be found in [13].

Lemma 3.1. Let $k$ be a field, $R$ a commutative $k$-algebra, and $N_1, N_2$ be free $R$-modules. If $N_2 \xrightarrow{\delta_2} N_1 \xrightarrow{\delta_1} R$ is a complex, then so is

$$D^k(N_2) \xrightarrow{\Delta_3} N_1 \otimes_k N_2 \xrightarrow{\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)} N_2 \oplus \Lambda^2_k N_1 \xrightarrow{\delta_2 \otimes \text{Koszul}} N_1 \xrightarrow{\delta_1} R,$$  \hspace{1cm} (3.1)

where $\text{Kos}^{\delta_1} : \Lambda^2_k N_1 \rightarrow N_1$ is the Koszul map associated to $\delta_1$, $q : N_1 \otimes_k N_1 \rightarrow \Lambda^2_k N_1$ is the natural quotient map, and $\Delta_3(\theta^{(2)}) = \delta_2(\theta) \otimes \theta$ for $\theta \in N_2$.

By Remark 1.5, the above definition of $\Delta_3$ does define $\Delta_3$ on all of $D^k_2(N_2)$. The Koszul map $\text{Kos}^{\delta_1} : \Lambda^2_k N_1 \rightarrow N_1$ is the map defined by $\text{Kos}^{\delta_1}(m \wedge n) = \delta_1(m) \cdot n - \delta_1(n) \cdot m$ for $m, n \in N_1$; that is, the action of $\Lambda^*_k N_1^*$ on $\Lambda^*_k N_1$ arising from the homomorphism $\delta_1 : N_1 \rightarrow R$. 

15
Proof of Lemma 3.1. The Tate technique of killing cycles shows that the complex $N_2 \xrightarrow{\delta_2} N_1 \xrightarrow{\delta_1} R$ induces a Differential Graded $R$-Algebra $\wedge^*_R N_1 \otimes D^*_R N_2$. We give a hands-on argument for the relevant part of this technique.

The fact that $\delta_1 \circ (\delta_2 \oplus \text{Kos} \delta_1) = 0$ is immediate by assumption and the definition of Koszul maps. Let $n_1 \in N_1$ and $n_2 \in N_2$. Then

$$\left( (\delta_2 \oplus \text{Kos} \delta_1) \circ (\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)) \right)(n_1 \otimes n_2)$$

$$= (\delta_2 \oplus \text{Kos} \delta_1) (\delta_1(n_1) \cdot n_2 - n_1 \wedge \delta_2(n_2))$$

$$= \delta_2 (\delta_1(n_1) \cdot n_2) - \text{Kos} \delta_1 (n_1 \wedge \delta_2(n_2))$$

$$= \delta_1(n_1)\delta_2(n_2) - \delta_1(n_1)\delta_2(n_2) + \delta_1(\delta_2(n_2)) \cdot n_1$$

$$= 0$$

since $\delta_1 \circ \delta_2 = 0$ by assumption.

Finally, since all the modules in (3.1) are free $R$-modules, Lemma 1.3 from [19] implies that we need only show that $(\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)) \circ \Delta_3$ is zero on elements of the form $\theta^{(2)} \in D^k_2(N_2)$ to show that $(\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)) \circ \Delta_3$ is identically zero. So, let $\theta \in N_2$. Then, since $\delta_1 \circ \delta_2 = 0$,

$$\left( (\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)) \circ \Delta_3 \right)(\theta^{(2)}) = (\delta_1 \otimes 1 - q \circ (1 \otimes \delta_2)) (\delta_2(\theta) \otimes \theta)$$

$$= \delta_1(\delta_2(\theta)) \cdot \theta - \delta_2(\theta) \wedge \delta_2(\theta)$$

$$= 0.$$

Thus, (3.1) is, in fact, a complex. \qed

3.1 Set-Up

The following definitions give the basic structure we will use to define our complex, as well as the ring over which the complex will be defined.

(a) Let $k$ be a field, $S$ be a set of 4 variables, $V$ be the vector space spanned by $S$, $\text{sf}$ be the set of square free quadratic monomials in the variables $S$, $\text{ps}$ be the set
of perfect square quadratic monomials in the variables S, SF be the vector space spanned by sf, PS be the vector space spanned by ps. Observe that
\[ \text{Sym}^k_2(V) = SF \oplus PS. \]

(b) Of course, \( S = \{ s | s \in S \} \) and \( S^* = \{ s^* | s \in S \} \) are dual bases for \( V \) and \( V^* \).

c) Let \( \text{one} : V \to k \) be defined by \( \text{one}(s) = 1 \) for all \( s \in S \). So, \( \text{one} \in V^* \) and
\[ \text{one} = \sum_{s \in S} s^*. \]

d) Observe that \( \text{one}^{(2)} \) is a well-defined element in \( D^k_2(V^*) \). Furthermore, \( \text{one}^{(2)}(m) = 1 \) for all monomials \( m \) of degree two in \( S \). Furthermore,
\[ \text{one}^{(2)} = \sum_{s_1 \neq s_2 \in S} s_1^* s_2^* + \sum_{s \in S} s^{*^{(2)}}. \]

e) The modules \( D^k_2(V^*) \) and \( \text{Sym}^k_2(V) \) are dual to one another; hence, \( \land^\bullet_k D^k_2(V^*) \)
acts on \( \land^\bullet_k \text{Sym}^k_2(V) \). In particular, \( \text{one}^{(2)} : \land^3_k PS \to \land^2_k PS \) behaves as follows:
\[ \text{one}^{(2)}(s_1^2 \land s_2^2 \land s_3^2) = s_2^2 \land s_3^2 - s_1^2 \land s_3^2 + s_1^2 \land s_2^2, \]
for \( s_1, s_2, s_3 \in S \).

(f) Let \( Q = \text{Sym}^k_4 \left( \text{SF} \oplus \frac{\land^2 PS}{(\text{one}^{(2)}(\land^3 PS))} \right) \).

The complex that we will define in this chapter and the following be will a \( Q \)
module complex. Notice that \( Q \) is isomorphic to a polynomial ring in 9 variables.

g) Let \( \Omega \in \text{Sym}^k_4(V) \) be the product of the four elements of \( S \).

(h) If \( m \in \text{sf} \), then let \( \overline{m} \) be the unique element of \( \text{sf} \) with \( m\overline{m} = \Omega \).

(i) Recall the definition of \( \overline{m} \) for \( m \in \text{sf} \) as given in (h). Define
\[ \text{SF} = \frac{\text{SF}}{\langle \{ m - \overline{m} | m \in \text{sf} \} \rangle}. \]
If \( m \in \text{sf} \), then let \( \langle m \rangle \) be the image of \( m \) in \( \overline{\text{SF}} \); in particular, for \( m \in \text{sf} \), \( \langle m \rangle = \langle \overline{m} \rangle \). Furthermore, \( \langle m \rangle \) is a coset which consists of the two elements \( m \) and \( \overline{m} \) of \( \text{sf} \). Observe that \( \overline{\text{SF}} \) is a vector space of dimension 3.

(j) Recall the element \( \text{one}^{(2)} \in D^k_2(V^*) \). That is, \( \text{one}^{(2)} \) is a homomorphism

\[
\text{one}^{(2)} : \text{Sym}^k_2(V) \to k.
\]

Let \( \text{one}^{(2)}|_{\text{SF}} \) be the restriction of \( \text{one}^{(2)} \) to \( \text{SF} \). Observe that \( \text{one}^{(2)}|_{\text{SF}} : \text{SF} \to k \) factors through \( \text{SF} \to \overline{\text{SF}} \). Let \( \overline{\text{one}^{(2)}|_{\text{SF}}} : \overline{\text{SF}} \to k \) be the induced map. Of course,

\[
\overline{\text{one}^{(2)}|_{\text{SF}}} \in \overline{\text{SF}}^*.
\]

(k) Observe that \( \langle xy \rangle, \langle xz \rangle, \langle xw \rangle \) is a basis for \( \overline{\text{SF}} \) if \( S = \{x, y, z, w\} \). (One gets the same basis for any choice of names for the elements of \( S \).) So, \( \langle xy \rangle^*, \langle xz \rangle^*, \langle xw \rangle^* \) is the basis for \( \overline{\text{SF}}^* \) which is dual to the basis \( \langle xy \rangle, \langle xz \rangle, \langle xw \rangle \) for \( \overline{\text{SF}} \). When \( \overline{\text{one}^{(2)}|_{\text{SF}}} \) is written in terms of the basis \( \langle xy \rangle^*, \langle xz \rangle^*, \langle xw \rangle^* \), one obtains

\[
\overline{\text{one}^{(2)}|_{\text{SF}}} = \langle xy \rangle^* + \langle xz \rangle^* + \langle xw \rangle^* \in \overline{\text{SF}}^*.
\]

3.1.1 The \( k[\text{Sym}(S)] \)-module structure of the defined spaces

As stated previously, our complex will be a \( Q \)-module complex. In addition, the complex will also have the structure of \( k[\text{Sym}(S)] \)-module complex.

We can observe that the \( k \)-vector spaces we have so far defined are \( k[\text{Sym}(S)] \)-modules. The fact that \( V \), \( \text{SF} \), and \( \text{PS} \) are is clear. To see that \( \overline{\text{SF}} \) is as well, recall the definition of \( \overline{\text{SF}} \): \( \overline{\text{SF}} = \frac{\text{SF}}{\langle \{m - \overline{m} \mid m \in \text{sf} \} \rangle} \). Let \( S = \{s_1, s_2, s_3, s_4\} \). Then \( \text{sf} = \{s_1s_2, s_3s_4, s_1s_3, s_2s_4, s_1s_4, s_2s_3\} \). It is easy to check that each of the elements \( (s_1s_2) \) and \( (s_1s_2s_3s_4) \) of \( \text{Sym}(S) \) send every element of \( \{m - \overline{m} \mid m \in \text{sf} \} \) into \( \{m - \overline{m} \mid m \in \text{sf} \} \). Since \( (s_1s_2) \) and \( (s_1s_2s_3s_4) \) generate \( \text{Sym}(S) \), we have that

\[
\sigma \cdot (m_1 - \overline{m_1}) \in \{m - \overline{m} \mid m \in \text{sf} \}
\]
for any \( m_1 \in \text{sf} \) and any \( \sigma \in \text{Sym}(S) \). It follows that \( \overline{\text{SF}} \) is a \( \mathbf{k}[\text{Sym}(S)] \)-module. Since \( V, \text{PS}, \) and \( \overline{\text{SF}} \) are now \( \mathbf{k}[\text{Sym}(S)] \)-modules, so are \( \Lambda^i_{\mathbf{k}} V, \Lambda^i_{\mathbf{k}} \overline{\text{SF}}, \Lambda^i_{\mathbf{k}} \text{PS}, \text{Sym}^i_{\mathbf{k}}(V) \), and \( D^i_{\mathbf{k}}(V^*) \) for each \( i > 0 \).

For the remainder of the section, we consider the map \( \text{one} : V \to \mathbf{k} \) and associated maps and modules. Both \( \text{one} \in D^1_{\mathbf{k}}(V^*) = V^* \) and \( \text{one}^{(2)} \in D^2_{\mathbf{k}}(V^*) \) are \( \text{Sym}(S) \)-equivariant: this follows from the facts that \( \text{one} \) is constant on \( S \), \( \text{one}^{(2)} \) is constant on \( \text{ps} \cup \text{sf} \), and \( \text{Sym}(S) \) acts trivially on elements of \( \mathbf{k} \). Alternatively, writing \( \text{one} = \sum_{s \in S} s^* \) and \( \text{one}^{(2)} = \sum_{s_1 \neq s_2 \in S} s_1^* s_2^* + \sum_{s \in S} s^{* (2)} \) as in (c) and (d) of the previous section makes the \( \text{Sym}(S) \)-equivariance clear.

Note that the decomposition \( \text{Sym}^2_{\mathbf{k}}(V) = \text{PS} \oplus \text{SF} \) is a decomposition not just of vector spaces, but also of \( \mathbf{k}[\text{Sym}(S)] \)-modules and that

\[
D^2_{\mathbf{k}}(V^*) = \text{Hom}_{\mathbf{k}}(\text{Sym}^2_{\mathbf{k}}(V), \mathbf{k}) = \text{Hom}_{\mathbf{k}}(\text{PS} \oplus \text{SF}, \mathbf{k}) = \text{Hom}_{\mathbf{k}}(\text{PS}, \mathbf{k}) \oplus \text{Hom}_{\mathbf{k}}(\text{SF}, \mathbf{k}).
\]

As a consequence, both \( \text{one}^{(2)} \mid_{\text{PS}} \) and \( \text{one}^{(2)} \mid_{\text{SF}} \) are \( \text{Sym}(S) \)-equivariant. Additionally, \( \overline{\text{one}^{(2)}} \mid_{\text{SF}} : \overline{\text{SF}} \to \mathbf{k} \) is \( \text{Sym}(S) \)-equivariant, which again follows easily from either of the facts that \( \text{one}^{(2)} \mid_{\text{SF}} \) is constant on \( \text{sf} \) or from writing the map in the form of part (k) of the previous section.

Now, for any positive integer \( n \), the Koszul maps \( \Lambda^n_{\mathbf{k}} \text{PS} \to \Lambda^{n-1}_{\mathbf{k}} \text{PS}, \Lambda^n_{\mathbf{k}} \text{SF} \to \Lambda^{n-1}_{\mathbf{k}} \text{SF}, \) and \( \Lambda^n_{\mathbf{k}} \overline{\text{SF}} \to \Lambda^{n-1}_{\mathbf{k}} \overline{\text{SF}} \) induced by \( \text{one}^{(2)} \mid_{\text{PS}}, \text{one}^{(2)} \mid_{\text{SF}}, \) and \( \overline{\text{one}^{(2)}} \mid_{\text{SF}} \), respectively, are \( \text{Sym}(S) \)-equivariant. To verify this assertion, \( m_1 \land m_2 \land \cdots \land m_n \in \Lambda^n_{\mathbf{k}} \overline{\text{SF}} \) and \( \sigma \in \text{Sym}(S) \). Then

\[
\text{one}^{(2)} \mid_{\overline{\text{SF}}} (\sigma \cdot (m_1 \land m_2 \land \cdots \land m_n))
\]

\[
= \text{one}^{(2)} \mid_{\overline{\text{SF}}} (\sigma \cdot m_1 \land \sigma \cdot m_2 \land \cdots \land \sigma \cdot m_n)
\]

\[
= \sum_{i=1}^{n} (-1)^{i+1} \text{one}^{(2)} (\sigma \cdot m_i) \land m_1 \land \cdots \land \sigma m_{i-1} \land \sigma \cdot m_{i+1} \land \cdots \land \sigma \cdot m_n
\]

\[
= \sum_{i=1}^{n} (-1)^{i+1} \sigma \cdot \text{one}^{(2)} (m_i) \land m_1 \land \cdots \land \sigma m_{i-1} \land \sigma \cdot m_{i+1} \land \cdots \land \sigma \cdot m_n
\]

\[
= \sum_{i=1}^{n} (-1)^{i+1} \text{one}^{(2)} (m_i) \land m_1 \land \cdots \land \sigma m_{i-1} \land \sigma \cdot m_{i+1} \land \cdots \land \sigma \cdot m_n
\]
\[ \sigma \cdot \sum_{i=1}^{n} (-1)^{i+1} \text{one}^{(2)}(m_i) m_1 \land \cdots \land m_{i-1} \land m_{i+1} \land \cdots \land m_n \]

\[ = \sigma \cdot \text{one}^{(2)}|_{\text{SF}}(m_1 \land m_2 \land \cdots \land m_n). \]

The fourth equality above is due to the fact that \( \text{Sym}(S) \) acts trivially on \( k \) and \( \text{one}^{(2)}(m_i) \in k \) for any \( m_i \in \text{SF} \). The argument for \( \wedge^n \text{PS} \) or \( \wedge^n \text{SF} \) is the exactly the same.

Knowing now that \( \text{SF}, \wedge^2_k \text{PS} \) and \( \wedge^3_k \text{PS} \) are \( k[\text{Sym}(S)] \)-modules and that the map the induced map \( \text{one}^{(2)} : \wedge^3_k \text{PS} \to \wedge^2_k \text{PS} \) is \( \text{Sym}(S) \)-equivariant, we can conclude that

\[ Q = \text{Sym}_k^k \left( \text{SF} \oplus \frac{\wedge^2_{k} \text{PS}}{(\text{one}^{(2)}|_{\text{SF}}) (\wedge^3_{k} \text{SF})} \right) \]

is \( k[\text{Sym}(S)] \)-module as well.

Although we have not yet mentioned the module \( \frac{\wedge^2_{k} \text{SF}}{(\text{one}^{(2)}|_{\text{SF}}) (\wedge^3_{k} \text{SF})} \), it will be used later, so we note now that the above shows that it is a module over \( k[\text{Sym}(S)] \).

### 3.1.2 Preliminary Map Definitions

As the final bit of set-up before we define the first complex, we define a few homomorphisms that will be relevant.

