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Turán Problems and Spectral Theory on Hypergraphs and Tensors

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Turán Problems and Spectral Theory on Hypergraphs and Tensors

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Abstract

Turán problems on uniform hypergraphs have been actively studied for many decades. However, on non-uniform hypergraphs, these problems are rarely considered. We refer a non-uniform hypergraph as an R-hypergraph where R is the set of cardinalities of all edges. An R-graph H is called degenerate if it has the smallest Turán density $|R(H)|-1$. What do the degenerate R-graphs look like? For the special case $R = \{r\}$, the answer to this question is simple: they are r-partite r-uniform hypergraphs. However, it is more intrigue for the other cases. A degenerate hypergraph is called trivial if it is contained in the blow-up of a single edge or a chain. In this thesis, we characterize the degenerate $\{1, 3\}$-hypergraphs, we proved that there always exist non-trivial degenerate R-graphs except the cases $R = \{r\}$ and $R = \{1, 2\}$. We extend our work to the Turán density of k-edge-colored r-uniform hypergraphs. We determine the Turán densities of all 2-edge-colored bipartite graphs, also give an important application of the study of 2-edge-colored graphs on Turán problems of $\{2, 3\}$-hypergraphs.

Hypergraphs are simply generalization of graphs. Tensors are simply generalization of matrices. Over the past decade, the spectral theory of hypergraphs and tensors has been developed. Many properties of graphs and matrices have been generalized to hypergraphs and tensors. The second main concern of this thesis is to study the methods for finding the largest eigenvalue of uniform hypergraphs and determining the maximum high-order tensors when fix some parameters. We give a tight upper bound on the spectral radius of an r-uniform hypergraph H with fixed number of edges. Our result generalizes the classical Stanley's theorem on graphs. We extend the problem to general r-order $\{0,1\}$-tensor A, we prove that the spectral radius of A with e ones is at most $e^{\frac{r-1}{r}}$ with the equality holds if and only if $e = k^r$ for some integer k and all ones forms a principle sub-tensor $1_{k\times k\times \ldots k}$. We also prove a stability result for general tensor A with e ones where $e = k^r + l$ with relatively small l. Using the stability result, we completely characterized the maximum tensors among all r-order $\{0,1\}$-tensor A with $k^r + l$ ones, for $-r \le l \le r+1$, and k sufficiently large. To prove above results, we generalize several important theorems from matrices to tensors.

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Chapter 1

Introduction

1.1 Introduction for Turán theory

Given a graph H, the Turán problem asks for the maximum possible number of edges (denoted as $ex(H,n)$) in a graph G on n vertices without a copy of H as a sub-graph. The graph achieving $ex(H,n)$ is called an extremal graph with respect to H. A reletively easier version of Turán problem is to determine the aysmptotic result of extremal number, called the Turán density. For a family $\pi(\mathcal{H})$ of finite forbidden graphs, the Turán density is defined as $\pi(\mathcal{H}) = \lim_{n\to\infty} ex(\mathcal{H},n)/\binom{n}{2}$. Turán problems started with the Mantel's theorem [41] which states that any graph on n vertices with no triangle contains at most $\lfloor n^2/4 \rfloor$ edges, and the complete bipartite graph on n vertices achieves this maximum. In 1941, Turán [58] generalized the Mantel's theorem, he determined the graph with maximum number of edges among all simple graphs on n vertices that doesn't contain the complete graph K_ℓ as a sub-graph, which gives that $\pi(K_\ell) = 1 - \frac{1}{\ell-1}$. Finally, the famed Erdős-Stone-Simonovits Theorem [14, 15] proved that the Turán density of any graph H is $\pi(H) = 1 - \frac{1}{\mathcal{X}(H)-1}$, where $\mathcal{X}(H)$ is the chromatic number of H.

By contrast with the graph case, the Turán density of uniform hypergraphs is defined in a similar way. But the situation for uniform hypergraph even for 3-uniform hypergraph is far more complex and still has mysterious questions. Let K_k^r denote the complete r-graph on k vertices. Turán [59] posed the natural question of determining $ex(n, K_k^r)$, for $r \geq 3$. However, no case is solved for any $k > r > 3$. The well known conjecture by Turán [58] is if $\pi(K_4^3) = 5/9$. Erdős [17] offered \$500 for the solution of any case and \$1000 for a general solution. At first, de Caen proved that $\pi(K_4^3) \leq 0.6213$, about 10 years later, Chung-Lu gave a stronger result that $\pi(K_4^3) \leq 0.5936$. Another 10 years later, Razborov [51] proved that $\pi(K_4^3)$ is 0.56167 using the flag algebra method. Currently this is the best known bounds.

As an important and active area in the extremal combinatorics, Turán problems on uniform hypergraphs have been actively studied for many decades. However, on non-uniform hypergraphs, there are only few results. Johnston and Lu [25] established the framework of the Turán theory for non-uniform hypergraphs.

A hypergraph $H = (V, E)$ consists of a vertex set V and an edge set $E \subseteq 2^V$. Here the edges of E could have different cardinalities. The set of all the cardinalities of edges in H is denoted by $R(H)$, the set of edge types. Let us fix a finite set R of positive integers and refer all the simple hypergraphs H with $R(H) \subseteq R$ as R-hypergraphs (or R-graphs, for short).

For example, $\{2\}$-graphs are just graphs and $\{r\}$-graphs are just r-uniform hypergraphs. An R-graph H on n vertices is denoted as H_n^R. We denote H^r as the rth *level hypergraph* of H which consists of all edges of cardinality r of H. We denote K_n^R as the complete hypergraph on n vertices with edge set $\cup_{i \in R} \binom{[n]}{i}$. We say H' is a *subgraph* of H, denoted by $H' \subseteq H$, if there exists a 1-1 map $f : V(H') \to V(H)$ so that $f(e) \in E(H)$ for any $e \in E(H')$. A necessary condition for $H' \subseteq H$ is $R(H') \subseteq R(H)$.

Since the non-uniform hypergraphs are quite different from uniform hypergraphs, we have to use a different way to measure the edge density of a non-uniform hypergraph, which is the Lubell function, it is the expected number of edges in the hypergraph hit by a random full chain [25]. For a non-uniform hypergraph G on n vertices, the *Lubell function* of G is defined by

$$h_n(G) := \sum_{e \in E(G)} \frac{1}{\binom{n}{|e|}} = \sum_{r \in R(G)} \frac{|E(G^r)|}{\binom{n}{r}}.$$

Given a family of hypergraphs $\mathcal{H}$ with common set of edge types R, we say G is $\mathcal{H}$*-free* if G doesn't contain any member of $\mathcal{H}$ as a sub-graph. Let $\pi_n(\mathcal{H})$ be the maximum edge density of any $\mathcal{H}$-free R-graph on n vertices. The *Turán density* of $\mathcal{H}$ is defined to be:

$$\begin{aligned}\pi(\mathcal{H}) &= \lim_{n\to\infty} \pi_n(\mathcal{H}) \\ &= \lim_{n\to\infty} \max\left\{h_n(G) : |v(G)| = n, G \subseteq K_n^R, \text{and } G \text{ is } \mathcal{H}\text{-free}\right\}.\end{aligned}$$

A hypergraph $G := G_n^R$ is *extremal* with respect to the family $\mathcal{H}$ if G is $\mathcal{H}$*-free* and $h_n(G)$ is maximized.

Lu and Johnston [25] proved that this limit always exists by a simple average argument of Katona-Nemetz-Simonovits theorem [27].

Theorem 1.1.1 (Lu and Johnston [25])**.** *For any family $\mathcal{H}$ of R-graphs, $\pi(\mathcal{H})$ is well-defined, i.e. the limit $\lim_{n\to\infty} \pi_n(\mathcal{H})$ exists.*

For any non-uniform hypergraph H, it is trivial that $\pi(H) \leq |R(H)|$ and it is easy to see that $\pi(H) \geq |R(H)|-1$, since we can take an $(|R(H)|-1)$-complete hypergraph $K_n^{|R(H)|-1}$ without the appearance of H. We are interested in these R-graphs with the smallest Turán density, and all these as *degenerate* R-graphs.

What do the degenerate R-graphs look like? For the special case $R = \{r\}$, Erdős [16] showed that an r-uniform hypergraph H is degenerate if and only if it is r-partite, that is, a sub-graph of a blow-up of a single edge of cardinality r. As a natural extension of a single edge, the chain C^R for any set R is degenerate. Thus every sub-graph of a blow-up of a chain is also degenerate. We say a degenerate R-graph is *trivial* if it is a sub-graph of a blow-up of the chain C^R. For $R = \{1, 2\}$, the authors in [25] completely classified the Turán densities in this class, their results tell us that all degenerate $\{1, 2\}$-graphs are trivial.

Theorem 1.1.2 (Lu and Johnston [25]). *For any hypergraph H with $R(H) = \{1, 2\}$, we have*

$$\pi(H) = \begin{cases} 2 - \frac{1}{\mathcal{X}(H^2)-1} & \text{if } H^2 \text{ is not bipartite;} \\ \frac{5}{4} & \text{if } H^2 \text{ is bipartite and } \min\{k : \overline{P}_{2k} \subseteq H\} = 1; \\ \frac{9}{8} & \text{if } H^2 \text{ is bipartite and } \min\{k : \overline{P}_{2k} \subseteq H\} \geq 2; \\ 1 & \text{if } H^2 \text{ is bipartite and } \overline{P}_{2k} \nsubseteq H \text{ for any } k \geq 1. \end{cases}$$

where $H^2 \in H$ is the graph with all edges of cardinality 2. $\overline{P}_{2k}$ is a closed path of length $2k$, and $\mathcal{X}(H^2)$ is the chromatic number of H^2.

In Chapter 3, we will consider the Turán problems on $\{1, 3\}$-graphs. We will characterize the degenerate $\{1, 3\}$-graph and further prove that for any finite set R of distinct positive integers, except the case $R = \{1, 2\}$, there always exist non-trivial degenerate R-graphs. We also compute the Turán densities of some small $\{1, 3\}$-graphs.

When we are trying to characterize the degenerate $\{2, 3\}$-graph, we extend the Turán theory to 2-edge-colored graphs. Generally, given positive integers $k \geq r \geq 2$, and a set of colors C, with $|C| = k$, a k-edge-colored r-uniform hypergraph H (for short, k-colored r-graph) is an r-uniform hypergraph that allows k different colors on each hyperedge. We name each color by an integer $i \in [k]$, then H can be expressed as $H = (V, E_1, E_2, \ldots, E_k)$ where E_i denotes the set of hyperedges colored by number i. We say H' is a sub-graph of H, denoted by $H' \subseteq H$, if $V(H') \subseteq V(H)$, $E_i(H') \subseteq E_i(H)$ for every i. Given a family of k-colored r-graphs $\mathcal{H}$, we say G is *$\mathcal{H}$-free* if it doesn't contain any member of $\mathcal{H}$ as a sub-graph. To measure the edge density of G, we use $h_n(G)$, which is defined by

$$h_n(G) := \sum_{i=1}^{k} \frac{|E_i(G)|}{\binom{n}{r}}.$$

Then we define the Turán density of $\mathcal{H}$ as

$$\pi(\mathcal{H}) := \lim_{n\to\infty} \pi_n(\mathcal{H}) = \lim_{n\to\infty} \max_{G_n} h_n(G_n),$$

where the maximum is taken over all $\mathcal{H}$-free k-colored r-graphs G_n on n vertices.

By a simple average argument of Katona-Nemetz-Simonovits theorem[27], this limit always exists.

Theorem 1.1.3. *[4] For any fixed family $\mathcal{H}$ of k-colored r-graphs, $\pi(\mathcal{H})$ is well-defined, i.e.* $\lim_{n\to\infty} \pi_n(\mathcal{H})$ *exists.*

When $\mathcal{H} = \{H\}$, we simply write $\pi(\{H\})$ as $\pi(H)$. Note that $\pi(\mathcal{H})$ agrees with the definition of

$$\pi(\mathcal{H}) = \frac{ex(\mathcal{H}, n)}{\binom{n}{r}},$$

where $ex(\mathcal{H}, n)$ is the maximum number of hyperedges in an n-vertex $\mathcal{H}$-free k-colored r-graph.

In Chapter 4, we let $k = 2$. A 2-edge-colored graph is a simple graph (without loops) where each edge is colored either red or blue, or both. We call an edge a double-colored edge if it is colored with both colors. For short, we call the 2-edge-colored graphs simply as 2-colored graphs. A 2-colored graph H can be written as a triple $H = (V, E_r, E_b)$ where V is the vertex set, $E_r \subseteq \binom{V}{2}$ is the set of red edges and $E_b \subseteq \binom{V}{2}$ is the set of blue edges. Denote $|E_r|$ and $|E_b|$ as the size of each set, denote H_r, H_b as the induced sub-graphs of H generated by all the red edges and all the blue edges respectively. A graph can be considered as a special 2-colored graph with only one color. We say H is *proper* if there exists at least one edge in each class E_r and E_b. Throughout the paper, we consider the proper 2-colored graphs.

About the 2-colored graphs, several different perspective were studied. Like the problems on homomorphisms of 2-colored graphs [42, 18]; problems on partitioning the vertices of 2-colored graphs into monochromatic paths and cycles; problems on finding a longest alternating cycle in such a colored graph [53, 1] In this paper, we study the 2-colored graphs in a different perspective. Like graphs, we can ask the Turán density for 2-colored graphs.

It is easy to see that $\pi(H) \geq 1$ for any proper 2-colored graph H, since we can take a complete graph with all edges a single color that does not contain a copy of H. For a 2-colored graph H, we say H is *degenerate* if $\pi(H) = 1$. We say H is *bipartite* if neither H_r nor H_b contains an odd cycle with all edges colored by red or blue. Note that if H is degenerate, then it must be bipartite. Otherwise, say $H_b = (V, E_b)$ is not a bipartite graph, one may consider the union of the red complete graph and a blue H_b-free graph with positive density at least $\frac{1}{2} + o(1)$, the resulting graph forms a H-free 2-colored graph with edge density at least $\frac{3}{2} + o(1)$, a contradiction.

In Chapter 4, we will determine the Turán density of all 2-colored bipartite graphs and characterize the 2-colored graphs achieving these Turán values. Our consideration on 2-colored graphs is motivated by the study of Turán density of non-uniform hypergraphs. In the last section of this chapter, we will study the degenerate $\{2, 3\}$-graphs and show an application of the Turán density of of 2-colored graphs on $\{2, 3\}$-graphs.

Given an r-uniform hypergraph H on n vertices, the *Lagrangian* of H is the maximum value of $\sum_{\{i_1,\ldots,i_r\}\in E(H)} x_{i_1} \cdots x_{i_r}$ where the non-negative $n-$vector $\mathbf{x} = (x_1 \ldots, x_n)$ satisfies $\sum\limits_{i=1}^{n} |x_i| = 1$. In 1965, Motzkin and Straus [43] provided a new proof of Turán's theorem based on a continuous characterization of the clique number of a graph using the Lagrangian of a graph. This new proof aroused interests in the study of Lagrangians of r-uniform graphs. The Lagrangian of a hypergraph has been a useful tool in hypergraph extremal problems. For example, Sidorenko and Frankl-Fűredi [57] applied Lagrangians of hypergraphs in finding Turán densities of uniform hypergraphs. Peng et al. in [48] gave some applications of Lagrangian method in determining Turán densities of non-uniform hypergraphs. The Lagrangian of uniform hypergraph is one case of the p-spectral radius, in the following section, we will introduce spectral theory of graphs and uniform hypergraphs.

1.2 Introduction for spectral theory

Spectral graph theory is a well developed field which has received considerable attention over the last several decades. It studies the properties of a graph in relationship to its characteristic polynomial, eigenvalues, and eigenvectors of its adjacency matrix, Laplacian matrix, etc. In 2005, Lim [35] and Qi [50] independently introduced eigenvalues for higher order tensors. Cooper and Dutle [13] extended numerous results from spectral graph theory in the case of uniform hypergraphs. In many ways spectral hypergraph theory is a relatively new and exciting field which is being actively studied. Since hypergraphs are simply generalization of graphs, tensors are simply generalization of matrices. Many properties of graphs and matrices have been generalized to hypergraphs and tensors.

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. Its adjacency matrix $A(G) = (a_{ij})$ is defined to be the $n \times n$ matrix, where $a_{ij} = 1$ if v_i is adjacent to v_j, otherwise $a_{i,j} = 0$. The largest eigenvalue $\lambda_1(A(G))$ is called the spectral radius of G, we refer it as $\rho(G)$.

Let H be an r-uniform hypergraph, Cooper and Dutle [13] defined the adjacency tensor $A(H)$ of H to be the r-order n-dimensional tensor $A(H) = (a_{i_1 \cdots i_r})$ by

$$a_{i_1 \cdots i_r} = \begin{cases} \frac{1}{(r-1)!} & \text{if } \{i_1, \ldots, i_r\} \text{ is an edge of } H \ , \\ 0 & \text{otherwise,} \end{cases}$$

where each i_j runs from 1 to n for $j \in [r]$. The spectral radius $\rho(H)$ is defined to be the largest modulus of eigenvalues of $A(H)$.

A typical question of spectral graph theory is: which graph has the maximum spectral radius among all graphs with e edges? Beside of this question, it is interesting to know how large the spectral radius can be if we fixed the number of edges? This problem dates back to the year 1987. When $r = 2$, Stanley [54] proved the following results.

Theorem 1.2.1. *[54]*

$$\lambda_1(G) \geq \frac{\sqrt{1+8e}-1}{2},$$

where the equality holds if and only if $e = \binom{k}{2}$ *and* G *is the union of the complete graph* K_k *and some isolated vertices.*

If $e = \binom{k}{2}$, Brualdi and Hoffman [7] proved that the maximum of $\rho(G)$ is reached by the union of a complete graph on k vertices and some possible isolated vertices. They conjectured that the maximum spectral radius of a graph G with $e = \binom{k}{2} + s$ edges is attained by the graph G_e, which is obtained from complete graph K_k by adding a new vertex and s new edges. In 1987, Stanley [54] proved that the spectral radius of a graph G with e edges is at most $\frac{\sqrt{1+8e}-1}{2}$. The equality holds if and only if $e = \binom{k}{2}$ and G is the union of the complete graph K_k and some isolated vertices. Friedland [20] proved a bound which is tight on the complete graph with one, two, or three edges removed or the complete graph with one edge added. Rowlinson [52] finally confirmed Brualdi and Hoffman's conjecture, and proved that G_e attains the maximum spectral radius among all graphs with e edges.

In keeping with the tradition of extending results from spectral graph theory to spectral hypergraph theory, in Chapter 5 we will generalize Stanley's theorem to hypergraphs, that is, maximizing the spectral radius of r-uniform hypergraphs among all r-uniform hypergraphs with a given number of edges. The main tool that we used is the α-normal labeling method, which was first developed by the second author and Dr. Man to classifying all connected r-uniform hypergraphs with spectral radius at most $\sqrt[r]{4}$ in the paper [39]. This method is used in [29] and is generalized in [63].

Since the adjacency matrix of graph and adjacency tensor of hypergraph are both non-negative and symmetric, the definition of spectral radius from the eigenvalue perspective is equivalent to the following definitions: for $r \geq 2$ and $p \geq 1$, the p-spectral radius of an r-uniform hypergraph H on n vertices is

$$\rho_p(H) = \max_{\|\mathbf{x}\|_p=1} r \sum_{\{i_1,\ldots,i_r\}\in E(H)} x_{i_1} \cdots x_{i_r},$$

where $\|\mathbf{x}\|_r = \left(\sum\limits_{i=1}^{n} |x_i|^p\right)^{1/p}$. When $p = r$, the p-spectral radius is just the spectral radius of H. When $p = 1$, the p-spectral radius is just the Lagrangian of H. Recently, Nikiforov [46] conjectured that for any $r \geq 3$ and any r-uniform hypergraph H with m edges, if $t \geq r-1$ is the unique real number such that $m = \binom{t}{r}$, then $\rho_1(H) \leq mt^{-r}$, with equality holding if and only if t is an integer. The author also confirmed the cases $3 \leq r \leq 5$. Very recently, Lu [38] not only completely settled this conjecture also proved that $\rho_p(H) \leq rm/t^{r/p}$ for any $p \geq 1$ and $r \geq 2$.

For a real nonnegative square matrix A the spectral radius $\rho(A)$ is the largest eigenvalue of A in modulus, which is really as guaranteed by the Perron-Frobenius theorem. The problem of finding the maximal spectral radius for all $\{0, 1\}$-matrices with prescribed number of ones was introduced by Brualdi and Hoffman [7] in 1985. Let $g(e)$ be the maximal spectral radius of A among all $\{0, 1\}$-matrices A with e ones. They proved that for each positive integer k, $g(k^2) = g(k^2 + 1) = k$. When $e = k^2$, the equality holds if A is essentially a $k \times k$ all-1-matrix (inserted by possibly extra rows/columns of 0's). When $e = k^2 + 1$ and $k \geq 3$, the equality is attained for only when a useless additional 1 is put at any place else to a $k \times k$ all-1-matrix. (But for $k = 1$, or 2, there is another A with $\rho(A) = k$.) Friedland [20] solved another cases when $e = k^2 - 1$, $e = k^2 - 4$, or $e = k^2 + l$ for a fixed l and k sufficiently large. In all cases, the matrices with maximum spectral radius are characterized.

We consider a similar problem for $\{0, 1\}$-tensor (of order $r > 2$) with a fixed number of 1's. We ask which tensor attains the maximum spectral radius. A $\{0,1\}$-tensor is always nonnegative, thus the spectral radius $\rho(A)$ is an H^+-eigenvalue, and the associated eigenvector $\mathbf{x} \in \mathbb{R}^n_+$. Consider the set of all $\{0,1\}$-tensors with a fixed number of 1's. For fixed integer $r \geq 3$ and $e \geq 1$, let

$$\mathcal{T}^r_{n,e} = \{ \text{ all } \{0,1\}\text{-tensors of order } r \text{ and dimension } n \text{ with exactly } e \text{ 1's}\},$$

and

$$\mathcal{T}^r_e = \cup_n \mathcal{T}^r_{n,e}.$$

Now we consider the objective function

$$g_r(e) = \max_{A \in \mathcal{T}^r_e} \rho(A).$$

For a fixed r and e, we say $A \in \mathcal{T}^r_e$ is a *maximum tensor* if $\rho(A) = g_r(e)$.

In Chapter 6, we will prove some important lemmas on nonnegative tensors and give a bound on $g_r(e)$. We will also show the structure of the maximum $\{0,1\}$-tensor when $e = k^r + l$ with relatively small l and determine the maximum tensors for $-r-1 \leq l \leq r$.

Chapter 2

Notations and Lemmas

2.1 Turán theory for non-uniform hypergraphs

In this dissertation, the notation $[n]$ represents the set of $\{1, \ldots, n\}$. For convenience, we represent an edge $\{a, b\}$ by ab. For a fixed set $R = \{k_1, k_2, \ldots, k_r\}$, with $(k_1 < k_2 < \ldots < k_r)$, *R-flag* is an R-graph containing exactly one edge of each size. The *chain* C^R is the special R-flag with the edge set $E(C^R) = \{[k_1], [k_2], \ldots, [k_r]\}$, where $[k_i]$ is the set of all positive integers from 1 to k_i for each $i \in [r]$. For $R = \{1, 3\}$, the chain $C^{\{1,3\}} = \{1, 123\}$. For any R-flag L, we have $\pi(L) = |R| - 1$ (see [25]). The degenerate R-graphs has the smallest Turán density, i.e.

Definition 2.1.1 (Degenerate hypergraphs)**.** *A hypergraph H is called* degenerate *if $\pi(H) = |R(H)| - 1$.*

Thus the chain $C^{\{1,3\}}$ is a degenerate $\{1, 3\}$-graph.

To determine the Turán densities of uniform hypergraphs, many theorems such as supersaturation, blow-up were set up and play important roles. These tools can also be generalized to non-uniform hypergraphs by the early work in [25] and can be used for non-degenerate non-uniform hypergraphs.

Blow-up and Suspension

Definition 2.1.2 (Blow-up hypergraphs)**.** *[25] For any hypergraph H on n vertices and positive integers $s_1, s_2, \ldots, s_n$, the* blow-up *of H is a new hypergraph (V, E), denoted by $H_n(s_1, s_2, \ldots, s_n)$, satisfying*

- *$V := \bigsqcup_{i=1}^{n} V_i$, where $|V_i| = s_i$,*

- *$E := \bigcup_{F \in E(H)} \prod_{i \in F} V_i$.*

When $s_1 = s_2 = \ldots = s_n = s$, we simply write it as $H(s)$.

The blow-up operation does not change the Turán density.

Theorem 2.1.1 (Blow-up Families)**.** *[25] Let $\mathcal{H}$ be a finite family of hypergraphs and let $s \geq 2$. Then $\pi(\mathcal{H}(s)) = \pi(\mathcal{H})$.*

A direct corollary of Theorem 2.1.1 is the following result.

Theorem 2.1.2 (Squeeze Theorem)**.** *[25] Let H be any hypergraph. If there exists a hypergraph H' and an integer $s \geq 2$ such that $H' \subseteq H \subseteq H'(s)$, then $\pi(H) = \pi(H')$.*

It is easy to generalize the concepts of homomorphisms and H-coloring to general R-graphs.

Definition 2.1.3. *Given two R-graphs G and H, a hypergraph* homomorphism *is a vertex map $f : V(G) \to V(H)$ such that, if $\{v_1, \ldots, v_r\} \in E(G)$, then*

$$\{f(v_1), \ldots, f(v_r)\} \in E(H), \text{ for all } r \in R.$$

Definition 2.1.4. *A hypergraph G is called H-*colorable *if and only if there exists a homomorphism from G to H.*

Note that, if there exists a homomorphism from G to H, then G is isomorphic to a sub-graph of a blow-up of H. Thus we have:

Lemma 2.1.1. *If G is H-colorable, then $\pi(G) \leq \pi(H)$.*

Another tool used in evaluating the Turán densities of non-uniform hypergraphs is called *suspension*.

Definition 2.1.5. *[25] The suspension of a hypergraph H, denoted by $S(H)$, is the hypergraph with $V = V(H) \cup \{v\}$ where $\{v\}$ is a new vertex not in $V(H)$, and the edge set $E = \{e \cup \{v\} : e \in E(H)\}$. We write $S^t(H)$ to denote the hypergraph obtained by iterating the suspension operation t-times, i.e. $S^2(H) = S(S(H))$ and $S^3(H) = S(S(S(H)))$, etc.*

The relationship between $\pi(H)$ and $\pi(S(H))$ was investigated in [25].

Proposition 2.1.1. *[25] For any family of hypergraphs $\mathcal{H}$ we have that $\pi(S(\mathcal{H})) \leq \pi(\mathcal{H})$.*

R-graphs with loops and Lagrangian

We say a hypergraph is simple if there is at most one edge connecting any collection of vertices. A general hypergraph allows every edge to be a multi-set of vertices. A loop edge is a multiset of vertices. Sometimes we need to enlarge the concept of R-graphs to *R-graphs with loops*. For example, consider a $\{1, 3\}$-graph H_1 with the edge set $\{x, xyy, yyy\}$. Here xyy is a loop edge with vertex x occurring once and vertex y twice. In general, a loop edge $e = x_1^{m_1} \cdots x_l^{m_l}$ consists of m_1 copies of vertex x_1, m_2 copies of vertex x_2, and so on. For a loop edge $e = x_1^{m_1} \cdots x_l^{m_l}$, the *cardinality* of e is $|e| = \sum_i m_i$. We also define a multinomial coefficient c_e to be

$$c_e := \binom{|e|}{m_1, m_2, \ldots, m_l} = \frac{|e|!}{m_1!m_2! \cdots m_l!}.$$

Definition 2.1.6. *The* polynomial form *of an R-graph H with loops on n vertices, denoted by $\lambda(H, \vec{x})$ with $\vec{x} = (x_1, x_2, \ldots, x_n)$ is defined as*

$$\lambda(H, \vec{x}) := \sum_{e \in E(H)} c_e \prod_{i \in e} x_i.$$

The Lagrangian *of H, denoted by $\lambda(H)$, is the maximum value of the polynomial $\lambda(H, \vec{x})$ over the simplex $S_n = \{(x_1, x_2, \ldots, x_n) \in [0, 1]^n : \sum_{i=1}^n x_i = 1\}$.*

For any R-graph H (with possible loops), one can construct the family of H-colorable R-graph by blowing up H in a certain way. The Lagrangian of H is the maximum edge density of the H-colorable R-graphs that one can get in this way. This definition of Lagrangian is the same as the one in [26]; but differs from the classical Lagrangian for r-uniform hypergraphs such as in [48] by a constant multiplicative factor. This is not essential. This is a special case of more general Lagrangian of non-uniform hypergraphs introduced by Peng-Wu-Yao [49].

Let's define the product of R-graphs (with loops):

Definition 2.1.7. *For any two general R-graphs H_1 and H_2 with vertices set V_1 and V_2 respectively, we define the product of H_1 and H_2, which is denoted by $H_1 \times H_2 = (V,\ E)$, where*

$$V = V_1 \times V_2,\ E = \cup_{r \in R} E(H_1^r) \times E(H_2^r),$$

the $E(H_i^r)$ denotes the set of all edges of cardinality r in H_i for $i = 1, 2$. Here $E(H_1^r) \times E(H_2^r)$ consists of all products of $e \times_\sigma f$, where $\sigma = (\sigma(1), \ldots, \sigma(r))$ takes over all permutations of $[r]$. For example, given $e = \{v_1, \ldots, v_r\} \in E(H_1)$, $f = \{u_1, \ldots, u_r\} \in E(H_2)$, then $e \times_\sigma f = \{(v_1, u_{\sigma(1)}), (v_2, u_{\sigma(2)}), \ldots, (v_r, u_{\sigma(r)})\}$ is an edge in $E(H_1^r) \times E(H_2^r)$.

