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# Classical and Quantum Kac's Chaos

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CLASSICAL AND QUANTUM KAC'S CHAOS

by

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## ABSTRACT

In 1956 Kac studied the Boltzmann equation, an integro-differential equation which describes the density function of the distribution of the velocities of the molecules of dilute monoatomic gases under the assumption that the energy is only transferred via collisions between the molecules. In an attempt at a solution to the Boltzmann equation, Kac introduced a property of the density function that he called the "Boltzmann property" which describes the behavior of the density function at a given fixed time as the number of particles tends to infinity. The Boltzmann property has been studied extensively since then, and has been abstracted to a property simply called chaos, or Kac's chaos in classical mechanics. Kac's chaos has attracted the attention of mathematicians and physicists such as Carlen, Grunbaum, McKean, Mischler, Mouhot, and Sznitman.

On the other hand, in ergodic theory, chaos usually refers to the mixing properties of a dynamical system as time tends to infinity. In this thesis, a relationship is derived between classical Kac's chaos and the notion of mixing. This relationship provides examples of Kac's chaos built from dynamical systems with certain mixing-type properties. In order to prove this relationship, a famous result of Sznitman is used, which states that the classical form of Kac's chaos is equivalent to a certain convergence of empirical measures.

Further, the quantum version of Kac's chaos is studied in this thesis. This form of chaos was implicitly introduced by Spohn and explicitly formulated by Gottlieb. The quantum analogue of the result of Sznitman which gives the equivalence of Kac's chaos to 2-chaoticity and to a certain convergence of empirical measures is proven. Finally,

a simple, different proof of a result of Spohn which states that chaos propagates with respect to certain Hamiltonians that define the evolution of the mean field limit for interacting quantum systems is proven.

# TABLE OF CONTENTS

ACKNOWLEDGMENTS . . . . .	ii
ABSTRACT . . . . .	iii
CHAPTER 1 INTRODUCTION AND MATHEMATICAL BACKGROUND . . . . .	1
1.1 Origins of Kac’s Chaos . . . . .	1
1.2 Mixing as a Form of Chaos . . . . .	6
1.3 The Partial Trace and the Quantum Version of Kac’s Chaos . . . . .	8
CHAPTER 2 A CONNECTION BETWEEN MIXING AND CLASSICAL KAC’S CHAOS . . . . .	18
2.1 Mixing Dynamical Systems . . . . .	18
2.2 An Example of Classical Kacs Chaos . . . . .	26
CHAPTER 3 QUANTUM KAC’S CHAOS . . . . .	33
3.1 Definitions and Examples . . . . .	33
3.2 A Quantum Version of Sznitman’s Result . . . . .	40
CHAPTER 4 PROPAGATION OF QUANTUM KAC’S CHAOS . . . . .	45
4.1 Definitions and Set-Up . . . . .	45
4.2 Two Hierarchies of Differential Equations . . . . .	47
4.3 Propagation of Quantum Kac’s Chaos with respect to $(H_N)_{N=1}^\infty$ . . . . .	55

BIBLIOGRAPHY . . . . . 61

# CHAPTER 1

## INTRODUCTION AND MATHEMATICAL BACKGROUND

### 1.1 ORIGINS OF KAC'S CHAOS

In this chapter, we present an overview and motivation for the results presented in this thesis. We also give mathematical background which will be used throughout this thesis. In this section the historical significance of Kac's chaos is presented. Basic definitions related to Kac's chaos in its classical form also appear. In 1956, Kac [21] was interested in solving the non-linear integro-differential equation known as the Boltzmann equation [21, Equation (1.1)] originally devised by Boltzmann [4]. The solution to the Boltzmann equation is a family  $(f^{(N)})_{N=1}^{\infty}$  of probability density functions, where  $f^{(N)}$  describes the velocities and positions of  $N$  dilute gas molecules moving in  $\mathbb{R}^3$ , interacting via elastic binary collisions. The non-linearity of the Boltzmann equation provided difficulty in obtaining the existence of its solution.

If the gas is restricted to a container, there are no external forces, and the number  $N$  of molecules is assumed to be equidistributed, then  $f^{(N)}$  depends on the velocities of the  $N$  gas molecules and time, thus having  $3N+1$  real variables. Then the Boltzmann equation takes a simplified reduced form [21, Equation (1.3)] which is still a non-linear integro-differential equation. Further assuming that the kinetic energy of the system remains constant proportional to  $N$ , the  $3N$  variables representing velocity lie on a sphere of radius  $\sqrt{N}$  in  $\mathbb{R}^{3N}$ , and in order to obtain a further simplified version of the Boltzmann equation, one can replace the  $3N$  real variables by one real variable  $x$ . This further reduces the Boltzmann equation to the reduced Boltzmann equation



[21, Equation (3.5)]:

$$\frac{\partial f(x, t)}{\partial t} = \frac{\nu}{2\pi} \int_{-\infty}^{\infty} \int_0^{2\pi} \{f(x \cos \theta + y \sin \theta, t) f(-x \sin \theta + y \cos \theta, t) - f(x, t) f(y, t)\} d\theta dy. \quad (1.1)$$

Kac further introduced a linear differential equation which he called the “Master Equation” [21, Equation (2.6)]. Kac’s master equation depends on a positive integer  $N$  and its solution has  $N + 1$  real variables  $(x_1, \dots, x_N, t)$ . The  $N$  real variables  $(x_1, \dots, x_N)$  belong on the “Kac’s sphere”  $\mathbb{K}^N$  which stands for the sphere in  $\mathbb{R}^N$  centered at the origin whose radius is equal to  $\sqrt{N}$ , (i.e.  $\mathbb{K}^N = \{(x_1, \dots, x_N) \in \mathbb{R}^N : x_1^2 + \dots + x_N^2 = N\}$ ), while the extra variable  $t$  represents time. Kac sought solutions  $\phi^{(N)}$  to his master equation that are symmetric in the variables  $(x_1, \dots, x_N)$  for every  $t \geq 0$ , i.e.

$$\phi^{(N)}(x_1 \dots, x_N, t) = \phi^{(N)}(x_{\pi(1)}, \dots, x_{\pi(N)}, t), \text{ for every permutation } \pi \text{ of } \{1, \dots, N\}. \quad (1.2)$$

Moreover for any set  $E$ , a function  $g : E^N \rightarrow \mathbb{C}$  is called symmetric if

$$g^\pi(x_1, x_2, \dots, x_N) = g(x_1, x_2, \dots, x_N) \quad (1.3)$$

for all permutations  $\pi$  of  $\{1, \dots, N\}$  and for all  $(x_1, x_2, \dots, x_N) \in E^N$ , where for each permutation  $\pi$  of  $\{1, \dots, N\}$  we define  $g^\pi : E^N \rightarrow \mathbb{C}$  by

$$g^\pi(x_1, x_2, \dots, x_N) := g(x_{\pi(1)}, x_{\pi(2)}, \dots, x_{\pi(N)}). \quad (1.4)$$

We also assume that the solution  $\phi^{(N)}$  to Kac’s master equation is a *density function on  $\mathbb{K}^N$*  i.e. we assume that  $\phi^{(N)} d\sigma^N$  is a probability measure on the Borel subsets of  $\mathbb{K}^N$  where  $\sigma^N$  denotes the normalized uniform measure on  $\mathbb{K}^N$ . For each  $1 \leq m < N$ , we can define a probability measure  $(\phi^{(N)} d\sigma^N)_m$  on the Borel subsets of  $\mathbb{R}^m$  by

$$(\phi^{(N)} d\sigma^N)_m(A) = \int_{P_m^{-1}(A)} \phi^{(N)} d\sigma^N$$

where  $P_m : \mathbb{K}^N \rightarrow \mathbb{R}^m$  is the canonical projection into the first  $m$  copies of  $\mathbb{R}$ . It is clear that  $(\phi^{(N)} d\sigma^N)_m$  is absolutely continuous with respect to the  $m$ -dimensional Lebesgue measure  $\lambda^m$  on  $\mathbb{R}^m$ , and thus by the Radon-Nikodym Theorem there exists a function  $\phi_m^{(N)} \in L^1(\mathbb{R}^m)$ , called the  $m^{\text{th}}$  marginal function of  $\phi^{(N)}$ , such that

$$\int_A \phi_m^{(N)} d\lambda^m = (\phi^{(N)} d\sigma^N)_m(A) = \int_{P_m^{-1}(A)} \phi^{(N)} d\sigma^N \quad (1.5)$$

for every Borel subset  $A$  of  $\mathbb{R}^m$ . Hence  $\phi_m^{(N)} d\lambda^m$  is a Borel probability measure on  $\mathbb{R}^m$  for every  $1 \leq m < N$ .

Kac [21] noticed that if  $\lim_{N \rightarrow \infty} \phi_1^{(N)}(x, 0)$  exists weakly in  $L^1(\mathbb{R})$  and  $\lim_{N \rightarrow \infty} \phi_2^{(N)}(x, y, 0)$  exists weakly in  $L^1(\mathbb{R}^2)$ , and

$$\lim_{N \rightarrow \infty} \phi_2^{(N)}(x, y, 0) = \lim_{N \rightarrow \infty} \phi_1^{(N)}(x, 0) \lim_{N \rightarrow \infty} \phi_1^{(N)}(y, 0), \quad (1.6)$$

then the same limits exist at any later time  $t$ , and satisfy

$$\lim_{N \rightarrow \infty} \phi_2^{(N)}(x, y, t) = \lim_{N \rightarrow \infty} \phi_1^{(N)}(x, t) \lim_{N \rightarrow \infty} \phi_1^{(N)}(y, t). \quad (1.7)$$

Then equation (1.7) implies that the function  $f$  defined by

$$f(x, t) := \lim_{N \rightarrow \infty} \phi_1^{(N)}(x, t)$$

satisfies equation (1.1). Hence Kac proved the existence of the solution to the reduced Boltzmann equation.

Whenever equation (1.6) implies equation (1.7) for all times  $t > 0$ , we say that the ‘‘Boltzmann property propagates in time’’. Hence Kac [21] proved that the Boltzmann property propagates in time for his ‘‘Master Equation’’.

Equation (1.7) motivated Kac to introduce the following definition: For all  $N \in \mathbb{N}$  let  $\phi^{(N)}$  be a symmetric, (as in (1.2)), probability density function defined on  $\mathbb{K}^N$  (i.e.  $\phi^{(N)} d\sigma^N$  is a Borel probability measure on  $\mathbb{K}^N$ ). For  $1 \leq k < N$  let  $\phi_k^{(N)}$  denote the  $k^{\text{th}}$  marginal of  $\phi^{(N)}$ . The sequence  $(\phi^{(N)})$  is said to have the ‘‘Boltzmann property’’ if

for all  $k \in \mathbb{N}$  the limit  $\lim_{N \rightarrow \infty} \phi_k^{(N)}$  exists weakly in  $L^1(\mathbb{R}^k)$ , and moreover, if  $\phi_1$  denotes the weak  $L^1(\mathbb{R})$  limit of  $\phi_1^{(N)}$ , then for all  $k \in \mathbb{N}$  and for almost all  $x_1, \dots, x_k \in \mathbb{R}$ :

$$\lim_{N \rightarrow \infty} \phi_k^{(N)}(x_1, \dots, x_k) = \prod_{i=1}^k \phi_1(x_i). \quad (1.8)$$

Many authors including McKean [26], Johnson [20], Tanaka [34], Ueno [35], Grünbaum [17], Murata [29], Graham and Méléard [16], Sznitman [32], [33], Mischler [27], Carlen, Carvalho and Loss [9], Mischler and Mouhot [28] have abstracted the idea of the “Boltzmann property” to a sequence of probability measures on a topological space. Instead of having the “Boltzmann property”, the sequence of probability measures nowadays are said to be chaotic. In order to discuss chaotic sequences of probability measures, these authors first define the notion of a symmetric probability measure.

**Definition 1.1.1.** *Let  $E$  be a topological space,  $N$  be a positive integer,  $\mu_N$  be a probability measure on the Borel subsets of  $E^N$ . Then  $\mu_N$  is called **symmetric** if for any  $N$ -many continuous real-valued bounded functions on  $E$ ,  $g_1, g_2, \dots, g_N$ ,*

$$\int_{E^N} g_1(x_1)g_2(x_2) \cdots g_N(x_N)d\mu_N = \int_{E^N} g_1(x_{\pi(1)})g_2(x_{\pi(2)}) \cdots g_N(x_{\pi(N)})d\mu_N$$

*for any permutation  $\pi$  of  $\{1, \dots, N\}$ .*

Note that if  $\phi^{(N)}$  is a symmetric (in the sense of (1.2)) density function on the Kac’s sphere  $\mathbb{K}^N$ ,  $\sigma^N$  denotes, as above, the uniform Borel probability measure on  $\mathbb{K}^N$ , and  $\widetilde{\sigma}^N$  denotes the extension of  $\sigma^N$  to the Borel subsets of  $\mathbb{R}^N$  such that the support of  $\widetilde{\sigma}^N$  is equal to  $\mathbb{K}^N$  (this is possible since  $\mathbb{K}^N$  is a Borel subset of  $\mathbb{R}^N$ ), then  $\phi^{(N)}d\widetilde{\sigma}^N$  is a symmetric probability measure on  $\mathbb{R}^N$ , (in the sense of Definition 1.1.1). Thus Definition 1.1.1 gives a more general notion of symmetry than that of Equation (1.2) that was considered by Kac.

Now we define the Boltzmann property, or Kac’s chaos, but following the above mentioned literature, we use a more descriptive terminology:

**Definition 1.1.2.** Let  $E$  be a topological space,  $\mu$  be a Borel probability measure on  $E$ , and for every  $N \in \mathbb{N}$  let  $\mu_N$  be a symmetric Borel probability measure on  $E^N$ . For  $k \in \mathbb{N}$ , we say that  $(\mu_N)_{N=1}^\infty$  is  $k - \mu$ -**chaotic** if for every choice  $g_1, g_2, \dots, g_k$  of continuous bounded real-valued functions on  $E$ , we have

$$\lim_{N \rightarrow \infty} \int_{E^N} g_1(x_1)g_2(x_2) \cdots g_k(x_k) d\mu_N = \prod_{j=1}^k \int_E g_j(x) d\mu(x).$$

We say that  $(\mu_N)_{N=1}^\infty$  is  $\mu$ -**chaotic** if  $(\mu_N)_{N=1}^\infty$  is  $k - \mu$ -chaotic for all  $k \geq 1$ .

Note that Definition 1.1.2 of chaotic sequences of measures defined on products  $E^N$  of a topological space  $E$  extends the notion of the Boltzmann property which was defined by Kac for a sequence of symmetric density functions defined on the Kac's sphere  $\mathbb{K}^N$ . Indeed, it is easy to see that to every sequence  $(\phi^{(N)})_N$  of symmetric (in the sense of (1.2)) density functions on  $\mathbb{K}^N$ , which satisfy the Boltzmann property (in the sense of (1.8)), corresponds a sequence  $(d\mu_N)_N$  of Borel measures on  $\mathbb{R}^N$ , which is chaotic in the sense of Definition 1.1.2. Indeed, if  $\phi^{(N)} : \mathbb{K}^N \rightarrow [0, \infty)$  is a symmetric density function for every  $N \in \mathbb{N}$ , then the sequence  $d\mu_N = \phi^{(N)} d\widetilde{\sigma}^N$  which was considered in the paragraph above the Definition 1.1.2 is  $\nu$ -chaotic where  $d\nu = \phi_1 dx$  and  $dx$  is the Lebesgue measure on  $\mathbb{R}$ . Hence Definition 1.1.2 gives a more general notion of chaoticity than the Boltzmann property defined by Kac.

Two other related forms of chaoticity that exist in literature are the chaoticity in the sense of Boltzmann entropy and the chaoticity in the sense of Fisher information. These two notions were introduced by Carlen, Carvalho, Le Roux, Loss, and Villani [8]. Hauray and Mischler has shown that chaoticity in the sense of Fisher information implies chaoticity in the sense of Boltzmann entropy, which in turn implies Kac's chaoticity [18, Theorem 1.4]. Carrapatoso [10] has extended the results of [18] to probability measures with support on the Boltzmann spheres.

## 1.2 MIXING AS A FORM OF CHAOS

In ergodic theory, chaos usually refers to the mixing properties of dynamical systems as time tends to infinity. In this section we recall several definitions from ergodic theory which will be useful in Chapter 2. In particular, we provide an introduction to dynamical systems, mixing, and stationary limits.

The notion of “chaos” in ergodic theory, has its origins in the works of Poincare at the end of the 19th century. Its meaning is dynamical randomness of physical quantities that evolve with time. The set up for measure theoretic dynamical systems consists of a probability space  $(\Omega, \Sigma, \mu)$  which is called the *phase space*, and either a measurable map  $S : \Omega \rightarrow \Omega$  (in the case of discrete time dynamical systems), or a family of measurable maps  $S_t : \Omega \rightarrow \Omega$  for  $t \geq 0$  (in the case of continuous time dynamical systems) satisfying  $S_t \circ S_s = S_{t+s}$  for all  $s, t \in [0, \infty)$  (semigroup property). Such tuple  $(\Omega, \Sigma, \mu, S)$  or  $(\Omega, \Sigma, \mu, (S_t)_{t \geq 0})$  is called a *measure theoretic dynamical system*, or simply a *dynamical system*. In the case of discrete time dynamical systems, the composition of the map  $S$  with itself  $n$  many times, (where  $n$  is a non-negative integer), is usually denoted as  $S^n$ , (a notation which resembles powers of  $S$ ), and plays the role of  $S_n$  that appears in the above definition of continuous time dynamical systems. For simplicity we only consider discrete time dynamical systems and we keep in mind that the “exponent”  $n$  that appears in the compositions  $S^n$  represents time. The maps  $S^n$  can be thought to act on  $\Omega$ , (by the formula  $\Omega \ni \omega \mapsto S^n \omega$ ), or on real valued functions on  $\Omega$ , (where the action of  $S^n$  on such function  $f$  produces the real valued function  $\Omega \ni \omega \mapsto f(S^n \omega)$ ), or on probability measures on  $\Sigma$  (where the action of  $S^n$  on such measure  $\nu$  produces the measure  $\Sigma \ni A \mapsto \nu(S^{-n} A)$ ). Thus one can study orbits of points of  $\Omega$ , (i.e. the sequence of points  $(S^n(\omega))_{n \in \mathbb{N} \cup \{0\}}$ ), or orbits of real valued functions on  $\Omega$ , or orbits of probability measures on  $\Sigma$ . The property of chaos in ergodic theory refers to the randomness of these orbits and it is explicitly quantified and studied in the books of ergodic theory. An excellent book

on this subject is the book of Arnold and Avez, [1], or the short survey of Sinai [30]. Two quantifications of the notions of chaos in the measure theoretic ergodic theory are the notions of the “stationary limit” and “mixing”:

**Definition 1.2.1.** (i) We say that a dynamical system  $(\Omega, \Sigma, \mu, S)$  is asymptotically stationary with stationary limit  $\nu$  if  $\nu(A) = \lim_{k \rightarrow \infty} \mu(S^{-k}A)$  for each  $A \in \Sigma$ .

(ii) We say that a dynamical system  $(\Omega, \Sigma, \mu, S)$  is mixing if

$$\lim_{k \rightarrow \infty} \left| \mu(S^{-k}A \cap B) - \mu(S^{-k}A)\mu(B) \right| = 0 \quad \text{for all } A, B \in \Sigma.$$

Note that if the members of a sequence of probability measures are defined on a common  $\sigma$ -algebra  $\Sigma$  and converge at every fixed element of  $\Sigma$  then the limit is also a probability measure [5, Theorem 4.6.3(i)]. Thus the limit  $\nu$  that is obtained in Definition 1.2.1(i) is a probability measure, since obviously, for every  $k \in \mathbb{N}$ , the map  $\Sigma \ni A \mapsto \mu(S^{-k}A)$  defines a probability measure on  $\Sigma$ . Obviously, if a dynamical system  $(\Omega, \Sigma, \mu, S)$  is asymptotically stationary with stationary limit  $\nu$  then  $\nu$  is *invariant under  $S$* , (or equivalently,  *$S$  is  $\nu$ -measure preserving*), i.e.

$$\nu(S^{-1}(A)) = \nu(A) \quad \text{for all } A \in \Sigma. \tag{1.9}$$

It is also obvious that if the dynamical system  $(\Omega, \Sigma, \mu, S)$  is asymptotically stationary with stationary limit  $\nu$  then it is mixing if and only if

$$\lim_{k \rightarrow \infty} \left| \mu(S^{-k}A \cap B) - \nu(A)\mu(B) \right| = 0 \quad \text{for all } A, B \in \Sigma.$$

In particular, if  $(\Omega, \Sigma, \mu, S)$  is a dynamical system and the map  $S$  is  $\mu$ -measure preserving, then  $(\Omega, \Sigma, \mu, S)$  is mixing if and only if

$$\lim_{k \rightarrow \infty} \left| \mu(S^{-k}A \cap B) - \mu(A)\mu(B) \right| = 0 \quad \text{for all } A, B \in \Sigma.$$

In Chapter 2, we will examine a connection between the ergodic notion of mixing and the classical form of Kac’s chaos introduced in Definition 1.1.2. In particular,

we will show that dynamical systems with certain mixing-type properties can be used to construct sequences of chaotic probability measures. In order to obtain this result, we will rely on a famous result of Sznitman [33, Proposition 2.2] giving various formulations of Kac's chaos.

### 1.3 THE PARTIAL TRACE AND THE QUANTUM VERSION OF KAC'S CHAOS

In this section, we introduce the notions of the partial trace of a trace class operator on a Hilbert space and quantum Kac's chaos. We also give some of the historical background for the quantum version of Kac's chaos. Boltzmann's equation and Equation (1.1) describe evolutions in models of classical mechanics. Corresponding quantum mechanical models are described in [31, V. Quantum Mechanical Models]. In such models, density functions are replaced by density operators, which via the trace duality define states on algebras of bounded linear operators acting on Hilbert spaces. The following is the definition of a density operator.

**Definition 1.3.1.** *Let  $\mathbb{H}$  be a Hilbert space. A trace class operator  $\rho$  on  $\mathbb{H}$  is a density operator if and only if  $\rho$  is positive and  $\text{tr}(\rho) = 1$ .*

The corresponding notion to the marginals of density functions defined in line (1.5) is the partial trace of a density operator. We will recall the definition of the partial trace using the formulation outlined in Attal [2]. In the following discussion, for a Hilbert space  $\mathbb{H}$ , the norm on  $\mathbb{H}$  will be denoted by  $\|\cdot\|_{\mathbb{H}}$  and the inner product on  $\mathbb{H}$  will be denoted by  $\langle \cdot, \cdot \rangle_{\mathbb{H}}$ . The set of bounded operators on  $\mathbb{H}$  will be denoted by  $\mathcal{B}(\mathbb{H})$ , and the identity operator on  $\mathbb{H}$  will be denoted by  $1_{\mathbb{H}}$ . The trace class norm will be denoted by  $\|\cdot\|_1$ . We will mean by  $(x_j)_{j \in J} \subset \mathbb{H}$  a sequence of elements  $x_j \in \mathbb{H}$  with index set  $J$ .