(a) Recall that

\[ \text{Sym}_2^k(V) = \text{SF} \oplus \text{PS} \]

and

\[ \wedge^2_k \text{Sym}_2^k(V) = \wedge^2_k (\text{SF} \oplus \text{PS}) = \wedge^2_k \text{SF} \oplus (\text{SF} \otimes_k \text{PS}) \oplus \wedge^2_k \text{PS}. \]

Let \( T : \text{Sym}_2^k V \to Q \) be defined by

\[ T|_{\text{SF}} \text{ is the natural injection} \]

and

\[ T|_{\text{PS}} \text{ is the zero map.} \]

Define \( U : \wedge^2_k \text{Sym}_2^k V \to Q \) by

\[ U|_{\wedge^2_k \text{SF} \oplus (\text{SF} \otimes_k \text{PS})} \text{ is the zero map} \]
and

\[ U|_{\Lambda^2_k \text{PS}} \] is the natural quotient map.

**Observation 3.2.** Recall that \(Q = \text{Sym}_k \left( SF \oplus \frac{\Lambda^2_{\text{PS}}}{\text{one}(2)(\Lambda^2_{\text{PS}})} \right)\). With the above definitions of \(T\) and \(U\), we may now think of \(Q\) as the ring of polynomials in the variables \(\{T(s_is_j)\}\) and \(\{U(s^2_is_j^2)\}\) for \(s_i, s_j\) distinct elements of \(S\) subject to:

\[ U(s^2_i \wedge s^2_j) - U(s^2_i \wedge s^2_k) + U(s^2_j \wedge s^2_k) = 0. \] (3.2)

(b) Define the \(Q\)-module augmentation map \(\varepsilon : Q \to \text{Sym}_k V\) by

\[ \varepsilon(T(m)) = m \quad \text{for} \ m \in \text{sf} \quad \text{and} \]

\[ \varepsilon(U(x^2 \wedge y^2)) = \text{one}(2)(x^2 \wedge y^2) = y^2 - x^2 \quad \text{for} \ x, y \in S. \]

(c) Define \(\alpha : \Lambda^4_k V \to \Lambda^2_k SF\) to be the \(k\)-isomorphism such that

\[ \alpha(s_1 \wedge s_2 \wedge s_3 \wedge s_4) = \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle. \]

We need to see that \(\alpha\) makes sense. From subsection 3.1.1, we know that both \(\Lambda^4_k V\) and \(\Lambda^2_k SF\) are \(k[\text{Sym}(S)]\)-modules.

Consider the action of the cycles \((s_1s_2)\) and \((s_1s_2s_3s_4)\):

\[ (s_1s_2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle = \langle s_2s_1 \rangle \wedge \langle s_2s_3 \rangle \wedge \langle s_2s_4 \rangle \]

\[ = \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle \]

\[ = -\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle \]

\[ (s_1s_2s_3s_4) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle = \langle s_2s_3 \rangle \wedge \langle s_2s_4 \rangle \wedge \langle s_2s_1 \rangle \]

\[ = \langle s_1s_4 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_2 \rangle \]

\[ = -\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle \]
Since \( \{(s_1s_2), (s_1s_2s_3s_4)\} \) is a generating set for \( \text{Sym}(S) \), we conclude that, for any \( \sigma \in \text{Sym}(S) \),
\[
\sigma \cdot (s_1s_2) \land (s_1s_3) \land (s_1s_4) = \text{sign}(\sigma)(s_1s_2) \land (s_1s_3) \land (s_1s_4),
\]
where \( \text{sign} : \text{Sym}(S) \to \{\pm 1\} \) is the sign map. Hence, \( \alpha \) is well-defined and \( \text{Sym}(S) \)-equivariant.

(d) If \( M \) is a module with submodules \( A \) and \( B \) so that \( M = A \oplus B \), let \( \pi_A : M \to A \) be the projection of \( M \) onto \( A \).

3.2 Module Definitions

Now we are ready to define the modules of our complex. Call the complex \( X_\bullet \), which will be a \( \mathbb{Q} \)-module complex
\[
X_\bullet : \mathbb{Q} \otimes_k C_4 \to \mathbb{Q} \otimes_k C_3 \to \mathbb{Q} \otimes_k C_2 \to \mathbb{Q} \otimes_k C_1 \to \mathbb{Q}.
\]

(a) Define \( C_1 = C_{1,1} \oplus C_{1,2} \oplus C_{1,3} \), where:

(i) \( C_{1,1} = \frac{\Lambda^2_{\mathbb{K}} \text{SF}}{\left(\text{one}^{(2)} \left(\Lambda^2_{\mathbb{K}} \text{SF}\right)\right)} \).

We note the following for future reference:
\[
\langle s_is_k \rangle \land \langle s_is_l \rangle - \langle s_is_j \rangle \land \langle s_is_l \rangle + \langle s_is_j \rangle \land \langle s_is_k \rangle = 0 \quad (3.3)
\]
in \( C_{1,1} \) for any labeling \( S = \{s_i, s_j, s_k, s_l\} \).

(ii) \( C_{1,2} = \Lambda^2_{\mathbb{K}} V \).

(iii) \( C_{1,3} \) is the subvector space of \( \text{Sym}^2_{\mathbb{K}}(V) = \mathbb{K}[S]_2 \) spanned by
\[
\{(s_1 - s_2)(s_3 - s_4) | \{s_1, s_2, s_3, s_4\} = S\}.
\]
Notice that if \( S = \{s_1, s_2, s_3, s_4\} \), then
\[
(s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3) = 0. \quad (3.4)
\]
Hence, \( C_{1,3} \) can be generated by \( \{(s_1 - s_2)(s_3 - s_4), (s_1 - s_3)(s_2 - s_4)\} \).
(b) Define \( C_2 = C_{2,1} \oplus C_{2,2} \oplus \Lambda^2_k C_1 \), where

\[
\begin{align*}
(i) \quad C_{2,1} &= \frac{V}{\sum_{s \in S} s} \otimes_k \Lambda^4_k V \quad \text{and} \\
(ii) \quad C_{2,2} &= \text{the subvector space of } \Lambda^2_k V \otimes_k \Lambda^2_k V \text{ spanned by} \\
&\quad \{s_1 \wedge s_2 \otimes s_3 \wedge s_4 \mid \{s_1, s_2, s_3, s_4\} = S\}.
\end{align*}
\]

(c) Define \( C_3 = C_{3,1} \oplus (C_1 \otimes_k (C_{2,1} \oplus C_{2,2})) \), where \( C_{3,1} = \Lambda^2_k V \).

(d) Define \( C_4 = C_{4,1} \oplus C_{4,2} \oplus D^k_2(C_{2,1} \oplus C_{2,2}) \), where

\[
(i) \quad C_{4,1} = \frac{V}{\sum_{s \in S} s} \quad \text{and} \\
(ii) \quad C_{4,2} = SF.
\]

Thus, \( X_* \) will look like \( Q \) tensored over \( k \) with

\[
\begin{pmatrix}
C_{4,1} \oplus C_{4,2} \\
\oplus \\
D^k_2(C_{2,1} \oplus C_{2,2})
\end{pmatrix} \rightarrow 
\begin{pmatrix}
C_{3,1} \\
\oplus \\
C_1 \otimes_k (C_{2,1} \oplus C_{2,2})
\end{pmatrix} \rightarrow 
\begin{pmatrix}
C_{2,1} \oplus C_{2,2} \\
\oplus \\
\Lambda^2_k C_1
\end{pmatrix} \rightarrow C_1 \rightarrow k
\]

Notice that, if we ignore \( C_{4,1} \oplus C_{4,2} \) and \( C_{3,1} \), the above looks to be of the form (3.1).

**Observation 3.3.** Since \( C_1 = C_{1,1} \oplus C_{1,2} \oplus C_{1,3} \), there is an isomorphism

\[
\Lambda^2_k C_1 \cong \left\{ \left( \Lambda^2_k C_{1,1} \right) \oplus \left( C_{1,1} \otimes_k C_{1,2} \right) \oplus \left( C_{1,1} \otimes_k C_{1,3} \right) \right\}
\]

given by

\[
\begin{bmatrix}
a_1 \\
a_2 \\
a_3
\end{bmatrix} \wedge 
\begin{bmatrix}
b_1 \\
b_2 \\
b_3
\end{bmatrix} \mapsto 
\begin{bmatrix}
a_1 \wedge b_1 \\
a_1 \otimes_k b_2 - b_1 \otimes_k a_2 \\
a_1 \otimes_k b_3 - b_1 \otimes_k a_3 \\
a_2 \wedge b_2 \\
a_2 \otimes_k b_3 - b_2 \otimes_k a_3 \\
a_3 \wedge b_3
\end{bmatrix} \in \Lambda^2_k C_{1,1} \\
\text{etc.}
\]

(3.5)
We will use this isomorphism, writing elements of $\bigwedge_k^2 C_1$ as elements of $\bigwedge_k^2 C_{1,1} \oplus (C_{1,1} \otimes_k C_{1,2}) \oplus (C_{1,2} \otimes_k C_{1,3}) \oplus \bigwedge_k^2 C_{1,2} \oplus (C_{1,2} \otimes_k C_{1,3}) \oplus \bigwedge_k^2 C_{1,3}$ instead.

3.2.1 The $C_i$ modules are $k[\text{Sym}(S)]$-modules

From section 1.3 and subsection 3.1.1, we know that $C_{1,1}, C_{1,2}, C_{1,3}, C_{2,1}, C_{3,1}, C_{4,1},$ and $C_{4,2}$ are $k[\text{Sym}(S)]$-modules. Since $C_1 = C_{1,1} \oplus C_{1,2} \oplus C_{1,3}, C_2 = C_{2,1} \oplus C_{2,2} \oplus \bigwedge_k^2 C_{1,1}, C_3 = C_{3,1} \oplus (C_1 \otimes_k (C_{2,1} \oplus C_{2,2})), and C_4 = C_{4,1} \oplus C_{4,2} \oplus D_2^k(C_{2,1} \oplus C_{2,2}),$ to show that $C_1, C_2, C_3,$ and $C_4$ are modules over $k[\text{Sym}(S)],$ it suffices to show that $C_{2,2}$ is such.

Let $S = \{s_1, s_2, s_3, s_4\}.$ Remember that $C_{2,2}$ is the subvector space of $\bigwedge_k^2 V \otimes_k \bigwedge_k^2 V$ spanned by

$$\{s_i \wedge s_j \otimes s_k \wedge s_l \mid \{i, j, k, l\} = \{1, 2, 3, 4\}\}.$$

Any permutation of $S$ will clearly send an element of this set to either an element of the set or the negative of an element of the set, so $C_{2,2}$ is a $k[\text{Sym}(S)]$-module.

Now we want to investigate how the modules we have are related to indecomposable $k[\text{Sym}(S)]$-modules. Using the same notation for $\tau_i$ and $\tau_{2,2}$ as in subsection 1.3,
computation of the characters gives:

<table>
<thead>
<tr>
<th></th>
<th>id</th>
<th>τ₂</th>
<th>τ₃</th>
<th>τ₄</th>
<th>τ₂,₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\bigwedge^2 V)</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
</tr>
<tr>
<td>(\bigwedge^4 V)</td>
<td>1</td>
<td>-1</td>
<td>1</td>
<td>-1</td>
<td>1</td>
</tr>
<tr>
<td>(C_{4,1} = V/\langle \sum s \rangle)</td>
<td>3</td>
<td>1</td>
<td>0</td>
<td>-1</td>
<td>-1</td>
</tr>
<tr>
<td>(C_{2,1} = V/\langle \sum s \rangle \otimes_k \bigwedge^4 V)</td>
<td>3</td>
<td>-1</td>
<td>0</td>
<td>1</td>
<td>-1</td>
</tr>
<tr>
<td>(C_{1,3} = \langle (s_1 - s_2)(s_3 - s_4)</td>
<td>S = {s_1, s_2, s_3, s_4}\rangle)</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
</tr>
<tr>
<td>(C_{1,1})</td>
<td>2</td>
<td>0</td>
<td>-1</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(C_{1,2})</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>(C_{2,2})</td>
<td>6</td>
<td>-2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(C_{3,1})</td>
<td>6</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>(C_{4,2})</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(\bigwedge^2 C_{1,2})</td>
<td>15</td>
<td>-3</td>
<td>0</td>
<td>1</td>
<td>-1.</td>
</tr>
</tbody>
</table>

Using the above and Facts 1.9 from subsection 1.3, we can compute the following:

<table>
<thead>
<tr>
<th></th>
<th>id</th>
<th>τ₂</th>
<th>τ₃</th>
<th>τ₄</th>
<th>τ₂,₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>(C_1 \otimes_k C_{2,1})</td>
<td>30</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>(C_1 \otimes_k C_{2,2})</td>
<td>60</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>4</td>
</tr>
</tbody>
</table>

Since there is an isomorphism \(D_2(C_{2,1} \oplus C_{2,2}) \cong D_2(C_{2,1}) \oplus (C_{2,1} \otimes_k C_{2,2}) \oplus D_2(C_{2,2})\),
we find the characters for the summands:

<table>
<thead>
<tr>
<th></th>
<th>id</th>
<th>τ₂</th>
<th>τ₃</th>
<th>τ₄</th>
<th>τ₂,₂</th>
</tr>
</thead>
<tbody>
<tr>
<td>(D_2(C_{2,1}))</td>
<td>6</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>2</td>
</tr>
<tr>
<td>(C_{2,1} \otimes_k C_{2,2})</td>
<td>18</td>
<td>2</td>
<td>0</td>
<td>0</td>
<td>-2</td>
</tr>
<tr>
<td>(D_2(C_{2,2}))</td>
<td>21</td>
<td>5</td>
<td>0</td>
<td>1</td>
<td>5</td>
</tr>
</tbody>
</table>

Therefore, we can write each of the the \(C_i\) modules as direct sums of indecomposable \(k[\text{Sym}(S)]\)-modules:

\[
C_1 \cong \frac{V}{\langle \sum s \rangle} \oplus \left[ \frac{V}{\langle \sum s \rangle} \otimes_k \bigwedge^4 V \right] \oplus C_{1,3}^2
\]
3.3 Definitions of the Differential Maps

We now will define $d_1, d_2, d_3, d_4$ for $X_*$:

$$X_*: Q \otimes_k C_4 \xrightarrow{d_4} Q \otimes_k C_3 \xrightarrow{d_3} Q \otimes_k C_2 \xrightarrow{d_2} Q \otimes_k C_1 \xrightarrow{d_1} Q.$$ 

We noted before that we can consider $Q$ the polynomial ring in the variables $T(s_is_j)$ and $U(s_i^2 \land s_j^2)$ for $s_i, s_j$ distinct elements of $S$. In the modules $Q \otimes_k C_i$ for $i \in \{1, 2, 3, 4\}$, we will think of elements of $Q$ as being coefficients of elements of $C_i$, so we will use multiplicative "." or juxtaposition notation between elements of $Q$ and elements of $C_\ast$, as well as identifying $\gamma = 1 \cdot \gamma = 1 \otimes \gamma \in Q \otimes C_i$ for an element $\gamma \in C_i$, $i \in \{1, 2, 3, 4\}$.

The complex $X_\ast$ will have a subcomplex of the form (3.1) if we remove the $C_{1,1} \oplus C_{4,2}$ summands of $C_4$ and the $C_{3,1}$ summand of $C_3$. We will refer to the subsequence of maps and modules of the form (3.1) as the "Tate portion" of our sequence of maps and modules. In order to make the homomorphisms below more understandable, we define many of the maps in pieces, giving the definition on and to various summands of the $C_i$ modules.

(a) Define $d_1: Q \otimes_k C_1 \rightarrow Q$ to be the $Q$-module map such that:

1.) For $m_1, m_2 \in s$, $d_1|_{Q \otimes_k C_{1,1}}: Q \otimes_k C_{1,1} \rightarrow Q$ is

$$d_1|_{Q \otimes_k C_{1,1}}(\langle m_1 \rangle \land \langle m_2 \rangle) = T(m_1)T(m_1') - T(m_2)T(m_2').$$
The facts that the map above is antisymmetric in \(m_1\) and \(m_2\) and well-defined for \(\langle m_1 \rangle = \langle m_2 \rangle\) and \(\langle m_2 \rangle = \langle m_1 \rangle\) are immediate. Applying the map to an expression of the form \(\langle s_i s_k \rangle \land \langle s_i s_j \rangle - \langle s_i s_j \rangle \land \langle s_i s_l \rangle + \langle s_i s_j \rangle \land \langle s_i s_k \rangle\) for \(\{s_i, s_j, s_k, s_l\} = S\) (see formula (3.3)) does give zero, so the above is a well-defined map on \(Q \otimes_k C_{1,1} = Q \otimes_k \Lambda^2_{k SF} \odot \left(\Lambda^2_{k SF}\right)\). See Proposition 3.4 for the proof that the above is \(k[\text{Sym}(S)\text{-equivariant}].\)

2.) For \(\{s_1, s_2, s_3, s_4\} = S\), \(d_1|_{Q \otimes_k C_{1,1}} : Q \otimes_k C_{1,1} \to Q\) is

\[
d_1|_{Q \otimes_k C_{1,1}}(s_1 \land s_2) = T(s_3 s_4)U(s_1^2 \land s_2^2) + T(s_1 s_3)T(s_1 s_4) - T(s_2 s_3)T(s_2 s_4).
\]

The definition is antisymmetric in \(s_1\) and \(s_2\) and symmetric in \(s_3\) and \(s_4\), and is therefore well-defined on \(Q \otimes_k C_{1,1} = Q \otimes_k \Lambda^2_k V\). The \(\text{Sym}(S)\text{-equivariance}\) is clear.