Lemma 2.1.2. *For any two R-graphs H_1 and H_2, if hypergraph H is H_1 and H_2-colorable, then it's $(H_1 \times H_2)$-colorable.*

Proof. By definition, there exist two graph homomorphisms $f_1 : V(H) \mapsto V(H_1)$ and $f_2 : V(H) \mapsto V(H_2)$. Note that H could be an R-graph. Then for any $r \in R$, if edge $e = \{v_1, \ldots, v_r\} \in E(H)$, we have

$$f_1(e) = \{f_1(v_1), \ldots, f_1(v_r)\} \in E(H_1)$$

and

$$f_2(e) = \{f_2(v_1), \ldots, f_2(v_r)\} \in E(H_2).$$

Define a map $f := f_1 \times f_2$ from $V(H)$ to $V(H_1) \times V(H_2)$, such that $f(v) = (f_1(v), f_2(v))$. Then we have

$$f(e) = \{(f_1(v_1), f_2(v_1)), \ldots, (f_1(v_r), f_2(v_r))\} \in f_1(e) \times f_2(e) \subseteq E(H_1 \times H_2).$$

Thus the map f takes edges in H to edges in $H_1 \times H_2$, it is a graph homomorphism. Therefore, H is $(H_1 \times H_2)$-colorable. $\square$

2.2 Turán theory for k-colored r-graphs

In this section, we give some definitions and lemmas related to the k-colored r-graphs for $k \geq r \geq 2$. These are natural generalizations from the Turán theory of graphs.

Similarly, we start with the definition of *blow-up* of a k-colored r-graph.

Definition 2.2.1 (*Blow-up Families*)**.** *For any k-colored r-graph H on n vertices and positive integers $s_1, s_2, \ldots, s_n$, the* blow-up *of H is a new k-colored r-graph, denoted by $H(s_1, s_2, \ldots, s_n) = (V,\ E_1, \ldots,\ E_k)$, satisfying*

- $V := \bigsqcup_{i=1}^{n} V_i$, *where* $|V_i| = s_i$,
- $E_j = \bigcup_{F \in E_j(H)} \prod_{i \in F} V_i$, *for each* $j \in [k]$.

When $s_1 = s_2 = \cdots = s_n = s$, we simply write it as $H(s)$.

There is also a natural generalization of the supersaturation lemma and blow-up in k-colored r-graphs.

Lemma 2.2.1 (Supersaturation)**.** *[4] For any k-colored r-graph H and $a > 0$, then there are $b, n_0 > 0$ so that if G is a k-colored r-graph on $n > n_0$ vertices with $h_n(G) > \pi(H) + a$ then G contains at least $b\binom{n}{v(H)}$ copies of H.*

Proof. Since we have $\lim_{n\to\infty} \pi_n(H) = \pi(H)$, there exists an $n_0 > 0$ so that if $t > n_0$ then $\pi_t(H) < \pi(H) + \frac{a}{r}$. Suppose $n > t$, and G is a k-colored r-graph on n vertices with $h_n(G) > \pi(H)+a$. Let T represent any t-set, then G must contain at least $\frac{a}{2}\binom{n}{t}$ t-sets $T \subseteq V(G)$ satisfying $h_t(G[T]) > (\pi(H)+\frac{a}{2})$. Otherwise, we would have

$$\begin{aligned}\sum_T h_t(G[T]) &\le \binom{n}{t}(\pi(H)+\frac{a}{2}) + \frac{a}{2}\binom{n}{t}\\ &= (\pi(H)+a)\binom{n}{t}.\end{aligned}$$

But we also have

$$\begin{aligned}\binom{t}{r}\sum_T h_t(G[T]) &= \binom{n-r}{t-r}\binom{n}{r}h_n(G)\\ &> \binom{n-r}{t-r}\binom{n}{r}(\pi(H)+a)\\ &= (\pi(H)+a)\binom{t}{r}\binom{n}{t}.\end{aligned}$$

A contradiction. Since $t > n_0$, it follows that each of the $\frac{a}{2}\binom{n}{t}$ t-sets $T \subseteq V(G)$ satisfying $h_t(G[T]) > (\pi(H)+\frac{a}{r})$ contains a copy of H, so the number of copies of H in G is at least $\frac{a}{2}\binom{n}{t}/\binom{n-v(H)}{t-v(H)} = \frac{a}{2}\binom{n}{v(H)}/\binom{t}{v(H)}$. Let $b = \frac{a}{2}/\binom{t}{v(H)}$, the result follows. $\square$

The 'blow-up' does not change the Turán density of k-colored r-graphs.

Lemma 2.2.2. *[4] For any $s > 1$ and any k-colored r-graph H, $\pi(H(s)) = \pi(H)$.*

Proof. First, since any H-free r-graph G is also $H(s)$-free, we have $\pi(H) \le \pi(H(s))$. We will show that for any $a > 0$, $\pi(H(s)) < \pi(H)+a$.

By the supersaturation lemma, for any $a > 0$, there are $b, n_0 > 0$ so that if G is a k-colored r-graph on $n > n_0$ vertices with $h_n(G) > \pi(H)+a$ then G contains at least $b\binom{n}{v(H)}$ copies of H. Consider an auxiliary $v(H)$-graph U on the same vertex set as G such that the edges of U correspond to copies of H in G. Note that U contains at least $b\binom{n}{v(H)}$ edges. For any $S > 0$, if n is large enough we can find a copy K of $K_{v(H)}^{v(H)}(S)$ in U. Note that K is the complete $v(H)$-partite v(H)-graph with S vertices in each part, then by Erdős's result $\pi(K) = 0$. Fix one such K in U. Color each edge of K with one of the $v(H)!$ colors corresponding to the possible orderings with which the vertices of H are mapped into the parts of K. By Ramsey theory, one of the color classes contains at least $S^v/v!$ edges. For large enough S (such that $S^v/v! \geq s$) it follows that U contains a monochromatic copy of $K_{v(H)}^{v(H)}(s)$, which gives a copy of $H(s)$ in G. Thus $\pi(H(s)) < \pi(H) + a$.

$\square$

Given two k-colored r-graphs G and H, a *graph homomporhism* between them is a map $f\colon V(G) \to V(H)$ which keeps the colored edges, that is, $f(e) \in E_i(H)$ whenever $e \in E_i(G)$ for $i \in [k]$. We say G is *H-colorable* if there is a graph homomorphism from G to H. Equivalently, G is a sub-graph of a blow-up of H. It is easy to prove the following lemmas.

Lemma 2.2.3. *[4] Let $\mathcal{H}$ be a family of k-colored r-graphs. If G is H-colorable for any $H \in \mathcal{H}$, then $\pi(G) \leq \pi(\mathcal{H})$.*

Definition 2.2.2. *Given two k-colored r-graphs G_1 and G_2 with vertices set V_1 and V_2, we define the product of G_1 and G_2, denoted by $G_1 \times G_2 = (V_1 \times V_2,\ E_1, \ldots, E_k)$, where for any $i \in [k]$,*

$$E_i = E_i(G_1) \times E_i(G_2) = \{e \times f \mid e \in E_i(G_1), f \in E_i(G_2)\}.$$

For example, if $e = \{v_1, \ldots, v_r\} \in E_i(G_1)$, $f = \{u_1, \ldots, u_r\} \in E_i(G_2)$, then $e \times f = \cup_{\sigma \in S_r}\{(v_1, u_{\sigma(1)}), \ldots, (v_r, u_{\sigma(r)})\}$, where $\sigma = (\sigma(1), \cdots, \sigma(r))$ takes over all permutations of $[r]$.

Lemma 2.2.4. *[4] A k-colored r-graph G is G_1 and G_2 colorable, then it's $(G_1 \times G_2)$-colorable.*

Proof. There exist two graph homomorphisms $f_1 : V(G) \mapsto V(G_1)$ and $f_2 : V(G) \mapsto V(G_2)$ such that for any edge $e = \{v_1, \ldots, v_r\} \in E(G)$ (wlog, suppose $e \in E_1(G)$),

we have

$$f_1(e) = \{f_1(v_1), \ldots, f_1(v_r)\} \in E_1(G_1),$$

and

$$f_2(e) = \{f_2(v_1), \ldots, f_2(v_r)\} \in E_1(G_2).$$

Define a map $f := f_1 \times f_2$ from $V(G)$ to $V(G_1) \times V(G_2)$, such that $f(v) = (f_1(v), f_2(v))$ for any $v \in V(G)$. Then we have

$$f(e) = \{(f_1(v_1), f_2(v_1)), \ldots, (f_1(v_r), f_2(v_r))\} \in f_1(e) \times f_2(e) \subseteq E_1(G_1 \times G_2).$$

Thus the map f is a graph homomorphism. Hence G is $(G_1 \times G_2)$-colorable. $\square$

2.3 Spectral radius of graphs and matrices

Let G be a simple graph with vertex set $\{v_1, \ldots, v_n\}$. Its adjacency matrix $A(G) = (a_{ij})$ is defined to be the $n \times n$ matrix, where $a_{ij} = 1$ if v_i is adjacent to v_j, otherwise $a_{i,j} = 0$. It follows immediately that if G is a simple graph, then $A(G)$ is a symmetric $(0, 1)$ matrix in which every diagonal entry is zero. We shall denote the characteristic polynomial of G by

$$\phi(G) = Det(\lambda I - A(G)).$$

Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as

$$\lambda_1(A(G)) \geq \lambda_2(A(G)) \cdots \geq \lambda_n(A(G)).$$

The sequence of n eigenvalues is called the *spectrum* of G. The largest eigenvalue $\lambda_1(A(G))$ is called the spectral radius of G, we refer it as $\rho(G)$.

In general, let $A = (a_{ij})$ be a matrix, an eigenvector of A is a vector such that $Ax = \lambda\mathbf{x}$ for some real or complex number λ. This number λ is called the eigenvalue of A belonging to eigenvector $\mathbf{x}$. Clearly λ is an eigenvalue if and only if the matrix

$A - \lambda I$ is singular, equivalently, iff $det(A - \lambda I) = 0$. We say A is *non-negative* if $a_{ij} \geq 0$; we say A is *irreducible* if there exists a m such that A^m is positive; we say A is *aperiodic* if the greatest common divisor of all natural numbers m such that $(A^m)_{ii} > 0$ is 1.

Theorem 2.3.1 (Perron-Frobenius theorem)**.** *If A is an aperiodic irreducible non-negative $n \times n$ matrix with spectral radius $\rho(A)$, then $\rho(A)$ is the largest eigenvalue in absolute value of A, and A has an eigenvector α with eigenvalue $\rho(A)$ whose components are all positive.*

Apply Perron-Frobenius theorem to the adjacency matrix of a connected graph G, and let $\rho(G)$ be the spectral radius of G, we have the following facts:

1. There exists an eigenvector $\mathbf{x}$ for $\rho(G)$ so that all entries are positive.

2. If there exist a positive vector an eigenvector α corresponding to the eigenvalue λ, then $\rho(G) = \lambda$.

2.4 Spectral radius of hypergraphs and tensors

For $r \geq 3$, an r-uniform hypergraph H on n vertices consists of a vertex set V and an edge set $E \subseteq \binom{V}{r}$. Cooper and Dutle [13] defined the adjacency tensor A of H to be the r-order n-dimensional tensor $A = (a_{i_1 \cdots i_r})$ by

$$a_{i_1 \cdots i_r} = \begin{cases} \frac{1}{(r-1)!} & \text{if } \{i_1, \ldots, i_r\} \text{ is an edge of } H \ , \\ 0 & \text{otherwise,} \end{cases}$$

where each i_j runs from 1 to n for $j \in [r]$. The adjacency tensor A of r-uniform hypergraph is always nonnegative and symmetric.

Given an r-uniform hypergraph H, the polynomial form $P_H(\mathbf{x}) \colon \mathbb{R}^n \to \mathbb{R}$ is defined for any vector $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ as

$$P_H(\mathbf{x}) = \sum_{i_1, \ldots, i_r = 1}^{n} a_{i_1 \cdots i_r} x_{i_1} \cdots x_{i_r} = r \sum_{\{i_1, \ldots, i_r\} \in E(H)} x_{i_1} \cdots x_{i_r}.$$

Then the spectral radius of an r-uniform hypergraph H is

$$\rho(H) = \max_{\|\mathbf{x}\|_r=1} P_H(\mathbf{x}) = \max_{\|\mathbf{x}\|_r=1} r \sum_{\{i_1,\ldots,i_r\}\in E(H)} x_{i_1}\cdots x_{i_r},$$

where $\|\mathbf{x}\|_r = \left(\sum\limits_{i=1}^{n} |x_i|^r\right)^{1/r}$.

In general, one can also define the spectral radius of any tensor A using eigenvalues. An n-dimension r-order tensor A in real field $\mathbb{R}$ is a multi-dimensional array consisting of n^r entries:

$$a_{i_1\cdots i_r} \in \mathbb{R}, \quad \text{where indexes } i_1, i_2, \ldots, i_r \text{ ranges from 1 to n.}$$

A is called *nonnegative* if every element $a_{i_1\cdots i_r} \geq 0$; it is called *symmetric* if its entries are invariant under any permutation of their indices, i.e. $a_{i_1\cdots i_r} = a_{i_{\sigma(1)}\cdots i_{\sigma(r)}}$ for all $\sigma \in \mathfrak{S}_r$, where $\mathfrak{S}_r$ is a symmetric group on $[r]$. For every $i \in [n]$, the *ith slice* A_i is an sub-tensor of A consisting of all elements $a_{ii_2\cdots i_r}$ with the first index being fixed to i.

For a tensor A of order $r \geq 2$ and dimension $n \geq 2$, a pair $(\lambda, \mathbf{x}) \in \mathbb{C} \times (\mathbb{C}^n \setminus \{\mathbf{0}\})$ is called an *eigenvalue* and an *eigenvector* of A, if they satisfy

$$A\mathbf{x}^{r-1} = \lambda \mathbf{x}^{[r-1]} \quad \text{where} \quad \mathbf{x}^{[r-1]} = (x_1^{r-1}, \ldots, x_n^{r-1})^T.$$

That is, for all $i = 1, 2, \ldots, n$,

$$\sum_{i_2,\ldots,i_r=1}^{n} a_{ii_2\cdots i_r} x_{i_2} \cdots x_{i_r} = \lambda x_i^{r-1}. \tag{2.1}$$

The spectral radius $\rho(A)$ is defined to be the largest modulus of eigenvalues of A.

$$\rho(A) = \max\{|\lambda| : \ \lambda \text{ is an eigenvalue of } A\}.$$

If $\mathbf{x}$ is a real eigenvector of A, clearly the corresponding eigenvalue λ is also real. In this case, $\mathbf{x}$ is called an H-eigenvector and λ an H-eigenvalue. Furthermore, if $\mathbf{x} \in \mathbb{R}^n_+$, where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x \geq 0\}$, then λ is an H^+-eigenvalue of A. If $\mathbf{x} \in \mathbb{R}^n_{++}$, where $\mathbb{R}^n_{++} = \{x \in \mathbb{R}^n : x > 0\}$, then λ is said to be an H^{++}-eigenvalue of A.

The classical Perron-Frobenius theorem for matrices has been generalized to nonnegative tensors:

Theorem 2.4.1. ***(Perron-Frobenius theorem for nonnegative tensors)***

1. *(Yang and Yang 2010 [60]) If A is nonnegative tensor of order r and dimension n, then the spectral radius $\rho(A)$ is an H^+-eigenvalue of A.*

2. *(Friedland, Gaubert and Han 2011 [21]) If furthermore A is weakly irreducible, then $\rho(A)$ is the unique H^{++}-eigenvalue of A, with the unique eigenvector $\mathbf{x} \in \mathbb{R}^n_{++}$, up to a positive scaling coefficient.*

3. *(Chang, Pearson and Zhang 2008 [9]) If moreover A is irreducible, then $\rho(A)$ is the unique H^+-eigenvalue of A, with the unique eigenvector $\mathbf{x} \in \mathbb{R}^n_+$, up to a positive scaling coefficient.*

Perason and Zhang[47] also proved that the adjacency tensor A of a connected hypergraph H is weakly irreducible, thus by Perron-Frobenius theorem, there exists a unique positive eigenvector up to scales corresponding to $\rho(H)$. And this eigenvector is called *Perron-Frobenius vector*.

Below is the **Perron-Frobenius theorem for hypergraphs**.

Theorem 2.4.2. *(Cooper-Dutle 2012, Friedland-Gaubert-Han 2011) If H is a connected r-uniform hypergraph, then $\rho(H)$ is the unique positive eigenvalue with a positive eigenvector $\mathbf{x}$, up to a positive scaling coefficient.*

When it comes to the spectral radius of uniform hypergraphs, the edge-shifting operation can be used to increase the spectral radius. By edge-shifting, we mean that for $k \geq 1$, we can move k edges $(e_1, \ldots, e_k)$ from $(v_1, \ldots, v_k)$ to v , i.e. replace each edge e_i by new edge $e_i \setminus \{v_i\} \cup \{v\}$ for $i = 1, 2, \ldots, k$. Here v_i is a vertex incident to e_i.

Lemma 2.4.1. *[33] Let $k \geq 1$ and let H be a connected r-hypergraph. Let H' be the hypergraph obtained from H by moving edges $(e_1, \ldots, e_k)$ from $(v_1, \ldots, v_k)$ to v. Assume that H' contains no multiple edges. If $\mathbf{x}$ is a Perron vector of H and $x_v \geq \max_{1\leq i\leq k} x_{v_i}$, then $\rho(H') > \rho(H)$.*

By Lemma 2.4.1, we can increase the spectral radius by doing the edge-shifting operations stated in the lemma; this process will end until no potential edge can be moved in H. The resulting graph only has one non-trivial connected component.

Lemma 2.4.2. *[19] If H is a maximum hypergraph among the connected hypergraphs with fixed number edges, then H contains a vertex v adjacent to all other vertices.*

Remark: In fact, from the proof of the above Lemma, one can choose v to be any one of the vertices where the Perron-Frobenius vector achieves the maximum value.

In [39], Lu and Man discovered a novel way to link the spectral radius to *α-normal labeling* of any connected hypergraph, which is a very efficient method to compute or approximate the spectral radius.

Definition 2.4.1. *[39] A weighted incidence matrix B of a hypergraph H is a $|V|\times|E|$ matrix such that for any vertex v and any edge e, the entry $B(v,e) > 0$ if $v \in e$ and $B(v,e) = 0$ if $v \notin e$.*

Definition 2.4.2. *[39]*

1. *A hypergraph H is called α-normal if there exists a weighted incidence matrix B satisfying*

    a) *$\sum_{e:v\in e} B(v,e) = 1$, for any $v \in V(H)$.*

    b) *$\prod_{v\in e} B(v,e) = \alpha$, for any $e \in E(H)$.*

    Moreover, the incidence matrix B is called consistent if for any cycle $v_0e_1v_1\cdots v_l$,

$$\prod_{i=1}^{l} \frac{B(v_i, e_i)}{B(v_{i-1}, e_i)} = 1.$$

In this case, H is called consistently α-normal.

2. *A hypergraph H is called α-subnormal if there exists a weighted incidence matrix B satisfying*

   a) $\sum_{e:v\in e} B(v,e) \leq 1$, *for any* $v \in V(H)$.

   b) $\prod_{v\in e} B(v,e) \geq \alpha$, *for any* $e \in E(H)$.

   Moreover, H is called strictly α-subnormal if it is α-subnormal but not α-normal.

3. *A hypergraph H is called α-supernormal if there exists a weighted incidence matrix B satisfying*

   a) $\sum_{e:v\in e} B(v,e) \geq 1$, *for any* $v \in V(H)$.

   b) $\prod_{v\in e} B(v,e) \leq \alpha$, *for any* $e \in E(H)$.

   Moreover, H is called strictly α-supernormal if it is α-supernormal but not α-normal.

Lemma 2.4.3. *[39] Let H be a connected r-uniform hypergraph. Then the spectral radius of H is $\rho(H)$ if and only if H is consistently α-normal with $\alpha = (1/\rho(H))^r$.*

Lemma 2.4.4. *[39] Let H be an r-uniform hypergraph.*

1. *If H is consistently α-normal, then the spectral radius of H satisfies*

$$\rho(H) = \alpha^{-\frac{1}{r}}.$$

2. *If H is α-subnormal, then the spectral radius of H satisfies*

$$\rho(H) \leq \alpha^{-\frac{1}{r}}.$$

3. *If H is α-supernormal, then the spectral radius of H satisfies*

$$\rho(H) \geq \alpha^{-\frac{1}{r}}.$$

Chapter 3

Turán Theory for $\{1,3\}$-graphs

3.1 Degenerate $\{1,3\}$-graphs

In this chapter, we focus on the Turán densities of R-graphs for $R = \{1,3\}$. We will first give a necessary and sufficient condition for the degenerate $\{1,3\}$-graphs, then study the non-degenerate $\{1,3\}$-graphs. Let us call an edge of cardinality i as an i-edge, for each $i \in R$. For convenience, we call a vertex that forms a 1-edge as "black vertex", otherwise, "white vertex". We use notations of form $H_n^{\bullet}$ to represent a hypergraph on n vertices that contains only one "black vertex", similarly, $H_n^{\bullet\bullet}$ represents a hypergraph on n vertices that contains two "black vertices", and so on. To simplify our notations for $\{1,3\}$-graphs, we use form of abc to denote the edge $\{a,b,c\}$.

For $\{1,3\}$-graph H, we have $1 \leq \pi(H) \leq 2$. Observe that the product of two general R-graphs could be an R-graph. This is very useful in determining the Turán density. A degenerate R-graph H must be G-colorable for any R-graph G with $\lambda(G) > 1$. By Lemma 2.2.4, it must be colorable by the product of these R-graphs.

In the following, we introduce some constructions for $\{1,3\}$-graphs.

Construction A: Consider a $\{1,3\}$-graph (with loops) H_A on two vertices $\{x,y\}$ with edges $\{x, xyy, yyy\}$. The polynomial form of H_A is

$$\lambda(H_A, \vec{x}) = x_1 + 3x_1x_2^2 + x_2^3.$$

It can be shown that $\lambda(H, \vec{x})$ reaches the maximum $1 + \frac{\sqrt{3}}{18}$ over the simplex $S_2 =$

$\{(x_1, x_2) \in [0,1]^2, x_1 + x_2 = 1\}$ at $x_1 = \frac{1}{2} - \frac{\sqrt{3}}{6}$. Thus, we have

$$\lambda(H_A) = 1 + \frac{\sqrt{3}}{18} \approx 1.096225.$$

A $\{1,3\}$-graph G_A on n vertices is generated by blowing-up H_A as follows: set a vertex partition $V(G_A) = X \cup Y$ with $|X| \approx (\frac{1}{2} - \frac{\sqrt{3}}{6})n$ such that all 1-edges are in X (drawn by a black point), and all 3-edges are either formed by three vertices in Y or by one vertex in X plus two vertices in Y. In another words,

$$E(G_A) = \binom{X}{1} \cup \binom{X}{1} \times \binom{Y}{2} \cup \binom{Y}{3}.$$

We have

$$\begin{aligned} h_n(G_A) &= \frac{|X|}{n} + \frac{|X|\binom{|Y|}{2} + \binom{|Y|}{3}}{\binom{n}{3}} \\ &= \lambda(H_A, \vec{x}) + O(\frac{1}{n}). \\ &= \lambda(H_A) + O(\frac{1}{n}). \end{aligned}$$

Here $\vec{x} = (\frac{|X|}{n}, \frac{|Y|}{n}) = (\frac{1}{2} - \frac{\sqrt{3}}{6}, \frac{1}{2} + \frac{\sqrt{3}}{6})$.

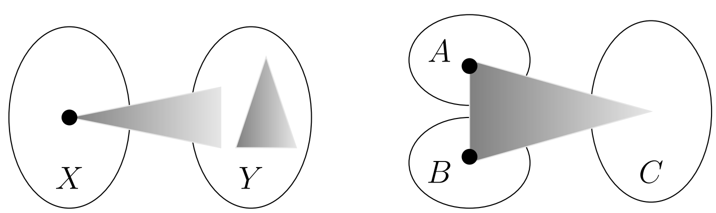

Figure 3.1 The $\{1,3\}$-graphs G_A with max $h_n(G_A) = 1 + \frac{\sqrt{3}}{18} + o_n(1)$ and and G_B with max $h_n(G_B) = \frac{4}{9} + \frac{\sqrt{3}}{3} + o_n(1)$.

Construction B: Let H_B be a general $\{1,3\}$-graph on three vertices $\{a,b,c\}$ with the edge set $\{a, b, abc\}$. We have

$$\lambda(H_B, \vec{x}) = x_1 + x_2 + 6x_1x_2x_3.$$

It is easy to check

$$\lambda(H_B) = \frac{4}{9} + \frac{\sqrt{3}}{3} \approx 1.021794714,$$

which is reached at $x_1 = x_2 = \frac{1+\sqrt{3}}{6}$, and $x_3 = \frac{2-\sqrt{3}}{3}$. A $\{1,3\}$-graph G_B on n vertices is generated by blowing-up H_B as follows: set a vertex partition $V(G_B) = A \cup B \cup C$. All 1-edges are in A and B(drawn by black points), all 3-edges are formed by exactly one vertex in each partition. We have

$$E(G_B) = \binom{A}{1} \cup \binom{B}{1} \cup \binom{A}{1} \times \binom{B}{1} \times \binom{C}{1}.$$

Note that G_B is H_B-colorable. Thus $h_n(G_B) = \lambda(H_B) + O(\frac{1}{n})$.

The following is an example of product of two constructions:

Example 3.1.1. *The product of two $\{1,3\}$-graphs H_A and H_B is given below. Let ax stand for (a,x), similar for other labels, then the vertex set is $V(H_A \times H_B) = \{ax, ay, bx, by, cx, cy\}$ and the edge set is*

$$E(H_A \times H_B) = \{\{ax\}, \{bx\}, \{cy, bx, ay\}, \{cy, ay, by\}, \{cy, by, ax\}, \{cx, ay, by\}\}.$$

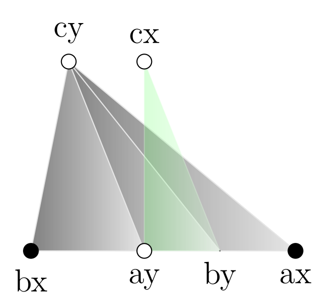

Figure 3.2 Product of H_A and H_B.

In this section, we will characterize the degenerate $\{1,3\}$-graph. Let's consider $H_5^{\{1,3\}}$, the 4-vertex $\{1,3\}$-graph with edge set $\{2, 3, 124, 135, 145\}$.

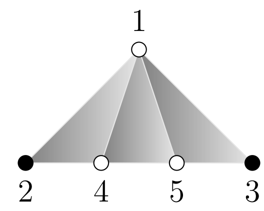

Figure 3.3
$\{1,3\}$-graph
$H_5^{\{1,3\}}$.

We first prove the following lemma.

Lemma 3.1.1. *[3] Any degenerate $\{1,3\}$-graph is $H_5^{\{1,3\}}$-colorable.*

Proof. Observe that a degenerate $\{1,3\}$-graph H must be contained in G_A and G_B. Equivalently, H is both H_A and H_B-colorable, then it must be colorable by the product $H_A \times H_B$. We define a map $f : V(H_A \times H_B) \to [5]$ such that: $f(cx) = f(cy) = 1$, $f(bx) = 2$, $f(ax) = 3$, $f(ay) = 4$, $f(by) = 5$. Obviously, f is a graph homomorphism from $H_A \times H_B$ to $H_5^{\{1,3\}}$. The result follows. $\square$

Let $K_3^{\bullet\bullet}$ be a $\{1,3\}$-graph on 3 vertices with edges $\{1, 2, 123\}$, and $G_4^{\bullet}$ be a $\{1,3\}$-graph on 4 vertices with edges $\{1, 123, 134, 234\}$.

Remark 3.1.1. *[3] $K_3^{\bullet\bullet}$ is not contained in G_A whose edge density reaches $1 + \frac{\sqrt{3}}{18}$, and $G_4^{\bullet}$ is not contained in G_B whose edge density reaches $\frac{4}{9} + \frac{\sqrt{3}}{3}$. Thus both $K_3^{\bullet\bullet}$ and $G_4^{\bullet}$ are non-degenerate $\{1,3\}$-graphs.*

Lemma 3.1.2. *[3] $\pi(\{K_3^{\bullet\bullet}, G_4^{\bullet}\}) = 1$.*

Proof. For any positive integer n, let G be a $\{K_3^{\bullet\bullet}, G_4^{\bullet}\}$-free $\{1,3\}$-graph on n vertices. Denote S as the set of all 1-edges of G, i.e. $S = \{v \in V(G) : \{v\} \in E(G)\}$, and let $|S| = xn$ for some $x \in (0,1)$. Let $\overline{S}$ be the complement of S, i.e. $\overline{S} = V(G) \setminus S$, then $|\overline{S}| = (1-x)n$.

