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Quick Trips: On the Oriented Diameter of Graphs

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Quick Trips: On the Oriented Diameter of Graphs

by

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Dedication

This dissertation is dedicated to the memory of my grandparents: Jane Wisness, Osmund Wisness, and Samuel Cochran.

Acknowledgments

Firstly, Prof. Éva Czabarka, thank you so much for all your help and understanding throughout the process of graduate school. I have learned so much about what it means to be a mathematical researcher from you. I also thank you for helping me through the process of applying for jobs, even if it meant keeping my feet to the fire some of the time! You are an advisor in all forms of the word, and I appreciate all the help and guidance in taking me from where I was to where I wanted to go.

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To the extended Wisness and Cochran families, thank you for the support. Nana, thank you for always having an open home with a warm embrace there for me. Thank you Karen and David for allowing Ballground to be a holiday home for me throughout graduate school.

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Abstract

In this dissertation, I will discuss two results on the oriented diameter of graphs with certain properties. In the first problem, I studied the oriented diameter of a graph G. Erdős et al. in 1989 showed that for any graph with $|V| = n$ and $\delta(G) = \delta$ the maximum the diameter could possibly be was $3\frac{n}{\delta+1}$. I considered whether there exists an orientation on a given graph with $|G| = n$ and $\delta(G) = \delta$ that has a small diameter. Bau and Dankelmann (2015) showed that there is an orientation of diameter $11\frac{n}{\delta+1} + O(1)$, and showed that there is a graph which the best orientation admitted is $3\frac{n}{\delta+1} + O(1)$. It was left as an open question whether the factor of 11 in the first result could be reduced to 3. The result above was improved to $7\frac{n}{\delta+1} + O(1)$ by Surmacs (2017) and I will present a proof of a further improvement of this bound to $5\frac{n}{\delta-1} + O(1)$. It remains open whether 3 is the best answer.

In the second problem, I studied the oriented diameter of the complete graph K_n with some edges removed. We will show that given K_n with $n \geq 5$ and any collection of edges E', with $|E'| = n - 5$, that there is an orientation of this graph with diameter 2. It remains a question how many edges we can remove to guarantee larger diameters.

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Chapter 1

Background And Introduction

1.1 Quick Trips on Networks

Many real world objects can be described in terms of graph theory, as graphs are essentially networks. An example of finding small distances in graphs is well known as the "Six Degrees of Kevin Bacon" problem. In this example, one wishes to show the the "distance" of any actor to Kevin Bacon is less than six. This can be described as a network, with the actors as nodes, where connections are made between nodes if the actors have participated in a movie together. Most actors are within 4 steps of Kevin Bacon. For a more thorough investigation of this problem, see the work by Collins and Chow (1998). This phenomenon, the "Small World Phenomenon" can be detected in different types of networks, and consequently is well studied in the area of Network Science. The internet (Albert, Jeong, and Barabási 1999; Barabási, Albert, and Jeong 2000), social networks (Perliger and Pedahzur 2011; Ugander et al. 2011; Myers et al. 2014), brain neural networks(Sporns et al. 2004), and protein-protein interaction networks (Bork et al. 2004; Van Noort, Snel, and Huynen 2004) are just a few examples. The study of these social networks has given rise to a better understanding of terrorist networks (Perliger and Pedahzur 2011) and the spread of disease(Klovdahl et al. 1994). Distance in a graph is also used to form the base of Google's Page Rank algorithm (Page et al. 1999), which decides the order in which pages show up in a Google search. In this case, two pages are connected if there is a link on one page that goes to the other page, and the number of clicks needed to reach one from the other represents the distance of two pages. The Page Rank algorithm uses this parameter to decide in what order to show webpages to answer a user search. Other examples of networks where it is important to have the diameter, i.e. the largest distance within the network, to be small include transportation networks (Woolley-Meza et al. 2011; Kurant and Thiran 2006) and computer networks (Pandurangan, Raghavan, and Upfal 2003; Royer and Toh 1999).

In unoriented graphs one can traverse the connections between the vertices in any directions, a model that is clearly not appropriate for all applications, including some mentioned before. Oriented graphs have the direction of traverse specified on these connections; and this changes the nature of the problem of finding the (oriented) distance and diameter in a network. The question of whether a bridgeless unoriented graph of small diameter has an orientation with small oriented diameter has been thoroughly investigated (Chvátal and Thomassen 1978; Kwok, Liu, and West 2010; Chung, Garey, and Tarjan 1985). Many results have also been found about the oriented diameter of certain classes of graphs (Gutin 1994; Koh and Tan 1996a; Koh and Tan 1996b; Šoltés 1986). For a relatively comprehensive look at problems about the oriented diameter of a graph, the interested reader can consult a survey by Koh and Tay (2002). More current results on the oriented diameter can be found in the following papers (Fomin, Matamala, and Rapaport 2004; Gutin et al. 2002; Gutin and Yeo 2002; Koh and Ng 2005; Lakshmi 2011; Lakshmi and Paulraja 2007; Lakshmi and Paulraja 2009). A more thorough explanation of these results can be found in Chapter 2.

1.2 Initial Definitions

Definition 1.1. We write $f(x) = O(g(x))$ if there is a positive real number M and a a real number x_0 such that

$$|f(x)| \leq M|g(x)| \text{ for all } x \geq x_0.$$

Definition 1.2. Given a set A, let $\binom{A}{k}$ denote all k element subsets of A.

Definition 1.3. Let $G = (V, E)$ denote a finite graph with vertex set V and edge set $E \subseteq \binom{V}{2}$. Let us only consider graphs without loops and multiple edges.

By $|G|$ we mean the order of G, $|V(G)|$. We sometimes will denote $|G| = n$ and $|E| = m$.

Definition 1.4. Let K_n denote the complete graph on n vertices. That is, a graph with $|G| = n$ and $G = (V, \binom{V}{2})$.

Notation 1.5. Given an integer $k > 0$ let $[k] = \{1, 2, \ldots, k\}$.

Definition 1.6. Let M denote a perfect matching of a graph on $2n$ vertices. That is $M = ([2n], \{\{2i-1, 2i\} | i \in [n]\})$.

Definition 1.7. Given two graphs, G and H with $V(H) \subseteq V(G)$, let $G - H = (V(G), E(G) \setminus E(H))$.

Definition 1.8. Let P_n denote the path on n vertices.

Definition 1.9. Let $K_{a,b}$ denote the complete bipartite graph with partite sets of size a and b respectively.

Definition 1.10. Given $G = (V, E)$, a subgraph H of G, denoted $H \leq G$, is a graph $H = (V', E')$ for which $V' \subseteq V$ and $E' \subseteq E \cap \binom{V'}{2}$.

Definition 1.11. Given $G = (V, E)$, we consider the complement of the graph G, denoted $\overline{G}$, as the graph $\overline{G} = (V, \overline{E})$. That is, every edge not in the original graph is in the complement. Note that $E(G) \cap E(\overline{G}) = \emptyset$ and if $|G| = n$, $K_n = (V, E(G) \cup E(\overline{G}))$.

Definition 1.12. If G and H are graphs, then $G \cup H$ means the disjoint union of G and H. The edgeless graph on n vertices is denoted by nk_1.

Definition 1.13. Given G and $\overline{G}$, it may be easier to consider these graphs as a coloring of the edges of a graph with red and blue. With the red edges representing G, $R = G$, and the blue edges as the complementary edges, $B = \overline{G}$.

Definition 1.14. If $W \subseteq V$, then the red and blue subgraph induced by W in R and B, respectively, is denoted by $R[W]$ and $B[W]$.

Definition 1.15. An orientation of the graph G, denoted $\overrightarrow{G} = (V, A)$ is a digraph with the same vertex set as G, where the arc set A is obtained from $E(G)$ by assigning a single direction to an edge. If the edge $e = uv$ is oriented from u to v, we will denote this arc $\overrightarrow{uv}$.

Notation 1.16. Given a path $P = v_0v_1 \dots v_\ell$, we mean by $\overrightarrow{P}$ the directed path (or the orientation of P) with arcs $\overrightarrow{v_0v_1}, \overrightarrow{v_1v_2}, \dots, \overrightarrow{v_{\ell-1}v_\ell}$. If $e_i = v_{i-1}v_i$, we may also use $\overrightarrow{P} = \overrightarrow{e_1e_2 \dots e_\ell}$ for the same object.

Notation 1.17. If U and W are disjoint subsets of V then we indicate by $U \rightarrow W$ that for all $x \in U$ and $y \in W$ that are adjacent in R we orient the edge xy as $\overrightarrow{xy}$, i.e., from x to y. If U or W consist of a single vertex u or w, respectively then we write $u \rightarrow W$ instead of $\{u\} \rightarrow W$, and similarly $U \rightarrow w$ and $u \rightarrow w$.

Definition 1.18. Given a graph G and an edge set $E' \subseteq E$, define $G \backslash E' = (V, E \backslash E')$, let $G \setminus e = (V, E \setminus \{e\})$.

Definition 1.19. For a set $A \subseteq V(G)$, the induced subgraph of G on the vertex set A is denoted by $G[A]$. That is, $G[A] = (A, \binom{A}{2} \cap E)$.

Definition 1.20. Given a graph G, we call vertices u and v such that $u, v \in V(G)$ adjacent if $uv \in E(G)$. Given $v \in V(G)$, the degree of v in G is the number of vertices adjacent to v. We denote this $\deg(v)$.

Symbolically $\deg(v) = |\{uv : u \in V(G), u \neq v, uv \in E(G)\}|$.

Definition 1.21. The minimum degree of a graph G is $\delta(G) = \min\{\deg(v) : v \in V(G)\}$. If no ambiguity arises, we let $\delta(G) = \delta$.

Definition 1.22. The closed neighborhood of a vertex v, i.e., the set comprising v and all its neighbors, is denoted $N_G[v]$. The open neighborhood, i.e., just the set of neighbors of v is denoted $N_G(v)$. We may also consider $N[v]$ and $N(v)$ if no ambiguity arises.

Definition 1.23. Given a graph G and a set of vertices $A \subseteq V(G)$ we let $N_G(A) = \cup_{v\in A} N_G(v)$ and analogously $N_G[A] = \cup_{v\in A} N_G[v]$. Again, we can consider $N[A]$ and $N(A)$ if no ambiguity arises.

Notation 1.24. Given a path $Q = v_0 v_1 \dots v_k$, we say Q has length $\ell(Q) = k$, so $\ell(P_n) = n - 1$.

Definition 1.25. A walk on k vertices is a sequence of vertices in which each adjacent pair of vertices is adjacent to each other in the graph. A walk is considered closed if it starts and ends at the same vertex.

Definition 1.26. A cycle on n vertices, denoted C_n is a closed walk with no repetitions of edges or vertices allowed.

Definition 1.27. We define the distance between u and v in a graph G or digraph $\overrightarrow{G}$ as the minimum number of edges or arcs on a path from u to v. We denote this as $\rho_G(u, v)$ or $\rho_{\overrightarrow{G}}(u, v)$. If there does not exist a path from u to v, we say that $\rho_G(u, v) = \infty$ or $\rho_{\overrightarrow{G}}(u, v) = \infty$.

Definition 1.28. The eccentricity of a vertex, denoted $\epsilon(v)$, is the maximum distance from v to any other vertex. Symbolically $\epsilon(v) = \max\{\rho_G(v, u) : u \in V(G)\}$ or $\epsilon(v) = \max\{\rho_{\overrightarrow{G}}(v, u) : u \in V(G)\}$

Definition 1.29. A component of G is a maximal set of vertices A for which for all $u, v \in A$, $\rho_G(u, v) < \infty$.

Definition 1.30. A graph is connected if it is comprised of one component.

Definition 1.31. A bridge of a connected graph G is an edge whose removal from $E(G)$ disconnects G into two components.

Definition 1.32. A graph G is bridgeless if it is connected and there is no edge which is a bridge.

Definition 1.33. We call the maximum of all distances between two vertices in a graph or digraph of G the diameter of G, denoted $\text{diam}(G)$ or $\text{diam}(\overrightarrow{G})$. In symbolic terms, $\text{diam}(G) = \max\{\rho_G(u,v) : u,v \in V(G)\}$ and $\text{diam}(\overrightarrow{G}) = \max\{\rho_{\overrightarrow{G}}(u,v) : u,v \in V(\overrightarrow{G})\}$.

Remark 1.34. Note that diameter is related to eccentricity. In particular, $\text{diam}(G) = \max\{\epsilon(v) : v \in V(G)\}$ and $\text{diam}(\overrightarrow{G}) = \max\{\epsilon(v) : v \in V(\overrightarrow{G})\}$.

Definition 1.35. If $\text{diam}(G) < \infty$, we call G connected. If $\text{diam}(\overrightarrow{G}) < \infty$, then we call $\overrightarrow{G}$ strongly connected.

Definition 1.36. We call the minimum of the eccentricities of v for all $v \in G$ or $v \in \overrightarrow{G}$ the radius of G, denoted $\text{rad}(G)$ or $\text{rad}(\overrightarrow{G})$. In symbolic terms, $\text{rad}(G) = \min\{\epsilon(v) : v \in G\}$ and $\text{rad}(\overrightarrow{G}) = \min\{\epsilon(v) : v \in V(\overrightarrow{G})\}$.

Definition 1.37. Given subgraphs H and H' for which $H \leq G$, $H' \leq G$ and $V(H) \subsetneq V(H')$ and strongly connected orientations $\overrightarrow{H} < \overrightarrow{H'}$, we call $\overrightarrow{H'}$ an extension of the orientation $\overrightarrow{H}$.

Notation 1.38. Given H, H', and H'' with $H, H', H'' \leq G$, $A = V(H')$, $B = V(H'')$, $A \subseteq V(H)$, and $B \subseteq V(H)$, let

$$\rho_H(A,B) = \min\{\rho_H(a,b) : a \in A,\, b \in B\} = \rho_H(H',H'') = \rho_H(A,H'') = \rho_H(H',B)$$

. If $A = \{v\}$, let $\rho_H(v,B) = \rho_H(A,B)$. If $B = \{u\}$, consider an analogous definition.

Notation 1.39. Given H, H', and H'' with $H, H', H'' \leq G$, $A = V(H')$, $B = V(H'')$, $A \subseteq V(H)$, and $B \subseteq V(H)$, we define $\text{diam}_H(A, B) = \max\{\rho_H(u, v) : u \in A, v \in B\}$ and consider similar definitions for $\text{diam}_H(H', B)$, $\text{diam}_H(A, H'')$, and $\text{diam}_H(H', H'')$. If $A = \{v\}$, let $\text{diam}_H(v, B) = \text{diam}_H(A, B)$. If $B = \{u\}$, consider an analogous definition.

Remark 1.40. Note that in either of the previous definitions, we could replace H with $\overrightarrow{H}$.

Remark 1.41. Note that since the distance function is symmetric for undirected graphs, we have $\rho_G(A, B) = \rho_G(B, A)$. Note that it is not necessarily true that $\rho_{\overrightarrow{G}}(A, B) = \rho_{\overrightarrow{G}}(B, A)$. To see this consider the cycle on n vertices C_n and an oriented cycle on n vertices with $n \geq 3$

Definition 1.42. We define the oriented diameter of a graph G as the following:

$$\overrightarrow{\text{diam}}(G) = \min\{\text{diam}(\overrightarrow{G}) : \overrightarrow{G} \text{is strongly connected}\}.$$

1.3 Outline of Results

In the rest of this dissertation, I will focus on two main results, both pertaining to the oriented diameter of graphs. The first problem investigates the oriented diameter of a graph G with $|G| = n$ and $\delta(G) = \delta$.

It was shown by Bau and Dankelmann (2015) that for all bridgeless graphs G with $|G| = n$, $\delta(G) = \delta$, that

$$\overrightarrow{\text{diam}}(G) \leq 11\frac{n}{\delta + 1} + O(1).$$

Bau and Dankelmann (2015) also showed that there is a construction of a bridgeless graph G with $|G| = n$, $\delta(G) = \delta$, and

$$\overrightarrow{\text{diam}}(G) \geq 3\frac{n}{\delta + 1} + O(1).$$

This means that for all G with $|G| = n$, $\delta(G) = \delta$, we have that

$$3\frac{n}{\delta + 1} + O(1) \leq \overrightarrow{\text{diam}}(G) \leq 11\frac{n}{\delta + 1} + O(1).$$

It was believed that $11\frac{n}{\delta+1} + O(1)$ was not the best possible upper bound by Bau and Dankelmann (2015). I had preliminary results that suggested an upper bound of $9\frac{n}{\delta-1} + O(1)$. I was writing up this proof when I discovered a paper by Surmacs (2017) that improved the bound to $7\frac{n}{\delta+1} + O(1)$. In Chapter 3 I will show a proof of a new bound of $5\frac{n}{\delta-1} + O(1)$. It is still an open problem what the constant factor on the upper bound should be.

The second problem I will consider will be an investigation of the oriented diameter of the complete graph K_n with some edges removed. We will show that given $K_n \geq 5$ and any collection of edges E', with $|E'| = n - 5$, that $\overrightarrow{\text{diam}}(K_n \setminus E') \leq 2$.

Chapter 2

Previous Results on the Diameter of Graphs

2.1 The Oriented Diameter of Given Graphs

A classical result by Robbins (1939) states that every bridgeless graph has a strongly connected orientation. This result unfortunately gives no information on distances in this orientation. It may be advantageous to know the diameter of these orientations. In particular it can help to know if there is a graph with a small orientation. The minimum diameter over all orientations of a given graph is referred to as the oriented diameter of a graph. A formal definition of this is given in Section 1.2.

The first natural question that was posed was whether a bridgeless graph of diameter d has an orientation of small diameter was shown to be affirmative by Chvátal and Thomassen (1978), who showed the following theorem.

Theorem 2.1. *Given a bridgeless graph G for which $diam(G) = d$,*

$$\overrightarrow{diam}(G) \leq 2d^2 + 2d.$$

Chvátal and Thomassen (1978) also showed that there exist graphs of diameter d for which every orientation has diameter at least $\frac{1}{2}d^2 + d$. This implies a gap between $\frac{1}{2}d^2 + d$ and $2d^2 + 2d$, and it is unknown for $d \geq 3$ what the maximum oriented diameter is.

Chvátal and Thomassen (1978) also showed that finding the oriented diameter of a bridgeless graph is NP-complete. While this is unfortunate, algorithms can be found that give approximate answers. A linear time algorithm showing that a bridgeless graph of diameter d has a strongly connected orientation of diameter at most $8d^2+8d$ was given by Chung, Garey, and Tarjan (1985).

Definition 2.2. A complete bipartite graph is a graph where given vertex sets V_1, V_2, $|V_1| = n$, $|V_2| = m$, $V_1 \cap V_2 = \emptyset$ and $E = \{v_1v_2 : v_1 \in V_1, v_2 \in V_2\}$, $G = K_{n,m} = (V_1 \cup V_2, E)$.

Definition 2.3. A complete k-partite graph is a generalization of a complete bipartite graph where $V = V_1 \cup \cdots \cup V_k$, $V_i \cap V_j = \emptyset$ for $i \neq j$, and $E = \{uw : u \in V_i, w \in V_j, i \neq j\}$.

The oriented diameter of a complete bipartite graph was shown to be between 3 and 4 by Šoltés (1986). It was shown that complete k-partite graphs with $k \geq 3$ always have an oriented diameter of between 2 and 3 in the following papers (Gutin et al. 2002; Plesník 1985).

Definition 2.4. In an undirected graph G, a dominating set is a set of vertices $S \subseteq V(G)$ for which every vertex is in S or is adjacent to S.

Definition 2.5. The domination number of a graph G, denoted $\gamma(G)$, is the minimum cardinality of such a set S.

The domination number is another parameter of interest to those who study oriented diameter. In particular, Campan, Truta, and Beckerich (2015) showed the following theorem:

Theorem 2.6. *Every bridgeless graph with domination number γ has an orientation of diameter at most $9\gamma - 5$.*

This was improved in the following theorem by Laetsch and Kurz 2012:

Theorem 2.7. *Every bridgeless graph with domination number γ has an orientation of diameter at most $\lceil \frac{7\gamma+2}{2} \rceil$.*

Definition 2.8. Given the problem of finding an orientation of small diameter, we call an algorithm an (a, b) approximation algorithm, if for every graph G, the algorithm outputs an orientation $\overrightarrow{H}$ of G for which

$$\text{diam}(\overrightarrow{H}) \leq a \cdot \overrightarrow{\text{diam}}(G) + b.$$

Definition 2.9. Given a cycle C, a chord is an edge e for which both endpoints are in C, yet the edge is not in C. A chordless or induced cycle in a graph G is a cycle of length more than 3 which has no chord. A graph G is chordal if it has no chordless cycles.

Fomin, Matamala, and Rapaport (2004) proved that there exists an approximation algorithm for the problem of finding an orientation of small diameter of any chordal graph.

Theorem 2.10. *There is a linear time $(2, 1)$-approximation algorithm for finding the oriented diameter on the class of chordal graph.*

In particular Fomin, Matamala, and Rapaport (2004) proved the stronger result below. The authors also proved that this was the best possible result by showing the construction below.

Theorem 2.11. *There is a linear-time algorithm such that, given a chordal graph G, it computes an orientation $\overrightarrow{G}$ of G such that for all pairs of vertices $u, v \in V(G)$,*

$$\rho_{\overrightarrow{G}}(u, v) \leq 2\rho_G(u, v) + 1.$$

Theorem 2.12. *There exists a chordal graph G^n for which diam$(G^n) = 2n + 1$ and diam$(\overrightarrow{G^n}) = 2\,$diam$(G^n) + 1$ for every strongly connected orientation $\overrightarrow{G^n}$ of G^n.*

More results on the oriented diameter of graphs can be found in the following papers Gutin 1994; Gutin et al. 2002; Gutin and Yeo 2002; Koh and Ng 2005; Lakshmi 2011; Lakshmi and Paulraja 2007; Lakshmi and Paulraja 2009.

Noticing that the complete graph, K_n, for $n \geq 5$ admits an orientation of diameter at most 2, the natural question of how many edges you can remove from the graph and have that there still exists an orientation of diameter 2. We discuss this problem in Chapter 4.

2.2 The Oriented Diameter of a Graph with Minimum Degree

Given a graph with large minimum degree, it would be expected that the oriented diameter of the graph would be small, as large minimum degree means that a lot of the vertices are connected to each other. A well known result by Erdős et al. 1989 shows that the diameter of a connected graph of order n and minimum degree δ is at most $3\frac{n}{\delta+1} + O(1)$. Note that this result is for the diameter of an unoriented graph. We wish to consider results of similar form for oriented diameter. That is, we wish to consider if given a graph G, there is an orientation of diameter $c\frac{n}{\delta+1} + O(1)$.

A reminder that Bau and Dankelmann (2015) showed that there is an orientation of diameter $11\frac{n}{\delta+1} + O(1)$, and showed that there is a graph which the best orientation admitted is $3\frac{n}{\delta+1} + O(1)$. It was left as an open question whether the factor of 11 in the first result could be reduced to 3.

The result above was improved to $7\frac{n}{\delta+1} + O(1)$ by Surmacs (2017) and I will present a proof in chapter 3 of a further improvement of this bound to $5\frac{n}{\delta-1} + O(1)$.

Chapter 3

The Oriented Diameter of a Graph with Minimum Degree

In this chapter I will prove the following theorem:

Theorem 3.1. *Given a graph G, with $|G| = n$ and $\delta(G) = \delta$,*

$$\overrightarrow{diam}(G) \leq 5\frac{n}{\delta - 1} + O(1).$$

The proof of this theorem is quite involved, so it is split into different sections. Section 3.1 will introduce the requisite definitions and the main lemma we will need to prove the theorem. This main lemma will be proved in two distinct stages, outlined in sections 3.2 and 3.3. Finally, in section 3.4 we will prove the theorem using the main lemma.

3.1 Introduction to the Main Lemma

Let $H \leq G$, $A \subseteq V(H)$ and $\delta(G) = \delta$.

Definition 3.2. Given $A \subseteq V(G)$, call A a δ-set if $|N[A]| \geq (\delta - 1)|A|$.

Definition 3.3. Given $H \leq G$ and $A \subseteq V(H)$, if

P1 A is a δ-set, and

P2 there exists an orientation $\overrightarrow{H}$ of H of diameter at most $5|A|$.

we call $(\overrightarrow{H}, A)$ a target pair.

Definition 3.4. Given a target pair $(\overrightarrow{H_1}, A_1)$, if there exists a vertex v of G with $\rho(v, H_1) = 6$, we call $(\overrightarrow{H_1}, A_1)$ an extendable target pair. We will denote by v_6 a vertex such that $\rho_G(v_6, H_1) = 6$, $P = v_0v_1v_2v_3v_4v_5v_6$ is a shortest path from $V(H)$ to v_6. We further let $e_i = v_{i-1}v_i$. See 3.1 for an example of this labelling of edges and vertices.

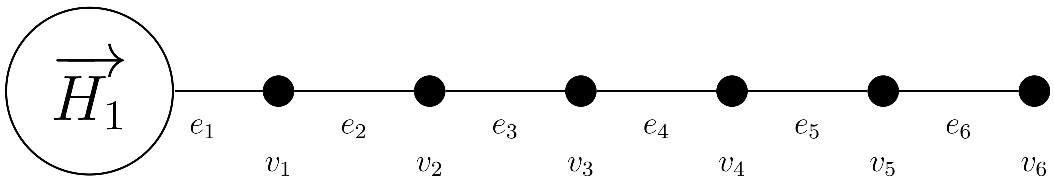

Figure 3.1 An example of an extendable target pair and a path of length 6.