3.) For \(\{s_1, s_2, s_3, s_4\} = S\), \(d_1|_{Q \otimes_k C_{1,1}} : Q \otimes_k C_{1,1} \to Q\) is

\[
d_1|_{Q \otimes_k C_{1,1}}((s_1 - s_2)(s_3 - s_4)) = \begin{cases} 
+ U(s_1^2 \land s_2^2)U(s_3^2 \land s_4^2) \\
- T(s_1 s_3)^2 + T(s_1 s_4)^2 \\
+ T(s_2 s_3)^2 - T(s_2 s_4)^2.
\end{cases}
\]

The definition above satisfies

\[
d_1|_{Q \otimes_k C_{1,1}}((s_1 - s_2)(s_3 - s_4)) = -d_1|_{Q \otimes_k C_{1,1}}((s_2 - s_1)(s_3 - s_4)),
\]

\[
d_1|_{Q \otimes_k C_{1,1}}((s_1 - s_2)(s_3 - s_4)) = -d_1|_{Q \otimes_k C_{1,1}}((s_1 - s_2)(s_4 - s_3)),
\]

and

\[
d_1|_{Q \otimes_k C_{1,1}}((s_1 - s_2)(s_3 - s_4)) = d_1|_{Q \otimes_k C_{1,1}}((s_3 - s_4)(s_1 - s_2)).
\]

The fact that the map is zero on the expression

\[
(s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3)
\]

is immediate. Applying the map to an expression of the form \(\langle s_i s_k \rangle \land \langle s_i s_j \rangle - \langle s_i s_j \rangle \land \langle s_i s_l \rangle + \langle s_i s_j \rangle \land \langle s_i s_k \rangle\) for \(\{s_i, s_j, s_k, s_l\} = S\) (see formula (3.3)) does give zero, so the above is a well-defined map on \(Q \otimes_k C_{1,1} = Q \otimes_k \Lambda^2_{k SF} \odot \left(\Lambda^2_{k SF}\right)\). See Proposition 3.4 for the proof that the above is \(k[\text{Sym}(S)\text{-equivariant}].\)
is proven in Proposition 3.4. Therefore, \( d_1|_{Q \otimes_k C_{1,3}} \) is well-defined. The Sym(S)-equivariance is easy to check.

(b) Define \( d_2 : Q \otimes_k C_2 \to Q \otimes_k C_1 \) to be the \( Q \)-module homomorphism described below.

1.) For \( s \in S \) and \( \omega \in \wedge^3 V \), there is a basis \( \{ t \otimes w \mid t \in S \} \) of \( C_{2,1} \). Define \( d_2|_{Q \otimes_k C_{2,1}} \) on this basis by the following and then extend the definitions linearly to get a homomorphism.

\[
\pi_{Q \otimes_k C_{1,1}} \circ d_2|_{Q \otimes_k C_{2,1}} (s \otimes \omega) = \sum_{t \in S} U(t^2 \wedge s^2) \cdot \langle st \rangle^*(\alpha(\omega)),
\]

\[
\pi_{Q \otimes_k C_{1,2}} \circ d_2|_{Q \otimes_k C_{2,1}} (s \otimes \omega) = \sum_{t \in S} T(\overline{ts}) \cdot (t^* \wedge s^*)(\omega),
\]

and

\[
\pi_{Q \otimes_k C_{1,3}} \circ d_2|_{Q \otimes_k C_{2,1}} = 0.
\]

Notice that the above do make sense, as if \( t = s \), then \( U(t^2 \wedge s^2) = 0 \) and \( t^* \wedge s^* = 0 \), so that we do not use any \( \langle st \rangle \) or \( \overline{st} \) with \( st \in ps \), which we have not defined. The fact that

\[
\sum_{s \in S} \pi_{Q \otimes_k C_{1,1}} \circ d_2|_{Q \otimes_k C_{2,1}} (s \otimes \omega)
\]

\[
= \sum_{s \in S} \sum_{t \in S} U(t^2 \wedge s^2) \cdot \langle st \rangle^*(\alpha(\omega))
\]

\[
= \sum_{\{s,t\} \subset S} \left( U(t^2 \wedge s^2) \cdot \langle st \rangle^*(\alpha(\omega)) + U(s^2 \wedge t^2) \cdot \langle ts \rangle^*(\alpha(\omega)) \right)
\]

\[
= 0
\]

shows that \( \pi_{Q \otimes_k C_{1,1}} \circ d_2|_{Q \otimes_k C_{2,1}} \) is well-defined. The argument for \( \pi_{Q \otimes_k C_{1,2}} \circ d_2|_{Q \otimes_k C_{2,1}} \) is similar. It is again quickly seen that the definition is Sym(S)-equivariant.
2.) For \( \{s_1, s_2, s_3, s_4\} = S \), \( d_2|_{Q \otimes_k C_{2,2}} : Q \otimes_k C_{2,2} \to Q \otimes_k C_{1,1} \) is defined on the basis \( \{s_i \otimes s_j \otimes s_k \otimes s_l \mid \{i, j, k, l\} = \{1, 2, 3, 4\}, i < j, k < l \} \) of \( C_{2,2} \) by

\[
\pi_{Q \otimes_k C_{1,1}} \circ d_2|_{Q \otimes_k C_{2,2}}(s_1 \otimes s_2 \otimes s_3 \otimes s_4) = -2T(s_3s_4) \cdot (s_1s_4) \wedge (s_1s_3).
\]

The antisymmetry in \( s_1 \) and \( s_2 \) and in \( s_3 \) and \( s_4 \) follows from the fact that \( \langle m \rangle = \langle \overline{m} \rangle \) in \( SF \). Otherwise, well-definedness and \( \text{Sym}(S) \)-equivariance is easily seen.

3.) For \( \{s_1, s_2, s_3, s_4\} = S \), define \( d_2|_{Q \otimes_k C_{2,2}} : Q \otimes_k C_{2,2} \to Q \otimes_k C_{1,2} \) by

\[
\pi_{Q \otimes_k C_{1,2}} \circ d_2|_{Q \otimes_k C_{2,2}}(s_1 \otimes s_2 \otimes s_3 \otimes s_4) = \begin{cases} 
-U(s_1^2 \wedge s_2^2) \cdot s_3 \wedge s_4 \\
-T(s_1s_4) \cdot s_1 \wedge s_3 + T(s_1s_3) \cdot s_1 \wedge s_4 \\
+T(s_2s_4) \cdot s_2 \wedge s_3 - T(s_2s_3) \cdot s_2 \wedge s_4.
\end{cases}
\]

Again, the required antisymmetry and \( \text{Sym}(S) \)-equivariance is clear and the definition above is the definition on the basis \( \{s_i \otimes s_j \otimes s_k \otimes s_l \mid \{i, j, k, l\} = \{1, 2, 3, 4\}, i < j, k < l \} \) of \( C_{2,2} \) and then extended linearly.

4.) For \( \{s_1, s_2, s_3, s_4\} = S \), \( d_2|_{Q \otimes_k C_{2,2}} : Q \otimes_k C_{2,2} \to Q \otimes_k C_{1,3} \) is

\[
\pi_{Q \otimes_k C_{1,3}} \circ d_2|_{Q \otimes_k C_{2,2}}(s_1 \otimes s_2 \otimes s_3 \otimes s_4) = T(s_1s_2) \cdot (s_1 - s_2)(s_3 - s_4).
\]

The definition above is well-defined and equivariant with respect to the \( \text{Sym}(S) \) action.

5.) On \( Q \otimes_k \Lambda^2 C_1 \), define

\[
d_2|_{Q \otimes_k \Lambda^2 C_1} = \text{Kos}^{d_1},
\]

the Koszul map \( \Lambda^2 C_1 \to C_1 \) induced by \( d_1 \). This is part of the Tate portion of our sequence of maps and modules. Since \( d_1 \) is \( \text{Sym}(S) \)-equivariant, so is \( \text{Kos}^{d_1} \).
(c) Define $d_3 : Q \otimes_k C_3 \to Q \otimes_k C_2$ to be the $Q$-module homomorphism that is given below. For parts c1) through c5), the $\mathrm{Sym}(S)$-equivariance, the antisymmetry in $s_1$ and $s_2$, and the symmetry in $s_3$ and $s_4$, are clear, so those maps are $\mathrm{Sym}(S)$-equivariant and well-defined on $Q \otimes_k C_3 = Q \otimes_k \Lambda^2 V$.

1.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_k C_{3,1}} : Q \otimes_k C_{3,1} \to Q \otimes_k C_{2,1}$ is defined by

$$\pi_{Q \otimes_k C_{2,1}} \circ d_3|_{Q \otimes_k C_{3,1}}(s_1 \wedge s_2) = 2T(s_3 s_4) \cdot (s_3 - s_4) \otimes s_1 \wedge s_2 \wedge s_3 \wedge s_4.$$ 

2.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_k C_{3,1}} : Q \otimes_k C_{3,1} \to Q \otimes_k C_{2,2}$ is defined by

$$\pi_{Q \otimes_k C_{2,2}} \circ d_3|_{Q \otimes_k C_{3,1}}(s_1 \wedge s_2) = \begin{cases} +U(s_3^2 \wedge s_4^2) \cdot (s_1 \wedge s_2 \otimes s_3 \wedge s_4) \\ +T(s_2 s_3) \cdot (s_1 \wedge s_3 \otimes s_2 \wedge s_4) \\ +T(s_2 s_4) \cdot (s_1 \wedge s_4 \otimes s_2 \wedge s_3) \\ -T(s_1 s_3) \cdot (s_2 \wedge s_3 \otimes s_1 \wedge s_4) \\ -T(s_1 s_4) \cdot (s_2 \wedge s_4 \otimes s_1 \wedge s_3). \end{cases}$$

3.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_k C_{3,1}} : Q \otimes_k C_{3,1} \to Q \otimes_k C_{1,1} \otimes_k C_{1,2}$ is defined by

$$\pi_{Q \otimes_k (C_{1,1} \otimes_k C_{1,2})} \circ d_3|_{Q \otimes_k C_{3,1}}(s_1 \wedge s_2) = \begin{cases} -2\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \otimes s_1 \wedge s_2 \\ -2\langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \otimes s_1 \wedge s_2. \end{cases}$$

4.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_k C_{3,1}} : Q \otimes_k C_{3,1} \to Q \otimes_k \Lambda^2_{k} C_{1,2}$ is defined by

$$\pi_{Q \otimes_k \Lambda^2_{k} C_{1,2}} \circ d_3|_{Q \otimes_k C_{3,1}}(s_1 \wedge s_2) = \begin{cases} +(s_1 \wedge s_3) \wedge (s_2 \wedge s_3) \\ +(s_1 \wedge s_4) \wedge (s_2 \wedge s_4). \end{cases}$$

5.) For $S = \{s_1, s_2, s_3, s_4\}$, $d_3|_{Q \otimes_k C_{3,1}} : Q \otimes_k C_{3,1} \to Q \otimes_k C_{1,2} \otimes_k C_{1,3}$ is defined by

$$\pi_{Q \otimes_k (C_{1,2} \otimes_k C_{1,3})} \circ d_3|_{Q \otimes_k C_{3,1}}(s_1 \wedge s_2) = -(s_3 \wedge s_4) \cdot (s_1 - s_2)(s_3 - s_4).$$
6.) The images of $d_3|_{Q \otimes_k C_{3,1}}$ in the $Q \otimes_k \Lambda^2_k C_{1,1}$, $Q \otimes_k C_{1,1} \otimes_k C_{1,3}$, and $Q \otimes_k \Lambda^2_k C_{1,3}$ summands of $Q \otimes_k \Lambda^2_k C_1$ are all zero.

7.) Finally, $d_3|_{Q \otimes_k (C_1 \otimes (C_2,1 \oplus C_2,2))} : Q \otimes_k (C_1 \otimes (C_2,1 \oplus C_2,2)) \to Q \otimes_k C_2$ is the map

$$d_1 \otimes 1 - \text{quot} \circ (1 \otimes d_2|_{Q \otimes_k C_{2,1} \oplus C_{2,2}}),$$

where $\text{quot} : C_1 \otimes_k C_1 \to \Lambda^2_k C_1$ is the natural quotient map. This piece of $d_3$ is part of the Tate portion. It is $\text{Sym}(S)$-equivariant by the $\text{Sym}(S)$-equivariance of $d_1$, $d_2$, and $\text{quot}$.

(d) Define $d_4 : Q \otimes_k C_4 \to Q \otimes_k C_3$ by:

1.) For $s \in S$ define $d_4|_{Q \otimes_k C_{4,1}} : Q \otimes_k C_{4,1} \to Q \otimes_k C_{3,1}$ by

$$\pi_{Q \otimes_k C_{3,1}} \circ d_4|_{Q \otimes_k C_{4,1}}(s) = \sum_{t \in S} T(st) \cdot t \wedge s.$$

Extending the definition linearly gives a homomorphism on $Q \otimes_k V$. It gives a map on $Q \otimes_k C_4 = Q \otimes_k V/\langle \sum t \rangle$ since

$$\pi_{Q \otimes_k C_{3,1}} \left( \sum_{s \in S} d_4|_{Q \otimes_k C_{4,1}}(s) \right) = \sum_{s \in S} \sum_{t \in S} T(st) \cdot t \wedge s = \sum_{\{s,t\} \subseteq S} \left( T(st) \cdot t \wedge s + T(st) \cdot s \wedge t \right) = 0.$$

The equivariance with respect to $\text{Sym}(S)$ is easy to check. The fact that $d_4|_{Q \otimes_k C_{4,1}}$ is independent of a particular enumeration of $S \setminus \{s\}$, see Proposition 3.4.

2.) For $s \in S$ and $S = \{s, s_1, s_2, s_3\}$, $d_4|_{Q \otimes_k C_{4,1}} : Q \otimes_k C_{4,1} \to Q \otimes_k C_{1,1} \otimes_k C_{2,1}$ is

$$\pi_{Q \otimes_k C_{1,1} \otimes C_{2,1}} \circ d_4|_{Q \otimes_k C_{4,1}}(s) = \begin{cases} -2(\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle) \otimes (s_1 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s) \\ +2(\langle s_1 s_2 \rangle \wedge \langle s_2 s_3 \rangle) \otimes (s_2 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s) \\ -2(\langle s_1 s_3 \rangle \wedge \langle s_2 s_3 \rangle) \otimes (s_3 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s). \end{cases}$$
The above is Sym(S)-equivariant by inspection. See Proposition 3.4 to see that it is well-defined on $Q \otimes_k C_4 = Q \otimes_k V / \langle \sum t \rangle$.

3.) For $s \in S$ and $S = \{s, s_1, s_2, s_3\}$, define $d_4|_{Q \otimes_k C_{4,1}} : Q \otimes_k C_{4,1} \to Q \otimes_k C_{1,2} \otimes_k C_{2,2}$ by

$$
\pi_{Q \otimes_k C_{1,2} \otimes_k C_{2,2}} \circ d_4|_{Q \otimes_k C_{4,1}}(s) = \begin{cases} 
+(s_1 \wedge s_2) \otimes (s \wedge s_3 \otimes s_1 \wedge s_2) \\
+(s_1 \wedge s_3) \otimes (s \wedge s_2 \otimes s_1 \wedge s_3) \\
+(s_2 \wedge s_3) \otimes (s \wedge s_1 \otimes s_2 \wedge s_3).
\end{cases}
$$

Again, we leave the proof that the map is well-defined on $Q \otimes_k C_4 = Q \otimes_k V / \langle \sum t \rangle$ to Proposition 3.4 and claim that Sym(S)-equivariance is clear.

4.) The images of $d_4|_{Q \otimes_k C_{4,1}}$ in the summands $Q \otimes_k C_{1,2} \otimes_k C_{2,1}$, $Q \otimes_k C_{1,3} \otimes_k C_{2,1}$, $Q \otimes_k C_{1,1} \otimes_k C_{2,2}$, and $Q \otimes_k C_{1,3} \otimes_k C_{2,2}$ are all zero.