We first need a lemma about the trace class norm. This lemma will be used throughout the construction of the partial trace. Attal [2] gives the following con-

struction of the partial trace for separable Hilbert spaces. We will follow the method and proofs of Attal, but extend this construction to arbitrary Hilbert spaces.

**Lemma 1.3.2.** [2, Theorem 2.10] *A bounded operator  $T$  on a Hilbert space  $\mathbb{H}$  is trace class if and only if*

$$\sum_{j \in J} |\langle x_j, T y_j \rangle_{\mathbb{H}}| < \infty$$

for all orthonormal families  $(x_j)_{j \in J}$  and  $(y_j)_{j \in J}$  in  $\mathbb{H}$ .

Moreover there exists orthonormal sequences  $(x_n)_{n \in \mathbb{N}}$  and  $(y_n)_{n \in \mathbb{N}}$  in  $\mathbb{H}$  such that

$$\sum_{n \in \mathbb{N}} |\langle x_n, T y_n \rangle_{\mathbb{H}}| = \|T\|_1.$$

*Proof.* We will first assume that  $T$  is a trace class operator on  $\mathbb{H}$ . Then  $T$  can be written as

$$T = \sum_{n \in \mathbb{N}} \lambda_n |u_n\rangle \langle v_n|$$

for some orthonormal families  $(u_n)_{n \in \mathbb{N}}$  and  $(v_n)_{n \in \mathbb{N}}$  and some positive summable sequence  $(\lambda_n)_{n \in \mathbb{N}}$  such that  $\sum_{n \in \mathbb{N}} \lambda_n = \|T\|_1$ . Thus for any orthonormal families  $(x_j)_{j \in J}$  and  $(y_j)_{j \in J}$  in  $\mathbb{H}$ , we have

$$\begin{aligned} \sum_{j \in J} |\langle x_j, T y_j \rangle_{\mathbb{H}}| &= \sum_{j \in J} \left| \left\langle x_j, \sum_{n \in \mathbb{N}} \lambda_n |u_n\rangle \langle v_n| y_j \right\rangle_{\mathbb{H}} \right| \\ &\leq \sum_{j \in J} \sum_{n \in \mathbb{N}} \lambda_n |\langle x_j, u_n \rangle_{\mathbb{H}}| |\langle v_n, y_j \rangle_{\mathbb{H}}| = \sum_{n \in \mathbb{N}} \sum_{j \in J} \lambda_n |\langle x_j, u_n \rangle_{\mathbb{H}}| |\langle v_n, y_j \rangle_{\mathbb{H}}| \\ &\leq \sum_{n \in \mathbb{N}} \lambda_n \left( \sum_{j \in J} |\langle x_j, u_n \rangle_{\mathbb{H}}|^2 \right)^{1/2} \left( \sum_{j \in J} |\langle v_n, y_j \rangle_{\mathbb{H}}|^2 \right)^{1/2} \leq \sum_{n \in \mathbb{N}} \lambda_n \|u_n\|_{\mathbb{H}} \|v_n\|_{\mathbb{H}} \\ &= \sum_{n \in \mathbb{N}} \lambda_n < \infty. \end{aligned}$$

Note that for the orthonormal families  $(u_n)_{n \in \mathbb{N}}$  and  $(v_n)_{n \in \mathbb{N}}$  in  $\mathbb{H}$  we have

$$\sum_{n \in \mathbb{N}} |\langle u_n, T v_n \rangle_{\mathbb{H}}| = \sum_{n \in \mathbb{N}} \left| \sum_{m \in \mathbb{N}} \lambda_m \langle u_n, u_m \rangle_{\mathbb{H}} \langle v_m, v_n \rangle_{\mathbb{H}} \right| = \sum_{m \in \mathbb{N}} \lambda_m = \|T\|_1.$$



Assume now that  $T$  is a bounded operator on  $\mathbb{H}$  such that

$$\sum_{j \in J} |\langle x_j, Ty_j \rangle_{\mathbb{H}}| < \infty$$

for all orthonormal families  $(x_j)_{j \in J}$  and  $(y_j)_{j \in J}$  in  $\mathbb{H}$ . Let the polar decomposition of  $T$  be given by  $T = U|T|$  for some partial isometry  $U$ . Choose an orthonormal family  $(y_j)_{j \in J}$  in  $\overline{\text{Ran}|T|}$  and define the family  $(x_j)_{j \in J}$  by  $x_j := Uy_j$  for each  $j \in J$ . Since  $U$  is isometric on  $\overline{\text{Ran}|T|}$ , we have that  $U^*x_j = y_j$  for all  $j \in J$ . Thus

$$\sum_{j \in J} |\langle x_j, Ty_j \rangle_{\mathbb{H}}| = \sum_{j \in J} |\langle x_j, U|T|y_j \rangle_{\mathbb{H}}| = \sum_{j \in J} |\langle y_j, |T|y_j \rangle_{\mathbb{H}}| = \sum_{j \in J} \langle y_j, |T|y_j \rangle_{\mathbb{H}}. \quad (1.10)$$

By hypothesis, the left-hand side of line (1.10) is finite, so  $\sum_{j \in J} \langle y_j, |T|y_j \rangle_{\mathbb{H}}$  is finite. Let  $(\widetilde{y}_j)_{j \in J'}$  be an orthonormal basis extending  $(y_j)_{j \in J}$  by including orthonormal vectors in  $\overline{\text{Ran}|T|}^{\perp} = \text{Ker}|T|$ . Then

$$\text{tr}|T| = \sum_{j \in J'} \langle \widetilde{y}_j, |T|\widetilde{y}_j \rangle_{\mathbb{H}} = \sum_{j \in J} \langle y_j, |T|y_j \rangle_{\mathbb{H}} < \infty,$$

i.e.  $T$  is trace class on  $\mathbb{H}$ . □

With this lemma, we can introduce the operators needed to define the partial trace. We then present a lemma and theorem detailing the construction.

**Definition 1.3.3.** [2, Definition 2.26] *Let  $\mathbb{H}$  and  $\mathbb{K}$  be Hilbert spaces. For any  $k \in \mathbb{K}$  consider the operator  $|k\rangle_{\mathbb{K}} : \mathbb{H} \rightarrow \mathbb{H} \otimes \mathbb{K}$  defined by*

$$|k\rangle_{\mathbb{K}}h = h \otimes k.$$

*It is clear that  $|k\rangle_{\mathbb{K}}$  is a bounded operator from  $\mathbb{H}$  to  $\mathbb{H} \otimes \mathbb{K}$  of norm  $\|k\|_{\mathbb{K}}$ . Its adjoint is the operator  ${}_{\mathbb{K}}\langle k| : \mathbb{H} \otimes \mathbb{K} \rightarrow \mathbb{H}$  defined by*

$${}_{\mathbb{K}}\langle k|u \otimes v = \langle k, v \rangle_{\mathbb{K}}u$$

*on simple tensors, and is extended linearly and continuously to be defined on all of  $\mathbb{H} \otimes \mathbb{K}$ .*

If  $T \in \mathcal{B}(\mathbb{H} \otimes \mathbb{K})$  then it is clear that  ${}_{\mathbb{K}}\langle k|T|k\rangle_{\mathbb{K}} \in \mathcal{B}(\mathbb{H})$ . The next lemma gives conditions for this operator to be trace class on  $\mathbb{H}$ .

**Lemma 1.3.4.** *[2, Lemma 2.27] Let  $\mathbb{H}$  and  $\mathbb{K}$  be Hilbert spaces. If  $T$  is a trace-class operator on  $\mathbb{H} \otimes \mathbb{K}$ , then for any  $k \in \mathbb{K}$  the operator*

$${}_{\mathbb{K}}\langle k|T|k\rangle_{\mathbb{K}}$$

*is a trace class operator on  $\mathbb{H}$ .*

*Proof.* Let  $k \in \mathbb{K}$ . Without loss of generality, we may assume  $\|k\|_{\mathbb{K}} = 1$ . Let  $(g_j)_{j \in J}$  and  $(h_j)_{j \in J}$  be any orthonormal family in  $\mathbb{H}$  for some index set  $J$ . Then

$$\begin{aligned} \sum_{j \in J} |\langle g_j, {}_{\mathbb{K}}\langle k|T|k\rangle_{\mathbb{K}} h_j \rangle_{\mathbb{H}}| &= \sum_{j \in J} |\langle |k\rangle_{\mathbb{K}} g_j, T|k\rangle_{\mathbb{K}} h_j \rangle_{\mathbb{H} \otimes \mathbb{K}}| \\ &= \sum_{j \in J} |\langle g_j \otimes k, T(h_j \otimes k) \rangle_{\mathbb{H} \otimes \mathbb{K}}|. \end{aligned} \quad (1.11)$$

Notice that  $(g_j \otimes k)_{j \in J}$  and  $(h_j \otimes k)_{j \in J}$  are orthonormal families in  $\mathbb{H} \otimes \mathbb{K}$ . Since  $T$  is a trace class operator on  $\mathbb{H} \otimes \mathbb{K}$ , the right-hand side of line (1.11) is finite by Lemma 1.3.2. Thus, the left-hand side of line (1.11) is finite, and  ${}_{\mathbb{K}}\langle k|T|k\rangle_{\mathbb{K}}$  is a trace class operator on  $\mathbb{H}$  using Lemma 1.3.2 again.  $\square$

We will need the following lemma about absolutely summable series for the next theorem.

**Lemma 1.3.5.** *If  $\sum_{n \in \mathbb{N}} a_n$  is absolutely summable and for every  $n \in \mathbb{N}$ , the series  $a_n = \sum_{m \in \mathbb{N}} b_{n,m}$  is absolutely summable, then  $\sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} |b_{n,m}| < \infty$ .*

*Proof.* For each  $n, m \in \mathbb{N}$ , define  $\epsilon_{n,m} := e^{-i \operatorname{Arg} b_{n,m}}$ . By assumption, for each  $n$ ,  $\sum_{m \in \mathbb{N}} b_{n,m}$  is absolutely convergent, so the series is unconditionally convergent. Thus, by [23, Proposition 1.c.7], there exists a constant  $K > 0$  such that

$$\left| \sum_{m \in \mathbb{N}} \epsilon_{n,m} b_{n,m} \right| \leq K \left| \sum_{m \in \mathbb{N}} b_{n,m} \right|.$$

Hence

$$\sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} |b_{n,m}| = \sum_{n \in \mathbb{N}} \sum_{m \in \mathbb{N}} \epsilon_{n,m} b_{n,m} = \sum_{n \in \mathbb{N}} \left| \sum_{m \in \mathbb{N}} \epsilon_{n,m} b_{n,m} \right| \leq \sum_{n \in \mathbb{N}} K \left| \sum_{m \in \mathbb{N}} b_{n,m} \right| = K \sum_{n \in \mathbb{N}} |a_n|$$

which by assumption is finite.  $\square$

**Theorem 1.3.6.** [2, Theorem 2.28] *Let  $\mathbb{H}$  and  $\mathbb{K}$  be two Hilbert spaces. Let  $T$  be a trace class operator on  $\mathbb{H} \otimes \mathbb{K}$ . Then for any orthonormal basis  $(k_j)_{j \in J}$  of  $\mathbb{K}$ , the series*

$$\text{tr}_{\mathbb{K}}(T) = \sum_{j \in J} \mathbb{K} \langle k_j | T | k_j \rangle_{\mathbb{K}}$$

is  $\|\cdot\|_1$ -convergent. The operator  $\text{tr}_{\mathbb{K}}(T)$  defined this way does not depend on the choice of orthonormal basis  $(k_j)_{j \in J}$ .

The operator  $\text{tr}_{\mathbb{K}}(T)$  is the unique trace class operator on  $\mathbb{H}$  such that

$$\text{tr}(\text{tr}_{\mathbb{K}}(T)B) = \text{tr}(T(B \otimes \mathbf{1}_{\mathbb{K}})) \quad (1.12)$$

for all  $B \in \mathcal{B}(\mathbb{H})$ .

*Proof.* Let  $(k_j)_{j \in J}$  be an orthonormal basis of  $\mathbb{K}$ . By Lemma 1.3.4, for each  $j \in J$ ,  $\mathbb{K} \langle k_j | T | k_j \rangle_{\mathbb{K}}$  is trace class on  $\mathbb{H}$ . Hence, for each  $j \in J$ , there exists orthonormal sequences  $(x_n^j)_{n \in \mathbb{N}}$  and  $(y_n^j)_{n \in \mathbb{N}}$  in  $\mathbb{H}$  such that

$$\|\mathbb{K} \langle k_j | T | k_j \rangle_{\mathbb{K}}\|_1 = \sum_{n \in \mathbb{N}} \left| \langle x_n^j, \mathbb{K} \langle k_j | T | k_j \rangle_{\mathbb{K}} y_n^j \rangle_{\mathbb{H}} \right|$$

by Lemma 1.3.2. This means that we have

$$\begin{aligned} \sum_{j \in J} \|\mathbb{K} \langle k_j | T | k_j \rangle_{\mathbb{K}}\|_1 &= \sum_{j \in J} \sum_{n \in \mathbb{N}} \left| \langle x_n^j, \mathbb{K} \langle k_j | T | k_j \rangle_{\mathbb{K}} y_n^j \rangle_{\mathbb{H}} \right| \\ &= \sum_{j \in J} \sum_{n \in \mathbb{N}} \left| \langle |k_j\rangle_{\mathbb{K}} x_n^j, T |k_j\rangle_{\mathbb{K}} y_n^j \rangle_{\mathbb{H} \otimes \mathbb{K}} \right| = \sum_{j \in J} \sum_{n \in \mathbb{N}} \left| \langle x_n^j \otimes k_j, T(y_n^j \otimes k_j) \rangle_{\mathbb{H} \otimes \mathbb{K}} \right|. \end{aligned}$$

Since  $(x_n^j \otimes k_j)_{j \in J, n \in \mathbb{N}}$  and  $(y_n^j \otimes k_j)_{j \in J, n \in \mathbb{N}}$  are orthonormal families in  $\mathbb{H} \otimes \mathbb{K}$ , and  $T$  is a trace class operator on  $\mathbb{H} \otimes \mathbb{K}$ , we have that the right-hand side of the equation

above is finite by Lemma 1.3.2. Thus,  $\sum_{j \in J} \mathbb{K} \langle k_j | T | k_j \rangle_{\mathbb{K}}$  is  $\|\cdot\|_1$ -convergent, and the operator

$$\mathrm{tr}_{\mathbb{K}}(T) = \sum_{j \in J} \mathbb{K} \langle k_j | T | k_j \rangle_{\mathbb{K}}$$

is a well-defined trace class operator on  $\mathbb{H}$ .

We will now check that this operator is independent of the choice of orthonormal basis  $(k_j)_{j \in J}$  of  $\mathbb{K}$ . Let  $(\tilde{k}_i)_{i \in J}$  be another orthonormal basis of  $\mathbb{K}$  (notice that we can use the same index set  $J$  because orthonormal bases of a Hilbert space have the same cardinality). Then for any  $x, y \in \mathbb{H}$

$$\langle x, \mathrm{tr}_{\mathbb{K}}(T)y \rangle_{\mathbb{H}} = \sum_{j \in J} \langle x \otimes k_j, T(y \otimes k_j) \rangle_{\mathbb{H} \otimes \mathbb{K}} \quad (1.13)$$

$$= \sum_{j \in J} \sum_{i \in J} \langle k_j, \tilde{k}_i \rangle_{\mathbb{K}} \langle x \otimes \tilde{k}_i, T(y \otimes k_j) \rangle_{\mathbb{H} \otimes \mathbb{K}} \quad (1.14)$$

$$= \sum_{j \in J} \sum_{i \in J} \sum_{l \in J} \langle k_j, \tilde{k}_i \rangle_{\mathbb{K}} \langle \tilde{k}_l, k_j \rangle_{\mathbb{K}} \langle x \otimes \tilde{k}_i, T(y \otimes \tilde{k}_l) \rangle_{\mathbb{H} \otimes \mathbb{K}} \quad (1.15)$$

$$= \sum_{i \in J} \sum_{l \in J} \langle \tilde{k}_l, \tilde{k}_i \rangle_{\mathbb{K}} \langle x \otimes \tilde{k}_i, T(y \otimes \tilde{k}_l) \rangle_{\mathbb{H} \otimes \mathbb{K}} \quad (1.16)$$

$$= \sum_{i \in J} \langle x \otimes \tilde{k}_i, T(y \otimes \tilde{k}_i) \rangle_{\mathbb{H} \otimes \mathbb{K}} = \langle x, \sum_{j \in J'} \mathbb{K} \langle \tilde{k}_i | T | \tilde{k}_i \rangle_{\mathbb{K}} y \rangle_{\mathbb{H}},$$

i.e.  $\mathrm{tr}_{\mathbb{K}}(T) = \sum_{i \in J} \mathbb{K} \langle \tilde{k}_i | T | \tilde{k}_i \rangle_{\mathbb{K}}$ , and the independence is proven.

In order to make sense of the above calculations, we observe that each summation has at most countably many non-zero terms, and by Fubini's Theorem the summations in line (1.15) can be permuted to obtain line (1.16). Indeed, by Lemma 1.3.2

$$\begin{aligned} & \sum_{j \in J} \left| \langle x \otimes k_j, T(y \otimes k_j) \rangle_{\mathbb{H} \otimes \mathbb{K}} \right| \\ &= \|x\|_{\mathbb{H}} \|y\|_{\mathbb{H}} \sum_{j \in J} \left| \left\langle \frac{x}{\|x\|_{\mathbb{H}}} \otimes k_j, T \left( \frac{y}{\|y\|_{\mathbb{H}}} \otimes k_j \right) \right\rangle_{\mathbb{H} \otimes \mathbb{K}} \right| < \infty \end{aligned}$$

since  $\left( \frac{x}{\|x\|_{\mathbb{H}}} \otimes k_j \right)_{j \in J}$  and  $\left( \frac{y}{\|y\|_{\mathbb{H}}} \otimes k_j \right)_{j \in J}$  are orthonormal families in  $\mathbb{H} \otimes \mathbb{K}$  and  $T$  is trace class. This proves that there are at most countably many non-zero terms in the sums over  $j \in J$  and that the series in line (1.13) is absolutely summable. Also,

for each  $j \in J$ , by the Pythagorean Theorem,

$$\sum_{i \in J} |\langle \tilde{k}_i, k_j \rangle_{\mathbb{K}}|^2 = \sum_{i \in J} \|\langle \tilde{k}_i, k_j \rangle_{\mathbb{K}} \tilde{k}_i\|_{\mathbb{K}}^2 = \left\| \sum_{i \in J} \langle \tilde{k}_i, k_j \rangle_{\mathbb{K}} \tilde{k}_i \right\|_{\mathbb{K}}^2 = \|k_j\|_{\mathbb{K}}^2 < \infty,$$

so there are at most countably many non-zero terms in the sums over  $i \in J$  and the series  $\sum_{i \in J} \langle \tilde{k}_i, k_j \rangle_{\mathbb{K}} \tilde{k}_i$  converges unconditionally, and thus, so does

$\sum_{i \in J} \langle k_j, \tilde{k}_i \rangle_{\mathbb{K}} \langle x \otimes \tilde{k}_i, T(y \otimes k_j) \rangle_{\mathbb{H} \otimes \mathbb{K}}$ . Since unconditional summability and absolute summability are equivalent in  $\mathbb{C}$ , we have that  $\sum_{i \in J} \langle k_j, \tilde{k}_i \rangle_{\mathbb{K}} \langle x \otimes \tilde{k}_i, T(y \otimes k_j) \rangle_{\mathbb{H} \otimes \mathbb{K}}$  is absolutely summable for each  $j \in J$ . By Lemma 1.3.5, this means the double series in line (1.14) is absolutely summable. A similar argument using Lemma 1.3.5 again will show that the triple series in line (1.15) is absolutely summable as well and there are all but countably many non-zero terms in the sums over  $l \in J$ .

We will now prove the characterization in line (1.12). Let  $(h_i)_{i \in I}$  be an orthonormal basis of  $\mathbb{H}$ , and let  $(k_j)_{j \in J}$  be an orthonormal basis of  $\mathbb{K}$ . Then for any  $B \in \mathcal{B}(\mathbb{H})$ ,

$$\begin{aligned} \operatorname{tr}(T(B \otimes 1_{\mathbb{K}})) &= \sum_{i \in I} \sum_{j \in J} \langle h_i \otimes k_j, T(B \otimes 1_{\mathbb{K}}) h_i \otimes k_j \rangle_{\mathbb{H} \otimes \mathbb{K}} \\ &= \sum_{i \in I} \sum_{j \in J} \langle h_i \otimes k_j, T(B h_i \otimes k_j) \rangle_{\mathbb{H} \otimes \mathbb{K}} \\ &= \sum_{i \in I} \sum_{j \in J} \langle h_i, {}_{\mathbb{K}} \langle k_j | T | k_j \rangle_{\mathbb{K}} B h_i \rangle_{\mathbb{H}} = \operatorname{tr}(\operatorname{tr}_{\mathbb{K}}(T) B) \end{aligned}$$

where the last equality holds because the double-sum in the last line has at most countably many nonzero terms (since it is absolutely convergent), and the sum  $\sum_{j \in J} {}_{\mathbb{K}} \langle k_j | T | k_j \rangle_{\mathbb{K}}$  has at most countably many nonzero terms because it is  $\|\cdot\|_1$ -convergent.

If  $S$  is a trace class operator on  $\mathbb{H}$  such that  $\operatorname{tr}(SB) = \operatorname{tr}(T(B \otimes 1))$  for every  $B \in \mathcal{B}(\mathbb{H})$ , then  $\operatorname{tr}((S - \operatorname{tr}_{\mathbb{K}}(T))B) = 0$  for every  $B \in \mathcal{B}(\mathbb{H})$ , i.e.  $S = \operatorname{tr}_{\mathbb{K}}(T)$ .  $\square$

**Definition 1.3.7.** *Let  $\mathbb{H}$  and  $\mathbb{K}$  be Hilbert spaces and let  $T$  be a trace class operator on  $\mathbb{H} \otimes \mathbb{K}$ . The operator  $\operatorname{tr}_{\mathbb{K}}(T)$  as defined above is called the partial trace of  $T$  over  $\mathbb{K}$ .*

The following theorem gives some useful and insightful properties of the partial trace.

**Theorem 1.3.8.** [2, Theorem 2.29] Let  $\mathbb{H}$  and  $\mathbb{K}$  be two Hilbert spaces and let  $T$  be a trace class operator on  $\mathbb{H} \otimes \mathbb{K}$ .

1. If  $T$  is of the form  $A \otimes B$ , with  $A$  being a trace class operator on  $\mathbb{H}$  and  $B$  being a trace class operator on  $\mathbb{K}$ , then

$$\text{tr}_{\mathbb{K}}(T) = \text{tr}(B)A.$$

2. We always have

$$\text{tr}(\text{tr}_{\mathbb{K}}(T)) = \text{tr}(T).$$

3. If  $A, B \in \mathcal{B}(\mathbb{H})$ , then

$$\text{tr}_{\mathbb{K}}((A \otimes 1_{\mathbb{K}})T(B \otimes 1_{\mathbb{K}})) = A \text{tr}_{\mathbb{K}}(T)B.$$

The following theorem gives another useful result about the partial trace which we will use throughout this thesis.