Definition 3.5. We call $(\overrightarrow{H_2}, A_2)$ an extension of a target pair $(\overrightarrow{H_1}, A_1)$ if $(\overrightarrow{H_2}, A_2)$ is a target pair itself, $A_2 \supset A_1$, and $\overrightarrow{H_2}$ is an extension of $\overrightarrow{H_1}$.

Definition 3.6. Let $\ell_i(a, b) = \rho_{G \setminus \{e_1, \ldots, e_i\}}(a, b)$.

Definition 3.7. If $\ell_i(v_i, H_1) < \infty$, let $P_i = v_{i,0}v_{i,1} \ldots v_{i,\ell_i(v_i,H_1)}$ be a shortest path from v_i to H_1 in $G \setminus \{e_1, \ldots, e_i\}$.

Remark 3.8. Notice that for $k \leq i$, $G \setminus \{e_1 \ldots e_k\} \leq G \setminus \{e_1 \ldots e_i\}$, which implies $\ell_k(a, b) \leq \ell_i(a, b)$. We also have for all i that $\ell_i(v_i, H_1) \geq \rho(v_i, H_1) = i$.

See Figure 3.2 for an example of an extendable target pair $(\overrightarrow{H_1}, A_1)$ with three vertices v_1, v_2 and v_3 for which $\ell_1(v_1, H_1) = 2$, $\ell_2(v_2, H_1) = 2$, and $\ell_3(v_3, H_1) = 8$ and a labeling of the vertices on those paths.

Definition 3.9. Let j be the smallest index for $v_i \in V(P)$ such that $|N_G[v_i] \cap N_G[H_1]| \leq 2$. Let $s_1 := v_j$.

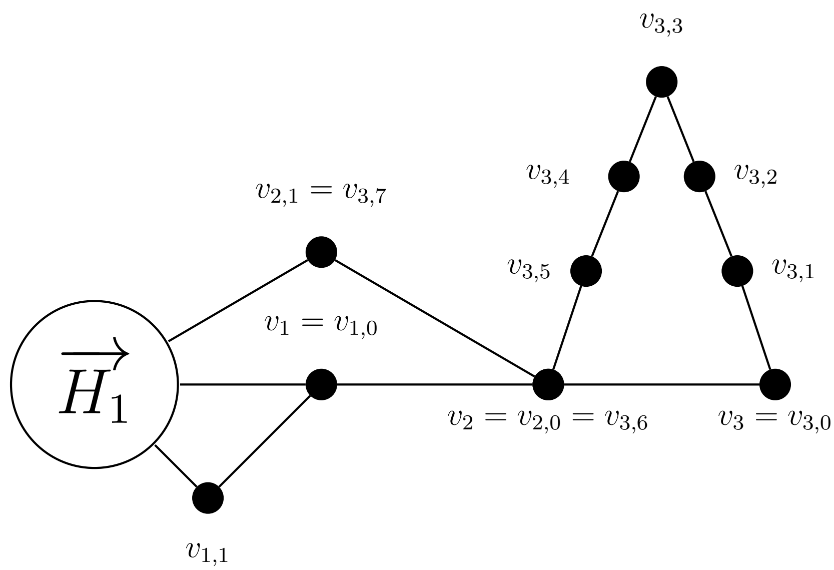

Figure 3.2 Labels of vertices on paths back to the original graph.

Remark 3.10. Notice that $1 \leq j \leq 3$, since for $i \geq 3$, $\rho_G(v_i, H_1) = i \geq 3$, so $N_G[v_3] \cap N_G[H_1] = \emptyset$.

Consider Figure 3.3 for some examples of how j is found. Note that these examples are drawn as if every path of length $\ell_i(v_i, H_1)$ is included in the figure.

Definition 3.11. Given an extendable target pair $(\overrightarrow{H_1}, A_1)$. If $\ell_j(s_1, H_1) \geq 5$, let s_2 be the last vertex on the path P_j such that $|N[s_2] \cap N[s_1]| \geq 3$. If $\ell_j(s_1, H_1) \leq 4$, let $s_2 = v_{j,1}$.

Note that it is possible that $s_2 = s_1$, so s_2 exists.

Definition 3.12. Let $\mathcal{L}_j := \ell_j(v_j, H_1)$.

Lemma 3.13. *Given a bridgeless graph G of minimum degree $\delta = \delta(G)$, there exists a target pair $(\overrightarrow{H_1}, A_1)$. Moreover, if $(\overrightarrow{H_1}, A_1)$ is an extendable target pair, then there exists an extension $(\overrightarrow{H_2}, A_2)$ of $(\overrightarrow{H_1}, A_1)$.*

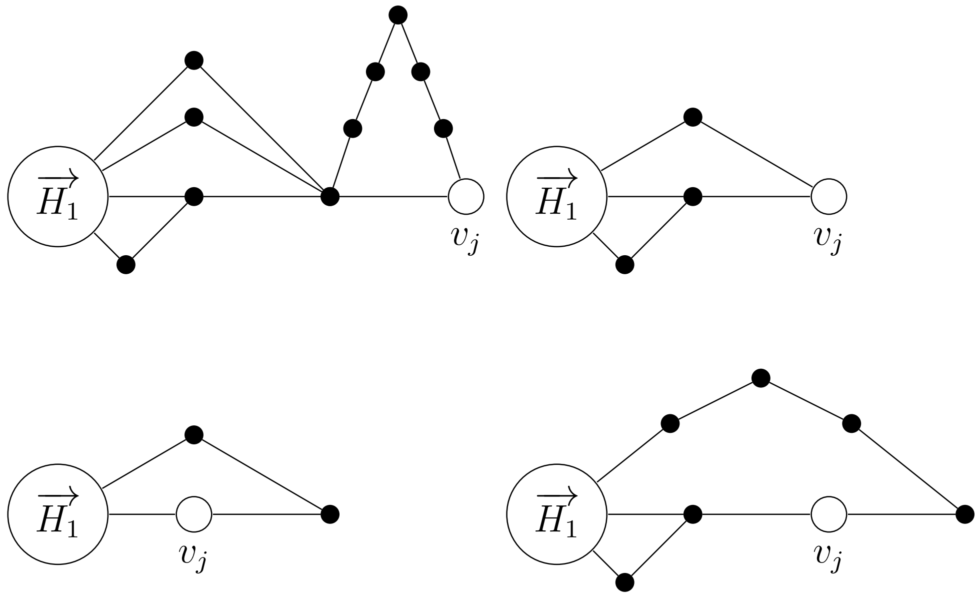

Figure 3.3 Examples of graphs with the first special vertex labelled.

Choose any vertex $v \in G$ and let $A_1 = \{v\}$ and $H = (\{v\}, \emptyset)$. Since $\deg(v) \geq \delta$ we have $|N[A_1]| \geq \delta + 1 \geq \delta - 1$, so property **P1** holds. H_1 has no edges that need to be oriented, and $\overrightarrow{\text{diam}}(H) = 0 \leq 5$, hence property **P2** holds. We now show that given an extendable target pair, $(\overrightarrow{H_1}, A_1)$, we can find an extension to another target pair $(\overrightarrow{H_2}, A_2)$. We will prove this in two stages with several lemmas.

3.2 Stage 1

Note that in all figures after Figure 3.5, the labelling of a vertex as a special vertex of some kind may imply other edges and vertices exist in the graph. These may sometimes be left out of a figure for simplicity even though they may still exist in the underlying graph G.

Let $(\overrightarrow{H_1}, A_1)$ be an extendable target pair.

Lemma 3.14. *$\ell_j(v_j, H_1) < \infty$.*

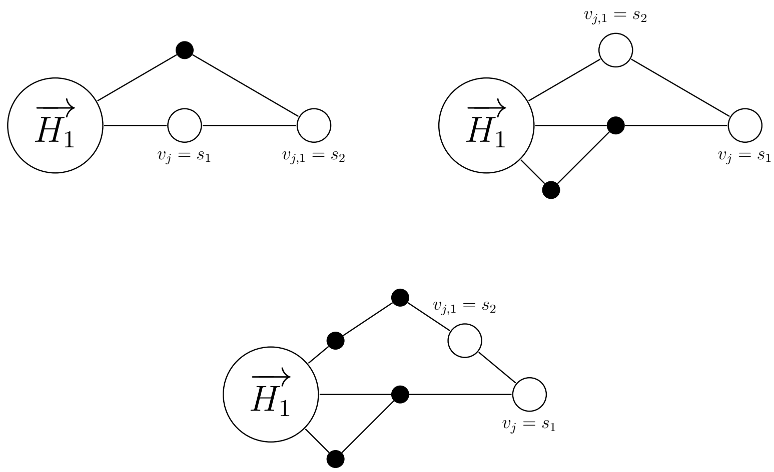

Figure 3.4 Examples of graphs with both the first and second special vertex labelled where the length is less than 4.

Proof. We first have that for $1 \leq p < j$, $|N_G[v_p] \cap N_G[H_1]| \geq 3$, so $\ell_p(v_p, H_1) \leq 2$ and any path of length $\ell_p(v_p, H_1) \leq p+1$ from v_p to H_1 must be disjoint from $\{e_{p+1}, \dots, e_j\}$. If such a path contained an edge $e_q \in \{e_{p+1}, \dots, e_j\}$ then such a path would be a path of length at least $q+1 \geq p+2 \geq 2$. Hence $\ell_j(v_p, H_1) \leq p+1$ for all $p \in [j-1]$.

Since G is bridgeless, $G - e_j$ contains a path from v_j to $V(H_1) \cup \{v_1, \dots, v_{j-1}\}$. Let Q be a shortest such path. Q does not contain any of the edges $e_1, \dots, e_{j-1}$. Hence Q is a path in $G - \{e_1, \dots, e_j\}$. If Q ends in a vertex of H_1, then $\ell_j(v_j, H_1) = \ell(Q) < \infty$. If Q ends at v_p, $p \in \{1, \dots, j-1\}$, then Q together with a shortest (v_p, H_1)-path form a (v_j, H_1)-walk in $G \setminus \{e_1, \dots, e_j\}$ of length at most $\ell(Q) + p + 1$. In both cases we conclude that $\ell_j(v_j, H_1) < \infty$. □

Lemma 3.15. *Given an extendable target pair $(\overrightarrow{H_1}, A)$, and a set $A' \subseteq V(G) \setminus A$ such that $|N[A']| - |N[H_1] \cap N[A']| \geq (\delta - 1)|A'|$, $A \cup A'$ is a δ-set of G.*

Proof. Let $A \subseteq V(H_1)$ with $|N[A]| \geq (\delta - 1)|A|$, $A' \subseteq V(G) \setminus A$, and $|N[A']| -$

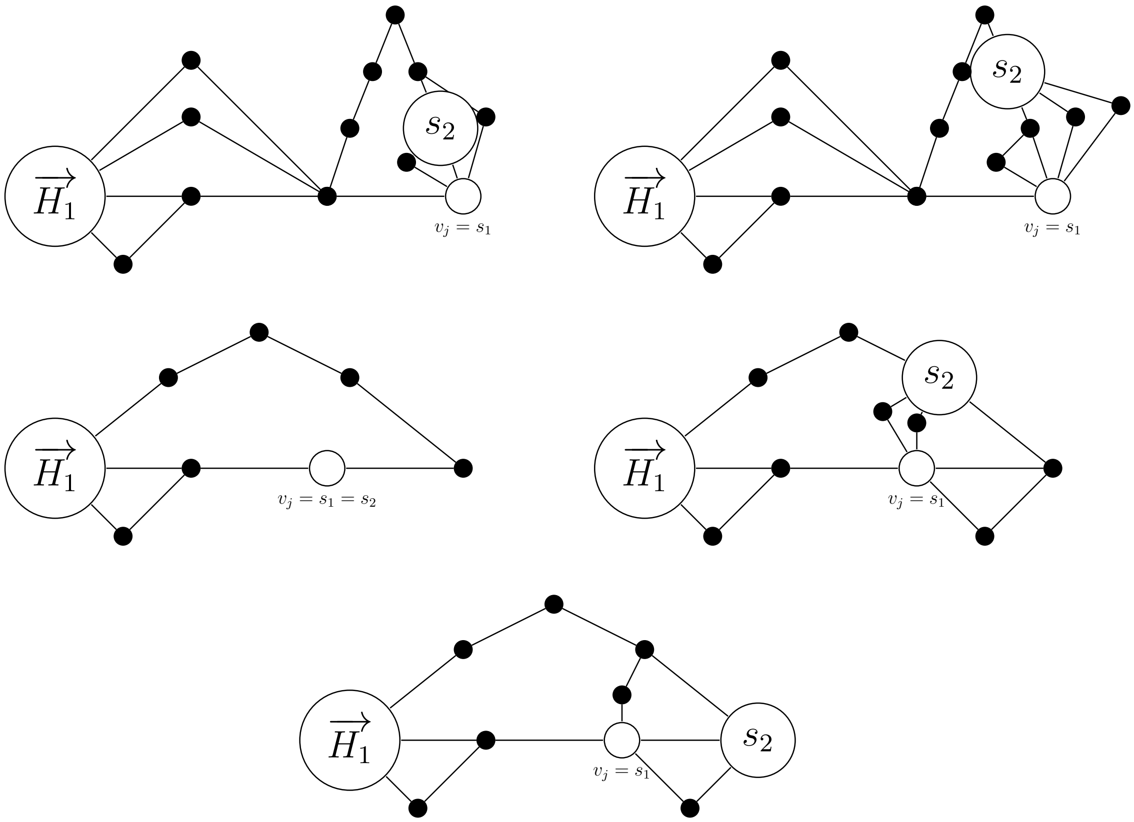

Figure 3.5 Examples of graphs with both the first and second special vertex labelled where the length is greater than or equal to 4.

$|N[A'] \cap N[H_1]| \geq (\delta - 1)|A'|$.

$$\begin{aligned}
|N[A \cup A']| &= |N[A] \cup N[A']| \\
&= |N[A]| + |N[A']| - |N[A] \cap N[A']| \\
&\geq |N[A]| + |N[A']| - |N[H_1] \cap N[A']| \\
&\geq (\delta - 1)|A| + (\delta - 1)|A'| \\
&= (\delta - 1)|A \cup A'|.
\end{aligned}$$

□

This means that in any case where we want to extend a δ-set A to a new δ-set $A' \cup A$ with $A \cap A' = \emptyset$ it is sufficient to prove that $|N[A']| - |N[H_1] \cap N[A']| \geq (\delta - 1)|A'|$

for it to be a δ-set.

Definition 3.16. Let $V' = V(H_2) \setminus V(H_1)$.

Lemma 3.17. *Given $H_1 \leq G$, a strong orientation $\overrightarrow{H_1}$, and $\overrightarrow{H_2}$, an extension of $\overrightarrow{H_1}$,*

$$\mathit{diam}(\overrightarrow{H_2}) \leq \max\{\mathit{diam}_{\overrightarrow{H_2}}(\overrightarrow{H_1}, V'), \mathit{diam}_{\overrightarrow{H_2}}(V', \overrightarrow{H_1}), \mathit{diam}_{\overrightarrow{H_2}}(V', V'), \mathit{diam}(\overrightarrow{H_1})\}.$$

Proof. Since $\overrightarrow{H_1}$ is a strongly connected orientation and $V(H_2) = V(H_1) \cup V'$

$$\begin{aligned}
\operatorname{diam}(\overrightarrow{H_2}) &= \max\{\rho_{\overrightarrow{H_2}}(u,v) : u, v \in V(\overrightarrow{H_2})\} \\
&= \max\{\max\{\rho_{\overrightarrow{H_2}}(u,v) : u \in V(\overrightarrow{H_1}), v \in V'\}, \\
&\qquad \max\{\rho_{\overrightarrow{H_2}}(u,v) : u \in V', v \in V(\overrightarrow{H_1})\}, \\
&\qquad \max\{\rho_{\overrightarrow{H_2}}(u,v) : u \in V', v \in V'\}, \\
&\qquad \max\{\rho_{\overrightarrow{H_2}}(u,v) : u \in V(\overrightarrow{H_1}), v \in V(\overrightarrow{H_1})\}\} \\
&\leq \max\{\operatorname{diam}_{\overrightarrow{H_2}}(u,v), \operatorname{diam}_{\overrightarrow{H_2}}(V', \overrightarrow{H_1}), \operatorname{diam}_{\overrightarrow{H_2}}(u,v), \operatorname{diam}(\overrightarrow{H_1})\}. \square
\end{aligned}$$

Lemma 3.18. *Given a graph H_1 with a strongly connected orientation $\overrightarrow{H_1}$, and an extension $\overrightarrow{H_2}$ of $\overrightarrow{H_1}$ such that $E(\overrightarrow{H_2}) \setminus E(\overrightarrow{H_1})$ constitutes a trail of length q that starts and ends in $V(H_1)$, $\mathit{diam}(\overrightarrow{H_2}) \leq \mathit{diam}(\overrightarrow{H_1}) + q - 1$.*

Proof. Let $V' = V(H_2) \setminus V(H_1)$.

Notice that the following inequalities hold:

$$\begin{aligned}
\operatorname{diam}_{\overrightarrow{H_2}}(\overrightarrow{H_1}, V') &\leq \operatorname{diam}(\overrightarrow{H_1}) + (q-1) \\
\operatorname{diam}_{\overrightarrow{H_2}}(V', \overrightarrow{H_1}) &\leq \operatorname{diam}(\overrightarrow{H_1}) + (q-1) \\
\operatorname{diam}_{\overrightarrow{H_2}}(\overrightarrow{H_1}, \overrightarrow{H_1}) &\leq \operatorname{diam}(\overrightarrow{H_1}) \\
\operatorname{diam}_{\overrightarrow{H_2}}(V', V') &\leq \operatorname{diam}(\overrightarrow{H_1}) + (q-1).
\end{aligned}$$

So by Lemma 3.17 we find that $\operatorname{diam}(\overrightarrow{H_2}) \leq \operatorname{diam}(\overrightarrow{H_1}) + q - 1$. □

Recall $s_1 := v_j$, and if $\ell_j(v_j, H_1) \geq 5$, then s_2 is the last vertex on the path P_j such that $|N[s_2] \cap N[s_1]| \geq 3$. If $\ell_j(v_j, H_1) \leq 4$, then $s_2 = v_{j,1}$.

Lemma 3.19. *Let $(\overrightarrow{H_1}, A_1)$ be an extendable target pair, $k = \lfloor \frac{\ell_j(s_2,H)-1}{3} \rfloor$, $A' = \{v_{j,\mathcal{L}_j-3s} | s \in [k]\} \cup \{v_j\}$, and $A_2 = A' \cup A_1$. Then A_2 is a δ-set.*

Proof. Let $(\overrightarrow{H_1}, A_1)$ be an extendable target pair, $k = \lfloor \frac{\ell_j(s_2,H)-1}{3} \rfloor$, $A' = \bigcup_{s\in[k]}\{v_{j,\mathcal{L}_j-3s}\} \cup \{v_j\}$, and $A_2 = A_1 \cup A'$.

If $k = 0$, $A' = \{v_j\}$, so $|N[A']| \geq \delta+1$. By the definition of v_j, $|N[H_1] \cap N[A']| \leq 2$. Hence, $|N[A']| - |N[H] \cap N[A']| \geq (\delta - 1)|A'|$, so by Lemma 3.15, A_2 is a δ-set.

If $k > 0$, let $A' = \bigcup_{s\in[k]}\{v_{j,\ell-3s}\} \cup \{v_j\}$. For $s \in [k] \setminus \{1\}$, and $0 < p < j$ there must be no edges $v_{j,\mathcal{L}_j-3s}v_p$. If there were such an edge, then this edge $v_{j,\ell-3s}v_p$ together with P_p would form a $(v_{j,\ell} - 3s, H)$-path Q of length at most 3 in $G - \{e_1, \ldots, e_j\}$, and since $s > 1$, the $(v_{j,\mathcal{L}_j-3s}, H)$-section of P_j has length at least six. Hence P_j would not be a shortest (v_j, H)-path in $G - \{e_1, \ldots, e_j\}$, a contradiction.

For all $s, t \in [k]$, $s \neq t$, $\ell_j(v_{j,\mathcal{L}_j-3s}, v_{j,\mathcal{L}_j-3t}) \geq 3$, since P_j is a shortest (v_j, H) path. Hence, $N[v_j, \ell - 3s]$, $s \in [k]$ are pairwise disjoint. Notice that $N[v_{j,\mathcal{L}_j-3}] \cap N[A_1] \subseteq \{v_1\}$. If there were another vertex, there would be a length 2 path edge disjoint from $e_1 \ldots e_j$ from $v_{j,\mathcal{L}_j-3}$ to H a contradiction to the definition of P_j. Notice that $N[v_{j,\mathcal{L}_j-3}] \cap N[v_j] \subseteq \{v_{j-1}\}$ for a similar reason. By our arguments above $|N[A']| \geq |\bigcup_{a\in A'} N[a] \setminus \{v_{j-1}\}| \geq |A'|(\delta+1) - 1$. Since $N[v_{j,\mathcal{L}_j(v_j,H)-3}] \cap N[A_1] \subseteq \{v_1\}$, $|N[H] \cap N[A']| \leq |N[v_j] \cap V(H)| \leq 2$ and either $\{v_1\} \subseteq N[v_j] \cap N[H]$, or $N[v_j] \cap N[H] = \emptyset$. Since $|A'| \geq k + 1 \geq 2$, $|N[A']| - |N[H] \cap N[A']| \geq (\delta + 1)|A'| - 3 \geq (\delta - 1)|A'|$, so by Lemma 3.15, A_2 is a δ-set in G. □

See Figure 3.6 to see an example of this with $k > 0$. In this figure, dashed lines represent edges that do not exist in the graph. Note that A' is comprised of the vertices that are represented as diamonds.

Lemma 3.20. *If $\ell_j(v_j, H_1) \geq 6$, $\ell_j(s_2, H_1) \geq 4$.*

Proof. Let $\ell_j(v_j, H_1) \geq 6$. This implies $\ell_j(v_j, v_{j,3}) \geq 3$. Note that if $j \geq 2$ and $v_{j,3} = v_{j-1}$, then $\ell_j(v_{j-1}, H_1) \geq 3$. This implies that $|N[v_{j-1}] \cap N[H_1]| \leq 2$, a

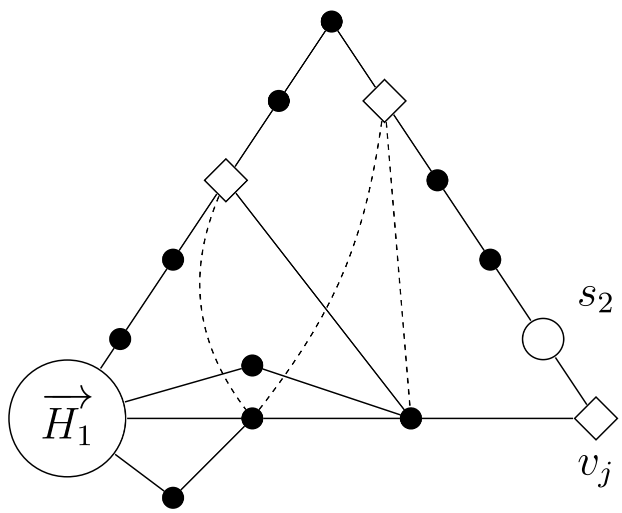

Figure 3.6 An example of a long path and the possible connections.

contradiction to the definition of v_j, so $v_{j-1} \neq v_{j,3}$. If there was a vertex $v \neq v_{j-1}$ such that $v \in N[v_{j,3}] \cap N[v_j]$, then $\ell_j(v_{j,3}, v_j) = 2$, a contradiction to the definition of P_j, so $N[v_{j,3}] \cap N[v_j] \subseteq \{v_{j-1}\}$. Hence, $|N[v_{j,3}] \cap N[v_j]| \leq 1$. Hence, $s_2 \in \{v_j, v_{j,1}, v_{j,2}\}$, which implies $\ell_j(s_2, H_1) \geq \ell_j(v_{j,2}, H_1) = \ell_j(v_j, H_1) - 2 \geq 4$. □

Lemma 3.21. *Given an extendable target pair* $(\overrightarrow{H_1}, A_1)$*, if* $j + \mathcal{L}_j \leq 6$*, there exists an extension* $(\overrightarrow{H_2}, A_2)$ *of* $(\overrightarrow{H_1}, A_1)$*.*

Proof. Let $j + \mathcal{L}_j \leq 6$. Let $A_2 = A_1 \cup \{v_j\}$, and $\overrightarrow{H_2} = \overrightarrow{H_1} \cup \{\overrightarrow{e_1 \ldots e_j}\} \cup \overrightarrow{P_j}$.

By Lemma 3.19, Property **P1** holds for A_2. Since the edges we added to $\overrightarrow{H_1}$ to get $\overrightarrow{H_2}$ make a directed trail from $\overrightarrow{H_1}$ to $\overrightarrow{H_1}$ of length at most 6, we have by Lemma 3.18

$$\mathrm{diam}(\overrightarrow{H_2}) \leq \mathrm{diam}(\overrightarrow{H_1}) + (6) - 1 = \mathrm{diam}(\overrightarrow{H_1}) + 5.$$

Since $|A_2 \setminus A_1| = 1$

$$\operatorname{diam}(\overrightarrow{H_2}) \leq \operatorname{diam}(\overrightarrow{H_1}) + 5 \leq 5|A_1| + 5 = 5|A_2|.$$

Hence, property **P2** holds. □

See Figure 3.7 for examples of orientations in this case. We again represent as diamonds any vertex which is in A'.