5.) For $s_1 s_2 \in \text{sf}$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q \otimes_k C_{4,2}} : Q \otimes_k C_{4,2} \to Q \otimes_k C_{3,1}$ is defined by

$$
\pi_{Q \otimes_k C_{3,1}} \circ d_4|_{Q \otimes_k C_{4,2}}(s_1 s_2) = \begin{cases} 
+T(s_1 s_3) \cdot s_1 \wedge s_4 + T(s_1 s_4) \otimes s_1 \wedge s_3 \\
+T(s_2 s_3) \cdot s_2 \wedge s_4 + T(s_2 s_4) \otimes s_2 \wedge s_3 \\
-U(s_3^2 \wedge s_4^2) \cdot s_3 \wedge s_4.
\end{cases}
$$

Then extend this definition from the basis sf of SF to all of $Q \otimes_k C_{4,2} = Q \otimes_k SF$. The definition is symmetric in $s_1$ and $s_2$, as well as in $s_3$ and $s_4$, so it is well-defined on sf and is independent of enumeration of $S \setminus \{s_1, s_2\}$, so it is well-defined. The fact that the map is Sym(S)-equivariant is obvious.

6.) For $s_1 s_2 \in \text{sf}$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q \otimes_k C_{4,2}} : Q \otimes_k C_{4,2} \to Q \otimes_k C_{1,1} \otimes_k C_{2,2}$ is defined to be

$$
\pi_{Q \otimes_k C_{1,1} \otimes_k C_{2,2}} \circ d_4|_{Q \otimes_k C_{4,2}}(s_1 s_2) = 2((s_1 s_3) \wedge (s_1 s_4)) \otimes (s_1 \wedge s_2 \otimes s_3 \wedge s_4).
$$
Again, the map is well-defined and Sym($S$)-equivariant, as long as one recalls that $\langle m \rangle = \langle \overline{m} \rangle$ in $S \overline{F}$. We claim that the Sym($S$)-equivariance and well-definedness is also obvious for parts d7) through d9) below.

7.) For $s_1 s_2 \in sf$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q \otimes_k C_{4,2}} : Q \otimes_k C_{4,2} \to Q \otimes_k C_{1,2} \otimes_k C_{2,1}$ is defined to be

$$\pi_{Q \otimes_k C_{1,2} \otimes_k C_{2,1}} \circ d_4|_{Q \otimes_k C_{4,2}}(s_1 s_2) = \begin{cases} +2(s_3 \wedge s_4) \otimes s_1 \otimes (s_1 \wedge s_2 \wedge s_3 \wedge s_4) \\ -2(s_3 \wedge s_4) \otimes s_2 \otimes (s_1 \wedge s_2 \wedge s_3 \wedge s_4). \end{cases}$$

8.) For $s_1 s_2 \in sf$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q \otimes_k C_{4,2}} : Q \otimes_k C_{4,2} \to Q \otimes_k C_{1,2} \otimes_k C_{2,2}$ is defined to be

$$\pi_{Q \otimes_k C_{1,2} \otimes_k C_{2,2}} \circ d_4|_{Q \otimes_k C_{4,2}}(s_1 s_2) = \begin{cases} -(s_1 \wedge s_3) \otimes (s_2 \wedge s_3 \otimes s_1 \wedge s_4) \\ -(s_2 \wedge s_3) \otimes (s_1 \wedge s_3 \otimes s_2 \wedge s_4) \\ -(s_1 \wedge s_4) \otimes (s_2 \wedge s_4 \otimes s_1 \wedge s_3) \\ -(s_2 \wedge s_4) \otimes (s_1 \wedge s_4 \otimes s_2 \wedge s_3). \end{cases}$$

9.) For $s_1 s_2 \in sf$ and $S = \{s_1, s_2, s_3, s_4\}$, $d_4|_{Q \otimes_k C_{4,2}} : Q \otimes_k C_{4,2} \to Q \otimes_k C_{1,3} \otimes_k C_{2,2}$ is defined to be

$$\pi_{Q \otimes_k C_{1,3} \otimes_k C_{2,2}} \circ d_4|_{Q \otimes_k C_{4,2}}(s_1 s_2) = (s_1 - s_2)(s_3 - s_4) \otimes (s_3 \wedge s_4 \otimes s_1 \wedge s_2).$$

10.) The images of $d_4|_{Q \otimes_k C_{4,2}}$ in the summands $Q \otimes_k C_{1,1} \otimes_k C_{2,1}$ and $Q \otimes_k C_{1,3} \otimes_k C_{2,1}$ are both zero.

11.) For $\theta \in C_{2,1} \oplus C_{2,2}$,

$$d_4|_{Q \otimes_k D_2(C_{2,1} \oplus C_{2,2})}(\theta^{(2)}) = d_2|_{Q \otimes_k C_{2,1} \oplus C_{2,2}}(\theta) \otimes \theta.$$ 

This map is part of the Tate portion and, by Remark 1.5, this is sufficient to understand the map on all of $Q \otimes_k D_2(C_{2,1} \oplus C_{2,2})$. 

33
Proposition 3.4. With maps as previously defined, we have the following three facts.

1. The map $d_1|_{Q^g_k C_{1,1}}$ is Sym($S$)-equivariant.

2. The element $(s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3)$ is in the kernel of $d_1|_{Q^g_k C_{1,3}}$.

3. The map $d_4|_{Q^g_k C_{4,1}}$ is zero on the $k$-subvector space $\left\{ \sum t \right\}$ of $V$ and the definition of $d_4|_{Q^g_k C_{4,1}}(s)$ for $s \in S$ does not depend on a particular enumeration of $S \setminus \{s\}$.

Proof. First, we show that $d_1|_{Q^g_k C_{1,1}} : Q \otimes_k C_{1,1} \rightarrow Q$ is Sym($S$)-equivariant. Let $S = \{s_1, s_2, s_3, s_4\}$. The elements $\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle$ and $\langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle$ generate $C_{1,1}$ and $\langle s_1 s_2 \rangle$ and $\langle s_1 s_2 s_3 s_4 \rangle$ generate Sym($S$). It is sufficient to check that the action of $\langle s_1 s_2 \rangle$ and $\langle s_1 s_2 s_3 s_4 \rangle$ commute with applying $d_1|_{Q^g_k C_{1,1}}$ to each of $\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle$ and $\langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle$:

\[
d_1|_{Q^g_k C_{1,1}}(\langle s_1 s_2 \rangle \cdot (\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle))
\]

\[
= d_1|_{Q^g_k C_{1,1}}(\langle s_1 s_2 \rangle \wedge \langle s_2 s_3 \rangle)
\]

\[
= T(s_1 s_2)T(s_3 s_4) - T(s_2 s_3)T(s_1 s_4)
\]

\[
= (s_1 s_2) \cdot (T(s_1 s_2)T(s_3 s_4) - T(s_1 s_2)T(s_2 s_4))
\]

\[
= (s_1 s_2) \cdot d_1|_{Q^g_k C_{1,1}}(\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle).
\]

\[
d_1|_{Q^g_k C_{1,1}}(\langle s_1 s_2 s_3 s_4 \rangle \cdot (\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle))
\]

\[
= d_1|_{Q^g_k C_{1,1}}(\langle s_2 s_3 \rangle \wedge \langle s_2 s_4 \rangle)
\]

\[
= T(s_2 s_3)T(s_1 s_4) - T(s_2 s_4)T(s_1 s_3)
\]

\[
= (s_1 s_2 s_3 s_4) \cdot (T(s_1 s_2)T(s_3 s_4) - T(s_1 s_3)T(s_2 s_4))
\]
\[ = (s_1 s_2 s_3 s_4) \cdot d_1|_{Q \otimes k C_{1,1}}((s_1 s_2) \wedge (s_1 s_3)).\]

\[ d_1|_{Q \otimes k C_{1,1}}((s_1 s_2) \cdot ((s_1 s_2) \wedge (s_1 s_4))) \]
\[ = d_1|_{Q \otimes k C_{1,1}}((s_1 s_2) \wedge (s_2 s_4)) \]
\[ = T(s_1 s_2)T(s_3 s_4) - T(s_2 s_4)T(s_1 s_3) \]
\[ = (s_1 s_2) \cdot (T(s_1 s_2)T(s_3 s_4) - T(s_1 s_4)T(s_2 s_3)) \]
\[ = (s_1 s_2) \cdot d_1|_{Q \otimes k C_{1,1}}((s_1 s_2) \wedge (s_1 s_4)).\]

\[ d_1|_{Q \otimes k C_{1,1}}((s_1 s_2 s_3 s_4) \cdot ((s_1 s_2) \wedge (s_1 s_4))) \]
\[ = d_1|_{Q \otimes k C_{1,1}}((s_2 s_3) \wedge (s_2 s_1)) \]
\[ = T(s_2 s_3)T(s_1 s_4) - T(s_1 s_2)T(s_3 s_4) \]
\[ = (s_1 s_2 s_3 s_4) \cdot (T(s_1 s_2)T(s_3 s_4) - T(s_1 s_4)T(s_2 s_3)) \]
\[ = (s_1 s_2 s_3 s_4) \cdot d_1|_{Q \otimes k C_{1,1}}((s_1 s_2) \wedge (s_1 s_4)).\]

Now, still letting \( S = \{s_1, s_2, s_3, s_4\} \) we see that \( d_1|_{Q \otimes k C_{1,3}} \) is zero on
\[ (s_1 - s_2)(s_3 - s_4) - (s_1 - s_3)(s_2 - s_4) + (s_1 - s_4)(s_2 - s_3).\]

Indeed,

\[
\begin{align*}
+d_1|_{Q \otimes k C_{1,3}}((s_1 - s_2)(s_3 - s_4)) \\
-d_1|_{Q \otimes k C_{1,3}}((s_1 - s_3)(s_2 - s_4)) \\
+ +d_1|_{Q \otimes k C_{1,3}}((s_1 - s_4)(s_2 - s_3)) \\
= +U(s_1^2 \wedge s_2^2)U(s_2^2 \wedge s_3^2) - T(s_1 s_3)^2 + T(s_1 s_4)^2 + T(s_2 s_3)^2 - T(s_2 s_4)^2 \\
- (U(s_1^2 \wedge s_3^2)U(s_2^2 \wedge s_4^2) - T(s_1 s_2)^2 + T(s_1 s_4)^2 + T(s_2 s_3)^2 - T(s_3 s_4)^2) \\
+ U(s_1^2 \wedge s_4^2)U(s_2^2 \wedge s_3^2) - T(s_1 s_2)^2 + T(s_1 s_3)^2 + T(s_2 s_4)^2 - T(s_3 s_4)^2
\end{align*}
\]
where the last equality comes from the formula (3.2) from Observation 3.2. By combining terms and then using formula (3.2) again, we get that the above is equal to

\[
\begin{align*}
\{ & U(s_1^2 \land s_2^2)U(s_3^2 \land s_4^2) \\
& - U(s_1^2 \land s_3^2)U(s_2^2 \land s_4^2) \\
& + U(s_1^2 \land s_4^2)U(s_2^2 \land s_3^2) \\
& + U(s_2^2 \land s_3^2)U(s_4^2 \land s_1^2) \\
& - (U(s_1^2 \land s_2^2) + U(s_2^2 \land s_3^2))U(s_2^2 \land s_3^2) \\
& + U(s_1^2 \land s_4^2)U(s_2^2 \land s_4^2)
\end{align*}
\]

Next, given \(s \in S\), we prove that the definition of \(d_4|_{Q \otimes_k C_{4,1}}(s)\) does not depend on a particular enumeration of \(S \setminus \{s\}\). Since

\[
d_4|_{Q \otimes_k C_{4,1}}(s) = \begin{cases} 
+T(ss_1) \cdot s_1 \land s + T(ss_2) \cdot s_2 \land s + T(ss_3) \cdot s_3 \land s \\
-2((s_1s_2) \land (s_1s_3)) \otimes (s_1 \otimes s_1 \land s_2 \land s_3 \land s) \\
+2((s_1s_2) \land (s_2s_3)) \otimes (s_2 \otimes s_1 \land s_2 \land s_3 \land s) \\
-2((s_1s_3) \land (s_2s_3)) \otimes (s_3 \otimes s_1 \land s_2 \land s_3 \land s) \\
+(s_1 \land s_2) \otimes (s \land s_3 \otimes s_1 \land s_2) \\
+(s_1 \land s_3) \otimes (s \land s_2 \otimes s_1 \land s_3) \\
+(s_2 \land s_3) \otimes (s \land s_1 \otimes s_2 \land s_3)
\end{cases}
\]

one can see that the elements \((s_1s_2), (s_1s_3)\), and \((s_2s_3) = (s_1s_2)(s_1s_3)(s_1s_2)\) of Sym(S) will leave the formula invariant by properties of exterior powers and the fact that \(\langle m \rangle = \langle m \rangle\) in \(C_{1,1}\) for all \(m \in sf\). Thus, the formula does not depend on a particular choice of names \(\{s_1, s_2, s_3\} = S \setminus \{s\}\).
For \( d_4|_{Q\otimes_k C_{4,1}} \), it remains to show that

\[
\sum_{s \in S} \pi_{Q\otimes_k C_1 \otimes_k C_{2,1}} \circ d_4|_{Q\otimes_k C_{4,1}}(s) = 0
\]

and

\[
\sum_{s \in S} \pi_{Q\otimes_k C_1 \otimes_k C_{2,2}} \circ d_4|_{Q\otimes_k C_{4,1}}(s) = 0.
\]

Now, for \( \sum_{s \in S} \pi_{Q\otimes_k C_1 \otimes_k C_{2,1}} \circ d_4|_{Q\otimes_k C_{4,1}}(s) \), recall that in SF, \( \langle m \rangle = \langle \overline{m} \rangle \) for any \( m \in \text{sf} \). Thus,

\[
\sum_{s \in S} \pi_{Q\otimes_k C_1 \otimes_k C_{2,1}} \circ \sum_{s \in S} d_4|_{Q\otimes_k C_{4,1}}(s)
\]

\[
= \left\{ -2(\langle s_2 s_3 \rangle \land (s_2 s_4) \rangle \otimes (s_2 \otimes s_2 \land s_3 \land s_4 \land s_1) +2(\langle s_2 s_3 \rangle \land (s_3 s_4) \rangle \otimes (s_3 \otimes s_2 \land s_3 \land s_4 \land s_1) -2(\langle s_2 s_4 \rangle \land (s_3 s_4) \rangle \otimes (s_4 \otimes s_2 \land s_3 \land s_4 \land s_1) -2(\langle s_1 s_3 \rangle \land (s_1 s_4) \rangle \otimes (s_1 \otimes s_1 \land s_3 \land s_4 \land s_2) +2(\langle s_1 s_3 \rangle \land (s_3 s_4) \rangle \otimes (s_3 \otimes s_1 \land s_3 \land s_4 \land s_2) -2(\langle s_1 s_4 \rangle \land (s_3 s_4) \rangle \otimes (s_4 \otimes s_1 \land s_3 \land s_4 \land s_2) -2(\langle s_1 s_3 \rangle \land (s_2 s_4) \rangle \otimes (s_2 \otimes s_1 \land s_2 \land s_4 \land s_3) -2(\langle s_1 s_4 \rangle \land (s_2 s_4) \rangle \otimes (s_4 \otimes s_1 \land s_2 \land s_4 \land s_3) -2(\langle s_1 s_2 \rangle \land (s_2 s_3) \rangle \otimes (s_2 \otimes s_1 \land s_2 \land s_3 \land s_4) -2(\langle s_1 s_3 \rangle \land (s_2 s_3) \rangle \otimes (s_3 \otimes s_1 \land s_2 \land s_3 \land s_4) \right\}
\]

\[
= \left\{ -2(\langle s_1 s_2 \rangle \land (s_1 s_3) \rangle \otimes (s_1 \otimes s_1 \land s_2 \land s_3 \land s_4) +2(\langle s_1 s_2 \rangle \land (s_1 s_4) \rangle \otimes (s_1 \otimes s_1 \land s_2 \land s_3 \land s_4) -2(\langle s_1 s_3 \rangle \land (s_1 s_4) \rangle \otimes (s_1 \otimes s_1 \land s_2 \land s_3 \land s_4) \right\}
\]

\[
= 0
\]

37
by formula (3.3).