**Theorem 1.3.9.** Let  $\mathbb{H}$  and  $\mathbb{K}$  be two Hilbert spaces and let  $T$  be a trace class operator on  $\mathbb{H} \otimes \mathbb{K}$ . Then  $\|\text{tr}_{\mathbb{K}}(T)\|_1 \leq \|T\|_1$ , i.e. the partial trace is a contraction from the trace class operators on  $\mathbb{H} \otimes \mathbb{K}$  to the trace class operators on  $\mathbb{H}$ .

*Proof.* Let the Polar Decomposition of  $\text{tr}_{\mathbb{K}}(T)$  be  $\text{tr}_{\mathbb{K}}(T) = U |\text{tr}_{\mathbb{K}}(T)|$  for some partial isometry  $U \in \mathcal{B}(\mathbb{H})$ . We have that  $U^* \in \mathcal{B}(\mathbb{H})$  is also a partial isometry and  $|\text{tr}_{\mathbb{K}}(T)| = U^* \text{tr}_{\mathbb{K}}(T)$ . Thus by property (1.12),

$$\begin{aligned} \|\text{tr}_{\mathbb{K}}(T)\|_1 &= \text{tr} |\text{tr}_{\mathbb{K}}(T)| = \text{tr}(U^* \text{tr}_{\mathbb{K}}(T)) = \text{tr}(U^* \otimes 1_{\mathbb{K}} T) \\ &\leq \|U^* \otimes 1_{\mathbb{K}}\|_{\infty} \|T\|_1 \leq \|T\|_1 \end{aligned}$$

because partial isometries are contractions, where  $\|\cdot\|_{\infty}$  denotes the  $\mathcal{B}(\mathbb{H} \otimes \mathbb{K})$  norm.

□

The following theorem states an important property of the partial trace which we will use throughout this thesis.

**Theorem 1.3.10.** *Let  $\mathbb{H}$  and  $\mathbb{K}$  be Hilbert spaces, and let  $\rho$  be a density operator on  $\mathbb{H} \otimes \mathbb{K}$ . Then  $\text{tr}_{\mathbb{K}}(\rho)$  is a density operator on  $\mathbb{H}$ .*

*Proof.* By part 2 of Theorem 1.3.8,

$$\text{tr}(\text{tr}_{\mathbb{K}}(\rho)) = \text{tr}(\rho) = 1.$$

Let  $(k_j)_{j \in J}$  be an orthonormal basis of  $\mathbb{K}$ . Then for every  $h \in \mathbb{H}$ ,

$$\langle h, \text{tr}_{\mathbb{K}}(\rho)h \rangle_{\mathbb{H}} = \langle h, \sum_{j \in J} \langle k_j | \rho | k_j \rangle_{\mathbb{K}} h \rangle_{\mathbb{H}} = \sum_{j \in J} \langle h \otimes k_j, \rho(h \otimes k_j) \rangle_{\mathbb{H} \otimes \mathbb{K}} \geq 0,$$

i.e.  $\rho$  is positive. Therefore  $\text{tr}_{\mathbb{K}}(\rho)$  is a density operator on  $\mathbb{H}$ . □

The corresponding quantum mechanical notion to the chaotic sequences of probability measures, as well as the corresponding notion to the propagation of chaos appears in [31, Theorem 5.7] where the time evolution is given by a specific family of Hamiltonians. Gottlieb [15] formulated the notion of chaotic sequences of density operators. The partial trace will play a role in the quantum definition of chaos as marginals played a corresponding role in the definition of a chaotic sequence of probability measures. In Section 3.1, we study the notion of chaos which was introduced by Spohn and formalized by Gottlieb. To honor the fact that the definition of chaos was originated by the work of Kac for classical models, we refer to its quantum version as “quantum Kac’s chaos”.

It is worth mentioning that there is a vast literature on the notion of “quantum chaos”, where notions of ergodic theory are extended to quantum physical models. Without attempting to give detailed references to quantum chaos, we refer the interested reader to the books [11], [13], and [19], and to the article [15] where some of these notions are presented.

The main result of Chapter 3 is Theorem 3.2.1 which is the analogue of [33, Proposition 2.2(i)]. This will be the quantum version of the result we used to prove the connection between mixing and Kac's chaos with in section 2.2. In Chapter 4, the main result is Theorem 4.3.1 which is a simpler, different proof of the propagation of chaos result of Spohn [31, Theorem 5.7]. This result shows that chaos propagates in the mean field limit for interacting quantum systems.



## CHAPTER 2

# A CONNECTION BETWEEN MIXING AND CLASSICAL KAC'S CHAOS

### 2.1 MIXING DYNAMICAL SYSTEMS

The main result of this chapter is Theorem 2.1.1 which gives a relationship between the classical form of chaos defined in Definition 1.1.2 and the ergodic theory notion of mixing defined in Definition 1.2.1(ii). In particular, we will show that dynamical systems with certain mixing-type properties can be used to construct sequences of chaotic probability measures.

Before stating the main result we introduce some notation. If  $E$  is a topological space then  $\mathcal{B}(E)$  will denote the  $\sigma$ -algebra of the Borel subsets of  $E$ , and  $M(E)$  will denote the set of probability measures on  $\mathcal{B}(E)$ . Also  $\Sigma_n$  will denote the set of permutations of  $\{1, \dots, n\}$  for each  $n \in \mathbb{N}$ .

We now present the main result of the chapter.

**Theorem 2.1.1.** *Let  $E$  be a separable metric space,  $\mu$  be a probability measure on  $\mathcal{B}(E)$ , and  $S : E \rightarrow E$  be a Borel measurable map. Assume that*

1. *For every  $A \in \mathcal{B}(E)$ ,*

$$\sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}A) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}A)| \xrightarrow[k \rightarrow \infty]{} 0,$$

*and*

2.  *$(E, \mathcal{B}(E), \mu, S)$  is asymptotically stationary with stationary limit  $\nu$ .*

For every  $n \in \mathbb{N}$  define  $\mu_n : \mathcal{B}(E^n) \rightarrow [0, 1]$  by

$$\mu_n(A) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mu\{x \in E : (S^{\sigma(1)}(x), \dots, S^{\sigma(n)}(x)) \in A\}.$$

Then  $(\mu_n)_{n \in \mathbb{N}}$  is  $\nu$ -chaotic.

Note that assumption 1 of the main result is related to the mixing property that was introduced in Definition 1.2.1(ii). The differences between the two properties are: The limit in Definition 1.2.1(ii) is taken for any Borel sets  $A$  and  $B$  while in assumption 1, the sets  $A$  and  $B$  are equal and they belong to the  $\sigma$ -algebra  $\{S^{-i}A : A \in \mathcal{B}(E)\}$ . In that sense, assumption 1 is weaker than Definition 1.2.1(ii). On the other hand, Definition 1.2.1(ii) lacks the uniformity which is manifested in assumption 1 by the presence of the supremum. In that sense, assumption 1 is stronger than Definition 1.2.1(ii).

One way to guarantee that a dynamical system satisfies assumption 1 of Theorem 2.1.1 is by means of the next lemma.

**Lemma 2.1.2.** *Let  $(\Omega, \Sigma, \mu, S)$  be a dynamical system which is asymptotically stationary. Let  $\Pi \subset \Sigma$  be a  $\pi$ -system such that  $\sigma(\Pi) = \Sigma$  and*

$$\limsup_{k \rightarrow \infty} \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B)| = 0 \quad (2.1)$$

for all  $A, B \in \Pi$ . Then (2.1) is satisfied for all  $A, B \in \Sigma$  (hence assumption 1 of Theorem 2.1.1 is satisfied as well).

*Proof.* The proof will be by way of the Dynkin  $\pi - \lambda$  Theorem. Fix  $A \in \Pi$ , and define  $\Lambda_A := \{B \in \Sigma : \limsup_{k \rightarrow \infty} \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B)| = 0\}$ . It

is clear that  $\Omega \in \Lambda_A$ . Now, assume  $B_1, B_2 \in \Lambda_A$  such that  $B_1 \subset B_2$ . Then we have

$$\begin{aligned}
0 &\leq \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}(B_2 \setminus B_1)) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}(B_2 \setminus B_1))| \\
&= \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B_2) - \mu(S^{-i}A \cap S^{-k}S^{-i}B_1) \\
&\quad - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B_2) + \mu(S^{-i}A)\mu(S^{-k}S^{-i}B_1)| \\
&\leq \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B_2) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B_2)| \\
&\quad + \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B_1) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B_1)|
\end{aligned}$$

Taking limits on both sides as  $k \rightarrow \infty$ , we get that  $B_2 \setminus B_1 \in \Lambda_A$ .

Lastly, let  $(B_n)_{n=1}^\infty \subset \Lambda_A$  be any monotone increasing sequence with limit  $B \in \Sigma$ .

We need to show that  $B \in \Lambda_A$ . Let  $\epsilon > 0$ . Denote by  $\nu$  the stationary limit of  $(\Omega, \Sigma, \mu, S)$ . We have that

$$\begin{aligned}
&\sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B)| \\
&\leq \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B) - \mu(S^{-i}A)\nu(S^{-i}B)| \tag{2.2}
\end{aligned}$$

$$+ \sup_{i \in \mathbb{N}} |\mu(S^{-i}A)| |\nu(B) - \mu(S^{-k}S^{-i}B)| \tag{2.3}$$

By assumption, there exists a  $k_0 \in \mathbb{N}$  such that  $|\nu(B) - \mu(S^{-k}B)| < \epsilon$  for all  $k \geq k_0$ , and thus, line (2.3) can be made small. Notice that there exists an  $n_0 \in \mathbb{N}$  such that  $|\nu(B_n) - \nu(B)| < \epsilon$  for all  $n \geq n_0$ . Using this information, we focus on line (2.2),

$$\begin{aligned}
&\sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B) - \mu(S^{-i}A)\nu(S^{-i}B)| \\
&= \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B_{n_0}) \\
&\quad + \mu(S^{-i}A \cap S^{-k}S^{-i}(B \setminus B_{n_0})) - \mu(S^{-i}A)\nu(S^{-i}B)| \\
&\leq \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B_{n_0}) - \mu(S^{-i}A)\nu(S^{-i}B)| \tag{2.4}
\end{aligned}$$

$$+ \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}(B \setminus B_{n_0}))| \tag{2.5}$$

Line (2.4) can be estimated as

$$\begin{aligned}
& \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B_{n_0}) - \mu(S^{-i}A)\nu(S^{-i}B)| \\
& \leq \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B_{n_0}) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B_{n_0})| \\
& + \sup_{i \in \mathbb{N}} |\mu(S^{-i}A)| |\mu(S^{-k}S^{-i}B_{n_0}) - \nu(B_{n_0})| + \sup_{i \in \mathbb{N}} |\mu(S^{-i}A)| |\nu(B_{n_0}) - \nu(B)|
\end{aligned}$$

There exists a  $k_1 \in \mathbb{N}$  such that  $|\mu(S^{-k}S^{-i}B_{n_0}) - \nu(B_{n_0})| < \epsilon$  for all  $k \geq k_1$ , and there exists a  $k_2 \in \mathbb{N}$  such that  $\sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B_{n_0}) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B_{n_0})| < \epsilon$  for all  $k \geq k_2$ . Thus, for all  $k \geq \max\{k_1, k_2\}$ , line (2.4) is small.

Line (2.5) can be estimated as

$$\begin{aligned}
& \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}(B \setminus B_{n_0}))| \leq \sup_{i \in \mathbb{N}} |\mu(S^{-k}S^{-i}(B \setminus B_{n_0}))| \\
& \leq \sup_{i \in \mathbb{N}} |\mu(S^{-k}S^{-i}B) - \nu(B)| + |\nu(B) - \nu(B_{n_0})| + \sup_{i \in \mathbb{N}} |\nu(B_{n_0}) - \mu(S^{-k}S^{-i}B_{n_0})|
\end{aligned}$$

which is small for all  $k \geq \max\{k_0, k_1\}$ . Hence, we have that

$$\sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B)| < 6\epsilon \text{ for all } k \geq \max\{k_0, k_1, k_2\}.$$

Thus  $B \in \Lambda_A$ . By the Dynkin  $\pi - \lambda$  Theorem, for all  $A \in \Pi$  and all  $B \in \Sigma$  we have

$\lim_{k \rightarrow \infty} \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B)| = 0$ . The same argument can be turned around to show that for a fixed  $B \in \Sigma$  (2.1) holds for all  $A \in \Sigma$ .  $\square$

Using Lemma 2.1.2, we obtain the following corollary of Theorem 2.1.1.

**Corollary 2.1.3.** *Let  $E$  be a separable metric space,  $\mu$  be a probability measure on  $\mathcal{B}(E)$ , and  $S : E \rightarrow E$  be Borel measurable. Let  $\Pi \subset \Sigma$  be a  $\pi$ -system such that  $\sigma(\Pi) = \Sigma$ . Assume that*

1. *For every  $A, B \in \Pi$ ,*

$$\sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}B) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}B)| \xrightarrow[k \rightarrow \infty]{} 0,$$

*and*

2.  $(E, \mathcal{B}(E), \mu, S)$  is asymptotically stationary with stationary limit  $\nu$ .

For every  $n \in \mathbb{N}$  define  $\mu_n : \mathcal{B}(E^n) \rightarrow [0, 1]$  by

$$\mu_n(A) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mu\{x \in E : (S^{\sigma(1)}(x), \dots, S^{\sigma(n)}(x)) \in A\}.$$

Then  $(\mu_n)_{n \in \mathbb{N}}$  is  $\nu$ -chaotic.

The next two remarks give sufficient conditions for the assumptions of Theorem 2.1.1 to be met.

**Remark 2.1.4.** Let  $(\Omega, \Sigma, \mu, S)$  be a dynamical system which is mixing and  $S$  is  $\mu$ -measure preserving (as in line (1.9)). Then the assumptions 1 and 2 of Theorem 2.1.1 are satisfied for this dynamical system.

Indeed, for every  $A \in \Sigma$  we have

$$\begin{aligned} & \sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}A) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}A)| \\ &= \sup_{i \in \mathbb{N}} |\mu(S^{-i}(A \cap S^{-k}A)) - \mu(A)\mu(S^{-k}A)| = |\mu(A \cap S^{-k}A) - (\mu(A))^2| \xrightarrow[k \rightarrow \infty]{} 0 \end{aligned}$$

where the second equality is valid because  $S$  is  $\mu$ -measure preserving and the limit is valid because the dynamical system is mixing. Thus assumption 1 of Theorem 2.1.1 is satisfied. Also,  $(\Omega, \Sigma, \mu, S)$  is asymptotically stationary with stationary limit  $\mu$  since  $S$  is  $\mu$ -measure preserving. Thus assumption 2 of Theorem 2.1.1 is satisfied as well.

**Remark 2.1.5.** Let  $(\Omega, \Sigma, \mu, S)$  be a dynamical system and let  $\Pi \subset \Sigma$  be a  $\pi$ -system such that  $\sigma(\Pi) = \Sigma$ .

(i) If  $\lim_{k \rightarrow \infty} |\mu(S^{-k}A \cap B) - \mu(S^{-k}A)\mu(B)| = 0$  holds for every  $A, B \in \Pi$ , then it holds for every  $A, B \in \Sigma$ .

(ii) If  $\mu(S^{-1}(A)) = \mu(A)$  holds for all  $A \in \Pi$  then it holds for all  $A \in \Sigma$ .

See Shalizi and Kontorovich [24, Theorem 384] for the proof of part (i). The proof of part (ii) is very easy using Dynkin's  $\pi - \lambda$  theorem.

We now present three examples of dynamical systems that satisfy the assumptions of Remark 2.1.4. The first example is called the "baker's map". The measure space for the baker's map is  $([0, 1]^2 := [0, 1] \times [0, 1], \mathcal{B}([0, 1]^2), \mu)$  where  $\mu$  is the Lebesgue measure restricted to  $\mathcal{B}([0, 1]^2)$ . The map  $S : [0, 1]^2 \rightarrow [0, 1]^2$  of the dynamical system is defined by

$$S(x, y) = \begin{cases} (2x, \frac{1}{2}y) & 0 \leq x < \frac{1}{2}, 0 \leq y \leq 1 \\ (2x - 1, \frac{1}{2}y + \frac{1}{2}) & \frac{1}{2} \leq x \leq 1, 0 \leq y \leq 1 \end{cases}.$$

Lasota and Mackey [25, Example 4.3.1] prove that the baker's map is mixing by verifying Definition 1.2.1(ii) for all rectangles  $A, B$  with sides parallel to  $x$  and  $y$  axes. These rectangles form a  $\pi$ -system that generates the  $\sigma$ -algebra  $\mathcal{B}([0, 1] \times [0, 1])$ . Thus by Remark 2.1.5(i) the baker's map is mixing. Using the same  $\pi$ -system and Remark 2.1.5(ii) it is easy to verify that the baker's map is measure preserving.

Another example of a dynamical system which satisfies the assumptions of Remark 2.1.4 is the Anosov map, (also called the cat map). The measure space for the Anosov map is  $([0, 1]^2 := [0, 1] \times [0, 1], \mathcal{B}([0, 1]^2), \mu)$  where  $\mu$  is the Lebesgue measure restricted to  $\mathcal{B}([0, 1]^2)$ . The map  $S : [0, 1]^2 \rightarrow [0, 1]^2$  for the Anosov map is defined by

$$S(x, y) = (x + y, x + 2y) \pmod{1}.$$

Lasota and Mackey prove that the cat map is mixing by using the Fibonacci sequence and Fourier transforms [25, Example 4.4.3]. Arnold and Avez show that the Anosov Map is measure preserving [1, Example 1.16].

We now present a construction of an infinite product of probability measures satisfying the assumptions of Remark 2.1.4. If  $(\Omega_n, \Sigma_n, \mu_n)_{n \in \mathbb{N}}$  is a sequence of probability spaces then the Cartesian product  $\prod_{n=1}^{\infty} \Omega_n$  can be naturally equipped with an infinite product of these measures, as defined by Kakutani [22]. Denote this infinite

product probability space by  $(\prod_{n=1}^{\infty} \Omega_n, \prod_{n=1}^{\infty} \Sigma_n, \prod_{n=1}^{\infty} \mu_n)$ . The  $\sigma$ -algebra  $\prod_{n=1}^{\infty} \Sigma_n$  is generated by the  $\pi$ -system  $\prod_{n=1}^{<\infty} \Sigma_n$  consisting of all sets of the form  $\prod_{n=1}^{\infty} A_n$  where  $A_n \in \Sigma_n$  for all  $n \in \mathbb{N}$  and  $A_n = \Omega_n$  for all but finitely many  $n$ 's. If  $\Sigma_n = \Sigma$  for all  $n \in \mathbb{N}$  then let  $\Sigma^{<\infty}$  denote the  $\pi$ -system  $\prod_{n=1}^{<\infty} \Sigma$ . If  $A = \prod_{n=1}^{\infty} A_n$  is such a set then we define  $(\prod_{n=1}^{\infty} \mu_n)(A) = \prod_{n=1}^{\infty} (\mu_n(A_n))$ . If  $\Omega_n = \Omega$  and  $\Sigma_n = \Sigma$  for all  $n \in \mathbb{N}$  then  $\prod_{n=1}^{\infty} \Omega_n$  is denoted by  $\Omega^{\mathbb{N}}$ , and  $\prod_{n=1}^{\infty} \Sigma_n$  is denoted by  $\Sigma^{\mathbb{N}}$ . If moreover  $\mu_n = \mu$  for all  $n \in \mathbb{N}$  then  $\prod_{n=1}^{\infty} \mu_n$  is denoted by  $\mu^{\mathbb{N}}$ . Assume that for every  $n \in \mathbb{N}$   $(\Omega, \Sigma, \mu_n, S)$  is a dynamical system (i.e. in general we may allow different measures to be considered on the same  $\sigma$ -algebra  $\Sigma$ ). Define  $S^{\mathbb{N}} : \Omega^{\mathbb{N}} \rightarrow \Omega^{\mathbb{N}}$  by

$$S^{\mathbb{N}}((\omega_n)_{n \in \mathbb{N}}) = (S(\omega_{n+1}))_{n \in \mathbb{N}}.$$

Then  $S^{\mathbb{N}}$  is measurable i.e.  $(\Omega^{\mathbb{N}}, \Sigma^{\mathbb{N}}, \prod_{n=1}^{\infty} \mu_n, S^{\mathbb{N}})$  is a dynamical system. Indeed, it is enough and easy to check that  $(S^{\mathbb{N}})^{-1}(A) \in \Sigma^{\mathbb{N}}$  for every set  $A$  in the  $\pi$ -system  $\Sigma^{<\infty}$ . Obviously, if  $(\Omega, \Sigma, \mu, S)$  is a dynamical system and  $S$  is  $\mu$ -measure preserving, then  $S^{\mathbb{N}}$  is  $\mu^{\mathbb{N}}$ -measure preserving, (it is enough to be verified on sets of the  $\pi$ -system  $\prod_{n=1}^{<\infty} \Sigma_n$ , which is an easy task). Thus, if  $(\Omega, \Sigma, \mu, S)$  denotes either the baker's dynamical system, or the Anosov dynamical system defined above, then  $S^{\mathbb{N}}$  is  $\mu^{\mathbb{N}}$ -measure preserving. We claim that for any dynamical system  $(\Omega, \Sigma, \mu, S)$ ,  $(\Omega^{\mathbb{N}}, \Sigma^{\mathbb{N}}, \mu^{\mathbb{N}}, S^{\mathbb{N}})$  is always mixing. By Remark 2.1.5(i) this claim is enough and easy to be verified for sets  $A, B$  in the  $\pi$ -system  $\Sigma^{<\infty}$ . Indeed if  $A = \prod_{n=1}^{\infty} A_n, B = \prod_{n=1}^{\infty} B_n$  where  $A_n, B_n \in \Sigma$  for all  $n$  and  $A_n = B_n = \Omega$  for all  $n > m$ , then

$$\mu^{\mathbb{N}}((S^{\mathbb{N}})^{-k} A) = \prod_{n=1}^m \mu(S^{-k} A_n), \quad \mu^{\mathbb{N}}(B) = \prod_{i=1}^m \mu(B_i),$$

and

$$\mu^{\mathbb{N}}((S^{\mathbb{N}})^{-k} A \cap B) = \prod_{i=1}^m \mu(B_i) \prod_{n=1}^m \mu(S^{-k} A_n) \quad \text{for all } k > m.$$

Hence

$$|\mu^{\mathbb{N}}((S^{\mathbb{N}})^{-k} A \cap B) - \mu^{\mathbb{N}}((S^{\mathbb{N}})^{-k} A) \mu^{\mathbb{N}}(B)| = 0 \quad \text{for all } k > m.$$

Thus if  $(\Omega, \Sigma, \mu, S)$  is a dynamical system such that  $S$  is  $\mu$ -measure preserving, then we obtain that  $(\Omega^{\mathbb{N}}, \Sigma^{\mathbb{N}}, \mu^{\mathbb{N}}, S^{\mathbb{N}})$  is a dynamical system that satisfies the assumptions of Remark 2.1.4.