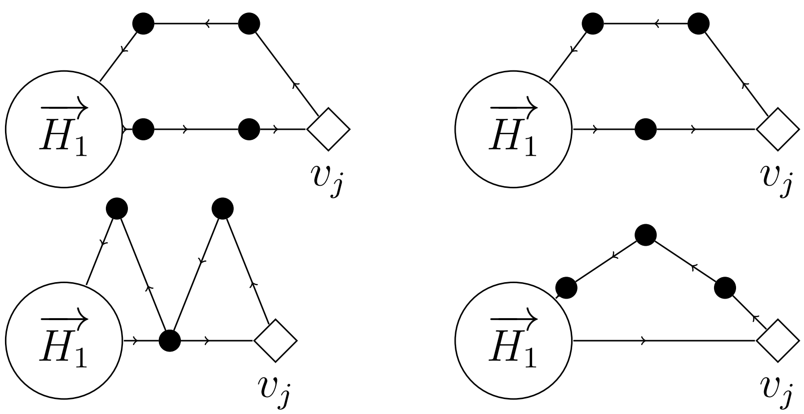

Figure 3.7 Examples of extensions with a short directed trails and one special vertex added.

Lemma 3.22. *Given an extendable target pair $(\overrightarrow{H_1}, A_1)$, if $\ell_j(s_2, H_1) \geq 4$, there exists an extension $(\overrightarrow{H_2}, A_2)$ of $(\overrightarrow{H_1}, A_1)$.*

Proof. Let $\ell_j(s_2, H_1) \geq 4$, and $k = \lfloor \frac{\ell_j(s_2,H_1)-1}{3} \rfloor$. Note that $k \geq \lfloor \frac{\ell_j(s_2,H_1)-1}{3} \rfloor \geq \lfloor \frac{4-1}{3} \rfloor \geq 1$. Let $A_2 = A_1 \cup \bigcup_{p \in [k]} \{v_{j,\mathcal{L}_j - 3p}\} \cup \{v_j\}$. By Lemma 3.19, and since $j \leq 3$, A_2 satisfies Property **P1**. Since s_2 is the first vertex on P_j for which $|N[v_j] \cap N[s_2]| \leq 2$, we have that $1 \leq \ell_j(v_j, s_2) \leq 3$.

Let $\overrightarrow{H_2} = \overrightarrow{H_1} \cup \overrightarrow{e_1 e_2 \ldots e_j} \cup \overrightarrow{P_j}$. Note that $\overrightarrow{e_1 e_2 \ldots e_j} \cup \overrightarrow{P_j}$ is a trail of length at most $j + 3 + 3(k+1)$. By Lemma 3.18 and since $j \leq 3$,

$$\operatorname{diam}(\overrightarrow{H_2}) \leq \operatorname{diam}(\overrightarrow{H_1} \cup \overrightarrow{e_1 e_2 \ldots e_j} \cup \overrightarrow{P_j})$$

$$\leq \operatorname{diam}(\overrightarrow{H_1}) + j + 3 + 3(k+1) - 1$$

$$\leq \operatorname{diam}(\overrightarrow{H_1}) + 3(k+1) + 7.$$

Noticing that $|A_2| - |A_1| = (k+2) \geq 2$,

$$\begin{aligned}
\operatorname{diam}(\overrightarrow{H_2}) &\leq \operatorname{diam}(\overrightarrow{H_1}) + 3(k+1) + 7 \\
&\leq 5|A_1| + 3(k+2) + 4 \\
&\leq 5|A_1| + 3(|A_2 \setminus A_1|) + 2(|A_2 \setminus A_1|) \\
&\leq 5|A_1| + 5|A_2 \setminus A_1| \\
&\leq 5|A_2|.
\end{aligned}$$

Hence, property **P2** holds. □

See Figure 3.8 for some examples of orientations of this case. Again, the vertices that are represented by diamonds are added to A'.

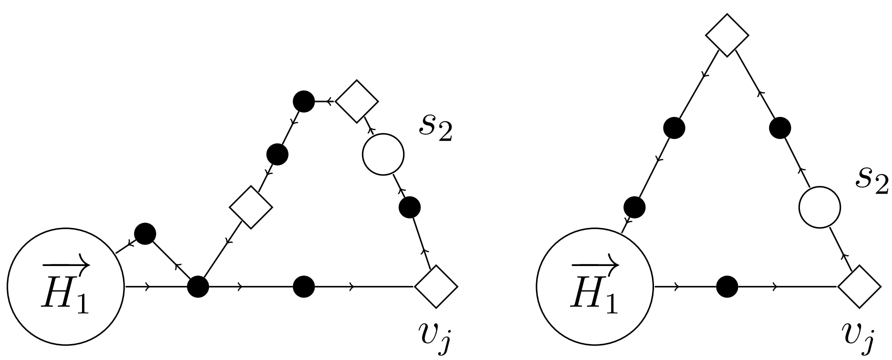

Figure 3.8 Examples of extensions with a long paths back to the original subgraph and two or more special vertices added.

Theorem 3.23. *Given an extendable target pair* $(\overrightarrow{H_1}, A_1)$*, assume that one of the following holds:*

1. $j = 1$,
2. $j = 2$ *and* $\mathcal{L}_j \neq 5$,
3. $j = 2$, $\mathcal{L}_j = 5$, *and* $\ell_j(s_2, H_1) \geq 4$
4. $j = 3$, $\mathcal{L}_j \notin \{4, 5\}$, *or*
5. $j = 3$, $\mathcal{L}_j \in \{4, 5\}$, *and* $\ell_j(s_2, H_1) \geq 4$.

Then there exists an extension $(\overrightarrow{H_2}, A_2)$ *of* $(\overrightarrow{H_1}, A_1)$.

Proof. Case 1: $j = 1$

If $\mathcal{L}_j \leq 5$ use Lemma 3.21 to extend the target pair $(\overrightarrow{H_1}, A_1)$ to the target pair $(\overrightarrow{H_2}, A_2)$. If $\mathcal{L}_j \geq 6$, use Lemmas 3.20 and 3.22 to extend the target pair $(\overrightarrow{H_1}, A_1)$ to the target pair $(\overrightarrow{H_2}, A_2)$. By Lemma 3.14, $\mathcal{L}_j < \infty$, hence we have considered all cases where $j = 1$.

Case 2: $j = 2$ and and $\mathcal{L}_j \neq 5$,

If $\mathcal{L}_j \leq 4$ use Lemma 3.21 to extend the target pair $(\overrightarrow{H_1}, A_1)$ to the target pair $(\overrightarrow{H_2}, A_2)$. If $\mathcal{L}_j \geq 6$, use Lemmas 3.20 and 3.22 to extend the target pair $(\overrightarrow{H_1}, A_1)$ to the target pair $(\overrightarrow{H_2}, A_2)$. By Lemma 3.14 we find that $\mathcal{L}_j < \infty$.

Case 3: $j = 2$, $\mathcal{L}_j = 5$, and $\ell_j(s_2, H_1) \geq 4$

Use Lemma 3.22 to extend the target pair $(\overrightarrow{H_1}, A_1)$ to the target pair $(\overrightarrow{H_2}, A_2)$.

Case 4: $j = 3$, $\mathcal{L}_j \notin \{4, 5\}$

If $\mathcal{L}_j \leq 3$ use Lemma 3.21 to extend the target pair $(\overrightarrow{H_1}, A_1)$ to the target pair $(\overrightarrow{H_2}, A_2)$. If $\mathcal{L}_j \geq 6$ use Lemmas 3.20 and 3.22 to extend the target pair $(\overrightarrow{H_1}, A_1)$ to the target pair $(\overrightarrow{H_2}, A_2)$. By Lemma 3.14 we find that $\mathcal{L}_j < \infty$. So we are only left to consider the case of $4 \leq \mathcal{L}_j \leq 5$.

Case 5: $j = 3$, $\mathcal{L}_j \in \{4, 5\}$, and $\ell_j(s_2, H_1) \geq 4$

If $\ell_j(s_2, H_1) \geq 4$, use Lemma 3.22 to extend the target pair $(\overrightarrow{H_1}, A_1)$ to the target pair $(\overrightarrow{H_2}, A_2)$. $\square$

The only cases left to prove in Stage II are:

1. $j = 2$, $\ell_j(v_j, H_1) = 5$ and $\ell_j(s_2, H_1) \leq 3$, or

2. $j = 3$, $4 \leq \ell_j(v_j, H_1) \leq 5$ and $\ell_j(s_2, H_1) = 3$.

Note that if $\ell_j(v_j, H_1) = 5$, since $N|[s_2] \cap N[s_1]| \geq 3$, we have that $\ell_j(s_1, s_2) \leq 2$, so $\ell_j(s_2, H_1) \geq \ell_j(s_1, H_1) - \ell_j(s_1, s_2) \geq 3$. So $\ell_j(s_2, H_1) = 3$. So in the case that $j = 2$, $\ell_j(v_j, H_1) = 5$ and $\ell_j(s_2, H_1) \leq 3$, we may assume $\ell_j(s_2, H_1) = 3$.

3.3 Stage 2

For Stage II, assume that one of the following hold:

Q1 $j = 2$, $\ell_j(v_j, H_1) = 5$ and $\ell_j(s_2, H_1) = 3$, or

Q2 $j = 3$, $4 \leq \ell_j(v_j, H_1) \leq 5$ and $\ell_j(s_2, H_1) = 3$.

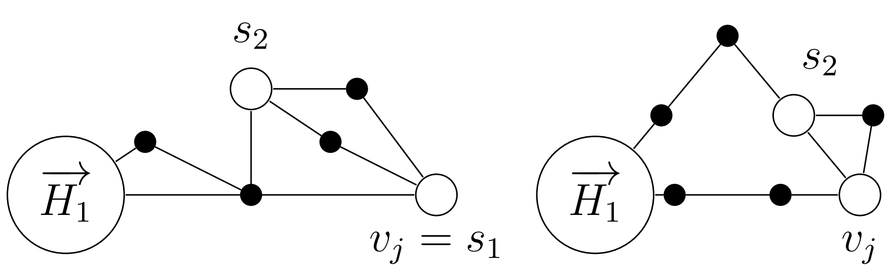

Figure 3.9 Examples of the structures we will consider in Stage 2.

If there is a path of length 2 between s_1 and s_2 that includes e_{j+1}, redefine $P_j := v_j v_{j+1} s_2 \cup P_j[s_2 \ldots v_{j,\mathcal{L}_j}]$, even if $s_1 s_2 \in E(G)$. Notice that this increases $\ell(P_j)$ by at most 1.

Definition 3.24. Let $\overrightarrow{H_1'}$ be the extension of $\overrightarrow{H_1}$ defined by $\overrightarrow{H_1'} := \overrightarrow{H_1} \cup \{\overrightarrow{e_1, \ldots, e_j}\} \cup \overrightarrow{P_j}$. Define $v_0' = v_j$, $v_1' = v_{j+1}$, $\ldots$,$v_{6-j}' = v_6$. Let $e_i' = v_{i-1}' v_i'$ for $i = 1, 2, \ldots, 6-j$. Let $\ell_i'(a,b) = \rho_{G \setminus \{e_1', \ldots, e_i'\}}(a,b)$. Let $\mathcal{L}_i' = \ell_i'(v_i', H_1')$. If $\mathcal{L}_i' < \infty$, let $P_i' = v_{i,0}' v_{i,1}' \ldots v_{i,\mathcal{L}_i}'$ be a shortest path from v_i' to H_1' in $G \setminus \{e_1', \ldots, e_i'\}$. Given an extendable target pair $(\overrightarrow{H_1}, A_1)$ and the extension of $\overrightarrow{H_1}$ to $\overrightarrow{H_1}'$, call $(\overrightarrow{H_1'}, A_1)$ an augmented extendable target pair.

The labeling does not necessarily imply the existence of other vertices in these figures. In Figure 3.10 you will find an example of an augmented extendable target pair. In this figure labels will imply the existence of other vertices for simplicity. Consider Figure 3.11 for examples of how the vertex v_m' is defined. Here we have added all the possible necessary vertices for $v_1', v_2', \ldots, v_m'$ to make the definitions more clear.

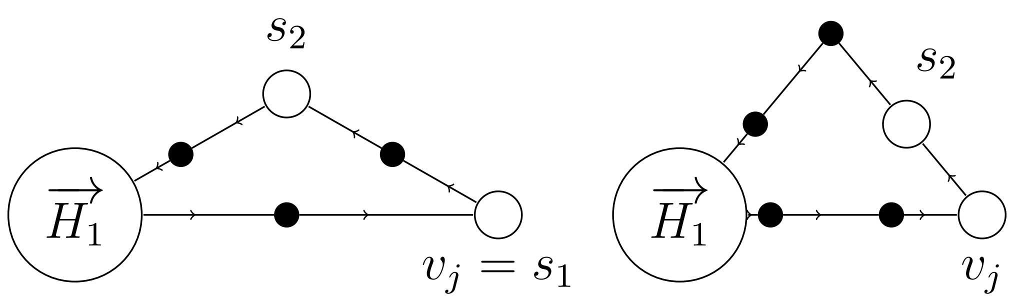

Figure 3.10 Example of augmented extendable target pairs.

Definition 3.25. Let $m \in \{1, 2, \ldots, 6-j\}$ be the smallest value for which $|N[v_m'] \cap N[s_1]| \leq 2$ and $|N[v_m'] \cap N[s_2]| \leq 2$. Since $\rho(s_1, H_1) = j \leq 3$ and $\rho(s_2, H) \leq$

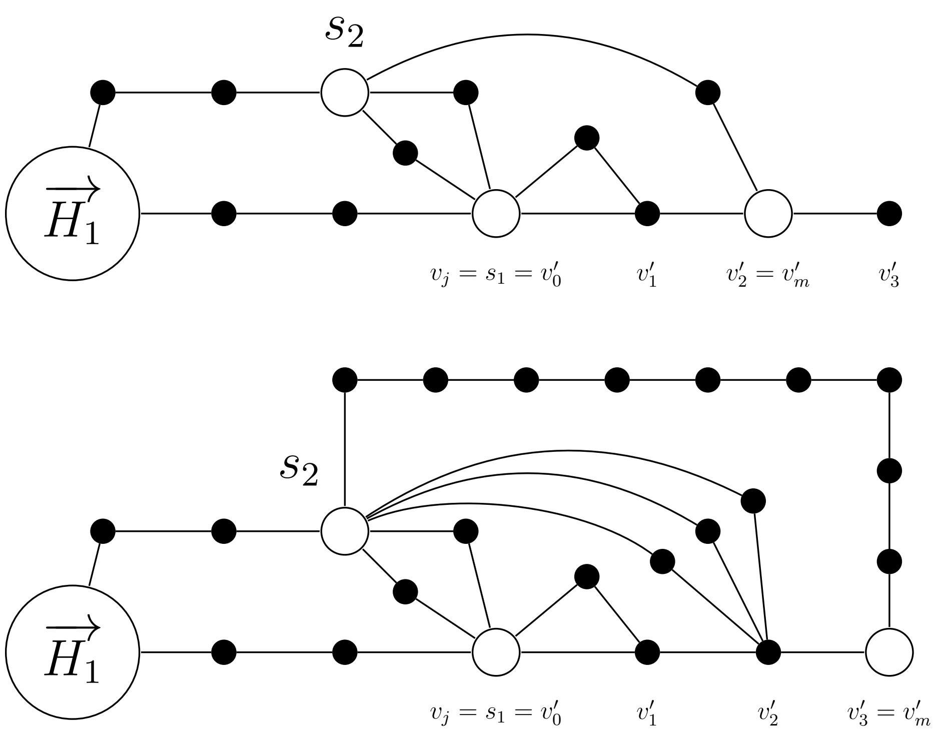

Figure 3.11 Examples giving the definition of the second special path vertex in Stage 2.

$\ell_j(s_2, H_1) \leq 3$, we have by $\rho(v'_{6-j}, H_1) = 6$ and the triangle inequality that

$$\rho_G(v'_{6-j}, \{s_1, s_2\}) \geq 3.$$

So there exists such an m with this property. A similar proof to that of Lemma 3.14 shows that $\mathcal{L}'_m < \infty$.

Definition 3.26. Let the vertex $s_3 := v'_{m,t}$ be defined as the first vertex on P'_m for which $\ell'_m(v'_m, s_3) \geq 3$ and either:

1. $|N[s_3] \cap N[s_2]| \geq 3$, or

2. $|N[s_3] \cap N[s_1]| \geq 3$.

If such a vertex does not exist, let $t = \mathcal{L}'_m$.

Lemma 3.27. *Given an augmented extendable target pair* $(\overrightarrow{H_1}', A_1)$ *and*

$$q = \lfloor \frac{\ell'_m(v'_m, v'_{m,t}) - 1}{3} \rfloor$$

,

1. *if* $q \geq 1$*, let* $A' = \{v_j, v'_m\} \cup \bigcup_{s \in [q]} v'_{m,3s}$*,*
2. *if* $q = 0$ *and* $\mathcal{L}'_m \leq 3$*, let* $A' = \{v_j, v'_m\}$*,*
3. *if* $q = 0$ *and* $4 \leq \mathcal{L}'_m \leq 5$*, and* $|N[s_3] \cap N[s_2]| \geq 3$ *and* $|N[s_3] \cap N[s_1]| \geq 3$*, let* $A' = \{v_j, v'_m\}$*,*
4. *if* $q = 0$ *and* $4 \leq \mathcal{L}'_m \leq 5$*,* $\ell_{j+m}(s_3, H_1) \leq 2$*, let* $A' = \{v_j, v'_m\}$*, and*
5. *if* $q = 0$ *and* $4 \leq \mathcal{L}'_m \leq 5$*,* $\ell_{j+m}(s_3, H_1) \geq 3$ *and there exists* $s_l \in \{s_1, s_2\}$ *for which* $|N[s_3] \cap N[s_l]| \leq 2$*, let* $A' = \{s_l, v'_m, s_3\}$*.*

In all cases $A_2 := A' \cup A_1$ *is a* δ*-set.*

Proof. Let $(\overrightarrow{H_1}', A_1)$ be an augmented extendable target pair and $q = \lfloor \frac{\ell'_m(v'_m, v'_{m,t}) - 1}{3} \rfloor$.

Case 1: $q \geq 1$.

Then $A' = \{v_j, v'_m\} \cup \bigcup_{s \in [q]} \{v'_{m,3s}\}$.

Claim 1: $|N[A'] \cap N[H_1]| \leq 2$.

Since $j + m \geq 3$, $N[v'_m] \cap N[H_1] = \emptyset$. Since $j \in \{2, 3\}$ and $\ell_j(v_j, H_1) \geq 4$, we have that $|N[v_j] \cap N[H_1]| \leq 1$. Also, for any $c < q$, $N[v'_{m,3c}] \cap N[H_1] = \emptyset$. If not, $\ell'_m(v'_m, H'_1) \leq 3c + 3 \leq 3q$, a contradiction to the fact that $3q < \ell'_m(v'_m, s_3)$. Also, $N[v'_{m,3q}] \cap N[H_1] \subseteq \{v_1\}$, otherwise $\ell'_m(v'_{m,3q}, H'_1) \leq 2$, and $t = 3q$, a contradiction. Hence, $|N[A'] \cap N[H_1]| \leq 2$.

To consider $|N[A']|$ we will count the number of times a vertex shows up in the pairwise intersection of any $v, w \in A'$ and subtract this from $(\delta+1)|A'|$. The pairwise intersections are described in the following claim.

Claim 2: Let $x \neq y \in A'$.

(a) If $\{x, y\} \neq \{v_j, v'_m\}, \{v_j, v'_{m,3q}\}, \{v'_m, v'_{m,3q'}\}$, then

$$N[x] \cap N[y] = \emptyset.$$

(b) If $\{x, y\}$ is one of $\{v'_m, v'_{m,3q}\}$, $\{v_j, v'_m\}$, or $\{v_j, v'_{m,3q}\}$, then

$$N[v'_m] \cap N[v'_{m,3q}] \subseteq \begin{cases} \emptyset & \text{if } m = 1 \\ \{v'_{m-1}\} & \text{if } m \geq 2, \end{cases}$$

$$|N[v_j] \cap N[v'_m]]| \leq \begin{cases} 2 & \text{if } m \leq 2 \\ 0 & \text{if } m \geq 3, \end{cases}$$

$$|N[v_j] \cap N[v'_{m,3q}]| \leq 2.$$

Considering case (a), we find that for all $c, d \in [q]$, $c \neq d$, $\ell'_m(v'_{m,3c}, v'_{m,3d}) \geq 3$, since $\ell(P'_m) = \mathcal{L}'_m$. Hence for all $c \in [q]$, $N[v'_{m,3c}]$ are all pairwise disjoint. For $c \in [q-1]$, $|N[v'_{m,3c}] \cap N[v'_m]| = \emptyset$, since otherwise either $\ell'_m(v'_{m,3c}, v'_m) = 2$ a contradiction to the definition of $v'_{m,3c}$ or $v'_i \in N[v'_{m,3c}] \cap N[v'_m]$ for some i such that $1 \leq i < m$ in which case $\mathcal{L}'_m \leq 3c + 2 < 3q$ a contradiction. A similar argument shows that the intersection $N[v'_{m,3c}] \cap N[v_j]$ is empty for $c < q$.

Considering case (b), first we will consider $N[v'_m] \cap N[v'_{m,3q}]$. If there exists a vertex $v \in V(H'_1)$ such that $v \in N[v'_m] \cap N[v'_{m,3q}]$, then $\ell'_m(v'_m, H'_1) \leq 4$ a contradiction. If there exists a vertex $v \notin V(P'_m) \cup V(H'_1)$ such that $v \in N[v'_m] \cap N[v'_{m,3q}]$, then $\ell'_m(v'_m, v'_{m,3q}) \leq 2$ a contradiction to the definition of $v'_{m,3q}$. Hence we find that if $m = 1$, then $N[v'_m] \cap N[v'_{m,3q}] = \emptyset$. If $m \geq 2$, and there exists a vertex $v \in N[v'_m] \cap N[v'_{m,3q}]$ such that $v \in V(P'_m) \setminus (V(H'_1) \cup \{v'_m, v'_{m-1}\})$, then we must have that $m \geq 3$ and this implies that $\ell(v'_m, v) = 1$ a contradiction to the fact that $v \neq v'_{m-1}$.

It remains to consider $N[v_j] \cap N[v'_m]$ and $N[v_j] \cap N[v'_{m,3q}]$.

For $N[v_j] \cap N[v'_m]$, if $m \geq 3$, since $\rho_G(v_j, v'_m) = m$, we have $N[v_j] \cap N[v'_m] = \emptyset$. If $m \leq 2$, by the definition of m we have $|N[v'_m] \cap N[v_j]| \leq 2$. If $N[v_j] \cap N[v'_{m,3q}] > 2$, then

q would be a vertex such that $|N[s_3] \cap N[s_1]| \geq 3$, yet $q < t$ which is a contradiction to the definition of t in Definition 3.26.

Claim 3: $|N[A']| \geq (\delta+1)|A'| - 4$.

It follows from Claim 2 by summation over all 2-element subsets of A' that

$$\sum_{\{x,y\} \subseteq A'} |N[x] \cap N[y]| \leq 5$$

. In order to prove Claim 3 it suffices to show that this inequality is strict since then in $\sum_{x \in A'} |N[x]|$ at most four vertices were counted twice, and so $|N[A']| = |\bigcup_{x \in A'} N[x]| \geq \sum_{x \in A'} |N[x]| - 4 \geq (\delta+1)|A'| - 4$.

Now suppose to the contrary that $\sum_{\{x,y\} \subseteq A'} |N[x] \cap N[y]| = 5$. Then the first and second part of Claim 2(b) yield that firstly $m \geq 2$ and $N[v_j] \cap N[v'_{m,3q}] = \{v'_{m-1}\}$, and secondly $m \leq 2$. Hence $m = 2$ and so $v'_1 \in N[v_j] \cap N[v'_m] \cap N[v'_{m,3q}]$. But then v'_1 is counted three times in $\sum_{x \in A'} |N[x]|$, while two other vertices are counted twice, which leads to an overcount of four. Hence $|N[A']| \geq (\delta+1)|A'| - 4$ follows.

Claim 4: A_2 is a δ-set.

We find that $|N[A']| - |N[H_1] \cap N[A']| \geq (\delta+1)(q+2) - 6 \geq (\delta-1)(q+2) \geq (\delta-1)|A'|$. So by Lemma 3.15 A_2 is a δ-set in G.

Case 2: $q = 0$, conditions (2,3,4)

Assume that $q = 0$ and conditions (2), (3) or (4) of Lemma 3.27 apply, so $A' = \{v_j, v'_m\}$. By definition of v'_m we have that $|N[v'_m] \cap N[v_j]| \leq 2$ so $|N[A']| \geq 2(\delta+1) - 2$. Since we have that $2 \leq j \leq 3$, because all other cases were considered in Stage I, $j + m \geq 3$, which implies $N[v'_m] \cap N[H_1] = \emptyset$. We also have by definition of v_j that $|N[v_j] \cap N[H_1]| \leq 2$. Hence we find that $|N[A'] \cap N[H_1]| \leq 2$. So we find that $|N[A']| - |N[H_1] \cap N[A']| \geq 2(\delta+1) - 4 \geq (\delta-1)2 \geq (\delta-1)|A'|$. So by Lemma 3.15 A_2 is a δ-set in G.