Lastly,

\[
\sum_{s \in S} \pi_{Q \otimes_k C_1 \otimes_k C_2 ; 2} \circ d_4 (Q \otimes_k C_{4 , 1} (s)) = 0,
\]

which is easy to see.

\[\square\]

### 3.4 The maps and modules of \(X_*\) form a complex

**Proposition 3.5.** The sequence of maps and modules

\[
Q \otimes_k C_4 \xrightarrow{d_4} Q \otimes_k C_3 \xrightarrow{d_3} Q \otimes_k C_2 \xrightarrow{d_2} Q \otimes_k C_1 \xrightarrow{d_1} Q \xrightarrow{\varphi} \Sym^k_k (V)
\]

is a complex.
Proof. If \( m_1, m_2 \in \mathsf{sf} \), then
\[
(\varepsilon \circ d_1)((m_1) \wedge (m_2)) = \varepsilon(T(m_1)T(m_1) - T(m_2)T(m_2))
\]
\[
= m_1 m_1 - m_2 m_2
\]
\[
= \Omega - \Omega = 0.
\]

Now assume that \( S = \{s_1, s_2, s_3, s_4\} \). Then
\[
(\varepsilon \circ d_1)(s_1 \wedge s_2) = \varepsilon(U(s_1 \wedge s_2) + T(s_1 s_3) - T(s_2 s_3)T(s_2 s_4))
\]
\[
= s_3 s_4 \cdot (s_2 - s_1) + s_1 s_3 \cdot s_1 s_4 - s_2 s_3 \cdot s_2 s_4 \in \text{Sym}^k(V)
\]
\[
= 0
\]
and
\[
(\varepsilon \circ d_1)((s_1 - s_2)(s_3 - s_4))
\]
\[
(\varepsilon(U(s_1 \wedge s_2)U(s_1 \wedge s_3) - T(s_1 s_3)^2 + T(s_2 s_3)^2 - T(s_2 s_4)^2)
\]
\[
= (s_2 - s_1)(s_3 - s_4) - s_1 s_3 \cdot s_1 s_4 + s_2 s_3 \cdot s_2 s_4
\]
\[
= 0.
\]

Thus,
\[
\varepsilon \circ d_1 = 0.
\]

Now consider \( d_1 \circ d_2 \). Since \( d_2 \) on \( Q \otimes \Lambda^2 C_1 \) is the Koszul map induced by \( d_1 \), it is immediate that \( d_1 \circ d_2|_{Q \otimes \Lambda^2 C_1} = 0 \), so we only need show that \( d_1 \circ d_2|_{Q \otimes \Lambda^2 C_{2,1}} \) and \( d_1 \circ d_2|_{Q \otimes \Lambda^2 C_{2,2}} \) are zero.

For \( d_1 \circ d_2|_{Q \otimes \Lambda^2 C_{2,1}} \), let \( s \in S \) and \( S = \{s, t_1, t_2, t_3\} \). Then
\[
(d_1 \circ d_2|_{Q \otimes \Lambda^2 C_{2,1}})(s \otimes (s \wedge t_1 \wedge t_2 \wedge t_3))
\]

39
\[
\begin{cases}
+d_1 \left( U(t_1^2 \wedge s^2) \cdot \langle st_1 \rangle^* (\langle st_1 \rangle \wedge \langle st_2 \rangle \wedge \langle st_3 \rangle) \right) \\
+d_1 \left( U(t_2^2 \wedge s^2) \cdot \langle st_2 \rangle^* (\langle st_1 \rangle \wedge \langle st_2 \rangle \wedge \langle st_3 \rangle) \right) \\
+d_1 \left( U(t_3^2 \wedge s^2) \cdot \langle st_3 \rangle^* (\langle st_1 \rangle \wedge \langle st_2 \rangle \wedge \langle st_3 \rangle) \right) \\
+d_1 \left( T(t_1 s) \cdot (t_1^* \wedge s^*) (s \wedge t_1 \wedge t_2 \wedge t_3) \right) \\
+d_1 \left( T(t_2 s) \cdot (t_2^* \wedge s^*) (s \wedge t_1 \wedge t_2 \wedge t_3) \right) \\
+d_1 \left( T(t_3 s) \cdot (t_3^* \wedge s^*) (s \wedge t_1 \wedge t_2 \wedge t_3) \right)
\end{cases}
= \begin{cases}
+U(t_1^2 \wedge s^2) d_1 (\langle st_2 \rangle \wedge \langle st_3 \rangle) \\
-U(t_2^2 \wedge s^2) d_1 (\langle st_1 \rangle \wedge \langle st_3 \rangle) \\
+U(t_3^2 \wedge s^2) d_1 (\langle st_1 \rangle \wedge \langle st_2 \rangle) \\
+T(t_2 t_3) d_1 (t_2 \wedge t_3) \\
-T(t_1 t_3) d_1 (t_1 \wedge t_3) \\
+T(t_1 t_2) d_1 (t_1 \wedge t_2)
\end{cases}
\]

Expanding all of the \( d_1 \) terms, the above is seen to be zero, as long as one recalls formula (3.2): \( U(s_i^2 \wedge s_j^2) - U(s_i^2 \wedge s_k^2) + U(s_j^2 \wedge s_k^2) = 0 \) for any three distinct \( s_i, s_j, s_k \in S \).

This is sufficient to show that \( d_1 \circ d_2 |_{Q \otimes_k C_{2,1}} = 0 \) since any element of \( Q \otimes_k C_{2,1} \) is a multiple of an element of the form \( s \otimes (s \wedge t_1 \wedge t_2 \wedge t_3) \) for \( s, t_1, t_2, t_3 \in S \).
For the remaining summand of $Q \otimes_k C_2$, let $S = \{s_1, s_2, s_3, s_4\}$. Then

$$(d_1 \circ d_2|_{Q \otimes_k C_2})(s_1 \wedge s_2 \otimes s_3 \wedge s_4) = \begin{cases} 
-2d_1(T(s_3s_4) \cdot \langle s_1s_4 \rangle \wedge \langle s_1s_3 \rangle) \\
-d_1(U(s_1^2 \wedge s_2^2) \cdot s_3 \wedge s_4) \\
-d_1(T(s_1s_4) \cdot s_1 \wedge s_3) \\
+d_1(T(s_1s_3) \cdot s_1 \wedge s_4) \\
+d_1(T(s_2s_4) \cdot s_2 \wedge s_3) \\
-d_1(T(s_2s_3) \cdot s_2 \wedge s_4) \\
+d_1(T(s_1s_2) \cdot (s_1 - s_2)(s_3 - s_4)).
\end{cases}$$

Again to see that the above is zero requires only the use of the definition of $d_1$ and formula (3.2).

Hence,

$$d_1 \circ d_2 = 0.$$ 

Since we now know $d_1 \circ d_2 = 0$, we can invoke Lemma 3.1 to conclude that the Tate portion of the defined sequence of maps and modules is a complex. That is,

$$d_2 \circ d_3|_{Q \otimes_k C_1 \otimes_k (C_{2,1} \oplus C_{2,2})} = 0$$

and

$$d_3 \circ d_4|_{Q \otimes_k C_2 \otimes_k (C_{2,1} \oplus C_{2,2})} = 0,$$

so, to complete the proof, it remains to show that $d_2 \circ d_3|_{Q \otimes_k C_3,1}$, $d_3 \circ d_4|_{Q \otimes_k C_4,1}$, and $d_3 \circ d_4|_{Q \otimes_k C_4,2}$ are all zero.
Let \( s_1, s_2 \in S \) and \( S \setminus \{ s_1, s_2 \} = \{ s_3, s_4 \} \). Then

\[
(d_2 \circ d_3)|_{Q \otimes_k C_3, 1} (s_1 \wedge s_2) = \begin{cases} 
+2d_2 \left( T(s_3 s_4) \cdot (s_3 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s_4) \right) \\
-2d_2 \left( T(s_3 s_4) \cdot (s_4 \otimes s_1 \wedge s_2 \wedge s_3 \wedge s_4) \right) \\
+d_2 \left( U(s_3^2 \wedge s_4^2) \cdot (s_1 \wedge s_2 \otimes s_3 \wedge s_4) \right) \\
+d_2 \left( T(s_2 s_3) \cdot (s_1 \wedge s_3 \otimes s_2 \wedge s_4) \right) \\
+d_2 \left( T(s_2 s_4) \cdot (s_1 \wedge s_4 \otimes s_2 \wedge s_3) \right) \\
-d_2 \left( T(s_1 s_3) \cdot (s_2 \wedge s_3 \otimes s_1 \wedge s_4) \right) \\
-d_2 \left( T(s_1 s_4) \cdot (s_2 \wedge s_4 \otimes s_1 \wedge s_3) \right) \\
-2d_2 \left( \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \otimes s_1 \wedge s_2 \right) \\
-2d_2 \left( \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \otimes s_1 \wedge s_2 \right) \\
+d_2 \left( (s_1 \wedge s_3) \wedge (s_2 \wedge s_3) \right) \\
+d_2 \left( (s_1 \wedge s_4) \wedge (s_2 \wedge s_4) \right) \\
-d_2 \left( (s_3 \wedge s_4) \otimes (s_1 - s_2)(s_3 - s_4) \right). 
\end{cases}
\]

We consider the projections to the direct summands of \( Q \otimes_k C_1 \). To start, the
image of \((d_2 \circ d_3)|_{\mathcal{Q} \otimes_k \mathcal{C}_{3,1}}(s_1 \wedge s_2)\) in \(\mathcal{Q} \otimes_k \mathcal{C}_{1,1}\) is

\[
\begin{aligned}
+2d_1(s_1 \wedge s_2) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \\
+2d_1(s_1 \wedge s_2) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \\
+2T(s_3 s_4) \sum_{t \in S} U(t^2 \wedge s_3^2) \cdot \langle s_3 t \rangle^*(\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \\
-2T(s_3 s_4) \sum_{t \in S} U(t^2 \wedge s_3^2) \cdot \langle s_4 t \rangle^*(\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \\
-2U(s_3^2 \wedge s_4^2) T(s_3 s_4) \cdot \langle s_1 s_3 \rangle \\
+2T(s_2 s_3) T(s_2 s_4) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \\
-2T(s_1 s_3) T(s_1 s_4) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \\
+2T(s_2 s_3) T(s_2 s_4) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \\
-2T(s_1 s_3) T(s_1 s_4) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle.
\end{aligned}
\]

Collecting the same \(\langle m_1 \rangle \wedge \langle m_2 \rangle\) terms for \(m_1, m_2 \in \text{sf}\), we get

\[
\left. (\pi_{\mathcal{Q} \otimes_k \mathcal{C}_{1,1}} \circ d_2 \circ d_3) \right|_{\mathcal{Q} \otimes_k \mathcal{C}_{3,1}}(s_1 \wedge s_2) =
\]

\[
\begin{aligned}
+2(T(s_2 s_3) T(s_2 s_4) - T(s_1 s_3) T(s_1 s_4)) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \\
+2d_1(s_1 \wedge s_2) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \\
+2T(s_2 s_3) T(s_2 s_4) - T(s_1 s_3) T(s_1 s_4)) \cdot \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle \\
+2T(s_3 s_4) \sum_{t \in S} U(t^2 \wedge s_3^2) \cdot \langle s_3 t \rangle^*(\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \\
-2T(s_3 s_4) \sum_{t \in S} U(t^2 \wedge s_3^2) \cdot \langle s_4 t \rangle^*(\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \\
-2U(s_3^2 \wedge s_4^2) T(s_3 s_4) \cdot \langle s_1 s_3 \rangle.
\end{aligned}
\]

By definition of \(d_1\), we have

\[
\left. (\pi_{\mathcal{Q} \otimes_k \mathcal{C}_{1,1}} \circ d_2 \circ d_3) \right|_{\mathcal{Q} \otimes_k \mathcal{C}_{3,1}}(s_1 \wedge s_2) =
\]
Using (3.2) followed by (3.3), the above becomes

\[
\begin{align*}
&+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
&+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle \\
&+2T(s_3s_4) \sum_{t \in S} U(t^2 \wedge s_3^2) \cdot \langle s_3t \rangle^* (\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \\
&-2T(s_3s_4) \sum_{t \in S} U(t^2 \wedge s_3^2) \cdot \langle s_4t \rangle^* (\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \\
&-2U(s_3^2 \wedge s_4^2)T(s_3s_4) \cdot \langle s_1s_4 \rangle \wedge \langle s_1s_3 \rangle.
\end{align*}
\]

After expanding the summations and collecting terms, we get

\[
\begin{align*}
&+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
&+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
&-2T(s_3s_4)U(s_1^2 \wedge s_4^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
&-2T(s_3s_4)U(s_2^2 \wedge s_4^2) \cdot \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle \\
&-2T(s_3s_4)U(s_2^2 \wedge s_4^2) \cdot \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle.
\end{align*}
\]

Using (3.2) followed by (3.3), the above becomes

\[
\begin{align*}
&+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
&+2T(s_3s_4)U(s_1^2 \wedge s_2^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle \\
&-2T(s_3s_4)U(s_1^2 \wedge s_4^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \\
&+2T(s_3s_4)U(s_2^2 \wedge s_4^2) \cdot \langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle \\
&-2T(s_3s_4)U(s_2^2 \wedge s_4^2) \cdot \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle = 0.
\end{align*}
\]

44
Moving on, the portion of \((d_2 \circ d_3)|_{Q \otimes_k C_{3,1}}\) \((s_1 \land s_2)\) contained in \(Q \otimes_k C_{1,2}\) is

\[
\begin{align*}
-2d_1((s_1s_2) \land (s_1s_3)) \cdot s_1 \land s_2 & - 2d_1((s_1s_2) \land (s_1s_4)) \cdot s_1 \land s_2 \\
+d_1((s_1 - s_2)(s_3 - s_4)) \cdot s_3 \land s_4 & \\
+d_1(s_1 \land s_3) \cdot s_2 \land s_3 & - d_1(s_2 \land s_3) \cdot s_1 \land s_3 \\
+d_1(s_1 \land s_4) \cdot s_2 \land s_4 & - d_1(s_2 \land s_4) \cdot s_1 \land s_4 \\
+2T(s_3s_4) \sum_{i \in S} T(q_{3i}) \cdot (t^* \land s_4^i)(s_1 \land s_2 \land s_3 \land s_4) & \\
-2T(s_3s_4) \sum_{i \in S} T(q_{3i}) \cdot (t^* \land s_4^i)(s_1 \land s_2 \land s_3 \land s_4) & \\
-U(s_3^2 \land s_4^2)T(s_1s_4) \cdot s_1 \land s_3 + U(s_2^2 \land s_4^2)T(s_1s_4) \cdot s_1 \land s_3 & \\
+U(s_3^2 \land s_4^2)T(s_1s_3) \cdot s_1 \land s_4 & - U(s_2^2 \land s_4^2)T(s_1s_3) \cdot s_1 \land s_4 \\
+U(s_2^2 \land s_3^2)T(s_2s_4) \cdot s_2 \land s_3 & - U(s_1^2 \land s_3^2)T(s_2s_4) \cdot s_2 \land s_3 \\
-U(s_2^2 \land s_3^2)T(s_2s_3) \cdot s_2 \land s_4 & - U(s_1^2 \land s_3^2)T(s_2s_3) \cdot s_2 \land s_4 \\
-U(s_2^2 \land s_3^2)U(s_3^2 \land s_4^2) \cdot s_3 \land s_4 & \\
-2T(s_1s_4)T(s_2s_4) \cdot s_1 \land s_2 & - 2T(s_1s_4)T(s_2s_3) \cdot s_1 \land s_2 \\
+T(s_1s_2)T(s_2s_4) \cdot s_1 \land s_3 & + T(s_1s_3)T(s_3s_4) \cdot s_1 \land s_3 \\
+T(s_1s_4)T(s_4s_3) \cdot s_1 \land s_4 & + T(s_1s_2)T(s_2s_3) \cdot s_1 \land s_4 \\
-T(s_2s_3)T(s_3s_4) \cdot s_2 \land s_3 & - T(s_1s_2)T(s_1s_4) \cdot s_2 \land s_3 \\
-T(s_2s_4)T(s_3s_4) \cdot s_2 \land s_4 & - T(s_1s_2)T(s_1s_3) \cdot s_2 \land s_4 \\
-T(s_2s_3)^2 \cdot s_3 \land s_4 & + T(s_2s_4)^2 \cdot s_3 \land s_4 \\
-T(s_1s_4)^2 \cdot s_3 \land s_4 & + T(s_1s_3)^2 \cdot s_3 \land s_4.
\end{align*}
\]

