2017

Polynomials Of Small Mahler Measure With no Newman Multiples

Spencer Victoria Saunders
University of South Carolina

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by

Spencer Victoria Saunders

Bachelor of Science
Regent University 2014

Submitted in Partial Fulfillment of the Requirements

for the Degree of Master of Science in

Mathematics

College of Arts and Sciences

University of South Carolina

2017

Accepted by:

Michael Filaseta, Director of Thesis

Ognian Trifonov, Reader

Cheryl L. Addy, Vice Provost and Dean of the Graduate School

ABSTRACT

A Newman polynomial is a polynomial with coefficients in $\{0,1\}$ and with constant term 1. It is known that the roots of a Newman polynomial must lie in the slit annulus $\{z \in \mathbb{C} : \phi^{-1} < |z| < \phi\} \setminus \mathbb{R}^+$ where ϕ denotes the golden ratio; however, it is not guaranteed that all polynomials whose roots lie in this slit annulus divide a Newman polynomial. The Mahler measure of a monic polynomial is defined to be the product of the absolute values of those roots of the polynomial which are greater than 1. K. Hare and M. Mossinghoff have asked whether there is a $\sigma > 1$ such that if a polynomial $f(z) \in \mathbb{Z}[z]$ has Mahler measure less than σ and has no nonnegative real roots, then it must divide a Newman polynomial. In this thesis, we present a new upper bound on such a σ if it exists. We also show that there are infinitely many monic polynomials that have distinct Mahler measures which all lie below ϕ, have no nonnegative real roots, and have no Newman multiples. Finally, we consider a more general notion in which multiples of polynomials are considered in $\mathbb{R}[z]$ instead of $\mathbb{Z}[z]$.

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CHAPTER 1

INTRODUCTION

A $0,1$-polynomial is a monic, polynomial with all its coefficients in $\{0,1\}$. A *Newman polynomial* is a univariate $0,1$-polynomial with constant term 1. We will denote the set of all Newman polynomials by $\mathscr{N}$. Odlyzko and Poonen (1993) showed that if $\alpha \in \mathbb{C}$ is a root of a polynomial $F \in \mathscr{N}$, then α lies in the slit annulus

$$A_\phi \setminus \mathbb{R}^+ = \{z \in \mathbb{C} : \phi^{-1} < |z| < \phi\} \setminus \mathbb{R}^+$$

where $\phi = (1+\sqrt{5})/2$ denotes the golden ratio.

In this text, we will explore the cases when a given polynomial has or, more appropriately, does not have a multiple that is Newman. It is important to note that a polynomial with a root outside the annulus A_ϕ cannot have a Newman multiple. Moreover, we note that no polynomial with a positive real root can have a Newman multiple.

The following result, due to K. Hare and M. J. Mossinghoff (2014), answers a question about a particular class of polynomials which satisfy the above conditions.

Theorem 1.1. *If a real number $\beta \in (-\phi, -1)$ is the only root of a monic polynomial $f(z) \in \mathbb{Z}[z]$ which lies outside the open unit disk $\mathscr{D} = \{z \in \mathbb{C} : |z| < 1\}$ and $f(z)$ has no nonnegative real roots, then there exists a Newman polynomial $F(z)$ with $F(\beta) = 0$.*

Before giving their proof, we will first make a few more observations. In the theorem above, β represents a real algebraic integer with $|\beta| > 1$ whose other conjugates all lie in $\mathscr{D}$ (that is, having modulus less than 1). A $\beta > 0$ with this property is called a *Pisot number*. Bertin, Decomps-Guilloux, Grandet-Hugot, Pathiaux-Delefosse, and Schreiber

(1992) show that every Pisot number less than ϕ may be expressed as a root of one of the following polynomials for some positive integer n:

$$
\begin{aligned}
p_{2n}(z) &= z^{2n+1} - z^{2n-1} - z^{2n-2} - \cdots - z - 1, \\
q_{2n+1}(z) &= z^{2n+1} - z^{2n} - z^{2n-2} - \cdots - z^2 - 1, \\
r_n(z) &= z^n(z^2 - z - 1) + z^2 - 1, \\
g(z) &= z^6 - 2z^5 + z^4 - z^2 + z - 1.
\end{aligned}
$$

Because a Newman polynomial cannot have positive real roots, no Pisot number will be a root of a Newman polynomial. We say β is a *negative Pisot number* if $-\beta$ is a Pisot number and consider instead such numbers. Observe that Theorem 1.1 is a statement about negative Pisot numbers.