Case 3: $q = 0$, condition (5).

In order to bound $|N[A']|$ from below, we consider the intersections $N[v'_m] \cap N[s_3]$, $N[v'_m] \cap N[s_\ell]$ and $N[s_\ell] \cap N[s_3]$.

Claim 5: $N[v'_m] \cap N[s_3] \subseteq \{v'_{m-1}\}$.

If there were an $x \neq v'_{m-1}$ such that $x \in N[v'_m] \cap N[s_3]$, then we would have that $\ell'_m(v'_m, s_3) \leq 2$ a contradiction to the definition of s_3, so $N[v'_m] \cap N[s_3] \subseteq \{v'_{m-1}\}$.

Claim 6 (a): If $\ell = 1$ then $s_\ell = v_j$ and

$$|N[v_j] \cap N[v'_m]| \leq \begin{cases} 2 & \text{if } m \leq 2, \\ 0 & \text{if } m \geq 3, \end{cases}$$

$$|N[v_j] \cap N[s_3]| \leq 2.$$

(b) If $\ell = 2$ then

$$N[v'_m] \cap N[s_2] \subseteq \{v'_{m-1}\},$$

$$|N[s_2] \cap N[s_3]| \leq 2.$$

(a) Since $\ell = 1$ and since by condition (5) of this lemma we have $|N[v_j] \cap N[s_3]| = |N[s_1] \cap N[s_3]| \leq 2$. We have $N[v_j] \cap N[v'_m] = \emptyset$ if $m \geq 3$ since otherwise, if v_j and v'_m had a common neighbor, we would have $m = \rho_G(v_j, v'_m) \leq 2$, a contradiction. If $m \leq 2$ then $|N[v_j] \cap N[v'_m]| \leq 2$ by the definition of v'_m.

(b) Since $\ell = 2$ and by condition (5) of this lemma we have $|N[s_2] \cap N[s_3]| \leq 2$. The inclusion $N[v'_m] \cap N[s_2] \subseteq \{v'_{m-1}\}$ follows from the fact that if there were a vertex $x \neq v'_{m-1}$ with $x \in N[v'_m] \cap N[s_2]$ then $\ell'_m(v'_m, s_2) \leq 2$, a contradiction to $\mathcal{L}'_m \geq 4$.

Claim 7: $|N[A']| \geq |A'|(\delta + 1) - 4$.

It follows from Claim 6 by summation over the three 2-element subsets of A' that $\sum_{\{x,y\} \subseteq A'} |N[x] \cap N[y]| \leq 5$. In order to prove Claim 7 if suffices to show that this inequality is strict since then in $\sum_{x \in A'} |N[x]|$ at most four vertices were counted twice, and so $|N[A']| = |\bigcup_{x\in A'} N[x]| \geq \sum_{x\in A'} |N[x]| - 4 \geq (\delta+1)|A'| - 4$.

Now suppose to the contrary that $\sum_{\{x,y\}\subseteq A'} |N[x] \cap N[y]| = 5$. Then $\ell = 1$, so $A' = \{v_j, v'_m, s_3\}$. By Claim 6(a) we have firstly $m \leq 2$ and so $v'_1 \in N[v_j] \cap N[v'_m]$, and secondly $N[v'_m] \cap N[s_3] = \{v'_{m-1}\}$. This implies $m = 2$ and so $v'_1 \in N[v_j] \cap N[v'_m] \cap N[v'_{m,3q}]$. But then v'_1 is counted three times in $\sum_{x\in A'} |N[x]|$, while two other vertices are counted twice, which leads to an overcount of four. Hence $|N[A']| \geq (\delta+1)|A'| - 4$ follows.

Claim 8: $|N[A'] \cap N[H_1]| \leq 3$.

If $\ell = 1$, then $A' = \{v_j, v'_m, s_3\}$, and if $\ell = 2$, then $A' = \{s_2, v'_m, s_3\}$. For each of the vertices in A' we consider its joint closed neighborhood with H_1.

We have $N[v'_m] \cap N[H_1] = \emptyset$ since $\rho(v'_m, H_1) = j + m \geq 3$. We also have $N[s_3] \cap N[H_1] \subseteq \{v_1\}$ since by condition (v) of this lemma we have $\ell_{j+m}(s_3, H_1) \geq 3$, and if there were a vertex $x \in N[s_3] \cap N[H_1]$ with $x \neq v_1$, then x would give rise to a path of length at most two from s_3 to H_1 not containing any edge of P, a contradiction. We also find similarly that $N[s_2] \cap N[H_1] \subseteq \{v_1\}$, since by conditions **Q1** and **Q2** we have $\ell_j(s_2, H_1) = 3$.

In the case that $\ell = 1$, by the definition of v_j we have $|N[v_j] \cap N[H_1]| \leq 2$. If $N[v_j] \cap N[H_1] = \emptyset$, we find that $N[\{v_j, v'_m, s_3\}] \cap N[H_1] \subseteq \{v_1\}$. If $N[v_j] \cap N[H_1] \neq \emptyset$, then $j = 2$, so $\{v_1\} \subseteq N[v_j] \cap N[H_1]$, so $N[\{v_j, v'_m, s_3\}] \cap N[H_1] = N[v_j] \cap N[H_1]$, hence $|N[v_j] \cap N[H_1]| = |N[\{v_j, v'_m, s_3\}] \cap N[H_1]| \leq 2$. In the case that $\ell = 2$, we find that $N[\{s_2, v'_m, s_3\}] \cap N[H_1] \subseteq \{v_1\}$.

From the above we conclude that $|N[A'] \cap N[H_1]| \leq 2$ in both cases $\ell = 1$ and $\ell = 2$.

Claim 9: A_2 is a δ-set

Since $|A'| = 3$, we find $|N[A']| \geq (\delta+1)|A'| - 4$ and $|N[A']| - |N[H_1] \cap N[A']| \geq (\delta+1)|A'| - 6 \geq (\delta-1)|A'|$. By Lemma 3.15, A_2 is a δ-set in G.

□

We will consider the following cases given an augmented extendable target pair $(\overrightarrow{H_1}', A_1)$. These cases will be considered in order, so going to the next item implies none of the previous items occur.

- $s_3 \neq v'_{m,\mathcal{L}'_m}$ or
- $s_3 = v'_{m,\mathcal{L}'_m}$ and $s_3 \in \{s_2, s_1, v_{j,1}\}$ or
- $s_3 = v'_{m,\mathcal{L}'_m}$ and $s_3 \in V(P_j) \setminus \{s_2, s_1, v_{j,1}\}$
- $s_3 = v'_{m,\mathcal{L}'_m}$ and $s_3 \in \{v_1, \ldots, v_{j-1}\}$
- $s_3 = v'_{m,\mathcal{L}'_m}$ and $s_3 \in V(H_1)$

We will first prove some properties about some subgraphs that arise in the following cases:

- $s_3 \neq v'_{m,\mathcal{L}'_m}$ or
- $s_3 = v'_{m,\mathcal{L}'_m}$ and $s_3 \in \{s_2, s_1, v_{j,1}\}$

Note that if we have that $s_3 \neq v'_{m,\mathcal{L}'_m}$, this implies that either

- $|N[s_3] \cap N[s_1]| \geq 3$ and $|N[s_3] \cap N[s_2]| \geq 3$ or
- $|N[s_3] \cap N[s_1]| \geq 3$ and $|N[s_3] \cap N[s_2]| \leq 2$ or
- $|N[s_3] \cap N[s_2]| \geq 3$ and $|N[s_3] \cap N[s_1]| \leq 2$.

Lemma 3.28. *If either*

1. $s_3 = v'_{m,\mathcal{L}'_m}$ *and* $s_3 \in \{s_1, s_2\} \cup \{v_{j,1}\}$ *or*

2. $|N[s_3] \cap N[s_1]| \geq 3$ *and* $|N[s_3] \cap N[s_2]| \geq 3$,

there exists an oriented subgraph $\overrightarrow{W}$ *that is well defined given* $\overrightarrow{H_1}'$ *and has the following properties:*

1. $\rho_{\overrightarrow{W}}(s_3, s_1) \leq 4$,

2. $\rho_{\overrightarrow{W}}(s_3, s_2) \leq 2$,

3. $\rho_{\overrightarrow{W}}(s_1, s_2) \leq 2$,

4. $diam(\overrightarrow{W}, s_2) \leq 3$,

5. $diam(\overrightarrow{W}) \leq 6$,

6. $diam(s_1, \overrightarrow{W}) \leq 5$, *and*

7. $diam(\overrightarrow{W}, s_1) \leq 5$.

If either

1. $|N[s_3] \cap N[s_1]| \geq 3$ *and* $|N[s_3] \cap N[s_2]| \leq 2$ *or*

2. $|N[s_3] \cap N[s_2]| \geq 3$ *and* $|N[s_3] \cap N[s_1]| \leq 2$,

there exists an oriented subgraph $\overrightarrow{W}$ *that is well defined given* $\overrightarrow{H_1}'$ *and has the following properties:*

1. $\rho_{\overrightarrow{W}}(s_3, s_1) \leq 4$,

2. $\rho_{\overrightarrow{W}}(s_3, s_2) \leq 4$,

3. $\rho_{\overrightarrow{W}}(s_1, s_2) \leq 2$,

4. $diam(\overrightarrow{W}, s_2) \leq 5$,

5. $diam(\overrightarrow{W}) \leq 6$,

6. *diam*$(s_1, \overrightarrow{W}) \leq 5$, *and*

7. *diam*$(\overrightarrow{W}, s_1) \leq 5$.

Notice that when pairwise comparing each of these parameters, the maximum in the second case can only increase by 2 when comparing to the maximum in the first case.

Proof. Since $|N[s_2] \cap N[s_1]| \geq 3$, we find that there exist at least two edge disjoint paths of length at most 2 from s_1 to s_2, call them R_1 and R_2. If one of these is a subgraph of P_j, let it be R_1 and orient it as $\overrightarrow{R_1} = \overrightarrow{s_1 \ldots s_2}$, and orient $\overrightarrow{R_2} = \overrightarrow{s_2 \ldots s_1}$. If one of the two includes e_j (note that since R_1 is edge disjoint from $e_1, \ldots, e_j$ this can't be R_1) label it as $\overrightarrow{R_2} = \overrightarrow{s_2 \ldots s_1}$, and orient the other one $\overrightarrow{R_1} = \overrightarrow{s_1 \ldots s_2}$ if it is not already. We will eventually orient P as $\overrightarrow{P}$, so the only conflicts will be if one of the paths includes the edge e_j or e_{j+1}. Consider the following cases:

1. Let $s_3 = v'_{m,\mathcal{L}'_m}$ and $s_3 \in \{s_1, s_2, v_{j,1}\}$. In this case, let $W := W_1$. Notice by examination that all of the properties hold.

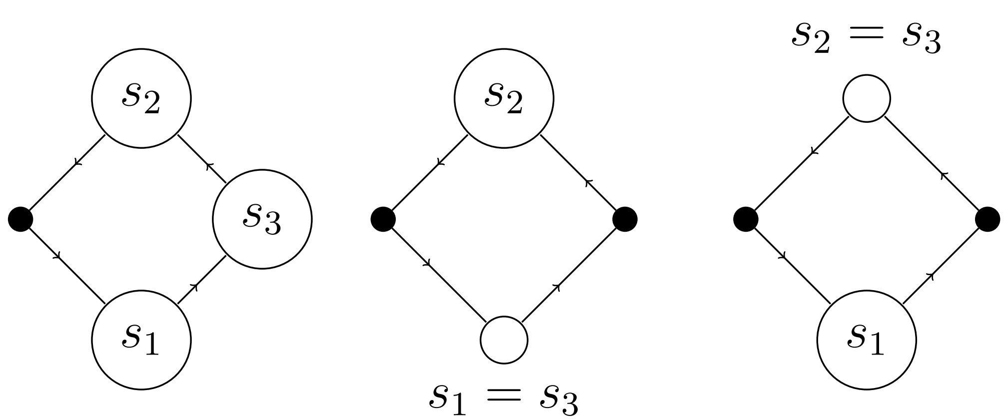

Figure 3.12 Example of widgets in H_1 case 1.

2. Let $s_3 \neq v'_{m,\mathcal{L}'_m}$ and $|N[s_3] \cap N[s_1]| \geq 3$ and $|N[s_3] \cap N[s_2]| \geq 3$.

Notice that in this case, because W_1 is a cycle of order $2, 3$ or 4 that $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| \le 2$.

2.1. Let $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| = 2$, then there exist vertices $u, v \in W_1$ such that $W_1 = \overrightarrow{s_2 v s_1 u s_2}$, and $s_3u, s_3v \in E(G)$. Orient s_3u as $\overrightarrow{s_3u}$, and orient s_3v as $\overrightarrow{vs_3}$. Notice by examination that the properties hold.

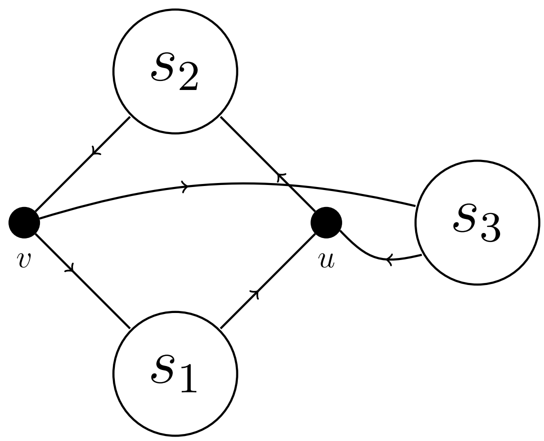

Figure 3.13 Example of a widget in case 2.1.

2.2. Let $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| \le 1$. Since $|N[s_3] \cap N[s_1]| \ge 3$, we must have that either $s_3s_1 \in E(G)$, $s_3v_{j+1} \in E(G)$, or if not, there exists some path of length two from s_1 to s_3 which is edge disjoint from $P \cup W_1 \cup P_j[\{s_2 \dots v_{j,\ell_j}]$. If $s_1s_3 \in E(G)$ orient this edge as $\overrightarrow{s_1s_3}$. If $s_3s_1 \notin E(G)$, then we must have that either $s_3v_{j+1} \in E(G)$ or there exists some path of length two from s_1 to s_3 which is edge disjoint from $P \cup W_1 \cup P_j[\{s_2 \dots v_{j,\ell_j}]$. In either case orient such a path as $\overrightarrow{s_1 \dots s_3}$. Since $|N[s_3] \cap N[s_2]| \ge 3$, either $s_3s_2 \in E(G)$, in which case orient this edge as $\overrightarrow{s_3s_2}$. Or if $s_3s_2 \notin E(G)$, then we must have that there exists some path of length two from s_2 to s_3 which is edge disjoint from $P \cup W_1 \cup P_j[\{s_2 \dots v_{j,\ell_j}]$. Orient such a path as $\overrightarrow{s_3 \dots s_2}$.

Let $W := \overrightarrow{W_1} \cup \overrightarrow{s_1 \ldots s_3 \ldots s_2}$, where $\overrightarrow{s_1 \ldots s_3 \ldots s_2}$ are the paths we found in the two cases above. Notice by examination that the properties hold.

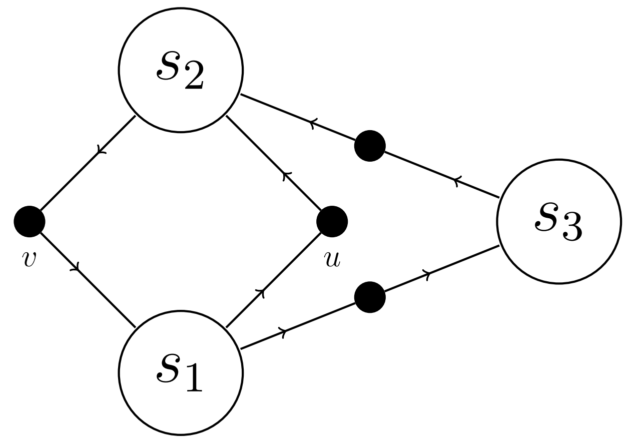

Figure 3.14 Example of a widget in case 2.2.

3. Let $s_3 \neq v'_m$ and $s_3 \neq v'_{m,\mathcal{L}'_m}$ and either

- $|N[s_3] \cap N[s_1]| \geq 3$ and $|N[s_3] \cap N[s_2]| \leq 2$ or
- $|N[s_3] \cap N[s_2]| \geq 3$ and $|N[s_3] \cap N[s_1]| \leq 2$

3.1. Let $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| = 2$, do the same as in the case of $|N[s_3] \cap N[s_1]| \geq 3$ and $|N[s_3] \cap N[s_2]| \geq 3$. See Figure 3.13 for an example of this case.

3.2. Let $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| \leq 1$, we split into two cases:

3.2.1. Let $|N[s_3] \cap N[s_1]| \geq 3$ and $|N[s_3] \cap N[s_2]| \leq 2$.

If $\{v_{j-1}, v_{j+1}\} \subseteq N[s_3] \cap N[s_1]$, then let $W = W_1 \cup \overrightarrow{s_3 v_{j-1} v_j v_{j+1} s_3}$. Notice by examination that the properties hold. If $v_{j-1} \in N[s_3] \cap N[s_1]$ and $v_{j+1} \notin N[s_3] \cap N[s_1]$, since $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| \leq 1$, there is a path edge disjoint from W_1, e_j, and e_{j+1} of length at most 2 from s_1 to s_3, call it R_3. In this case, let $W := W_1 \cup \overrightarrow{s_3 v_{j-1} s_1} \cup \overrightarrow{R_3}$.

Notice by examination that the properties hold. If $v_{j+1} \in N[s_3] \cap N[s_1]$ and $v_{j-1} \notin N[s_3] \cap N[s_1]$, since $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| \leq 1$, there is a path edge disjoint from W_1, e_j, and e_{j+1} of length at most 2 from s_3 to s_1, call it R_4. In this case, let $W := W_1 \cup \overrightarrow{s_3 v_{j-1} s_1} \cup \overrightarrow{R_4}$. Notice by examination that the properties hold. If $v_{j-1} \notin N[s_3] \cap N[s_1]$ and $v_{j+1} \notin N[s_3] \cap N[s_1]$, since $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| \leq 1$, there are two paths edge disjoint from W_1, e_j, and e_{j+1} of length at most 2 from s_1 to s_3, call them R_5 and R_6. Orient $\overrightarrow{R_5} = \overrightarrow{s_3 \ldots s_1}$ and $\overrightarrow{R_6} = \overrightarrow{s_1 \ldots s_3}$. In this case, let $W := W_1 \cup \overrightarrow{s_3 v_{j-1} s_1} \cup \overrightarrow{R_5} \cup \overrightarrow{R_6}$. Notice by examination that the properties hold.

3.2.2. Let $|N[s_3] \cap N[s_2]| \geq 3$ and $|N[s_3] \cap N[s_1]| \leq 2$.

Let $s_2 = v_{j,a}$. If $v_{j,a+1} \in N[s_3] \cap N[s_2]$, since $|N[s_3] \cap N[s_2]| \geq 3$ and $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| \leq 1$, there is a path of length at most 2 from s_3 to s_2 that is edge disjoint from W_1 and $v_{j,a}v_{j,a+1}$, call it R. Let $\overrightarrow{W} := \overrightarrow{W_1} \cup \overrightarrow{s_2 v_{j,a+1} s_3} \cup \overrightarrow{R}$. Notice by examination that the properties hold. If $v_{j,a+1} \notin N[s_3] \cap N[s_2]$, since $|N[s_3] \cap N[s_2]| \geq 3$ and $|(N(s_1) \cap N(s_2) \cap V(W_1)) \cap N(s_3)| \leq 1$, there are two paths of length at most 2 from s_3 to s_2 that are edge disjoint from W_1 and $v_{j,a}v_{j,a+1}$, call them R_1 and R_2. Orient $\overrightarrow{R_1} = \overrightarrow{s_2 \ldots s_3}$ and $\overrightarrow{R_2} = \overrightarrow{s_3 \ldots s_2}$.

□

Lemma 3.29. *Given an augmented extendable target pair* $(\overrightarrow{H_1'}, A_1)$*, if*

1. $s_3 \neq v'_{m,\mathcal{L}'_m}$ *or*
2. $s_3 = v'_{m,\mathcal{L}'_m}$ *and* $s_3 \in \{s_2, s_1, v_{j,1}\}$

then we can extend $(\overrightarrow{H_1}, A_1)$ *to* $(\overrightarrow{H_2}, A_2)$*.*

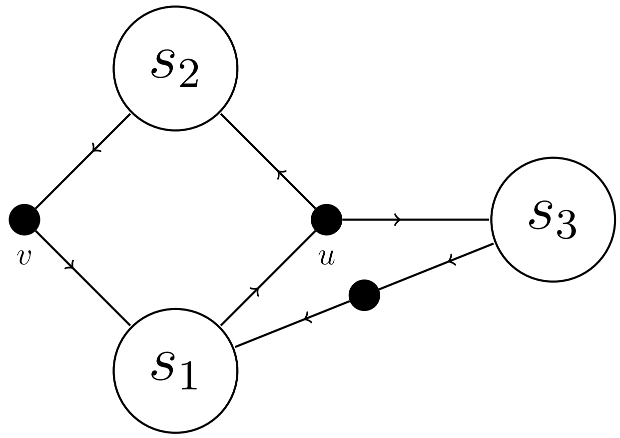

Figure 3.15 Example of a widget in case 3.2.1.

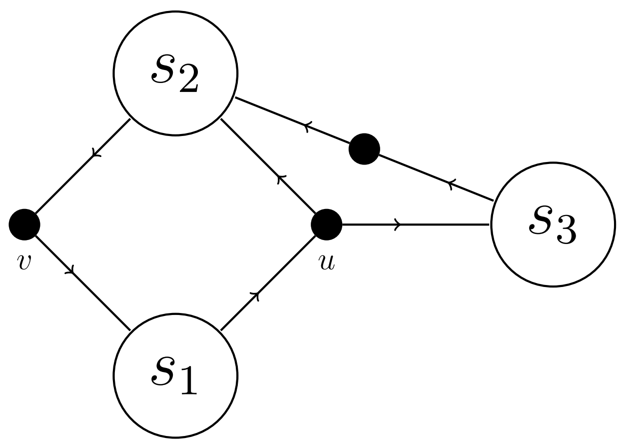

Figure 3.16 Example of a widget in case 3.2.2.

Proof. Let $\overrightarrow{Q_2} = \overrightarrow{H_1} \cup \overrightarrow{e_1 \dots e_{j+m}} \cup \overrightarrow{P_j[s_2 v_{v,t+1} \dots v_{j,\ell_j}]} \cup \overrightarrow{P'_m[v_{m,0} \dots v_{m,t-1} s_3]} \cup \overrightarrow{W}$. Where $\overrightarrow{W}$ is defined as in the lemma above. For $1 \leq p < m$, $|N[v'_p] \cap N[s_1]| \geq 3$, so there exists a path P''_p from v'_p to $\overrightarrow{H'_1} \cup \overrightarrow{W}$, of length at most two which is edge disjoint from $\overrightarrow{H'_1} \cup \overrightarrow{W}$, or there exists some edge not in $\overrightarrow{H'_1} \cup \overrightarrow{W}$, call it $v'_p w$ such that $\mathrm{diam}_{\overrightarrow{Q_2}}(w, \{s_1, s_2, s_3\}) = 1$. Let $\overrightarrow{H_2} = \overrightarrow{Q_2} \cup \overrightarrow{P''_1} \cup \dots \cup \overrightarrow{P''_{m-1}} \cup \overrightarrow{W}$.

Let $V''_m := \overrightarrow{H_2} \setminus V(\overrightarrow{H'_1} \cup \overrightarrow{W})$. Let $V''_j := \overrightarrow{H'_1} \setminus \overrightarrow{W}$. We have that $\mathcal{L}'_m \geq 1$, so $\ell'_m(v'_m, s_3) = 3q + r + 1$, where $q \geq 0$ and $0 \leq r \leq 2$.

Let

1. $s_3 \neq v'_m$ and $|N[s_3] \cap N[s_1]| \geq 3$ and $|N[s_3] \cap N[s_2]| \geq 3$ or

2. $s_3 = v'_{m,\mathcal{L}'_m}$ and $s_3 \in \{s_1, s_2\} \cup \{v_{j,1}\}$

let $A_2 = \{v_j, v'_m\} \cup \bigcup_{s \in [q]} v'_{m,3s}$. By lemma 3.27 we have that property **P1** of 3.13 holds for A_2.