Denote $E(G^3)$ as the set of all 3-edges of G. To forbidden $K_3^{\bullet\bullet}$, there is at most one black vertex in any 3-edges of G, thus we have

$$E(G^3) \subseteq \binom{S}{1} \times \binom{\overline{S}}{2} \cup \binom{\overline{S}}{3}.$$

We consider the 3-edges of G in edge set $\binom{S}{1} \times \binom{\overline{S}}{2}$. Define y as the average edge density of such 3-edges in G. Thus

$$y = \frac{|E(G^3) \cap (S \times \binom{\overline{S}}{2})|}{|S| \times \binom{|\overline{S}|}{2}}.$$

Note that there exists one vertex $s_0 \in S$ such that $|C(s_0)| \geq y \times \binom{|\overline{S}|}{2}$, where $C(s_0)$ is the set of 3-edges that contain the black vertex s_0. For any vertex $u \in \overline{S}$, define

$$W_u := \{v \in \overline{S} | s_0uv \in E(G)\}.$$

We then have

$$\sum_{u \in \overline{S}} |W_u| \leq |\overline{S}| \times (|\overline{S}| - 1)$$

and

$$|C(s_0)| = \frac{1}{2} \sum_{u \in \overline{S}} |W_u|,$$

which implies

$$\sum_{u \in \overline{S}} |W_u| \geq 2y \times \binom{|\overline{S}|}{2}.$$

To forbidden $G_4^{\bullet}$, if $s_0uv, s_0uk \in E(G)$, then $uvk \notin E(G)$. Since for each $u \in \overline{S}$, there are $\binom{|W_u|}{2}$ pair of vertices each can form a 3-edge with u, we need to remove these edges in $\binom{\overline{S}}{3}$. Let N be the number of 3-edges in $\binom{\bar{S}}{2}$ but not in G, by Cauchy-Schwarz inequality, we have

$$\begin{aligned} N &\geq \frac{1}{3} \sum_{u \in \overline{S}} \binom{|W_u|}{2} \\ &\geq \frac{1}{6} \frac{1}{|\overline{S}|} (\sum_{u \in \overline{S}} |W_u|)^2 - \frac{1}{6} \sum_{u \in \overline{S}} |W_u| \\ &\geq \frac{1}{6} y^2 |\overline{S}|^3 - \frac{1}{6} |\overline{S}| \times (|\overline{S}| - 1). \end{aligned} \tag{3.1}$$

Thus we have

$$h_n(G) \le x + y \times 3x \times (1-x)^2 + (1-x)^3 - y^2(1-x)^3 + o_n(1).$$

When $x \le \frac{2}{5}$, the above expression reaches the maximum value when $y = \frac{3}{2}\frac{x}{1-x} \le 1$. When $x \ge \frac{2}{5}$, the above expression reaches the maximum value when $y = 1$. Thus we obtain

$$h_n(G) \le \begin{cases} x + (1-x)^3 + \frac{9}{4}x^2(1-x) & \text{for } x \le \frac{2}{5}; \\ x + 3x(1-x)^2 & \text{for } x \ge \frac{2}{5}. \end{cases}$$

When $x \le \frac{2}{5}$, by solving $f(x) = x + (1-x)^3 + \frac{9}{4}x^2(1-x) \le 1$, we get $x \le \frac{8}{13}$. This always holds since $x \le \frac{2}{5} \le \frac{8}{13}$. When $x \ge \frac{2}{5}$, $g(x) = x + 3x(1-x)^2 \le 1$ is equivalent to $3x(1-x) \le 1$ which is always true. Thus in both cases, we have $h_n(G) \le 1 + o_n(1)$, implies that $\pi(\{K_3^{\bullet\bullet}, G_4^{\bullet}\}) = \lim_{n\to\infty} h_n(G) = 1$. The proof is complete. $\square$

Theorem 3.1.1. *[3] A $\{1,3\}$-hypergraph is degenerate if and only if it's $H_5^{\{1,3\}}$-colorable, where $H_5^{\{1,3\}}$ is a hypergraph with vertex set $V = [5]$ and edge set*

$$E = \{\{2\}, \{3\}, \{1,2,4\}, \{1,3,5\}, \{1,4,5\}\}.$$

Proof. Note that $H_5^{\{1,3\}}$ is $K_3^{\bullet\bullet}$ and $G_4^{\bullet}$-colorable respectively, we have $\pi(H_5^{\{1,3\}}) \le \pi(\{K_3^{\bullet\bullet}, G_4^{\bullet}\})$. By Lemma 3.1.1 and Lemma 3.1.2, the result follows. $\square$

3.2 Non-degenerate $\{1,3\}$-graphs

A hypergraph is called 3-*partite* if its vertex set V can be partitioned into 3 different classes V_1, V_2, V_3 such that every edge intersects each class in exactly one vertex. A 3-partite $\{1,3\}$-graph is $K_3^{\bullet\bullet\bullet}$-colorable, where $K_3^{\bullet\bullet\bullet}$ is a $\{1,3\}$-graph on 3 vertices with edge set $\{1,2,3,123\}$. So far we know that the chain $C^{\{1,3\}} = \{1,123\}$ is 3-partite and it is degenerate, while a slightly larger 3-partite $\{1,3\}$-graph $K_3^{\bullet\bullet} = \{1,2,123\}$ is not degenerate. We have $\pi(K_3^{\bullet\bullet}) \ge 1 + \frac{\sqrt{3}}{18}$ since it's not contained in the G_A.

Theorem 3.2.1. *For any $K_3^{\bullet\bullet}$-colorable $\{1,3\}$-graph H, if $K_3^{\bullet\bullet} \not\subseteq H$, H must be $H_5^{\{1,3\}}$-colarable, then $\pi(H) = 1$.*

Proof. Since H satisfies $K_3^{\bullet\bullet} \not\subseteq H \subseteq K_3^{\bullet\bullet}(s)$ for $s \geq 2$, then H is $K_3^{\bullet\bullet}$-colorable, which means there exists a vertex partition $V(H) = V_1 \cup V_2 \cup V_3$ so that H is 3-partite and the level-graph H^1 only appears in at most two vertex partitions (say V_2 and V_3). Since H does not contain $K_3^{\bullet\bullet}$ as sub-graph, then each edge of the level-graph H^3 can only intersect one vertex in V_1, plus one white (black) vertex in V_2 and one black (white) vertex in V_3, or intersect one vertex in V_1 plus two white vertices in V_2 and V_3. Let f be a map such that $f(v) = 1$ if $v \in V_1$, $f(v) = 2$ if v is a black vertex in V_2, $f(v) = 5$ if v is a white vertex in V_2, and $f(v) = 3$ if v is a black vertex in V_3, $f(v) = 4$ if v is a white vertex in V_3. One can check that f is a hypergraph homomorphism from H to $H_5^{\{1,3\}}$. Thus H is $H_5^{\{1,3\}}$-colarable, we have $\pi(H) = 1$. $\square$

Theorem 3.2.2. *[3] For any $K_3^{\bullet\bullet}$-colorable $\{1,3\}$-graph H, if $K_3^{\bullet\bullet} \subseteq H$, then $\pi(H) = \pi(K_3^{\bullet\bullet}) = 1 + \frac{\sqrt{3}}{18}$.*

Proof. Since $K_3^{\bullet\bullet} \subseteq H \subseteq K_3^{\bullet\bullet}(s)$ for $s \geq 2$, by Theorem 2.1.2, we have $\pi(H) = \pi(K_3^{\bullet\bullet})$. For any $K_3^{\bullet\bullet}$-free $\{1,3\}$-graph G on n vertices, let X be the set of all 1-edges in G, and $Y \subseteq V(G)$ be the complement of X. On one hand, since it is $K_3^{\bullet\bullet}$-free, there is no 3-edge of form $\binom{X}{3}$ or form $\binom{X}{2} \times \binom{Y}{1}$. Thus G is H_A-colorable. Therefore $\lim_{n\to\infty} h_n(G) \leq \lambda(H_A) = 1+\frac{\sqrt{3}}{18}$. On the other hand, the construction G_A is $K_3^{\bullet\bullet}$-free. We have $\pi(K_3^{\bullet\bullet}) \geq \lambda(H_A) = 1 + \frac{\sqrt{3}}{18}$. Thus, we have $\pi(H) = \pi(K_3^{\bullet\bullet}) = 1 + \frac{\sqrt{3}}{18}$. $\square$

A result following above theorem indicates a break for the Turán density of $\{1,3\}$-graphs:

Corollary 3.2.1. *[3] Let α be a real value in $[1, \frac{4}{9} + \frac{\sqrt{3}}{3})$. For any $\{1,3\}$-graph H with $\pi(H) \leq \alpha$, it must be the case that $\pi(H) = 1$.*

Proof. Let H be any $\{1,3\}$-graph with $\pi(H) < \lambda(H_B) = \frac{4}{9} + \frac{\sqrt{3}}{3}$. Then H must be H_B-colorable, hence $K_3^{\bullet\bullet}$-colorable. By Theorem 3.2.2, $\pi(H)$ is either 1 or $1 + \frac{\sqrt{3}}{18}$, thus we must have $\pi(H) = 1$. $\square$

3.2.1 The 3-partite $\{1,3\}$-graphs

In previous part, all $\{1,3\}$-graphs we studied are 3-partite. In this section, we continue to study the Turán densities of 3-partite $\{1,3\}$-graphs.

Lemma 3.2.1. *[3] Let H be a 3-partite $\{1,3\}$-graph such that $K_3^{\bullet\bullet\bullet} \subseteq H$, where $K_3^{\bullet\bullet\bullet} = \{1,2,3,123\}$. Then $\pi(H) = 1 + \frac{2\sqrt{3}}{9}$.*

Proof. Since any 3-partite $\{1,3\}$-graph is $K_3^{\bullet\bullet\bullet}$-colorable, we only need to prove $\pi(K_3^{\bullet\bullet\bullet}) = 1 + \frac{2\sqrt{3}}{9}$.

On one hand, consider an extremal $K_3^{\bullet\bullet\bullet}$-free $\{1,3\}$-graph G_n. Let X be the set vertices of 1-edges in G_n. Projecting all the vertices in X into a single vertex x and all the vertices not in X into a single vertex y, we get an $\{1,3\}$-graph (with loops) H_C: where $E(H_c) = \{x, xxy, xyy, yyy\}$. This projection is a hypergraph homomorphism from G_n to H_C since G is $K_3^{\bullet\bullet\bullet}$-free. Thus G is H_c-colorable. In particular, we have

$$\pi(K_3^{\bullet\bullet\bullet)} = \lim_{n\to\infty} h_n(G_n) \leq \lambda(H_c) = 1 + \frac{2\sqrt{3}}{9}.$$

On the other hand, any blow-up of H_c does not contain the sub-graph $K_3^{\bullet\bullet\bullet}$. The blow-up graph G_C has the maximal edge density $1 + \frac{2\sqrt{3}}{9}$:

Hence, $\pi(K_3^{\bullet\bullet\bullet)} = \lambda(H_c) = 1 + \frac{2\sqrt{3}}{9}$.

$\square$

Let us restrict H to be 3-partite but containing no $K_3^{\bullet\bullet\bullet}$ as sub-graph. One can check such H must be $H_6^{\{1,3\}}$-colorable, where

$$H_6^{\{1,3\}} = \{1, 2, 3, 124, 145, 135, 236, 246, 356, 456\}.$$

$H_6^{\{1,3\}}$ is not contained in G_D with maximal edge density $\frac{4}{3}$.

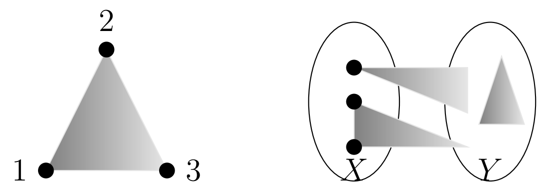

Figure 3.4 $\{1,3\}$-graph $K_3^{\bullet\bullet\bullet}$ and its extremal configution G_C.

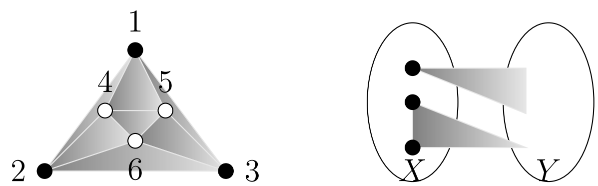

Figure 3.5 $\{1,3\}$-graph $H_6^{\{1,3\}}$ and its extremal configuration G_D.

So far we couldn't determine the upper bound of $\pi(H_6^{\{1,3\}})$. We leave this open. Let's turn our attention to the sub-graphs of $H_6^{\{1,3\}}$ and we aim to determine their Turán densities. Now let us first consider two sub-graphs of $H_6^{\{1,3\}}$:

$$H_5^* = \{1,2,3,124,145,135\},\ H_6^* = \{1,2,3,124,135,236\}.$$

For both of them, the Turán density is greater than $1+\frac{\sqrt{3}}{9}$, since they are not contained in G_E and G_F respectively($\lim_{n\to\infty} h_n(G_E) = \lim_{n\to\infty} h_n(G_F)$).

To calculate the upper bounds of $\pi(H_5^*)$, we need the following lemma.

Lemma 3.2.2. *[3] Let* $H_4^{\bullet\bullet} = \{1,2,123,124,134\}$*, then* $\pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}) = 1+\frac{\sqrt{3}}{9}$.

Proof. To see the lower bound, observe that both $H_4^{\bullet\bullet}$ and $K_3^{\bullet\bullet\bullet}$ are not contained in G_E.

To see the upper bound, let G represent a $\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}$-free graph on n vertices, let $X \subseteq V(G)$ be the set of all 1-edges of G, $|X| = xn$ for some real $x \in (0,1)$, and

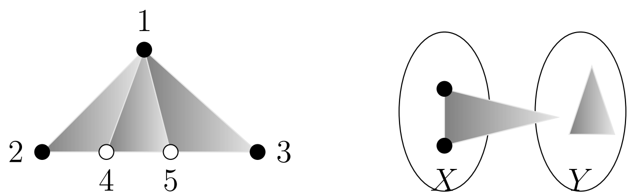

Figure 3.6 $\{1,3\}$-graph H_5^* and its extremal configuration G_E.

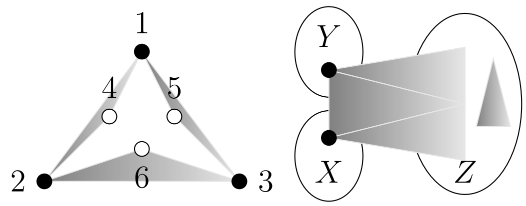

Figure 3.7 $\{1,3\}$-graph H_6^* and its extremal configuration G_F.

let $Y = V(G) \setminus X$, then $|Y| = (1-x)n$. To forbidden $K_3^{\bullet\bullet\bullet}$, there is no 3-edge of form $\binom{X}{3}$. Let y be the density of 3-edges in G among all edges of form $\binom{X}{2} \times \binom{Y}{1}$. For any pair of vertices (i,j) in X, denote d_{ij} as the number of vertices $k \in Y$ so that $\{ijk\} \in E(G)$. Then

$$y = \frac{\sum_{(i,j)\in\binom{X}{2}} d_{ij}}{\binom{|X|}{2} \times \binom{|Y|}{1}}.$$

To forbidden $H_4^{\bullet\bullet}$, for each pair of vertices $(i,j) \in \binom{X}{2}$, if ijk and ijl are in $E(G)$, neither kli nor klj can be contained in $E(G)$. Thus for every pair of $\{i,j\}$, the number of 3-edges not shown in G is at least $2\binom{d_{ij}}{2}$. Let M be the total number of 3-edges of form $\binom{X}{1} \times \binom{Y}{2}$ not shown in G, then by Cauchy-Schwarz inequality, we have

$$\begin{aligned} M &\geq \frac{\sum_{i,j\in\binom{X}{2}} 2\binom{d_{ij}}{2}}{|X|} \\ &\geq \frac{\left(\sum_{i,j\in X} d_{ij}\right)^2}{\binom{xn}{2}(xn)} - \frac{\sum_{i,j\in X} d_{ij}}{xn} \\ &\geq \frac{1}{2}y^2x(1-x)^2n^3 - \frac{1}{2}yx(1-x)n^2. \end{aligned}$$

Thus

$$h_n(G) \leq x + (1-x)^3 + 3x^2(1-x)y + 3x(1-x)^2 - 3x(1-x)^2y^2 + o_n(1).$$

A simple calculation can show that $h_n(G)$ achieves maximum value at $y = 1$, which implies that for any positive integer n, any extremal $\{1,3\}$-graph of $\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}$ is H_E-colorable where $E(H_E) = \{x, xxy, yyy\}$. Therefore,

$$\pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}) = \lim_{n\to\infty} h_n(G_n) \leq \lambda(H_E) = 1 + \frac{\sqrt{3}}{9}$$

. The result follows. □

Lemma 3.2.3. *[3]* $\pi(H_5^*) = 1 + \frac{\sqrt{3}}{9}$.

Proof. On one hand, H_5^* is not contained in G_E, then $\pi(H_5^*) \geq 1 + \frac{\sqrt{3}}{9}$. On the other hand, H_5^* is $K_3^{\bullet\bullet\bullet}$ and $H_4^{\bullet\bullet}$-colorable, thus $\pi(H_5^*) \leq \pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet}\}) \leq 1 + \frac{\sqrt{3}}{9}$. The result follows. □

Corollary 3.2.2. *[3] The proper sub-graphs of H_5^* can be classified to two different sets: either the sub-graph contains $K_3^{\bullet\bullet}$ and is $K_3^{\bullet\bullet}$-colorable, in this case the Turán density is $1 + \frac{\sqrt{3}}{18}$; or the sub-graph does not contain $K_3^{\bullet\bullet}$, then it is $H_5^{\{1,3\}}$-colorable, in this case the Turán density is* 1.

To calculate the upper bounds of $\pi(H_6^*)$, we need the following lemma.

Lemma 3.2.4. *[3] Let $H_4^{\bullet\bullet\bullet} = \{1, 2, 3, 124, 134, 234\}$, then $\pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet\bullet}\}) = 1+\frac{\sqrt{3}}{9}$.*

Proof. To see the lower bound, observe that both $K_3^{\bullet\bullet\bullet}$ and $H_4^{\bullet\bullet\bullet}$ are not contained in G_F.

To see the upper bound, let G represent a $\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet\bullet}\}$-free graph on n vertices, let $X \subseteq V(G)$ be the set of all 1-edges of G, $|X| = xn$ for some real $x \in (0, 1)$, let $Y = V(G) \setminus X$, then $|Y| = (1-x)n$. To forbidden $K_3^{\bullet\bullet\bullet}$, there is no 3-edge of form $\binom{X}{3}$.

Let y be the density of 3-edges in G among all edges of form $\binom{X}{2} \times \binom{Y}{1}$. For each $i \in Y$, let $D_i = \{\{j,k\} \in \binom{X}{2} | \ ijk \in E(G)\}$, denote $d_i = |D_i|$. Then

$$y = \frac{\sum_{i \in Y} d_i}{\binom{|X|}{2} \times \binom{|Y|}{1}}.$$

Suppose $y > \frac{1}{2}$, then there exists $i \in Y$, such that $d_i > \frac{1}{2}\binom{|X|}{2}$. By the fact that the Turán density of a triangle graph is $\frac{1}{2}$, there must exist a triple $\{j,k,l\} \in \binom{X}{3}$ such that $\{ijk, ijl, ikl\} \subseteq E(G)$, which is a copy of $H_4^{\bullet\bullet\bullet}$, a contradiction. Note that the existence of 3-edge of form $\binom{Y}{3}$ or $\binom{X}{1} \times \binom{Y}{2}$ does not result in an occurrence of $H_4^{\bullet\bullet\bullet}$ or $K_3^{\bullet\bullet\bullet}$ in G. Thus we can take all such edges. Thus we have $y \leq \frac{1}{2}$, then

$$h_n(G) \leq x + (1-x)^3 + \frac{3}{2}x^2(1-x) + 3x(1-x)^2 + o_n(1),$$

which achieves the maximum $1 + \frac{\sqrt{3}}{9}$ at $x = 1 - \frac{\sqrt{3}}{3}$.

Hence, we have $\pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet\bullet}\}) = \lim_{n\to\infty} h_n(G_n) \leq 1 + \frac{\sqrt{3}}{9}$. The result follows. $\square$

Lemma 3.2.5. *[3]* $\pi(H_6^*) = 1 + \frac{\sqrt{3}}{9}$.

Proof. On one hand, H_6^* is not contained in G_F, then $\pi(H_6^*) \geq 1 + \frac{\sqrt{3}}{9}$. On the other hand, H_6^* is $K_3^{\bullet\bullet\bullet}$ and $H_4^{\bullet\bullet\bullet}$-colorable, thus $\pi(H_6^*) \leq \pi(\{K_3^{\bullet\bullet\bullet}, H_4^{\bullet\bullet\bullet}\}) \leq 1 + \frac{\sqrt{3}}{9}$. The result follows. $\square$

Since we are considering all sub-graphs of $H_6^{\{1,3\}}$, we start this by looking at the larger sub-graphs then the smaller sub-graphs. Using above two lemmas, we are able to determine the Turán density for a list of sub-graphs of $H_6^{\{1,3\}}$. Now we let H be a sub-graph of $H_6^{\{1,3\}}$, we have:

1. If H is $K_3^{\bullet\bullet}$-colorable, thus $\pi(H) = 1$ or $\pi(H) = 1 + \frac{\sqrt{3}}{18}$.

2. If H is not above case, then H must contain all 1-edges: $1, 2, 3$, and none of them is isolated. Then we have several different cases:

   a) Suppose H is obtained from $H_6^{\{1,3\}}$ by removing one 3-edge consisting of two black vertices and one white vertex (say 236 or equivalence), then one can check H is H_5^*-colorable. Note that $H_5^* \subseteq H$, by Lemmas 3.2.3, we have $\pi(H) = 1 + \frac{\sqrt{3}}{9}$. Similarly, for any sub-graph H' of H, if H' contains H_5^* or a H_5^*-colorable graph as sub-graph, then $\pi(H') = 1 + \frac{\sqrt{3}}{9}$. If H' is not above case, by trial and error, there is only one situation: H' contains the following sub-graph H^* (or its equivalence) which is not contained in G_E. Thus $\pi(H') = 1 + \frac{\sqrt{3}}{9}$:

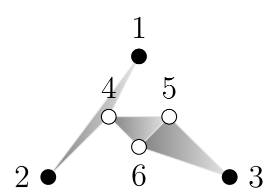

Figure 3.8
$\{1, 3\}$-graph H^*.

b) Let H be sub-graph of $H_6^{\{1,3\}}$ by removing edges 145, 246, 456 and 356, the resulting graph is H_6^*. By Lemma 3.2.5, $\pi(H) = \pi(H_6^*) = 1 + \frac{\sqrt{3}}{9}$.

3. The following graphs $H_6^{\{1,3\}}$, $H_6^a, H_6^b, , H_6^c, H_6^d$ and H_6^e are unsolved. We conjecture that the extremal configuration of $H_6^{\{1,3\}}$ and H_6^b is Construction G_D: thus we conjecture $\pi(H_6^{\{1,3\}}) = \pi(H_6^b) = \frac{4}{3}$.

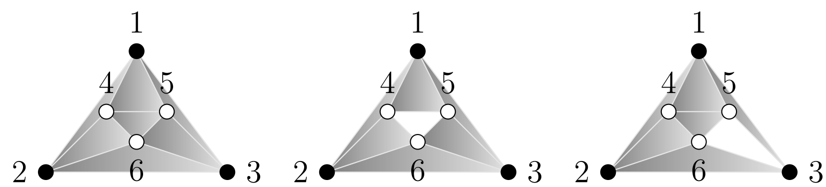

Figure 3.9 $\{1,3\}$-graphs: $H_6^{\{1,3\}}$, H_6^a and H_6^b.

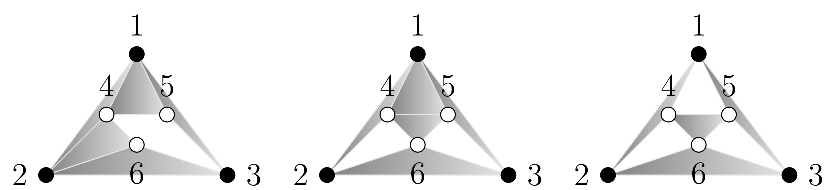

Figure 3.10 $\{1,3\}$-graphs: H_6^c, H_6^d and H_6^e.

3.3 Non-trivial degenerate R-graphs

Recall that a degenerate R-graph H is trivial if it is contained in a blow-up of the chain C^R, otherwise, we say H is non-trivial. In this section, we will prove the following theorem with the use of *suspension* operations on hypergraphs.

Theorem 3.3.1. *[3] Let R be a set of distinct positive integers with $|R| \geq 2$ and $R \neq \{1,2\}$. Then a non-trivial degenerate R-graph always exists.*

Definition 3.3.1. *[25] The suspension of a hypergraph H, denoted by $S(H)$, is the hypergraph with $V = V(H) \cup \{v\}$ where $\{v\}$ is a new vertex not in $V(H)$, and the edge set $E = \{e \cup \{v\} : e \in E(H)\}$. We write $S^t(H)$ to denote the hypergraph obtained by iterating the suspension operation t-times, i.e. $S^2(H) = S(S(H))$ and $S^3(H) = S(S(S(H)))$, etc.*

The relationship between $\pi(H)$ and $\pi(S(H))$ was investigated in [25].

Proposition 3.3.1. *[25] For any family of hypergraphs $\mathcal{H}$ we have that $\pi(S(\mathcal{H})) \leq \pi(\mathcal{H})$.*

Given a general set R and positive integer t, we denote $(R+t)$ as the set obtained from R by adding t to each element of R. Note that if the R-graph H is not contained in a blow-up of chain C^R, then $S^t(H)$ is not contained in a blow-up of the chain $C^{(R+t)}$. Thus we have the following fact:

Corollary 3.3.1. *[3] Let H be a non-trivial degenerate R-graph, let t be any positive integer. Then the t-times suspension $S^t(H)$ is a non-trivial degenerate $(R+t)$-graph.*

Lemma 3.3.1. *[3] Given a positive integer $t \geq 2$, and a $\{1, t\}$-graph H, let $T(H)$ be the $\{1, t+1\}$-graph obtained from H by adding a new vertex $v \notin V(H)$ such that $V = V(H) \cup \{v\}$, $T(H)^1 = H^1$ and $T(H)^{1+t} = \{e \cup \{v\} : e \in E(H^t)\}$. Then we have $\pi(T(H)) \leq \pi(H)$.*

Proof. Let n be a positive integer and $G = (V, E)$ be an extremal $T(H)$-free $\{1, t+1\}$-graph on n vertices. We have $\pi_n(T(H)) = h_n(G)$. Denote E_i as the set of i-edges of G, for $i = 1, t+1$. For any vertex $v \in V(G)$, denote G_v as the hypergraph obtained from G with the vertex set $V(G_v) = V \setminus \{v\}$ and the edge sets $E(G_v) = E_{v,1} \cup E_{v,t}$, where $E_{v,1} = \{u \in V(G_v) : u \in E_1\}$ and $E_{v,t} = \{\{u_1, \ldots, u_t\} : \{v, u_1, \ldots, u_t\} \in E_{t+1}\}$. Observe that G_v is an H-free $\{1, t\}$-graph on $n - 1$ vertices. Thus $h_{n-1}(G_v) \leq \pi_{n-1}(H)$.

Since

$$|E_1| = \frac{1}{n-1} \sum_{v \in V(G)} |E_{v,1}| \quad \text{and} \quad |E_{t+1}| = \frac{1}{(t+1)} \sum_{v \in V(G)} |E_{v,t}|,$$

then

$$\begin{aligned} h_n(G) &= \frac{|E_1|}{\binom{n}{1}} + \frac{|E_{1+t}|}{\binom{n}{1+t}} \\ &= \sum_{v \in V(G)} \frac{|E_{v,1}|}{(n-1)\binom{n}{1}} + \sum_{v \in V(G)} \frac{|E_{v,t}|}{(t+1)\binom{n}{1+t}} \\ &= \frac{1}{n} \sum_{v \in V(G)} \left(\frac{|E_{v,1}|}{\binom{n-1}{1}} + \frac{|E_{v,t}|}{\binom{n-1}{t}} \right) \\ &= \frac{1}{n} \sum_{v \in V(G)} h_{n-1}(G_v) \\ &\leq \pi_{n-1}(H). \end{aligned}$$

Thus $\pi(T(H)) = \lim_{n\to\infty} \pi_n(T(H)) = \lim_{n\to\infty} h_n(G) \leq \pi(H)$. $\square$

Lemma 3.3.2. *[3] Let R be a set of two distinct positive integers, $R \neq \{1, 2\}$. Then there exist non-trivial degenerate R-graphs.*

Proof. By Corollary 3.3.1, for every positive integer k, one can take the suspension of $H_5^{\{1,3\}}$ k-times, the resulting graph $S^k(H_5^{\{1,3\}})$ is a non-trivial degenerate $\{1 + k, 3 + k\}$-graph. Thus there are non-trivial degenerate hypergraphs of edge types: $\{1, 3\}, \{2, 4\}, \{3, 5\}, \ldots, \{k, k + 2\}, \ldots$.

In [25], the authors found a non-trivial degenerate $\{2, 3\}$-graph: $H_4^{\{2,3\}}$ with edges $\{12, 13, 234\}$. Similarly, by Corollary 3.3.1, there are non-trivial degenerate hypergraphs of edge types: $\{2, 3\}, \{3, 4\}, \{4, 5\}, \ldots, \{k, k + 1\}, \ldots$.

Using Lemma 3.3.1 on $H_5^{\{1,3\}}$, there are non-trivial degenerate hypergraphs of edge types: $\{1, 4\}, \{1, 5\}, \ldots, \{1, t\}, \ldots$, for integer $t \geq 4$. For each of these non-trivial degenerate $\{1 + t\}$-graphs, applying Corollary 3.3.1, there are non-trivial degenerate hypergraphs of edge types: $\{2, 1 + t\}, \{3, 2 + t\}, \ldots, \{k, k - 1 + t\}, \ldots$.

To summarize, for each integer $k \geq 2$ and each integer $t \geq 3$, we have non-trivial degenerate hypergraphs of edge types $\{1, t\}, \{k, k+1\}, \{k, k+2\}, \{k, k+t\}$, which cover all sets of two distinct positive integers, except $\{1, 2\}$. $\square$

Lemma 3.3.3. *[3] Let R be a set of distinct positive integers with $|R| \geq 2$ and $1 \notin R$. If there exist non-trivial degenerate R-graphs, then there exist non-trivial degenerate $\{1\} \cup R$-graphs.*

Proof. For each R stated in the lemma, let H be the non-trivial degenerate R-graph. Let H' be the disjoint union of H with a single 1-edge $v \notin H$. Clearly, H' is not contained in a blow-up of chain $C^{\{1\} \cup R}$. We will prove that H' is also degenerate.