In all the examples that we have mentioned so far, the map of the dynamical system is measure preserving. Such maps are trivially asymptotically stationary with stationary limits being equal to the original measure. We now describe how infinite product probability measures can be used to give examples of dynamical systems that satisfy the assumptions of Theorem 2.1.1 without the map of the dynamical system being measure preserving. In this example the stationary limit of the dynamical system is different than the original measure. Let  $\Omega = [0, 1]$  and  $\Sigma = \mathcal{B}([0, 1])$ . For every  $k \in \mathbb{N}$  define a density function  $\phi_k : [0, 1] \rightarrow \{0, 1, 2\}$  by

$$\phi_k(x) = \chi_{[0, 1 - \frac{1}{2^{k-1}}]}(x) + 2\chi_{(1 - \frac{1}{2^{k-1}}, 1 - \frac{1}{2^k}]}(x)$$

and the probability measure  $\mu_k : \Sigma \rightarrow [0, 1]$  by

$$\mu_k(A) := \int_A \phi_k(x) dx \quad \text{for all } A \in \Sigma.$$

Consider the probability space  $(\Omega^{\mathbb{N}}, \Sigma^{\mathbb{N}}, \prod_{k=1}^{\infty} \mu_k)$  and in order to make the notation easier let  $\mathbb{M} = \prod_{k=1}^{\infty} \mu_k$  and  $\mathbb{L} = \lambda^{\mathbb{N}}$  where  $\lambda$  is the Lebesgue measure on  $[0, 1]$ . Consider a map  $S : \Omega \rightarrow \Omega$  such that  $S^{\mathbb{N}}$  is  $\mathbb{L}$ -measure preserving but not  $\mathbb{M}$ -measure preserving (this is valid for example when  $S$  is the identity map). In order to make the notation easier let  $\mathbb{S} = S^{\mathbb{N}}$ . We claim that the dynamical system  $(\Omega^{\mathbb{N}}, \Sigma^{\mathbb{N}}, \mathbb{M}, \mathbb{S})$  satisfies the assumptions of Theorem 2.1.1. Indeed, in order to verify assumption 2 of Theorem 2.1.1, we prove that  $\mathbb{M}$  is asymptotically stationary with stationary limit equal to  $\mathbb{L}$ .

Fix any  $A \in \Sigma^{\mathbb{N}}$ . Define for each  $k \in \mathbb{N}$  the set

$$C_k := [0, 1]^k \times [0, 1 - \frac{1}{2^k}] \times [0, 1 - \frac{1}{2^{k+1}}] \times [0, 1 - \frac{1}{2^{k+2}}] \times \dots$$

Since  $C_k = \bigcap_{n=0}^{\infty} ([0, 1]^k \times [0, 1 - \frac{1}{2^k}] \times \dots \times [0, 1 - \frac{1}{2^{k+n}}] \times [0, 1]^{\mathbb{N}})$ , we have that  $C_k \in \Sigma^{\mathbb{N}}$ , and if  $B \in \Sigma^{\mathbb{N}}$  with  $B \subset C_k$  then  $\mathbb{M}(B) = \mathbb{L}(B)$ . Thus for any  $A \in \Sigma^{\mathbb{N}}$  we have



that

$$\begin{aligned}
& |\mathbb{M}(\mathbb{S}^{-k}A) - \mathbb{L}(A)| = |\mathbb{M}(\mathbb{S}^{-k}A) - \mathbb{L}(\mathbb{S}^{-k}B)| \\
&= |\mathbb{M}((\mathbb{S}^{-k}A) \cap C_k) + \mathbb{M}((\mathbb{S}^{-k}A) \setminus C_k) - \mathbb{L}((\mathbb{S}^{-k}A) \cap C_k) - \mathbb{L}((\mathbb{S}^{-k}A) \setminus C_k)| \\
&= |\mathbb{M}((\mathbb{S}^{-k}A) \setminus C_k) - \mathbb{L}((\mathbb{S}^{-k}A) \setminus C_k)| \leq 2 \left( 1 - \prod_{s=k}^{\infty} \left( 1 - \frac{1}{2^s} \right) \right) \xrightarrow{k \rightarrow \infty} 0,
\end{aligned}$$

where the last inequality is valid because  $\mathbb{M}(C_k) = \mathbb{L}(C_k) = \prod_{s=k}^{\infty} \left( 1 - \frac{1}{2^s} \right)$ , hence  $\mathbb{M}([0, 1]^{\mathbb{N}} \setminus C_k) = \mathbb{L}([0, 1]^{\mathbb{N}} \setminus C_k) = 1 - \prod_{s=k}^{\infty} \left( 1 - \frac{1}{2^s} \right)$ . This verifies assumption 2 of Theorem 2.1.1.

Now, in order to verify assumption 1 of Theorem 2.1.1 we use Lemma 2.1.2. Fix sets  $A, B \in \prod_{n=1}^{<\infty} \mathcal{B}([0, 1])$ . Then  $A = \prod_{n=1}^{\infty} A_n$  where  $A_n \in \mathcal{B}([0, 1])$  and there exists  $N \in \mathbb{N}$  such that  $A_n = [0, 1]$  for all  $n > N$ . Then by the definition of the infinite product measure we have that for all  $k > N$

$$\mathbb{M}((\mathbb{S}^{-i}A) \cap (\mathbb{S}^{-k}\mathbb{S}^{-i}B)) = \mathbb{M}(\mathbb{S}^{-i}A)\mathbb{M}(\mathbb{S}^{-k}\mathbb{S}^{-i}B) \text{ for all } i \in \mathbb{N}.$$

Hence (2.1) is valid, and the assumptions of Lemma 2.1.2 are met. This means assumption 1 of Theorem 2.1.1 is valid.

## 2.2 AN EXAMPLE OF CLASSICAL KACS CHAOS

In this section, we will prove the main result of the chapter, Theorem 2.1.1. Many times when deciding whether a sequence of measures is  $\nu$ -chaotic, it is easier to show one of the equivalent formulations of chaos. Sznitman proves various equivalences to the definition of chaos which we list below.

**Theorem 2.2.1.** *[33, Proposition 2.2] Let  $E$  be a separable metric space,  $(\mu_n)_{n=1}^{\infty}$  a sequence of symmetric probability measures on  $E^n$  (as in Definition 1.1.1), and  $\nu$  be a probability measure on  $E$ . The following are equivalent:*

1. *The sequence  $(\mu_n)_{n=1}^{\infty}$  is  $\nu$ -chaotic (as in Definition 1.1.2).*

2. The function  $X_n : E^n \rightarrow M(E)$  defined by  $X_n := \frac{1}{n} \sum_{i=1}^n \delta_{x_i}$  (where  $\delta_x$  stands for the Dirac measure at  $x$ ) converges in law with respect to  $\mu_n$  to the constant random variable  $\nu$ , i.e. for every  $g \in C_b(E)$  we have that

$$\int_{E^n} |(X_n - \nu)g|^2 d\mu_n \xrightarrow{n \rightarrow \infty} 0,$$

where  $C_b(E)$  stands for the space of bounded continuous scalar valued functions on  $E$ .

3. The sequence  $(\mu_n)_{n=1}^\infty$  satisfies Definition 1.1.2 with  $k = 2$ .

In order to construct examples of sequences of symmetric probability measures satisfying condition 2 of Theorem 2.2.1 we will show that it is sufficient to construct a sequence of probability measures (not necessarily symmetric) which satisfy the same condition. Given a measurable space  $(E, \Sigma)$ ,  $n \in \mathbb{N}$ , and a probability measure  $\mu_n$  on the product space  $(E^n, \Sigma^n)$ , we define a symmetric probability measure  $\mu_n^{\text{sym}}$  on  $(E^n, \Sigma^n)$  in the following way: For each  $\sigma \in \Sigma_n$  define  $\Pi_\sigma : E^n \rightarrow E^n$  by  $\Pi_\sigma(x_1, \dots, x_n) = (x_{\sigma(1)}, \dots, x_{\sigma(n)})$ , and define the probability measure  $\mu_n^\sigma : \Sigma^n \rightarrow [0, 1]$  by  $\mu_n^\sigma(A) = \mu_n(\Pi_\sigma(A))$  for each  $A \in \Sigma^n$ . It is easy to verify that for any bounded and measurable function  $f : E^n \rightarrow \mathbb{C}$  we have that

$$\int_{E^n} f d\mu_n^\sigma = \int_{E^n} f^\sigma d\mu_n$$

where  $f^\sigma$  is defined as in (1.4). The symmetric probability measure  $\mu_n^{\text{sym}} : \Sigma^n \rightarrow [0, 1]$  is then defined by

$$\mu_n^{\text{sym}}(A) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mu_n^\sigma(A) \text{ for each } A \in \Sigma^n.$$

For any bounded and measurable function  $f : E^n \rightarrow \mathbb{C}$  which is symmetric (as in (1.3)),

$$\int_{E^n} f d\mu_n^{\text{sym}} = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \int_{E^n} f d\mu_n^\sigma = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \int_{E^n} f^\sigma d\mu_n = \int_{E^n} f d\mu_n. \quad (2.6)$$

For a fixed  $g \in C_b(E)$  and  $\nu \in M(E)$  if we apply (2.6) for  $f := |(X_n - \nu)(g)|^2 : E^n \rightarrow \mathbb{C}$  (which is obviously bounded, measurable, and symmetric), we obtain the following.

**Remark 2.2.2.** *Let  $E$  be a separable metric space,  $\mu_n$  be a probability measure on  $\mathcal{B}(E^n)$ , and  $\nu$  be a probability measure on  $\mathcal{B}(E)$ . For any fixed  $g \in C_b(E)$  we have that*

$$\int_{E^n} |(X_n - \nu)g|^2 d\mu_n \xrightarrow{n \rightarrow \infty} 0 \quad \text{if and only if} \quad \int_{E^n} |(X_n - \nu)g|^2 d\mu_n^{sym} \xrightarrow{n \rightarrow \infty} 0.$$

In order to prove Theorem 2.1.1, we will also need the following.

**Proposition 2.2.3.** *Let  $E$  be a separable metric space,  $\mu$  be a probability measure on  $\mathcal{B}(E)$ , and  $S : E \rightarrow E$  be a Borel measurable map. Assume that*

1. *For every  $A \in \mathcal{B}(E)$ ,*

$$\sup_{i \in \mathbb{N}} |\mu(S^{-i}A \cap S^{-k}S^{-i}A) - \mu(S^{-i}A)\mu(S^{-k}S^{-i}A)| \xrightarrow{k \rightarrow \infty} 0,$$

*and*

2.  *$(E, \mathcal{B}(E), \mu, S)$  is asymptotically stationary with stationary limit  $\nu$ .*

*For every  $n \in \mathbb{N}$  define  $\mu_n : \mathcal{B}(E^n) \rightarrow [0, 1]$  by*

$$\mu_n(A) = \mu\{x \in E : (S(x), \dots, S^n(x)) \in A\}.$$

*Then for every  $g \in C_b(E)$ ,*

$$\int_{E^n} |(X_n - \nu)g|^2 d\mu_n \xrightarrow{n \rightarrow \infty} 0.$$

Notice that for the measure  $\mu_n$  defined in Proposition 2.2.3, and for each  $A \in \mathcal{B}(E^n)$ ,

$$\begin{aligned}
\mu_n^{\text{sym}}(A) &= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mu_n^\sigma(A) = \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mu_n(\Pi_\sigma(A)) \\
&= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mu\{x \in E : (S(x), \dots, S^n(x)) \in \Pi_\sigma(A)\} \\
&= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mu\{x \in E : (S^{\sigma^{-1}(1)}(x), \dots, S^{\sigma^{-1}(n)}(x)) \in A\} \\
&= \frac{1}{n!} \sum_{\sigma \in \Sigma_n} \mu\{x \in E : (S^{\sigma(1)}(x), \dots, S^{\sigma(n)}(x)) \in A\}. \tag{2.7}
\end{aligned}$$

Now the proof of Theorem 2.1.1 follows immediately from Proposition 2.2.3, Remark 2.2.2, the fact that  $\mu_n^{\text{sym}}$  is a symmetric probability measure, Theorem 2.2.1, and (2.7). It remains to prove Proposition 2.2.3.

*Proof of Proposition 2.2.3.* First, let  $g := \chi_{E_1}$  for some  $E_1 \in \mathcal{B}(E)$ . Then, using that for  $1 \leq i < j \leq n$ , we have

$$\int_{E^n} g(x_i)g(x_j)d\mu_n = \mu_n(E^{i-1} \times E_1 \times E^{j-i-1} \times E_1 \times E^{n-j}) = \mu(S^{-i}(E_1) \cap S^{-j}(E_1))$$

we can write

$$\begin{aligned}
&\int_{E^n} |(X_n - \nu)(g)|^2 d\mu_n = \int_{E^n} \left| \frac{1}{n} \sum_{i=1}^n g(x_i) - \int_E g d\nu \right|^2 d\mu_n \\
&= \frac{1}{n^2} \sum_{1 \leq i, j \leq n} \int_{E^n} g(x_i)g(x_j)d\mu_n - \frac{2\nu(E_1)}{n} \sum_{i=1}^n \int_{E^n} g(x_i)d\mu_n + (\nu(E_1))^2 \\
&= \frac{1}{n^2} \sum_{1 \leq i, j \leq n} \mu(S^{-i}(E_1) \cap S^{-j}(E_1)) - \frac{2\nu(E_1)}{n} \sum_{1 \leq i \leq n} \mu(S^{-i}(E_1)) + (\nu(E_1))^2 \\
&= \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \mu(S^{-i}(E_1) \cap S^{-(j-i)}(S^{-i}(E_1))) \tag{2.8} \\
&+ \frac{1}{n^2} \sum_{1 \leq i \leq n} \mu(S^{-i}(E_1)) - \frac{2\nu(E_1)}{n} \sum_{1 \leq i \leq n} \mu(S^{-i}(E_1)) + (\nu(E_1))^2
\end{aligned}$$

We have

$$\frac{1}{n^2} \sum_{1 \leq i \leq n} \mu(S^{-i}(E_1)) \leq \frac{1}{n^2} \sum_{1 \leq i \leq n} 1 \xrightarrow{n \rightarrow \infty} 0$$

and by assumption 2,

$$\frac{2\nu(E_1)}{n} \sum_{1 \leq i \leq n} \mu(S^{-i}(E_1)) \xrightarrow{n \rightarrow \infty} 2(\nu(E_1))^2$$

Also, line (2.8) can be written as

$$\begin{aligned} & \frac{2}{n^2} \sum_{1 \leq i < j \leq n} \mu(S^{-i}(E_1) \cap S^{-(j-i)}(S^{-i}(E_1))) \\ &= \frac{2}{n^2} \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \left[ \mu(S^{-i}(E_1) \cap S^{-k}(S^{-i}(E_1))) - \mu(S^{-i}(E_1))\mu(S^{-k}(S^{-i}(E_1))) \right] \quad (2.9) \\ &+ \frac{2}{n^2} \sum_{i=1}^{n-1} \sum_{k=1}^{n-i} \mu(S^{-i}(E_1))\mu(S^{-k}(S^{-i}(E_1))). \quad (2.10) \end{aligned}$$

First, let us focus on line (2.9). Let  $\epsilon > 0$ . By assumption 1 of Theorem 2.1.1, there exists  $K_0 \in \mathbb{N}$  such that if  $k \geq K_0$  then

$$|\mu(S^{-i}(E_1) \cap S^{-k}(S^{-i}(E_1))) - \mu(S^{-i}(E_1))\mu(S^{-k}(S^{-i}(E_1)))| < \epsilon$$

for every  $i$ . Thus for  $n > K_0 + 1$  we have that line (2.9) is less than or equal to

$$\begin{aligned} & \frac{2}{n^2} \sum_{k=1}^{K_0} \sum_{i=1}^{n-k} \left| \mu(S^{-i}(E_1) \cap S^{-k}(S^{-i}(E_1))) - \mu(S^{-i}(E_1))\mu(S^{-k}(S^{-i}(E_1))) \right| \\ &+ \frac{2}{n^2} \sum_{k=K_0+1}^{n-1} \sum_{i=1}^{n-k} \epsilon. \end{aligned}$$

Since the first double sum has at most  $K_0^2 + \frac{K_0 n}{2}$  terms, the second double sum has at most  $\frac{(n-K_0)n}{2}$  terms, and

$$0 \leq \left| \mu(S^{-i}(E_1) \cap S^{-k}(S^{-i}(E_1))) - \mu(S^{-i}(E_1))\mu(S^{-k}(S^{-i}(E_1))) \right| \leq 2,$$

we have that line (2.9) is less than or equal to

$$\frac{4(K_0^2 + K_0 n/2)}{n^2} + \frac{\epsilon(n - K_0)n}{n^2} \xrightarrow{n \rightarrow \infty} \epsilon.$$

Now we will focus on line (2.10). By assumption 2, there exists  $N_0 \in \mathbb{N}$  such that if  $n \geq N_0$  then

$$|\mu(S^{-n}(E_1) - \nu(E_1))| < \epsilon.$$

Hence, line (2.10) is equal to

$$\frac{2}{n^2} \sum_{i=1}^{N_0} \sum_{k=1}^{n-i} \mu(S^{-i}(E_1))\mu(S^{-k}(S^{-i}(E_1))) + \frac{2}{n^2} \sum_{i=N_0+1}^{n-1} \sum_{k=1}^{n-i} \mu(S^{-i}(E_1))\mu(S^{-k}(S^{-i}(E_1)))$$

The first double sum has at most  $N_0^2 + \frac{N_0 n}{2}$  terms, and thus,

$$\frac{2}{n^2} \sum_{i=1}^{N_0} \sum_{k=1}^{n-i} \mu(S^{-i}(E_1))\mu(S^{-k}(S^{-i}(E_1))) \leq \frac{2(N_0^2 + N_0 n/2)}{n^2} \xrightarrow{n \rightarrow \infty} 0$$

The second double sum can be rewritten as

$$\begin{aligned} & \frac{2}{n^2} \sum_{i=N_0+1}^{n-1} \sum_{k=1}^{n-i} \mu(S^{-i}(E_1))\mu(S^{-k}(S^{-i}(E_1))) \\ &= \frac{2}{n^2} \frac{(n-1-N_0)^2}{2} (\nu(E_1))^2 \\ &+ \frac{2}{n^2} \sum_{i=N_0+1}^{n-1} \sum_{k=1}^{n-i} \left[ \mu(S^{-i}E_1)[\mu(S^{-k}S^{-i}E_1) - \nu(E_1)] + [\mu(S^{-i}E_1) - \nu(E_1)]\nu(E_1) \right]. \end{aligned}$$

We have

$$\frac{2}{n^2} \frac{(n-1-N_0)^2}{2} (\nu(E_1))^2 \xrightarrow{n \rightarrow \infty} (\nu(E_1))^2$$

and

$$\begin{aligned} & \frac{2}{n^2} \sum_{i=N_0+1}^{n-1} \sum_{k=1}^{n-i} \left[ \mu(S^{-i}E_1)[\mu(S^{-k}S^{-i}E_1) - \nu(E_1)] + [\mu(S^{-i}E_1) - \nu(E_1)]\nu(E_1) \right] \\ & \leq \frac{2}{n^2} \frac{(n-1)^2}{2} \epsilon + \frac{2}{n^2} \frac{(n-1)^2}{2} \epsilon \xrightarrow{n \rightarrow \infty} (\nu(E_1))^2 + 2\epsilon. \end{aligned}$$

Hence line (2.10) converges to  $(\nu(E_1))^2$  as  $n \rightarrow \infty$ . This shows

$$\int_{E^n} |(X_n - \nu)(g)|^2 d\mu_n \xrightarrow{n \rightarrow \infty} 0$$

for  $g = \chi_{E_1}$ .

Now, consider the simple function  $g := \sum_{k=1}^K \alpha_k \chi_{E_k}$ . We have

$$\begin{aligned} & \left( \int_{E^n} |(X_n - \nu)(g)|^2 d\mu_n \right)^{1/2} = \left( \int_{E^n} \left| (X_n - \nu) \left( \sum_{k=1}^K \alpha_k \chi_{E_k} \right) \right|^2 d\mu_n \right)^{1/2} \\ &= \left( \int_{E^n} \left| \sum_{k=1}^K \alpha_k [(X_n - \nu)\chi_{E_k}] \right|^2 d\mu_n \right)^{1/2} \\ &\leq \sum_{k=1}^K |\alpha_k| \left( \int_{E^n} |(X_n - \nu)\chi_{E_k}|^2 d\mu_n \right)^{1/2} \xrightarrow{n \rightarrow \infty} 0 \end{aligned}$$

since the limit is zero for each characteristic function and we have a finite sum.

Finally, let  $g \in C_b(E)$  and  $\epsilon > 0$ . There exists a simple function  $G$  such that  $|g(x) - G(x)| < \epsilon$  for all  $x \in E$ . We have that

$$\begin{aligned}
& \left( \int_{E^n} |(X_n - \nu)(g - G)|^2 d\mu_n \right)^{1/2} \\
& \leq \left( \int_{E^n} |X_n(g - G)|^2 d\mu_n \right)^{1/2} + \left( \int_{E^n} |\nu(g - G)|^2 d\mu_n \right)^{1/2} \\
& = \left( \int_{E^n} \left| \frac{1}{n} \sum_{i=1}^n (g - G)(x_i) \right|^2 d\mu_n \right)^{1/2} + |\nu(g - G)| \\
& \leq \frac{1}{n} \sum_{i=1}^n \left( \int_{E^n} |(g - G)(x_i)|^2 d\mu_n \right)^{1/2} + \left| \int_E (g - G) d\nu \right| \leq \sum_{i=1}^n \frac{\epsilon}{n} + \epsilon = 2\epsilon
\end{aligned}$$

and since  $G$  is a simple function,

$$\left( \int_{E^n} |(X_n - \nu)G|^2 d\mu_n \right)^{1/2} \xrightarrow{n \rightarrow \infty} 0.$$

Therefore by the triangle inequality on  $L^2(E^n, d\mu_n)$  norm we obtain since  $\epsilon$  is arbitrary,

$$\left( \int_{E^n} |(X_n - \nu)g|^2 d\mu_n \right)^{1/2} \xrightarrow{n \rightarrow \infty} 0.$$

□

# CHAPTER 3

## QUANTUM KAC'S CHAOS

### 3.1 DEFINITIONS AND EXAMPLES

Sznitman used probabilistic methods to show existence [32] and uniqueness [33] to the homogeneous Boltzmann equation. Theorem 2.2.1 [33, Proposition 2.2] was important in his proofs. The main result of this chapter is Theorem 3.2.1 which is a quantum version of Theorem 2.2.1 [33, Proposition 2.2]. In this section, we introduce the quantum versions of Definition 1.1.1 and Definition 1.1.2. Instead of considering probability measures on a measure space, we consider density operators on a Hilbert space.

Throughout this chapter and the next,  $\mathbb{H}$  will denote an arbitrary Hilbert space,  $\mathcal{B}(\mathbb{H})$  will denote the set of bounded operators on  $\mathbb{H}$ , and  $\mathcal{D}(\mathbb{H})$  will denote the set of density operators on  $\mathbb{H}$ . The identity operator on  $\mathcal{B}(\mathbb{H})$  will be denoted by 1. For any operator  $A \in \mathcal{B}(\mathbb{H})$  and  $k \in \mathbb{N}$ ,  $A^{\otimes k}$  will denote the tensor product of  $A$  with itself  $k$  times. In addition, for any  $A \in \mathcal{B}(\mathbb{H})$ ,  $\|A\|_\infty$  will denote the  $\mathcal{B}(\mathbb{H})$  norm of  $A$ . If  $A$  is a trace class operator on  $\mathbb{H}$ , then  $\|A\|_1$  will denote the trace class norm of  $A$ .