Given the properties for the above cases in Lemma 3.28, we find the following inequalities:

$$\begin{aligned}
\mathrm{diam}_{\overrightarrow{H_2}}(V''_j, V''_m) \leq& \max\{2 + \mathrm{diam}(\overrightarrow{H_1}) + j + m + 3q + 2,\\
& 2 + \mathrm{diam}(\overrightarrow{H_1}) + j + m - 1 + 1\}\\
\leq& 10 + \mathrm{diam}(\overrightarrow{H_1}) + 3q\\
\mathrm{diam}_{\overrightarrow{H_2}}(V''_j, V''_j) \leq& \mathrm{diam}(\overrightarrow{H_1}) + j + \rho_{\overrightarrow{W}}(s_1, s_2) + 2\\
\leq& \mathrm{diam}(\overrightarrow{H_1}) + 7\\
\mathrm{diam}_{\overrightarrow{H_2}}(V''_m, V''_j) \leq& \max\{3q + 2 + 1 + \rho_{\overrightarrow{W}}(s_3, s_2) + 3 + j - 1,\\
& 2 + \mathrm{diam}_{\overrightarrow{W}}(\{s_3, s_1\}, s_2) + 3 + \mathrm{diam}(\overrightarrow{H_1}) + j - 1\}\\
\leq& \mathrm{diam}(\overrightarrow{H_1}) + 10 + 3q\\
\mathrm{diam}_{\overrightarrow{H_2}}(V''_m, V''_m) \leq& \max\{2 + \mathrm{diam}_{\overrightarrow{W}}(\{s_3, s_1\}, s_1) + m - 1 + 1,\\
& 2 + \mathrm{diam}_{\overrightarrow{W}}(\{s_3, s_1\}, s_1) + m + 3q + 2,\\
& 3q + 2 + \rho_{\overrightarrow{W}}(s_3, s_1) + m - 1 + 1\}\\
\leq& \mathrm{diam}(\overrightarrow{H_1}) + 10 + 3q\\
\mathrm{diam}_{\overrightarrow{H_2}}(\overrightarrow{W}, \overrightarrow{W}) \leq& 6\\
\mathrm{diam}_{\overrightarrow{H_2}}(\overrightarrow{W}, V''_j) \leq& \mathrm{diam}(\overrightarrow{W}, s_2) + 3 + \mathrm{diam}(\overrightarrow{H_1}) + j - 1\\
\leq& 10 + \mathrm{diam}(\overrightarrow{H_1})
\end{aligned}$$

$$
\begin{aligned}
\operatorname{diam}_{\overrightarrow{H_2}}(V_j'', \overrightarrow{W}) &\leq 2 + \operatorname{diam}(\overrightarrow{H_1}) + j + \operatorname{diam}(s_1, \overrightarrow{W}) \\
&\leq 10 + \operatorname{diam}(\overrightarrow{H_1}) \\
\operatorname{diam}_{\overrightarrow{H_2}}(V_m'', W) &\leq \max\{3q + 2 + \operatorname{diam}(\overrightarrow{W}), 2 + \operatorname{diam}(\overrightarrow{W})\} \\
&\leq 3q + 7 \\
\operatorname{diam}_{\overrightarrow{H_2}}(W, V_m'') &\leq \max\{\operatorname{diam}(\overrightarrow{W}, s_1) + m + 3q + 1, \operatorname{diam}(\overrightarrow{W}, s_1) + m + -1 + 1\} \\
&\leq 3q + 10.
\end{aligned}
$$

Putting these inequalities together with Lemma 3.17, we find that in these cases that $\operatorname{diam}(\overrightarrow{H_2}) \leq \operatorname{diam}(H_1) + 3q + 10$.

Noticing that $|A_2| - |A_1| = (q + 2) \geq 2$, we find that:

$$
\begin{aligned}
\operatorname{diam}(\overrightarrow{H_2}) &\leq \operatorname{diam}(\overrightarrow{H_1}) + 3q + 10 \\
&\leq 5|A_1| + 3(q + 2) + 4 \\
&\leq 5|A_1| + 3(|A_2 \setminus A_1|) + 2(|A_2 \setminus A_1|) \\
&\leq 5|A_1| + 5|A_2 \setminus A_1| \\
&\leq 5|A_2|.
\end{aligned}
$$

Hence, property **P2** of Lemma 3.13 holds. □

See Figure 3.17 for an example of such an orientation that is of low diameter. Here we represent all the possible subgraphs W that could be plugged in, with s_1, s_2 and s_3 on the edge of the subgraph W.

Lemma 3.30. *Given an augmented extendable target pair* $(\overrightarrow{H_1'}, A_1)$*, if* $s_3 \neq v'_{m,\mathcal{L}'_m}$ *and either*

- $|N[s_3] \cap N[s_1]| \geq 3$ *and* $|N[s_3] \cap N[s_2]| \leq 2$ *or*
- $|N[s_3] \cap N[s_2]| \geq 3$ *and* $|N[s_3] \cap N[s_1]| \leq 2$,

then we can extend $(\overrightarrow{H_1}, A_1)$ *to* $(\overrightarrow{H_2}, A_2)$.

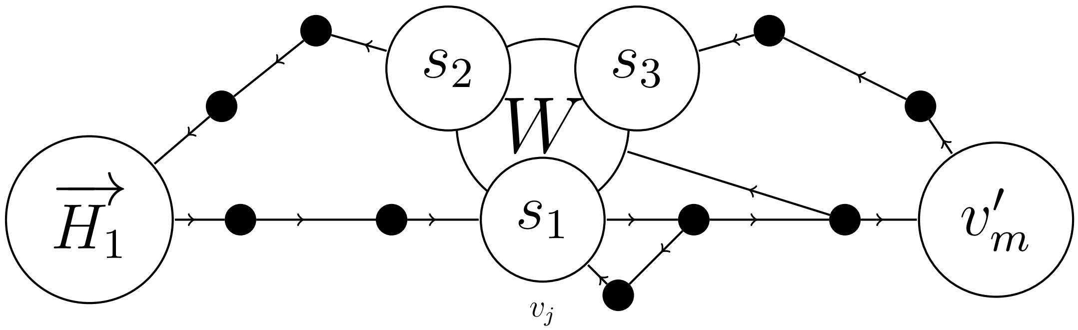

Figure 3.17 An orientation of the edges in the extended orientation.

Proof. We consider a similar proof to 3.29, using the note that because in 3.28 we noted that we only increased the maximum of any value by 2 in this case. Noting that we only used one term from the properties lists in any of our calculations for $\mathrm{diam}(\overrightarrow{H_2})$, we notice that $\mathrm{diam}(\overrightarrow{H_2}) \leq \mathrm{diam}(\overrightarrow{H_1}) + 3q + 10$. If $q \geq 1$, let A_2 remain the same, and since $(q+2) \geq 3$ notice that $3(q+2) + 6 \leq 5(|A_2 \setminus A_1|)$. If $q = 0$, add s_3 to A_2. By 3.27, we have that Property **P1** of Lemma 3.13 still holds. Since $\mathrm{diam}(\overrightarrow{H_1}) \leq 12$, $|A_2| - |A_1| = 3$, and $12 \leq 5(|A_2| - |A_1|)$, property **P2** of Lemma 3.13 holds. □

Lemma 3.31. *Given an augmented extendable target pair* $(\overrightarrow{H_1'}, A_1)$, *if*

- $s_3 = v'_{m,\mathcal{L}'_m}$ *and* $s_3 \in V(P_j) \setminus \{s_2, s_1, v_{j,1}\}$ *or*
- $s_3 = v'_{m,\mathcal{L}'_m}$ *and* $s_3 \in \{v_1, \ldots, v_{j-1}\}$ *or*
- $s_3 = v'_{m,\mathcal{L}'_m}$ *and* $s_3 \in V(H_1)$,

then we can extend $(\overrightarrow{H_1}, A_1)$ *to a new target pair* $(\overrightarrow{H_2}, A_2)$.

Proof. If $s_3 \in \{v_1, \ldots, v_{j-1}\}$, since $|N[v_i] \cap N[H_1]| \geq 3$, there exists a path Q of length at most 2 from s_3 to $V(\overrightarrow{H_1})$ edge disjoint from P. If $s_3 \in V(P_j) \setminus \{s_2, s_1, v_{j,1}\}$, then let $Q = s_3 \ldots v_{j,\mathcal{L}_j}$, where Q follows P_j. Note that Q is a path of length at most 2 from s_3 to $V(\overrightarrow{H_1})$ which is edge disjoint from P.

Let $P''_m := P'_m[\{v'_m, v'_{m,1}, \dots, s_3\}] \cup Q$.

Let $A_2 = A_1 \cup \{v_j, v_{j+m}\} \cup \{v'_{m,3s} | s \in [q]\}$. By Lemma 3.27 we have that property **P1** of 3.13 holds for A_2.

Let $\overrightarrow{H_2} = \overrightarrow{H_1} \cup \overrightarrow{e_1 \dots e_{j+m}} \cup \overrightarrow{P''_m}$. Since we have $q = \lfloor \frac{\ell'_m(v'_m, s_3)-1}{3} \rfloor$, this implies that $\ell'_m(v'_m, s_3) \le 3q + 3$.

Note that in all of these cases we add to $\overrightarrow{H_1}$ an oriented trail of length at most $(j+m) + \ell'_m(v'_m, s_3) + \ell_{j+m}(s_3, H_1) \le (j+m) + 3q + 3 + 2 \le 6 + 3q + 3 + 2 \le 3q + 11$.

By Lemma 3.18 we find the following:

$$\begin{aligned} \operatorname{diam}(\overrightarrow{H_2}) &\le \operatorname{diam}(\overrightarrow{H_1} \cup \overrightarrow{e_1 e_2 \dots e_{j+m}} \cup \overrightarrow{P''_m}) \\ &\le \operatorname{diam}(\overrightarrow{H_1}) + 3q + 11 - 1 \\ &\le \operatorname{diam}(\overrightarrow{H_1}) + 3(q+2) + 4. \end{aligned}$$

Noticing that $|A_2| - |A_1| = (q+2) \ge 2$, we find that:

$$\begin{aligned} \operatorname{diam}(\overrightarrow{H_2}) \le& \operatorname{diam}(\overrightarrow{H_1}) + 3(q+2) + 4 \\ \le& 5|A_1| + 3(q+2) + 4 \\ \le& 5|A_1| + 3(|A_2 \setminus A_1|) + 2(|A_2 \setminus A_1|) \\ \le& 5|A_1| + 5|A_2 \setminus A_1| \\ \le& 5|A_2|. \end{aligned}$$

Hence, property **P2** of Lemma 3.13 holds. □

3.4 Proof of Theorem

First consider the following theorem by Chvatal and Thomassen (cite Chvatal and Thomassen).

Theorem 3.32. *Every bridgeless graph of radius r admits an orientation of radius at most $r^2 + r$.*

Let $\text{rad}(\overrightarrow{G})$ represent the radius of $\overrightarrow{G}$. We will use this theorem with Lemma 3.13 to prove our theorem.

Theorem 3.33. *Given $G = (V, E)$, a bridgeless graph of order n and minimum degree δ, we have that*

$$\overrightarrow{diam}(G) \leq 5\frac{n}{\delta - 1} + 60.$$

Proof. In Lemma 3.13, we showed that there is target pair $(\overrightarrow{H}, A)$ such that $\forall v \in V(G)$, $\rho_G(v, H) \leq 5$.

Since A is a δ set, we have that $|N[A]| \geq (\delta - 1)|A|$, since $N[A] \subseteq V(G)$, we find that $(\delta - 1)|A| \leq n$, so $|A| \leq \frac{n}{\delta-1}$. Hence, $\overrightarrow{\text{diam}}(H) \leq 5\frac{n}{\delta-1}$.

Contract $V(H)$ to one vertex. Call this multi/loopy graph G^*. Theorem 3.32 shows that there is an orientation of G^*, $\overrightarrow{G^*}$, such that $\text{rad}(\overrightarrow{G^*}) \leq r^2 + r$. Since $\text{rad}(\overrightarrow{G^*}) \leq \text{diam}(\overrightarrow{G^*}) \leq 2\text{rad}(\overrightarrow{G^*}$, we have $\text{diam}(\overrightarrow{G^*}) \leq 2(r^2 + r)$. By expanding $V(H)$ we find that for two vertices in $V(G) \setminus V(H)$, if in G^*, a shortest path between them did not pass through $V(H)$, then they are at most distance $2(r^2 + r)$ apart. If the shortest paths between them only pass through $V(H)$, then they are at most distance $\overrightarrow{\text{diam}}(H) + 2(r^2 + r) = 5\frac{n}{\delta-1} + 2(r^2 + r)$. Since $\text{rad}(G^*) \leq 5$, we find that:

$$\overrightarrow{\text{diam}}(G) \leq 5\frac{n}{\delta - 1} + 60.$$

□

Chapter 4

The Oriented Diameter of a Complete Graph with Some Edges Removed

4.1 Introduction

In this section we relate the existence of an orientation of diameter two of graph of given order to its size. Füredi et al. (1998) gave an asymptotically sharp lower bound on the number of edges in a graph of given order that admits an orientation of diameter two. The purpose of this section is to determine for every $n \geq 5$ the minimum value $m(n) = |E(G)|$ such that every simple graph of order $n = |G|$ and size at least $m(n)$ has an orientation of diameter two.

Proposition 4.1 (Koh and Tay 2002)**.** *For $n \geq 5$, the graph obtained from a complete graph on $n-1$ vertices by adding a new vertex and edges joining it to three vertices in the complete graph does not have an orientation of diameter two. Hence $m(n) \geq \binom{n}{2} - n + 5$ for $n \geq 5$.*

Conjecture 4.2 (Koh and Tay 2002)**.** *This construction is best possible, so $m(n) = \binom{n}{2} - n + 5$ for $n \geq 5$.*

For $n \geq 5$, the graph G_n, obtained from a complete graph on $n-1$ vertices by adding a new vertex v and edges joining v to three vertices in the complete graph, does not have an orientation of diameter two. Indeed, suppose to the contrary that G_n has an orientation D of diameter two. Then v has either two in-neighbors and one out-neighbor, or vice versa. We may assume the former. Let u be the out-neighbor of v in D. Since every vertex is at distance at most two from v, every vertex in $V(D) - \{u, v\}$ is adjacent from u. Hence, if $x \in V(D) - \{u, v\}$ is a vertex not adjacent to v in D, a shortest (x, u)-path goes through v and has thus length at least three, a contradiction to D having diameter two. Hence G_n has no orientation of diameter two. It follows that $m(n) \geq m(G_n) + 1 = \binom{n}{2} - n + 5$ for $n \geq 5$. This was observed by Koh and Tay (2002), who conjectured that this construction is best possible, and so $m(n) = \binom{n}{2} - n + 5$ for $n \geq 5$. It is the aim of this paper to show that this conjecture is true by proving the following theorem.

Theorem 4.3. *Let G be a simple graph of order n, where $n \geq 5$, and size at least $\binom{n}{2} - n + 5$. Then G has an orientation of diameter two.*

Our proof of Theorem 4.3 consists of a sequence of Lemmas. An outline of the proof is as follows. We suppose to the contrary that the theorem is false and that G is a counterexample of minimum order and size. Our proof focuses on the complement $\overline{G}$ of G, defined as the graph on the same vertex set as G, where two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G.

In Section 4.3 we give some sufficient conditions for graphs to have an orientation of diameter two, and we present a list of several graphs that have an orientation of diameter two. In Section 4.4 we present some useful properties of the graph $\overline{G}$ that will be useful later; in particular we show that each component of $\overline{G}$ contains neither three independent vertices nor two non-adjacent vertices that share more than one neighbour. These results, together with some results in Section 4.5 on the components of $\overline{G}$ that are trees, will be used in Section 4.6 to show that each component of $\overline{G}$ is either a short path or one of four types of graphs. We show that the presence of any of these four types of graphs either allows us to apply certain reductions to the graph G to obtain a smaller counterexample G', or that G is one of the graphs in the list of graphs with an orientation of diameter two presented in Section 4.3, so G is not a counterexample. Finally, we conclude the proof by dealing with the case that all components of $\overline{G}$ are trees.

4.2 Notation

All graphs and digraphs in this paper have neither loops nor multiple edges. Let G be a graph of order $n = n(G)$ and size $m = m(G)$. We define the *excess* of G by $\text{ex}(G) = m(G) - n(G)$. We find it convenient to consider G and $\overline{G}$ as obtained by colouring the edges of a complete graph on n vertices either red or blue, with the edges of G being the red, and the edges of $\overline{G}$ as blue edges. Accordingly, we usually denote G as R, and $\overline{G}$ as B. We denote the vertex set common to R and B by V. If $W \subseteq V$, then the red and blue subgraph induced by W in R and B, respectively, is denoted by $R[W]$ and $B[W]$.

Let u, v be vertices of a graph G or digraph D. If $uv \in E(G)$ then we say that u and v are adjacent in G and that u is a neighbor of v. The set of all neighbors of v is the neighborhood of v in G, denoted by $N_G(v)$. The closed neighborhood $N_G[v]$ of v in G is defined as $N_G(v) \cup \{v\}$. If $\overrightarrow{uv}$ is a directed edge of D, then we say that v is an out-neighbor of u and that u is an in-neighbor of v. The *degree* of vertex v in G is the number of neighbors of v, it is denoted by $\text{deg}G(v)$.

By K_n, P_n, C_n, and $K_{a,b}$ we mean the complete graph on n vertices, the path on n vertices, the cycle on n vertices, and the complete bipartite graph whose partite sets have a and b vertices, respectively. If n is even, then $K_n - M$ denotes a complete graph of order n with a perfect matching removed. If G and H are graphs, then $G \cup H$ means the disjoint union of G and H. If a is a positive integer, then aG means the disjoint union of a copies of G, so the edgeless graph on n vertices is denoted by nK_1.

If U and W are disjoint subsets of V then we indicate by $U \to W$ that for all $x \in U$ and $y \in W$ that are adjacent in R we orient the edge xy as $\overrightarrow{xy}$, i.e., from x

to y. If U or W consist of a single vertex u or w, respectively then we write $u \to W$ instead of $\{u\} \to W$, and similarly $U \to w$ and $u \to w$.

If A, B are sets of vertices in H, then $d_H(A, B)$ is defined as $\min_{u \in A, v \in B} d_H(u, v)$, and $d_H(u, B)$ and $d_H(A, v)$ are defined analogously.

As usual, $[n] = \{1, 2, 3, \ldots, n\}$ and for a set A and $k \in \mathbb{N}$, $\binom{A}{k}$ denotes the collection of k-element subsets of A.

4.3 Some Sufficient Conditions for an Orientation of Diameter two

In this section we present some sufficient conditions for the existence of an orientation of diameter two of a graph. Using these conditions we obtain a list of several graphs that have an orientation of diameter two. This list will be used extensively in later sections.

Definition 4.4. Let $W \subseteq V$. An orientation D of $R[W]$ is *good* if there exists a partition of W into two sets U_1 and V_1, which we call the *partition classes* of W (or of D), such that

(i) $d_D(x, y) \leq 2$ whenever x and y are both in U_1 or both in V_1,

If in addition

(ii) every vertex in U_1 has an in-neighbor and an out-neighbor in V_1 and vice versa,

then D is a *non-trivial good orientation.* If $R[W]$ has a (non-trivial) good orientation, then we sometimes say simply that W has a (non-trivial) good orientation.

The following lemma is based on a construction of digraphs of diameter two with no 2-cycles having close to the minimum number or edges by Füredi et al. (1998).

Lemma 4.5. *Let $a, b \in \mathbb{N}$ with $2 \leq a \leq b \leq \binom{a}{\lfloor a/2 \rfloor}$. If $R[W]$ contains $K_{a,b}$ as a spanning subgraph, then $R[W]$ has a non-trivial good orientation. If $R[W]$ is isomorphic to $K_{1,1}$, then $R[W]$ has a good orientation.*

See Figure 4.1 for an example of an orientation of $K_{4,6}$ using Lemma 4.5.

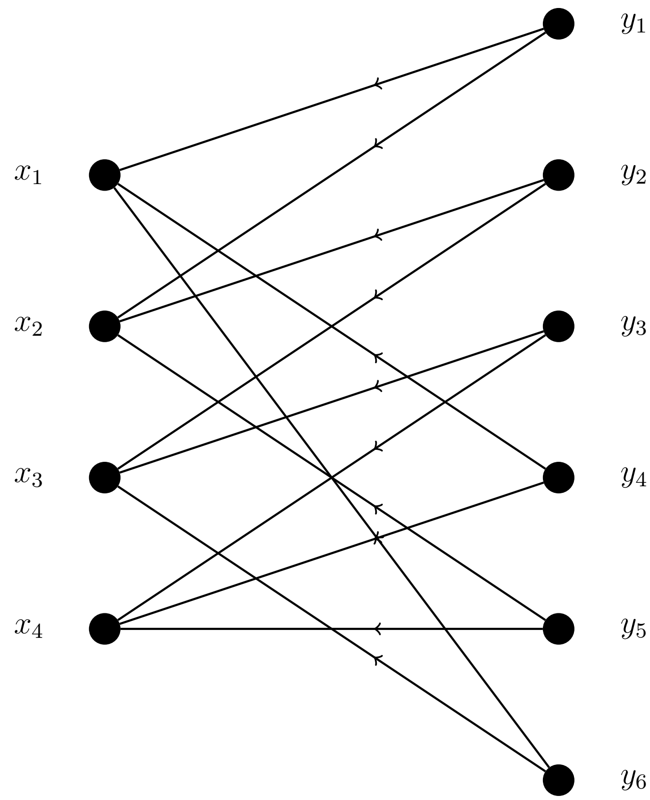

Figure 4.1 A good orientation of $K_{4,6}$ using Lemma 4.5. Missing edges in $K_{a,b}$ are oriented from x_i to y_j.

Proof. Clearly it suffices to prove the lemma for the case that $R[W]$ is isomorphic to $K_{a,b}$. Any orientation of $K_{1,1}$ is vacuously good, so assume $2 \leq a \leq b$.

Assume the vertices of the partite classes of $K_{a,b}$ are $x_1, \ldots, x_a$ and $y_1, \ldots, y_b$. Let $U_1 = \{x_1, \ldots, x_a\}$ and $V_1 = \{y_1, \ldots, y_b\}$. Let $c = \lfloor \frac{a}{2} \rfloor$ and consider an injection $f : [b] \to \binom{[a]}{c}$ such that for $i \in [a] \subseteq [b]$ we have $f(i) = \{i+1, \ldots, i+c\}$ (where numbers in the set are taken modulo a). Such an injection exists by the conditions on a and b. Orient the edge $y_i x_j$ as $\overrightarrow{y_i x_j}$ if $j \in f(i)$, and as $\overrightarrow{x_i y_i}$ otherwise. For $i \neq k, i, k \in [b]$, both $f(i) \setminus f(k)$ and $f(k) \setminus f(i)$ are nonempty, ensuring a directed path of length 2 in both directions between y_i and y_k. For $1 \leq i < k \leq a$, let $\ell \in [a]$

such that $\ell \equiv k + c \mod a$, then we have that $i \in f(i) \setminus f(k)$ an $\ell \in f(k) \setminus f(i)$, which ensures a directed path of length 2 in both directions between x_i and x_k. Hence condition (i) is satisfied.

Clearly every vertex $y_i \in V_1$ has $\lfloor \frac{a}{2} \rfloor$ in-neighbors and $\lceil \frac{a}{2} \rceil$ out-neighbors in U_1. Every vertex $x_i \in U_1$ is adjacent from y_i and to y_{i-1}. Hence condition (ii) is satisfied. □

Definition 4.6. Given two complete graphs K_ℓ and K_k with $\ell \leq k$. Label the vertices of K_ℓ with $[\ell]$ and the vertices of K_k with i' where $i \in [k]$. We define $K_\ell \boxplus K_k$ as the disjoint union of K_ℓ and K_k with the edges ii' included for $i \in [\ell]$.

See Figure 4.2 for an example of $K_3 \boxplus K_5$.

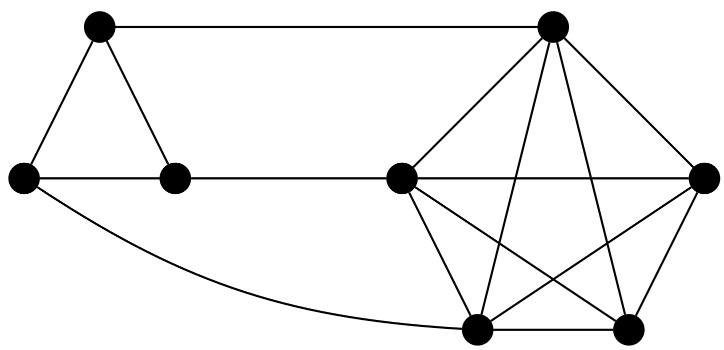

Figure 4.2 A drawing of $K_3 \boxplus K_5$.

Lemma 4.7. *Let $a, b \in \mathbb{N}$ with $3 \leq a \leq b \leq 2a$. Assume $R[W]$ contains $K_{a,b}$ as a spanning subgraph with partite sets X and Y, $B[X] \leq K_a$, and $B[Y] \leq K_a \boxplus K_{b-a}$, then $R[W]$ has a non-trivial good orientation.*

See Figure 4.3 for an example with $B[V] = K_3 \boxplus K_3$ and an orientation of the edges of $R[V]$.

See Figure 4.4 for an example of an orientation of the red graph associated with $B = K_3 \cup K_3 \boxplus K_3$.