Let n be a positive integer and $G = (V, E)$ be an extremal H'-free $\{1\} \cup R$-graph on n vertices. We have $\pi_n(H') = h_n(G)$. Denote E_i as the set of i-edges of G, for each $i \in \{1\} \cup R$. For any 1-edge $v \in E_1$, consider the sub-graph G_v of G by removing all 1-edges (keep the vertices of these 1-edges in G_v). Then the vertex set $V(G_v) = V$, set of i-edges $E_i(G_v) = E_i(G)$ for each $i \in R$. Then we have

$$|E_i(G)| = \frac{1}{|E_1|} \sum_{v \in E_1} |E_i(G_v)|, \ \forall i \in R.$$

Observe that G_v is an H-free R-graph on n vertices, so $\pi_n(H) \geq h_n(G_v)$. Then we have

$$\begin{aligned}
h_n(G) &= \sum_{i \in \{1\} \cup R} \frac{|E_i|}{\binom{n}{i}} \\
&= \frac{|E_1|}{\binom{n}{1}} + \sum_{i \in R} \sum_{v \in E_1} \frac{|E_i(G_v)|}{|E_1|\binom{n}{i}} \\
&\leq 1 + \frac{1}{|E_1|} \sum_{v \in E_1} h_n(G_v) \\
&\leq 1 + \pi_n(H).
\end{aligned}$$

Thus $\pi(H') = \lim_{n\to\infty} \pi_n(H') = \lim_{n\to\infty} h_n(G) \leq 1 + \pi(H) = |R|$, then $\pi(H') = |R|$. Therefore, H' is a non-trivial degenerate $\{1\} \cup R$-graph. $\square$

Proof of Theorem 3.3.1. Using the non-trivial degenerate R-graph for R stated in Lemma 3.3.2, then apply Lemma 3.3.3, we obtain non-trivial degenerate R-graphs for $|R| = 3$ and $1 \in R$. Apply Corollary 3.3.1, we then obtain all other non-trivial degenerate R-graphs for $|R| = 3$. Repeatedly apply Lemma 3.3.3 and Corollary 3.3.1, we can obtain all R-graphs for $|R| \geq 4$, the result follows. □

We conjecture that for any set R, there exists an R-graph H^R such that if G^R is R-degenerate if and only if G^R is H^R-colorable. This conjecture is true for the case $R = \{r\}$ with $r \geq 2$ and $R = \{1, 2\}$ and is confirmed for $R = \{1, 3\}$ in this paper. Perhaps the next interesting degenerate non-uniform hypergraphs are $\{2, 3\}$-graphs, however, it is more difficult to determine their structures. In the next chapter, we will extend our work to the class of 2-edge-colored graphs whose Turán density has a relation with the $\{2, 3\}$-graphs.

Chapter 4

Turán Theory for 2-edge-colored graphs

In this chapter, we will prove the following theorem and show an application on $\{2,3\}$-graphs in our study. The notation $[n]$ is the set of $\{1,\ldots,n\}$. For convenience, we represent an edge $\{a,b\}$ by ab.

Theorem 4.0.2. *[4] The Turán densities of all bipartite 2-colored graphs are in the set* $\left\{1,\frac{4}{3},\frac{3}{2}\right\}$.

1. *A 2-colored graph H is degenerate if and only if it is T-colorable, where T is the 2-colored graph with vertices $[4]$ and red edges $\{12,13,34\}$, blue edges $\{12,23,34\}$.*

2. *A 2-colored graph H satisfies $\pi(H)=\frac{4}{3}$, then H must be H_8-colorable but not T-colorable, where H_8 is the 2-colored graph with vertices $[8]$ and red edges*

$$E_r(H_8)=\{12,13,24,34,16,37,48,25,35,18,46,27\}$$

blue edges

$$E_b(H_8)=\{56,57,68,78,26,15,47,38,35,18,46,27\}.$$

3. *A 2-colored bipartite graph H satisfies $\pi(H)=\frac{3}{2}$, then H is not H_8-colorable.*

The following theorem shows a relation between such $\{2,3\}$-graphs and the 2-colored graphs.

Theorem 4.0.3. *[4] Let $H = (V,\ E_r,\ E_b)$ be a 2-colored graph, and $H' = (V', E^2,\ E^3)$ be a $\{2,3\}$-graph obtained from H by adding a new vertex $v \notin (V)$ such that $V' = V \cup \{v\}$ and $E^2 = E_r$, and $E^3 = \{v\} \times E_b$. Then $\pi(H') \leq \pi(H)$.*

4.1 Turán density of bipartite 2-colored graphs

Given a 2-colored graph H, to compute the lower bound of $\pi(H)$, we need to construct a family of H-free 2-colored graphs G_n with $h_n(G_n)$ as large as possible. Here are three useful constructions.

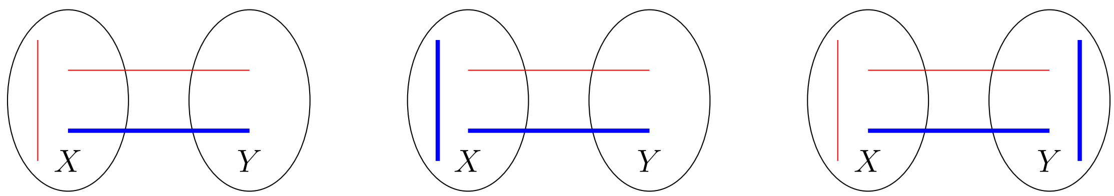

Figure 4.1 G_A, G_B and G_C with $h_n(G_A) = h_n(G_B) = \frac{4}{3} + o_n(1)$ at $|X| = \frac{2}{3}n$ and $h_n(G_C) = \frac{3}{2} + o_n(1)$ at $|X| = \frac{1}{2}n$.

G_A: A 2-colored graph G_A is generated by partitioning the vertex set into two parts such that $V(G_A) = X \cup Y$ and the red edges either meet two vertices in X or meet one vertex in X plus the other in Y, the blue edges meet one vertex in X plus the other in Y. In another words, the red edges are $E_r(G_A) = \binom{X}{2} \cup \binom{X}{1} \times \binom{Y}{1}$ and blue edges are $E_b(G_A) = \binom{X}{1} \times \binom{Y}{1}$. Let $|V(G_A)| = n$, $|X| = xn$ and $|Y| = (1-x)n$ for some real number $x \in (0,1)$. We have

$$h_n(G_A) = \frac{\binom{|X|}{2} + 2\binom{|X|}{1}\binom{|Y|}{1}}{\binom{n}{2}}$$
$$= 4x - 3x^2 + o_n(1),$$

which reaches the maximum $\frac{4}{3}$ at $x = \frac{2}{3}$.

G_B: It is the dual of G_A by simply exchanging red edges with blue edges. In another words, the red edges are $E_r(G_B) = \binom{X}{1} \times \binom{Y}{1}$ and blue edges are $E_b(G_B) = \binom{X}{2} \cup \binom{X}{1} \times \binom{Y}{1}$.

G_C: A 2-colored graph G_C is generated by partitioning the vertex set into two parts such that $V(G_C) = X \cup Y$ and the red edges either meet two vertices in X or meet one vertex in X plus the other in Y, the blue edges either meet two vertices in Y or meet one vertex in X plus the other in Y. In another words, the red edges are $E_r(G_C) = \binom{X}{2} \cup \binom{X}{1} \times \binom{Y}{1}$ and blue edges are $E_b(G_C) = \binom{X}{1} \times \binom{Y}{1} \cup \binom{Y}{2}$.

Example 4.1.1. *The product of G_A and G_B is a blow-up of T, with $V(T) = [4]$, and red edges $\{12, 13, 34\}$, blue edges $\{12, 23, 34\}$:*

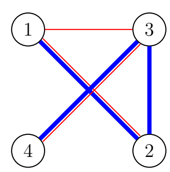

Figure 4.2
2-colored
graph T.

We define a map $f : V(H) \to \{1, 2, 3, 4\}$ as follows:

1. If v appears in X of G_A and in Y of G_B, set $f(v) = 1$.
2. If v appears in Y of G_A and in X of G_B, set $f(v) = 2$.
3. If v appears in X of G_A and in X of G_B, set $f(v) = 3$.
4. If v appears in Y of G_A and in Y of G_B, set $f(v) = 4$.

One can check f is a graph homomorphism from the product $G_A \times G_B$ to T.

We first give a boundary to divide the Turán densities of 2-colored non-bipartite graphs and 2-colored bipartite graphs.

Lemma 4.1.1. *[4]*

1. *For any* 2*-colored non-bipartite graph* H*,* $\pi(H) \geq \frac{3}{2}$*.*

2. *For any* 2*-colored bipartite graph* H*,* $\pi(H) \leq \frac{3}{2}$*.*

Before proceeding to the proof, let us see several important 2-colored graphs that achieve the value $\frac{3}{2}$.

Lemma 4.1.2. *[4] Let* K_3 *be a triangle with three double-colored edges, i.e.* $K_3 = ([3], \{12, 13, 23\}, \{12, 13, 23\})$*. Then* $\pi(K_3) = \frac{3}{2}$*.*

Proof. Observe that H is not contained in G_C, thus $\pi(K_3) \geq \frac{3}{2}$. Now we prove the other direction. Let n be a positive integer and G be any K_3-free 2-colored graph on n vertices. By induction on n we will prove $|E(G)| \leq \frac{3}{2}\binom{n}{2} + cn$ for some finite number c. The cases for $n \leq 7$ are easy to check. Let $n \geq 7$. We assume the statement holds for any K_3-free graphs on less than n vertices. If G contains no double-colored edges, then $|E_r(G)| + |E_b(G)| \leq \binom{n}{2} < \binom{n}{2}$, we are done. So we assume G contains at least one double-colored edge and call it uv. Then G is one of the following cases.

Case 1: There exist no vertex w so that both uv and vw are double-colored edges. Then there are at most two edges from each of the rest vertices to $\{u, v\}$. By inductive hypothesis, when G is restricted to the complement set of $\{u, v\}$, the number of edges of $G[V \setminus \{u, v\}]$ is at most $\frac{3}{2}\binom{n-2}{2} + c'(n-2)$ edges. Thus,simply taking $c = \frac{c'}{4}$, we have

$$|E(G)| \leq 2 + 2(n-2) + \frac{3}{2}\binom{n-2}{2} + c'(n-2) \leq \frac{3}{2}\binom{n}{2} + cn.$$

Case 2: There exists a vertex w so that both uv and vw are double-colored edges. Let $V_1 = \{u, v, w\}$, we can divide the rest of vertices into four distinct subsets V_2, V_3, V_4, V_5 so that

$$V_2 = \{v' \in V | v'u, v'w \text{ are double-colored }\},$$

$$V_3 = \{v' \in V | v'v \text{ is double-colored }\},$$

$$V_4 = \{v' \in V | v'u \text{ is double-colored but } v'w \text{ is not doule-colored}\},$$

$$V_5 = \{v' \in V | v'w \text{ is double-colored but } v'u \text{ is not doule-colored}\}.$$

By the definitions of V_i and the fact that G is H-free, V_2, V_3, V_4, V_5 are disjoint. Furthermore, we can also obtain the following facts: there are at most one edge from each vertex in V_2 to each vertex in V_4(or V_5); there are at most five edges from each vertex in V_2 to V_1; there are at most four edges from each vertex in V_i to V_1, for $i = \{3, 4, 5\}$; there are at most one edge between any pair of vertices in V_i, for $i = \{2, 3, 4, 5\}$. Thus the number of edges in G is

$$\begin{aligned}
|E(G)| \leq 5 &+ 2(|V_2||V_3| + |V_3||V_4| + |V_3||V_5| + |V_4||V_5|) \\
&+ |V_2||V_4| + |V_2||V_5| + 5|V_2| + 4(|V_3| + |V_4| + |V_5|) \\
&+ \binom{|V_2|}{2} + \binom{|V_3|}{2} + \binom{|V_4|}{2} + \binom{|V_5|}{2} \\
= 5 &+ |V_2||V_3| + |V_3||V_4| + |V_3||V_5| + |V_4||V_5| \\
&+ 5|V_2| + 4(|V_3| + |V_4| + |V_5|) + \binom{n-3}{2}.
\end{aligned}$$

To get the maximum of $|E(G)|$, let $f(x, y, z, w) = 5 + xy + yz + yw + zw + 5x + 4(y + z + w) + \binom{n-3}{2}$, such that $x + y + z + w = n - 3$. Using the Lagrangian method, $f(x, y, z, w)$ achieves maximum $f(x, y, z, w) = \frac{3n^2}{4} - \frac{n}{2} - 1$ at $x = \frac{n}{2} - 3, y = \frac{n}{2} - 2, z = w = 1$. By taking $c \geq 1/4$, we have $f(x, y, z, w) \leq \frac{3}{2}\binom{n}{2} + cn$. The result follows.

$\square$

Corollary 4.1.1. *Let* $K_3^- = ([3], \{12, 13, 23\}, \{12, 13\})$, *then* $\pi(K_3^-) = \frac{3}{2}$.

Proof. Since K_3^- is a sub-graph of K_3, then $\pi(K_3^-) \leq \frac{3}{2}$. By Lemma 4.1.1, $\pi(K_3^-) \geq \frac{3}{2}$. The result follows. $\square$

Except the 2-colored non-bipartite graph, some bipartite graphs also achieves $\pi(H) = \frac{3}{2}$. See the following 2-colored graph on four vertices $\{1, 2, 3, 4\}$:

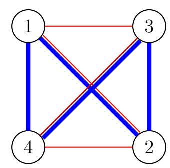

Figure 4.3
2-colored
graph T_1.

Lemma 4.1.3. *[4]* $T_1 = ([4], \{12, 34, 13, 24\}, \{12, 34, 14, 23\}$. *Then* $\pi(T_1) = \frac{3}{2}$.

Proof. Since T_1 is not contained in G_C, we have $\pi(T_1) \geq \frac{3}{2}$. Now we prove the other direction. Let n be a positive integer and G be any T_1-free 2-colored graph on n vertices. We will prove $h_n(G) \leq \frac{3}{2}$ by induction on n, i.e. prove $|E(G)| \leq \frac{3}{2}\binom{n}{2} + cn$ for some finite number c which depends on n. The cases for $n \leq 4$ are trivial. Let $n \geq 5$. We assume the statement holds for any T_1-free graphs on less than n vertices. With same reason as in Lemma 4.1.2, we can assume G contains at least one double-colored edge. Then G is one of the following cases.

Case 1: G doesn't contain K_3 as a subgraph, by Lemma 4.1.2 $h_n(G) \leq \frac{3}{2}$.

Case 2: G contains a isomorphic subgraph K_3, let $V_1 = \{a, b, c\}$ be the vertices of this triangle and $V_2 = V(G) \setminus V_1$. Then there are at most 4 edges from any vertex in V_2 to V_1. To see this, suppose there are 5 edges from the vertex $w \in V_2$

to V_1, then there are only two possible graphs on $V_1 \cup \{w\}$ and each of them contains a copy of T_1. A contradiction.

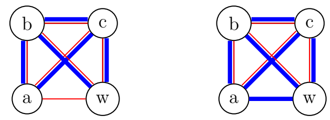

Figure 4.4 2-colored graphs for Proof 4.1 Case 2.

Applying the inductive hypothesis to $G[V_2]$, we have

$$|E(G[V_2])| \leq \frac{3}{2}\binom{|V_2|}{2} + c'|V_2|,$$

for some finite constant c'. Then the number of edges in G is :

$$\begin{aligned} |E(G)| &= |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)| \\ &\leq 6 + \frac{3}{2}\binom{|V_2|}{2} + c'|V_2| + 4|V_2|. \end{aligned} \tag{4.1}$$

While

$$\begin{aligned} \frac{3}{2}\binom{n}{2} + cn &= \frac{3}{2}\binom{|V_1| + |V_2|}{2} + cn \\ &= 3 \times \frac{3}{2} + 3c + \frac{3}{2}\binom{|V_2|}{2} + c|V_2| + \frac{3}{2} \times 3 \times |V_2| \\ &= \frac{9}{2} + 3c + \frac{3}{2}\binom{|V_2|}{2} + c|V_2| + \frac{9}{2}|V_2|. \end{aligned} \tag{4.2}$$

By taking $c = c'$, then (4.1) is always less than (4.2). The induction step is finished. It follows that $h_n(G) \leq \frac{3}{2}$. Therefore, $\pi(T_1) = \frac{3}{2}$. $\square$

Proof of Lemma 4.1.1. For Item 1, let H be a 2-colored non-bipartite graph, without loss of generality, assume H contains a triangle with three red edges. For n large enough, let G be a 2-colored graph on n vertices such that G_b is a complete graph and G_r is a balanced complete bipartite graph both on n vertices. We have $h_n(G) = \frac{3}{2} + o_n(1)$. Note that G is H-free. Thus $\pi(H) \geq \frac{3}{2}$.

For Item 2, it is sufficient to prove that any 2-colored bipartite graph H is T_1-colorable. For any 2-colored bipartite graph H, the sub-graph H_r can be partitioned into two disjoint parts $V_1(H_r)$ and $V_2(H_r)$ such that the red edges form a bipartite graph between $V_1(H_r)$ and $V_2(H_r)$. Similarly for the sub-graph H_b, the blue edges form a bipartite graph between $V_1(H_b)$ and $V_2(H_b)$. Let S be the set of vertices incidents to double colored edges, then S can be divided into four classes: $V_1(H_r) \cap V_1(H_b), V_1(H_r) \cap V_2(H_b), V_2(H_r) \cap V_1(H_b)$ and $V_2(H_r) \cap V_2(H_b)$. We define a map $f : V(H) \to \{1, 2, 3, 4\}$ as follows:

1. If $v \in V_1(H_r) \cap V_1(H_b)$, set $f(v) = 1$.
2. If $v \in V_1(H_r) \cap V_2(H_b)$, set $f(v) = 4$.
3. If $v \in V_2(H_r) \cap V_1(H_b)$, set $f(v) = 3$.
4. If $v \in V_2(H_r) \cap V_2(H_b)$, set $f(v) = 2$.
5. If $uv \in E_r(H) \setminus E_b(H)$, set $f(u) = 1, f(v) = 2$.
6. If $uv \in E_b(H) \setminus E_r(H)$, set $f(u) = 3, f(v) = 4$.

One can verify that this map f is a graph homomorphism from H to T_1. Hence, $\pi(H) \leq \frac{3}{2}$. □

4.1.1 The degenerate 2-colored graphs

In this part, we will determine the degenerate 2-colored graphs. Recall Example 4.1.1 the 2-colored bipartite graph $T = (\{1, 2, 3, 4\}, \{12, 13, 34\}, \{12, 23, 34\})$, it plays an important role.

Lemma 4.1.4. *[4] Let n be a positive integer, for any T-free 2-colored graph G on n vertices, G has at most $\binom{n+1}{2}$ edges. Thus T is degenerate.*

Proof. We will prove this lemma by induction on n. It is trivial for $n = 1, 2, 3, 4$. Assume $n \geq 5$. We assume that the statement holds for any T-free 2-colored graphs on less than n vertices.

Let $G = (V, E_r, E_b)$ be a T-free 2-colored graph on n vertices. We also assume G contains at least one double-colored edge uv, or else $|E_r(G)| + |E_b(G)| \leq \binom{n}{2} < \binom{n}{2} < \binom{n+1}{2}$. Then G is one of the following cases.

Case 1: There exists a vertex w so that both uw and vw are double-colored edges. Since G is T-free, there is no double-colored edges from u, v, w to the rest of the vertices. By inductive hypothesis, when G is restricted to the complement set of $\{u, v, w\}$, the number of edges of $G[V \setminus \{u, v, w\}]$ is at most $\binom{n-2}{2}$ edges. Thus, G has at most

$$6 + 3(n-3) + \binom{n-2}{2} = \binom{n+1}{2}.$$

Case 2: Now we assume no such w exists. Let $X = \{x \in V : |E(\{x\}, \{u, v\})| \geq 3\}$. That is, for each vertex $x \in X$, x has exactly 3 edges connecting to u and v. Since G is T-free, for each $x \in X$, x has no double-colored edges to any vertex not in $\{u, v, x\}$. In particular, the induced sub-graph $G[X]$ of G has no double-colored edge. Let $V_1 = \{u, v\} \cup X$ and V_2 be the complement set. Then the induced sub-graph $G[V_1]$ has at most

$$2 + 3|X| + \binom{|X|}{2} < \binom{|X|+3}{2} = \binom{|V_1|+1}{2}$$

edges. Applying the inductive hypothesis to $G[V_2]$, then $G[V_2]$ has at most $\binom{|V_2|+1}{2}$ edges. Note that all edges from X to V_2 are single colored and the number of edges from $\{u, v\}$ to each vertex in V_2 is at most 2. Thus the total number of edges from V_1 to V_2 is at most $|V_1||V_2|$ edges. Combining these facts together, we have G has at most N edges, where

$$N = \binom{|V_1|+1}{2} + |V_1||V_2| + \binom{|V_2|+1}{2} = \binom{|V|+1}{2}.$$

We finish the inductive step. Then we have

$$\pi(T) = \lim_{n\to\infty} \max_{G_n} h_n(G_n) \le \lim_{n\to\infty} \frac{\binom{n+1}{2}}{\binom{n}{2}} = 1,$$

implying $\pi(T) = 1$. T is degenerate. $\square$

Proof of Item 1 of Theorem 4.0.2. Assume H is a degenerate 2-colored graph, then it must be G_A and G_B-colorable. By Lemma 2.2.4, it must be $G_A \times G_B$-colorable. Note that the product of this two graphs is T-colorable. Thus H is T-colorable. By Lemma 4.1.4, the result follows. $\square$

Remark 4.1.1. *Above proof implies that for any* 2*-colored graph* H *with* $\pi(H) < \frac{4}{3}$*, then* $\pi(H) = 1$*.*

4.1.2 Non-degenerate 2-colored bipartite graphs

In this part, we will further classify the non-degenerate 2-colored bipartite graphs. By Lemma 4.1.1, the largest possible Turán density of a 2-colored bipartite graph H is $\frac{3}{2}$, so if $\pi(H) < \frac{3}{2}$, it must be contained in the construction G_c and its variations, thus it must be colored by the product of three constructions:

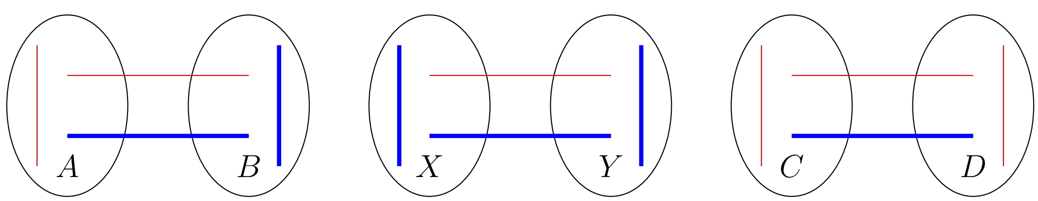

Figure 4.5 Construction G_c and its variations.

The product of above three graphs is a blow-up of following graph H_8. Let ACX stand for vertex in $A \cap C \cap X$, similar for other labels:

To compute the Turán density of H_8, we need the following 2-colored graph $T_2 = ([4], \{12, 14, 23, 24, 34\}, \{12, 13, 14, 23, 34\})$. T_2 is not contained in a variation of G_C, thus $\pi(T_2) \ge \frac{3}{2}$.

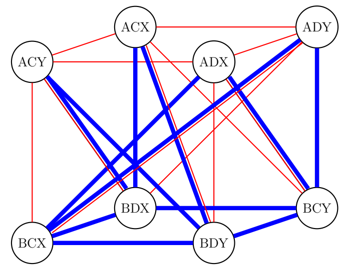

Figure 4.6 2-colored graph H_8.

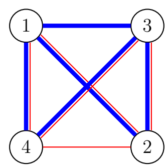

Figure 4.7 2-colored graph T_2.

Lemma 4.1.5. *[4]* $\pi(\{T_1, T_2\}) \leq \frac{4}{3}$.

Proof. For any positive integer n, let G be a $\{T_1, T_2\}$-free 2-colored graph on n vertices. We will prove $h_n(G) \leq \frac{4}{3}$ by induction on n, i.e. $E(G) \leq \frac{3}{2}\binom{n}{2} + cn$ for some finite number c. It is not hard to check the cases for $n < 6$. Assume $n \geq 6$ and the statement holds for any $\{T_1, T_2\}$-free graph on less than n vertices. With same reason as in Lemma 4.1.2, we can assume G contains at least one double-colored edge. Then G is one of the following cases.

Case 1: G contains a triangle consisting of three double-colored edges, let $V_1 = \{a, b, c\}$ be the vertices of this triangle and $V_2 = V(G) \setminus V_1$. By Lemma 4.1.3 "Case 2", for any vertex $w \in V_2$, there are at most 4 edges form w to V_1.

Case 2: G contains $V_1 = \{a, b, c\}$ such that $|E(G[V_1])| = 5$. Let $V_2 = V(G) \setminus V_1$. For any vertex $w \in V_2$, there are at most 4 edges to V_1. One can check the following graphs with 5 edges from w to V_1 contain T_1, T_2 and T_1 respectively.

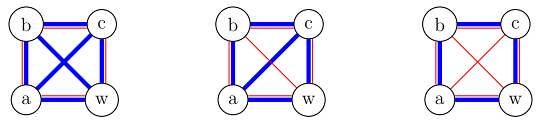

Figure 4.8 2-colored graphs for Proof 4.1.2 Case 2.

Case 3: G contains two incident double-colored edges ab and bc, but no edge connecting a and c. Let $V_1 = \{a, b, c\}$, $V_2 = V(G) \setminus V_1$. Then there cannot be 5 edges from any vertex $w \in V_2$ to V_1, otherwise, G is a graph either in Case 1 or in Case 2. Thus there are at most 4 edges from any vertex in V_2 to V_1.

Case 4: If G is not the above three cases, then for any double-colored edge connecting a and b, there are at most 2 edges from any other vertex to $\{a, b\}$.

For the first three cases, applying the inductive hypothesis to $G[V_2]$, we have

$$|E(G[V_2])| \leq \frac{4}{3}\binom{|V_2|}{2} + c'|V_2|,$$

for some finite constant c'.

Then the number of edges in G is:

$$\begin{aligned} |E(G)| &= |E(G[V_1])| + |E(G[V_2])| + |E(V_1, V_2)| \\ &\leq 6 + \frac{4}{3}\binom{|V_2|}{2} + c'|V_2| + 4|V_2|. \end{aligned} \tag{4.3}$$

While

$$\begin{aligned} \frac{4}{3}\binom{n}{2} + cn &= \frac{4}{3}\binom{|V_1| + |V_2|}{2} + cn \\ &= \frac{4}{3} \times 3 + 3c + \frac{4}{3}\binom{|V_2|}{2} + c|V_2| + \frac{4}{3} \times 3|V_2| \\ &= 4 + 3c + \frac{4}{3}\binom{|V_2|}{2} + c|V_2| + 4|V_2|. \end{aligned} \tag{4.4}$$

By taking a finite number $c \geq c' + 1$, then (4.3) is always less than (4.4).

For Case 4,

$$|E(G)| \leq 2 + \frac{4}{3}\binom{n-2}{2} + c(n-2) + 2(n-2) \leq \frac{4}{3}\binom{n}{2} + cn \tag{4.5}$$

for any $c > 0$.

Hence, we conclude that $h_n(G) \leq \frac{4}{3}$. The induction step is finished. □

Lemma 4.1.6. *[4]* $\pi(H_8) = \frac{4}{3}$.

Proof. We first prove $\pi(H_8) \leq \frac{4}{3}$. To show this, we prove that H_8 is T_1 and T_2-colorable, i.e there are graph homomorphisms from H_8 to T_1 and from H_8 to T_2.

For T_1: We define a map f by $f(ACX) = f(BCX) = 4, f(ADY) = f(BDY) = 3, f(ACY) = f(BCY) = 2, f(ADX) = f(BDX) = 1$. One can check that f is a graph homomorphism from H_8 to T_1.

For T_2: We define a map g by $g(ACX) = g(ADX) = 1, g(ADY) = g(ACY) = 3, g(BDX) = g(BDY) = 2, g(BCY) = g(BCX) = 4$. It is easy to check that g is a graph homomorphism from H_8 to T_2.

For any positive integer n, let G_n be a 2-colored graph on n vertices such that $h_n(G_n) \geq \pi(T_1, T_2) + \epsilon = \pi(T_1(s), T_2(s)) + \epsilon$, for any $s \geq 2$ and $\epsilon > 0$. Then G_n contains $T_1(s)$ or $T_2(s)$ as subgraph, further G_n contains H_8 as subgraph. Then $\pi(H_8) \leq \pi(\{T_1, T_2\})$. By Lemma 4.1.5, $\pi(H_8) \leq \frac{4}{3}$. By Remark 4.1.1, if $\pi(H_8) < \frac{4}{3}$, then $\pi(H_8) = 1$, while H_8 is not T-colorable, a contradiction. Thus it must be the case $\pi(H_8) = \frac{4}{3}$. □

Remark 4.1.2. *As we know, if $\pi(H) < \frac{3}{2}$, it must be colorable by G_c and its variations, then it must be be colorable by H_8 according to Lemma 2.2.4. Thus $\pi(H) \leq \frac{4}{3}$.*

For convenience, we use numbers to represent vertices: $ACX = 1, ADY = 2, ACY = 3, ADX = 4, BDX = 5, BCY = 6, BCX = 7, BDY = 8$. Then H_8

has edges:

$$E_r(H_8) = \{12, 13, 24, 34, 16, 37, 48, 25, 35, 18, 46, 27\};$$

$$E_b(H_8) = \{56, 57, 68, 78, 26, 15, 47, 38, 35, 18, 46, 27\}.$$

Now we are ready to finish the proof of Theorem 4.0.2.

Proof of Items 2 and 3 in Theorem 4.0.2. By Remark 4.1.1, Remark 4.1.2 and Lemma 4.1.1, the Turán densities of all bipartite 2-colored graphs are in the set $\left\{1, \frac{4}{3}, \frac{3}{2}\right\}$. To show Item 2, let H be a 2-colored graph with $\pi(H) = \frac{4}{3}$, then H must be H_8-colorable. One can check if H does not contain T as a sub-graph, then H must be T-colorable, implying $\pi(H) = 1$, a contradiction. By excluding the bipartite 2-colored graphs in Item 2, we obtain the result in Item 3. $\square$

Example 4.1.2. *Let T_3 be the following 2-colored graph, T_3 is non-degenerate and* $\pi(T_3) = \frac{4}{3}$.