For any  $k, N \in \mathbb{N}$  with  $k \leq N$ , and  $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$ , we will denote by  $\rho_N^{(k)} \in \mathcal{D}(\mathbb{H}^{\otimes k})$  the partial trace of  $\rho_N$  where we trace out all but the first  $k$  copies of  $\mathbb{H}$  (as in Definition 1.3.7). In addition, for an index set  $A \subset \{1, \dots, N\}$ , we will denote by  $\text{tr}_A(\rho_N)$  the partial trace of  $\rho_N$  where we trace out the copies of  $\mathbb{H}$  indexed by elements of  $A$ . Notice that  $\text{tr}_{[k+1, N]}(\rho_N) = \rho_N^{(k)}$ .

We first have to extend the definition of symmetric measures (Definition 1.1.1)



to density operators. The following is the quantum version of symmetry (Definition 1.1.1) we will use in this thesis.

**Definition 3.1.1.** *Let  $N \in \mathbb{N}$ . A density operator  $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$  is **symmetric** if and only if for every  $A_1, \dots, A_N \in \mathcal{B}(\mathbb{H})$  and for every permutation  $\pi \in \Sigma_N$ ,*

$$\text{tr}(A_1 \otimes \dots \otimes A_N \rho_N) = \text{tr}(A_{\pi(1)} \otimes \dots \otimes A_{\pi(N)} \rho_N).$$

This is not the same formulation of the definition of symmetric density operators given by Gottlieb [15]. To obtain the formulation given by Gottlieb [15], for  $N \in \mathbb{N}$ , define for each  $\pi \in \Sigma_N$  the unitary operator  $U_\pi^{[N]} \in \mathcal{B}(H^{\otimes N})$  by

$$U_\pi^{[N]}(x_1 \otimes \dots \otimes x_N) = x_{\pi^{-1}(1)} \otimes \dots \otimes x_{\pi^{-1}(N)}. \quad (3.1)$$

A density operator  $\rho_N \in \mathcal{B}(\mathbb{H}^{\otimes N})$  is symmetric according to [15] if and only if  $U_\pi^{[N]} \rho_N = \rho_N U_\pi^{[N]}$  for every  $\pi \in \Sigma_N$ . However, Gottlieb's definition of symmetric densities is equivalent to Definition 3.1.1 as we show next.

**Proposition 3.1.2.** *Let  $N \in \mathbb{N}$  and  $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$ . Then  $\rho_N$  is symmetric (as in Definition 3.1.1) if and only if  $U_\pi^{[N]} \rho_N = \rho_N U_\pi^{[N]}$  for all  $\pi \in \Sigma_N$ .*

*Proof.* ( $\Rightarrow$ ) Let  $\pi \in \Sigma_N$ . Then

$$\begin{aligned} \text{tr}(A_1 \otimes \dots \otimes A_N \rho_N) &= \text{tr}(A_{\pi(1)} \otimes \dots \otimes A_{\pi(N)} \rho_N) \\ &= \text{tr}(U_{\pi^{-1}}^{[N]}(A_1 \otimes \dots \otimes A_N) U_\pi^{[N]} \rho_N) \\ &= \text{tr}((A_1 \otimes \dots \otimes A_N) U_\pi^{[N]} \rho_N U_{\pi^{-1}}^{[N]}) \end{aligned} \quad (3.2)$$

for any  $A_1, \dots, A_N \in \mathcal{B}(\mathbb{H}^{\otimes N})$ .

Let  $(e_i)_{i \in I}$  be an orthonormal basis of  $\mathbb{H}$ . Then  $(e_{i_1} \otimes e_{i_2} \otimes \dots \otimes e_{i_N})_{i_1, i_2, \dots, i_N \in I}$  is an orthonormal basis of  $\mathbb{H}^{\otimes N}$ . Now assume that  $U_\pi^{[N]} \rho_N U_{\pi^{-1}}^{[N]} \neq \rho_N$ . This implies that for some  $(j_1, \dots, j_k) \in I \times \dots \times I$  we have  $U_\pi^{[N]} \rho_N U_{\pi^{-1}}^{[N]}(e_{j_1} \otimes \dots \otimes e_{j_N}) \neq \rho_N(e_{j_1} \otimes \dots \otimes e_{j_N})$ , hence there exists  $(k_1, \dots, k_N) \in I \times \dots \times I$  such that

$$\langle e_{k_1} \otimes \dots \otimes e_{k_N}, U_\pi^{[N]} \rho_N U_{\pi^{-1}}^{[N]} e_{j_1} \otimes \dots \otimes e_{j_N} \rangle \neq \langle e_{k_1} \otimes \dots \otimes e_{k_N}, \rho_N e_{j_1} \otimes \dots \otimes e_{j_N} \rangle.$$

This implies that

$$\begin{aligned}
& \operatorname{tr} \left( U_{\pi}^{[N]} \rho_N U_{\pi^{-1}}^{[N]} |e_{j_1}\rangle \langle e_{k_1}| \otimes |e_{j_2}\rangle \langle e_{k_2}| \otimes \cdots \otimes |e_{j_N}\rangle \langle e_{k_N}| \right) \\
&= \langle e_{k_1} \otimes \cdots \otimes e_{k_N}, U_{\pi}^{[N]} \rho_N U_{\pi^{-1}}^{[N]} e_{j_1} \otimes \cdots \otimes e_{j_N} \rangle \\
&\neq \langle e_{k_1} \otimes \cdots \otimes e_{k_N}, \rho_N e_{j_1} \otimes \cdots \otimes e_{j_N} \rangle \\
&= \operatorname{tr} (\rho_N |e_{j_1}\rangle \langle e_{k_1}| \otimes |e_{j_2}\rangle \langle e_{k_2}| \otimes \cdots \otimes |e_{j_N}\rangle \langle e_{k_N}|)
\end{aligned}$$

which contradicts Equation (3.2). Therefore  $U_{\pi^{-1}}^{[N]} \rho_N U_{\pi}^{[N]} = \rho_N$ .

( $\Leftarrow$ ) For each  $\pi \in \Sigma_N$ ,

$$\begin{aligned}
& \operatorname{tr}(A_1 \otimes \cdots \otimes A_N \rho_N) = \operatorname{tr}(A_1 \otimes \cdots \otimes A_N U_{\pi}^{[N]} \rho_N U_{\pi^{-1}}^{[N]}) \\
&= \operatorname{tr}(U_{\pi^{-1}}^{[N]} (A_1 \otimes \cdots \otimes A_N) U_{\pi}^{[N]} \rho_N) = \operatorname{tr}(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(N)} \rho_N).
\end{aligned}$$

□

Some examples of symmetric density operators are as follows.

**Example 3.1.3.** Let  $\rho \in \mathcal{D}(\mathbb{H})$ . For any  $N \in \mathbb{N}$ , define  $\rho_N := \rho^{\otimes N}$ . It is clear that  $\rho_N$  is symmetric.

**Example 3.1.4.** Let  $N \in \mathbb{N}$  and  $B_1, \dots, B_N \in \mathcal{D}(\mathbb{H})$ . Then

$$\rho_N := \frac{1}{N!} \sum_{\sigma \in \Sigma_N} B_{\sigma(1)} \otimes \cdots \otimes B_{\sigma(N)} \in \mathcal{D}(\mathbb{H}^{\otimes N})$$

is symmetric.

**Example 3.1.5.** Let  $E$  be a separable topological space,  $D : E \rightarrow \mathcal{D}(\mathbb{H})$  be continuous,  $N \in \mathbb{N}$ , and  $\mu_N \in M(E^N)$  be symmetric. Then

$$D_N := \int_{E^N} D(\omega_1) \otimes D(\omega_2) \otimes \cdots \otimes D(\omega_N) d\mu_N(\omega_1, \omega_2, \dots, \omega_N)$$

exists as a Bochner integral and  $D_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$  is symmetric.

*Proof.* For each  $\pi \in \Sigma_N$  and  $A_1, \dots, A_N \in \mathcal{B}(\mathbb{H})$ ,

$$\begin{aligned}
& \text{tr}(A_1 \otimes \cdots \otimes A_N D_N) \\
&= \text{tr}(A_1 \otimes \cdots \otimes A_N \int_{E^N} D(\omega_1) \otimes D(\omega_2) \otimes \cdots \otimes D(\omega_N) d\mu_N) \\
&= \int_{E^N} \text{tr}(A_1 D(\omega_1)) \text{tr}(A_2 D(\omega_2)) \cdots \text{tr}(A_N D(\omega_N)) d\mu_N \\
&= \int_{E^N} \text{tr}(A_1 D(\omega_{\pi^{-1}(1)})) \text{tr}(A_2 D(\omega_{\pi^{-1}(2)})) \cdots \text{tr}(A_N D(\omega_{\pi^{-1}(N)})) d\mu_N \\
&= \int_{E^N} \text{tr}(A_{\pi(1)} D(\omega_1)) \text{tr}(A_{\pi(2)} D(\omega_2)) \cdots \text{tr}(A_{\pi(N)} D(\omega_N)) d\mu_N \\
&= \text{tr}(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(N)} \int_{E^N} D(\omega_1) \otimes D(\omega_2) \otimes \cdots \otimes D(\omega_N) d\mu_N) \\
&= \text{tr}(A_{\pi(1)} \otimes \cdots \otimes A_{\pi(N)} D_N).
\end{aligned}$$

□

The following is the quantum version of Definition 1.1.2 that we will use in this thesis.

**Definition 3.1.6.** Let  $(\rho_N)_{N=1}^\infty$  be a sequence of symmetric density operators such that  $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$  for each  $N \in \mathbb{N}$ ,  $\rho \in \mathcal{D}(\mathbb{H})$  be a density operator, and  $k \in \mathbb{N}$ . Then  $(\rho_N)_{N=1}^\infty$  is  $k - \rho$ -**chaotic** if and only if for all  $A_1, \dots, A_k \in \mathcal{B}(\mathbb{H})$ ,

$$\text{tr}(A_1 \otimes \cdots \otimes A_k \otimes 1^{\otimes(N-k)} \rho_N) \xrightarrow{N \rightarrow \infty} \prod_{j=1}^k \text{tr}(\rho A_j).$$

We say that  $(\rho_N)_{N=1}^\infty$  is  $\rho$ -**chaotic** if and only if  $(\rho_N)_{N=1}^\infty$  is  $k - \rho$ -chaotic for all  $k \geq 1$ .

Next we give many equivalent formulations of this definition. We will use the fact that the partial trace of a density operator is a density operator (Theorem 1.3.10).

**Proposition 3.1.7.** Let  $(\rho_N)_{N=1}^\infty$  be a sequence of symmetric density matrices such that  $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$  for each  $N \in \mathbb{N}$ ,  $\rho \in \mathcal{D}(\mathbb{H})$ , and  $k \in \mathbb{N}$ . The following are equivalent

1.  $(\rho_N)_{N=1}^\infty$  is  $k - \rho$ -chaotic,

2.  $\text{tr} \left( (\rho_N^{(k)} - \rho^{\otimes k}) A_1 \otimes \cdots \otimes A_k \right) \xrightarrow{N \rightarrow \infty} 0$  for all  $A_1, \dots, A_k \in \mathcal{B}(\mathbb{H})$ ,
3.  $\langle s, (\rho_N^{(k)} - \rho^{\otimes k}) t \rangle \xrightarrow{N \rightarrow \infty} 0$  for all  $s, t \in \mathbb{H}^{\otimes k}$ , and
4.  $\text{tr} |\rho_N^{(k)} - \rho^{\otimes k}| \xrightarrow{N \rightarrow \infty} 0$ .

*Proof.* ((1)  $\Leftrightarrow$  (2)) This is obvious. See property (1.12) and Attal [2, Theorem 2.28].

((2)  $\Rightarrow$  (3)) Let  $\epsilon > 0$ , and  $t, s \in \mathbb{H}^{\otimes k}$ . Without loss of generality, we can make  $\|t\| = \|s\| = 1$ . Fix an orthonormal basis  $(e_i)_{i \in I}$  of  $\mathbb{H}$ . Then  $(e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k})_{i_1, i_2, \dots, i_k \in I}$  is an orthonormal basis of  $\mathbb{H}^{\otimes k}$ , and we can write

$$\begin{aligned} t &= \sum_{i_1, i_2, \dots, i_k \in I} t_{i_1 i_2 \dots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \quad \text{and} \\ s &= \sum_{j_1, j_2, \dots, j_k \in I} s_{j_1 j_2 \dots j_k} e_{j_1} \otimes e_{j_2} \otimes \cdots \otimes e_{j_k} \end{aligned}$$

with

$$\sum_{i_1, i_2, \dots, i_k \in I} |t_{i_1 i_2 \dots i_k}|^2 = 1 \quad \text{and} \quad \sum_{j_1, j_2, \dots, j_k \in I} |s_{j_1 j_2 \dots j_k}|^2 = 1.$$

For every finite index set  $J \subset I$ , let

$$\begin{aligned} t_{I \setminus J} &= \sum_{i_1, i_2, \dots, i_k \in I \setminus J} t_{i_1 i_2 \dots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k}, \\ t_J &= \sum_{i_1, i_2, \dots, i_k \in J} t_{i_1 i_2 \dots i_k} e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes e_{i_k} \end{aligned}$$

and similarly define  $s_{I \setminus J}$  and  $s_J$ .

Choose a finite index set  $J \subset I$  such that  $\|t_{I \setminus J}\| < \epsilon$  and  $\|s_{I \setminus J}\| < \epsilon$ . Then we have

$$\begin{aligned} &\langle t, (\rho_N^{(k)} - \rho^{\otimes k}) s \rangle \\ &= \langle t_J, (\rho_N^{(k)} - \rho^{\otimes k}) s_J \rangle \end{aligned} \tag{3.3}$$

$$+ \langle t_J, (\rho_N^{(k)} - \rho^{\otimes k}) s_{I \setminus J} \rangle + \langle t_{I \setminus J}, (\rho_N^{(k)} - \rho^{\otimes k}) s_J \rangle + \langle t_{I \setminus J}, (\rho_N^{(k)} - \rho^{\otimes k}) s_{I \setminus J} \rangle. \tag{3.4}$$

We have that  $\|\rho_N^k - \rho^{\otimes k}\|_\infty \leq \|\rho_N^k - \rho^{\otimes k}\|_1 \leq 2$ . Thus, using Cauchy-Schwarz inequalities, line (3.4) is bounded independent of  $N$  by

$$\begin{aligned} & \|t_J\| \left\| \left( \rho_N^{(k)} - \rho^{\otimes k} \right) \right\|_\infty \|s_{I \setminus J}\| + \|t_{I \setminus J}\| \left\| \left( \rho_N^{(k)} - \rho^{\otimes k} \right) \right\|_\infty \|s_J\| \\ & + \|t_{I \setminus J}\| \left\| \left( \rho_N^{(k)} - \rho^{\otimes k} \right) \right\|_\infty \|s_{I \setminus J}\| < 2\epsilon + 2\epsilon + 2\epsilon^2 = 4\epsilon + 2\epsilon^2. \end{aligned}$$

Line (3.3) is equal to

$$\begin{aligned} & \sum_{\substack{i_1, i_2, \dots, i_k, \\ j_1, j_2, \dots, j_k \in J}} \overline{t_{i_1 i_2 \dots i_k} s_{j_1 j_2 \dots j_k}} \langle e_{i_1} \otimes \dots \otimes e_{i_k}, \left( \rho_N^{(k)} - \rho^{\otimes k} \right) e_{j_1} \otimes \dots \otimes e_{j_k} \rangle \\ & = \sum_{\substack{i_1, i_2, \dots, i_k, \\ j_1, j_2, \dots, j_k \in J}} \overline{t_{i_1 i_2 \dots i_k} s_{j_1 j_2 \dots j_k}} \operatorname{tr} \left( \left( \rho_N^{(k)} - \rho^{\otimes k} \right) |e_{j_1} \otimes \dots \otimes e_{j_k}\rangle \langle e_{i_1} \otimes \dots \otimes e_{i_k}| \right) \\ & = \sum_{\substack{i_1, i_2, \dots, i_k, \\ j_1, j_2, \dots, j_k \in J}} \overline{t_{i_1 i_2 \dots i_k} s_{j_1 j_2 \dots j_k}} \operatorname{tr} \left( \left( \rho_N^{(k)} - \rho^{\otimes k} \right) |e_{j_1}\rangle \langle e_{i_1}| \otimes \dots \otimes |e_{j_k}\rangle \langle e_{i_k}| \right). \quad (3.5) \end{aligned}$$

By assumption, there exists an  $N_1 \in \mathbb{N}$  such that for all  $N \geq N_1$ , line (3.5) is bounded by  $\epsilon$ . Therefore, for every  $N \geq N_1$ ,

$$\left| \langle t, \left( \rho_N^{(k)} - \rho^{\otimes k} \right) s \rangle \right| < \epsilon + 4\epsilon + 2\epsilon^2 = 5\epsilon + 2\epsilon^2.$$

((3)  $\Leftrightarrow$  (4)) Wehrl [36, Theorem 3] proved that a sequence,  $(D_N)_{N=1}^\infty \subset \mathcal{D}(\mathbb{K})$ , of density operators on a Hilbert space  $\mathbb{K}$  converges in the weak operator topology to a density operator  $D \in \mathcal{D}(\mathbb{K})$  if and only if it converges in norm, i.e.  $\operatorname{tr}|D_N - D| \xrightarrow{N \rightarrow \infty} 0$ . Our result follows by letting  $\mathbb{K} = \mathbb{H}^{\otimes k}$ ,  $D_N = \rho_N^{(k)}$  for each  $N$ , and  $D = \rho^{\otimes k}$ . (Wehrl [36] assumed in his paper that all Hilbert spaces considered are separable. However, the examination of his proof shows that the assumption of separability of the Hilbert space is not needed in Theorem 3.)

((4)  $\Rightarrow$  (2)) This is obvious. □

**Remark 3.1.8.** Notice that condition (3) of Proposition 3.1.7 is equivalent to

$$\operatorname{tr} \left( \left( \rho_N^{(k)} - \rho^{\otimes k} \right) A \right) \xrightarrow{N \rightarrow \infty} 0 \text{ for all finite rank operators } A \in \mathcal{B}(\mathbb{H}^{\otimes k}). \quad (3.6)$$

Furthermore since (3.6) is weaker than

$$\operatorname{tr} \left( \left( \rho_N^{(k)} - \rho^{\otimes k} \right) A \right) \xrightarrow{N \rightarrow \infty} 0 \text{ for all } A \in \mathcal{B}(\mathbb{H}^{\otimes k}) \quad (3.7)$$

which is further weaker than condition (4) of Proposition 3.1.7, we have that conditions (3.6) and (3.7) are equivalent to the conditions presented in Proposition 3.1.7.

Condition (4) in Proposition 3.1.7 appears in [31, Theorem 5.7]. Gottlieb [15] used this condition for all  $k$  to define that “ $\rho_N$  is  $\rho$ -chaotic”. Proposition 3.1.7 shows that Gottlieb’s definition agrees with ours. We will now give some examples of chaotic sequences.

**Example 3.1.9.** Let  $\rho \in \mathcal{B}(\mathbb{H})$ . For each  $N \in \mathbb{N}$ , define  $\rho_N := \rho^{\otimes N}$ . Then it is clear that  $(\rho_N)_{N=1}^{\infty}$  is  $\rho$ -chaotic.

The following example due to Gottlieb [15, Lemma 1.3.2] gives a way of constructing a chaotic sequence of density operators from any classically chaotic sequence of probability measures.

**Example 3.1.10.** Let  $E$  be a separable topological space, and for each  $N \in \mathbb{N}$  let  $\mu_N \in M(E^N)$  and  $\mu \in M(E)$  such that  $(\mu_N)_{N=1}^{\infty}$  is  $\mu$ -chaotic. Let  $D : E \rightarrow \mathcal{D}(\mathbb{H})$  be a continuous function. Then for  $N \in \mathbb{N}$ ,

$$D_N := \int_{E^N} D(\omega_1) \otimes D(\omega_2) \otimes \cdots \otimes D(\omega_N) d\mu_N(\omega_1, \omega_2, \dots, \omega_N)$$

and  $\bar{D} := \int_E D(\omega) d\mu$  exist as Bochner integrals,  $D_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$  is symmetric,  $\bar{D} \in \mathcal{D}(\mathbb{H})$ , and  $(D_N)_{N=1}^{\infty}$  is  $\bar{D}$ -chaotic.

*Proof.* For any  $k \geq 1$  and  $A_1, \dots, A_k \in \mathcal{B}(\mathbb{H})$ ,

$$\begin{aligned} & \operatorname{tr}(A_1 \otimes \cdots \otimes A_k \otimes 1^{\otimes(N-k)} \int D(\omega_1) \otimes \cdots \otimes D(\omega_N) d\mu_N) \\ &= \operatorname{tr} \left( \int A_1 D(\omega_1) \otimes \cdots \otimes A_k D(\omega_k) \otimes D(\omega_{k+1}) \otimes \cdots \otimes D(\omega_N) d\mu_N \right) \\ &= \int \operatorname{tr}(A_1 D(\omega_1)) \cdots \operatorname{tr}(A_k D(\omega_k)) d\mu_N \end{aligned}$$

which converges to

$$\int \operatorname{tr}(A_1 D(\omega_1)) \cdots \operatorname{tr}(A_k D(\omega_k)) d\mu^{\otimes k} = \prod_{j=1}^k \operatorname{tr} \left( A_j \int D(\omega) d\mu \right)$$

as  $N$  approaches infinity by Definition 1.1.2. □

### 3.2 A QUANTUM VERSION OF SZNITMAN'S RESULT

In this section, we are ready to prove the analogous statement to Theorem 2.2.1 [33, Proposition 2.2] for chaotic sequences of density operators.

**Theorem 3.2.1.** *Let  $(\rho_N)_{N=1}^\infty$  be a sequence of symmetric density operators such that  $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$  for each  $N \in \mathbb{N}$ , and let  $\rho \in \mathcal{D}(\mathbb{H})$ . Then the following are equivalent.*

(1)  $(\rho_N)_{N=1}^\infty$  is  $k - \rho$ -chaotic for all  $k \in \mathbb{N}$ ,

(2)  $(\rho_N)_{N=1}^\infty$  is  $2 - \rho$ -chaotic, and

(3) for each  $A \in \mathcal{B}(\mathbb{H})$ ,

$$\operatorname{tr} \left( \left| \frac{1}{N} \sum_{j=1}^N 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)} - \operatorname{tr}(A\rho) 1^{\otimes N} \right|^2 \rho_N \right) \xrightarrow{N \rightarrow \infty} 0.$$

**Definition 3.2.2.** For  $N \in \mathbb{N}$ , define  $X_N : \mathcal{B}(\mathbb{H}) \rightarrow \mathcal{B}(\mathbb{H}^{\otimes N})$  by

$$X_N(A) = \frac{1}{N} \sum_{j=1}^N 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)}.$$

The function  $X_N$  is studied in [14] and is called a **quantum empirical measure**. The above theorem and [33, Proposition 2.2] gives more justifications for the choice of this term.