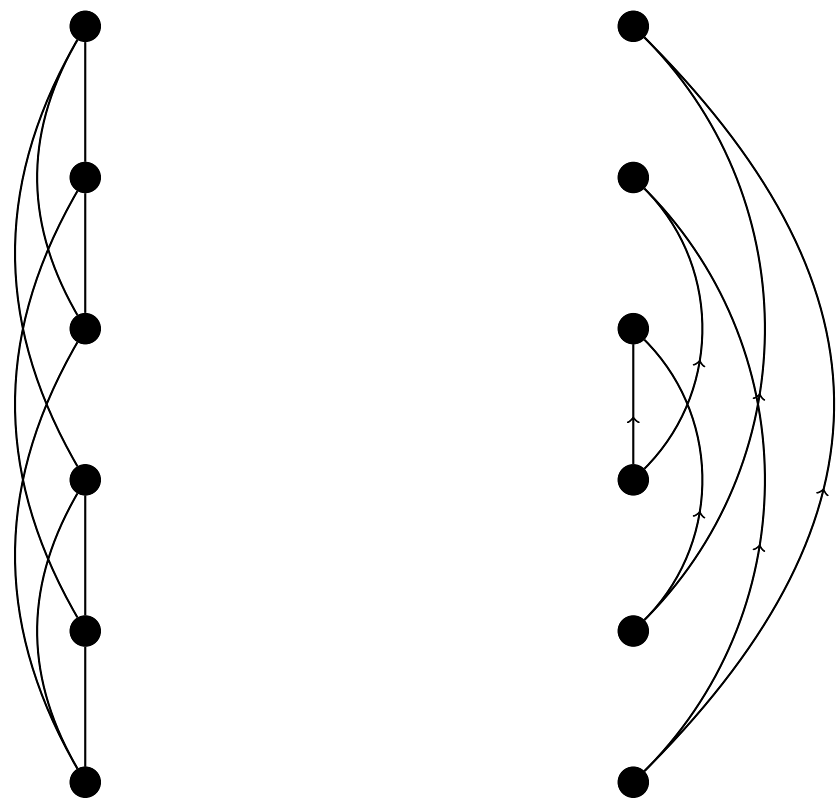

Figure 4.3 On the left, see a drawing of $K_3 \boxplus K_3$, on the right see the orientation of $\overline{K_3 \boxplus K_3}$ to be used in 4.7.

Proof. Assume $3 \leq a \leq b \leq 2a$ and $R[W]$ contains $K_{a,b}$ as a spanning subgraph with partite sets A and B, $R[X] \leq K_a$, and $R[Y] \leq K_a \boxplus K_{b-a}$. It suffices to prove the lemma for the case that $R[W]$ is isomorphic to $K_{a,b}$, and $R[Y] = K_a \boxplus K_{b-a}$. Label the vertices of X as $i \in [a]$. Label a of the vertices in Y with i' where $i \in [a]$. Label the other $b-a$ vertices in Y with i'' where $i \in [b-a]$.

For $i \in [a]$ orient the edges ii' as $\overrightarrow{ii'}$. For $i, j \in [a]$ where $i \neq j$, orient the edges ij' as $\overrightarrow{j'i}$. For $i \in [b-a]$ orient the edges ii'' as $\overrightarrow{i''i}$. For $i, j \in [b-a]$ orient the edges ij'' as $\overrightarrow{ij''}$. For $i, j \in [b-a]$ with $i \neq j$ orient the edges $i''j'$ as $\overrightarrow{i''j}$. Notice that the oriented pairs within the vertex set Y are exactly a $\overline{K_a \boxplus K_{b-a}}$.

Certainly all vertices in X have an in-neighbor and out-neighbor in Y and vice

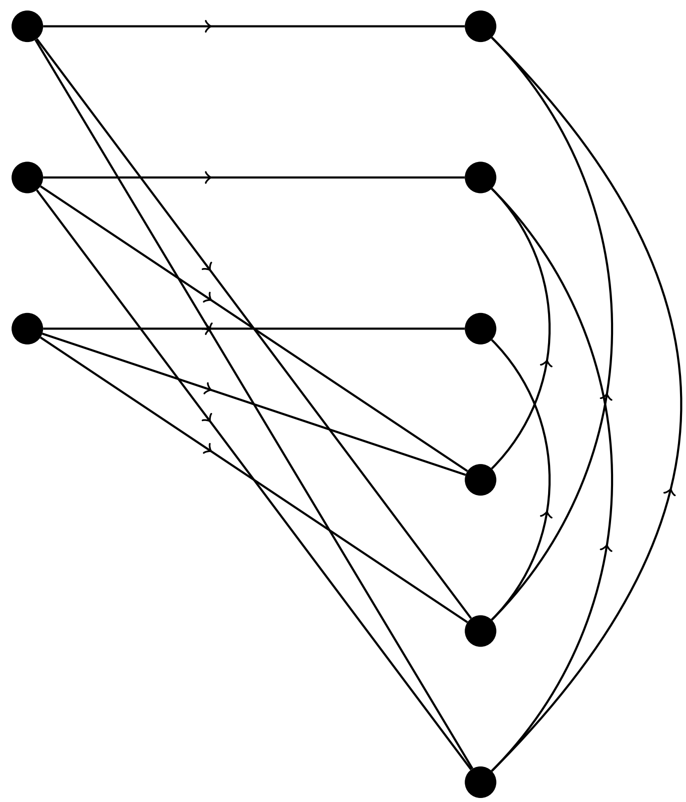

Figure 4.4 A good orientation using Lemma 4.7.
Missing edges in $K_{a,b}$ are oriented from y_i to x_j.

versa. It is left to show that for any vertices $v_1, v_2 \in X$ or $v_1, v_2 \in Y$ that $d_D(v_1, v_2) \leq 2$. Given two vertices $i, j \in X$ with $i \neq j$, consider a path of length two along the arcs $\overrightarrow{ii'}$ and $\overrightarrow{i'j}$. Given two vertices $i', j' \in Y$, consider the path of length two along the arcs $\overrightarrow{i'j}$ and $\overrightarrow{ji'}$. Given two vertices $i', j'' \in Y$, to get from i' to j'' consider the path of length two along the arcs $\overrightarrow{i'i}$ and $\overrightarrow{ij''}$. To get from j'' to i' consider the path of length one along the arc $\overrightarrow{j''i'}$. Hence the conditions for the existence of a non-trivial good orientation hold.

□

Corollary 4.8. *For a vertex set $W \subseteq V$, if $B[W]$ is a disoint union of paths, as long as the components of $B[W]$ can be partitioned into sets X and Y such that $|X| = a$*

and $|Y| = b$ for which $3 \leq a \leq b \leq 2a$, then $R[W]$ has a non-trivial good orientation.

Proof. Let $B[W]$ be the disjoint union of paths which can be partitioned into sets X and Y such that $|X| = a$ and $|Y| = b$ for which $3 \leq a \leq b \leq 2a$. Notice that since we have a partition of the components of $B[W]$, $R[W]$ has $K_{a,b}$ as spanning subgraph with partite sets X and Y. Also, note that $B[X] \leq P_a \leq K_a$ and $B[Y] \leq P_b \leq K_a \boxplus K_{b-a}$. □

Lemma 4.9. *If V can be partitioned into two disjoint sets W and Z so that ther is no edge in B joining a vertex in W to a vertex in Z, $R[W]$ has a non-trivial good orientation, and one of the following holds for Z:*
(i) Z has a non-trivial good orientation, or
(ii) $2 \leq |Z| \leq 3$ and the vertices in Z are isolated in B, or
(iii) $|Z| = 2$ and the two vertices of Z form a component of order 2 in B,
then R has an orientation of diameter 2.

Proof. (i) Let $R[W]$ and $R[Z]$ be non-trivially well orientable. Let D_1 be a non-trivial good orientation of $R[W]$ with a corresponding partition of W into sets U_1 and V_1. Let D_2 be a non-trivial good orientation of $R[Z]$ with a corresponding partition of Z into sets U_2 and V_2. We assign the orientation $U_1 \to U_2$, $U_2 \to V_1$, $V_1 \to V_2$, and $V_2 \to U_1$. We also include D_1 and D_2 in the orientation. It is easy to verify that this orientation indeed has diameter 2.

(ii) Assume that W has a non-trivial good orientation, and let $Z = \{y_1, y_2, \ldots, y_k\}$, with $k = n - |W|$, with $2 \leq k \leq 3$. Let D_1 be a non-trivial good orientation of $R[W]$ with a corresponding partition of W into sets U_1 and V_1. For y_1 orient $U_1 \to y_1$ and $y_1 \to V_1$, and further orient $\{y_2, y_3, \ldots, y_k\} \to U_1$ and $V_1 \to \{y_2, y_3, \ldots, y_k\}$. Finally, if $k = 3$ orient the edges in $R[Z]$ to form a tournament of diameter two and if $k = 2$ orient the edge y_1y_2 arbitrarily. It is easy to verify that this orientation indeed has diameter two.

(iii) The proof is analogous to (i) and (ii) and thus omitted. □

Lemma 4.10. *The following graphs have an orientation of diameter two:*

1. K_5,
2. $K_n - M$,
3. $\overline{K_4 \cup 7K_1}$,
4. $\overline{DB_{4,3} \cup 8K_1}$,
5. $\overline{DB_{4,2} \cup 7K_1}$,
6. $\overline{DB_{4,1} \cup 7K_1}$,
7. $\overline{DB_{3,3} \cup 6K_1}$ *or* $\overline{DB_{3,3} \cup K_2 \cup 5K_1}$,
8. $\overline{SDB_{3,3} \cup 6K_1}$ *or* $\overline{SDB_{3,3} \cup K_2 \cup 5K_1}$,
9. $\overline{DB_{3,2} \cup aP_1 \cup bP_2}$, *with* $a, b \geq 0$ *and* $a + b = 5$,
10. $\overline{C_5 \cup aP_1 \cup bP_2}$, *with* $a, b \geq 0$ *and* $a + b = 5$,
11. $\overline{DB_{3,1} \cup aP_1 \cup bP_2}$, *with* $a, b \geq 0$ *and* $a + b = 5$,
12. $\overline{K_3 \cup aP_1 \cup bP_2}$, *with* $a, b \geq 0$ *and* $a + b = 5$, *and*
13. $\overline{aP_1 \cup bP_2 \cup cP_3 \cup dP_4}$, *with* $a, b, c, d \geq 0$ *and* $a + b + c + d = 5$.

Proof. For many of these graphs, we will find a partition of V two disjoint sets W and Z so that ther is no edge in B joining a vertex in W to a vertex in Z for which the conditions of Lemma 4.9 hold. Since $R[W]$ is non-trivially well orientable, let its corresponding sets be U_1 and V_1. If $R[Z]$ is similarly non-trivially well orientable, let its corresponding sets be U_2 and V_2. In this case, we will notate this partition by (U_2, V_2, U_1, V_1). If $R[Z] = 2K_1$ or $R[Z] = P_2$, we will notate these partitions as (K_1, K_1, U_1, V_1) and (P_2, U_1, V_1) respectively. We will do case (4) in full detail.

1. Let $R = K_5$. Label the vertices of K_5 as $\{v_1, v_2, \ldots, v_5\}$ and consider the orientation $\overrightarrow{v_1v_2}, \overrightarrow{v_2v_3}, \overrightarrow{v_3v_4}, \overrightarrow{v_4v_5}, \overrightarrow{v_5v_1}, \overrightarrow{v_1v_3}, \overrightarrow{v_3v_5}, \overrightarrow{v_5v_2}, \overrightarrow{v_2v_4}$, and $\overrightarrow{v_4v_1}$. It is easy to verify that this orientation has diameter two.

2. Let $R = K_n - M$ with n even and $n \geq 6$.

   For $n = 8$, consider the partition (P_2, P_2, P_2, P_2), using Lemma 4.5 and Lemma 4.9 to find an orientation of diameter two. For $n = 2k$ with $k = 3$ or $k \geq 5$, consider the partition $(P_2, \lfloor \frac{k-1}{2} \rfloor P_2, \lceil \frac{k-1}{2} \rceil P_2)$, using Lemma 4.5 and Lemma 4.9 to find an orientation of diameter two.

3. Let $B = K_4 \cup 7K_1$. Consider the partition $(K_1, K_1, K_4, 5K_1)$ and use Lemmas 4.5 and 4.9 to find an orientation of diameter two.

4. Let $B = DB_{4,3} \cup 8K_1$. Consider the partition $(K_1, K_1, 6K_1, DB_{4,3})$. Note that $n(DB_{4,3}) = 7$ and $n(6K_1) = 6$. Since $6K_1$ and $DB_{4,3}$ form a partition into two independent graphs U_1 and V_1, with $|U_1| = a$ and $|V_1| = b$, where $2 \leq a \leq b \leq \binom{a}{\lfloor \frac{a}{2} \rfloor}$, then the condition for Lemma 4.5 holds. Noticing that $U_2 = K_1$ and $V_2 = K_1$ we find the conditions of Lemma 4.9 are satisfied, so there exists an orientation of diameter two..

5. Let $B = DB_{4,2} \cup 7K_1$. Consider the partition $(K_1, K_1, 5K_1, DB_{4,2})$ and use Lemmas 4.5 and 4.9 to find an orientation of diameter two.

6. Let $B = DB_{4,1} \cup 7K_1$. Consider the partition $(K_1, K_1, 5K_1, DB_{4,1})$ and use Lemmas 4.5 and 4.9 to find an orientation of diameter two.

7. Let $B = DB_{3,3} \cup 6K_1$ or $DB_{3,3} \cup K_2 \cup 5K_1$. Consider the partitions $(K_1, K_1, 4K_1, DB_{3,3})$ or $(K_1, K_1, 3K_1 \cup K_2, DB_{3,3})$ and use Lemmas 4.5 and 4.9 to find an orientation of diameter two.

8. Let $B = SDB_{3,3} \cup 6K_1$ or $SDB_{3,3} \cup K_2 \cup 5K_1$. Consider the partitions $(K_1, K_1, 4K_1, SDB_{3,3})$ or $(K_1, K_1, 3K_1 \cup K_2, SDB_{3,3})$ and use Lemmas 4.5 and 4.9 to find an orientation of diameter two.

9. Let $B = DB_{3,2} \cup aP_1 \cup bP_2$, with $a, b \geq 0$ and $a+b = 5$. There must be two paths of the same size, choose the pair of paths of shortest length and let them be P_i. Let H be the union of the remaining three paths. Note that $DB_{3,2} \leq K_3 \boxplus K_2$, $DB_{3,2} \leq K_4 \boxplus K_1$, and $DB_{3,2} \leq K_5$. Clearly, $3 \leq n(H) \leq 6$. If $3 \leq n(H) \leq 5$, then consider the partition $(P_i, P_i, H, DB_{3,2})$ and use Lemmas 4.7 and 4.9 to find an orientation of diameter two. If $n(H) = 6$, then consider the partition $(P_i, P_i, DB_{3,2}, H)$ and use Lemmas 4.5 and 4.9 to find an orientation of diameter two.

10. Let $B = C_5 \cup aP_1 \cup bP_2$, with $a, b \geq 0$ and $a + b = 5$. Let H and the pair of paths be as in case (9). Note that $C_5 \leq K_3 \boxplus K_2$. If $n(H) = 3$, consider the partition (P_i, P_i, H, C_5) and use Lemmas 4.7 and 4.9 to find an orientation of diameter two. If $4 \leq n(H) \leq 5$, consider the same partition, except use Lemma 4.5 instead of Lemma 4.7. If $n(H) = 6$, consider the partition (P_i, P_i, C_5, H), and use Lemmas 4.5 and 4.9 to find an orientation of diameter two.

11. Let $B = DB_{3,1} \cup aP_1 \cup bP_2$, with $a, b \geq 0$ and $a + b = 5$. Consider a similar pair of paths and graph H as in case (9). Note that $DB_{3,1} \leq K_3 \boxplus K_1$ and $DB_{3,1} \leq K_4$. If $3 \leq n(H) \leq 4$, consider the partition $(P_i, P_i, H, DB_{3,1})$ and use Lemmas 4.7 and 4.9 to find an orientation of diameter two. If $5 \leq n(H) \leq 6$, consider the partition $(P_i, P_i, DB_{3,1}, H)$ and use Lemmas 4.5 and 4.9 to find an orientation of diameter two.

12. Let $B = K_3 \cup aP_1 \cup bP_2$, with $a, b \geq 0$ and $a + b = 5$. Consider a similar pair of paths and a graph H as in case (9). Noting that $3 \leq n(H) \leq 6$, consider the

partition (P_i, P_i, K_3, H) and use Lemmas 4.7 and 4.9 to find an orientation of diameter two.

13. All cases where $n(G) < 10$ were considered using a computer search.

See Appendix A for an explanation of how this search was done.

Let $B = aP_1 \cup bP_2 \cup cP_3 \cup dP_4$, with $a, b, c, d \geq 0$ and $a+b+c+d = 5$. Let H be the union of two paths, P_i say, of equal length as in Case (9), let P_j, P_k, P_ℓ be the remaining paths, and let P_j be the longest of these paths.

Case 1: $i = 1$.

Let $i = 1$ and from the remaining paths consider the longest path, let it be P_j. Let the two remaining paths be P_k and P_ℓ. We only need to consider $n(G) \geq 10$ so we have $2i + j + k + \ell = 2 + j + k + \ell \geq 10$. If $j \leq 2$, we have that $k + \ell \geq 6$, so $\max\{k, \ell\} \geq 3$, a contradiction to the fact that j was the longest of the remaining paths, so we have that $3 \leq j \leq 4$. Since $j \leq 4$, we have that $k+\ell \geq 4$, so $j \leq k+\ell$. Since $j \geq k$ and $j \geq \ell$, it is also true that $j \leq k+\ell \leq 2j$. Consider the partition $(P_1, P_1, P_j, P_k \cup P_\ell)$ and use Lemmas 4.7 and 4.9 to find an orientation of diameter two.

Case 2: $i \geq 2$.

Let $i \geq 2$ and from the remaining paths consider the longest path, let it be P_j. Let the two remaining paths be P_k and P_ℓ.

Since $i \geq 2$, we have that it can not be that $k = \ell = 1$, otherwise we would be in case 1, so it must be true that $k + \ell \geq 3$.

Case 2a: $k + \ell \leq j$.

Since $3 \leq k + \ell \leq j \leq 4$, consider the partition $(P_i, P_i, P_k \cup P_\ell, P_j)$ and use Lemmas 4.7 and 4.9 to find an orientation of diameter two.

Case 2b: $j \leq k + \ell$.

If $i = 2$, $n = 10$, and $j = 2$, then either $k = \ell = 2$ i.e. $R = K_{10} - M$, a case we proved earlier, or $\max\{k, \ell\} \geq 3$, a contradiction to the fact that $j \geq k$ and $j \geq \ell$. If $i = 2$ and $n \geq 11$, we must have that $j = \max\{j, k, \ell\} \geq 3$. Since i is the order of the shortest pair of paths that have the same length, if $i \geq 3$ and $n \geq 10$, we have three paths P_j, P_k, P_ℓ for which $j = \max\{j, k, \ell\} \geq 3$. Otherwise, there would be a pair of paths of order 1 or a pair of order 2. We have that $3 \leq j$.

Since $j \geq k$ and $j \geq \ell$, we have that $3 \leq j \leq k + \ell \leq 2j$. Consider the partition $(P_i, P_i, P_j, P_k \cup P_\ell)$ and use Lemmas 4.7 and 4.9 to find an orientation of diameter two. □

Definition 4.11. Let $W \subseteq V$ such that $B[W]$ is the union of one or more components of B. We say that W is a *reducible unit* if $R[W]$ has a good orientation. We say that W is a *reduction* if $R[W]$ has a non-trivial good orientation. and $\mathrm{ex}(B[W]) \geq -1$.

Lemma 4.12. *If V can be partitioned into at least 3 reducible units, then R has an orientation of diameter 2*

Proof. Assume that V can be partitioned into k reducible units $W_1, W_2, \ldots, W_k$, where $k \geq 3$. Then for each $i \in \{1, 2, \ldots, k\}$, W_i has a good orientation D_i with partite classes U_i and V_i of W_i.

Consider an orientation D' of diameter two of the complete graph on the vertex set $\{u_1, v_1, u_2, v_2, \ldots, u_k, v_k\}$ with the perfect matching $\{u_i v_i \mid 1 \leq i \leq k\}$ removed. Such an orientation exists by Lemma 4.10.

We now combine D' and $\bigcup_{i=1}^{k} D_i$ to obtain an orientation of diameter two of R. The sets $U_1, V_1, U_2, V_2, \ldots, U_k, V_k$ form a partition of $V(G)$. Let xy be an edge of R. If x and y are in the same set W_i, then orient xy as in D_i. The remaining edges of R are oriented as follows: whenever $\overrightarrow{u_i u_j}$ ($\overrightarrow{u_i v_j}$, $\overrightarrow{v_i u_j}$, $\overrightarrow{v_i v_j}$) is an edge in D' then we assign the orientation $U_i \to U_j$ ($U_i \to V_j$, $V_i \to U_j$, $V_i \to V_j$). Let D be the resulting orientation.

To see that between any two vertices of D there is a path of length at most two note that for $x, y \in V(D)$ either both vertices are in the same set U_i (or V_i), in which case there is a path of length at most two in D_i, or they are in different sets, for example $x \in U_i$ and $y \in U_j$, in which case the (u_i, u_j)-path in D' gives rise to an (x, y)-path in D. $\square$

4.4 Some Properties of B

From now on we assume that G is a minimal counterexample, that is, G is a graph on n vertices, $n \geq 5$ and at least $\binom{n}{2} - (n-5)$ edges that has no orientation of diameter two, and among those graphs let G be a graph of minimum order and of minimum size. Clearly, if G has n vertices then G has exactly $\binom{n}{2} - (n-5)$ edges. Hence the corresponding graph B has order n and size $n-5$. Moreover, it follows from Lemma 4.10 that $n \neq 5, 6$, so $n \geq 7$.

In this section we show that a minimal counterexample cannot have a reduction. We also show that no components of B contains three independent vertices, and that no component has two independent vertices that have to common neighbors. These properties will be used extensively in the following sections.

Lemma 4.13. *Let G be a minimal counterexample. Then B has no reduction.*

Proof. Suppose to the contrary that B has a reduction W. Then $|W| > 2$ and, by $m(B[W]) \geq |W| - 1$, also $W \neq V$. Let D be a non-trivial good orientation of $R[W]$ and let U_1 and V_1 be the classes of D. Replace in B the vertices of W with two vertices, u_1, v_1 and a blue edge connecting $u_1 v_1$, to obtain the blue graph B^* on n^* vertices and m^* edges. Note that $B[W]$ is a union of components of B, so B contains no edges joining vertices in W to vertices in $V - W$. Then $n^* = n + 2 - |W| < n$ and by $m(B[W]) \geq |W| - 1$,

$$m^* = (n-5) - m(B[W]) + 1 \leq n - 3 - |W| = n^* - 5.$$

In particular, $1 \leq n^* - 5$, so $5 < n^*$. Since B was a minimal a counterexample, the red graph R^* corresponding to B^* has an orientation D^* of diameter 2.

We now make use of D and D^* to obtain an orientation of diameter 2 of R. Let $x, y \in V$. If $x, y \in W$ then orient xy as in D. If $x, y \in V - W$ then orient xy as in D^*. The remaining edges, joining a vertex in $x \in V - W$ to a vertex in $y \in W$ are oriented as follows. If xu_1 has received the orientation $\overrightarrow{xu_1}$ in D^* then we orient $x \to U_1$, and if xu_1 has received the orientation $\overrightarrow{u_1x}$ in D^* then we orient $U_1 \to x$. Similarly, if xv_1 has received the orientation $\overrightarrow{xv_1}$ in D^* then we orient $x \to V_1$, and if xv_1 has received the orientation $\overrightarrow{v_1x}$ in D^* then we orient $V_1 \to x$.

As in the proof of Lemma 4.12 we now conclude that the resulting orientation of R has diameter 2. But then G is not a counterexample, a contradiction. Hence G has no reduction. □

Lemma 4.14. *Let G be a minimal counterexample. If x_1, x_2, x_3 is an independent set of order* 3 *in B, and N_i is the set of vertices in $v \in V - \{x_1, x_2, x_3\}$ having exactly i neighbors (in B) in $\{x_1, x_2, x_3\}$ for $i \in \{2, 3\}$, then*

$$|N_2| \leq 1 \quad \textit{and} \quad N_3 = \emptyset. \tag{4.1}$$

Proof. Suppose to the contrary that there are three independent vertices x_1, x_2, x_3 in B such that (4.1) does not hold. Create a new blue graph B^* on $n' = n - 2$ vertices by identifying x_1, x_2 and x_3 to a new vertex x and removing multiple edges. Then $n(B^*) = n - 2$ and

$$m(B^*) = m(B) - |N_2| - 2|N_3| \leq m(B) - 2 = n - 7 = n(B^*) - 5.$$

Therefore, since G is a minimal counterexample, the red graph R^* corresponding to B^* has an orientation D^* of diameter 2.

We now make use of D^* to obtain an orientation D of R of diameter 2. Orient every edge uv with $u, v \notin \{x_1, x_2, x_3\}$ as in D^*. If an edge ux is present in R^*, then all edges ux_i, $i = 1, 2, 3$ are present in R, and depending on whether ux is oriented as $\overrightarrow{ux}$ or as $\overrightarrow{xu}$ in D^*, we orient them $u \to \{x_1, x_2, x_3\}$ or $\{x_1, x_2, x_3\} \to u$. If an edge ux_i is present in R, but ux is not present in R^*, then orient ux_i arbitrarily. Finally orient the edges x_1x_2, x_2x_3 and x_3x_1 as $\overrightarrow{x_1x_2}$, $\overrightarrow{x_2x_3}$ and $\overrightarrow{x_3x_1}$, respectively.