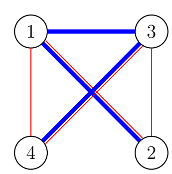

Figure 4.9
2-colored
graph T_3.

4.2 Application on the degenerate $\{2,3\}$-graphs

In this section, we study the degenerate $\{2,3\}$-graphs and show an application of the study of 2-edge-colored graphs on Turán density of $\{2,3\}$-graphs.

To show that an R-graph H is not degenerate, recall the method in Chapter 3 is to construct a family of H-free R-graphs G_n with $h_n(G_n) > (1+\epsilon)$ for some $\epsilon > 0$. Here are three $\{2,3\}$-graphs with edge density greater than 1.

Constructions:

$G_1^{\{2,3\}}$: A $\{2,3\}$-graph $G_1^{\{2,3\}}$ is generated by the general hypergraph H_1 with vertex set $\{a,b,c\}$ and edge set $\{aa, bb, cc, abc\}$, if there exists a partition of vertex set such that $V(G_1^{\{2,3\}}) = A \cup B \cup C$ and every 2-edge meets two vertices in A (or B, or C), every 3-edge meets A, B, C one vertex respectively. Actually $G_1^{\{2,3\}}$ is H_1-colorable. We express the edges of $G_1^{\{2,3\}}$ simply by

$$E(G_1^{\{2,3\}}) = \binom{A}{2} \cup \binom{A}{1}\binom{B}{1} \cup \binom{A}{1}\binom{C}{1} \cup \binom{A}{1}\binom{B}{1}\binom{C}{1}.$$

To calculate $h_n(G_1^{\{2,3\}})$, let $|A| = xn$ and $|B| = |C| = \frac{1-x}{2}n$ for some value $x \in (0,1)$. We have

$$\begin{aligned} h_n(G_1^{\{2,3\}}) &= \frac{\binom{xn}{2} + \binom{xn}{1}\binom{(1-x)n}{1}}{\binom{n}{2}} + \frac{xn(\frac{(1-x)n}{2})^2}{\binom{n}{3}} \\ &= x^2 + 2x(1-x) + \frac{3}{2}x(1-x)^2 + o_n(1) \\ &= \frac{7}{2}x - 4x^2 + \frac{3}{2}x^3 + o_n(1). \end{aligned}$$

The above value reaches the maximum value $\frac{245}{243} + o_n(1)$ at $x = \frac{7}{9}$.

$G_2^{\{2,3\}}$: A $\{2,3\}$-graph $G_2^{\{2,3\}}$ is generated by the general hypergraph H_2 with vertex set $\{x,y\}$ and edge set $\{xy, xxx, xxy\}$, if there exists a partition of vertex set such that $V(G_2^{\{2,3\}}) = X \cup Y$ and every 2-edge meets one vertex in X and one vertex in Y, every 3-edge either meet three vertices in X or two vertices in X plus one vertex in Y. Actually $G_2^{\{2,3\}}$ is H_2-colorable. We express the edges of $G_2^{\{2,3\}}$ simply by

$$E(G_2^{\{2,3\}}) = \binom{X}{3} \cup \binom{X}{2}\binom{Y}{1} \cup \binom{X}{1}\binom{Y}{1}.$$

We have

$$h_n(G_2^{\{2,3\}}) = \frac{\binom{xn}{3} + \binom{xn}{2}\binom{(1-x)n}{1}}{\binom{n}{3}} + \frac{xn(1-x)n}{\binom{n}{2}}$$
$$= x^3 + 3x^2(1-x) + 2x(1-x) + o_n(1)$$
$$= 2x + x^2 - 2x^3 + o_n(1).$$

The above value reaches the maximum value $\frac{19+13\sqrt{13}}{54}+o_n(1) \approx 1.21985...+o_n(1)$ at $x = \frac{1+\sqrt{13}}{6}$.

$G_3^{\{2,3\}}$: A $\{2,3\}$-graph $G_3^{\{2,3\}}$ is generated by the general hypergraph H_3 with vertex set $\{e, f\}$ and edge set $\{ee, eef\}$, if there exists a partition of vertex set such that $V(G_2^{\{2,3\}}) = E \cup F$ and every 2-edge meets two vertices in E, every 3-edge meets two vertices in E plus one vertex in F. Actually $G_3^{\{2,3\}}$ is H_3-colorable. We express the edges of $G_3^{\{2,3\}}$ simply by

$$E(G_3^{\{2,3\}}) = \binom{E}{2} \cup \binom{E}{2}\binom{Y}{1}.$$

We have

$$h_n(G_3^{\{2,3\}}) = \frac{\binom{xn}{2}}{\binom{n}{2}} + \frac{\binom{xn}{2}\binom{(1-x)n}{1}}{\binom{n}{3}}$$
$$= x^2 + 3x^2(1-x) + o_n(1)$$
$$= 4x^2 - 3x^3 + o_n(1).$$

The above value reaches the maximum value $\frac{256}{243} + o_n(1)$ at $x = \frac{8}{9}$.

A degenerate $\{2,3\}$-graph must appear as sub-graphs in all above $\{2,3\}$-graphs $G_1^{\{2,3\}}, G_2^{\{2,3\}}$ and $G_3^{\{2,3\}}$, thus it must appear as sub-graph in the product of these hypergraphs. By taking this product, we get a 12-vertex $\{2,3\}$-graph which is $H_9^{\{2,3\}}$-colorable. Thus we have

Lemma 4.2.1. *[4] The degenerate $\{2,3\}$-graphs must be $H_9^{\{2,3\}}$-colorable.*

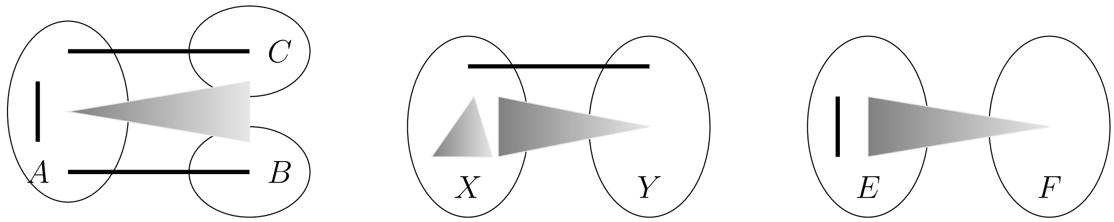

Figure 4.10 $G_1^{\{2,3\}}, G_2^{\{2,3\}}$ and $G_3^{\{2,3\}}$ with $h_n(G_1^{\{2,3\}}) = \frac{245}{243}$ at $|A| = \frac{7}{9}n$, $h_n(G_2^{\{2,3\}}) \approx 1.21985$ at $|X| = (\frac{1+\sqrt{13}}{6})n$ and $h_n(G_3^{\{2,3\}}) = \frac{256}{243}$ at $|E| = \frac{8}{9}n$.

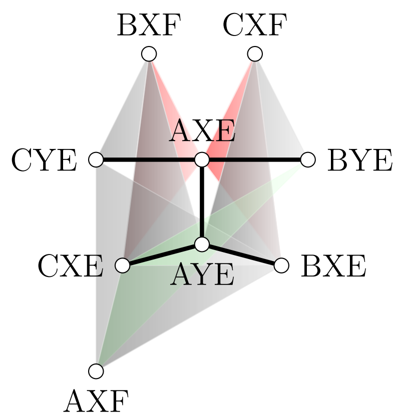

Figure 4.11 $\{2,3\}$-graph $H_9^{\{2,3\}}$.

If we remove vertex AXF and edges connecting to it, the resulting hypergraph is $H_5^{\{2,3\}}$-colorable, where $H_5^{\{2,3\}} = ([5], \{12, 13, 34, 125, 135, 345\})$.

Note there exists a single vertex $5 \in V(H_5^{\{2,3\}})$ connecting all 3-edges of $H_5^{\{2,3\}}$.

Proof of 4.0.3. Let n be positive integer, let $G = (V,\ E_2,\ E_3)$ be a H'-free $\{2,3\}$-graph on n vertices. For any vertex $v \in V(G)$, let $G_v = (V(G) \setminus \{v\}, E_{v,2}, E_{v,3})$ be a 2-colored graph obtained form G, such that, the set of red edges $E_{v,2}$ consists of 2-edges in G_v, the set of blue edges $E_{v,3}$ consists of pairs u, w in G_v so that $\{vuw\} \in E_3$ of G. Observe that G_v is H-free since $G = (V,\ E_2,\ E_3)$ is H'-free. Thus $h_n(G_v) \leq \pi_n(H)$.

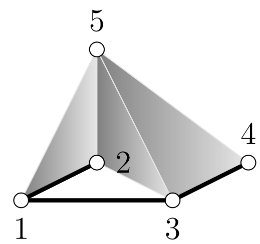

Figure 4.12
$\{2,3\}$-graph
$H_5^{\{2,3\}}$.

Since

$$|E_2| = \frac{1}{n-2}\sum_{v\in V(G)} |E_{v,2}|,\ |E_3| = \frac{1}{3}\sum_{v\in V(G)} |E_{v,3}|,$$

Then

$$\begin{aligned}
h_n(G) &= \frac{|E_2|}{\binom{n}{2}} + \frac{|E_3|}{\binom{n}{3}} \\
&= \sum_{v\in V(G)} \frac{|E_{v,2}|}{(n-2)\binom{n}{2}} + \sum_{v\in V(G)} \frac{|E_{v,3}|}{3\binom{n}{3}} \\
&= \frac{1}{n}\sum_{v\in V(G)} \frac{|E_{v,2}|}{\binom{n-1}{2}} + \frac{1}{n}\sum_{v\in V(G)} \frac{|E_{v,3}|}{\binom{n-1}{2}} \\
&= \frac{1}{n}\sum_{v\in V(G)} \left(\frac{|E_{v,2}|}{\binom{n-1}{2}} + \frac{|E_{v,3}|}{\binom{n-1}{2}}\right) \\
&\le \frac{1}{n}\sum_{v\in V(G)} h_n(G_v) \\
&\le \pi(H).
\end{aligned}$$

Therefore $\pi(H') \le \pi(H)$. $\square$

Observe that we can obtain $H_5^{\{2,3\}}$ from the 2-colored graph T by adding vertex 5, and connect it with blue edges to form 3-edges. Thus we have $\pi(H_5^{\{2,3\}}) = 1$, $H_5^{\{2,3\}}$ is a degenerate $\{2,3\}$-graph. So far we couldn't give an upper bound of $\pi(H_9^{\{2,3\}})$, we make the following conjecture:

Conjecture 4.2.1. *A* $\{2,3\}$*-graph is degenerate if and only if it is* $H_9^{\{2,3\}}$*-colorable.*

Chapter 5

Spectral Radius of Hypergraphs

In this chapter, we will focus on the spectral radius of uniform hypergraph with fixed number of edges. Note that the spectral radius of the complete hypergraph K_n^r is $\binom{n-1}{r-1}$. This motivated us to define an analytic function $f_r \colon [0, \infty) \to [1, \infty)$ so that

$$f_r\left(\binom{n}{r}\right) = \binom{n-1}{r-1}. \tag{5.1}$$

We will prove the following result.

Theorem 5.0.1. *[5] For $r \geq 2$, suppose that H is an r-uniform hypergraph with e edges. Then its spectral radius $\rho(H)$ is at most $f_r(e)$. The equality holds if and only if $e = \binom{k}{r}$ for an integer k and H is the complete r-uniform hypergraph K_k^r possibly with some isolated vertices added.*

Note that $f_2(x)$ satisfies $f_2(\binom{n}{2}) = n - 1$. Let $e = \binom{n}{2}$ and solve for n. We get

$$f_2(e) = \frac{\sqrt{8e+1} - 1}{2}. \tag{5.2}$$

Stanley's theorem is just a special case with $r = 2$.

5.1 Properties of the function $f_r(x)$

We first see the formal definition and some properties on function $f_r(x)$. For a fixed positive integer r, consider the polynomial $p_r(x) = \frac{x(x-1)\cdots(x-r+1)}{r!}$. Since the binomial coefficient $\binom{n}{r} = p_r(n)$, we view $\binom{x}{r}$ as the polynomial $p_r(x)$. Note that $p_r(x)$ is an increasing function over the interval $[r-1, \infty)$ so that the inverse function exists. Let

$p_r^{-1}\colon [0,\infty) \to [r-1,\infty)$ denote the inverse function of $p_r(x)$ (when restricted to the interval $[r-1,\infty)$. We define a function $f_r\colon [0,\infty) \to [1,\infty)$ as follows:

$$f_r(x) := p_{r-1}(p_r^{-1}(x) - 1).$$

Thus $f_r(x)$ satisfies Equation (5.1). This function plays an essential role in this chapter. It has the following properties.

Lemma 5.1.1. *[5] Suppose $y = \binom{\eta}{r}$ with $\eta \geq r-1$. Then we have*

1. *$\eta = \frac{ry}{f_r(y)}$.*
2. *$f_r(y)$ is an increasing function on $[0,\infty)$.*
3. *The derivative of $f_r(y)$ is given by:*

$$f_r'(y) = \frac{r}{\eta} \frac{\sum_{i=1}^{r-1} \frac{1}{\eta - i}}{\sum_{i=0}^{r-1} \frac{1}{\eta - i}}.$$

Proof. The formula $f_r(y) = \binom{\eta-1}{r-1} = \frac{r}{\eta}\binom{\eta}{r}$ implies item 1. Note

$$\ln p_r(x) = \sum_{j=0}^{r-1} \ln(x-j) - \ln(r!).$$

We have

$$p_r(x)' = p_r(x) \cdot \frac{d}{dx}\left[\sum_{j=0}^{r-1} \ln(x-j) - \ln(r!)\right] = \binom{x}{r} \sum_{j=0}^{r-1} \frac{1}{x-j}.$$

View η as a function of y and apply the Chain rule. We have

$$f_r'(y) = \binom{\eta-1}{r-1}' = \frac{df_r}{d\eta}\frac{d\eta}{dy} = \frac{\frac{df_r}{d\eta}}{\frac{dy}{d\eta}} = \frac{\binom{\eta-1}{r-1}\sum_{i=1}^{r-1}\frac{1}{\eta-i}}{\binom{\eta}{r}\sum_{i=0}^{r-1}\frac{1}{\eta-i}} = \frac{r}{\eta}\frac{\sum_{i=1}^{r-1}\frac{1}{\eta-i}}{\sum_{i=0}^{r-1}\frac{1}{\eta-i}}.$$

Since $\eta > r-1$, the right-hand side of $f_r'(y)$ is positive. Thus $f_r(y)$ is an increasing function. $\square$

For convenience, we also define f_1 to be the constant function $f_1(x) \equiv 1$.

Lemma 5.1.2. *[5] For an integer $r \geq 2$ and any two reals e and x with $e \geq x \geq f_r(e)$, we have*

$$\frac{x^{1/(r-1)} f_{r-1}(x)}{f_r(e)^{r/(r-1)}} + \frac{f_r(e-x)}{f_r(e)} \leq 1. \tag{5.3}$$

Proof. Let $F(x) = \frac{x^{1/(r-1)} f_{r-1}(x)}{f_r(e)^{r/(r-1)}} + \frac{f_r(e-x)}{f_r(e)}$. Note $F(x)$ is a smooth function. To show $F(x) \leq 1$ for all $x \in [f_r(e), e]$, it is sufficient to prove the following facts:

1. $F(f_r(e)) = 1$.

2. $F'(f_r(e)) < 0$.

3. $F''(x) \leq 0$ for any $x \in [f_r(e), e]$.

Note that item 3 indicates that $F'(x)$ is a decreasing function on $[f_r(e), e)$, together with item 2, we get $F'(x) < 0$, implies that $F(x)$ is a strictly decreasing function on $[f_r(e), e]$. By item 1, we conclude that $F(x) \leq 1$ for $x \in [f_r(e), e]$, with the inequality holds if and only if $x = f_r(e)$.

Let s, t, and u be three positive reals satisfying $e = \binom{s}{r}$, $x = \binom{t}{r-1}$, and $e - x = \binom{u}{r}$. Since $x = \binom{t}{r-1} \geq f_r(e) = \binom{s-1}{r-1}$, we have $t \geq s - 1$. Similarly, we have

$$\binom{u}{r} = e - x = \binom{s}{r} - \binom{t}{r-1} \leq \binom{s}{r} - \binom{s-1}{r-1} = \binom{s-1}{r}.$$

It implies that $u \leq s - 1$. Thus, we have

$$t \geq s - 1 \geq u, \tag{5.4}$$

with the equality holds if and only if $x = f_r(e) = \binom{s-1}{r-1}$.

We have

$$
\begin{aligned}
F(f_r(e)) &= \frac{\binom{s-1}{r-1}^{1/(r-1)}\binom{s-2}{r-2}}{\binom{s-1}{r-1}^{r/(r-1)}} + \frac{\binom{s-2}{r-1}\binom{s-1}{r-1}^{1/(r-1)}}{\binom{s-1}{r-1}^{r/(r-1)}} \\
&= \frac{\binom{s-1}{r-1}^{1/(r-1)}\left[\binom{s-2}{r-2}+\binom{s-2}{r-1}\right]}{\binom{s-1}{r-1}^{r/(r-1)}} \\
&= \frac{\binom{s-1}{r-1}^{1/(r-1)}\binom{s-1}{r-1}}{\binom{s-1}{r-1}^{r/(r-1)}} \\
&= 1.
\end{aligned}
$$

Proof of item 1 is finished.

Now we compute the derivative of $F(x)$. Note that e and s are constants while t and u are functions of x. Applying item 1 of Lemma 5.1.1 to $x = \binom{t}{r-1}$ and $x = -\binom{u}{r} + e$, we get

$$
\begin{aligned}
f'_{r-1}(x) &= \frac{(r-1)\sum_{i=1}^{r-2}\frac{1}{t-i}}{t\sum_{i=0}^{r-2}\frac{1}{t-i}}, \\
f'_r(e-x) &= -\frac{r\sum_{i=1}^{r-1}\frac{1}{u-i}}{u\sum_{i=0}^{r-1}\frac{1}{u-i}}.
\end{aligned}
$$

Thus, we have

$$
\begin{aligned}
F'(x) =& \frac{\frac{1}{r-1}x^{1/(r-1)-1}f_{r-1}(x) + x^{1/(r-1)}\frac{(r-1)\sum_{i=1}^{r-2}\frac{1}{t-i}}{t\sum_{i=0}^{r-2}\frac{1}{t-i}}}{f_r(e)^{r/(r-1)}} - \frac{r\sum_{i=1}^{r-1}\frac{1}{u-i}}{f_r(e)u\sum_{i=0}^{r-1}\frac{1}{u-i}} \\
=& \frac{\frac{1}{r-1}x^{1/(r-1)-1}\frac{x(r-1)}{t} + x^{1/(r-1)}\frac{(r-1)\sum_{i=1}^{r-2}\frac{1}{t-i}}{t\sum_{i=0}^{r-2}\frac{1}{t-i}}}{f_r(e)^{r/(r-1)}} - \frac{r}{f_r(e)}\frac{\sum_{i=1}^{r-1}\frac{1}{u-i}}{u\sum_{i=0}^{r-1}\frac{1}{u-i}} \\
=& \frac{x^{1/(r-1)}}{f_r(e)^{r/(r-1)}}\left(\frac{1}{t} + \frac{(r-1)\sum_{i=1}^{r-2}\frac{1}{t-i}}{t\sum_{i=0}^{r-2}\frac{1}{t-i}}\right) - \frac{r}{f_r(e)}\frac{\sum_{i=1}^{r-1}\frac{1}{u-i}}{u\sum_{i=0}^{r-1}\frac{1}{u-i}} \\
=& \frac{x^{1/(r-1)}}{f_r(e)^{r/(r-1)}}\left[\frac{r}{t} - \frac{r-1}{t^2\sum_{i=0}^{r-2}\frac{1}{t-i}}\right] - \frac{1}{f_r(e)}\left[\frac{r}{u} - \frac{r}{u^2\sum_{i=0}^{r-1}\frac{1}{u-i}}\right].
\end{aligned}
$$

When $x = f_r(e)$, by equations $t = u = s - 1$, we replace u by t for convenience.

$$
\begin{aligned}
F'(f_r(e)) &= \frac{f_r(e)^{1/(r-1)}}{f_r(e)^{r/(r-1)}} \left[\frac{r}{t} - \frac{r-1}{t^2 \sum_{i=0}^{r-2} \frac{1}{t-i}}\right] - \frac{1}{f_r(e)} \left[\frac{r}{t} - \frac{r}{t^2 \sum_{i=0}^{r-1} \frac{1}{t-i}}\right] \\
&= \frac{1}{f_r(e)} \left[\frac{r}{t} - \frac{r-1}{t^2 \sum_{i=0}^{r-2} \frac{1}{t-i}} - \frac{r}{t} + \frac{r}{t^2 \sum_{i=0}^{r-1} \frac{1}{t-i}}\right] \\
&= \frac{1}{f_r(e)} \left[\frac{1}{t^2} \left(\frac{r}{\sum_{i=0}^{r-1} \frac{1}{t-i}} - \frac{r-1}{\sum_{i=0}^{r-2} \frac{1}{t-i}}\right)\right] \\
&= \frac{1}{f_r(e) t^2} \frac{\sum_{i=0}^{r-2} \left(\frac{1}{t-i} - \frac{1}{t-r+1}\right)}{(\sum_{i=0}^{r-2} \frac{1}{t-i})(\sum_{i=0}^{r-1} \frac{1}{t-i})} \\
&< 0.
\end{aligned}
$$

Proof of item 2 is finished.

Let us compute the second derivative. Since $x = \binom{t}{r-1}$, we have

$$\frac{dt}{dx} = \frac{1}{\frac{dx}{dt}} = \frac{1}{x \sum_{i=0}^{r-2} \frac{1}{t-i}}. \tag{5.5}$$

Similarly, from $e - x = \binom{u}{r}$, we get

$$\frac{du}{dx} = -\frac{1}{(e-x) \sum_{i=0}^{r-1} \frac{1}{u-i}}. \tag{5.6}$$

To simplify the above equation, we compute the derivative of each main term separately, the derivative of first main term $x^{1/(r-1)} \left[\frac{r}{t} - \frac{r-1}{t^2 \sum_{i=0}^{r-2} \frac{1}{t-i}}\right]$ is

$$
\begin{aligned}
&\left(x^{1/(r-1)} \left[\frac{r}{t} - \frac{r-1}{t^2 \sum_{i=0}^{r-2} \frac{1}{t-i}}\right]\right)' \\
&= \frac{x^{1/(r-1)-1}}{r-1} \left[\frac{r}{t} - \frac{r-1}{t^2 \sum_{i=0}^{r-2} \frac{1}{t-i}}\right] + x^{1/(r-1)} \frac{dt}{dx} \left[-\frac{r}{t^2} + (r-1) \frac{1}{t^4 (\sum_{i=0}^{r-2} \frac{1}{t-i})^2}\right. \\
&\quad \left.\left(2t \sum_{i=0}^{r-2} \frac{1}{t-i} - t^2 \sum_{i=0}^{r-2} \frac{1}{(t-i)^2}\right)\right] \\
&= \frac{x^{1/(r-1)-1}}{t^2} \left[\frac{rt}{r-1} - \frac{r+1}{\sum_{i=0}^{r-2} \frac{1}{t-i}} + \frac{2t(r-1)}{t(\sum_{i=0}^{r-2} \frac{1}{t-i})^2} - \frac{(r-1) \sum_{i=0}^{r-2} \frac{1}{(t-i)^2}}{(\sum_{i=0}^{r-2} \frac{1}{t-i})^3}\right].
\end{aligned}
$$

Similar work for the derivative of the second main term $\frac{1}{u} - \frac{1}{u^2 \sum_{i=0}^{r-1} \frac{1}{u-i}}$, we have:

$$\left(\frac{1}{u} - \frac{1}{u^2 \sum_{i=0}^{r-1} \frac{1}{u-i}}\right)'$$
$$= -\frac{1}{(e-x)u^2 \sum_{i=0}^{r-1} \frac{1}{u-i}} \left[-1 + \frac{2}{u \sum_{i=0}^{r-1} \frac{1}{u-i}} - \frac{\sum_{i=0}^{r-1} \frac{1}{(u-i)^2}}{\left(\sum_{i=0}^{r-1} \frac{1}{u-i}\right)^2}\right].$$

After simplification, we have

$$F''(x) = \frac{x^{1/(r-1)-1}}{t^2 f_r(e)^{r/(r-1)}} \left[\frac{rt}{r-1} - \frac{r+1}{\sum_{i=0}^{r-2} \frac{1}{t-i}} + \frac{2(r-1)}{t\left(\sum_{i=0}^{r-2} \frac{1}{t-i}\right)^2} - \frac{(r-1)\sum_{i=0}^{r-2} \frac{1}{(t-i)^2}}{\left(\sum_{i=0}^{r-2} \frac{1}{t-i}\right)^3}\right]$$
$$+ \frac{1}{f_r(e)(e-x)u^2 \sum_{i=0}^{r-1} \frac{1}{u-i}} \left[-r + \frac{2r}{u \sum_{i=0}^{r-1} \frac{1}{u-i}} - \frac{r\sum_{i=0}^{r-1} \frac{1}{(u-i)^2}}{\left(\sum_{i=0}^{r-1} \frac{1}{u-i}\right)^2}\right].$$

Applying these two inequalities,

$$\sum_{i=0}^{r-2} \frac{1}{(t-i)^2} \geq \frac{(r-1)}{t^2} \qquad \text{and} \qquad \sum_{i=0}^{r-1} \frac{1}{(u-i)^2} \geq \frac{r}{u^2},$$

we have

$$-\frac{1}{\sum_{i=0}^{r-2} \frac{1}{t-i}} + \frac{2(r-1)}{t\left(\sum_{i=0}^{r-2} \frac{1}{t-i}\right)^2} - \frac{(r-1)\sum_{i=0}^{r-2} \frac{1}{(t-i)^2}}{\left(\sum_{i=0}^{r-2} \frac{1}{t-i}\right)^3}$$
$$= -\frac{1}{\sum_{i=0}^{r-2} \frac{1}{t-i}} \left[1 - \frac{2(r-1)}{t\sum_{i=0}^{r-2} \frac{1}{t-i}} + \frac{(r-1)\sum_{i=0}^{r-2} \frac{1}{(t-i)^2}}{(\sum_{i=0}^{r-2} \frac{1}{t-i})^2}\right]$$
$$\leq -\frac{1}{\sum_{i=0}^{r-2} \frac{1}{t-i}} \left[1 - \frac{2(r-1)}{t\sum_{i=0}^{r-2} \frac{1}{t-i}} + \frac{(r-1)^2}{t^2(\sum_{i=0}^{r-2} \frac{1}{t-i})^2}\right]$$
$$= -\frac{1}{\sum_{i=0}^{r-2} \frac{1}{t-i}} \left[1 - \frac{r-1}{t\sum_{i=0}^{r-2} \frac{1}{t-i}}\right]^2$$
$$\leq 0,$$

and similarly

$$-1 + \frac{2r}{u\sum_{i=0}^{r-1} \frac{1}{u-i}} - \frac{r\sum_{i=0}^{r-1} \frac{1}{(u-i)^2}}{\left(\sum_{i=0}^{r-1} \frac{1}{u-i}\right)^2} \leq -\left[1 - \frac{r}{u\sum_{i=0}^{r-1} \frac{1}{u-i}}\right]^2 \leq 0.$$

Thus, we have

$$F''(x) \leq \frac{x^{1/(r-1)-1}}{t^2 f_r(e)^{r/(r-1)}} \left[\frac{rt}{r-1} - \frac{r}{\sum_{i=0}^{r-2} \frac{1}{t-i}}\right] - \frac{r-1}{f_r(e)(e-x)u^2 \sum_{i=0}^{r-1} \frac{1}{u-i}}$$
$$= \frac{x^{1/(r-1)-1}}{t^2 f_r(e)^{r/(r-1)}} \frac{r\sum_{i=0}^{r-2} \frac{i}{t-i}}{(r-1)\sum_{i=0}^{r-2} \frac{1}{t-i}} - \frac{r-1}{f_r(e)(e-x)u^2 \sum_{i=0}^{r-1} \frac{1}{u-i}}.$$

To show the right side is negative, it is sufficient to prove

$$\frac{x^{1/(r-1)-1}}{t^2 f_r(e)^{1/(r-1)}} \frac{r\sum_{i=0}^{r-2}\frac{i}{t-i}}{(r-1)\sum_{i=0}^{r-2}\frac{1}{t-i}} \leq \frac{r-1}{(e-x)u^2\sum_{i=0}^{r-1}\frac{1}{u-i}}. \tag{5.7}$$

Equivalently,

$$\frac{u^2}{t^2}\cdot\frac{e-x}{x^{1-1/(r-1)}f_r(e)^{1/(r-1)}}\cdot\frac{r}{(r-1)^2}\cdot\frac{\sum_{i=0}^{r-2}\frac{i}{t-i}}{\sum_{i=0}^{r-2}\frac{1}{t-i}}\cdot\sum_{i=0}^{r-1}\frac{1}{u-i} \leq 1. \tag{5.8}$$

Since $\frac{\sum_{i=0}^{r-2}\frac{i}{t-i}}{\sum_{i=0}^{r-2}\frac{1}{t-i}} \leq r-2$ and $\sum_{i=0}^{r-1}\frac{1}{u-i} \leq \frac{r}{u-r+1}$, it is sufficient to prove

$$\frac{u^2}{t^2}\cdot\frac{e-x}{x^{1-1/(r-1)}f_r(e)^{1/(r-1)}}\cdot\frac{r(r-2)}{(r-1)^2}\cdot\frac{r}{u-r+1} \leq 1. \tag{5.9}$$

Replacing $x = \binom{t}{r-1}$, $e - x = \binom{u}{r} = \binom{u}{r-1}\frac{u-r+1}{r}$, and $f_r(e) = \binom{s-1}{r-1}$, the left side of Equation (5.9) becomes

$$\begin{aligned}
LHS &= \frac{u^2}{t^2}\cdot\frac{e-x}{f_r(e)^{1/(r-1)}x^{1-1/(r-1)}}\cdot\frac{r(r-2)}{(r-1)^2}\cdot\frac{r}{u-r+1}\\
&= \frac{u^2}{t^2}\cdot\frac{\binom{u}{r-1}}{\binom{s-1}{r-1}^{1/(r-1)}\binom{t}{r-1}^{1-1/(r-1)}}\frac{r(r-2)}{(r-1)^2}\\
&\leq \frac{r(r-2)}{(r-1)^2}\\
&< 1.
\end{aligned}$$

The second last inequality is due to the fact (5.4) that $t \geq s-1 \geq u$. Thus, $F''(x) \leq 0$. Proof of item 3 is finished.