With the above in mind, we consider similar families of polynomials with negative Pisot numbers as roots. Specifically, we consider the monic polynomials $-p_{2n}(-z)$, $-q_{2n+1}(-z)$, $r_{2n}(-z)$, $-r_{2n+1}(-z)$, and $g(-z)$. From the above, it follows that any negative Pisot number $\beta > -\phi$ occurs as a root of one of these monic polynomials for some n. We proceed now to a proof of Theorem 1.1 based on the work of Hare and Mossinghoff (2014).

Lemma 1.2. *For every positive integer n, the polynomial $r_n(z)$ has exactly one root β outside of $\mathscr{D}$ with $\beta \in \mathbb{R}$ and $\beta > 1$. Furthermore, the polynomial $r_n(z)$ is irreducible over $\mathbb{Q}$.*

Proof. For $z \in \mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$, we first observe that by a direct computation we have

$$|z^2 - z - 1|^2 = (z^2 - z - 1)(\overline{z}^2 - \overline{z} - 1) = 3 - z^2 - \overline{z}^2$$

and

$$|z^2 - 1|^2 = (z^2 - 1)(\overline{z}^2 - 1) = 2 - z^2 - \overline{z}^2,$$

where $\overline{z}$ denote the complex conjugate of z. In particular, we obtain

$$|z^2 - z - 1| > |z^2 - 1| \quad \text{for all } z \in \mathscr{C}. \tag{1.1}$$

Let $\rho_n(z) = z^n(z^2 - z - 1)$. From the above, we deduce that for $z \in \mathscr{C}$, we have

$$|r_n(z) - \rho_n(z)| = |z^2 - 1| < |z^2 - z - 1| = |\rho_n(z)|.$$

By Rouché's Theorem, we see that $r_n(z)$ and $\rho_n(z)$ have the same number of roots, counted to their multiplicity, inside $\mathscr{C}$. As $z^2 - z - 1$ has exactly one root in $\mathscr{C}$, we deduce $r_n(z)$ has exactly $n+1$ roots inside $\mathscr{C}$ and exactly one root outside $\mathscr{C}$. Call this root β. As $r_n(1) = -1$ and $r_n(2) = 2^n + 3 > 0$, we obtain β is a real root in the interval $(1,2)$. Thus, the first statement of the lemma follows.

Observe that $r_n(z)$ is monic. Assume $r_n(z) = u(z)v(z)$ for some monic $u(z)$ and $v(z)$ in $\mathbb{Z}[z]$ each of degree ≥ 1 with $u(\beta) = 0$. Since β is the only root of $r_n(z)$ outside of $\mathscr{D}$, the roots of $v(z)$ each have absolute value < 1. Since $|v(0)|$ is the absolute value of the product of the roots of $v(z)$, we obtain $|v(0)| < 1$. Note that $r_n(0) = -1$ implies $v(0) \neq 0$. We obtain a contradiction as $|v(0)|$ is a non-zero integer < 1, which is impossible. We deduce that $r_n(z)$ is irreducible, completing the proof. □

Proof of Theorem 1.1. Let β be a real number in $(-\phi, -1)$, and let $f(z) \in \mathbb{Z}[z]$ be as in the theorem so that $f(\beta) = 0$. This immediately implies that $f(z)$ is the minimal polynomial of β. Otherwise, there would exist monic $g(z)$ and $h(z) \in \mathbb{Z}[z]$ of degrees ≥ 1 satisfying $f(z) = g(z)h(z)$, $g(\beta) = 0$ and all of the roots of $h(z)$ lie inside $\mathscr{D}$. But this is a contradiction, as it implies $|h(0)|$, which is the absolute value of the product of the roots of $h(z)$, is not an integer.

Now, since $\beta \in (-\phi, -1)$, we have that β is a root of one of the polynomials $-p_{2n}(-z)$, $-q_{2n+1}(-z)$, $r_{2n}(-z)$, $-r_{2n+1}(-z)$, and $g(-z)$ for some positive integer n. Hence, the minimal polynomial for β, namely $f(z)$, must divide one of these polynomials for some positive integer n. Fix such an n. As a consequence of Lemma 1.2, the polynomial $r_{2n}(-z)$ is irreducible. Thus, if $f(z)$ divides $r_{2n}(-z)$, then $f(z) = r_{2n}(-z)$. Assume this is the case. Then $f(0) = r_{2n}(0) = -1$ and $f(1) = r_{2n}(-1) = 1$. By the Intermediate Value Theorem, $f(z)$ has a positive real root, contradicting the conditions of $f(z)$ in the statement of Theorem 1.1. Hence, $f(z)$ does not divide $r_{2n}(-z)$. Similarly, one checks that $g(-z)$ is irreducible, $g(0)=-1$, and $g(-1)=1$, so an analogous argument gives that $f(z)$ does not divide $g(-z)$. Therefore, $f(z)$ divides one of the polynomials $-p_{2n}(-z)$, $-q_{2n+1}(-z)$, and $-r_{2n+1}(-z)$.