Collecting terms containing the same \(s_i \land s_j\) for pairs \(\{i, j\}\), we get that the above
expression is equal to

\[
\begin{align*}
-2d_1((s_1s_2) & \land (s_1s_3)) \cdot s_1 \land s_2 - 2d_1((s_1s_2) \land (s_1s_4)) \cdot s_1 \land s_2 \\
-2T(s_1s_3)T(s_2s_4) \cdot s_1 \land s_2 - 2T(s_1s_4)T(s_2s_3) \cdot s_1 \land s_2 \\
-d_1(s_2 \land s_3) \cdot s_1 \land s_3 \\
-U(s_3^2 \land s_4^2)T(s_1s_4) \cdot s_1 \land s_3 + U(s_2^2 \land s_4^2)T(s_1s_4) \cdot s_1 \land s_3 \\
+T(s_1s_2)T(s_2s_4) \cdot s_1 \land s_3 + T(s_1s_3)T(s_3s_4) \cdot s_1 \land s_3 \\
-d_1(s_2 \land s_4) \cdot s_1 \land s_4 \\
+U(s_3^2 \land s_4^2)T(s_1s_3) \cdot s_1 \land s_4 - U(s_3^2 \land s_2^2)T(s_1s_3) \cdot s_1 \land s_4 \\
+T(s_1s_4)T(s_4s_3) \cdot s_1 \land s_4 + T(s_1s_2)T(s_2s_3) \cdot s_1 \land s_4 \\
+d_1(s_1 \land s_3) \cdot s_2 \land s_3 \\
+U(s_3^2 \land s_4^2)T(s_2s_4) \cdot s_2 \land s_3 - U(s_3^2 \land s_4^2)T(s_2s_4) \cdot s_2 \land s_3 \\
-T(s_2s_3)T(s_3s_4) \cdot s_2 \land s_3 - T(s_1s_2)T(s_1s_4) \cdot s_2 \land s_3 \\
+d_1(s_1 \land s_4) \cdot s_2 \land s_4 \\
-U(s_3^2 \land s_4^2)T(s_2s_3) \cdot s_2 \land s_4 - U(s_3^2 \land s_4^2)T(s_2s_3) \cdot s_2 \land s_4 \\
-T(s_2s_4)T(s_3s_4) \cdot s_2 \land s_4 - T(s_1s_2)T(s_1s_3) \cdot s_2 \land s_4 \\
+d_1((s_1 - s_2)(s_3 - s_4)) \cdot s_3 \land s_4 \\
-U(s_1^2 \land s_2^2)U(s_3^2 \land s_4^2) \cdot s_3 \land s_4 \\
-T(s_2s_3)^2 \cdot s_3 \land s_4 + T(s_2s_4)^2 \cdot s_3 \land s_4 \\
-T(s_1s_4)^2 \cdot s_3 \land s_4 + T(s_1s_3)^2 \cdot s_3 \land s_4 \\
+2T(s_3s_4) \sum_{t \in S} T(\overline{t}t_3) \cdot (t^* \land s_3^*)(s_1 \land s_2 \land s_3 \land s_4) \\
-2T(s_3s_4) \sum_{t \in S} T(\overline{t}t_4) \cdot (t^* \land s_4^*)(s_1 \land s_2 \land s_3 \land s_4)
\end{align*}
\]

46
\[
\begin{align*}
-4T(s_1s_2)T(s_3s_4) \cdot s_1 \land s_2 \\
+2T(s_1s_3)T(s_3s_4) \cdot s_1 \land s_3 \\
+2T(s_1s_4)T(s_4s_3) \cdot s_1 \land s_4 \\
-2T(s_2s_3)T(s_3s_4) \cdot s_2 \land s_3 \\
-2T(s_2s_4)T(s_3s_4) \cdot s_2 \land s_4 \\
+0 \cdot s_3 \land s_4 \\
+2T(s_3s_4) \sum_{t \in S} T(\overline{t}s_3) \cdot (t^* \land s_3^*)(s_1 \land s_2 \land s_3 \land s_4) \\
-2T(s_3s_4) \sum_{t \in S} T(\overline{t}s_4) \cdot (t^* \land s_4^*)(s_1 \land s_2 \land s_3 \land s_4)
\end{align*}
\]

\[
= 0,
\]

where the second-to-last equality is simply from the definition of \(d_1\). Thus,

\[
\pi_{Q \otimes_k C_{1,2}} \circ d_2 \circ d_3|_{Q \otimes_k C_{3,1}}(s_1 \land s_2) = 0.
\]

For the final summand of \(Q \otimes_k C_{1,3}\),

\[
(\pi_{Q \otimes_k C_{1,3}} \circ d_2 \circ d_3)|_{Q \otimes_k C_{5,1}}(s_1 \land s_2) = \begin{cases} 
-d_1(s_3 \land s_4) \cdot (s_1 - s_2)(s_3 - s_4) \\
+U(s_3^2 \land s_1^2)T(s_1s_2) \cdot (s_1 - s_2)(s_3 - s_4) \\
+T(s_1s_3)T(s_2s_3) \cdot (s_1 - s_3)(s_2 - s_4) \\
-T(s_1s_3)T(s_3s_2) \cdot (s_1 - s_4)(s_2 - s_3) \\
-T(s_1s_4)T(s_2s_4) \cdot (s_1 - s_3)(s_2 - s_4) \\
+T(s_1s_4)T(s_2s_4) \cdot (s_1 - s_4)(s_2 - s_3)
\end{cases}
\]
Recalling (3.4) the previous expression becomes

$$
\begin{align*}
-d_1(s_3 \wedge s_4) \cdot (s_1 - s_2)(s_3 - s_4) \\
+U(s_3^2 \wedge s_4^2)T(s_1 s_2) \cdot (s_1 - s_2)(s_3 - s_4) \\
+T(s_1 s_3)T(s_2 s_3) \cdot (s_1 - s_2)(s_3 - s_4) \\
-T(s_1 s_4)T(s_2 s_4) \cdot (s_1 - s_2)(s_3 - s_4),
\end{align*}
$$

which is zero by expanding $d_1(s_3 \wedge s_4)$. Therefore, we have finished showing

$$d_2 \circ d_3 = 0.$$

Next consider $d_3 \circ d_4|_{Q \otimes_k C_{4,1}}$. For $s \in S$ and $\{s, t_1, t_2, t_3\} = S$,

$$
(d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s) = \begin{cases}
+d_3\left(T(st_1) \cdot t_1 \wedge s\right) \\
+d_3\left(T(st_2) \cdot t_2 \wedge s\right) \\
+d_3\left(T(st_3) \cdot t_3 \wedge s\right) \\
-2d_3\left((\langle t_1 t_2 \rangle \wedge \langle t_1 t_3 \rangle) \otimes (t_1 \otimes t_1 \wedge t_2 \wedge t_3 \wedge s)\right) \\
+2d_3\left((\langle t_1 t_2 \rangle \wedge \langle t_1 s \rangle) \otimes (t_2 \otimes t_1 \wedge t_2 \wedge t_3 \wedge s)\right) \\
-2d_3\left((\langle t_1 t_3 \rangle \wedge \langle t_1 s \rangle) \otimes (t_3 \otimes t_1 \wedge t_2 \wedge t_3 \wedge s)\right) \\
+d_3\left(t_1 \wedge t_2 \otimes (s \otimes t_3 \wedge t_2 \wedge t_2)\right) \\
+d_3\left(t_1 \wedge t_3 \otimes (s \otimes t_2 \otimes t_1 \wedge t_3)\right) \\
+d_3\left(t_2 \wedge t_3 \otimes (s \otimes t_1 \otimes t_2 \wedge t_3)\right).
\end{cases}
$$

Again, we find the projections of $(d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s)$ onto the direct summands of $Q \otimes_k C_2$ and show these are all zero. Recall that we use the isomorphism (3.5) to
express $\wedge^2 C_1$ as a direct sum. Specifically,

\[
C_2 = C_{2,1} \oplus C_{2,2} \oplus \wedge^2_k C_1 = \left\{ \begin{array}{c} C_{2,1} \oplus C_{2,2} \\ \oplus \\ \wedge^2_k C_{1,1} \oplus (C_{1,1} \otimes_k C_{1,2}) \oplus (C_{1,1} \otimes_k C_{1,3}) \\ \oplus \\ \wedge^2_k C_{1,2} \oplus (C_{1,1} \otimes_k C_{1,3}) \oplus \wedge^2_k C_{1,3} \end{array} \right. 
\]

Two of the projections of $(d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s)$ are immediately zero:

\[
(\pi_{Q \otimes_k (C_{1,1} \otimes_k C_{1,3})} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s) = 0 
\]

and

\[
(\pi_{Q \otimes_k \wedge^2_k C_{1,3}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s) = 0. 
\]

We now calculate the remaining six projections.

\[
(\pi_{Q \otimes_k C_{2,1}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s) = \left\{ \begin{array}{c} +2T(st_1)T(t_2t_3) \cdot ((t_2 - t_3) \otimes t_1 \wedge t_2 \wedge t_3) \\ +2T(st_2)T(t_1t_3) \cdot ((t_1 - t_3) \otimes t_2 \wedge s \wedge t_1 \wedge t_3) \\ +2T(st_3)T(t_1t_2) \cdot ((t_1 - t_2) \otimes t_3 \wedge s \wedge t_1 \wedge t_2) \\ +2d_1(\langle t_1t_2 \rangle \wedge \langle st_2 \rangle) \cdot (t_1 \otimes s \wedge t_1 \wedge t_2 \wedge t_3) \\ -2d_1(\langle t_1t_2 \rangle \wedge \langle st_1 \rangle) \cdot (t_2 \otimes s \wedge t_1 \wedge t_2 \wedge t_3) \\ +2d_1(\langle st_2 \rangle \wedge \langle st_1 \rangle) \cdot (t_3 \otimes s \wedge t_1 \wedge t_2 \wedge t_3). \end{array} \right. 
\]

Using the definition of $d_1$ shows easily that $(\pi_{Q \otimes_k C_{2,1}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s) = 0$. 

49
For the image of $d_3 \circ d_4|_{Q \otimes_k C_{4,1}}(s)$ in the $Q \otimes_k C_{2,2}$ summand is

$$
\begin{align*}
+U(t_2^2 \land t_3^2)T(st_1) \cdot (t_1 \land s \otimes t_2 \land t_3) &+ U(t_1^2 \land t_3^2)T(st_2) \cdot (t_2 \land s \otimes t_1 \land t_3) \\
+U(t_1^2 \land t_3^2)T(st_3) \cdot (t_3 \land s \otimes t_1 \land t_2) \\
+T(st_1)T(st_2) \cdot (t_1 \land t_2 \otimes s \land t_3) &+ T(st_1)T(st_3) \cdot (t_1 \land t_3 \otimes s \land t_2) \\
-T(st_1)T(t_1t_2) \cdot (s \land t_2 \otimes t_1 \land t_3) &- T(st_1)T(t_1t_3) \cdot (s \land t_3 \otimes t_1 \land t_2) \\
+T(st_1)T(st_2) \cdot (t_2 \land t_1 \otimes s \land t_3) &+ T(st_2)T(st_3) \cdot (t_2 \land t_3 \otimes s \land t_1) \\
-T(st_2)T(t_2t_1) \cdot (s \land t_1 \otimes t_2 \land t_3) &- T(st_2)T(t_2t_3) \cdot (s \land t_3 \otimes t_2 \land t_1) \\
+T(st_1)T(st_3) \cdot (t_3 \land t_1 \otimes s \land t_2) &+ T(st_2)T(st_3) \cdot (t_3 \land t_2 \otimes s \land t_1) \\
-T(st_3)T(t_3t_1) \cdot (s \land t_1 \otimes t_3 \land t_2) &- T(st_3)T(t_3t_2) \cdot (s \land t_2 \otimes t_3 \land t_1) \\
+d_1(t_1 \land t_2) \cdot (s \land t_3 \otimes t_1 \land t_2) &+ d_1(t_1 \land t_3) \cdot (s \land t_2 \otimes t_1 \land t_3) \\
+d_1(t_2 \land t_3) \cdot (s \land t_1 \otimes t_2 \land t_3).
\end{align*}
$$

Collecting terms by the generators of $C_{2,2}$ gives

$$
(\pi_{Q \otimes_k C_{2,2}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s)
$$

$$
\begin{align*}
+&d_1(t_2 \land t_3) \cdot (s \land t_1 \otimes t_2 \land t_3) + U(t_2^2 \land t_3^2)T(st_1) \cdot (t_1 \land s \otimes t_2 \land t_3) \\
- &T(st_2)T(t_2t_1) \cdot (s \land t_1 \otimes t_2 \land t_3) - T(st_3)T(t_3t_1) \cdot (s \land t_1 \otimes t_3 \land t_2) \\
+ &d_1(t_1 \land t_3) \cdot (s \land t_2 \otimes t_1 \land t_3) + U(t_1^2 \land t_3^2)T(st_2) \cdot (t_2 \land s \otimes t_1 \land t_3) \\
- &T(st_1)T(t_1t_2) \cdot (s \land t_2 \otimes t_1 \land t_3) - T(st_3)T(t_3t_2) \cdot (s \land t_2 \otimes t_3 \land t_1) \\
= &+d_1(t_1 \land t_2) \cdot (s \land t_3 \otimes t_1 \land t_2) + U(t_1^2 \land t_2^2)T(st_3) \cdot (t_3 \land s \otimes t_1 \land t_2) \\
- &T(st_1)T(t_1t_3) \cdot (s \land t_3 \otimes t_1 \land t_2) - T(st_2)T(t_2t_3) \cdot (s \land t_3 \otimes t_2 \land t_1) \\
+ &T(st_1)T(st_2) \cdot (t_1 \land t_2 \otimes s \land t_3) + T(st_1)T(st_3) \cdot (t_2 \land t_1 \otimes s \land t_3) \\
+ &T(st_2)T(st_3) \cdot (t_1 \land t_3 \otimes s \land t_2) + T(st_1)T(st_3) \cdot (t_3 \land t_1 \otimes s \land t_2) \\
+ &T(st_2)T(st_3) \cdot (t_2 \land t_3 \otimes s \land t_1) + T(st_2)T(st_3) \cdot (t_3 \land t_2 \otimes s \land t_1),
\end{align*}
$$

50
which is zero by the definition of $d_1$.

For the $Q \otimes_k \Lambda^2_k C_{1,1}$ part,

$$(\pi_{Q \otimes_k \Lambda^2_k C_{1,1}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s) = \begin{cases} 
-2 \sum_{t \in S} U(t^2 \wedge t_1^2) \cdot \left( \langle t_1 t_2 \rangle \wedge \langle s t_2 \rangle \right) \wedge \left( \langle t_1 t \rangle \wedge (\alpha(s \wedge t_1 \wedge t_2 \wedge t_3)) \right) \\
+2 \sum_{t \in S} U(t^2 \wedge t_3^2) \cdot \left( \langle t_1 t_2 \rangle \wedge \langle s t_1 \rangle \right) \wedge \left( \langle t_1 t \rangle \wedge (\alpha(s \wedge t_1 \wedge t_2 \wedge t_3)) \right) \\
-2 \sum_{t \in S} U(t^2 \wedge t_3^2) \cdot \left( \langle s t_2 \rangle \wedge \langle s t_1 \rangle \right) \wedge \left( \langle t_3 t \rangle \wedge (\alpha(s \wedge t_1 \wedge t_2 \wedge t_3)) \right) 
\end{cases}$$

due to the fact that $\langle m \rangle = \langle m \rangle \in \overline{SF}$ for any $m \in sf$. Therefore,

$$(\pi_{Q \otimes_k \Lambda^2_k C_{1,1}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s) = \begin{cases} 
-2 \sum_{t \in S} U(t^2 \wedge t_1^2) \cdot \left( \langle t_1 t_2 \rangle \wedge \langle t_1 t_3 \rangle \right) \wedge \left( \langle t_1 t \rangle \wedge \langle t_2 t \rangle \right) \\
+2 \sum_{t \in S} U(t^2 \wedge t_3^2) \cdot \left( \langle t_1 t_2 \rangle \wedge \langle t_2 t_3 \rangle \right) \wedge \left( \langle t_1 t \rangle \wedge \langle t_2 t \rangle \right) 
\end{cases}$$

From here, using formula (3.3) gives that

$$(\pi_{Q \otimes_k \Lambda^2_k C_{1,1}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,1}})(s) = \begin{cases} 
-2 \sum_{t \in S} U(t^2 \wedge t_1^2) \cdot \left( \langle t_1 t_2 \rangle \wedge \langle t_1 t_3 \rangle \right) \wedge \left( \langle t_1 t \rangle \wedge \langle t_2 t \rangle \right) \\
+2 \sum_{t \in S} U(t^2 \wedge t_3^2) \cdot \left( \langle t_1 t_2 \rangle \wedge \langle t_2 t_3 \rangle \right) \wedge \left( \langle t_1 t \rangle \wedge \langle t_2 t \rangle \right) \\
+2 \sum_{t \in S} U(t^2 \wedge t_3^2) \cdot \left( \langle t_1 t_2 \rangle \wedge \langle t_1 t_3 \rangle \right) \wedge \left( \langle t_1 t \rangle \wedge \langle t_2 t \rangle \right) 
\end{cases}$$
\[
\begin{aligned}
&= \begin{cases} 
-2U(t_2^2 \wedge t_3^2) \cdot ((t_1t_2) \wedge (t_1t_3)) \wedge ((t_1t_3) \wedge (t_2t_3)) \\
+2U(t_2^2 \wedge t_3^2) \cdot ((t_1t_2) \wedge (t_1t_3)) \wedge ((t_1t_3) \wedge (t_2t_3)) 
\end{cases} \\
&= 0.
\end{aligned}
\]