*Proof.* ((1)  $\Rightarrow$  (2)) This is obvious.

((2)  $\Rightarrow$  (3)) Let  $A \in \mathcal{B}(\mathbb{H})$ . Notice that

$$\begin{aligned} & \operatorname{tr} \left( \left| \frac{1}{N} \sum_{j=1}^N 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)} - \operatorname{tr}(A\rho) 1^{\otimes N} \right|^2 \rho_N \right) \\ &= \operatorname{tr} \left( \left( \frac{1}{N} \sum_{i=1}^N 1^{\otimes(i-1)} \otimes A^* \otimes 1^{\otimes(N-i)} - \overline{\operatorname{tr}(A\rho)} 1^{\otimes N} \right) \right. \\ & \quad \left. \left( \frac{1}{N} \sum_{j=1}^N 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)} - \operatorname{tr}(A\rho) 1^{\otimes N} \right) \rho_N \right). \end{aligned} \tag{3.8}$$

By distributing, we obtain that the last expression is equal to

$$\frac{1}{N^2} \text{tr} \left( \sum_{i,j=1}^N (1^{\otimes(i-1)} \otimes A^* \otimes 1^{\otimes(N-i)}) (1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)}) \rho_N \right) \quad (3.9)$$

$$- \frac{\text{tr}(A\rho)}{N} \text{tr} \left( \sum_{j=1}^N 1^{\otimes(j-1)} \otimes A^* \otimes 1^{\otimes(N-j)} \rho_N \right) \quad (3.10)$$

$$- \frac{\overline{\text{tr}(A\rho)}}{N} \text{tr} \left( \sum_{j=1}^N 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j)} \rho_N \right) \quad (3.11)$$

$$+ |\text{tr}(A\rho)|^2. \quad (3.12)$$

We will obtain that the sum of lines (3.9), (3.10), (3.11), and (3.12) goes to zero as  $N$  approaches infinity. To evaluate (3.9), we consider three cases: when  $i = j$ , when  $i < j$ , and when  $j < i$ . If  $i = j$ , then by symmetry of  $\rho_N$ ,

$$\begin{aligned} \frac{1}{N^2} \sum_{j=1}^N \text{tr}(1^{\otimes(j-1)} \otimes |A|^2 \otimes 1^{\otimes(N-j)} \rho_N) &= \frac{1}{N} \text{tr}(|A|^2 \otimes 1^{\otimes(N-1)} \rho_N) \\ &\leq \frac{1}{N} \| |A|^2 \otimes 1^{\otimes(N-1)} \|_{\infty} \| \rho_N \|_1 \leq \frac{\| |A|^2 \|_{\infty}}{N}, \end{aligned}$$

which goes to zero as  $N$  approaches infinity. If  $i < j$ , then by symmetry of  $\rho_N$ ,

$$\begin{aligned} \frac{1}{N^2} \sum_{i < j} \text{tr} (1^{\otimes(i-1)} \otimes A^* \otimes 1^{\otimes(j-i-1)} \otimes A \otimes 1^{\otimes(N-j)} \rho_N) \\ &= \frac{1}{N^2} \frac{N!}{2(N-2)!} \text{tr}(A^* \otimes A \otimes 1^{\otimes(N-2)} \rho_N) \\ &= \frac{N-1}{2N} \text{tr}(A^* \otimes A \otimes 1^{\otimes(N-2)} \rho_N) \xrightarrow[N \rightarrow \infty]{2-\rho\text{-chaotic}} \frac{1}{2} \text{tr}(A\rho) \text{tr}(A^* \rho) = \frac{1}{2} |\text{tr}(A\rho)|^2. \end{aligned}$$

If  $j < i$  we obtain exactly the same limit. Thus, we have that line (3.9) converges to  $|\text{tr}(A\rho)|^2$  as  $N$  approaches infinity.

Using symmetry of  $\rho_N$  and by assumption, we obtain the limit of line (3.10),

$$\begin{aligned} \frac{-\text{tr}(A\rho)}{N} \text{tr} \left( \sum_{j=1}^N 1^{\otimes(j-1)} \otimes A^* \otimes 1^{\otimes(N-j)} \rho_N \right) \\ = -\text{tr}(A\rho) \text{tr}(A^* \otimes 1^{\otimes(N-1)} \rho_N) \xrightarrow[N \rightarrow \infty]{} -\text{tr}(A\rho) \text{tr}(A^* \rho) = -|\text{tr}(A\rho)|^2, \end{aligned}$$

where in the last limit, we used the obvious fact that if  $\rho_N$  is  $2 - \rho$ -chaotic then it is  $1 - \rho$ -chaotic. Similarly, line (3.11) converges to  $-|\text{tr}(A\rho)|^2$  as  $N$  approaches infinity.



Therefore, the sum of lines (3.9), (3.10), (3.11), and (3.12) converge to

$$|\operatorname{tr}(A\rho)|^2 - |\operatorname{tr}(A\rho)|^2 - |\operatorname{tr}(A\rho)|^2 + |\operatorname{tr}(A\rho)|^2 = 0,$$

and line (3.8) converges to 0 as  $N$  approaches infinity.

((3)  $\Rightarrow$  (1)) Let  $k \in \mathbb{N}$  and  $A_1, \dots, A_k \in \mathcal{B}(\mathbb{H})$ . Then

$$\left| \operatorname{tr} \left( A_1 \otimes \cdots \otimes A_k \otimes 1^{\otimes(N-k)} \rho_N \right) - \prod_{j=1}^k \operatorname{tr}(\rho A_j) \right| \leq \quad (3.13)$$

$$\left| \operatorname{tr} \left( A_1 \otimes \cdots \otimes A_k \otimes 1^{\otimes(N-k)} \rho_N \right) \right. \quad (3.14)$$

$$\left. - \operatorname{tr} \left( \prod_{j=1}^k \frac{1}{N} \left( A_j \otimes 1^{\otimes(N-1)} + 1 \otimes A_j \otimes 1^{\otimes(N-2)} + \cdots + 1^{\otimes(N-1)} \otimes A_j \right) \rho_N \right) \right|$$

$$+ \left| \operatorname{tr} \left( \prod_{j=1}^k \frac{1}{N} \left( A_j \otimes 1^{\otimes(N-1)} + \cdots + 1^{\otimes(N-1)} \otimes A_j \right) \rho_N \right) - \prod_{j=1}^k \operatorname{tr}(\rho A_j) \right|. \quad (3.15)$$

We label the first and second lines after the inequality by (3.14). Our goal will be to show that the sum of lines (3.14) and (3.15) goes to 0 as  $N$  approaches infinity.

For lines (3.14), for  $k \leq N$  we define  $E_{k,N}$  to be the set of embeddings (i.e. one-to-one maps)

$\sigma : \{1, \dots, k\} \rightarrow \{1, \dots, N\}$ . Notice that  $\#E_{k,N} = \frac{N!}{(N-k)!}$ . Furthermore, for  $\sigma \in E_{k,N}$  and  $i \in \{1, \dots, N\}$ , define

$$A_{\sigma,i} := \begin{cases} A_j & \text{when } \sigma(j) = i \\ 1 & \text{otherwise.} \end{cases}$$

Then, by symmetry of  $\rho_N$ , we can rewrite lines (3.14) as

$$\left| \operatorname{tr} \left( \left( \frac{(N-k)!}{N!} \sum_{\sigma \in E_{k,N}} [A_{\sigma,1} \otimes A_{\sigma,2} \otimes \cdots \otimes A_{\sigma,N}] \right) \right. \quad (3.16)$$

$$\left. - \frac{1}{N^k} \prod_{j=1}^k \left( A_j \otimes 1^{\otimes(N-1)} + 1 \otimes A_j \otimes 1^{\otimes(N-2)} + \cdots + 1^{\otimes(N-1)} \otimes A_j \right) \rho_N \right|. \quad (3.17)$$

In line (3.17), there are two types of terms: the terms with  $N-k$  1's in the expanded form which we call the off-diagonal terms, and all the other terms which we call the

diagonal terms. There are  $\frac{N!}{(N-k)!}$  off-diagonal terms and  $N^k - \frac{N!}{(N-k)!}$  diagonal terms. Let  $M := \max_{j=1,\dots,k} \|A_j\|_\infty$ . The off-diagonal terms are exactly the terms of line (3.16). Thus, the addition of line (3.16) and the off-diagonal terms of line (3.17) is bounded by

$$\begin{aligned} & \left\| \left( \frac{(N-k)!}{N!} - \frac{1}{N^k} \right) \sum_{\sigma \in E_{k,N}} A_{\sigma,1} \otimes \cdots \otimes A_{\sigma,N} \right\|_\infty \|\rho_N\|_1 \\ & \leq \frac{N!}{(N-k)!} \left( \frac{(N-k)!}{N!} - \frac{1}{N^k} \right) M^k. \end{aligned}$$

Each diagonal term is also bounded by  $M^k$ . Thus, the diagonal terms of line (3.17) are bounded by  $\frac{1}{N^k} \left( N^k - \frac{N!}{(N-k)!} \right) M^k$ . Hence, we can bound lines (3.14) and take the limit as  $N$  approaches  $\infty$ ,

$$\begin{aligned} & \frac{N!}{(N-k)!} \left( \frac{(N-k)!}{N!} - \frac{1}{N^k} \right) M^k + \frac{1}{N^k} \left( N^k - \frac{N!}{(N-k)!} \right) M^k \\ & = M^k \left[ \left( \frac{N^k(N-k)!}{N!} - 1 \right) \frac{N!}{N^k(N-k)!} + \frac{N!}{N^k(N-k)!} \left( \frac{N^k(N-k)!}{N!} - 1 \right) \right] \\ & = 2M^k \left[ \frac{N!}{N^k(N-k)!} \left( \frac{N^k(N-k)!}{N!} - 1 \right) \right] = 2M^k \left[ 1 - \frac{N!}{N^k(N-k)!} \right] \xrightarrow{N \rightarrow \infty} 0. \end{aligned}$$

So line (3.14) goes to 0 as  $N$  approaches infinity.

For line (3.15) can be rewritten as

$$\begin{aligned} & \left| \operatorname{tr} \left[ \left( \prod_{j=1}^k X_N(A_j) - \prod_{j=1}^k \operatorname{tr}(\rho A_j) 1 \right) \rho_N \right] \right| \\ & = \left| \sum_{l=0}^{k-1} \operatorname{tr} \left[ \left( \prod_{j=1}^l \operatorname{tr}(\rho A_j) \prod_{j=l+1}^k X_N(A_j) - \prod_{j=1}^{l+1} \operatorname{tr}(\rho A_j) \prod_{j=l+2}^k X_N(A_j) 1^{\otimes N} \right) \rho_N \right] \right| \\ & = \left| \sum_{l=0}^{k-1} \operatorname{tr} \left[ \left( X_N(A_{l+1}) - \operatorname{tr}(\rho A_{l+1}) 1^{\otimes N} \right) \prod_{j=1}^l \operatorname{tr}(\rho A_j) \prod_{j=l+2}^k X_N(A_j) \rho_N \right] \right|, \end{aligned}$$

and can be bounded by

$$\begin{aligned} & \sum_{l=0}^{k-1} \left| \operatorname{tr} \left[ \left( X_N(A_{l+1}) - \operatorname{tr}(\rho A_{l+1}) 1^{\otimes N} \right) \prod_{j=1}^l \operatorname{tr}(\rho A_j) \prod_{j=l+2}^k X_N(A_j) \rho_N \right] \right| \\ & \leq \sum_{l=0}^{k-1} \left( \sqrt{\operatorname{tr} \left[ \left| X_N(A_{l+1}) - \operatorname{tr}(\rho A_{l+1}) 1^{\otimes N} \right|^2 \rho_N \right]} \sqrt{\operatorname{tr} \left[ \left| \prod_{j=1}^l \operatorname{tr}(\rho A_j) \prod_{j=l+2}^k X_N(A_j) \right|^2 \rho_N \right]} \right) \end{aligned}$$

since by [7, Lemma 2.3.10] applied to the positive linear functional  $\omega(\cdot) := \text{tr}(\cdot \rho_N)$  we obtain  $|\text{tr}(BC\rho_N)| \leq \sqrt{\text{tr}(|B^*|^2 \rho_N)} \sqrt{\text{tr}(|C|^2 \rho_N)}$  for any  $B, C \in \mathcal{B}(\mathbb{H}^{\otimes N})$ .

By assumption, for each  $l$ ,

$$\sqrt{\text{tr} \left[ \left| X_N(A_{l+1}^*) - \text{tr}(\rho A_{l+1}^*) 1^{\otimes N} \right|^2 \rho_N \right]} \xrightarrow{N \rightarrow \infty} 0,$$

and if  $M := \max_{j=1, \dots, k} \|A_j\|_\infty$ , since  $\|X_N(A_j)\|_\infty \leq \|A_j\|_\infty \leq M$ , we have that

$$\begin{aligned} & \text{tr} \left[ \left| \prod_{j=1}^l \text{tr}(\rho A_j) \prod_{j=l+2}^k X_N(A_j) \right|^2 \rho_N \right] \leq \prod_{j=1}^l |\text{tr}(\rho A_j)|^2 \text{tr} \left[ \left| \prod_{j=l+2}^k X_N(A_j) \right|^2 \rho_N \right] \\ & \leq \prod_{j=1}^l |\text{tr}(\rho A_j)|^2 \left\| \prod_{j=l+2}^k X_N(A_j) \right\|_\infty^2 \|\rho_N\|_1 \leq M^{2k} \prod_{j=1}^l |\text{tr}(\rho A_j)|^2 \end{aligned}$$

which is bounded independent of  $N$ . Hence, line (3.15) converges to 0 as  $N$  goes to infinity. Therefore, line (3.13) converges to 0 as  $N$  approaches infinity.  $\square$

The next corollary follows from Theorem 3.2.1 and Proposition 3.1.7.

**Corollary 3.2.3.** *Let  $(\rho_N)_{N=1}^\infty$  be a sequence of symmetric density operators such that  $\rho_N \in \mathcal{D}(\mathbb{H}^{\otimes N})$  for each  $N \in \mathbb{N}$ , and let  $\rho \in \mathcal{D}(\mathbb{H})$ . Then the following are equivalent.*

1.  $(\rho_N)_{N=1}^\infty$  is  $\rho$ -chaotic,
2.  $\text{tr} \left| \rho_N^{(k)} - \rho^{\otimes k} \right| \xrightarrow{N \rightarrow \infty} 0$  for all  $k \in \mathbb{N}$ , and
3.  $\text{tr} \left| \rho_N^{(2)} - \rho^{\otimes 2} \right| \xrightarrow{N \rightarrow \infty} 0$ .

## CHAPTER 4

### PROPAGATION OF QUANTUM KAC'S CHAOS

#### 4.1 DEFINITIONS AND SET-UP

Spohn proved that under evolutions governed by certain families of Hamiltonians, chaotic sequences of density operators propagate in time [31, Theorem 5.7]. In this chapter, we will use the ideas of the proofs of Ducomet [12, Theorem 3.1], and Bardos, Golse, Gottlieb, and Mauser [3, Theorem 3.1] to give a simple, different proof to the result of Spohn. The main result of this chapter is Theorem 4.3.1. First, we define propagation of chaos. In this chapter we will continue to use the same notation as outlined in the beginning of chapter 3.

**Definition 4.1.1.** *Let  $(\rho_N(0))_{N=1}^\infty$  be a sequence of density operators and let  $(H_N)_{N=1}^\infty$  be a sequence of Hamiltonians where  $\rho_N(0) \in \mathcal{D}(\mathbb{H}^{\otimes N})$  and  $H_N \in \mathcal{B}(\mathbb{H}^{\otimes N})$  for every  $N \in \mathbb{N}$ . For each  $t \geq 0$  and  $N \in \mathbb{N}$ , define the density operator*

$$\rho_N(t) := e^{-itH_N} \rho_N(0) e^{itH_N} \in \mathcal{D}(\mathbb{H}^{\otimes N}). \quad (4.1)$$

*If, for each fixed  $t \geq 0$ , the sequence  $(\rho_N(t))_{N=1}^\infty$  is  $\rho(t)$ -chaotic for some  $\rho(t) \in \mathcal{D}(\mathbb{H})$ , then we say that chaos **propagates with respect to**  $(H_N)_{N=1}^\infty$ .*

We will now construct, as in Spohn [31], examples of propagation of chaos. We will examine the mean field limit for interacting quantum particles, see [31, pages 609 - 613]. For each  $N \in \mathbb{N}$  and  $\pi \in \Sigma_N$ , define the unitary operator  $U_\pi^{[N]} \in \mathcal{B}(\mathbb{H}^{\otimes N})$  by equation (3.1). For  $A \in \mathcal{B}(\mathbb{H})$ ,  $V \in \mathcal{B}(\mathbb{H} \otimes \mathbb{H})$ ,  $N \in \mathbb{N}$ , and  $j \in \{1, \dots, N\}$ , define

$$A_j^{[N]} := 1^{\otimes(j-1)} \otimes A \otimes 1^{\otimes(N-j-1)} \in \mathcal{B}(\mathbb{H}^{\otimes N}),$$

$$V_{12}^{[N]} := V \otimes 1^{\otimes(N-2)},$$

and

$$V_{ij}^{[N]} = U_{\pi^{-1}}^{[N]} V_{12}^{[N]} U_{\pi}^{[N]}$$

where  $\pi$  is any permutation where  $\pi(i) = 1$  and  $\pi(j) = 2$ . Notice that this operator is well defined and independent of the permutation  $\pi$  that we use, (as long as  $\pi(i) = 1$  and  $\pi(j) = 2$ ) because when applied to a simple tensor  $x_1 \otimes \cdots \otimes x_N$  all but the  $x_i$  and  $x_j$  spots are left invariant. For any self-adjoint  $A \in \mathcal{B}(\mathbb{H})$ , any self-adjoint  $V \in \mathcal{B}(\mathbb{H} \otimes \mathbb{H})$ , and each  $N \in \mathbb{N}$ , consider the Hamiltonian

$$H_N = \sum_{j=1}^N A_j^{[N]} + \frac{1}{N} \sum_{i \neq j; i, j=1}^N V_{ij}^{[N]}. \quad (4.2)$$

Also, define

$$H_{n,N} := \sum_{j=1}^n A_j^{[n]} + \frac{1}{N} \sum_{i \neq j; i, j=1}^n V_{ij}^{[n]} \quad (4.3)$$

for each  $n, N \in \mathbb{N}$ ,  $n \leq N$ .

The main result of this section is Theorem 4.3.1. In this theorem, we will assume that a sequence of density operators  $(\rho_N(0))_{N=1}^{\infty}$  is  $\rho(0)$ -chaotic and we will show that if  $(H_N)_{N=1}^{\infty}$  is defined by equation (4.2) and for all  $t \geq 0$ ,  $(\rho_N(t))_{N=1}^{\infty}$  is defined by equation (4.1), then for all  $t \geq 0$  the sequence  $(\rho_N(t))_{N=1}^{\infty}$  is  $\rho(t)$ -chaotic for some  $\rho(t) \in \mathcal{D}(\mathbb{H})$ , i.e. chaos propagates with respect to  $(H_N)_{N=1}^{\infty}$ . Before proving our main result (Theorem 4.3.1), we need to establish some preliminary results. The first preliminary result consists of proving that  $\rho_N(t)$  is symmetric for each  $N \in \mathbb{N}$  and  $t \geq 0$ .

**Proposition 4.1.2.** *For each  $N \in \mathbb{N}$  and  $t \geq 0$ ,  $\rho_N(t)$  (as defined in equation (4.1)) is symmetric.*

*Proof.* Let  $\pi \in \Sigma_N$ . By Proposition 3.1.2, we must show that

$$U_{\pi^{-1}}^{[N]} e^{-itH_N} \rho_N(0) e^{itH_N} U_{\pi}^{[N]} = U_{\pi^{-1}}^{[N]} \rho_N(t) U_{\pi}^{[N]} = \rho_N(t).$$

Since  $\rho_N(0)$  is symmetric, it is enough to show that  $U_{\pi^{-1}}^{[N]} e^{itH_N} U_{\pi}^{[N]} = e^{itH_N}$ . Furthermore, it is enough to show that  $U_{\pi^{-1}}^{[N]} H_N U_{\pi}^{[N]} = H_N$ .

First we prove that  $U_{\pi^{-1}}^{[N]} \sum_{j=1}^N A_j^{[N]} U_{\pi}^{[N]} = \sum_{j=1}^N A_j^{[N]}$ . Indeed,

$$\begin{aligned}
U_{\pi^{-1}}^{[N]} \sum_{j=1}^N A_j^{[N]} U_{\pi}^{[N]}(x_1 \otimes \cdots \otimes x_N) &= \sum_{j=1}^N U_{\pi^{-1}}^{[N]} A_j^{[N]}(x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(N)}) \\
&= \sum_{j=1}^N U_{\pi^{-1}}^{[N]}(x_{\pi^{-1}(1)} \otimes \cdots \otimes x_{\pi^{-1}(j-1)} \otimes A(x_{\pi^{-1}(j)}) \otimes x_{\pi^{-1}(j+1)} \otimes \cdots \otimes x_{\pi^{-1}(N)}) \\
&= \sum_{j=1}^N x_1 \otimes \cdots \otimes x_{\pi^{-1}(j)-1} \otimes A(x_{\pi^{-1}(j)}) \otimes x_{\pi^{-1}(j)+1} \otimes \cdots \otimes x_N \\
&= \sum_{j=1}^N x_1 \otimes \cdots \otimes x_{j-1} \otimes Ax_j \otimes x_{j+1} \otimes \cdots \otimes x_N = \sum_{j=1}^N A_j^{[N]}(x_1 \otimes \cdots \otimes x_N).
\end{aligned}$$

Next, we will show that  $U_{\pi^{-1}}^{[N]} \sum_{i \neq j; i, j=1}^N V_{ij}^{[N]} U_{\pi}^{[N]} = \sum_{i \neq j; i, j=1}^N V_{ij}^{[N]}$ . For each  $i, j \in \{1, \dots, N\}$  with  $i \neq j$ , choose  $\sigma_{ij} \in \Sigma_N$  with  $\sigma_{ij}(i) = 1$  and  $\sigma_{ij}(j) = 2$ . Then

$$\begin{aligned}
U_{\pi^{-1}}^{[N]} \sum_{i \neq j; i, j=1}^N V_{ij}^{[N]} U_{\pi}^{[N]} &= \sum_{i \neq j; i, j=1}^N U_{\pi^{-1}}^{[N]} V_{ij}^{[N]} U_{\pi}^{[N]} = \sum_{i \neq j; i, j=1}^N U_{\pi^{-1}}^{[N]} U_{\sigma_{ij}^{-1}}^{[N]} V_{12}^{[N]} U_{\sigma_{ij}}^{[N]} U_{\pi}^{[N]} \\
&= \sum_{i \neq j; i, j=1}^N U_{(\sigma_{ij}\pi)^{-1}}^{[N]} V_{12}^{[N]} U_{\sigma_{ij}\pi}^{[N]}. \tag{4.4}
\end{aligned}$$

Notice that  $(\sigma_{ij}\pi)(\pi^{-1}(i)) = 1$  and  $(\sigma_{ij}\pi)(\pi^{-1}(j)) = 2$ , and thus, line (4.4) is equal to

$$\sum_{i \neq j; i, j=1}^N V_{\pi^{-1}(i)\pi^{-1}(j)}^{[N]} = \sum_{i \neq j; i, j=1}^N V_{ij}^{[N]},$$

where the last equality is valid because  $i \neq j$  if and only if  $\pi^{-1}(i) \neq \pi^{-1}(j)$ . Thus, we obtain that  $U_{\pi^{-1}}^{[N]} \sum_{i \neq j=1}^N V_{ij}^{[N]} U_{\pi}^{[N]} = \sum_{i \neq j=1}^N V_{ij}^{[N]}$ . Hence, we have that  $U_{\pi^{-1}}^{[N]} H_N U_{\pi}^{[N]} = H_N$ , and  $\rho_N(t)$  is symmetric.  $\square$

## 4.2 TWO HIERARCHIES OF DIFFERENTIAL EQUATIONS

In this section, we construct two similar families of differential equations for  $(\rho_N^{(n)}(t))_{n=1}^{N-1}$  and  $(\rho(t)^{\otimes n})_{n=1}^{\infty}$ . The following Proposition gives a family of differential equations which is satisfied by  $(\rho_N^{(n)}(t))_{n=1}^{N-1}$ .