To finish the proof, it suffices now to show that each of the polynomials $-p_{2n}(-z)$, $-q_{2n+1}(-z)$, and $-r_{2n+1}(-z)$ divides a Newman polynomial. Observe that

$$\begin{aligned}
&-p_{2n}(-z)\big(z^{2n}+z^{2n-1}+\cdots+z+1\big)\\
&\qquad=\big(z^{2n+1}-z^{2n-1}+z^{2n-2}-\cdots-z+1\big)\big(z^{2n}+z^{2n-1}+\cdots+z+1\big)\\
&\qquad=z^{4n+1}+z^{4n}+z^{4n-2}+\cdots+z^{2n+4}+z^{2n+2}+z^{2n-2}+z^{2n-4}+\cdots+z^2+1,
\end{aligned}$$

which is a Newman polynomial. Also, we see immediately that

$$-q_{2n+1}(-z)=z^{2n+1}+z^{2n}+z^{2n-2}+\cdots+z^2+1,$$

so $-q_{2n+1}(-z)$ is a Newman polynomial. Next, we use that

$$\begin{aligned}
&-r_{2n+1}(-z)\big(z^{2n}+z^{2n-2}+\cdots+z^2+1\big)\\
&\qquad=\big(\big(z^{2n+1}-1\big)\big(z^2-1\big)+z^{2n+2}\big)\big(z^{2n}+z^{2n-2}+\cdots+z^2+1\big)\\
&\qquad=\big(z^{2n+1}-1\big)\big(z^{2n+2}-1\big)+z^{2n+2}\big(z^{2n}+z^{2n-2}+\cdots+z^2+1\big)\\
&\qquad=z^{4n+3}-z^{2n+1}+z^{2n+2}\big(z^{2n}+z^{2n-2}+\cdots+z^2\big)+1.
\end{aligned}$$

From this, we see that

$$-r_{2n+1}(-z)\big(z^{2n}+z^{2n-2}+\cdots+z^2+1\big)\cdot\big(z^{2n+1}+1\big)$$

is a Newman polynomial. Therefore, in any case, we see that $f(z)\mid F(z)$ for some $F(z)\in\mathscr{N}$, completing the proof. □

The *Mahler measure* of a polynomial

$$f(z)=\sum_{j=0}^{n}a_jz^j=a_n\prod_{k=1}^{n}(z-\beta_k),$$

for $f(z) \in \mathbb{Z}[z]$, is defined to be

$$M(f) = |a_n| \prod_{i=1}^{n} \max\{1, |\beta_i|\}.$$

If α is an algebraic integer, then $M(\alpha)$ is the Mahler measure of the minimal polynomial of α. If $f(z)$ is monic, then $M(f)$ is simply the product of the absolute values of all the roots of $f(z)$ which lie outside the unit circle $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$ in the complex plane. We also note that the Mahler measure of any cyclotomic polynomial or product of cyclotomic polynomials and a power of z is exactly 1. Furthermore, if $f(z)$ is the minimal polynomial of a Pisot number β, then $M(f) = |\beta|$.

In 1933, D. H. Lehmer posited that a lower bound greater than 1 for the Mahler measure of a polynomial $f(z) \in \mathbb{Z}[z]$ with $M(f) \neq 1$ exists, but this remains an open problem. The smallest measure greater than 1 he found is $1.17628\ldots$, occurring for the polynomial

$$\mathscr{L}(z) = z^{10} + z^9 - z^7 - z^6 - z^5 - z^4 - z^3 + z + 1.$$

This is still the smallest known Mahler measure larger than 1 of a polynomial in $\mathbb{Z}[z]$.