For the \( Q \otimes_k (C_{1,1} \otimes_k C_{1,2}) \) summand, recall that \( \text{quot} : C_1 \otimes_k C_1 \to \Lambda^2_k C_1 \) was defined (in Section 3.3) as the natural quotient map and that we are using the isomorphism (3.5) from Observation 3.3 to decompose the module \( \Lambda^2_k C_1 \) as a direct sum. For instance, if \( \gamma_1 \in C_{1,1} \) and \( \gamma_2 \in C_{1,2} \), then the element \( \text{quot}(\gamma_2 \otimes \gamma_1) \in \Lambda^2 C_1 \) is \( -\gamma_1 \otimes \gamma_2 \) in the \( C_{1,1} \otimes_k C_{1,2} \) summand of the decomposition of \( \Lambda^2_k C_1 \) under the isomorphism (3.5). Thus,

\[
(\pi_{Q \otimes_k (C_{1,1} \otimes_k C_{1,2})} \circ d_3 \circ d_4|_{Q \otimes_k C_{1,1}})(s)
\]

\[
\begin{aligned}
&= \begin{cases} 
-2T(st_1) \cdot ((t_1s) \wedge (t_1t_2) + (t_1s) \wedge (t_1t_3)) \otimes t_1 \wedge s \\
-2T(st_2) \cdot ((t_2s) \wedge (t_2t_1) + (t_2s) \wedge (t_2t_3)) \otimes t_2 \wedge s \\
-2T(st_3) \cdot ((t_3s) \wedge (t_3t_1) + (t_3s) \wedge (t_3t_2)) \otimes t_3 \wedge s \\
+2T(t_1t_2) \cdot \text{quot} ((t_1 \wedge t_2) \otimes (t_2s) \wedge (st_3)) \\
+2T(t_1t_3) \cdot \text{quot} ((t_1 \wedge t_3) \otimes (t_3s) \wedge (st_1)) \\
+2T(t_2t_3) \cdot \text{quot} ((t_2 \wedge t_3) \otimes (t_3s) \wedge (st_2)) \\
-2 \sum_{t \in S} T(\overline{H_1}) \cdot \text{quot} (((t_1t_2) \wedge (st_2)) \otimes (t^* \wedge t_1^*)(s \wedge t_1 \wedge t_2 \wedge t_3)) \\
+2 \sum_{t \in S} T(\overline{H_2}) \cdot \text{quot} (((t_1t_2) \wedge (st_1)) \otimes (t^* \wedge t_2^*)(s \wedge t_1 \wedge t_2 \wedge t_3)) \\
-2 \sum_{t \in S} T(\overline{H_3}) \cdot \text{quot} (((st_2) \wedge (st_1)) \otimes (t^* \wedge t_3^*)(s \wedge t_1 \wedge t_2 \wedge t_3)) 
\end{cases}
\end{aligned}
\]
\[-2T(st_1) \cdot (\langle t_1s \rangle \land \langle t_1t_2 \rangle + \langle t_1s \rangle \land \langle t_1t_3 \rangle) \otimes t_1 \land s\]
\[-2T(st_2) \cdot (\langle t_2s \rangle \land \langle t_2t_1 \rangle + \langle t_2s \rangle \land \langle t_2t_3 \rangle) \otimes t_2 \land s\]
\[-2T(st_3) \cdot (\langle t_3s \rangle \land \langle t_3t_1 \rangle + \langle t_3s \rangle \land \langle t_3t_2 \rangle) \otimes t_3 \land s\]
\[-2T(t_1t_2) \cdot \langle st_2 \rangle \land \langle st_1 \rangle \otimes (t_1 \land t_2)\]
\[-2T(t_1t_3) \cdot \langle st_3 \rangle \land \langle st_1 \rangle \otimes (t_1 \land t_3)\]
\[-2T(t_2t_3) \cdot \langle st_3 \rangle \land \langle st_2 \rangle \otimes (t_2 \land t_3)\]
\[-2 \sum_{t \in S} T(\overline{H}_1) \cdot \langle t_1t_2 \rangle \land \langle st_2 \rangle \otimes (t^* \land t^*_1)(s \land t_1 \land t_2 \land t_3)\]
\[+2 \sum_{t \in S} T(\overline{H}_2) \cdot \langle t_1t_2 \rangle \land \langle st_1 \rangle \otimes (t^* \land t^*_2)(s \land t_1 \land t_2 \land t_3)\]
\[-2 \sum_{t \in S} T(\overline{H}_3) \cdot \langle st_2 \rangle \land \langle st_1 \rangle \otimes (t^* \land t^*_3)(s \land t_1 \land t_2 \land t_3)\]

= 0,

where the last equality comes from noticing that the three summations at the end are exactly what is needed to make the expression zero. It is straightforward to see that the remaining two pieces of $d_3 \circ d_4|_{Q \otimes_{k} C_{4,1}}(s)$ are zero:
\[
\begin{aligned}
\pi_{Q \otimes \Lambda^2_{C_{1,2}}} \circ d_3 \circ d_4 |_{Q \otimes \Lambda^2_{C_{3,4}}} (s) &= 0 \\
&= +T(st_2) \cdot (t_2 \land t_1) \land (s \land t_1) \\
&+ T(st_2) \cdot (t_2 \land t_3) \land (s \land t_3) \\
&+ T(st_3) \cdot (t_3 \land t_1) \land (s \land t_1) \\
&+ T(st_3) \cdot (t_3 \land t_2) \land (s \land t_2) \\
&+ T(st_1) \cdot (t_1 \land t_2) \land (s \land t_2) \\
&+ T(st_1) \cdot (t_1 \land t_3) \land (s \land t_3) \\
&+ U(s^2 \land t_3^2) \cdot (t_1 \land t_2) \land (t_1 \land t_2) \\
&+ T(st_2) \cdot (t_1 \land t_2) \land (s \land t_1) \\
&- T(st_1) \cdot (t_1 \land t_2) \land (s \land t_2) \\
&- T(t_3t_2) \cdot (t_1 \land t_2) \land (t_3 \land t_1) \\
&+ T(t_3t_1) \cdot (t_1 \land t_2) \land (t_3 \land t_2) \\
&+ U(s^2 \land t_3^2) \cdot (t_1 \land t_3) \land (t_1 \land t_3) \\
&+ T(st_3) \cdot (t_1 \land t_3) \land (s \land t_1) \\
&- T(st_1) \cdot (t_1 \land t_3) \land (s \land t_3) \\
&- T(t_2t_3) \cdot (t_1 \land t_3) \land (t_2 \land t_1) \\
&+ T(t_2t_3) \cdot (t_1 \land t_3) \land (t_2 \land t_3) \\
&+ U(s^2 \land t_2^2) \cdot (t_2 \land t_3) \land (t_2 \land t_3) \\
&+ T(st_3) \cdot (t_2 \land t_3) \land (s \land t_2) \\
&- T(st_2) \cdot (t_2 \land t_3) \land (s \land t_3) \\
&- T(t_1t_3) \cdot (t_2 \land t_3) \land (t_1 \land t_2) \\
&+ T(t_1t_3) \cdot (t_2 \land t_3) \land (t_1 \land t_3) \\
\end{aligned}
\]
and

\[
(\pi_{Q \otimes k(C_{1,2} \otimes kC_{1,3})} \circ d_3 \circ d_4|_{Q \otimes kC_{4,1}})(s) = \begin{cases}
-T(st_1) \cdot (t_2 \land t_3) \otimes (t_1 - s)(t_2 - t_3) \\
-T(st_2) \cdot (t_1 \land t_3) \otimes (t_2 - s)(t_1 - t_3) \\
-T(st_3) \cdot (t_1 \land t_2) \otimes (t_3 - s)(t_1 - t_2) \\
-T(st_3) \cdot (t_1 \land t_2) \otimes (s - t_3)(t_1 - t_2) \\
-T(st_2) \cdot (t_1 \land t_3) \otimes (s - t_2)(t_1 - t_3) \\
-T(st_1) \cdot (t_2 \land t_3) \otimes (s - t_1)(t_2 - t_3)
\end{cases}
= 0.
\]

Hence,

\[d_3 \circ d_4|_{Q \otimes kC_{4,1}} = 0.\]

It remains to verify that \(d_3 \circ d_4|_{Q \otimes kC_{4,2}} = 0\). Let \(s_1s_2 \in sf\) and \(S = \{s_1, s_2, s_3, s_4\}\). Then

\[
(d_3 \circ d_4|_{Q \otimes kC_{4,2}})(s_1s_2) = \begin{cases}
+3(T(s_1s_3) \cdot s_1 \land s_4 + T(s_1s_4) \cdot s_1 \land s_3) \\
+3(T(s_2s_3) \cdot s_2 \land s_4 + T(s_2s_4) \cdot s_2 \land s_3) \\
-3(U(s_3^2 \land s_4^2) \cdot s_3 \land s_4) \\
+23 \left( (s_1s_3) \land (s_1s_4) \right) \otimes s_2 \land s_3 \otimes s_4 \\
+23 \left( s_3 \land s_4 \otimes (s_1 - s_2) \otimes (s_1 \land s_2 \land s_3 \land s_4) \right) \\
-3 \left( s_1 \land s_3 \otimes (s_2 \land s_3 \otimes s_1 \land s_4) \right) \\
-3 \left( s_2 \land s_3 \otimes (s_1 \land s_3 \otimes s_2 \land s_4) \right) \\
-3 \left( s_1 \land s_4 \otimes (s_2 \land s_4 \otimes s_1 \land s_3) \right) \\
-3 \left( s_2 \land s_4 \otimes (s_1 \land s_4 \otimes s_2 \land s_3) \right) \\
+3 \left( s_1 - s_2)(s_3 - s_4) \otimes (s_3 \land s_4 \otimes s_1 \land s_2) \right)
\end{cases}
\]
and we again look at the projections of the above to the various direct summands of \( Q \otimes_k C_2 \) and show that they are each zero. Four of the projections are short and easily seen to be zero:

\[
\begin{align*}
(\pi_{Q \otimes_k C_{2,1}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,2}})(s_1s_2) &= 0, \\
(\pi_{Q \otimes_k C_{1,1}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,2}})(s_1s_2) &= 0, \\
(\pi_{Q \otimes_k (C_{1,1} \otimes_k C_{1,3})} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,2}})(s_1s_2) &= 0, \\
(\pi_{Q \otimes_k (C_{1,3} \otimes_k C_{1,1})} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,2}})(s_1s_2) &= 0.
\end{align*}
\]
Now compute the image in $Q \otimes_k C_{2,2}$:

$\left\{ \begin{aligned}
(p_{Q \otimes_k C_{2,2}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,2}})(s_1 s_2) \\
+2d_1((s_1 s_3) \wedge (s_1 s_4)) \cdot (s_1 \wedge s_2 \otimes s_3 \wedge s_4) \\
+(U(s_2^2 \wedge s_3^2)T(s_1 s_4) + T(s_1 s_2)T(s_2 s_4) - T(s_1 s_3)T(s_3 s_4)) \cdot (s_1 \wedge s_3 \otimes s_2 \wedge s_4) \\
+(U(s_2^2 \wedge s_3^2)T(s_1 s_3) + T(s_1 s_2)T(s_2 s_3) - T(s_1 s_4)T(s_3 s_4)) \cdot (s_1 \wedge s_4 \otimes s_2 \wedge s_3) \\
+(U(s_2^2 \wedge s_3^2)T(s_2 s_4) + T(s_1 s_2)T(s_1 s_4) - T(s_2 s_3)T(s_3 s_4)) \cdot (s_2 \wedge s_3 \otimes s_1 \wedge s_4) \\
+(U(s_1^2 \wedge s_3^2)T(s_2 s_3) + T(s_1 s_2)T(s_1 s_3) - T(s_2 s_4)T(s_3 s_4)) \cdot (s_2 \wedge s_4 \otimes s_1 \wedge s_3) \\
+(T(s_1 s_3)^2 - T(s_1 s_4)^2 + T(s_2 s_4)^2 - T(s_2 s_3)^2) \cdot (s_3 \wedge s_4 \otimes s_1 \wedge s_2) \\
+2d_1((s_1 s_3) \wedge (s_1 s_4)) \cdot (s_1 \wedge s_2 \otimes s_3 \wedge s_4) \\
-d_1(s_1 \wedge s_3) \cdot (s_2 \wedge s_3 \otimes s_1 \wedge s_4) \\
-d_1(s_2 \wedge s_3) \cdot (s_1 \wedge s_3 \otimes s_2 \wedge s_4) \\
-d_1(s_1 \wedge s_4) \cdot (s_2 \wedge s_4 \otimes s_1 \wedge s_3) \\
-d_1(s_2 \wedge s_4) \cdot (s_1 \wedge s_4 \otimes s_2 \wedge s_3) \\
+d_1((s_1 - s_2)(s_3 - s_4)) \cdot (s_3 \wedge s_4 \otimes s_1 \wedge s_2) \\
-U(s_1^2 \wedge s_2^2)U(s_3^2 \wedge s_4^2) \cdot (s_3 \wedge s_4 \otimes s_1 \wedge s_2) \\
-T(s_4 s_1)U(s_3^2 \wedge s_2^2) \cdot (s_3 \wedge s_1 \otimes s_4 \wedge s_2) \\
-T(s_4 s_2)U(s_3^2 \wedge s_2^2) \cdot (s_3 \wedge s_2 \otimes s_4 \wedge s_1) \\
+T(s_3 s_1)U(s_3^2 \wedge s_2^2) \cdot (s_4 \wedge s_1 \otimes s_3 \wedge s_2) \\
+T(s_3 s_2)U(s_3^2 \wedge s_2^2) \cdot (s_4 \wedge s_2 \otimes s_3 \wedge s_1).
\end{aligned} \right.

Use the definition of $d_1$ and formula (3.2) to get $(p_{Q \otimes_k C_{2,2}} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,2}})(s_1 s_2) = 0.$
The projection of $d_3 \circ d_4|_{Q \otimes_k (C_{1,1} \otimes_k C_{1,2})}(s_1 s_2)$ onto $Q \otimes_k (C_{1,1} \otimes_k C_{1,2})$ is

$$
\begin{aligned}
-2T(s_1 s_4) \cdot (-\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle + \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \otimes s_1 \wedge s_3 \\
+2T(s_1 s_3) \cdot ((\langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle + \langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle) \otimes s_1 \wedge s_4 \\
+2T(s_2 s_4) \cdot ((\langle s_1 s_2 \rangle \wedge \langle s_1 s_4 \rangle + \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \otimes s_2 \wedge s_3 \\
-2T(s_2 s_3) \cdot (-\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle + \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \otimes s_2 \wedge s_4 \\
+2T(s_1 s_4) \cdot \text{quot} \left( (\langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \otimes s_1 \wedge s_3 \right) \\
-2T(s_1 s_3) \cdot \text{quot} \left( (\langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \otimes s_1 \wedge s_4 \right) \\
-2T(s_2 s_4) \cdot \text{quot} \left( (\langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \otimes s_2 \wedge s_3 \right) \\
+2T(s_2 s_3) \cdot \text{quot} \left( (\langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \otimes s_2 \wedge s_4 \right) \\
-2T(s_2 s_4) \cdot \text{quot} \left( (s_2 \wedge s_3) \otimes (s_1 s_4) \wedge (s_1 s_2) \right) \\
-2T(s_2 s_3) \cdot \text{quot} \left( (s_2 \wedge s_4) \otimes (s_1 s_3) \wedge (s_1 s_2) \right) \\
-2T(s_1 s_4) \cdot \text{quot} \left( (s_1 \wedge s_3) \otimes (s_2 s_4) \wedge (s_2 s_1) \right) \\
-2T(s_1 s_3) \cdot \text{quot} \left( (s_1 \wedge s_4) \otimes (s_2 s_3) \wedge (s_2 s_1) \right) \\
+2U(s_3^2 \wedge s_4^2) \cdot (\langle s_3 s_4 \rangle \wedge \langle s_3 s_1 \rangle) \otimes (s_3 \wedge s_4) \\
+2U(s_3^2 \wedge s_4^2) \cdot (\langle s_3 s_4 \rangle \wedge \langle s_3 s_2 \rangle) \otimes (s_3 \wedge s_4) \\
+2U(s_3^2 \wedge s_4^2) \cdot \text{quot} \left( (\langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle) \otimes s_3 \wedge s_4 \right) \\
-2 \sum_{t \in S} U(t^2 \wedge s_2^2) \cdot \text{quot} \left( (s_3 \wedge s_4) \otimes (s_1 t)^* ((\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle)) \right) \\
+2 \sum_{t \in S} U(t^2 \wedge s_2^2) \cdot \text{quot} \left( (s_3 \wedge s_4) \otimes (s_2 t)^* ((\langle s_1 s_2 \rangle \wedge \langle s_1 s_3 \rangle \wedge \langle s_1 s_4 \rangle)) \right).
\end{aligned}
$$