To see that D is an orientation of diameter 2, consider two vertices u and v of D. If $u, v \in \{x_1, x_2, x_3\}$, then clearly there exists a (u, v)-path of length at most two in $D[\{x_1, x_2, x_3\}]$, the subdigraph of D induced by $\{x_1, x_2, x_3\}$. If $u \in \{x_1, x_2, x_3\}$ and $v \in V - \{x_1, x_2, x_3\}$ or vice versa then the (x, v)-path of length at most two in D^* gives rise to a (u, v)-path of the same length in D. If $u, v \in V - \{x_1, x_2, x_3\}$ then the (u, v)-path of length at most two in D^* gives rise to a (u, v)-path of the same length in D. This shows that D is an orientation of R of diameter 2, a contradiction to G being a counterexample. □

Lemma 4.15. *Let G be a minimal counterexample. Then no component of B has three independent vertices.*

Proof. Suppose to the contrary that B has a component which contains three independent vertices x_1, x_2 and x_3. We may assume that

$$d_B(x_1, \{x_2, x_3\}) = 2. \tag{4.2}$$

Indeed, if $d_B(x_1, \{x_2, x_3\}) \geq 3$ then let x_1' be a vertex on a shortest path in B from x_1 to $\{x_2, x_3\}$ that is at distance two from $\{x_2, x_3\}$. The new set $\{x_1', x_2, x_3\}$ is independent and satisfies (4.2).

By (4.2) we may assume, possibly after renaming vertices, that $d_B(x_1, x_2) = 2$. A similar argument as above now yields that we can choose x_3 such that also

$$d_B(x_3, \{x_1, x_2\}) = 2. \tag{4.3}$$

Hence we can choose $\{x_1, x_2, x_3\}$ such that it contains at least two pairs of vertices at distance two in B. Hence, possibly after renaming the vertices, we have

$$d_B(x_1, x_2) = d_B(x_2, x_3) = 2. \tag{4.4}$$

Now (4.4) implies that there exists a common neighbor y_{12} of x_1 and x_2, and a common neighbor y_{23} of x_2 and x_3 in B. If $y_{12} = y_{23}$, then the set N_3 of vertices with three neighbors $\{x_1, x_2, x_3\}$ contains y_{12} and is thus not empty, a contradiction to Lemma 4.14. If $N_3 = \emptyset$, then $y_{12} \neq y_{23}$ and so the set N_2 of vertices with exactly two neighbors in $\{x_1, x_2, x_3\}$ contains y_{12} and y_{23}, again a contradiction to Lemma 4.14. □

Lemma 4.16. *Let G be a minimal counterexample. If x_1, x_2 are independent vertices in B, then x_1 and x_2 have at most one common blue neighbor.*

Proof. Suppose to the contrary that B has two vertices x_1 and x_2 that share two neighbors. Then x_1 and x_2 are in the same component of B. Choose a vertex x_3 from another component. Such a component exists since B has at least 5 components by Lemma 4.17. Then x_1, x_2, x_3 are independent vertices, for which the set N_2 of vertices having exactly two neighbors in $\{x_1, x_2, x_3\}$ has at least two elements, a contradiction to Lemma 4.14. □

4.5 On Tree Components of B

Since B has n vertices and $n - 5$ edges, B is not connected. In this section we give useful lower bounds on the number of components of B that are trees, and we show that for a given order t we can find a union F_t of tree components of B whose order is close to t. This will be useful in the case that B has a non-tree component B_1 of order t whose size is much greater than t. Since the order of B_1 and F_t are close to each other, we will often be able to apply Lemma 4.5 to show that $V(B_1) \cup V(F_t)$ has a non-trivial good orientation and, provided B_1 has sufficiently large excess, that they form a reduction. Recall that the *excess* of a graph H is defined as $\mathrm{ex}(H) = m(H) - n(H)$.

Lemma 4.17. *If B contains a component B_1 that is not a tree, then B has at least $ex(B_1) + 5$ components that are trees. B contains at least five components that are trees.*

Proof. Let $T_1, T_2, \ldots, T_k$ be the components of B that are trees, and $B_1, B_2, \ldots, B_\ell$ the components that are not trees. Then $\mathrm{ex}(T_i) = -1$ for all $i \in \{1, 2, \ldots, k\}$ and $\mathrm{ex}(B_i) \geq 0$ for all $i \in \{1, 2, \ldots, \ell\}$. Since $m(B) = n - 5$, we have $\mathrm{ex}(B) = -5$, and so

$$-5 = \mathrm{ex}(B) = \sum_{i=1}^{k} \mathrm{ex}(T_i) + \sum_{i=1}^{\ell} \mathrm{ex}(B_i) = -k + \sum_{i=1}^{\ell} \mathrm{ex}(B_i) \geq -k.$$

and so $k \geq 5$. Hence B has at least five components that are trees.

If B contains a component that is not a tree, B_1 say, then a similar argument yields that

$$-5 = \sum_{i=1}^{k} \mathrm{ex}(T_i) + \sum_{i=1}^{\ell} \mathrm{ex}(B_i) = -k + \sum_{i=1}^{\ell} \mathrm{ex}(B_i) \geq -k + \mathrm{ex}(B_1),$$

and so $k \geq 5 + \mathrm{ex}(B_1)$, as claimed. □

Lemma 4.18. *Let B' be a subgraph of B containing t or more tree components of B whose size does not exceed M_0. Then there exists t_0 with $t \leq t_0 \leq t + M_0$ such that some subset of the tree components in B' forms a forest F_t in B' satisfying $n(F_t) = t_0$ and $m(F_t) \geq t_0 - t$, and thus $ex(F_t) \geq -t$. If B' contains a tree of size m_0 and if $t > m_0$, then $m(F_t) \geq t_0 - t + m_0$ and thus $ex(F_t) \geq -t + m_0$.*

Proof. Let $T_1, T_2, \ldots, T_t$ the t tree components of B'. Since each B' has at least t tree components, $T_1 \cup T_2 \cup \cdots \cup T_t$ contains at least t vertices. Let j be the smallest positive integer such that $T_1 \cup T_2 \cup \cdots \cup T_j$ contains t or more vertices. Let $F_t = T_1 \cup T_2 \cup \cdots T_j$ and let $t_0 = n(F_t)$. Since T_j has size at most M_0 and thus order at most $M_0 + 1$, we have $t \leq t_0 \leq t + M_0$. Moreover, since $T_1 \cup T_2 \cup \cdots \cup T_{j-1}$ has less than t vertices, it follows that T_j has at least $t_0 - t + 1$ vertices and at least $t_0 - t$ edges. Hence

$m(F_1) \geq m(T_j) \geq t_0 - t$, and thus $\mathrm{ex}(T_t) \geq -t$.

If $t > m_0$, then we may assume that T_1 is a tree of size m_0. The same argument as above yields that $m(F_t) \geq m(T_1) + m(T_j) = m_0 + t_0 - t$ and thus $\mathrm{ex}(F_t) \geq -t + m_0$, as desired. □

We will see in the next section that the tree components of B have at most four vertices. Hence the following corollary, obtained from Lemma 4.18 by setting $M_0 = 3$, is useful.

Corollary 4.19. *Let B' be a subgraph of B containing t or more tree components of B which have order at most four. Then there exists t_0 with $t \leq t_0 \leq t + 3$ such that some subset of the tree components in B' forms a forest F_t in B' satisfying $n(F_t) = t_0$ and $m(F_t) \geq t_0 - t$, and so $ex(F_t) \geq -t$. If $t \geq m_0$ and B' contains a tree of size m_0, then $ex(F_t) \geq -t + m_0$.*

4.6 Describing the components of B

In this section we prove further properties of the graph B of a minimal counterexample. We show that each component of B is either a a path on at most four vertices, a complete graph, a proper dumbbell, a proper short dumbbell, or a 5-cycle. We further show that the order of a component of B cannot exceed six.

Lemma 4.20. *Let G be a minimal counterexample and B_1 a component of B.*

(a) If B_1 is a tree, then B_1 is a path P_i with $1 \leq i \leq 4$.

(b) If B_1 is not a tree, then B_1 is either

(i) a complete graph K_i with $i \geq 3$, or

(ii) a proper (k, ℓ)-dumbbell, or

(iii) a proper short dumbbell, or

(iv) a 5-cycle.

Proof. If B_1 is complete, then the lemma holds, so we assume that B_1 is not complete. Let x_1 and x_2 be two vertices of B_1 with $d_B(x_1, x_2) = \text{diam}(B_1) \geq 2$.

Case 1: $\text{diam}(B_1) \geq 3$.

Since B_1 does not have three independent vertices by Lemma 4.15, we conclude that $d_B(x_1, x_2) = 3$, that $V(B_1)$ is the disjoint union of $N_B[x_1]$ and $N_B[x_2]$, and that each $N_B[x_i]$ forms a clique.

Since B_1 is connected, B_1 has an edge joining a vertex $y_1 \in N_B(x_1)$ to a vertex $y_2 \in N_B(x_2)$. We show that B_1 does not contain a further edge joining a vertex $z_1 \in N_B(x_1)$ to a vertex $z_2 \in N_B(x_2)$. Indeed, if $y_1 = z_1$ then $\{y_1, x_2\}$ would be a set of two independent vertices that share two neighbors, if $y_2 = z_2$ then $\{y_2, x_1\}$ would be a set of two independent vertices that share two neighbors, and if $y_1 \neq z_1$ and $y_2 \neq z_2$ then $\{y_1, z_2\}$ would be a set of two independent vertices that share two neighbors, contradicting Lemma 4.16. It follows that B_1 is an (n_1, n_2)-dumbbell with $n_i = |N_B[x_i]| \geq 2$ for $i = 1, 2$. If B_1 is a tree, then this implies that $B_1 = P_4$. If B_1 is not a tree, then this implies that B_1 is a proper dumbbell.

Case 2: $\text{diam}(B_1) = 2$.

Then $N_B[x_1] \cup N_B[x_2] = V(B_1)$ since B_1 does not have three independent vertices by Lemma 4.15. Moreover, x_1 and x_2 have a common neighbor y in B_1. By Lemma 4.16, y is the only common neighbor of x_1 and x_2 in B_1. We consider two subcases:

Case 2a: $\deg_B(x_1) = 1$ or $\deg_B(x_2) = 1$.

Assume without loss of generality that $\deg_B(x_1) = 1$, so $N_B(x_1) = \{y\}$. Since $\text{diam}(B_1) = 2$, every vertex in $V(B_1) - \{x_1, y\}$ is adjacent to y in B_1. Since B_1 does not contain three independent vertices, $V(B_1) - \{x_1, y\}$ induces a complete graph in B_1. Therefore B_1 is a short $(2, n(B_1) - 1)$-dumbbell. If B_1 is a tree, then it follows that $B_1 = P_3$. If B_1 is not a tree, then it follows that B_1 is a proper dumbbell.

Case 2b: $\deg_B(x_1) \geq 2$ and $\deg_B(x_1) \geq 2$.

Since B_1 does not contain three independent vertices, $N_B[x_i] \setminus \{y\}$ induces a complete graph in B for $i \in \{1,2\}$. If y is adjacent to all vertices in C, then clearly B_1 is a short dumbbell, so assume that there is a vertex z_1 to which y is non-adjacent in B_1. We may assume that $z_1 \in N_B[x_1]$. Then $d_B(z_1, x_2) = 2$, so z_1 and x_2 have a common blue neighbor z_2. Since x_1 and z_2 are non-adjacent in B and thus cannot have two common neighbors, z_2 and y are non-adjacent in B. Since also the edges x_1x_2, x_1z_2 and x_2z_1 are not present in B, we conclude that $x_1, y, x_2, z_2, z_1, x_1$ is an induced 5-cycle in B_1. Hence B_1 contains an induced 5-cycle.

Rename the vertices of the 5-cycle as $v_0, v_1, v_2, v_3, v_4, v_0$. We show that B_1 contains only these five vertices. Suppose not. Then there exists a vertex w adjacent to a vertex in $\{v_0, v_1, v_2, v_3, v_4\}$ in B_1. If v is adjacent to only one or two vertices in $\{v_0, v_1, v_2, v_3, v_4\}$, then it is easy to see that v together with two suitably chosen vertices in $\{v_0, v_1, v_2, v_3, v_4\}$ forms an independent set of cardinality three, which is impossible. Hence v is adjacent to at least three vertices in $\{v_0, v_1, v_2, v_3, v_4\}$. But then v has two neighbors among these vertices that are not adjacent, without loss of generality v_1 and v_3, so that v_1 and v_3 are non-adjacent vertices with two common neighbors, a contradiction to Lemma 4.16. This proves that B_1 contains only $\{v_0, v_1, v_2, v_3, v_4\}$, and so B_1 is a 5-cycle. □

Lemma 4.21. *Let G be a minimal counterexample. Then B contains no component of order greater than six.*

Proof. Suppose to the contrary that B contains a component B_1 with more than six vertices. Let n_1 and m_1 be the order and size, respectively, of B_1. We first prove that

$$m_1 \geq \left\lceil \frac{1}{4}n_1^2 - \frac{1}{2}n_1 + 1 \right\rceil. \tag{4.5}$$

By Lemma 4.20, B_1 is a complete graph, a dumbbell, or a short dumbbell. It is easy to see that among all such graphs of order n_1 the dumbbell $DB_{\lceil n_1/2\rceil, \lfloor n_1/2 \rfloor}$ is the unique graph of minimum size. A simple calculation shows that $DB_{\lceil n_1/2\rceil, \lfloor n_1/2 \rfloor} = \lceil \frac{1}{4}n_1^2 - \frac{1}{2}n_1 + 1 \rceil$, and (4.5) follows.

CASE 1: $n_1 \geq 8$.

Let $t := n_1 - 2$. Then B contains at least t tree components since by Lemma 4.17 B contains at least $\text{ex}(B_1) + 5$ tree components, and by (4.5) we have $\text{ex}(B_1) + 5 \geq \frac{1}{4}n_1^2 - \frac{1}{2}n_1 + 6 \geq n_1 - 2$, as can easily be verified.

By Corollary 4.19, B contains a forest F_{n_1-2} of order t_0 and excess at least $-n_1 + 2$ for some t_0 with $n_1 - 2 \leq t_0 \leq n_1 + 1$ that is the union of tree components of B. Let $W := V(B_1) \cup V(F_{n_1-2})$. We show that W is a reduction. Clearly the graph $R[W]$ contains a spanning subgraph K_{n_1,t_0}. Since $n_1 - 2 \leq t_0 \leq n_1 + 1$ it is easy to verify that either $n_1 \leq t_0 \leq \binom{n_1}{2}$ or $t_0 < n_1 \leq \binom{t_0}{2}$ holds, so $R[W]$ has a non-trivial orientation by Lemma 4.5. By (4.5) we also have

$$\begin{aligned} \text{ex}(B[W]) &= \text{ex}(B_1) + \text{ex}(F_{n_1-2}) \\ &\geq \frac{1}{4}n_1^2 - \frac{1}{2}n_1 + 1 - n_1 + (-n_1 + 2) \\ &= \frac{1}{4}n_1^2 - \frac{5}{2}n_1 + 3 \\ &\geq -1, \end{aligned}$$

with the last inequality holding since the term $\frac{1}{4}n_1^2 - \frac{5}{2}n_1 + 3$ attains its minimum for $n_1 = 8$. Hence W is a reduction. This contradiction to Lemma 4.13 shows that Case 1 cannot occur.

CASE 2: $n_1 = 7$ and $B_1 \neq DB_{3,4}$.

It follows from (4.5) that $m_1 \geq 10$, with equality if and only if B_1 is the dumbbell $DB_{3,4}$. Since $B_1 \neq DB_{3,4}$ we have $m_1 \geq 11$ and thus $\text{ex}(B_1) \geq 4$. Now exactly the same argument as in Case 1 yields a contradiction.

CASE 3: $n_1 = 7$ and $B_1 = DB_{3,4}$.

Then $m_1 = 10$, so $\text{ex}(B_1) = 3$. If now $B - B_1$ contains only singleton components, then $m(B) = n - 5$ implies that $n = 15$ and that $B - B_1$ contains exactly eight components, so $B = DB_{3,4} \cup 8K_1$. But $\overline{DB_{3,4} \cup 8K_1}$ has an orientation of diameter two by Lemma 4.10, so B contains a component with at least one edge. Now Corollary

4.19 with $t = 5$ and $m_0 \geq 1$ yields that there exists a forest F_5 of order t_0, where $5 \leq t_0 \leq 8$, and excess at least $-5 + 1 = -4$. that is the union of tree components of B. Let $W = V(B_1) \cup V(F_5)$. As above we show that $R[W]$ has a non-trivial good orientation by Lemma 4.5. Moreover,

$$\text{ex}(R[W]) = \text{ex}(B_1) + \text{ex}(F_5) \geq 3 + (-4) = -1.$$

Hence W is a reduction, a contradiction to Lemma 4.13. □

Lemma 4.22. *Let G be a minimal counterexample. If B contains a component B_1 that is not a tree, then $B - B_1$ has exactly ex$(B_1) + 5$ components, and all of them are trees.*

Proof. Suppose to the contrary that $B - B_1$ contains a component B_2 that is also not a tree. Then $\text{ex}(B_1) \geq 0$ and $\text{ex}(B_2) \geq 0$. Let n_1 and n_2 be the order of B_1 and B_2, respectively. We may assume that $n_1 \geq n_2$. By Lemma 4.21 we have $n_1 \leq 6$. Also, $n_2 \geq 3$. If $n_1, n_2 \in \{4, 5, 6\}$, then $V(B_1) \cup V(B_2)$ has a non-trivial good orientation by Lemma 4.5 and is thus a reduction since $\text{ex}(B_1 \cup B_2) = \text{ex}(B_1) + \text{ex}(B_2) \geq 0$. If $n_1 \in \{4, 5, 6\}$ and $n_2 = 3$, then B contains no tree component B_3 or order 4 or 3 since otherwise $V(B_1) \cup V(B_3)$ or $V(B_2) \cup V(B_3)$ would form a reduction. Hence B contains a tree component B_3 of order $n_3 \in \{1, 2\}$. Then it is easy to verify that $V(B_1) \cup V(B_2) \cup V(B_3)$ form a reduction. In all cases we get a contradiction to Lemma 4.13. Hence all components of $B - B_1$ are trees.

Let $B - B_1$ have k components, $B_2, B_3, \ldots, B_{k+1}$ say. Since each tree has excess -1, and since $\text{ex}(B) = -5$, we have

$$5 = \text{ex}(B) = \sum_{i=1}^{k+1} \text{ex}(B_i) = \text{ex}(B_1) - k,$$

and so $k = \text{ex}(B_1) + 5$, as desired. □

Lemma 4.23. *Let G be a minimal counterexample. Then B contains no component that is a complete graph on three or more vertices.*

Proof. Suppose to the contrary that B contains a component B_1 that is a complete graph of order $n_1 \geq 3$.

CASE 1: $n_1 \geq 5$.

By Corollary 4.19, B contains a collection of tree components which form a forest F_{n_1} of order $n(F_{n_1}) = n_0$ and excess $\mathrm{ex}(F_{n_1}) \geq -n_1$ where $n_1 \leq n_0 \leq n_1 + 3$. Let $W = V(K_{n_1}) \cup V(F_{n_0})$. Then $R[W]$ contains K_{n_1,n_0}. Since $n_1 \geq 5$ we have $n_1 \leq n_0 \leq n_1 + 3 \leq \binom{n_1}{2}$, where the last inequality holds since $n_1 \geq 5$. Now Lemma 4.5 yields that $R[W]$ has a non-trivial good orientation. Since also

$$\begin{aligned} \mathrm{ex}(B[W]) &= \mathrm{ex}(K_{n_1}) + \mathrm{ex}(F_{n_1}) \\ &\geq \binom{n_1}{2} - n_1 - n_1 \\ &= \frac{1}{2} n_1(n_1 - 5) \\ &\geq -1, \end{aligned}$$

W is a reduction, a contradiction to Lemma 4.13.

CASE 2: $n_1 = 4$.

Then $B_1 = K_4$ and so $\mathrm{ex}(B_1) = 2$. If all components of $B - B_1$ are singletons, then $B = K_4 \cup 7K_1$ since $B - B_1$ has exactly seven components by Lemma 4.22. Hence $G = \overline{K_4 \cup 7K_1}$. But $\overline{K_4 \cup 7K_1}$ has an orientation of diameter two by Lemma 4.10, so G is not a counterexample. Hence we assume that $B - B_1$ has a component B_2 of size $m_0 \geq 1$. By Lemma 4.22, B_2 is a tree, and by Lemma 4.20 B_2 is a path on at most four vertices. If $B_2 = P_4$, then it is easy to verify that $V(B_1) \cup V(B_2)$ is a reduction, hence we may assume that $B - B_1$ contains only paths on at most three vertices. Letting $M_0 \leq 2$ and $m_0 \geq 1$ in Lemma 4.18 we get that there exists a forest F_4 of order t_0, where $4 \leq t_0 \leq 6$, with $\mathrm{ex}(F_4) \geq -3$. Since $V(B_1) \cup V(F_4)$ has a non-trivial good orientation by Lemma 4.5, and since $\mathrm{ex}(B[V(B_1) \cup V(F_4)]) \geq 2 + (-3) = -1$, it follows that $V(B_1) \cup V(F_4)$ is a reduction, a contradiction to Lemma 4.13.

CASE 3: $n_1 = 3$.

By Lemma 4.22 the graph $B - B_1$ has exactly $\mathrm{ex}(K_3) + 5 = 5$ components, which by Lemma 4.20 are paths on at most four vertices. If $B - B_1$ contains a component B_2 that is P_3, then it is easy to verify that $V(B_1) \cup V(B_2)$ form a reduction, a contradiction to Lemma 4.13. Hence $B - B_1$ contains only components that are K_1, K_2 or P_4. Hence we have $B = K_3 \cup aK_1 \cup bK_2 + cP_4$ for some nonnegative integers a, b, c with $a + b + c = 5$. But by Lemma 4.10, all such graphs have an orientation of diameter two. So G is not a counterexample, a contradiction. $\square$

Lemma 4.24. *Let G be a minimal counterexample. Then B contains no component that is a proper dumbbell.*

Proof. Suppose to the contrary that B contains a proper dumbbell B_1, and let n_1 be its order. Since B_1 is a proper dumbbell we have $n_1 \geq 4$, and by Lemma 4.21 we have $n_1 \leq 6$. Let B_2 be a largest component in $B - B_1$, and let n_2 be its order. B_2 is a tree by Lemma 4.22, and so B_2 is a path and $n_2 \leq 4$ by Lemma 4.20. We cannot have $n_2 = 4$ since in this case $V(B_1) \cup V(B_2)$ would form a reduction, a contraction to Lemma 4.13. Hence $n_2 \leq 3$.

Case 1: $n_1 = 6$.

B_1 is either a $(5, 1)$-dumbbell, a $(4, 2)$-dumbbell or a $(3, 3)$-dumbbell. We consider all three possibilities.

(i) Let B_1 be a $(5, 1)$-dumbbell. Then $m(B_1) = 11$ and $\mathrm{ex}(B_1) = 5$. Setting $M_0 \leq 2$ in Lemma 4.18 we get that there exists a forest F_4 of order t_0 and excess at least -4 for some $t_0 \in \{4, 5, 6\}$. $V(B_1) \cup V(F_4)$ has a non-trivial good orientation by Lemma 4.5, and since $\mathrm{ex}(B[V(B_1) \cup V(F_4)]) \geq 5 + (-1) \geq -1$, the set $V(B_1) \cup V(F_4)$ is a reduction, a contradiction to Lemma 4.13.

(ii) Let B_1 be a $(4, 2)$-dumbbell. Then $m(B_1) = 8$ and $\mathrm{ex}(B_1) = 2$. If all components of $B - B_1$ are singletons, then $B = DB_{4,2} \cup 7K_1$ since by Lemma 4.22 the graph $B - B_1$ has exactly $\mathrm{ex}(B_1) + 5 = 7$ components. But $\overline{DB_{4,2} \cup 7K_1}$ has an orientation of diameter two by Lemma 4.10, so it is not a counterexample. Hence we assume that $B - B_1$ contains a component of order at least two. Also, $B - B_1$ contains no component that is a path on three vertices since otherwise, if there was such a component B_2, then $V(B_1) \cup V(B_2)$ would be a reduction. Now setting $M_0 = 2$ and $m_0 = 1$ in Lemma 4.18, we get that $B - B_1$ contains a union of components that is a forest F_4 of order t_0 with $4 \leq t_0 \leq 6$ and $\text{ex}(F_4) \geq -3$. Now $\text{ex}(B[V(B_1) \cup V(F_4)]) \geq 2 + (-3) = -1$ and by Lemma 4.5, $V(B_1) \cup V(F_4)$ has a non-trivial good orientation. Hence $V(B_1) \cup V(F_4)$ is a reduction, a contradiction to Lemma 4.13.