Hare and Mossinghoff (2014) point out that some interesting results follow when one attempts to bound the Mahler measure of a polynomial. For instance, Pathiaux (1973) and Mignotte (1975) showed that if α is an algebraic number and $M(\alpha) < 2$, then there exists a polynomial $F(z)$ with coefficients from $\{-1, 0, 1\}$ such that $F(\alpha) = 0$. Thus, $F(z)$ has height 1. Moreover, Siegel's lemma, as referenced by Hare and Mossinghoff (2014), implies that if $f(z) \in \mathbb{Z}[z]$ has $M(f) < 2$, then there exists a polynomial $F(z)$ with height 1 such that $f(z) \mid F(z)$, even if $f(z)$ is not irreducible. Given their particular interest in the set of Newman polynomials, Hare and Mossinghoff pose the next natural question (which would follow if Lehmer's conjecture is true):

Problem 1. *Does there exist a real number $\sigma > 1$ such that if $f(z) \in \mathbb{Z}[z]$ has no nonnegative real roots and $M(f) < \sigma$, then $f(z) \mid F(z)$ for some $F(z) \in \mathscr{N}$?*

Note that we consider only $\sigma > 1$. A polynomial which has Mahler measure 1 is necessarily a product of cyclotomic polynomials times a power of z. For any such polynomial with $f(0) \neq 0$ and $f(1) \neq 0$, it can be shown that $f(z)$ has a multiple that is Newman. Given Theorem 1.1, one might like to immediately take σ to be ϕ, but Hare and Mossinghoff (2014) show that if such a σ exists, then $\sigma < \phi$. The polynomials enumerated in Table 1.1 are among the polynomials they encountered, each having no nonnegative real roots with Mahler measure less than ϕ and which do not divide a Newman polynomial. The polynomial of smallest measure they discovered like this is $z^6 - z^5 - z^3 + z^2 + 1$, giving as an upper bound for σ (assuming σ exists), the value $1.556014485\ldots$.

Table 1.1 Some polynomials from Hare and Mossinghoff (2014) with small measure and no Newman multiple

Polynomial	Mahler measure
$z^6 - z^5 - z^3 + z^2 + 1$	$1.556014485\ldots$
$z^7 - z^6 - z^5 + z^4 + z^3 - z^2 + 1$	$1.558378942\ldots$
$z^8 - z^7 + z^2 + 1$	$1.604364647\ldots$
$z^9 - z^8 - z^6 + z^2 + 1$	$1.615829244\ldots$
$z^8 - z^7 - z^5 + z^2 + 1$	$1.617538308\ldots$
$z^8 + z^7 + 2z^6 + z^5 + z^4 + z^3 + 2z^2 + z + 1$	$1.618530599\ldots$
$z^9 - z^8 + z^7 + z^5 + z^4 + z^2 + 1$	$1.621082531\ldots$
$z^8 - z^7 + z^5 + z^3 - z + 1$	$1.624147966\ldots$
$z^7 + z^5 - z^4 - z + 1$	$1.646642716\ldots$
$z^8 + z^7 + 2z^6 + 2z^5 + z^4 + z^3 + z^2 + z + 1$	$1.652235034\ldots$

P. Drungilas, J. Jankauskas, and J. Šiurys (2016) have improved upon this bound. They found 16 polynomials all of Mahler measure less than $1.55601\ldots$ which do not divide a Newman polynomial, with the ones of smallest Mahler measure listed in Table 1.2. The one with smallest Mahler measure is $z^9 + z^8 + z^7 - z^5 - z^4 - z^3 + 1$ with measure $1.436632261\ldots$. This now gives that if such a σ exists, then $\sigma \leq 1.436632261\ldots$.

In this thesis, we show that if σ exists, then $\sigma \leq 1.263095875\ldots$. Some of the resulting polynomials of small Mahler measure we found are listed in Table 1.3. Note, we also

Table 1.2 Some polynomials from Drungilas, Jankauskas, and Šiurys (2016) of small measure with with no Newman multiple

Polynomial	Mahler measure
$z^9+z^8+z^7-z^5-z^4-z^3+1$	$1.436632261\ldots$
$z^9+z^8-z^3-z^2+1$	$1.483444878\ldots$
$z^9-z^7-z^5+z^3+z+1$	$1.489581321\ldots$
$z^8-z^7-z^4+z^3+1$	$1.489581321\ldots$
$z^8+z^7-z^3-z^2+1$	$1.518690904\ldots$
$z^8+z^7+z^6-z^4-z^3-z^2+1$	$1.536566472\ldots$
$z^9-z^8-z^6+z^5+1$	$1.536913983\ldots$
$z^9+z^5-z^3-z^2+1$	$1.550687063\ldots$

found several more polynomials with no Newman multiple with Mahler measure less than $1.436632261\ldots$ as given by Drungilas, Jankauskas, and Šiurys (2016), but we do not list those here.