□

5.2 Proof of Theorem 5.0.1

Let $H = (V, E)$ be a connected r-uniform hypergraph whose spectral radius attains the maximum among all the r-uniform hypergraphs with e edges. We call H a *maximum* hypergraph. To use the α-normal labeling method, we will need the shadow graph and the link graph.

Definition 5.2.1. *Given a family $\mathcal{F}$ of r-sets, the shadow $\partial(\mathcal{F})$ is defined as*

$$\partial(\mathcal{F}) = \{e' : e' = e \setminus \{v\}, \text{for some } e \in \mathcal{F}, \text{ and } v \in e\}.$$

Definition 5.2.2. *Given an r-hypergraph H and a vertex v of H, the link graph G_v is the $(r-1)$-graph consisting of all $S \subset V(H)$ with $|S| = r-1$ and $S \cup \{v\} \in E(H)$.*

The celebrated Kruskal-Katona Theorem determines the minimum size of the shadow $\partial(\mathcal{F})$ given the size of $\mathcal{F}$.

Theorem 5.2.1. *(Kruskal [31] and Katona [30]) Any r-uniform set family $\mathcal{F}$ of size $m = \binom{a_r}{r} + \binom{a_{r-1}}{r-1} + \cdots + \binom{a_k}{k}$, where $a_r > a_{r-1} > \cdots > a_k \geq k \geq 1$, must have*

$$|\partial(F)| \geq \binom{a_r}{r-1} + \binom{a_{r-1}}{r-2} + \cdots + \binom{a_k}{k-1}.$$

Kruskal-Katona Theorem has many applications. However, it is not easy to apply directly. In this chapter, we use a slightly weaker version due to Lovász :

Theorem 5.2.2. *(Lovász [34]) Any r-uniform set family $\mathcal{F}$ of size $m = \binom{x}{r}$ where x is a real and $x \geq r$, must have*

$$|\partial(F)| \geq \binom{x}{r-1}.$$

In a hypergraph H, the degree $d(v)$ of a vertex v is the number of edges that contain v. Let H_v be the induced subgraph obtained from H by deleting the vertex v. Let G_v be the link graph of v. By definition of G_v, $d(v)$ is also the number of edges in G_v. We have the following lemma:

Lemma 5.2.1. *[5] Suppose that a connected hypergraph H reaches the maximum spectral radius $\rho(H)$ among all r-uniform hypergraphs with e edges. Suppose that the Perron-Frobenius vector of H reaches the maximum at a vertex v. Then we have the following properties:*

1. *The shadow graph $\partial(H_v)$ of H_v is a subgraph of the link graph G_v.*

2. *The link graph G_v is connected while H_v may be disconnected but has only one non-trivial connected component.*

3. *The link graph G_v has at least $f_r(e)$ edges.*

Proof. Let H be the maximum hypergraph on vertices $v_1, v_2, \ldots, v_n$. Let $\mathbf{x}$ be the Perron vector, with x_i be the *ith* entry of $\mathbf{x}$ corresponding to vertex v_i for $i = 1, 2, \ldots, n$. Then we have $x_v \geq x_u$ for any other vertex u.

We will prove Item 1 by contradiction. Suppose $\partial(H_v)$ is not a subgraph of G_v. Then there exists an $(r-1)$-subset $\{v_{i_1}, \ldots, v_{i_{r-1}}\}$ in $\partial(H_v)$ but not in $E(G_v)$. By the definition of the shadow $\partial(H_v)$, there is a vertex $u \neq v$ so that $\{u, v_{i_1}, \ldots, v_{i_{r-1}}\}$ is an edge of H_v. By moving this edge from u to v, we obtain a new hypergraph H' from H with larger spectral radius as guaranteed by Lemma 2.4.1, a contradiction.

For Item 2, removing all edges of H_v from H, the resulting hypergraph is still connected. Thus G_v has no isolated vertices. Now we will prove that G_v is connected. Otherwise, G_v has at least two non-trivial connected components. Let $\{v_{j_1}, \ldots, v_{j_{r-1}}\}$ and $\{v_{k_1}, \ldots, v_{k_{r-1}}\}$ be any two edges from different connected components of G_v. Here we assume the vertices are ordered non-increasingly according to the Perron-Fronenius vector $\mathbf{x}$; that is $x_{j_1} \geq x_{j_2} \geq \cdots \geq x_{j_{r-1}}$ and $x_{k_1} \geq x_{k_2} \geq \cdots \geq x_{k_{r-1}}$. We also assume that $x_{j_1} \geq x_{k_1}$. By Lemma 2.4.1, We can move the edge $\{v, v_{k_1}, \ldots, v_{k_{r-1}}\}$ from v_{k_1} to v_{j_1} to increase the spectral radius. Contradiction. A similar argument can show that H_v has only one non-trivial component.

For Item 3, we write $e = \binom{s}{r}$ and $|E(H_v)| = \binom{y}{r}$ for some real numbers $s, y \geq r-1$. By Theorem 5.2.2, we have

$$|\partial(H_v)| \geq \binom{y}{r-1}.$$

Applying Lemma 5.2.1 Item 1, we have

$$|E(G_v)| \geq |\partial(H_v)| \geq \binom{y}{r-1}.$$

Thus,

$$\begin{aligned}
\binom{s}{r} &= |E(H)| \\
&= |E(G_v)| + |E(H_v)| \\
&\geq \binom{y}{r-1} + \binom{y}{r} \\
&= \binom{y+1}{r}.
\end{aligned}$$

Thus, $s \geq y + 1$. It implies

$$\begin{aligned}
|E(G_v)| &= e - |E(H_v)| \\
&= \binom{s}{r} - \binom{y}{r} \\
&\geq \binom{s}{r} - \binom{s-1}{r} \\
&= \binom{s-1}{r-1} \\
&= f_r(e).
\end{aligned}$$

The proof is finished. $\square$

Proof of Theorem 5.0.1. We will use double inductions on r and e to prove the theorem. For $r = 2$ and any $e \geq 0$, Theorem 5.0.1 is just Stanley's theorem.

Inductively, we assume the statement is true for all $(r-1)$-hypergraphs. For r-hypergraph, clearly, the statement is trivial for the cases $e = 0, 1$. We assume the statement holds for all r-hypergraphs with less than e edges.

Let H be the maximum hypergraph among all r-hypergraphs of e edges. By Lemma 2.4.1, H has only one non-trivial connected component. By deleting isolated vertices if possible, we may assume that H is connected.

Let $\mathbf{x}$ be the Perron-Frobenius vector of H and v be a vertex such that $x_v = \max\{x_u, u \in V(H)\}$. By Lemma 5.2.1, the degree d of v is at least $f_r(e)$. Recall that

H_v is the induced hypergraph obtained from H by deleting the vertex v and G_v is the link graph of H at v.

The main idea is to construct an α-subnormal labeling for H by combining the α_1-normal labeling of G_v and the α_2-normal labeling of H_v properly. By Lemma 5.2.1, G_v is a connected $(r-1)$-hypergraph with d edges. By inductive hypothesis, we have

$$\rho(G_v) \leq f_{r-1}(d). \tag{5.10}$$

By Lemma 2.4.3, G_v has a consistent α_1-normal labeling with $\alpha_1 = \rho(G_v)^{-(r-1)}$. Let B_1 be the weighted incidence matrix of G_v corresponding to this α_1-normal labeling. We have

$$\sum_{f \in E(G_v):\, u \in f} B_1(u, f) = 1, \quad \text{for any vertex } u \in V(G_v), \tag{5.11}$$

$$\prod_{u \in f} B_1(u, f) = \alpha_1, \quad \text{for any edge } f \in E(G_v). \tag{5.12}$$

Let H'_v be the unique non-trivial connected component of H_v. Then H'_v has $|E(H_v)| = e - d$ edges. By inductive hypothesis, we have

$$\rho(H'_v) \leq f_r(e - d). \tag{5.13}$$

By Lemma 2.4.3, H'_v has a consistent α_2-normal labeling with $\alpha_2 = \rho(H'_v)^{-r}$. Let B_2 be the weighted incidence matrix of H'_v corresponding to this α_2-normal labeling. We have

$$\sum_{f \in E(H'_v):\, u \in f} B_2(u, f) = 1, \quad \text{for any vertex } u \in V(H'_v), \tag{5.14}$$

$$\prod_{u \in f} B_2(u, f) = \alpha_2, \quad \text{for any edge } f \in E(H'_v). \tag{5.15}$$

Now we define a weighed incidence matrix B of the hypergraph H. For any vertex

$u \in V(H)$ and any edge $f \in E(H)$, we have

$$B(u,f) = \begin{cases} 0 & \text{if } u \notin f; \\ 1/d & \text{else if } u = v; \\ xB_1(u, f - \{v\}) & \text{else if } u \in f \text{ and } u \neq v; \\ yB_2(u,f) & \text{otherwise.} \end{cases} \tag{5.16}$$

Here x, y are two real numbers in $[0,1]$ and will be chosen later.

Now consider the following two properties about B:

- For each vertex $u \in V(H)$, we estimate $\sum_{f \in E(H), u \in f} B(u,f)$ as follows:

  If $u = v$, we have

$$\sum_{f \in E(H), v \in f} B(v,f) = d \times \frac{1}{d} = 1. \tag{5.17}$$

  If $u \neq v$, we have

$$\begin{aligned} \sum_{f \in E(H), u \in f} B(u,f) &= x \sum_{f' \in E(G_v), u \in f'} B(u,f') + y \sum_{f \in E(H_v), u \in f} B(u,f) \\ &\leq x + y. \end{aligned} \tag{5.18}$$

  Here we applied Equations (5.11) and (5.14). Notice that if u is an isolated vertex of H_v, then the second sum is 0. Nevertheless, the above inequality holds.

- Now we estimate $\prod_{u \in f} B(u,f)$ for each edge $f \in E(H)$.

  If $v \in f$, then

$$\prod_{u \in f} B(u,f) = \frac{1}{d} \prod_{u \in f - \{v\}} xB_1(u, f \setminus \{v\}) = \frac{1}{d} x^{r-1} \alpha_1. \tag{5.19}$$

  If $v \notin f$, then

$$\prod_{u \in f} B(u,f) = \prod_{u \in f} yB_2(u,f) = y^r \alpha_2. \tag{5.20}$$

Set $\alpha = (f_r(e))^{-r}$, $x = \left(\frac{d\alpha}{\alpha_1}\right)^{1/(r-1)}$, and $y = \left(\frac{\alpha}{\alpha_2}\right)^{1/r}$. Then for each $f \in E(H)$, we have

$$\prod_{u \in f} B(u, f) = \alpha. \tag{5.21}$$

Recall $\alpha_1 = \rho(G_v)^{-(r-1)}$ and $\alpha_2 = \rho(H'_v)^{-r}$. Combining with Inequalities (5.10) and (5.13), we have

$$\begin{aligned} x + y &= \left(\frac{d\alpha}{\alpha_1}\right)^{1/(r-1)} + \left(\frac{\alpha}{\alpha_2}\right)^{1/r} \\ &= \frac{d^{\frac{1}{r-1}}\rho(G_v)}{f_r(e)^{\frac{r}{r-1}}} + \frac{\rho(H_v)}{f_r(e)} \\ &\le \frac{d^{\frac{1}{r-1}} f_{r-1}(d)}{f_r(e)^{\frac{r}{r-1}}} + \frac{f_r(e-d)}{f_r(e)} \\ &\le 1. \end{aligned}$$

The last inequality is due to Lemma 5.1.2 since $f_r(e) \le d \le e$.

Combining this with Equations (5.17) and (5.18), we have

$$\sum_{f \in E(H), u \in f} B(u, f) \le 1. \tag{5.22}$$

Equations (5.21) and (5.22) imply that H is α-subnormal with $\alpha = (f_r(e))^{-r}$. Hence, by Lemma 2.4.4, we have

$$\rho(H) \le f_r(e).$$

When the inequality holds, we must have $e = \binom{k}{r}$, $d = f_r(e) = \binom{k-1}{r-1}$, $\rho(G_v) = \binom{k-2}{r-2}$, and $\rho(H_v) = \binom{k-2}{r-1}$. By induction, G_v is the complete graph K_{k-1}^{r-1} and H_v is the complete graph K_{k-1}^{r}. Thus, H is the complete graph K_k^r. Since adding isolated vertices will not change the number of edges and the spectral radius, the inequality in Theorem 5.0.1 holds if and only if H is the complete hypergraph possibly with some isolated vertices added.

□

Chapter 6

Spectral Radius of $\{0,1\}$-Tensor

In this chapter, we prove the following theorem:

Theorem 6.0.3. *[6] For any r-order $\{0,1\}$-tensor A with e ones, the spectral radius $\rho(A)$ satisfies*

$$\rho(A) \le e^{\frac{r-1}{r}},$$

with the equality holds if and only if $e = k^r$ for some positive integer k and A is equivalent to J_k^r.

We also characterize the structure of maximum tensors for $e = k^r + l$ with sufficiently large k and $l \in \{-r-1, -r, \ldots, -1, 0, 1, 2, \ldots, r\}$.

Theorem 6.0.4. *[6] Let r, k be positive integers with $r \ge 3$ and k sufficiently large.*

1. *For $e = k^r + 1$, the maximum tensors in $\mathcal{T}_e^r$ are exactly the tensors which can be obtained from J_k^r by inserting an 1 to an arbitrary 0-position. All these maximum tensors have spectral radius k^{r-1}.*

2. *For $2 \le l \le r$, $e = k^r + l$, the maximum tensors in $\mathcal{T}_e^r$ is uniquely equivalent to the tensor obtained from J_k^r by inserting l ones at first l positions of the list:*

$$\{a_{(k+1)11\cdots 1}, a_{1(k+1)1\cdots 1}, a_{11(k+1)\cdots 1}, \ldots, a_{11\cdots(k+1)}\}.$$

3. *For $1 \le l \le r+1$, $e = k^r - l$, the maximum tensors in $\mathcal{T}_e^r$ is uniquely equivalent to the tensor obtained from J_k^r by placing l zeros at the first l positions from the list:*

$$\{a_{kk\cdots k}, a_{k(k-1)k\cdots k}, a_{kk(k-1)\cdots k}, \ldots, a_{kkk\cdots(k-1)}, a_{(k-1)k\cdots k}\}.$$

6.1 Lemmas on nonnegative tensors

Before proving our results, we need to prove important properties for nonnegative tensors. Let us start with some definitions and known facts.

Definition 6.1.1. *[35] An n-dimension r-order tensor $A = (a_{i_1 i_2 \cdots i_r})$ is called* reducible *if there exists a nonempty proper subset $I \subset \{1, \ldots, n\}$ such that $a_{i_1 i_2 \cdots i_r} = 0$ for all $i_1 \in I$ and $i_2, \ldots, i_r \notin I$. A tensor A is said to be irreducible if it is not reducible.*

Definition 6.1.2. *[23] A nonnegative matrix $G(A)$ is called the representation associated to the nonnegative tensor A, if the (i,j)-th element of $G(A)$ is defined to be the summation of $a_{ii_2 \cdots i_r}$ with indices $j \in \{i_2, \ldots, i_r\}$. A nonnegative tensor $A = (a_{i_1 i_2 \cdots i_r})$ is said to be* weakly reducible *if $G(A)$ is a reducible matrix. It is* weakly irreducible *if it is not weakly reducible.*

Theorem 6.1.1. *[21, 47] For an n-dimension r-order tensor $A = (a_{i_1 i_2 \cdots i_r})$, let $G_A = (V(G_A), E(G_A))$ be the digraph of the tensor A with vertex set $V(G_A) = \{1, 2, \ldots, n\}$ and arc set $E(G_A) = \{(i,j) | a_{ii_2 \cdots i_m} \neq 0, j \in i_2, \ldots, i_m\}$. A is weakly irreducible if the corresponding directed graph $G(A)$ is strongly connected. That is for any pair of vertices i and j, there exist directed paths from i to j and j to i.*

Theorem 6.1.2. *[56] Let A be an n-dimension r-order tensor, $r \geq 2$. Then there exists positive integers $k \geq 1$ and $n_1, \ldots, n_k$ with $n_1 + \cdots + n_k = n$ such that A is permutational similar to some $(n_1, \ldots, n_k)$-lower triangular block tensor, where all the diagonal blocks $A_1, \ldots, A_k$ are weakly irreducible. And we have:*

$$Det(A) = \prod_{i=1}^{r} (Det A_i)^{(r-1)^{n-n_i}},$$

and thus

$$\phi_A(\lambda) = \prod_{i=1}^{r} (\phi_{A_i}(\lambda))^{(r-1)^{n-n_i}}$$

where $\phi_A(\lambda)$ is the characteristic polynomial of the tensor A, that is $\phi_A(\lambda) = Det(\lambda I - A)$.

Please refer to [56] for more details on the definitions of determinants and the characteristic polynomial of tensor A. Since λ is an eigenvalue of A if and only if it is a root of the characteristic polynomial of A, Theorem 6.1.2 says that the spectral radius of tensor A is the spectral radius of lower triangular block tensor A_i for some i. This allows us to consider weakly irreducible tensor only.

There are several operations on $\mathcal{T}_e^r$ that keep both the spectral radius and the number of 1's.

Permutation on vertices ([36]): For any permutation $\varphi \in \mathfrak{S}_n$ and any tensor $A = (a_{i_1 i_2 \cdots i_r}) \in \mathcal{T}_n^r(e)$, define a new tensor as follows:

$$\varphi(A) = (a_{\varphi(i_1)\varphi(i_2)\cdots\varphi(i_r)}).$$

Transpose on indexes greater than 1: For any permutation τ on the index set $\{2, 3, \ldots, r\}$, define a new tensor A_τ as follows:

$$A_\tau = (a_{i_1 i_{\tau(2)} \cdots i_{\tau(2)}}).$$

Deleting/inserting isolated vertices: An index/vertex v is called *isolated* if $a_{i_1 i_2 \cdots i_r}$ is equal to 0 as long as v appears in the index $\{i_1, i_2, \ldots, i_r\}$. Deleting/Inserting an isolated vertex keeps the spectral radius.

We say two tensors in $\mathcal{T}_e^r$ are *equivalent* if one can be obtained from the other one by a sequence of the above operations. Denote J_k^r as the k-dimension r-order all-1-tensor $\mathbf{1}_{k \times \ldots \times k}$, it plays a special role in the maximum tensors.

We first prove the following lemma on general nonnegative tensors.

Lemma 6.1.1. *[6] Let A be an n-dimension r-order nonnegative tensor. If there exists a nonzero vector $\mathbf{x} \in \mathbb{R}_+^n$ and a scalar λ such that $A\mathbf{x}^{r-1} \geq \lambda \mathbf{x}^{[r-1]}$, then we* *have*

$$\rho(A) \geq \lambda.$$

Moreover, if A is weakly irreducible then the equality holds if and only if $\mathbf{x}$ is an eigenvector corresponding to $\rho(A)$.

Before proving this lemma, we have a simple corollary. Let A and B are two tensors of the same dimension and the same order. We say $A \geq B$ if $A - B$ is nonnegative. We also write $A > B$ if $A \geq B$ and $A \neq B$.

Corollary 6.1.1. *[6] For any two nonnegative tensors A and B, if $A \geq B$, then $\rho(A) \geq \rho(B)$. Furthermore, if B is weakly irreducible and $A > B$, then $\rho(A) > \rho(B)$.*

Proof. Let $\mathbf{x} \in \mathbb{R}^n_+$ be the Perron-Fronbenius vector of B. Observing

$$A\mathbf{x}^{r-1} \geq B\mathbf{x}^{r-1} = \rho(B)\mathbf{x}^{[r-1]}. \tag{6.1}$$

Applying Lemma 6.1.1, we have $\rho(A) \geq \rho(B)$.

If further B is weakly irreducible, then $\mathbf{x} \in \mathbb{R}^n_{++}$. Since $A > B$, one of Inequalities 6.1 is strict. In particular, $\mathbf{x}$ is not an eigenvector of A. Thus, we must have $\rho(A) > \rho(B)$. □

Proof of Lemma 6.1.1: When $\lambda = 0$, it is trivial. Without loss of generality, we assume $\lambda > 0$.

First we consider the case when A is weakly irreducible. We claim that we can modify $\mathbf{x}$ so that $\mathbf{x} \in \mathbb{R}^n_{++}$. That is, if there exists a nonzero vector $\mathbf{x} \in \mathbb{R}^n_+$ and a scalar λ such that $A\mathbf{x}^{r-1} \geq \lambda\mathbf{x}^{[r-1]}$, then there exists a new vector $\mathbf{y} \in \mathbb{R}^n_{++}$, such that $A\mathbf{y}^{r-1} \geq \lambda\mathbf{y}^{[r-1]}$.

If not, let $J = \{j \in [n] \mid x_j = 0\} \neq \emptyset$. Let $J_0 = J$ and for $i = 1, 2, \ldots$, define

$$J_i = \{j \in J_{i-1} \mid a_{jj_2\cdots j_r} > 0 \Rightarrow j_2, \ldots, j_r \in J_{i-1}\}.$$

We have

$$J = J_0 \supseteq J_1 \supseteq J_2 \supseteq \ldots.$$

Assume J_i is stabilized after s steps; i.e., $J_s = J_{s+1}$. Since A is weakly irreducible, $J_s = \emptyset$.

Let δ be the minimum among all positive entries of A. Let $\epsilon > 0$ be a tiny positive number satisfying $\frac{1}{\epsilon} \gg \log(1/\epsilon) \geq \frac{\lambda}{\delta}$. For $i = 1, 2, \ldots, s$, set $a_i = \sum_{j=1}^{i} (r-1)^{s-j}$, and $\epsilon_i = \frac{\epsilon}{\log^{a_i}(1/\epsilon)}$. We have

$$0 < \epsilon_s \lll \epsilon_{s-1} \lll \cdots \lll \epsilon_2 \lll \epsilon_1 \lll \epsilon \lll \frac{\delta}{\lambda}.$$

We define a new variable $\mathbf{y} = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n_{++}$ by

$$y_j = \begin{cases} x_j \text{ if } j \notin J; \\ \\ \epsilon_i \text{ if } j \in J_{i-1} \setminus J_i. \end{cases}$$

We claim that $A\mathbf{y}^{r-1} \geq \lambda \mathbf{y}^{[r-1]}$. When $j \notin J$, we have

$$(A\mathbf{y}^{r-1})_j \geq (A\mathbf{x}^{r-1})_j \geq \lambda x_j^{r-1} = \lambda y_j^{r-1}.$$

When $j \in J_{i-1} \setminus J_i$, then there exist an entry $a_{jj_2\ldots j_r} > 0$ and at least one index $j_l \notin J_{i-1}$ $(l \geq 2)$. Thus, we have

$$\begin{aligned}
(A\mathbf{y}^{r-1})_j &\geq a_{jj_2\ldots j_r} y_{j_2} \cdots y_{j_r} \\
&\geq \delta \epsilon_{i-1} \epsilon_s^{r-2} \\
&= \delta \frac{\epsilon^{r-1}}{\log^{a_{i-1}+(r-2)a_s}(1/\epsilon)} \\
&\geq \lambda \frac{\epsilon^{r-1}}{\log^{a_{i-1}+(r-2)a_s+1}(1/\epsilon)} \\
&= \lambda \frac{\epsilon^{r-1}}{\log^{a_i(r-1)}(1/\epsilon)} \\
&= \lambda \epsilon_i^{r-1} \\
&= \lambda y_j^{r-1}.
\end{aligned}$$

Here we applied the equality

$$a_{i-1} + (r-2)a_s + 1 = a_i(r-1),$$

which can be verified directly by the definition of a_i.

Hence, without loss of generality, we can assume $\mathbf{x} \in \mathbb{R}^n_{++}$. For any $\lambda > 0$, we define two sets S_λ and S_λ^+ as follows:

$$S_\lambda = \{\mathbf{x} \in \mathbb{R}^n_+ \colon A\mathbf{x}^{r-1} \geq \lambda \mathbf{x}^{[r-1]}\},$$
$$S_\lambda^+ = \{\mathbf{x} \in \mathbb{R}^n_{++} \colon A\mathbf{x}^{r-1} \geq \lambda \mathbf{x}^{[r-1]} \text{ and at least one inequality is strict}\}.$$

Let $\Lambda = \{\lambda \colon S_\lambda^+ \neq \emptyset\}$.

Claim 1: $\Lambda \subset \mathbb{R}$ is an open set.

For any $\lambda \in \Lambda$, there exists $\mathbf{x} \in \mathbb{R}^n_{++}$ satisfying the following system:

$$\sum_{i_2,\ldots,i_r=1}^{n} a_{ii_2\cdots i_r} x_{i_2} \cdots x_{i_r} \geq \lambda x_i^{r-1} \text{ for } i = 1, 2, \ldots, n. \tag{6.2}$$

Let A^i be the i-th equation in (6.2) and I be the index such that the equality holds at A^i. That is, $I = \{i \in [n] | \sum_{i_2,\ldots,i_r=1}^{n} a_{ii_2\cdots i_r} x_{i_2} \cdots x_{i_r} = \lambda x_i^{r-1}\}$.

Assume $I \neq \emptyset$. Since G_A is strongly connected, there exist at least one pair vertices $i \in I$ and $u \in [n] \setminus I$ such that $(i, u) \in E(G_A)$, for this to happen, we have $a_{ii_2\cdots i_r} \neq 0$ when $u = i_l$ for some $l \geq 2$. Then x_u appears in equation A^i. Since A^u is a strictly inequality, we can add appropriate positive tiny value ϵ_u to x_u so that A^u remains a strictly inequality. Now the i-th equation A^i becomes a strictly inequality while other strictly greater inequalities remain strict. By induction on $|I|$, after finite steps, we can obtain a new vector $\mathbf{x}'$ to replace $\mathbf{x}$ and we will have a new system with all strictly greater inequalities. That is, for all $i \in [n]$,

$$\sum_{i_2,\ldots,i_r=1}^{n} a_{ii_2\cdots i_r} x'_{i_2} \cdots x'_{i_r} > \lambda x'^{r-1}_i.$$

Therefore there exists an $\epsilon > 0$ such that $A\mathbf{x}'^{r-1} > (\lambda+\epsilon)\mathbf{x}'^{[r-1]}$. Thus $(\lambda-\epsilon, \lambda+\epsilon) \subset \Lambda$. Λ is an open set.

Since $\rho(A)$ exists, Λ is a bounded set. Let $\lambda_0 = \sup(\Lambda)$.

Claim 2: λ_0 is an eigenvalue of A. In particular, $\lambda_0 \le \rho(A)$.

In the definition of S_λ, the system of inequalities are homogeneous in $\mathbf{x}$. Without loss of generality, we can normalize $\mathbf{x}$ so that $\|\mathbf{x}\|_r = 1$. Note that the sphere in the first quadrant $\{x \in \mathbb{R}^n_+ \colon \|\mathbf{x}\|_r = 1\}$ is a compact set. Thus any sequence has a convergent subsequence and the limit point is also in this set. It implies that there is a $\mathbf{x} \in \mathbb{R}^n_+$ so that

$$A\mathbf{x}^{r-1} \ge \lambda_0 \mathbf{x}^{[r-1]}.$$

Now we show that $\mathbf{x} > 0$. Assume not, let $J = \{i \in [n] \colon x_i = 0\}$. By the previous argument, we can find a $y \in \mathbb{R}^n_{++}$ still satisfying

$$A\mathbf{y}^{r-1} \ge \lambda_0 \mathbf{y}^{[r-1]}.$$

Also notice that A_i are strictly inequality for all $i \in J$. Thus $\lambda_0 \in \Lambda$. Contradiction to the fact that Λ is an open set.

Hence $\mathbf{x} \in \mathbb{R}^n_{++}$ and $A\mathbf{x}^{r-1} = \lambda_0 \mathbf{x}^{[r-1]}$. Thus λ_0 is an eigenvalue of A. Therefore

$$\lambda \le \lambda_0 \le \rho(A).$$

If the inequality $\lambda = \rho(A)$ holds, then $\mathbf{x}$ is an eigenvector for $\rho(A)$.

Now we consider general A. By Theorem 6.1.2, A is permutationally similar to some $(n_1, n_2, \ldots, n_k)$-lower triangular block tensor, where all the diagonal blocks $A_1, \ldots, A_k$ are weakly irreducible. Denote by I_i the i-th block of indexes of size n_i. We have

$$A_1(\mathbf{x}|_{I_1})^{r-1} = (A\mathbf{x}^{r-1})|_{I_1} \ge (\lambda \mathbf{x}^{[r-1]})|_{I_1} = \lambda(\mathbf{x}|_{I_1})^{[r-1]}.$$

If $\mathbf{x}|_{I_1} \ne 0$, then the weakly irreducible tensor A_1 satisfies the condition of lemma. Thus by previous argument, we are done:

$$\rho(A) \ge \rho(A_1) \ge \lambda.$$

If $\mathbf{x}|_{I_1} = 0$, we consider I_2, and so on. Let j be the first indexes so that $\mathbf{x}|_{I_j} \neq 0$. We have

$$A_j(\mathbf{x}|_{I_j})^{r-1} = (A\mathbf{x}^{r-1})|_{I_j} \geq (\lambda \mathbf{x}^{[r-1]})|_{I_j} = \lambda(\mathbf{x}|_{I_j})^{[r-1]}.$$

Now the weakly irreducible tensor A_j satisfies the condition of lemma. We still have

$$\rho(A) \geq \rho(A_j) \geq \lambda.$$

□

Lemma 6.1.1 plays an important role in characterizing the largest eigenvalue and thus can be applied to determine the maximum tensors in the last section. In fact, this lemma gives another proof for the existence of the Perron-Frobenius vector for nonnegative tensor. Cooper and Dutle [13] proved a similar result on adjacency tensor of connected uniform hypergraph, that is, on a symmetric nonnegative weakly irreducible tensor.