Collect terms with the same $s_i \wedge s_j$ for each pair \{i, j\} to obtain

\[(\pi_{Q \otimes_k (C_{1,1} \otimes_k C_{1,2})} \circ d_3 \circ d_4|_{Q \otimes_k (C_{1,1} \otimes_k C_{1,2})})(s_1 s_2)\]
\[-2T(s_1s_4) \cdot (\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle + \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \otimes s_1 \wedge s_3 \]
\[+2T(s_1s_4) \cdot \text{quot} \left( (\langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \otimes s_1 \wedge s_3 \right) \]
\[-2T(s_1s_4) \cdot \text{quot} \left( (s_1 \wedge s_3) \otimes \langle s_2s_4 \rangle \wedge \langle s_2s_1 \rangle \right) \]
\[+2T(s_1s_3) \cdot (\langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle + \langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle) \otimes s_1 \wedge s_4 \]
\[-2T(s_1s_3) \cdot \text{quot} \left( (\langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \otimes s_1 \wedge s_4 \right) \]
\[-2T(s_1s_3) \cdot \text{quot} \left( (s_1 \wedge s_4) \otimes \langle s_2s_3 \rangle \wedge \langle s_2s_1 \rangle \right) \]
\[+2T(s_2s_4) \cdot (\langle s_1s_2 \rangle \wedge \langle s_1s_4 \rangle + \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \otimes s_2 \wedge s_3 \]
\[-2T(s_2s_4) \cdot \text{quot} \left( (\langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \otimes s_2 \wedge s_3 \right) \]
\[-2T(s_2s_4) \cdot \text{quot} \left( (s_2 \wedge s_3) \otimes \langle s_1s_4 \rangle \wedge \langle s_1s_2 \rangle \right) \]
\[-2T(s_2s_3) \cdot (\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle + \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \otimes s_2 \wedge s_4 \]
\[+2T(s_2s_3) \cdot \text{quot} \left( (\langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \otimes s_2 \wedge s_4 \right) \]
\[-2T(s_2s_3) \cdot \text{quot} \left( (s_2 \wedge s_4) \otimes \langle s_1s_3 \rangle \wedge \langle s_1s_2 \rangle \right) \]
\[+2U(s_1^2 \wedge s_2^2) \cdot (\langle s_1s_4 \rangle \wedge \langle s_3s_1 \rangle) \otimes (s_2 \wedge s_4) \]
\[+2U(s_1^2 \wedge s_2^2) \cdot (\langle s_1s_4 \rangle \wedge \langle s_3s_2 \rangle) \otimes (s_3 \wedge s_4) \]
\[+2U(s_1^2 \wedge s_2^2) \cdot \text{quot} \left( (\langle s_1s_4 \rangle \wedge \langle s_1s_4 \rangle) \otimes s_3 \wedge s_4 \right) \]
\[-2 \sum_{t \in S} U(t^2 \wedge s_1^2) \cdot \text{quot} \left( (s_3 \wedge s_4) \otimes (s_1s_2)^\ast(\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \right) \]
\[+2 \sum_{t \in S} U(t^2 \wedge s_2^2) \cdot \text{quot} \left( (s_3 \wedge s_4) \otimes (s_2s_1)^\ast(\langle s_1s_2 \rangle \wedge \langle s_1s_3 \rangle \wedge \langle s_1s_4 \rangle) \right) \]

As mentioned for the earlier calculation of \( \pi_{\otimes_k(C_{1,1} \otimes_k C_{1,2})} \circ d_3 \circ d_4 |_{\otimes_k C_{4,1}} \), \text{quot} \: C_1 \otimes_k C_1 \to \Lambda_k^2 C_1 \) is the natural quotient map and we use the isomorphism (3.5) to write elements of \( \Lambda_k^2 C_1 \). Thus, the terms above the final horizontal line above add to
zero. Therefore, the expression becomes

\[
\begin{align*}
+2U(s_3^2 \land s_4^2) \cdot ((s_3 s_4) \land (s_3 s_1)) \otimes (s_3 \land s_4) \\
+2U(s_3^2 \land s_4^2) \cdot ((s_3 s_4) \land (s_3 s_2)) \otimes (s_3 \land s_4) \\
+2U(s_1^2 \land s_2^2) \cdot \text{quot} \left( ((s_1 s_3) \land (s_1 s_4)) \otimes s_3 \land s_4 \right) \\
-2 \sum_{t \in S} U(t^2 \land s_1^2) \cdot \text{quot} \left( (s_3 \land s_4) \otimes (s_1 t)^* ((s_1 s_2) \land (s_1 s_3) \land (s_1 s_4)) \right) \\
+2 \sum_{t \in S} U(t^2 \land s_2^2) \cdot \text{quot} \left( (s_3 \land s_4) \otimes (s_2 t)^* ((s_1 s_2) \land (s_1 s_3) \land (s_1 s_4)) \right).
\end{align*}
\]

Expand the two summations to get

\[
\begin{align*}
(p_{Q \otimes k(c_{1,1} \otimes k_{1,2})} \circ d_3 \circ d_4 |_{Q \otimes k_{1,2}})(s_1 s_2)
\end{align*}
\]

\[
\begin{align*}
+2U(s_3^2 \land s_4^2) \cdot ((s_1 s_2) \land (s_1 s_3)) \otimes (s_3 \land s_4) \\
+2U(s_3^2 \land s_4^2) \cdot ((s_1 s_2) \land (s_1 s_3)) \otimes (s_3 \land s_4) \\
-2U(s_3^2 \land s_4^2) \cdot ((s_1 s_2) \land (s_1 s_3)) \otimes (s_3 \land s_4) \\
+2U(s_3^2 \land s_4^2) \cdot ((s_1 s_2) \land (s_1 s_4)) \otimes (s_3 \land s_4) \\
-2U(s_3^2 \land s_4^2) \cdot ((s_1 s_2) \land (s_1 s_4)) \otimes (s_3 \land s_4) \\
+2U(s_3^2 \land s_4^2) \cdot ((s_1 s_3) \land (s_1 s_4)) \otimes (s_3 \land s_4) \\
+2U(s_3^2 \land s_4^2) \cdot ((s_1 s_3) \land (s_1 s_4)) \otimes (s_3 \land s_4) \\
-2U(s_1^2 \land s_2^2) \cdot ((s_1 s_3) \land (s_1 s_4)) \otimes (s_3 \land s_4) \\
+2U(s_1^2 \land s_2^2) \cdot ((s_1 s_3) \land (s_1 s_4)) \otimes (s_3 \land s_4) \\
-2U(s_1^2 \land s_2^2) \cdot ((s_1 s_3) \land (s_1 s_4)) \otimes (s_3 \land s_4) \\
+2U(s_1^2 \land s_2^2) \cdot ((s_1 s_3) \land (s_1 s_4)) \otimes (s_3 \land s_4) \\
= 0,
\end{align*}
\]

the last equality following from (3.3) and the fact that \(U(s_2^2 \land s_1^2) = -U(s_1^2 \land s_2^2)\). The
next part is easy to check.

\[
\left( \pi_{Q \otimes \wedge} \lambda_{C_{1,2}} \circ d_3 \circ d_4 |_{Q \otimes \wedge C_{4,2}} \right)(s_1 s_2)
\]

\[
= \left\{
\begin{align*}
-T(s_1 s_3) \cdot (s_1 \land s_3) \land (s_3 \land s_4) - T(s_1 s_3) \cdot (s_1 \land s_2) \land (s_2 \land s_4) \\
-T(s_1 s_4) \cdot (s_1 \land s_2) \land (s_2 \land s_3) + T(s_1 s_4) \cdot (s_1 \land s_4) \land (s_3 \land s_4) \\
+ T(s_2 s_4) \cdot (s_1 \land s_2) \land (s_1 \land s_3) + T(s_2 s_4) \cdot (s_2 \land s_4) \land (s_3 \land s_4) \\
+ T(s_2 s_3) \cdot (s_1 \land s_2) \land (s_1 \land s_4) - T(s_2 s_3) \cdot (s_2 \land s_3) \land (s_3 \land s_4) \\
-T(s_1 s_4) \cdot (s_2 \land s_3) \land (s_1 \land s_2) + T(s_1 s_2) \cdot (s_2 \land s_3) \land (s_1 \land s_4) \\
+ T(s_3 s_4) \cdot (s_2 \land s_3) \land (s_3 \land s_2) - T(s_3 s_2) \cdot (s_2 \land s_3) \land (s_3 \land s_4) \\
-T(s_1 s_3) \cdot (s_2 \land s_4) \land (s_1 \land s_2) + T(s_1 s_2) \cdot (s_2 \land s_4) \land (s_1 \land s_3) \\
+ T(s_4 s_3) \cdot (s_2 \land s_4) \land (s_4 \land s_2) - T(s_4 s_2) \cdot (s_2 \land s_4) \land (s_4 \land s_3) \\
-T(s_2 s_4) \cdot (s_1 \land s_3) \land (s_2 \land s_1) + T(s_2 s_1) \cdot (s_1 \land s_3) \land (s_2 \land s_4) \\
+ T(s_3 s_4) \cdot (s_1 \land s_3) \land (s_3 \land s_1) - T(s_3 s_1) \cdot (s_1 \land s_3) \land (s_3 \land s_4) \\
-T(s_2 s_3) \cdot (s_1 \land s_4) \land (s_2 \land s_1) + T(s_2 s_1) \cdot (s_1 \land s_4) \land (s_2 \land s_3) \\
+ T(s_4 s_3) \cdot (s_1 \land s_4) \land (s_4 \land s_1) - T(s_4 s_1) \cdot (s_1 \land s_4) \land (s_4 \land s_3) \\
-U(s_3^2 \land s_4^2) \cdot (s_3 \land s_1) \land (s_4 \land s_1) \\
-U(s_3^2 \land s_4^2) \cdot (s_3 \land s_2) \land (s_4 \land s_2) \\
-U(s_1^2 \land s_3^2) \cdot (s_2 \land s_3) \land (s_2 \land s_4) \\
-U(s_1^2 \land s_3^2) \cdot (s_2 \land s_4) \land (s_2 \land s_3) \\
-U(s_2^2 \land s_3^2) \cdot (s_1 \land s_3) \land (s_1 \land s_4) \\
-U(s_2^2 \land s_3^2) \cdot (s_1 \land s_4) \land (s_1 \land s_3) \\
-2 \sum_{i \in \mathcal{S}} T(\overline{t s_1}) \cdot \left( (s_3 \land s_4) \land (t^* \land s_i^*)(s_1 \land s_2 \land s_3 \land s_4) \right) \\
+2 \sum_{i \in \mathcal{S}} T(\overline{t s_2}) \cdot \left( (s_3 \land s_4) \land (t^* \land s_i^*)(s_1 \land s_2 \land s_3 \land s_4) \right)
\end{align*}
\right.
\]

61
\begin{align*}
-2T(s_1s_3) \cdot (s_1 \land s_3) \land (s_3 \land s_4) + 2T(s_1s_4) \cdot (s_1 \land s_4) \land (s_3 \land s_4) \\
-2T(s_2s_3) \cdot (s_2 \land s_3) \land (s_3 \land s_4) + 2T(s_2s_4) \cdot (s_2 \land s_4) \land (s_3 \land s_4) \\
-2 \sum_{t \in S} T(\overline{t}s_1) \cdot \left((s_3 \land s_4) \land (t^* \land s_1^*) (s_1 \land s_2 \land s_3 \land s_4)\right) \\
+2 \sum_{t \in S} T(\overline{t}s_2) \cdot \left((s_3 \land s_4) \land (t^* \land s_2^*) (s_1 \land s_2 \land s_3 \land s_4)\right)
\end{align*}

= 0.

Finally, the one remaining projection of \(d_3 \circ d_4|_{Q \otimes_k C_{4,2}}(s_1s_2)\) is

\[
(\pi_{Q \otimes_k (C_{1,2} \otimes_k C_{1,3})} \circ d_3 \circ d_4|_{Q \otimes_k C_{4,2}})(s_1s_2)
\begin{align*}
+ U(s_3^2 \land s_4^2) \cdot (s_1 \land s_2) \cdot (s_3 - s_4)(s_1 - s_2) \\
+ T(s_1s_3) \cdot \text{quot} \left((s_2 \land s_3) \otimes (s_1 - s_3)(s_2 - s_4)\right) \\
+ T(s_1s_4) \cdot \text{quot} \left((s_2 \land s_4) \otimes (s_1 - s_4)(s_2 - s_3)\right) \\
+ T(s_2s_3) \cdot \text{quot} \left((s_1 \land s_3) \otimes (s_2 - s_3)(s_1 - s_4)\right) \\
+ T(s_2s_4) \cdot \text{quot} \left((s_1 \land s_4) \otimes (s_2 - s_4)(s_1 - s_3)\right) \\
+ T(s_1s_3) \cdot (s_2 \land s_3) \otimes (s_1 - s_4)(s_3 - s_2) \\
- T(s_1s_4) \cdot (s_2 \land s_4) \otimes (s_1 - s_3)(s_2 - s_4) \\
- T(s_2s_4) \cdot (s_1 \land s_4) \otimes (s_1 - s_4)(s_2 - s_3) \\
- T(s_2s_3) \cdot (s_1 \land s_3) \otimes (s_1 - s_3)(s_2 - s_4) \\
+ U(s_3^2 \land s_4^2) \cdot \text{quot} \left((s_1 - s_2)(s_3 - s_4) \otimes s_1 \land s_2\right) \\
+ T(s_3s_2) \cdot \text{quot} \left((s_1 - s_2)(s_3 - s_4) \otimes s_3 \land s_1\right) \\
- T(s_3s_1) \cdot \text{quot} \left((s_1 - s_2)(s_3 - s_4) \otimes s_3 \land s_2\right) \\
- T(s_4s_2) \cdot \text{quot} \left((s_1 - s_2)(s_3 - s_4) \otimes s_4 \land s_1\right) \\
+ T(s_4s_1) \cdot \text{quot} \left((s_1 - s_2)(s_3 - s_4) \otimes s_4 \land s_2\right).
\end{align*}

62
Collect all terms with the same $s_i \land s_j$, we get

\[
(\pi_{Q \otimes k(C_{1,2} \otimes kC_{1,3})} \circ d_3 \circ d_4|_{Q \otimes kC_{4,2}})(s_1s_2)
\]

\[
= \begin{aligned}
+T(s_1s_3) \cdot \text{quot} \left( (s_2 \land s_3) \otimes (s_1 - s_3)(s_2 - s_4) \right) \\
-T(s_3s_1) \cdot \text{quot} \left( (s_1 - s_2)(s_3 - s_4) \otimes s_3 \land s_2 \right) \\
+T(s_1s_3) \cdot (s_2 \land s_3) \otimes (s_1 - s_4)(s_3 - s_2) \\
+T(s_1s_4) \cdot \text{quot} \left( (s_2 \land s_4) \otimes (s_1 - s_4)(s_2 - s_3) \right) \\
+T(s_4s_1) \cdot \text{quot} \left( (s_1 - s_2)(s_3 - s_4) \otimes s_4 \land s_2 \right) \\
-T(s_1s_4) \cdot (s_2 \land s_4) \otimes (s_1 - s_3)(s_2 - s_4) \\
+T(s_2s_3) \cdot \text{quot} \left( (s_1 \land s_3) \otimes (s_2 - s_3)(s_1 - s_4) \right) \\
+T(s_3s_2) \cdot \text{quot} \left( (s_1 - s_2)(s_3 - s_4) \otimes s_3 \land s_1 \right) \\
-T(s_2s_3) \cdot (s_1 \land s_3) \otimes (s_1 - s_3)(s_2 - s_4) \\
+T(s_2s_4) \cdot \text{quot} \left( (s_1 \land s_4) \otimes (s_2 - s_4)(s_1 - s_3) \right) \\
-T(s_4s_2) \cdot \text{quot} \left( (s_1 - s_2)(s_3 - s_4) \otimes s_4 \land s_1 \right) \\
-T(s_2s_4) \cdot (s_1 \land s_4) \otimes (s_1 - s_4)(s_2 - s_3)
\end{aligned}
\]

= 0

by formula (3.4), as well as the same considerations mentioned for the calculations of

$\pi_{Q \otimes k \Lambda^2 C_{1,2}} \circ d_3 \circ d_4|_{Q \otimes kC_{4,1}}$ and $\pi_{Q \otimes k(C_{1,1} \otimes kC_{1,2})} \circ d_3 \circ d_4|_{Q \otimes kC_{4,2}}$.

We can now conclude that $d_3 \circ d_4 = 0$, and we have finished showing that $X_\bullet \rightarrow \text{Sym}_k^k(V)$ is indeed a complex. \(\square\)

3.5 The Homology of $X_\bullet$ by Macaulay2

The complex $X_\bullet$ we have defined is a free $Q$-module complex

\[
X_\bullet: Q(-5)^9 \oplus Q(-6)^{45} \xrightarrow{d_4} Q(-4)^6 \oplus Q(-5)^{90} \xrightarrow{d_3} Q(-3)^9 \oplus Q(-4)^{45} \xrightarrow{d_2} Q(-2)^{10} \xrightarrow{d_1} Q.
\]
Macaulay2 [12] indicates that $H_1(X_\bullet) = 0$ and that, if we refine $X_\bullet$ into a minimal homogeneous resolution of $k[I_2]$, then that resolution will look like:

$$
\xymatrix{
Q(-10) & Q(-9)^9 & Q(-8)^{36} & Q(-7)^{85} & Q(-6)^{125} \\
\leq & Q(-5)^{107} & Q(-3)^9 & Q(-2)^{10} & Q
}
$$
BIBLIOGRAPHY