**Proposition 4.2.1.** *Let  $N \in \mathbb{N}$ . For  $n \in \mathbb{N}$ ,  $n \leq N - 1$ , and  $t \geq 0$ , we have*

$$i \frac{d}{dt} \rho_N^{(n)}(t) = [H_{n,N}, \rho_N^{(n)}(t)] + \frac{N-n}{N} \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j_{n+1}}^{[n+1]} + V_{n+1,j}^{[n+1]}, \rho_N^{(n+1)}(t)] \quad (4.5)$$

where  $\rho_N(t)$  is given by (4.1) and  $H_{n,N}$  is given by (4.3).

*Proof.* We know

$$i \frac{d}{dt} \rho_N(t) = [H_N, \rho_N(t)].$$

Integrating both sides, we obtain

$$i (\rho_N(t) - \rho_N(0)) = \int_0^t [H_N, \rho_N(s)] ds.$$

Now, taking the partial trace of both sides, and using the fact that partial traces and integrals commute, we obtain

$$i (\rho_N^{(n)}(t) - \rho_N^{(n)}(0)) = \int_0^t \text{tr}_{[n+1,N]} ([H_N, \rho_N(s)]) ds. \quad (4.6)$$

We claim that

$$\begin{aligned} & \text{tr}_{[n+1,N]} ([H_N, \rho_N(s)]) \\ &= [H_{n,N}, \rho_N^{(n)}(s)] + \frac{N-n}{N} \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j_{n+1}}^{[n+1]} + V_{n+1,j}^{[n+1]}, \rho_N^{(n+1)}(s)] \end{aligned} \quad (4.7)$$

for each  $s \in [0, \infty)$ . In order to prove equation (4.7), fix  $s \in [0, \infty)$ , and by line [2, equation (2.11)], we need to prove that for every  $B \in \mathcal{B}(\mathbb{H}^{\otimes n})$

$$\begin{aligned} \text{tr} \left( [H_N, \rho_N(s)] B \otimes 1^{\otimes(N-n)} \right) &= \text{tr} \left( \left[ \sum_{j=1}^n A_j^{[n]} + \frac{1}{N} \sum_{i \neq j=1}^n V_{ij}^{[n]}, \rho_N^{(n)}(s) \right] B \right) \\ &+ \frac{N-n}{N} \sum_{j=1}^n \text{tr} \left( \text{tr}_{\{n+1\}} [V_{j_{n+1}}^{[n+1]} + V_{n+1,j}^{[n+1]}, \rho_N^{(n+1)}(s)] B \right). \end{aligned}$$

Let  $B \in \mathcal{B}(\mathbb{H}^{\otimes n})$ , and we have

$$\text{tr} \left( \left[ \sum_{j=1}^N A_j^{[N]} + \frac{1}{N} \sum_{i \neq j; i,j=1}^N V_{ij}^{[N]}, \rho_N(s) \right] B \otimes 1^{\otimes(N-n)} \right)$$

$$\begin{aligned}
&= \text{tr} \left( \sum_{j=1}^N A_j^{[N]} \rho_N(s) B \otimes 1^{\otimes(N-n)} + \frac{1}{N} \sum_{i \neq j; i, j=1}^N V_{ij}^{[N]} \rho_N(s) B \otimes 1^{\otimes(N-n)} \right. \\
&\quad \left. - \rho_N(s) \sum_{j=1}^N A_j^{[N]} B \otimes 1^{\otimes(N-n)} - \frac{1}{N} \rho_N(s) \sum_{i \neq j; i, j=1}^N V_{ij}^{[N]} B \otimes 1^{\otimes(N-n)} \right) \\
&= \text{tr} \left( B \otimes 1^{\otimes(N-n)} \sum_{j=1}^N A_j^{[N]} \rho_N(s) + \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{i \neq j; i, j=1}^N V_{ij}^{[N]} \rho_N(s) \right. \\
&\quad \left. - \rho_N(s) \sum_{j=1}^N A_j^{[N]} B \otimes 1^{\otimes(N-n)} - \frac{1}{N} \rho_N(s) \sum_{i \neq j; i, j=1}^N V_{ij}^{[N]} B \otimes 1^{\otimes(N-n)} \right) \\
&= \text{tr} \left( B \otimes 1^{\otimes(N-n)} \sum_{j=1}^N A_j^{[N]} \rho_N(s) - \rho_N(s) \sum_{j=1}^N A_j^{[N]} B \otimes 1^{\otimes(N-n)} \right) \quad (4.8) \\
&\quad + \text{tr} \left( \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{\substack{i \neq j; \\ i, j=1}}^N V_{ij}^{[N]} \rho_N(s) - \frac{1}{N} \rho_N(s) \sum_{\substack{i \neq j; \\ i, j=1}}^N V_{ij}^{[N]} B \otimes 1^{\otimes(N-n)} \right) \quad (4.9)
\end{aligned}$$

Line (4.8) can be rewritten as

$$\text{tr} \left( B \otimes 1^{\otimes(N-n)} \sum_{j=1}^n A_j^{[N]} \rho_N(s) \right) - \text{tr} \left( \rho_N(s) \sum_{j=1}^n A_j^{[N]} B \otimes 1^{\otimes(N-n)} \right) \quad (4.10)$$

$$+ \text{tr} \left( B \otimes 1^{\otimes(N-n)} \sum_{j=n+1}^N A_j^{[N]} \rho_N(s) \right) - \text{tr} \left( \sum_{j=n+1}^N A_j^{[N]} B \otimes 1^{\otimes(N-n)} \rho_N(s) \right). \quad (4.11)$$

Notice that  $B \otimes 1^{\otimes(N-n)} \sum_{j=n+1}^N A_j^{[N]} = \sum_{j=n+1}^N A_j^{[N]} B \otimes 1^{\otimes(N-n)}$ , and so line (4.11) is equal to zero (even without taking the trace into account). Notice that in line (4.10),  $A_j^{[N]} = A_j^{[n]} \otimes 1^{\otimes(N-n)}$  for  $j \leq n$ , thus line (4.10) can be written as

$$\begin{aligned}
&\text{tr} \left( B \sum_{j=1}^n A_j^{[n]} \otimes 1^{\otimes(N-n)} \rho_N(s) \right) - \text{tr} \left( \rho_N(s) \sum_{j=1}^n A_j^{[n]} B \otimes 1^{\otimes(N-n)} \right) \\
&= \text{tr} \left( \rho_N^{(n)}(s) B \sum_{j=1}^n A_j^{[n]} \right) - \text{tr} \left( \rho_N^{(n)}(s) \sum_{j=1}^n A_j^{[n]} B \right) \quad (\text{by [2, equation (2.11)]}) \\
&= \text{tr} \left( \left[ \sum_{j=1}^n A_j^{[n]}, \rho_N^{(n)}(s) \right] B \right).
\end{aligned}$$



Line (4.9) can be rewritten as

$$\begin{aligned}
& \operatorname{tr} \left( \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{\substack{i \neq j; \\ i, j=1}}^n V_{ij}^{[N]} \rho_N(s) \right) + \operatorname{tr} \left( \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{\substack{1 \leq i \leq n \\ < j \leq N}} V_{ij}^{[N]} \rho_N(s) \right) \\
+ & \operatorname{tr} \left( \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{\substack{1 \leq j \leq n \\ < i \leq N}} V_{ij}^{[N]} \rho_N(s) \right) + \operatorname{tr} \left( \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{\substack{i \neq j; \\ i, j=n+1}}^N V_{ij}^{[N]} \rho_N(s) \right) \\
- & \operatorname{tr} \left( \frac{1}{N} \rho_N(s) \sum_{\substack{i \neq j; \\ i, j=1}}^n V_{ij}^{[M]} B \otimes 1^{\otimes(N-n)} \right) - \operatorname{tr} \left( \frac{1}{N} \rho_N(s) \sum_{\substack{1 \leq i \leq n \\ < j \leq N}} V_{ij}^{[M]} B \otimes 1^{\otimes(N-n)} \right) \\
- & \operatorname{tr} \left( \frac{1}{N} \rho_N(s) \sum_{\substack{1 \leq j \leq n \\ < i \leq N}} V_{ij}^{[M]} B \otimes 1^{\otimes(N-n)} \right) - \operatorname{tr} \left( \frac{1}{N} \rho_N(s) \sum_{\substack{i \neq j; \\ i, j=n+1}}^N V_{ij}^{[M]} B \otimes 1^{\otimes(N-n)} \right).
\end{aligned}$$

The first and fifth terms of the above expression give

$$\operatorname{tr} \left( \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{\substack{i \neq j; \\ i, j=1}}^n V_{ij}^{[N]} \rho_N(s) - \frac{1}{N} \rho_N(s) \sum_{\substack{i \neq j; \\ i, j=1}}^n V_{ij}^{[M]} B \otimes 1^{\otimes(N-n)} \right). \quad (4.12)$$

The second and sixth terms of the same expression give

$$\operatorname{tr} \left( \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{\substack{1 \leq i \leq n \\ < j \leq N}} V_{ij}^{[N]} \rho_N(s) - \frac{1}{N} \rho_N(s) \sum_{\substack{1 \leq i \leq n \\ < j \leq N}} V_{ij}^{[M]} B \otimes 1^{\otimes(N-n)} \right). \quad (4.13)$$

The third and seventh terms of the same expression give

$$\operatorname{tr} \left( \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{\substack{1 \leq j \leq n \\ < i \leq N}} V_{ij}^{[N]} \rho_N(s) - \frac{1}{N} \rho_N(s) \sum_{\substack{1 \leq j \leq n \\ < i \leq N}} V_{ij}^{[M]} B \otimes 1^{\otimes(N-n)} \right). \quad (4.14)$$

The fourth and eighth terms of the same expression give

$$\operatorname{tr} \left( \frac{1}{N} B \otimes 1^{\otimes(N-n)} \sum_{\substack{i \neq j; \\ i, j=n+1}}^N V_{ij}^{[N]} \rho_N(s) - \frac{1}{N} \sum_{\substack{i \neq j; \\ i, j=n+1}}^N V_{ij}^{[M]} B \otimes 1^{\otimes(N-n)} \rho_N(s) \right). \quad (4.15)$$

Notice that  $B \otimes 1^{\otimes(N-n)} \sum_{i \neq j; i, j=n+1}^N V_{ij}^{[N]} = \sum_{i \neq j; i, j=n+1}^N V_{ij}^{[M]} B \otimes 1^{\otimes(N-n)}$ , and so line (4.15) is equal to zero (even without taking the trace into account).

Notice that  $V_{ij}^{[N]} = V_{ij}^{[n]} \otimes 1^{\otimes(N-n)}$  for  $i, j \leq n$ , thus line (4.12) can be rewritten

as

$$\begin{aligned}
& \frac{1}{N} \operatorname{tr} \left( B \sum_{\substack{i \neq j; \\ i, j=1}}^n V_{ij}^{[n]} \otimes 1^{\otimes(N-n)} \rho_N(s) \right) - \frac{1}{N} \operatorname{tr} \left( \sum_{\substack{i \neq j; \\ i, j=1}}^n V_{ij}^{[n]} B \otimes 1^{\otimes(N-n)} \rho_N(s) \right) \\
&= \frac{1}{N} \operatorname{tr} \left( \rho_N^{(n)}(s) B \sum_{\substack{i \neq j; \\ i, j=1}}^n V_{ij}^{[n]} \right) - \frac{1}{N} \operatorname{tr} \left( \rho_N^{(n)}(s) \sum_{\substack{i \neq j; \\ i, j=1}}^n V_{ij}^{[n]} B \right) \text{ (by [2, equation (2.11)])} \\
&= \operatorname{tr} \left( \left[ \frac{1}{N} \sum_{\substack{i \neq j; \\ i, j=1}}^n V_{ij}^{[n]}, \rho_N^{(n)}(s) \right] B \right).
\end{aligned}$$

There are  $N-n$  values of  $j$  in line (4.13) and by symmetry of  $\rho_N(s)$  we can replace all of these values of  $j$  by  $n+1$  and thus we have that line (4.13) can be rewritten as

$$\frac{N-n}{N} \operatorname{tr} \left( B \otimes 1^{\otimes(N-n)} \sum_{i=1}^n V_{i, n+1}^{[N]} \rho_N(s) \right) - \frac{N-n}{N} \operatorname{tr} \left( \sum_{i=1}^n V_{i, n+1}^{[N]} B \otimes 1^{\otimes(N-n)} \rho_N(s) \right).$$

Notice that  $V_{i, n+1}^{[N]} = V_{i, n+1}^{[n+1]} \otimes 1^{\otimes(N-(n+1))}$  for  $i \leq n$ , thus the last displayed equation is equal to

$$\begin{aligned}
& \frac{N-n}{N} \operatorname{tr} \left( \left( B \otimes 1 \sum_{i=1}^n V_{i, n+1}^{[n+1]} \right) \otimes 1^{\otimes(N-n-1)} \rho_N(s) \right) \\
& - \frac{N-n}{N} \operatorname{tr} \left( \left( \sum_{i=1}^n V_{i, n+1}^{[n+1]} B \otimes 1 \right) \otimes 1^{\otimes(N-n-1)} \rho_N(s) \right)
\end{aligned}$$

and therefore by [2, equation (2.11)] the last displayed expression is equal to

$$\begin{aligned}
& \frac{N-n}{N} \operatorname{tr} \left( B \otimes 1 \sum_{i=1}^n V_{i, n+1}^{[n+1]} \rho_N^{(n+1)}(s) \right) - \frac{N-n}{N} \operatorname{tr} \left( \sum_{i=1}^n V_{i, n+1}^{[n+1]} B \otimes 1 \rho_N^{(n+1)}(s) \right) \\
&= \frac{N-n}{N} \sum_{j=1}^n \operatorname{tr} \left( [V_{j, n+1}^{[n+1]}, \rho_N^{(n+1)}(s)] B \otimes 1 \right) \\
&= \frac{N-n}{N} \sum_{j=1}^n \operatorname{tr} \left( \operatorname{tr}_{\{n+1\}} [V_{j, n+1}^{[n+1]}, \rho_N^{(n+1)}(s)] B \right),
\end{aligned}$$

where again we used [2, equation (2.11)] to obtain the last equality.

Similarly, line (4.14) can be rewritten as

$$\frac{N-n}{N} \sum_{j=1}^n \operatorname{tr} \left( \operatorname{tr}_{\{n+1\}} [V_{n+1, j}^{[n+1]}, \rho_N^{(n+1)}(s)] B \right).$$

Thus, (4.6) and (4.7) lead to

$$\begin{aligned} & i \left( \rho_N^{(n)}(t) - \rho_N^{(n)}(0) \right) \\ &= \int_0^t \left( [H_{n,N}, \rho_N^{(n)}(s)] + \frac{N-n}{N} \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j,n+1}^{[n+1]} + V_{n+1,j}^{[n+1]}, \rho_N^{(n+1)}(s)] \right) ds. \end{aligned}$$

We take the derivative of both sides to obtain the result

$$i \frac{d}{dt} \rho_N^{(n)}(t) = [H_{n,N}, \rho_N^{(n)}(t)] + \frac{N-n}{N} \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j,n+1}^{[n+1]} + V_{n+1,j}^{[n+1]}, \rho_N^{(n+1)}(t)].$$

□

The next proposition concludes with a family of differential equations which is satisfied by  $(\rho(t)^{\otimes n})_{n=1}^{\infty}$ . This family of differential equations is similar to the ones displayed in equation (4.5).

**Proposition 4.2.2.** *Let  $\rho(0) \in \mathcal{D}(\mathbb{H})$ . If  $\rho(t)$  is the solution to the differential equation*

$$i \frac{d}{dt} \rho(t) = [A, \rho(t)] + \text{tr}_{\{2\}} [V_{12}^{[2]} + V_{21}^{[2]}, \rho(t) \otimes \rho(t)] \quad (4.16)$$

for  $t \geq 0$  (which is called the Hartree Equation), with initial condition  $\rho(0)$ , then we have that  $(\rho(t)^{\otimes n})_{n=1}^{\infty}$  satisfies the family of differential equations

$$i \frac{d}{dt} \rho(t)^{\otimes n} = \sum_{j=1}^n [A_j^{[n]}, \rho(t)^{\otimes n}] + \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j,n+1}^{[n+1]} + V_{n+1,j}^{[n+1]}, \rho(t)^{\otimes(n+1)}]. \quad (4.17)$$

*Proof.* We have

$$\begin{aligned} & i \frac{d}{dt} \rho(t)^{\otimes n} = \sum_{j=1}^n \rho(t)^{\otimes(j-1)} \otimes i \frac{d}{dt} \rho(t) \otimes \rho(t)^{\otimes(n-j)} \quad (\text{“product” rule}) \\ &= \sum_{j=1}^n \rho(t)^{\otimes(j-1)} \otimes \left( [A, \rho(t)] + \text{tr}_{\{2\}} [V_{12}^{[2]} + V_{21}^{[2]}, \rho(t) \otimes \rho(t)] \right) \otimes \rho(t)^{\otimes(n-j)} \end{aligned}$$

by assumption. The last expression splits into the following two parts

$$\sum_{j=1}^n \rho(t)^{\otimes(j-1)} \otimes [A, \rho(t)] \otimes \rho(t)^{\otimes(n-j)} \quad (4.18)$$

$$+ \sum_{j=1}^n \rho(t)^{\otimes(j-1)} \otimes \text{tr}_{\{2\}} [V_{12}^{[2]} + V_{21}^{[2]}, \rho(t) \otimes \rho(t)] \otimes \rho(t)^{\otimes(n-j)} \quad (4.19)$$

Line (4.18) can be rewritten as

$$\begin{aligned}
& [A, \rho(t)] \otimes \rho(t)^{\otimes(n-1)} + \rho(t) \otimes [A, \rho(t)] \otimes \rho(t)^{\otimes(n-1)} + \dots + \rho(t)^{\otimes(n-1)} \otimes [A, \rho(t)] \\
&= \sum_{j=1}^n [A_j^{[n]}, \rho(t)^{\otimes n}]. \tag{4.20}
\end{aligned}$$

We claim that for  $j \leq n$ ,

$$\text{tr}_{\{j+1\}} \left( \left( V_{jj+1}^{[n+1]} + V_{j+1j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) = \text{tr}_{\{n+1\}} \left( \left( V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right). \tag{4.21}$$

Indeed, by [2, equation (2.11)], for any  $B_1, \dots, B_n \in \mathcal{B}(\mathbb{H})$ , we have

$$\begin{aligned}
& \text{tr} \left( \text{tr}_{\{j+1\}} \left( \left( V_{jj+1}^{[n+1]} + V_{j+1j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) B_1 \otimes \dots \otimes B_n \right) \\
&= \text{tr} \left( \left( V_{jj+1}^{[n+1]} + V_{j+1j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} B_1 \otimes \dots \otimes B_j \otimes 1 \otimes B_{j+1} \otimes \dots \otimes B_n \right) \\
&= \text{tr} \left( B_1 \otimes \dots \otimes B_j \otimes 1 \otimes B_{j+1} \otimes \dots \otimes B_n \left( V_{jj+1}^{[n+1]} + V_{j+1j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) \\
&= \text{tr} \left( B_1 \otimes \dots \otimes B_n \otimes 1 \left( V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) \\
&= \text{tr} \left( \text{tr}_{\{n+1\}} \left( \left( V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) B_1 \otimes \dots \otimes B_n \right).
\end{aligned}$$

Similar to equation (4.21), we have that for  $j \leq n$ ,

$$\text{tr}_{\{j+1\}} \left( \rho(t)^{\otimes(n+1)} \left( V_{jj+1}^{[n+1]} + V_{j+1j}^{[n+1]} \right) \right) = \text{tr}_{\{n+1\}} \left( \rho(t)^{\otimes(n+1)} \left( V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]} \right) \right). \tag{4.22}$$

Line (4.19) can be rewritten as

$$\begin{aligned}
& \sum_{j=1}^n \rho(t)^{\otimes(j-1)} \otimes \text{tr}_{\{2\}} \left( \left( V_{12}^{[2]} + V_{21}^{[2]} \right) \rho(t) \otimes \rho(t) \right) \otimes \rho(t)^{\otimes(n-j)} \\
&- \sum_{j=1}^n \rho(t)^{\otimes(j-1)} \otimes \text{tr}_{\{2\}} \left( \rho(t) \otimes \rho(t) \left( V_{12}^{[2]} + V_{21}^{[2]} \right) \right) \otimes \rho(t)^{\otimes(n-j)} \\
&= \sum_{j=1}^n \text{tr}_{\{j+1\}} \left( \left( V_{jj+1}^{[n+1]} + V_{j+1j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right. \\
&\quad \left. - \rho(t)^{\otimes(n+1)} \left( V_{jj+1}^{[n+1]} + V_{j+1j}^{[n+1]} \right) \right) \tag{4.23}
\end{aligned}$$

By equations (4.21) and (4.22), lines (4.23) are equal to

$$\begin{aligned}
& \sum_{j=1}^n \left( \text{tr}_{\{n+1\}} \left( \left( V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]} \right) \rho(t)^{\otimes(n+1)} \right) \right. \\
& - \left. \text{tr}_{\{n+1\}} \left( \rho(t)^{\otimes(n+1)} \left( V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]} \right) \right) \right) \\
& = \sum_{j=1}^n \text{tr}_{\{n+1\}} \left[ V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]}, \rho(t)^{\otimes(n+1)} \right]. \tag{4.24}
\end{aligned}$$

Of course (4.20) and (4.24) complete the proof.  $\square$

Equation (4.16) has a unique solution. This solution  $\rho(t)$  is self-adjoint trace class for each  $t \geq 0$ , see [6, Theorem 4.1] and the next remark. We will see in the proof of Theorem 4.3.1 that  $\rho(t)$  is a density operator for each  $t \geq 0$ . The next remark checks that equation (4.16) satisfies the conditions of [6, Theorem 4.1].