Next, we will generalize a theorem of Schwarz[55] on general nonnegative r-order tensors with $r \geq 3$. For $n, r \geq 3$, let σ be a given set of n^r nonnegative real numbers (not necessarily pairwise distinct) and let $\mathcal{F}(\sigma)$ be the set of all n-dimension r-order tensors A for which σ is the set of their elements. Denote $f(\sigma)$ as the largest spectral radius among tensors in $\mathcal{F}(\sigma)$. Let $\mathcal{F}^*(\sigma)$ be the subset of $\mathcal{F}(\sigma)$ consisting of these tensors having the property that in each slice A_i the elements decrease according to the dictionary order; i.e. $a_{ii_2\cdots i_r} \geq a_{ij_2\cdots j_r}$ whenever $(i_2, \ldots, i_r) \leq (j_2, \ldots, j_r)$ under the dictionary order. Let $f^*(\sigma)$ be the largest spectral radius among tensors in $\mathcal{F}^*(\sigma)$. We first show that $f(\sigma)$ is attained by some tensor in $\mathcal{F}^*(\sigma)$.

Theorem 6.1.3. *[6] $f(\sigma) = f^*(\sigma)$.*

Proof. Let A be a tensor that attains the largest eigenvalue in $\mathcal{F}(\sigma)$, i.e. $\rho(A) = f(\sigma)$. Let $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}_+^n$ be the eigenvector associated to $f(\sigma)$. Since permutation on vertices keeps the spectral radius, without loss of generality, we can assume $x_1 \geq x_2 \geq \ldots \geq x_n \geq 0$. Now fix the vertex order of vertices.

Suppose that $A \notin \mathcal{F}^*(\sigma)$. Then A contains a pair of entries $a_{i_1 i_2 \cdots i_r}$ and $a_{i_1 j_2 \cdots j_r}$ satisfying

$$a_{i_1 i_2 \cdots i_r} < a_{i_1 j_2 \cdots j_r} \text{ but } (i_2, \ldots, i_r) < (j_2, \ldots, j_r).$$

We call such pair as a *disordered* pair.

By sequentially switching a disordered pair until no disordered pair is found, we create a sequence of tensors $B_0, B_1, B_2, \ldots, B_s \in \mathcal{F}(\sigma)$ satisfying

1. $B_0 = A$, and $B_s \in \mathcal{F}^*(\sigma)$.

2. For each k from 1 to s, B_k is created from B_{k-1} by switching one disordered pair.

We claim that for each k,

$$B_k \mathbf{x}^{r-1} \geq B_{k-1} \mathbf{x}^{r-1}.$$

Suppose that $(b_{i_1 i_2 \cdots i_r}, b_{i_1 j_2 \cdots j_r})$ is the disordered pair of B_{k-1}, which is switched to create B_k.

Then for any $i \neq i_1$, the i-th row is not affected by switching:

$$(B_k \mathbf{x}^{r-1})_i = (B_{k-1} \mathbf{x}^{r-1})_i. \tag{6.3}$$

Since $(b_{i_1 i_2 \cdots i_r}, b_{i_1 j_2 \cdots j_r})$ is a disordered pair, we have

$$b_{i_1 i_2 \cdots i_r} < b_{i_1 j_2 \cdots j_r}, \text{and} (i_2, \ldots, i_r) < (j_2, \ldots, j_r).$$

This implies $x_{j_2} \ldots x_{j_r} \leq x_{i_2} \ldots x_{i_r}$ since $x_1 \geq x_2 \geq \cdots \geq x_n > 0$. Thus, for the i_1-th row, we have

$$\begin{aligned} (B_k \mathbf{x}^{r-1})_{i_1} - (B_{k-1} \mathbf{x}^{r-1})_{i_1} \quad &= (b_{i_1 i_2 \cdots i_r} - b_{i_1 j_2 \cdots j_r})(x_{j_2} \ldots x_{j_r} - x_{i_2} \ldots x_{i_r}) \\ \geq 0. \end{aligned}$$

The claim is proved. Therefore, we have

$$B_s\mathbf{x}^{r-1} \geq B_{s-1}\mathbf{x}^{r-1} \geq \cdots \geq B_0\mathbf{x}^{r-1} = A\mathbf{x}^{r-1} = \rho(A)\mathbf{x}^{[r-1]}.$$

Applying Theorem 6.1.1, we get

$$\rho(B_s) \geq \rho(A).$$

Since A has the maximum spectral radius in $\mathcal{F}(\sigma)$, so is B_s. Thus $f^*(\sigma) = f(\sigma)$. The proof is finished. □

Remark: Note that if we restrict all tensors in $\mathcal{F}(\sigma)$ to be symmetric, we can get a stronger condition on the maximum tensor A: $a_{i_1 i_2 \cdots i_r} \geq a_{j_1 j_2 \cdots j_r}$ whenever $(i_1, \ldots, i_r) \leq (j_1, \ldots, j_r)$. The proof is easy, we only need to use the fact that the spectral radius of symmetric tensor is invariant under permutations of the indices $[r]$. Note there is a slightly different but similar fact on the adjacency tensor of uniform hypergraphs. In [33], Li-Shao-Qi introduced the operation of *moving edges* on uniform hypergraphs to increase the spectral radius. That is, for this special symmetric nonnegative tensor with zeros on the diagonals, we have, $a_{i_1 i_2 \cdots i_r} \geq a_{j_1 j_2 \cdots j_r}$ whenever $(i_1, \ldots, i_r) \leq (j_1, \ldots, j_r)$.

However, for non-symmetric tensor, the case is different. In [56], Shao-Shan-Zhang proved that determinant of a tensor could change after a transpose operation on indices. Here we provide an example to show that the even spectral radius could be changed under transpose operation.

Definition 6.1.3. *Let $A = (a_{i_1 i_2 \cdots i_r})$ be a tensor, we call $M = (a'_{i_1 i_2 \cdots i_r})$ a transpose of A if for all r-tuples $(i_1, i_2, \ldots, i_r)$, there exists a permutation τ on $[r]$, such that*

$$a'_{i_1 i_2 \cdots i_r} = a_{i_{\tau(1)} i_{\tau(2)} \cdots i_{\tau(r)}}.$$

When $r = 2$, $\rho(M) = \rho(A)$ is always true. However, when $r \geq 3$, it is not true generally. Here is an counter-example. Let A be an 2-dimension 3-order tensor with slices:

$$A_1 = \begin{bmatrix} 1 & 2 \\ 2 & 2 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 1 \\ 3 & 1 \end{bmatrix}.$$

The spectral radius $\rho(A) = 7$. Let $M = (a'_{i_1 i_2 \cdots i_r})$ be a transpose of A with permutation τ such that $\tau(i) = 4 - i$ for $i \in [3]$. That is $a'_{ijk} = a_{kji}$ for any tuple (i, j, k). Then

$$M_1 = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}, \qquad M_2 = \begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix}.$$

However $\rho(M) = 6.91618\ldots$.

6.1.1 Proof of Theorem 6.0.3

For a nonnegative tensor A, we can associate a multivariable polynomial p_A as follows:

$$p_A(x_1, \ldots, x_n) = \sum_{i_1, i_2, \ldots, i_r = 1}^{n} a_{i_1 i_2 \cdots i_r} x_{i_1} x_{i_2} \cdots x_{i_r}.$$

Let

$$\bar{\lambda}(A) = \max_{\mathbf{x} \in \mathbb{R}_+^n} \frac{p_A(\mathbf{x})}{\|x\|_r^r}.$$

This quality is well-defined and is closely related to $\rho(A)$. By taking $\mathbf{x}$ to be the Perron-Frobenius vector, we have

$$\rho(A) = \frac{p_A(\mathbf{x})}{\|\mathbf{x}\|_r^r} \leq \bar{\lambda}(A).$$

The equality holds if A is symmetric.

We call a lower dimensional tensor B a principal sub-tensor of A if B consists of m^r elements in A: for any set $\mathbb{N}$ that composed of m elements in $\{1, 2, \ldots, n\}$,

$$B = (a_{i_1 \cdots i_r}), \text{ for all } \ i_1, i_2, \ldots, i_r \in \mathbb{N}.$$

The concept was first introduced and used by Qi for the higher order symmetric tensor [50].

We will use several important inequalities with the first one:

Theorem 6.1.4 (Hölder's Inequality)**.** *Let a_i, b_i be nonnegative reals for $i = 1, 2, \ldots, n$, let $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\sum_{i=1}^{n} a_i b_i \leq \left(\sum_{i=1}^{n} a_i^p\right)^{1/p} \left(\sum_{i=1}^{n} b_i^q\right)^{1/q}.$$

Let us now prove Theorem 6.0.3.

Proof of Theorem 6.0.3. Suppose that $p_A(\mathbf{x})$ reaches the maximum $\bar{\lambda}(A)$ at $\mathbf{x} = (x_i, \ldots, x_n)^T$ on the unit sphere under r-norm. Then $\sum\limits_{i=1}^{n} x_i^r = 1$. Using Hölder's Inequality, we have

$$\begin{aligned}
\bar{\lambda}(A) &= \sum_{i_1, i_2, \ldots, i_r = 1}^{n} a_{i_1 i_2 \cdots i_r} x_{i_1} x_{i_2} \cdots x_{i_r} \\
&\leq \left(\sum_{i_1, i_2, \ldots, i_r = 1}^{n} (a_{i_1 i_2 \cdots i_r})^{\frac{r}{r-1}} \right)^{\frac{r-1}{r}} \left(\sum_{i_1, i_2, \ldots, i_r = 1}^{n} (x_{i_1} x_{i_2} \cdots x_{i_r})^r \right)^{\frac{1}{r}} \\
&= \left(\sum_{i_1, i_2, \ldots, i_r = 1}^{n} a_{i_1 i_2 \cdots i_r} \right)^{\frac{r-1}{r}} \times 1 \\
&= e^{\frac{r-1}{r}}.
\end{aligned} \tag{6.4}$$

The equality holds if all $a_{i_1 i_2 \cdots i_r}$ are nonzeros as long as $x_{i_1} \cdots x_{i_r} \neq 0$. Thus $A = J_k^r$, where J_k^r is a k-dimension r-order all-1-tensor, for any positive integer k. $\square$

Here is a lower bound on $\bar{\lambda}(A)$.

Lemma 6.1.2. *[6] If A is an n-dimension r-order $\{0,1\}$-tensor with e 1's, then*

$$\bar{\lambda}(A) \geq \frac{e}{n}.$$

Proof. Let $\mathbf{x} = (n^{-1/r}, \ldots, n^{-1/r})$. We have

$$\bar{\lambda}(A) \geq p_A(\mathbf{x}) = \frac{e}{n}.$$

$\square$

Corollary 6.1.2. *[6] If there is a symmetric k-dimension r-order $\{0,1\}$-tensor with at least e 1's, then we have*

$$g_r(e) \geq \frac{e}{k}.$$

For $e = k^r - l$, $l > 0$, there exists a symmetric k-dimension r-order $\{0,1\}$-tensor with at least $e - r!$ ones. Thus

$$g_r(k^r - l) \geq k^{r-1} - \frac{l + r!}{k}.$$

For sufficiently large $k > r!$, we have

$$g_r(k^r - l) \geq k^{r-1} - \frac{l}{k}.$$

This fact can be used to prove the structural theorem for $e = k^r - l$ with small l.

Lemma 6.1.3. *[6] If e is not form of $k^r + 1$ and $A \in \mathcal{T}_e^r$ is a maximum tensor, then A is weakly irreducible.*

Proof. For any integer k, it is easy to verify the case when $e = k^r$. Let $e \geq k^r + 2$, if A is not weakly irreducible, we can re-order the elements in $[n]$ so that A is a general lower-diagonal block tensor with weakly irreducible blocks $A_1, A_2, \ldots, A_s$ on the diagonal. Note that $\rho(A) = \rho(A_i)$ for some i by Theorem 6.1.2. If A_i is not the tensor of all 1's, we can move some 1 to A_i to get a new block A'_i, following by a new tensor $A' \in \mathcal{T}_e^r$. Applying Corollary 6.1.1, we have $\rho(A') \geq \rho(A'_i) > \rho(A_i) \geq \rho(A)$, a contradiction. If $A_i = J_k^r$, and A has at least two more 1's outside A_i, we have

$$\rho(A) = g_r(e) \geq g_r(k^r + 2) > k^{r-1} = \rho(A_i).$$

Contradiction. □

Remark: The reason we exclude the case for $e = k^r + 1$ is that $g_r(k^r + 1) = g_r(k^r) = k^{r-1}$, which will be proved in the last section.

Lemma 6.1.4. *[6] Suppose that A and B are two nonnegative n-dimension r-order tensors with same number of 1's. Let $\mathbf{x}$ be an H^+-eigenvector corresponding to $\rho(A)$. If B is symmetric and $p_A(\mathbf{x}) < p_B(\mathbf{x})$, we have*

$$\rho(A) < \rho(B).$$

Proof. Since $A\mathbf{x}^{r-1} = \rho(A)\mathbf{x}^{[r-1]}$, we have

$$\rho(A)\|\mathbf{x}\|_r^r = p_A(\mathbf{x}) < p_B(\mathbf{x}) < \bar{\lambda}(B)\|\mathbf{x}\|_r^r.$$

Thus $\rho(A) < \bar{\lambda}(B)$. Since B is symmetric, we have $\bar{\lambda}(B) = \rho(B)$. □

6.1.2 A stability result for maximum tensors

]

In this section, we will first prove a stability result; then apply it to obtain the structure of the maximum tensors. We will use the following inequalities:

Theorem 6.1.5 (Young's Inequality)**.** *Assume a and b are nonnegative real numbers, $p, q > 1$ and $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$ab \leq \frac{a^p}{p} + \frac{b^q}{q}.$$

Theorem 6.1.6 (Power Mean Inequality)**.** *For nonnegative real numbers $a_1, \dots, a_n$, if $k_1 \leq k_2$, then*

$$\left(\frac{\sum_{i=1}^n a_i^{k_1}}{n}\right)^{\frac{1}{k_1}} \leq \left(\frac{\sum_{i=1}^n a_i^{k_2}}{n}\right)^{\frac{1}{k_2}}.$$

Now let us prove the following lemma, which will be used to strengthen the Young's inequality.

Given the same r, e as in previous sections, we consider the following function:

$$f(x) = \frac{1}{r}x^r - \frac{1}{e^{\frac{r-1}{r}}}x + \frac{r-1}{re}.$$

We have the following lemma.

Lemma 6.1.5. *[6] Function $f(x)$ is continuous and r times differentiable in $(0,1)$, and has the following properties:*

1. *$f(x) \geq 0$. Equality holds if and only if $x = \frac{1}{e^{1/r}}$*
2. *$f(x)$ is a convex function.*
3. *$f(x) \geq \frac{(r-1)}{2} e^{-1+2/r}(x - e^{-1/r})^2$ for all $x > e^{-1/r}$.*

Proof. Since $f'(x) = x^{r-1} - \frac{1}{e^{\frac{r-1}{r}}}$, $f''(x) = (r-1)x^{r-2} \geq 0$. Thus $f(x)$ is a convex function. By solving $f'(x) = 0$ for x, we get the critical point $x_0 = \frac{1}{e^{1/r}}$, thus $f(x) \geq f(x_0) = 0$. For item 3, let $h(x) = f(x) - \frac{(r-1)}{2} e^{-1+2/r}(x - e^{-1/r})^2$. We have $h(e^{-1/r}) = h'(e^{-1/r}) = 0$ and $h''(x) = (r-1)x^{r-2} - (r-1)(e^{-1/r})^{r-2} > 0$ when $x > e^{-1/r}$. □

Throughout this section, we will consider $r \geq 3$ as a fixed constant, and let an integer k go to infinity.

Theorem 6.1.7. *[6] Let $e = k^r + l$ where $l = o(k^{\frac{2r-2}{r^2-r+2}})$ is allowed to be either positive or negative integer. Let $\epsilon = 0$ if $l \geq 0$ and $\epsilon = 1 + o_k(1)$ if $l < 0$. For any tensor $A \in \mathcal{T}_e^r$ with $\rho(A) \geq k^{r-1} - \epsilon \frac{l}{k}$, let v be the index where the Perron-Frobenius vector of A reaches the maximum. Suppose that the diagonal element $a_{v \cdots v} = 1$. Then A must contain a principal sub-tensor A_L such that*

(a) There are at most $O(|l|)$ zeros in A_L.

(b) There are at most $O(|l|)$ ones outside A_L.

(c) The dimension of A_L is k.

The proof of this theorem is the most difficult part of the paper. We will break it into several lemmas.

Let A be the tensor stated in the theorem, $\mathbf{x} = (x_1, \ldots, x_n)^T$ be the Perron-Frobenius eigenvector associated to the largest eigenvector $\rho(A)$. Assume $x_1 \geq x_2 \geq \cdots \geq x_n \geq 0$, and $\sum_{i=1}^{n} x_i^r = 1$.

Denote I as the index set of all ordered r-tuples $(i_1, i_2, \ldots, i_r)$ such that $a_{i_1 i_2 \cdots i_r} = 1$ (then $|I| = e$), and the complement $\bar{I} = [n]^r \setminus I$. I.e.

$$I = \Big\{(i_1, i_2, \ldots, i_r) \in [n]^r \mid a_{i_1 \cdots i_r} = 1\Big\};$$

$$\bar{I} = \Big\{(i_1, i_2, \ldots, i_r) \in [n]^r \mid a_{i_1 \cdots i_r} = 0\Big\}.$$

Setting $p = \frac{r}{r-1}$, and $q = r$, then $\frac{1}{p} + \frac{1}{q} = 1$. By Young's Inequality Theorem 6.1.5, for any ordered r-tuple $(i_1, i_2, \ldots, i_r) \in I$, we have

$$\begin{aligned}
&\frac{a_{i_1 \cdots i_r}}{\left(\sum\limits_{(j_1,\ldots,j_r)\in I} (a_{j_1 \cdots j_r})^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}}} \times \frac{x_{i_1} \cdots x_{i_r}}{\left(\sum\limits_{(j_1,\ldots,j_r)\in I} (x_{j_1} \cdots x_{j_r})^r\right)^{\frac{1}{r}}} \\
&= \frac{1}{e^{\frac{r-1}{r}}} \times \frac{x_{i_1} \cdots x_{i_r}}{\left(\sum\limits_{(j_1,\ldots,j_r)\in I} (x_{j_1} \cdots x_{j_r})^r\right)^{\frac{1}{r}}} \\
&\leq \frac{r-1}{re} + \frac{x_{i_1}^r \cdots x_{i_r}^r}{r \sum\limits_{(j_1,\ldots,j_r)\in I} (x_{j_1} \cdots x_{j_r})^r}.
\end{aligned}$$

Let

$$x_{i_1 i_2 \cdots i_r} = \frac{x_{i_1} \cdots x_{i_r}}{\left(\sum\limits_{(j_1,\ldots,j_r)\in I} (x_{j_1} \cdots x_{j_r})^r\right)^{\frac{1}{r}}},$$

then the difference of two sides in above inequality is exactly $f(x_{i_1 \cdots i_r})$:

$$f(x_{i_1 i_2 \cdots i_r}) = \frac{r-1}{re} + \frac{x_{i_1}^r \cdots x_{i_r}^r}{r \sum\limits_{(j_1,\ldots,j_r)\in I} (x_{i_1} \cdots x_{i_r})^r} - \frac{1}{e^{\frac{r-1}{r}}} \times \frac{x_{i_1} \cdots x_{i_r}}{\left(\sum\limits_{(j_1,\ldots,j_r)\in I} (x_{i_1} \cdots x_{i_r})^r\right)^{\frac{1}{r}}}.$$

Lemma 6.1.6. *[6] We have*

$$\sum_{(i_1,i_2,\ldots,i_r)\in I} f(x_{i_1 i_2 \cdots i_r}) \leq (\frac{r-1}{r} - \epsilon)\frac{l}{k^r} + O(l^2/k^{2r}). \tag{6.5}$$

Proof. Summing up $f(x_{i_1 i_2 \cdots i_r})$ over all indexes in I, we get

$$\sum_{(i_1,\ldots,i_r)\in I} f(x_{i_1 i_2 \cdots i_r}) = 1 - \frac{1}{e^{\frac{r-1}{r}}} \times \frac{\sum\limits_{(i_1,\ldots,i_r)\in I} x_{i_1} \cdots x_{i_r}}{\left(\sum\limits_{(i_1,\ldots,i_r)\in I} (x_{i_1} \cdots x_{i_r})^r\right)^{\frac{1}{r}}}$$

$$= 1 - \frac{1}{e^{\frac{r-1}{r}}} \times \frac{\rho(A)}{\left(\sum\limits_{(i_1,\cdots,i_r)\in I} (x_{i_1} \cdots x_{i_r})^r\right)^{\frac{1}{r}}}.$$

On the other hand,

$$\begin{aligned}
1 - \sum_{(i_1,\ldots,i_r)\in I} f(x_{i_1 i_2 \cdots i_r}) &= \frac{1}{e^{\frac{r-1}{r}}} \frac{\rho(A)}{\left(\sum\limits_{(i_1,\ldots,i_r)\in I} (x_{i_1} \cdots x_{i_r})^r\right)^{\frac{1}{r}}} \\
&\geq \frac{\rho(A)}{e^{\frac{r-1}{r}}} \\
&\geq \frac{k^{r-1} + \epsilon \frac{l}{k}}{(k^r + l)^{\frac{r-1}{r}}} \qquad (6.6) \\
&= \frac{1 + \frac{\epsilon l}{k^r}}{(1 + \frac{l}{k^r})^{\frac{r-1}{r}}} \\
&= 1 + (\epsilon - \frac{r-1}{r}) \frac{l}{k^r} - O(l^2/k^{2r}).
\end{aligned}$$

Therefore inequality (6.5) holds. ☐

Lemma 6.1.7. *[6] We have*

$$\sum_{(i_1,\ldots,i_r)\in \bar{I}} x_{i_1}^r \cdots x_{i_r}^r \leq \frac{(r - \epsilon r - 1)l}{k^r} + O(l^2/k^{2r}). \qquad (6.7)$$

Proof. By the Power Mean Inequality, we have

$$\frac{\sum\limits_{(i_1,\ldots,i_r)\in I} x_{i_1} \cdots x_{i_r}}{e} \leq \left(\frac{\sum\limits_{(i_1,\ldots,i_r)\in I} x_{i_1}^r \cdots x_{i_r}^r}{e}\right)^{\frac{1}{r}}.$$

It implies

$$\sum_{(i_1,\ldots,i_r)\in I} x_{j_1}^r \cdots x_{j_r}^r \geq \frac{\left(\sum_{(i_1,\ldots,i_r)\in I} x_{i_1}\cdots x_{i_r}\right)^r}{e^{r-1}}$$
$$\geq \frac{\rho(A)^r}{e^{r-1}}$$
$$\geq \frac{(k^{r-1}+\epsilon l/k)^r}{(k^r+l)^{r-1}}$$
$$\geq 1 - \frac{(r-\epsilon r-1)l}{k^r} - O(l^2/k^{2r}).$$

Thus,

$$\sum_{(i_1,\ldots,i_r)\in \bar{I}} x_{i_1}^r \cdots x_{i_r}^r = 1 - \sum_{(i_1,\ldots,i_r)\in I} x_{i_1}^r \cdots x_{i_r}^r \leq \frac{(r-\epsilon r-1)l}{k^r} + O(l^2/k^{2r}).$$

□

Now we are ready to prove Theorem 6.1.7.

Proof of Theorem 6.1.7. Let $c_2 = \sqrt{(\frac{2}{r} - \frac{2\epsilon}{r-1})l}$. We claim

$$x_1^r \leq \frac{1+c_2}{k}. \tag{6.8}$$

Otherwise, say $x_1^r > \frac{1+c_2}{k}$. We have

$$x_{11\cdots 1} = \frac{x_1^r}{\left(\sum_{(i_1,\ldots,i_r)\in I}(x_{i_1}\cdots x_{i_r})^r\right)^{\frac{1}{r}}} \geq x_1^r > e^{-1/r}.$$

Applying Item 3 of Lemma 6.1.5, we have

$$f(x_{11\cdots 1}) > \frac{(r-1)}{2} e^{-1+2/r}(x_{11\cdots 1} - e^{-1/r})^2$$
$$\geq \frac{(r-1)}{2}\frac{1}{(k^r+l)^{\frac{r-2}{r}}}\left(\frac{c_2}{k}\right)^2$$
$$= (\frac{r-1}{r} - \epsilon)\frac{l}{k^r} - O(l^2/k^{2r}).$$

Contradiction to inequality (6.5) by the choice of c_2 as k goes to infinity.

Now let c_1 be a constant such that :

$$c_1 = \frac{1}{2(c_2+1)^{r-1}}.$$

We separate the index set $\{1, 2, \ldots, n\}$ into two sets L and S, where L is called the large set that contains element i such that $x_i^r \geq \frac{c_1}{k}$, S is called the small set that contains the rest elements, i.e.

$$L = \left\{ i \in [n] \;|x_i^r \geq \frac{c_1}{k} \right\};$$
$$S = \left\{ i \in [n] \;|x_i^r < \frac{c_1}{k} \right\}.$$

Let $A_L = (a_{i_1 \cdots i_r})$ be the principal sub-tensor of A restricted to the large set L, i.e. for every element $a_{i_1 \cdots i_r} \in A_L$, the index r-tuple $(i_1, \ldots, i_r) \in L^r$.

Denote the number of zeros in A_L as N. By Lemma 6.1.7, we have

$$N \times \frac{c_1^r}{k^r} \leq \frac{(r - \epsilon r - 1)l}{k^r} + O(l^2/k^{2r}).$$

$$N \leq \frac{\frac{(r-\epsilon r-1)l}{k^r} + O(l^2/k^{2r})}{c_1^r/k^r} = 2^r(c_2+1)^{r(r-1)} \left((r - \epsilon r - 1)l + O(l^2/k^r) \right).$$

By the assumption $|l| = o(k^{\frac{2r-2}{r^2-r+2}})$, we have $N = o(k^{r-1})$.

Now consider the indexes outside of L. Let $(i_1 \cdots i_r) \in I \setminus L^r$, by Inequality (6.8) and the value of c_1, we have

$$\begin{aligned}
x_{i_1 \cdots i_r} &= \frac{x_{i_1} \cdots x_{i_r}}{\left(\sum_{(i_1,\ldots,i_r)\in I} (x_{i_1} \cdots x_{i_r})^r \right)^{\frac{1}{r}}} \\
&= (1+o(1)) x_{i_1} \cdots x_{i_r} \\
&\leq (1+o(1)) x_1^{r-1} \left(\frac{c_1}{k} \right)^{\frac{1}{r}} \\
&\leq (1+o(1)) \frac{1}{2^{\frac{1}{r}} k} \\
&< \frac{1}{e^{1/r}}.
\end{aligned}$$

Note that $f(x)$ is decreasing when $x \leq \frac{1}{e^{1/r}}$, thus

$$f(x_{i_1 \cdots i_r}) \geq f\left(\frac{1}{2^{\frac{1}{r}} k}\right) \approx \left(\frac{1}{2r} + \frac{r-1}{r} - \frac{1}{2^{\frac{1}{r}}}\right) \frac{1}{k^r} - O(l/k^{2r}).$$

Let $M = |I \setminus L^r|$, i.e. the number of 1's outside of A_L. By Inequality (6.5), we have

$$M \times f\left(\frac{1}{2^{\frac{1}{r}} k}\right) \leq \sum_{(i_1 \cdots i_r) \in I \setminus L^r} f(x_{i_1 \cdots i_r}) \leq (\frac{r-1}{r} - \epsilon) \frac{l}{k^r} + O(l^2/k^{2r}).$$

Solving M, we get

$$M \leq \frac{(\frac{r-1}{r} - \epsilon)\frac{l}{k^r} + O(l^2/k^{2r})}{\left(\frac{1}{2r} + \frac{r-1}{r} - \frac{1}{2^{\frac{1}{r}}}\right) \frac{1}{k^r} - O(l/k^{2r})} = O(|l|).$$

Since the total number of 1's in tensor A is $|L|^r + M - N = k^r + l$, we have

$$|L|^r = k^r + l + N - M \leq k^r + o(k^{r-1}).$$

Since both $|L|$ and k are integers, it implies $|L| = k$. Therefore, the dimension of A_L is k.

To finish Item (a), observe

$$N = M - l = O(|l|).$$

□

Next, we will further determine the number of zeros in A_L and number of ones outside of A_L for the maximum tensors A in $\mathcal{T}_e^r$.

Theorem 6.1.8. *[6] For fixed r, sufficiently large k, and $l > 0$ a constant, let $e = k^r + l$. Let A be the maximum tensor in $\mathcal{T}_e^r$, then A contains a principal subtensor J_k^r.*

Proof. Assume the dimension of A is n. Let $\rho(A)$ be the largest eigenvalue of A, $\mathbf{x}$ be the corresponding eigenvector, with $x_1 \geq x_2 \geq \cdots \geq x_n$. By Corollary 6.1.2, $\rho(A) \geq k^{r-1}$. By Theorem 6.1.7, A contains a principal subtensor A_k so that there are at most $O(l)$ zeros inside of A_k and at most $O(l)$ ones outside of A_k. This fact implies that $x_i = (1+o(1))k^{-1/r}$ for $1 \le i \le k$ and $x_j = O(k^{-1-1/r})$ for $i > k$.