(iii) Let B_1 be a $(3, 3)$-dumbbell, so $m(B_1) = 7$ and $\text{ex}(B_1) = 1$. By Lemma 4.22, $B - B_1$ contains exactly $\text{ex}(B_1) + 5 = 6$ components which are trees, and thus paths on at most four vertices by Lemma 4.20. If $B - B_1$ consists of six singleton components or of five singleton components and a K_2, then $G = \overline{DB_{3,3} \cup 6K_1}$ or $G = \overline{DB_{3,3} \cup K_2 \cup 5K_1}$. In both cases Lemma 4.10 shows that G has an orientation of diameter two. Hence we may assume that $B - B_1$ contains a path on three or four vertices, or two components K_2. In both cases we get that $B - B_1$ contains two tree components whose union F_4 is a forest of order t_0 with $4 \leq t_0 \leq 6$ and with $\text{ex}(F_4) \geq -2$. By Lemma 4.5, $V(B_1) \cup V(F_4)$ has a non-trivial good orientation, so it forms a reduction, a contradiction to Lemma 4.13.

Case 2: $n_1 = 5$.

Now B_1 is either a $(4, 1)$-dumbbell or a $(3, 2)$-dumbbell. We consider both possibilities.

(i) If B_1 is a $(4, 1)$-dumbbell, then precisely the same proof as in Case 1(ii) shows that either $G = \overline{DB_{4,1} \cup 7K_1}$, which by Lemma 4.10 is not a counterexample, or B contains a reduction, a contradiction to Lemma 4.13.

(ii) Let B_1 be a $(3, 2)$-dumbbell. Then $m(B_1) = 5$, so $\text{ex}(B_1) = 0$. Hence $B - B_1$ contains exactly five components, which are trees, by Lemma 4.22. If one of these, B_2, is a P_3 or P_4 notice since $P_3 \leq K_3$, $B_1 \leq K_3 \boxplus K_2$, $P_4 \leq K_4$, and $B_1 \leq K_4 \boxplus K_1$, that using 4.7 these are a reduction. Since also $\mathrm{ex}(B[V(B_1) \cup V(B_2)]) = -1$, the set $V(B_1) \cup V(B_2)$ is a reduction, a contradiction to Lemma 4.13. It follows that $n_2 \leq 2$. By Lemma 4.22 $B - B_1$ has exactly $\mathrm{ex}(B_1) + 5 = 5$ components, hence $B = DB_{3,2} \cup aK_1 + bK_2$ for some nonnegative integers a, b with $a + b = 5$. But all such graphs have an orientation of diameter two by Lemma 4.10, so G is not a counterexample, and we obtain a contradiction.

Case 3: $n_1 = 4$.

Since $DB_{3,1}$ is the only proper dumbbell of order 4 we have $B_1 = DB_{3,1}$ and thus $\mathrm{ex}(B_1) = 0$. By Lemma 4.22, $B - B_1$ has exactly $\mathrm{ex}(B_1) + 5 = 5$ components, each being a path of order at most four. If $B - B_1$ contains a component B_2 that is a path on four or three vertices, by a similar proof to case 2 we find that $V(B_1) \cup V(B_2)$ has a non-trivial good orientation by Lemma 4.7, and since $\mathrm{ex}(B[V(B_1) \cup V(B_2)]) = 0 + (-1) = -1$, $V(B_1) \cup V(B_2)$ is a reduction, a contradiction to Lemma 4.13. Hence $B - B_1$ contains only components that are paths of order at most two. It follows that $B = DB_{3,1} \cup aK_1 + bK_2$ for some nonnegative integers a, b with $a + b = 5$. But in all cases G has an orientation by Lemma 4.10, so G is not a counterexample, a contradiction. □

Lemma 4.25. *Let G be a minimal counterexample. Then B contains no component that is a proper short dumbbell.*

Proof. Suppose to the contrary that B contains a proper short dumbbell B_1 and let n_1 be its order. By Lemma 4.21 we have $n_1 \leq 6$. It is easy to check that the only proper short dumbbells of order not more than six are $SDB_{4,3}$ and $SDB_{3,3}$.

First let $B_1 = SDB_{4,3}$. Then $m(B_1) = 9$ and $\mathrm{ex}(B_1) = 3$. By Lemma 4.18 there exist tree components of $B - B_1$ whose union is a forest F_4 or order t_0 with $4 \leq t_0 \leq 7$ and $\mathrm{ex}(F_4) \geq -4$. By Lemma 4.5 $V(B_1) \cup V(F_4)$ has a non-trivial good orientation, and $\mathrm{ex}(B[V(B_1) \cup V(F_4)]) = \mathrm{ex}(B_1) + \mathrm{ex}(F_4) \geq -1$. Hence $V(B_1) \cup V(F_4)$ form a reduction, a contradiction to Lemma 4.13.

Now let $B_1 = SDB_{3,3}$. Then $m(B_1) = 6$ and $\text{ex}(B_1) = 1$. A proof identical to that of Lemma 4.24, Case 1(iii) shows that either $G = \overline{SDB_{3,3} \cup 6K_1}$ or $G = \overline{SDB_{3,3} \cup K_2 \cup 5K_1}$, or B has a reduction. But Lemma 4.10 shows that both, $\overline{SDB_{3,3} \cup 6K_1}$ and $\overline{SDB_{3,3} \cup K_2 \cup 5K_1}$, have an orientation of diameter two, so they are not counterexamples. Hence B has a reduction, a contradiction to Lemma 4.13. □

Lemma 4.26. *Let G be a minimal counterexample. Then B contains no component that is a 5-cycle.*

Proof. Suppose to the contrary that B contains a component B_1 that is a 5-cycle. Then $\text{ex}(B_1) = 0$, and by Lemma 4.22 the graph $B - B_1$ has exactly $\text{ex}(B_1) + 5 = 5$ components which are trees. By Lemma 4.20, these components are paths on at most four vertices. If $B - B_1$ contained a component, B_2 say, with $B_2 = P_4$, then it is easy to verify that $V(B_1) \cup V(B_2)$ form a reduction, contradicting Lemma 4.13. Hence $B - B_1$ does not contain P_4 as a component. If $B - B_1$ contained a component, B_2, with $B_2 = P_3$, notice that $P_3 \leq K_3$ and $C_5 \leq K_3 \boxplus K_2$, so the conditions of 4.7 hold, hence $V(B_1) \cup V(B_2)$ form a reduction, contradicting Lemma 4.13.

Therefore all components of $B - B_1$ are either P_2 or P_1. Hence $G = \overline{C_5 \cup aP_2 + bP_1}$ for some non-negative integers a, b with $a + b = 5$. But then Lemma 4.10 shows that G has an orientation of diameter two, so G is not a counterexample, a contradiction. □

We are now ready to complete the proof of Theorem 4.3.

Proof. Suppose to the contrary that Theorem 4.3 is false. Let G be a minimal counterexample, that is a graph of minimum order and minimum size for which the theorem does not hold. Clearly, $m(G) = n(G) - 5$. It follows from Lemma 4.10 that the theorem holds for K_5 and $K_6 - e$, the graph obtained from K_6 by removing a single edge. Hence $n(G) \geq 7$.

It follows from Lemma 4.20 that every component of B that is not a tree is either a complete graph on at least three vertices, a proper dumbbell, a proper short dumbbell, or a 5-cycle. B contains no component that is a complete graph on three or more vertices by Lemma 4.23, no component that is a proper dumbbell by Lemma 4.24, no component that is a proper short dumbbell by Lemma 4.25, and no component that is a 5-cycle by Lemma 4.26. Hence every component of B is a tree. Let $B_1, B_2, \ldots, B_k$ be the components of B. Since $m(B) = n-5$ we have $\mathrm{ex}(B) = -5$. Since $\mathrm{ex}(B_i) = -1$ for $i = 1, 2, \ldots, k$ we have $-5 = \mathrm{ex}(B) = \sum_{i=1}^{k} \mathrm{ex}(B_i) = -k$, so $k = 5$. Hence B has exactly five components B_1, B_2, B_3, B_4, B_5. By Lemma 4.20 each B_i is a path on at most four vertices. Hence $G = \overline{aP_4 \cup bP_3 \cup cK_2 \cup dK_1}$ for some non-negative integers a, b, c, d with $a + b + c + d = 5$. But each such graph has an orientation of diameter two by Lemma 4.10. Hence G is not a counterexample. This contradiction proves Theorem 4.3. □

Chapter 5

Future Directions

5.1 The Oriented Diameter of Graphs with Given Minimum Degree

The question of whether the upper bound on the oriented diameter of a graph with given order and minimum degree is closer to $3\frac{n}{\delta+1} + O(1)$ than $5\frac{n}{\delta-1} + O(1)$ is still open.

Conjecture 5.1. *Given a sufficiently large graph G with $|G| = n$ and $\delta(G) = \delta$, we can find an orientation of diameter $3\frac{n}{\delta+1} + O(1)$.*

We have also considered the same problem with different parameters added. Namely the girth of a graph and the connected domination number.

Definition 5.2. Given a graph G the girth of G, denoted g, is the smallest cycle in G.

I believe that a similar bound to our original exists for graphs of a certain order given minimum degree and girth as parameters. In particular, I conjecture the following.

Conjecture 5.3. *Given a sufficiently large graph G with $|G| = n$, $\delta(G) = \delta$, and girth g, we can find an orientation of diameter $g\frac{n}{\sum_{i=1}^{\lfloor \frac{g}{2} \rfloor} \delta^i} + O(1)$.*

Definition 5.4. A dominating set in a graph G is a set of vertices for which every vertex is either in the set or connected to a vertex in the set.

Definition 5.5. The domination number of a graph G, denoted γ is the minimum size dominating set.

The domination number and its variants are much studied, as they have important applications in social networks (Basuchowdhuri and Majumder 2014; Borgatti 2006).

Definition 5.6. A connected dominating set in a graph G is a dominating set of vertices that induces a connected graph.

Definition 5.7. The connected domination number is the minimum size connected dominating set in a graph G.

Question 5.8. Can we find an upper bound on the oriented diameter of a graph of a given order and connected domination number?

5.2 The Oriented Diameter of a Complete Graph with Some Edges Removed

We proved the following theorem.

Theorem 5.9. *Given K_n with $n \geq 5$ and any collection of edges E', with $|E'| = n-5$, $\overrightarrow{diam}(K_n \setminus E') \leq 2$.*

The following natural question arises.

Question 5.10. Let $k > 0$ be given. Is there a function $f(k, n)$ for which given any collection of edges E' with $|E'| \leq f(k, n)$ and the property that $K_n \setminus E'$ is bridgeless, that $\overrightarrow{\mathrm{diam}}(K_n \setminus E') \leq k$?

Bibliography

Albert, Réka, Hawoong Jeong, and Albert-László . L. Barabási (1999). “Internet: Diameter of the world-wide web”. In: *Nature* 401.6749, pp. 130–131. ISSN: 0028-0836.

Barabási, Albert-Lásó . L., Réka Albert, and Hawoong Jeong (2000). “Scale-free characteristics of random networks: the topology of the world-wide web”. In: *Physica A: Statistical Mechanics and its Applications* 281.1, pp. 69–77.

Basuchowdhuri, Partha and Subhashis Majumder (2014). “Finding Influential Nodes in Social Networks Using Minimum k-Hop Dominating Set”. In: *International Conference on Applied Algorithms*. Springer, pp. 137–151.

Bau, Sheng and Peter Dankelmann (2015). “Diameter of orientations of graphs with given minimum degree”. In: *European Journal of Combinatorics* 49, pp. 126–133.

Borgatti, Stephen P. (2006). “Identifying sets of key players in a social network”. In: *Computational & Mathematical Organization Theory* 12.1, pp. 21–34.

Bork, Peer et al. (2004). “Protein interaction networks from yeast to human”. In: *Current opinion in structural biology* 14.3, pp. 292–299.

Campan, Alina, Traian Marius Truta, and Matthew Beckerich (2015). “Fast Dominating Set Algorithms for Social Networks.” In: *MAICS*, pp. 55–62.

Chung, Fan RK, Michael R. Garey, and Robert E. Tarjan (1985). “Strongly connected orientations of mixed multigraphs”. In: *Networks* 15.4, pp. 477–484.

Chvátal, Vasek and Carsten Thomassen (1978). “Distances in orientations of graphs”. In: *Journal of Combinatorial Theory, Series B* 24.1, pp. 61–75.

Collins, J. J. and C. C. Chow (June 1998). *Nature*. England. DOI: 10.1038/30835.

Erdős, Paul et al. (1989). “Radius, diameter, and minimum degree”. In: *Journal of Combinatorial Theory, Series B* 47.1, pp. 73–79.

Fomin, Fedor V., MartÃŋn Matamala, and Ivan Rapaport (2004). “Complexity of approximating the oriented diameter of chordal graphs”. In: *Journal of Graph Theory* 45.4, pp. 255–269.

Füredi, Zoltán et al. (1998). “Minimal Oriented Graphs of Diameter 2”. In: *Graphs and Combinatorics* 14, pp. 345–350.

Gutin, Gregory (1994). “Minimizing and maximizing the diameter in orientations of graphs”. In: *Graphs and Combinatorics* 10.2-4, pp. 225–230.

Gutin, Gregory and Anders Yeo (2002). “Orientations of digraphs almost preserving diameter”. In: *Discrete applied mathematics* 121.1, pp. 129–138.

Gutin, Gregory et al. (2002). “Almost minimum diameter orientations of semicomplete multipartite and extended digraphs”. In: *Graphs and Combinatorics* 18.3, pp. 499–506.

Klovdahl, Alden S. et al. (1994). “Social networks and infectious disease: The Colorado Springs study”. In: *Social science & medicine* 38.1, pp. 79–88.

Koh, K. M. and B. P. Tan (1996a). “The diameter of an orientation of a complete multipartite graph”. In: *Discrete Mathematics* 149.1, pp. 131–139.

— (1996b). “The minimum diameter of orientations of complete multipartite graphs”. In: *Graphs and Combinatorics* 12.4, pp. 333–339.

Koh, K. M. and E. G. Tay (2002). “Optimal orientations of graphs and digraphs: a survey”. In: *Graphs and Combinatorics* 18.4, pp. 745–756.

Koh, Khee Meng and K. L. Ng (2005). “The orientation number of two complete graphs with linkages”. In: *Discrete mathematics* 295.1, pp. 91–106.

Kurant, Maciej and Patrick Thiran (Sept. 2006). “Extraction and analysis of traffic and topologies of transportation networks.” In: *Phys Rev E Stat Nonlin Soft Matter Phys* 74.3 Pt 2, p. 036114. ISSN: 1539-3755. DOI: 10.1103/PhysRevE.74.036114.

Kwok, Peter K., Qi Liu, and Douglas B. West (2010). “Oriented diameter of graphs with diameter 3”. In: *Journal of Combinatorial Theory, Series B* 100.3, pp. 265–274.

Laetsch, Martin and Sascha Kurz (2012). “Bounds for the minimum oriented diameter”. In: *Discrete Mathematics & Theoretical Computer Science* 14.

Lakshmi, R. (2011). “Optimal orientation of the tensor product of a small diameter graph and a complete graph”. In: *Australas. J. Combin* 50, pp. 165–169.

Lakshmi, R. and P. Paulraja (2007). “On optimal orientations of tensor product of complete graphs”. In: *Ars Combinatoria* 82, pp. 337–352.

— (2009). “On optimal orientations of tensor product of graphs and circulant graphs”. In: *Ars Combinatoria* 92, pp. 271–288.

Myers, Seth A. et al. (2014). “Information network or social network?: the structure of the twitter follow graph”. In: *Proceedings of the 23rd International Conference on World Wide Web*. ACM, pp. 493–498.

Page, Lawrence et al. (1999). “The PageRank citation ranking: bringing order to the web.” In: DOI: `pagerank`.

Pandurangan, Gopal, Prabhakar Raghavan, and Eli Upfal (2003). “Building low-diameter peer-to-peer networks”. In: *IEEE Journal on selected areas in communications* 21.6, pp. 995–1002.

Perliger, Arie and Ami Pedahzur (2011). “Social network analysis in the study of terrorism and political violence”. In: *PS: Political Science & Politics* 44.01, pp. 45–50.

Plesník, J. (1985). “Remarks on the diameters of orientations of graphs”. In: *Acta Math. Univ. Comenian* 36.3, pp. 225–236.

Robbins, H. E. (May 1939). “A Theorem on Graphs, with an Application to a Problem of Traffic Control”. In: *The American Mathematical Monthly* 46.5, pp. 281–283.

Royer, Elizabeth M. and Chai-Keong . K. Toh (1999). “A review of current routing protocols for ad hoc mobile wireless networks”. In: *IEEE personal communications* 6.2, pp. 46–55.

Sporns, Olaf et al. (2004). “Organization, development and function of complex brain networks”. In: *Trends in cognitive sciences* 8.9, pp. 418–425.

Surmacs, Michel (2017). “Improved bound on the oriented diameter of graphs with given minimum degree”. In: *European Journal of Combinatorics* 59, pp. 187–191.

Ugander, Johan et al. (2011). “The anatomy of the facebook social graph”. In: *arXiv preprint arXiv:1111.4503*.

Van Noort, Vera, Berend Snel, and Martijn A. Huynen (2004). "The yeast coexpression network has a small-world, scale-free architecture and can be explained by a simple model". In: *EMBO reports* 5.3, pp. 280–284.

Šoltés, L'ubomír (1986). "Orientations of graphs minimizing the radius or the diameter". In: *Mathematica Slovaca* 36.3, pp. 289–296.

Woolley-Meza, Olivia et al. (2011). "Complexity in human transportation networks: a comparative analysis of worldwide air transportation and global cargo-ship movements". In: *The European Physical Journal B* 84.4, pp. 589–600.

Appendix A

Sage Code

A.1 Introduction

In order to perform the calculations needed for the low cases in Case (13) of Lemma 4.10 and to find the diameter two orientation of K_5 I used SageMath 8.0. SageMath (Sage) is meant to be an open source replacement for traditional Mathematical Programming languages like Mathematica or Maple. It has a very robust set of graph theory functions and operators already linked in. I will notate the code below in comments so if someone else wanted to run it, they could. This code can also be found on the research page of my website: `http://math.garnercochran.com/research.html`.

I needed to use the `_strong_orientations_of_a_mixed_graph` function within the orientations library in Sage. Using the package `_strong_orientations_of_a_mixed_graph` gave me access to `strong_orientations_iterator()`, which is an iterator that starts with an undirected graph and can iterates through each of the possible strong orientations of that graph without having to enumerate a full list of them. This is advantageous, because it saves memory.

A.2 Code

```
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%This imports the strong_orientations_iterator() function that I need
```

```
%%to use to find diameter 2 orientations of these graphs.
%%
%%The package time allows me to see how long a computation takes to
%%complete.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

>>  from sage.graphs.orientations
import _strong_orientations_of_a_mixed_graph

>>  import time

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%This function takes in the generated iterator for strong
%%orientations, then runs through all orientations in order to find
%%one of diameter 2.
%%
%%It outputs a directed graph of diameter two if one exists. Note that
%%it may throw an error if one doesn't exist.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

>>  def giveMeDiamTwo(graphiterator):
        for graph in graphiterator:
            if graph.diameter()==2:
                return graph

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%This function takes two inputs, the order of the graph and the edges
```

```
%%of the graph to be removed. It creates a graph with edges
%%{0,1,2,\dots, n-1}, and deletededgesfromgraph should be a list of
%%lists where each inside list has length 2 and represents the pairs
%%that are missing from the graph.
%%It will output an orientation of diameter 2 if one exists.
%%Example: checkThisGraph(6,[[0,1],[2,3],[4,5]]) will return an
%%an orientation of the complete graph on 6 vertices minus a matching.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

>>  def checkThisGraph(n,deletededgesfromgraph):
        start_time = time.clock()
        BigGraph=graphs.CompleteGraph(n)
        for edges in deletededgesfromgraph:
            BigGraph.delete_edge(edges[0],edges[1])
        DiamTwoGraph=giveMeDiamTwo(
        BigGraph.strong_orientations_iterator())
        print time.clock() - start_time, "seconds"
        return DiamTwoGraph

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%This function checks all graphs of order n that we wished to find a
%%diameter two orientation. It will iterate through all the possible
%%collections of n-5 edges as unions of at most 5 paths of length 4
%%as blue graphs.
%%
%%It returns a list of the orientations these graphs.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
```

```
>>  def checkAllGraphs(n):
        PartsList=Partitions(n, length=5,max_part=4).list()
        diamTwoGraphs=[]
        for BlueGraph in PartsList:
            deletededges=[]
            pointer=0
            for Paths in BlueGraph:
                if Paths==1:
                    pointer+=1
                else:
                    for i in range(Paths-1):
                        deletededges.append([pointer,pointer+1])
                        pointer+=1
                    pointer+=1
            diamTwoGraphs.append(checkThisGraph(n,deletededges))
        return diamTwoGraphs

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%This checks all the possible graphs of order n that are missing
%%edges n-5 edges that make up a disjoint union of paths for 5<=n<10.
%%It makes a list called listOfOrientations that is a list of lists,
%%where listOfOrientations[i] is a list of the orientations of the
%%graphs that have n-5 edges as unions of at most 5 paths of length 4
%%as blue graphs.
%%
%%Directed adjacency matrices for these graphs can be found in
```

```
%%Appendix B.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

>>  listOfOrientations=[]
    for i in range(5,10):
        listOfOrientations.append(checkAllComplements(i))

%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%
%%The next two lines return orientations of diameter 2 of the complete
%%graphs on 6 and 8 vertices with a matching missing.
%%
%%Directed adjacency matrices for these graphs can be found in
%%Appendix B.
%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%%

>>  checkThisGraph(6,[[0,1],[2,3],[4,5]])

>>  checkThisGraph(8,[[0,1],[2,3],[4,5],[6,7]])
```

A.3 Outputs

In this section I will give the outputs that my code gives in the form of adjacency matrices. In particular, I ran the following code in order to print the adjacency matrices in LaTeXand import them into figures here. The adjacency matrices are defined where if on the vertex set $\{0,\ldots,n-1\}$, an edge is oriented $\overrightarrow{ij}$, then the entry $M_{i+1,j+1} = 1$ where $M_{i+1,j+1}$ represents the element in the $i+1$st row and $j+1$st column. For any other element $M_{k,\ell}$, we let $M_{k,\ell} = 0$.

The first outputs we need to consider are diameter two orientations of K_5, $K_6 - M$

and $K_8 - M$.

In order to do this we consider the following pairs of code and the matrix output.

```
>>  latex(checkThisGraph(5,[]).adjacency_matrix())
```

$$\left(\begin{array}{rrrrr}
0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 \\
1 & 0 & 0 & 0 & 0
\end{array}\right)$$

Figure A.1 The adjacency matrix for an orientation of diameter 2 of K_5.

```
>>  latex(checkThisGraph(6,[[0,1],[2,3],[4,5]]).adjacency_matrix())
```

$$\left(\begin{array}{rrrrrr}
0 & 0 & 1 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 1 & 0 \\
0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 1 & 1 \\
1 & 0 & 1 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0
\end{array}\right)$$

Figure A.2 The adjacency matrix for an orientation of diameter 2 of $K_6 - M$.

```
>>  latex(checkThisGraph(8,[[0,1],[2,3],[4,5],[6,7]])
.adjacency_matrix())
```

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.3 The adjacency matrix for an orientation of diameter 2 of $K_8 - M$.

Below you will find the outputs from the following, given the functions and variables from above in Section A.2. Note that in Lemma 4.10 we had reduced to the case where each path had at most 4 vertices.When we reach the case where $|G| = 9$, we need to remove 4 edges as a disjoint union of paths. We can not have all these edges be in one path. That would give a path with 5 vertices, so for $|G| = 9$ we only need to consider the cases where (if each union of paths is pairwise vertex disjoint) G is $K_9 - (P_4 \cup P_2)$, $K_9 - (P_3 \cup P_3)$, $K_9 - (P_3 \cup P_2 \cup P_2)$, or $K_9 - (P_2 \cup P_2 \cup P_2 \cup P_2)$.

```
for i in range(len(listOfOrientations)):
    for j in range(len(listOfOrientations[i])):
        print latex(listOfOrientations[i][j].adjacency_matrix())
```

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.4 The adjacency matrix for an orientation of diameter 2 of K_5.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.5 The adjacency matrix for an orientation of diameter 2 of $K_6 - P_2$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.6 The adjacency matrix for an orientation of diameter 2 of $K_7 - P_3$.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.7 The adjacency matrix for an orientation of diameter 2 of K_7 minus 2 edges.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.8 The adjacency matrix for an orientation of diameter 2 of $K_8 - P_4$.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.9 The adjacency matrix for an orientation of diameter 2 of $K_8 - (P_3 \cup P_2)$, where P_3 and P_2 are vertex disjoint.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.10 The adjacency matrix for an orientation of diameter 2 of $K_8 - (P_2 \cup P_2 \cup P_2)$, where each pair of P_2 are vertex disjoint.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.11 The adjacency matrix for an orientation of diameter 2 of $K_9 - (P_4 \cup P_2)$, where each pair of paths are vertex disjoint.

$$\begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.12 The adjacency matrix for an orientation of diameter 2 of $K_9 - (P_3 \cup P_3)$, where each pair of paths are vertex disjoint.

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.13 The adjacency matrix for an orientation of diameter 2 of $K_9 - (P_3 \cup P_2 \cup P_2)$, where each pair of paths are vertex disjoint.

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure A.14 The adjacency matrix for an orientation of diameter 2 of $K_9 - (P_2 \cup P_2 \cup P_2 \cup P_2)$, where each pair of paths are vertex disjoint.