**Remark 4.2.3.** *Let  $X$  be the real Banach space of self-adjoint trace class operators on  $\mathbb{H}$ . Define the mapping  $f : X \rightarrow X$  by  $f(T) = -i \text{tr}_{\{2\}} [V_{12}^{[2]} + V_{21}^{[2]}, T \otimes T]$ .*

*We will first prove that  $f$  is locally Lipschitzian. Let  $r \geq 0$ , and  $T, S \in X$  with  $\|T\|_1 \leq r$  and  $\|S\|_1 \leq r$ . Then*

$$\begin{aligned}
\|f(T) - f(S)\|_1 &= \left\| -i \text{tr}_{\{2\}} [V_{12}^{[2]} + V_{21}^{[2]}, T \otimes T] + i \text{tr}_{\{2\}} [V_{12}^{[2]} + V_{21}^{[2]}, S \otimes S] \right\|_1 \\
&= \left\| \text{tr}_{\{2\}} [V_{12}^{[2]} + V_{21}^{[2]}, T \otimes T - S \otimes S] \right\|_1 \\
&\leq \left\| [V_{12}^{[2]} + V_{21}^{[2]}, T \otimes T - S \otimes S] \right\|_1 \leq 4 \|V\|_\infty \|T \otimes T - S \otimes S\|_1 \\
&= 4 \|V\|_\infty \|T \otimes T - T \otimes S + T \otimes S - S \otimes S\|_1 \\
&\leq 4 \|V\|_\infty (\|T \otimes (T - S)\|_1 + \|(T - S) \otimes S\|_1) \\
&\leq 4r \|V\|_\infty \|T - S\|_1
\end{aligned}$$

*which shows that  $f$  is locally Lipschitzian.*

*We will now prove that  $f$  satisfies [6, inequality (4.1)]. Let  $\alpha \geq 0$ ,  $T \in X$ , and  $S = T - \alpha f(T)$ . We have that  $T = \sum_{i=1}^{\infty} \lambda_i |x_i\rangle \langle x_i|$  for some orthonormal sequence  $(x_i)_{i=1}^{\infty} \subset \mathbb{H}$  and some sequence  $(\lambda_i)_{i=1}^{\infty} \subset \mathbb{R}$ . Define  $\sigma = \sum_{i=1}^{\infty} \text{sign}(\lambda_i) |x_i\rangle \langle x_i|$ . Thus,*

$|T| = T\sigma = \sigma T$ , and

$$\begin{aligned}
\text{tr}(f(T)\sigma) &= -i\text{tr}\left(\text{tr}_{\{2\}}\left[V_{12}^{[2]} + V_{21}^{[2]}, T \otimes T\right]\sigma\right) = -i\text{tr}\left(\left[V_{12}^{[2]} + V_{21}^{[2]}, T \otimes T\right]\sigma \otimes 1\right) \\
&= -i\text{tr}\left(\left(V_{12}^{[2]} + V_{21}^{[2]}\right)(T \otimes T)(\sigma \otimes 1) - (T \otimes T)\left(V_{12}^{[2]} + V_{21}^{[2]}\right)(\sigma \otimes 1)\right) \\
&= -i\text{tr}\left(\left(V_{12}^{[2]} + V_{21}^{[2]}\right)|T| \otimes T - |T| \otimes T\left(V_{12}^{[2]} + V_{21}^{[2]}\right)\right) \\
&= -i\text{tr}\left(\left[V_{12}^{[2]} + V_{21}^{[2]}, |T| \otimes T\right]\right) = 0.
\end{aligned}$$

By the cyclicity of the trace, we also have  $\text{tr}(\sigma f(T)) = 0$ . Hence, we have

$$\begin{aligned}
\|T\|_1 &= \frac{1}{2}\text{tr}(T\sigma + \sigma T) = \frac{1}{2}\text{tr}\left((S + \alpha f(T))\sigma + \sigma(S + \alpha f(T))\right) \\
&= \frac{1}{2}\text{tr}(S\sigma + \sigma S) + \frac{\alpha}{2}\text{tr}(f(T)\sigma + \sigma f(T)) = \frac{1}{2}\text{tr}(S\sigma + \sigma S) \\
&\leq \frac{1}{2}\text{tr}|S\sigma + \sigma S| \leq \frac{1}{2}(\|S\|_1\|\sigma\|_\infty + \|\sigma\|_\infty\|S\|_1) = \|\sigma\|_\infty\|S\|_1 \\
&= \|S\|_1 = \|T - \alpha f(T)\|_1.
\end{aligned}$$

Finally, it is clear that  $M(\cdot) = -i[A, \cdot]$  defines the infinitesimal generator of the contraction semigroup  $T_t(\cdot) = e^{-itA}(\cdot)e^{itA}$  on  $X$ . Thus, all of the conditions of [6, Theorem 4.1] are satisfied.

### 4.3 PROPAGATION OF QUANTUM KAC'S CHAOS WITH RESPECT TO $(H_N)_{N=1}^\infty$

The similarity of the two equations (4.5) and (4.17) helps to prove the propagation of chaos presented in the following theorem. The idea of the proof of this theorem comes from Ducomet [12, Theorem 3.1], and Bardos, Golse, Gottlieb, and Mauser [3, Theorem 3.1]. The proof uses the result of Theorem 1.3.9 that the partial trace is a contraction.

**Theorem 4.3.1.** *Let a sequence  $(\rho_N(0))_{N=1}^\infty$  of density operators be  $\rho(0)$ -chaotic where  $\rho(0) \in \mathcal{D}(\mathbb{H})$ . Let  $(H_N)_{N=1}^\infty$  be a sequence of Hamiltonians defined by equation (4.2). Then, for each fixed  $t \geq 0$ , the sequence of density operators  $(\rho_N(t))_{N=1}^\infty$  defined in equation (4.1) is  $\rho(t)$ -chaotic where  $\rho(t)$  is the solution of the Hartree equa-*

tion (equation (4.16)) with initial condition  $\rho(0)$ . Thus chaos propagates with respect to the Hamiltonians  $(H_N)_{N=1}^\infty$ .

*Proof.* Fix  $T_0 \in \mathbb{N} \cup \{0\}$ . Let  $K_{T_0} := \sup \{ \|\rho(t)\|_1 : t \in [T_0, T_0 + 1] \} < \infty$  since by [6, Theorem 4.1] we know that the self-adjoint trace class solution  $\rho(t)$  to the Hartree equation is continuous with respect to time. Let  $\delta_{T_0} := \frac{1}{[8(\max\{\|V\|_\infty, 1\})K_{T_0}] + 1}$  where  $[8(\max\{\|V\|_\infty, 1\})K_{T_0}]$  is the integer part of  $8(\max\{\|V\|_\infty, 1\})K_{T_0}$ .

In order to prove Theorem 4.3.1 we will show the following:

Fix  $t_0 := T_0 + k\delta_{T_0}$  for some  $k \in \{0, 1, \dots, [8(\max\{\|V\|_\infty, 1\})K_{T_0}]\}$ . Assume that  $(\rho_N(t_0))_{N=1}^\infty$  is  $\rho(t_0)$  chaotic where  $\rho(t_0) \in \mathcal{D}(\mathbb{H})$ . Then for  $t \in [t_0, t_0 + \delta_{T_0}]$ ,  $(\rho_N(t))_{N=1}^\infty$  is  $\rho(t)$  chaotic where  $\rho(t) \in \mathcal{D}(\mathbb{H})$  is the solution to the Hartree equation (equation (4.16)) with initial condition  $\rho(t_0)$ .

For  $t \in [t_0, \infty)$ , by Proposition 4.2.2, for each  $n \in \mathbb{N}$ ,

$$i \frac{d}{dt} \rho(t)^{\otimes n} = \sum_{j=1}^n [A_j^{[n]}, \rho(t)^{\otimes n}] + \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j\ n+1}^{[n+1]} + V_{n+1\ j}^{[n+1]}, \rho(t)^{\otimes(n+1)}]. \quad (4.25)$$

Also notice that for each  $n, N \in \mathbb{N}$  with  $n \leq N - 1$ , by Proposition 4.2.1

$$\begin{aligned} i \frac{d}{dt} \rho_N^{(n)}(t) &= [H_{n,N}, \rho_N^{(n)}(t)] + \frac{N-n}{N} \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j\ n+1}^{[n+1]} + V_{n+1\ j}^{[n+1]}, \rho_N^{(n+1)}(t)] \\ &= \sum_{j=1}^n [A_j^{[n]}, \rho_N^{(n)}(t)] + \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j\ n+1}^{[n+1]} + V_{n+1\ j}^{[n+1]}, \rho_N^{(n+1)}(t)] \\ &+ \frac{1}{N} \sum_{i \neq j=1}^n [V_{i\ j}^{[n]}, \rho_N^{(n)}(t)] - \frac{n}{N} \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j\ n+1}^{[n+1]} + V_{n+1\ j}^{[n+1]}, \rho_N^{(n+1)}(t)] \\ &= \mathcal{L}_n(\rho_N^{(n)}(t)) + \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j\ n+1}^{[n+1]} + V_{n+1\ j}^{[n+1]}, \rho_N^{(n+1)}(t)] + \epsilon_n(t, N, \rho_N(t_0)) \end{aligned} \quad (4.26)$$

where  $\mathcal{L}_n(\cdot) := \sum_{j=1}^n [A_j^{[n]}, \cdot]$  and

$$\epsilon_n(t, N, \rho_N(t_0)) := \frac{1}{N} \sum_{i \neq j=1}^n [V_{i\ j}^{[n]}, \rho_N^{(n)}(t)] - \frac{n}{N} \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j\ n+1}^{[n+1]} + V_{n+1\ j}^{[n+1]}, \rho_N^{(n+1)}(t)].$$

Define  $E_{n,N}(t) := \rho_N^{(n)}(t) - \rho(t)^{\otimes n}$  for each  $n \leq N$ . Then, by subtracting (4.25)

from (4.26), we obtain that for each  $n, N \in \mathbb{N}$  with  $n \leq N - 1$ ,

$$\begin{aligned} i \frac{d}{dt} E_{n,N}(t) &= \sum_{j=1}^n [A_j^{[n]}, E_{n,N}(t)] + \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j,n+1}^{[n+1]} + V_{n+1,j}^{[n+1]}, E_{n+1,N}(t)] \\ &+ \epsilon_n(t, N, \rho_N(t_0)). \end{aligned}$$

The next step is to obtain an upper bound for the trace class norm of  $E_{n,N}(t)$  which does not involve  $\|A\|_\infty$ . In order to do this, we define a new evolution  $\mathcal{U}_{n,t}$  which will be evaluated at  $E_{n,N}(t)$ .

We define  $\mathcal{U}_{n,t}(\cdot) := e^{it\mathcal{L}_n(\cdot)} = e^{it \sum_{j=1}^n A_j^{[n]}} (\cdot) e^{-it \sum_{j=1}^n A_j^{[n]}}$ . We claim that  $\mathcal{U}_{n,t}$  is an isometry on the trace class operators on  $\mathbb{H}^{\otimes n}$  for each  $n \in \mathbb{N}$  and  $t \in [0, \infty)$ . Indeed, if  $T \in \mathcal{B}(\mathbb{H}^{\otimes n})$  is a trace class operator, then

$$\|\mathcal{U}_{n,t}(T)\|_1 = \|e^{it \sum_{j=1}^n A_j^{[n]}} T e^{-it \sum_{j=1}^n A_j^{[n]}}\|_1 \leq \|e^{it \sum_{j=1}^n A_j^{[n]}}\|_\infty \|T\|_1 \|e^{-it \sum_{j=1}^n A_j^{[n]}}\|_\infty = \|T\|_1,$$

and similarly, by observing that  $T = e^{-it \sum_{j=1}^n A_j^{[n]}} \mathcal{U}_{n,t}(T) e^{it \sum_{j=1}^n A_j^{[n]}}$ , we get the reverse inequality.

Now define  $Z_{n,N}(t) := \mathcal{U}_{n,t}(E_{n,N}(t))$ . Then

$$\begin{aligned} \frac{d}{dt} Z_{n,N}(t) &= i \sum_{j=1}^n A_j^{[n]} \mathcal{U}_{n,t}(E_{n,N}(t)) - i \mathcal{U}_{n,t}(E_{n,N}(t)) \sum_{j=1}^n A_j^{[n]} \\ &- i \mathcal{U}_{n,t} \left( \mathcal{L}_n(E_{n,N}(t)) + \sum_{j=1}^n \text{tr}_{\{n+1\}} [V_{j,n+1}^{[n+1]} + V_{n+1,j}^{[n+1]}, E_{n+1,N}(t)] + \epsilon_n(t, N, \rho_N(t_0)) \right) \\ &= -i \sum_{j=1}^n \mathcal{U}_{n,t} \left( \text{tr}_{\{n+1\}} [V_{j,n+1}^{[n+1]} + V_{n+1,j}^{[n+1]}, E_{n+1,N}(t)] \right) - i \mathcal{U}_{n,t}(\epsilon_n(t, N, \rho_N(t_0))) \quad (4.27) \end{aligned}$$

where the last equality follows because  $\mathcal{U}_{n,t}$  and  $\mathcal{L}_n$  commute hence

$$i \sum_{j=1}^n A_j^{[n]} \mathcal{U}_{n,t}(E_{n,N}(t)) - i \mathcal{U}_{n,t}(E_{n,N}(t)) \sum_{j=1}^n A_j^{[n]} - i \mathcal{U}_{n,t}(\mathcal{L}_n(E_{n,N}(t))) = 0.$$

By integrating both sides of equation (4.27), we obtain that, for each  $n, N \in \mathbb{N}$  with  $n \leq N - 1$ ,

$$\begin{aligned} Z_{n,N}(t) &= Z_{n,N}(t_0) - i \sum_{j=1}^n \int_{t_0}^t \mathcal{U}_{n,s} \left( \text{tr}_{\{n+1\}} [V_{j,n+1}^{[n+1]} + V_{n+1,j}^{[n+1]}, E_{n+1,N}(s)] \right) ds \\ &- i \int_{t_0}^t \mathcal{U}_{n,s}(\epsilon_n(s, N, \rho_N(t_0))) ds. \end{aligned}$$



We will aim to show that  $\lim_{N \rightarrow \infty} \|E_{n,N}(t)\|_1 = 0$ . We have

$$\begin{aligned}
\|E_{n,N}(t)\|_1 &= \|Z_{n,N}(t)\|_1 \leq \|Z_{n,N}(t_0)\|_1 + \left\| \int_{t_0}^t \mathcal{U}_{n,s}(\epsilon_n(s, N, \rho_N(t_0))) \right\|_1 \\
&+ \left\| \sum_{j=1}^n \int_{t_0}^t \mathcal{U}_{n,s}(\text{tr}_{\{n+1\}}[V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]}, E_{n+1,N}(s)]) ds \right\|_1 \\
&\leq \|E_{n,N}(t_0)\|_1 + (t - t_0) \|\epsilon_n(s, N, \rho_N(t_0))\|_1 \\
&+ 4\|V\|_\infty \sum_{j=1}^n \int_{t_0}^t \|E_{n+1,N}(s)\|_1 ds.
\end{aligned} \tag{4.28}$$

We notice that for every  $n, N \in \mathbb{N}$  with  $n \leq N - 1$  and  $s \in [0, \infty)$ ,

$$\begin{aligned}
&\|\epsilon_n(s, N, \rho_N(t_0))\|_1 \\
&= \left\| \frac{1}{N} \sum_{i \neq j=1}^n [V_{ij}^{[n]}, \rho_N^{(n)}(s)] - \frac{n}{N} \sum_{j=1}^n \text{tr}_{\{n+1\}}[V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]}, \rho_N^{(n+1)}(s)] \right\|_1 \\
&\leq \frac{1}{N} \left\| \sum_{i \neq j; i, j=1}^n [V_{ij}^{[n]}, \rho_N^{(n)}(s)] \right\|_1 + \frac{n}{N} \left\| \sum_{j=1}^n \text{tr}_{\{n+1\}}[V_{jn+1}^{[n+1]} + V_{n+1j}^{[n+1]}, \rho_N^{(n+1)}(s)] \right\|_1 \\
&\leq \frac{n(n-1)}{N} \|V\|_\infty + \frac{4n^2}{N} \|V\|_\infty \leq \frac{5n^2}{N} \|V\|_\infty.
\end{aligned} \tag{4.29}$$

Inequalities (4.28) and (4.29) give

$$\|E_{n,N}(t)\|_1 \leq \|E_{n,N}(t_0)\|_1 + \frac{5n^2}{N} \|V\|_\infty (t - t_0) + 4\|V\|_\infty \sum_{j=1}^n \int_{t_0}^t \|E_{n+1,N}(s)\|_1 ds.$$

Fixing  $n \in \mathbb{N}$  and iterating this inequality  $m$  more times for  $m \in \mathbb{N}$  and  $m \leq N - n - 1$ , we obtain

$$\begin{aligned}
\|E_{n,N}(t)\|_1 &\leq \|E_{n,N}(t_0)\|_1 + \frac{5n^2}{N} \|V\|_\infty (t - t_0) \\
&+ \sum_{k=1}^m (4\|V\|_\infty)^k \left[ \sum_{j_1=1}^n \sum_{j_2=1}^{n+1} \cdots \sum_{j_k=1}^{n+k-1} \left( \frac{(t - t_0)^k}{k!} \|E_{n+k,N}(t_0)\|_1 \right. \right. \\
&\quad \left. \left. + \frac{5(n+k)^2}{N} \|V\|_\infty \frac{(t - t_0)^{k+1}}{(k+1)!} \right) \right] \\
&+ (4\|V\|_\infty)^{m+1} \sum_{j_1=1}^n \sum_{j_2=1}^{n+1} \cdots \sum_{j_{m+1}=1}^{n+m} \int_{t_0}^t \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_m} \|E_{n+m+1,N}(t_{m+1})\|_1 dt_{m+1} dt_m \cdots dt_1.
\end{aligned} \tag{4.31}$$

Since  $\rho_N^{(n+m+1)}(t)$  is a density operator, by the definition of  $K_{T_0}$  we have that

$$\|E_{n+m+1,N}(s)\|_1 \leq 1 + K_{T_0}^{n+m+1} \leq 2K_{T_0}^{n+m+1}$$

for  $s \in [T_0, T_0 + 1]$  (since  $K \geq \|\rho(T_0)\|_1 = 1$ ). Thus the last line of inequality (4.30) can be bounded above by

$$\begin{aligned}
& (4\|V\|_\infty)^{m+1} \sum_{j_1=1}^n \sum_{j_2=1}^{n+1} \cdots \sum_{j_{m+1}=1}^{n+m} \int_{t_0}^t \int_{t_0}^{t_1} \cdots \int_{t_0}^{t_m} 2K_{T_0}^{n+m+1} dt_{m+1} dt_m \cdots dt_1 \\
&= 2K_{T_0}^{n+m+1} (4\|V\|_\infty)^{m+1} n(n+1) \cdots (n+m) \frac{(t-t_0)^{m+1}}{(m+1)!} \\
&= 2K_{T_0}^n \frac{n(n+1) \cdots (n+m)}{(m+1)!} (4\|V\|_\infty K_{T_0} (t-t_0))^{m+1} \\
&= 2K_{T_0}^n \binom{n+m}{n-1} (4\|V\|_\infty K_{T_0} (t-t_0))^{m+1} \\
&\leq \frac{2K_{T_0}^n}{n!} (n+m)^n (4\|V\|_\infty K_{T_0} (t-t_0))^{m+1}
\end{aligned}$$

where we used that

$$\binom{n+m}{n-1} = \frac{(n+m)!}{(n-1)!(m+1)!} \leq \frac{(n+m)^{n-1}}{(n-1)!} = \frac{n(n+m)^{n-1}}{n!} \leq \frac{(n+m)^n}{n!}.$$

Thus, by (4.30), we obtain

$$\begin{aligned}
& \|E_{n,N}(t)\|_1 \leq \|E_{n,N}(t_0)\|_1 + \frac{5n^2}{N} \|V\|_\infty (t-t_0) \\
&+ \sum_{k=1}^m (4\|V\|_\infty)^k \left[ \sum_{j_1=1}^n \sum_{j_2=1}^{n+1} \cdots \sum_{j_k=1}^{n+k-1} \left( \frac{(t-t_0)^k}{k!} \|E_{n+k,N}(t_0)\|_1 \right. \right. \\
&\quad \left. \left. + \frac{5(n+k)^2}{N} \|V\|_\infty \frac{(t-t_0)^{k+1}}{(k+1)!} \right) \right] \\
&+ \frac{2K_{T_0}^n}{n!} (n+m)^n (4\|V\|_\infty K_{T_0} (t-t_0))^{m+1}.
\end{aligned}$$

Let  $\epsilon > 0$ . Fix  $t \in [t_0, t_0 + \delta_{T_0}]$ . Choose  $m$  such that

$$\frac{2K_{T_0}^n}{n!} (n+m)^n (4\|V\|_\infty K_{T_0} (t-t_0))^{m+1} \leq \frac{2K_{T_0}^n}{n!} (n+m)^n \left(\frac{1}{2}\right)^{m+1} < \frac{\epsilon}{3}$$

where the first inequality is valid by the choice of  $\delta_{T_0}$ . Then since  $\lim_{N \rightarrow \infty} \|E_{n,N}(t_0)\|_1 = 0$  by Proposition 3.1.7, we can choose  $N_1 \in \mathbb{N}$  large enough such that

$$\|E_{n,N}(t_0)\|_1 + \sum_{k=1}^m (4\|V\|_\infty)^k \sum_{j_1=1}^n \sum_{j_2=1}^{n+1} \cdots \sum_{j_k=1}^{n+k-1} \frac{(t-t_0)^k}{k!} \|E_{n+k,N}(t_0)\|_1 < \frac{\epsilon}{3}$$

for all  $N \geq N_1$ .

Then choose  $N_2 \in \mathbb{N}$  such that

$$\frac{5n^2}{N} \|V\|_\infty (t - t_0) + \sum_{k=1}^m (4\|V\|_\infty)^k \sum_{j_1=1}^n \sum_{j_2=1}^{n+1} \cdots \sum_{j_k=1}^{n+k-1} \frac{5(n+k)^2}{N} \|V\|_\infty \frac{(t - t_0)^{k+1}}{(k+1)!} < \frac{\epsilon}{3}$$

for all  $N \geq N_2$ . For  $N \geq \max\{N_1, N_2\}$ ,

$$\|E_{n,N}(t)\|_1 < \epsilon,$$

i.e.  $\lim_{N \rightarrow \infty} \|E_{n,N}(t)\|_1 = 0$  for all  $t \in [t_0, t_0 + \delta_{T_0}]$ . Since, for  $n = 1$ ,  $\|E_{1,N}(t)\|_1 = \|\rho_N^{(1)}(t) - \rho(t)\|_1 \xrightarrow{N \rightarrow \infty} 0$  and  $\rho_N^{(1)}$  are density operators for all  $N$ , we now obtain that  $\rho(t) \in \mathcal{D}(\mathbb{H})$  for all  $t \in [t_0, t_0 + \delta_{T_0}]$ . Therefore  $\rho_N^{(n)}(t)$  is  $\rho(t)$ -chaotic for all  $t \in [t_0, t_0 + \delta_{T_0}]$ .

□

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