Here is the reason: for any $i \in \{1, 2, \cdots, n\}$, denote R_i as the summation of elements in ith slice A_i of A. Using HölderâĂŹs Inequality (Theorem 6.1.4), we have

$$\begin{aligned}
\rho(A)x_i^{r-1} &= \sum_{i_2,\ldots,i_r} a_{ii_2\cdots i_r} x_{i_2}\cdots x_{i_r} \\
&\le \left(\sum_{i_2,\ldots,i_r} (a_{ii_2\cdots i_r})^{\frac{r}{r-1}}\right)^{\frac{r-1}{r}} \left(\sum_{i_2,\ldots,i_r} (x_{i_2}\cdots x_{i_r})^r\right)^{\frac{1}{r}} \\
&\le \left(\sum_{i_2,\ldots,i_r} a_{ii_2\cdots i_r}\right)^{\frac{r-1}{r}} \times 1 \\
&= R_i^{\frac{r-1}{r}}.
\end{aligned}$$

For $i = 1$, we have $R_1 \approx k^{r-1} + o_k(1)$. Then

$$x_1 \le \frac{R_1^{\frac{1}{r}}}{\rho(A)^{\frac{1}{r-1}}} \approx \frac{k^{\frac{r-1}{r}} + o(1)}{k} = \frac{1+o(1)}{k^{\frac{1}{r}}}. \tag{6.9}$$

Let $s \ge k+1$, we have $R_s \le M \le O(l)$, and

$$\begin{aligned}
\rho(A)x_s^{r-1} &= \sum_{i_2\cdots i_r} a_{si_2\cdots i_r} x_{i_2}\cdots x_{i_r} \\
&\le \sum_{i_2\cdots i_r} a_{si_2\cdots i_r} x_1^{r-1} \\
&= R_s x_1^{r-1}.
\end{aligned}$$

Then

$$x_s \le \left(\frac{R_s}{\rho(A)}\right)^{\frac{1}{r-1}} x_1 \le \left(\frac{O(l)}{k^{r-1}}\right)^{\frac{1}{r-1}} x_1 = \frac{O(l^{\frac{1}{r-1}})}{k} x_1. \tag{6.10}$$

Sum on s, we have

$$\sum_{s\ge k+1}^{n} x_s^r \le O(l) \times \frac{O(l^{\frac{r}{r-1}})}{k^r} x_1^r = o(x_1^r).$$

Since

$$x_1^r + x_2^r + \cdots + x_k^r + \sum_{s\ge k+1} x_s^r = 1,$$

we get

$$x_1^r \geq \frac{1 - o(1)}{k},$$

together with (6.9),

$$x_1 \approx \frac{1 + o(1)}{k^{\frac{1}{r}}}.$$

We also have

$$\begin{aligned}
x_k^r &= 1 - x_1^r - \cdots - x_{k-1}^r - \sum_{s \geq k+1} x_s^r \\
&\geq 1 - (k - 1 + o(1))x_1^r \\
&\geq 1 - (k - 1 + o(1))\frac{1}{k} \\
&\geq \frac{1 - o(1)}{k}.
\end{aligned}$$

Then

$$x_k \geq \frac{1 - o(1)}{k^{\frac{1}{r}}},$$

together with (6.9) and $x_k \leq x_1$, we have for any $1 \leq i \leq k$,

$$x_k \approx \frac{1 + o(1)}{k^{\frac{1}{r}}}.$$

By (6.10), we have

$$x_s \leq \frac{x_1}{k}.$$

Since

$$\rho(A) x_s^{r-1} \geq x_1^{r-1},$$

by Theorem 6.0.3

$$x_s \geq \frac{x_1}{\rho(A)^{\frac{1}{r-1}}} \geq \frac{x_1}{e^{\frac{1}{r}}} \geq \frac{x_1}{k}.$$

Thus for $s \geq k+1$,

$$x_s \approx \frac{x_1 + o(1)}{k^{\frac{1}{r}}} \approx O(k^{-1-1/r}).$$

We observe that the contribution to $p_A(\mathbf{x})$ from the outside of A_k is at most

$$O(l) x_{k+1} x_1^{r-1} = \frac{O(l)}{k} x_1^r.$$

Then

$$p_A(\mathbf{x}) = p_{A_k}(\mathbf{x}) + p_{A-A_k}(\mathbf{x}) \le p_{A_k}(\mathbf{x}) + \frac{O(l)}{k^2}.$$

If A_k has some zeros, let $B = J_k^r$. We observe that

$$p_{A_k}(\mathbf{x}) < p_B(\mathbf{x}).$$

Applying Lemma 6.1.4, when k is sufficiently large, we have

$$\rho(A) < \rho(B) = k^{r-1}.$$

Contradiction! □

Still let $l > 0$, a similar argument can be applied to $e = k^r - l$. We have the following theorem.

Theorem 6.1.9. *[6] For fixed r, sufficiently large k, and $l > 0$ a constant, let $e = k^r - l$. Let A be a maximum tensor in $\mathcal{T}_e^r$ with no isolated vertices. Then the dimension of A is exactly k.*

Proof. Let $\rho(A)$ be the largest eigenvalue of A, $\mathbf{x}$ be the corresponding eigenvector, with $x_1 \ge x_2 \ge \cdots \ge x_n$. By Corollary 6.1.2, $\rho(A) \ge k^{r-1} - o_k(\frac{1}{k})$. By a similar argument as in above theorem, we have $x_i = (1 + o(1))k^{-1/r}$ for $1 \le i \le k$ and $x_j = O(k^{-1-1/r})$ for $i > k$.

Assume there are $M > 0$ ones outside of A_k, then there are at least $l + M$ zeros inside of A_k. Thus we have

$$\begin{aligned}
\rho_A(\mathbf{x}) &\le \rho_{A_k}(\mathbf{x}) + M x_{k+1} x_1^{r-1} \\
&\le k^{r-1} - (l + M)x_k^r + M x_{k+1} x_1^{r-1} \\
&= k^{r-1} - \frac{l + M}{k} + \frac{M}{k^2} \\
&\le k^{r-1} - \frac{l}{k}.
\end{aligned}$$

Contradicts to Corollary 6.1.2 for sufficiently large k. Since there is no one outside A_L, the dimension of A is exactly k.

□

6.2 Maximum tensors in $\mathcal{T}_e^r$ with small l

In this section, we will completely determine the maximum tensors A in $\mathcal{T}_e^r$ for $e = k^r + l$, $0 \leq l \leq r$, and $e = k^r - l$, $1 \leq l \leq r+1$.

Let $\mathbf{x} = (x_1, \ldots, x_n)$ be the eigenvector associated to $\rho(A)$. Without loss of generality, we assume that $x_1 \geq x_2 \cdots \geq x_n$. The tool in Theorem 6.1.3 allows to shift 1's to left in the same row to increase the spectral radius of tensor A. Although we couldn't shift 1's up across rows, for example, we cannot compare the elements $a_{ii_2\cdots i_r}$ and $a_{(i+1)i_2\cdots i_r}$ in a maximum tensor. But as to $\{0,1\}$-tensors, we have the following easy fact:

Corollary 6.2.1. *[6] Let A be a maximum n-dimension r-order $\{0,1\}$-tensors. If $a_{j1\cdots 1} = 1$ for some j, then $a_{i1\cdots 1} = 1$ for all $i < j$.*

Proof. Let $A_i = (a_{ii_2\cdots i_r})$ with $a_{ii_2\cdots i_r} \in A$. If there exist $j > i$ such that $a_{i1\cdots 1} = 0$ while $a_{j1\cdots 1} = 1$, by Theorem 6.1.3, every other element in A_i is 0. Thus we have

$$\rho(A)x_i^{r-1} = A_i\mathbf{x}^{r-1} = 0,$$

implying $x_i = 0$. However since $a_{j1\cdots 1} = 1$, we have

$$A_j\mathbf{x}^{r-1} = \rho(A)x_j^{r-1} > 0,$$

implying $x_j > 0$, contradiction to $x_j \leq x_i$. □

By Theorem 6.1.3 and Corollary 6.2.1, we have the following property for the maximum tensor A in $\mathcal{T}_e^r$: in each slice A_i, the '1' elements are always to the left and above of the '0' elements.

For $e = k^r + l$, we have proved that the maximum tensor A contains J_k^r as principal sub-tensor (see Theorem 6.1.8), so we just need to determine the positions for the rest of l ones outside of J_k^r.

For $l = 0$, $A = J_k^r$. For $l = 1$, the maximum tensor is not unique. No matter where to put the additional 1, the resulting tensor A is not weakly irreducible. Thus it will not increase the spectral radius. We have

$$g_r(k^r + 1) = g_r(k^r) = k^{r-1}.$$

For $2 \leq l \leq r$, it is sufficient to prove the following facts regarding the maximum tensor A:

Lemma 6.2.1. *[6] For the maximum tensor $A \in \mathcal{T}_e^r$ with $e = k^r + l$, $2 \leq l \leq r$, we have*

1. *There is no '1' element in slice A_{k+2}, i.e. $a_{(k+2)11\cdots 1}$ must be 0.*
2. *There is only one '1' element in slice A_{k+1}, which is $a_{(k+1)11\cdots 1} = 1$.*
3. *There is no '1' elements in slice A_i but outside J_k^r for $i \geq 2$, i.e. $a_{ii_2\cdots i_r} = 1$ if there exists $i_j \geq k + 1$.*

The details of the proof for Lemma 6.2.1 are in Appendix. By above analysis and Lemma 6.2.1, one can easily verify Item 1 and Item 2 in Theorem 6.0.4.

For $e = k^r - l$, we have proved that the dimension of A with e ones is exactly k (see Theorem 6.1.9), thus we only need to determine the positions for punching l 0's in A. For $l = 1$ or $l = r + 1$, we have the following results for part of Item 3 in Theorem 6.0.4.

Corollary 6.2.2. *[6] Let $r \geq 3$, $k \geq 1$ be positive integers.*

1. *Let $e = k^r - 1$, the maximum tensor in $\mathcal{T}_e^r$ is obtained from J_k^r by putting zero at $a_{kk\cdots k}$.*

2. *Let $e = k^r - r - 1$, the maximum tensor in $\mathcal{T}_e^r$ is obtained from J_k^r by placing zeros at $a_{(k-1)k\cdots k}$ and $a_{k(k-1)\cdots k}, \cdots, a_{kk\cdots(k-1)}$ and $a_{kk\cdots k}$.*

Proof. The indicated tensor A in each case is a symmetric tensor in $\mathcal{T}_e^r$, for any other tensor $A' \in \mathcal{T}_e^r$, let $\mathbf{x}$ be the vector corresponding to $\rho(A')$ with $x_1 \geq x_2 \geq \cdots \geq x_k$. By comparing two formulas $\rho_{A'}(x)$ and $\rho_A(x)$, we can see that $\rho_{A'}(x) \leq \rho_A(x)$. Thus by Lemma 6.1.4, $\rho(A') \leq \rho(A)$. □

For $2 \leq l \leq r$, we need to prove the following lemma:

Lemma 6.2.2. *[6] Let $A \in \mathcal{T}_e^r$ be the tensor that '0' elements appear at the end of slices A_{k-1} and A_k, let $B \in \mathcal{T}_e^r$ be the tensor that '0' elements only appear at the end of slice A_k. Then $\rho(B) \geq \rho(A)$.*

The details of the proof of Lemma 6.2.2 can be found at Appendix.

Now let us we prove Theorem 6.0.4, Item 3.

Proof of Theorem 6.0.4, Item 3. The idea in Lemma 6.2.2 is to compare the tensor when '0' elements only appear at the slice A_k with tensor when '0' elements also appear at slice A_{k-1}. Following this idea in Lemma 6.2.2, we repeatedly compare the tensor when '0' elements only appear at slice A_k with tensor when '0' elements also appear at slice A_{k-i}, for $i \geq 2$. There are only finite cases. It is tedious to include all computations here. The proof and result for each comparing is similar with the proof of Lemma 6.2.2. In the end we conclude: For a maximum tensor A, the l '0' elements can only appear at slice A_k. The proof is complete. □

6.3 Appendix

Proof of Lemma 6.2.1. **For Item 1:** Suppose $a_{(k+2)11\cdots 1} = 1$, by Corollary 6.2.1, $a_{(k+1)11\cdots 1} = 1$. By Lemma 6.1.3, A is a weakly irreducible tensor, then $a_{1(k+2)\cdots 1)}$ must be 1. Let

$$R = \{a_{1(k+1)\cdots 1}, a_{11(k+1)\cdots 1}, \ldots, a_{11\cdots(k+1)}\}.$$

The above assumption will force each element in set R is one, then the number of ones outside of J_k^r would be $3+r-1 = r+2 \geq l$, a contradiction. To see this, assume there are $s < |R| = r-1$ ones in set R. Let λ be the largest eigenvalue of A, $\mathbf{x}$ be the corresponding eigenvector, with $x_1 \geq x_2 \geq \cdots \geq x_{k+2}$. Then we have

$$\begin{aligned}
\lambda x_1^{r-1} &= (x_1 + \cdots + x_k)^{r-1} + s x_{k+1} x_1^{r-2} + x_{k+2} x_1^{r-2} \\
\lambda x_2^{r-1} &= (x_1 + \cdots + x_k)^{r-1} \\
&\cdots \\
\lambda x_k^{r-1} &= (x_1 + \cdots + x_k)^{r-1} \\
\lambda x_{k+1}^{r-1} &= x_1^{r-1} \\
\lambda x_{k+2}^{r-1} &= x_1^{r-1}.
\end{aligned}$$

Note that x_1 is strictly greater than x_2. Let B be a new tensor obtained from A by moving '1' from $a_{(k+2)1\cdots 1}$ to one of '0' elements in set R. Let $\mathbf{y}$ be a $(k+1)-$vector obtained from $\mathbf{x}$ such that $y_i = x_i$, for $1 \leq i \leq k+1$. By comparing $B\mathbf{y}^{r-1}$ and $\lambda \mathbf{y}^{r-1}$, we have the following system:

$$\begin{aligned}
\lambda y_1^{r-1} &< (y_1 + \cdots + y_k)^{r-1} + (s+1) y_{k+1} y_1^{r-2} \\
\lambda y_2^{r-1} &= (y_1 + \cdots + y_k)^{r-1} \\
\lambda y_3^{r-1} &= (y_1 + \cdots + y_k)^{r-1} \\
&\cdots \\
\lambda y_{k+1}^{r-1} &= y_1^{r-1}.
\end{aligned}$$

Thus $B\mathbf{y}^{r-1} \geq \lambda \mathbf{y}^{r-1}$. By Lemma 6.1.1, we have $\rho(B) > \lambda$, a contradiction. Therefore $a_{(k+2)11\cdots 1}$ must be 0, it follows the dimension of A is at most $k+1$.

For Item 2: Suppose $a_{(k+1)12\cdots 1} = 0$. Let λ be the largest eigenvalue of A, $\mathbf{x}$ be the corresponding eigenvector, with $x_1 \geq x_2 \geq \cdots \geq x_{k+1}$. Then

$$\lambda x_1^{r-1} = (x_1 + \cdots + x_k)^{r-1} + (l-2)x_{k+1}x_1^{r-2}$$
$$\lambda x_2^{r-1} = (x_1 + \cdots + x_k)^{r-1}$$
$$\cdots$$
$$\lambda x_k^{r-1} = (x_1 + \cdots + x_k)^{r-1}$$
$$\lambda x_{k+1}^{r-1} = x_1^{r-1} + x_1^{r-2}x_2.$$

Let B be a tensor obtained from A by moving '1' from $a_{(k+1)12\cdots 1}$ to some '0' elements in set R, here R is defined as above. Let $\mathbf{y}$ be a new vector obtained from $\mathbf{x}$ such that $y_i = x_i$ for $1 \leq i \leq k$, and $y_{k+1} = 2^{-\frac{1}{r-1}}x_{k+1}$. Then we have

$$\lambda y_1^{r-1} < (y_1 + \cdots + y_k)^{r-1} + (l-1)y_{k+1}y_1^{r-2}$$
$$\lambda y_2^{r-1} = (y_1 + \cdots + y_k)^{r-1}$$
$$\lambda y_3^{r-1} = (y_1 + \cdots + y_k)^{r-1}$$
$$\cdots$$
$$\lambda y_{k+1}^{r-1} < y_1^{r-1}.$$

To see above system, we only need to verify the first and the last inequalities. Note that $x_2 < x_1$, then

$$\lambda y_{k+1}^{r-1} = \lambda(2^{-\frac{1}{r-1}}x_{k+1})^{r-1} = \frac{1}{2}x_1^{r-1} + \frac{1}{2}x_1^{r-2}x_2 < x_1^{r-1} = y_1^{r-1}.$$

For the first inequality, we need to show that

$$\lambda y_1^{r-1} = (x_1 + \cdots + x_k)^{r-1} + (l-2)x_{k+1}x_1^{r-2} < (y_1 + \cdots + y_k)^{r-1} + (l-1)y_{k+1}y_1^{r-2}.$$

It is equivalent to show

$$l - 2 < 2^{-\frac{1}{r-1}}(l-1).$$

Let $f(r)=\frac{l-2}{l-1}-2^{-\frac{1}{r-1}}$, it is decreasing on r, then $f(r)\le f(l)$. Since $f(l)$ is increasing on l, and $\lim_{l\to\infty} f(l)=0$, then $f(r)<0$ for all $r\ge l$. Now we have $B\mathbf{y}^{r-1}\ge\lambda\mathbf{y}^{r-1}$, by Lemma 6.1.1, we get $\rho(B)>\lambda$. A contradiction.

For Item 3: Without loss of generality, we assume there are s ones in A_1 and t ones in A_2. Let λ be the largest eigenvalue of A, $\mathbf{x}$ be the corresponding eigenvector, with $x_1\ge x_2\ge\cdots\ge x_{k+1}$. Then

$$\begin{aligned}
\lambda x_1^{r-1} &= (x_1+\cdots+x_k)^{r-1}+s x_{k+1}x_1^{r-2}\\
\lambda x_2^{r-1} &= (x_1+\cdots+x_k)^{r-1}+t x_{k+1}x_1^{r-2}\\
\lambda x_3^{r-1} &= (x_1+\cdots+x_k)^{r-1}\\
&\cdots\\
\lambda x_{k+1}^{r-1} &= x_1^{r-1}.
\end{aligned}$$

Replace x_{k+1} by x_1, and let $z=(x_1+\cdots+x_k)$, the above system is equivalent to the following:

$$\begin{aligned}
\lambda x_1^{r-1} &= z^{r-1}+s\lambda^{-\frac{1}{r-1}}x_1^{r-1}\\
\lambda x_2^{r-1} &= z^{r-1}+t\lambda^{-\frac{1}{r-1}}x_1^{r-1}\\
\lambda x_3^{r-1} &= z^{r-1}\\
&\cdots\\
\lambda x_k^{r-1} &= z^{r-1}.
\end{aligned}$$

Solve x_1 and x_2 in above system, we get

$$x_1=(\lambda-s\lambda^{-\frac{1}{r-1}})^{-\frac{1}{r-1}}z,\ x_2=\Big(\frac{1-(s-t)\lambda^{-\frac{r}{r-1}}}{\lambda-s\lambda^{-\frac{1}{r-1}}}\Big)^{\frac{1}{r-1}}z.$$

Let B be a tensor obtained from A by moving these t '0' elements in A_2 to set R. Still apply Lemma 6.1.1, we want to find a new vector $\mathbf{y}$ such that $B\mathbf{y}^{r-1}\ge\lambda\mathbf{y}^{r-1}$.

I.e.

$$\begin{aligned}
&\lambda y_1^{r-1} \le z^{r-1} + (s+t)\lambda^{-\frac{1}{r-1}} y_1^{r-1} \\
&\lambda y_2^{r-1} = z^{r-1} \\
&\lambda y_3^{r-1} = z^{r-1} \\
&\cdots \\
&\lambda y_k^{r-1} = z^{r-1} \\
&\lambda y_{k+1}^{r-1} = y_1^{r-1}.
\end{aligned} \tag{6.11}$$

Let $y_i = x_i$ for $3 \le i \le k+1$. Let $y_2 = \lambda^{-\frac{1}{r-1}} z$, then $\lambda y_2^{r-1} = z^{r-1}$.

Let $y_1 = x_1 + x_2 - y_2$, to verify the first inequality we need to show $y_1 < (\lambda - (s+t)\lambda^{-\frac{1}{r-1}})^{-\frac{1}{r-1}} z$. I.e.

$$\begin{aligned}
&\left((\lambda - s\lambda^{-\frac{1}{r-1}})^{-\frac{1}{r-1}}\right) z + \left(\frac{1-(s-t)\lambda^{-\frac{1}{r-1}}}{\lambda - s\lambda^{-\frac{1}{r-1}}}\right)^{\frac{1}{r-1}} z - \lambda^{-\frac{1}{r-1}} z \\
&< \left(\lambda - (s+t)\lambda^{-\frac{1}{r-1}}\right)^{-\frac{1}{r-1}} z.
\end{aligned}$$

After divided by $\lambda^{-\frac{1}{r-1}} z$ from both sides and further simplification by letting $w = \lambda^{-\frac{r}{r-1}} > 0$, the above inequality is equivalent to

$$F(w) = \left\{(1-sw)^{-\frac{1}{r-1}} - (1-(s+t)w)^{-\frac{1}{r-1}}\right\} + \left\{\left(1 + \frac{tw}{1-sw}\right)^{\frac{1}{r-1}} - 1\right\} < 0,$$

By Cauchy's Mean Value Theorem, we have

$$F(w) = -\frac{1}{r-1}(1-\alpha)^{-\frac{1}{r-1}-1}(tw) + \frac{1}{r-1}(1+\beta)^{\frac{1}{r-1}-1}\frac{tw}{1-sw}.$$

where $sw < \alpha < (s+t)w$ and $0 < \beta < \frac{tw}{1-sw}$. Since $-(1-\alpha)^{-\frac{1}{r-1}-1}$ and $(1+\beta)^{\frac{1}{r-1}-1}$ are decreasing functions on α and β respectively, we have

$$\begin{aligned}
F(w) &< -\frac{1}{r-1}(1-sw)^{-\frac{1}{r-1}-1}(tw) + \frac{1}{r-1}(1+0)^{\frac{1}{r-1}-1}\frac{tw}{1-sw} \\
&= -\frac{1}{r-1}\frac{tw}{(1-sw)^{\frac{r}{r-1}}} + \frac{1}{r-1}\frac{tw}{1-sw} \\
&= \frac{tw}{(r-1)(1-sw)^{\frac{r}{r-1}}}\left((1-sw)^{\frac{1}{r-1}} - 1\right) \\
&< 0.
\end{aligned}$$

Now the system (6.11) is verified, by Lemma 6.1.1, we have $\rho(B) > \lambda$. A contradiction. $\square$

Proof of Lemma 6.2.2. Let A and B be given tensors as stated in the lemma. Suppose there are $t+1$ zeros in A_{k-1} and $s+1$ zeros in A_k, i.e. Suppose there are t zeros in the set of $\{a_{(k-1)(k-1)k\cdots k}, \ldots, a_{(k-1)kk\cdots(k-1)}\}$ and one zero at $a_{(k-1)kk\cdots k}$ in slice A_{k-1}; and there are s zeros in the set of $\{a_{k(k-1)k\cdots k}, \ldots, a_{kkk\cdots(k-1)}\}$ and one zero at $a_{kkk\cdots k}$ in slice A_k.

Let λ be largest eigenvalue of A, $\mathbf{x}$ be the corresponding eigenvector with $x_1 \geq x_2 \geq \cdots \geq x_k$. Then $s+t+2 = l$, and $s \geq t \geq 0$. We have

$$\begin{aligned}
\lambda x_1^{r-1} &= (x_1 + \cdots + x_k)^{r-1} \\
&\cdots \\
\lambda x_{k-2}^{r-1} &= (x_1 + \cdots + x_k)^{r-1} \\
\lambda x_{k-1}^{r-1} &= (x_1 + \cdots + x_k)^{r-1} - t x_{k-1} x_k^{r-2} - x_k^{r-1} \\
\lambda x_k^{r-1} &= (x_1 + \cdots + x_k)^{r-1} - s x_{k-1} x_k^{r-2} - x_k^{r-1}.
\end{aligned}$$

From the last two equations, we get $\lambda(x_{k-1}^{r-1} - x_k^{r-1}) = (s-t) x_{k-1} x_k^{r-2}$. Let $w = \frac{x_{k-1}}{x_k}$, we have $\lambda(w^{r-1} - 1) = (s-t)w$. Let $z = (x_1 + \cdots + x_k)$, since $\lambda x_k^{r-1} = z^{r-1} - (sw+1) x_k^{r-1}$, then $x_k = (\lambda + sw + 1)^{-\frac{1}{r-1}} z$ and $x_{k-1} = w(\lambda + sw + 1)^{-\frac{1}{r-1}} z$.

Note B is the tensor with all zeros in the following set

$$\{a_{k(k-1)k\cdots k}, \ldots, a_{kkk\cdots(k-1)}, a_{kkk\cdots k}\}.$$

Still apply Lemma 6.1.1, we want to find a new vector $\mathbf{y}$ such that $B\mathbf{y}^{r-1} \geq \lambda \mathbf{y}^{r-1}$.

Specifically,

$$\begin{aligned}
&\lambda y_1^{r-1} = (y_1 + \cdots + y_k)^{r-1} \\
&\cdots \\
&\lambda y_{k-2}^{r-1} = (y_1 + \cdots + y_k)^{r-1} \\
&\lambda y_{k-1}^{r-1} = (y_1 + \cdots + y_k)^{r-1} \\
&\lambda y_k^{r-1} < (y_1 + \cdots + y_k)^{r-1} - (s+t+1)y_{k-1}y_k^{r-2} - y_k^{r-1}.
\end{aligned} \tag{6.12}$$

Let $y_i = x_i$ for $1 \le i \le k-2$. Let $y_{k-1} = \lambda^{-\frac{1}{r-1}}z$, then $\lambda y_{k-1}^{r-1} = z^{r-1}$. Let $y_k = x_k + x_{k-1} - y_{k-1} = (\lambda + sw + 1)^{-\frac{1}{r-1}}z(1+w) - \lambda^{-\frac{1}{r-1}}z$.

Clearly, $\lambda y_i^{r-1} = \lambda x_i^{r-1} = x_1 + \cdots + x_k = y_1 + \cdots + y_k$, for $i \le k-1$.

We only need to verify the last inequality in system (6.12). I.e.

$$\begin{aligned}
&\lambda\left((\lambda + sw + 1)^{-\frac{1}{r-1}}z(1+w) - \lambda^{-\frac{1}{r-1}}z\right)^{r-1} \\
&+ (s+t+1)\lambda^{-\frac{1}{r-1}}z\left((\lambda + sw + 1)^{-\frac{1}{r-1}}z(1+w) - \lambda^{-\frac{1}{r-1}}z\right)^{r-2} \\
&< z^{r-1}.
\end{aligned}$$

After divided by z^{r-1}, we have

$$\begin{aligned}
&\lambda\left((\lambda + sw + 1)^{-\frac{1}{r-1}}(1+w) - \lambda^{-\frac{1}{r-1}}\right)^{r-1} \\
&+ (s+t+1)\lambda^{-\frac{1}{r-1}}\left((\lambda + sw + 1)^{-\frac{1}{r-1}}(1+w) - \lambda^{-\frac{1}{r-1}}\right)^{r-2} \\
&< 1.
\end{aligned} \tag{6.13}$$

Since $w = \frac{x_{k-1}}{x_k} \ge 1$, when $s = t$, $w = 1$, it is easy to verify that the left hand-side of (6.13) is increasing on λ and goes to 1 as $\lambda \to \infty$. Thus inequality (6.13) is verified. We consider the case $s > t$, so $w > 1$. Let $w = 1 + \epsilon$, we have

$$\lambda((1+\epsilon)^{r-1} - 1) = (s-t)(1+\epsilon)$$

Solve for ϵ, we get $\epsilon \approx \frac{s-t}{\lambda(r-1)}$, then $w \approx 1 + \frac{s-t}{\lambda(r-1)}$. Then

$$\begin{aligned}
Y &= (\lambda + sw + 1)^{-\frac{1}{r-1}}(1+w) - \lambda^{-\frac{1}{r-1}} \\
&= \lambda^{-\frac{1}{r-1}} \left(\frac{1+w}{(1+\frac{sw+1}{\lambda})^{\frac{1}{r-1}}} - 1 \right) \\
&\approx \lambda^{-\frac{1}{r-1}} \left(w - \frac{(sw+1)(1+w)}{\lambda(r-1)} \right) \\
&\leq \lambda^{-\frac{1}{r-1}} \left(w - \frac{2(s+1)}{\lambda(r-1)} \right) \\
&\approx \lambda^{-\frac{1}{r-1}} \left(1 - \frac{s+t+2}{\lambda(r-1)} \right).
\end{aligned}$$

Then insert Y to the left hand-side of (6.13), we get

$$\begin{aligned}
&\lambda Y^{r-1} + (s+t+1)\lambda^{-\frac{1}{r-1}} Y^{r-2} \\
&= \left(1 - \frac{s+t+2}{\lambda(r-1)}\right)^{r-1} + \frac{(s+t+1)}{\lambda} \left(1 - \frac{s+t+2}{\lambda(r-1)}\right)^{r-2} \\
&\approx \left(1 - \frac{s+t+2}{\lambda}\right) + \frac{(s+t+1)}{\lambda} \left(1 - \frac{(s+t+2)(r-2)}{\lambda(r-1)}\right) + O(\frac{1}{\lambda^2}) \\
&= 1 - \frac{1}{\lambda} + O(\frac{1}{\lambda^2}) \\
&< 1.
\end{aligned}$$

Thus inequality (6.13) is verified. By Lemma 6.1.1, we have $\rho(B) \geq \lambda$. A contradiction. □

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