We also show that there are infinitely many monic, irreducible polynomials $f(z)$ having exactly 2 roots outside $\mathscr{C}=\{z\in\mathbb{C}:|z|=1\}$ and no nonnegative real roots for which $M(f)<\phi$ and for which $f(z)$ has no Newman multiples. We use the example given by Hare and Mossinghoff (2014),

$$f(z)=z^6-z^5-z^3+z^2+1$$

with $M(f)=1.556014485\ldots$ to construct our infinite list. We define

$$\widetilde{f}(z)=z^{\deg f}f(1/z) \tag{1.2}$$

to be the *reciprocal* of $f(z)$ and construct the polynomial $F_n(z)=f(z)z^n+\widetilde{f}(z)$. A polynomial $f(z)\in\mathbb{Z}[z]$ is said to be reciprocal if $f(z)=\pm\tilde{f}(z)$ and non-reciprocal otherwise. The existence of our infinite list of polynomials as described above is a result of the following theorem.

Theorem 1.3. *Let $f(z) \in \mathbb{Z}[z]$ be monic and such that $f(z)$ has no roots on the unit circle $\mathscr{C}$. Suppose that $f(z)$ has no positive real roots and exactly two roots outside $\mathscr{C}$, both non-real and with multiplicity one. Suppose further that* $\gcd(f(z), \widetilde{f}(z)) = 1$. *For n a positive integer, define $h_n(z)$ as the largest degree monic factor of $f(z)z^n + \widetilde{f}(z)$ not divisible by a cyclotomic polynomial. Then the polynomials $h_n(z)$ include infinitely many distinct irreducible polynomials with distinct Mahler measures approaching the Mahler measure of $f(z)$ as n tends to infinity. Furthermore, these irreducible $h_n(z)$ have no positive real roots and each has exactly two roots outside $\mathscr{C}$. Also, if β is a root of $f(z)$ with $|\beta| > 1$ and if β is not a root of a Newman polynomial, then for n sufficiently large, no multiple of $h_n(z)$ in $\mathbb{Z}[z]$ is a Newman polynomial.*

Table 1.3 Some polynomials of small measure with no Newman multiple

Polynomial	Mahler measure
$z^{44} - z^{42} + z^{40} - z^{38} - z^{33} - z^{32} + z^{31} + z^{30} - 2z^{29} - z^{28} + 2z^{27} + z^{26} - z^{25} + z^{23} + z^{22} + z^{21} - z^{19} + z^{18} + 2z^{17} - z^{16} - 2z^{15} + z^{14} + z^{13} - z^{12} - z^{11} - z^{6} + z^{4} - z^{2} + 1$	1.263095875...
$z^{26} - z^{23} - z^{21} + z^{15} + z^{13} + z^{11} - z^{5} - z^{3} + 1$	1.272019269...
$z^{50} - z^{49} + z^{48} - z^{47} - z^{40} + z^{39} - z^{38} + z^{37} - z^{36} + z^{35} - z^{34} + z^{33} + z^{30} - z^{29} + z^{28} - z^{27} + z^{26} - z^{25} + z^{24} - z^{23} + z^{22} - z^{21} + z^{20} + z^{17} - z^{16} + z^{15} - z^{14} + z^{13} - z^{12} + z^{11} - z^{10} - z^{3} + z^{2} - z + 1$	1.273464959...
$z^{48} - z^{47} + z^{46} - z^{45} - z^{38} + z^{37} - z^{36} + z^{35} - z^{34} + z^{33} - z^{32} + z^{31} + z^{28} - z^{27} + z^{26} - z^{25} + z^{24} - z^{23} + z^{22} - z^{21} + z^{20} + z^{17} - z^{16} + z^{15} - z^{14} + z^{13} - z^{12} + z^{11} - z^{10} - z^{3} + z^{2} - z + 1$	1.279464310...
$z^{48} - z^{47} + z^{46} - z^{45} + z^{44} - z^{43} - z^{40} + z^{39} - 2z^{38} + 2z^{37} - 2z^{36} + 2z^{35} - z^{34} + z^{33} + z^{30} - z^{29} + z^{28} - z^{27} + z^{26} - z^{25} + z^{24} - z^{23} + z^{22} - z^{21} + z^{20} - z^{19} + z^{18} + z^{15} - z^{14} + 2z^{13} - 2z^{12} + 2z^{11} - 2z^{10} + z^{9} - z^{8} - z^{5} + z^{4} - z^{3} + z^{2} - z + 1$	1.279702474...
$z^{30} - z^{29} - z^{23} + z^{22} + z^{16} - z^{15} + z^{14} + z^{8} - z^{7} - z + 1$	1.299764321...
$z^{28} - z^{26} - z^{25} + z^{22} + z^{21} - z^{19} + z^{14} - z^{9} + z^{7} + z^{6} - z^{3} - z^{2} + 1$	1.309200435...

We end the thesis by showing that one can strengthen the notion of polynomials not having a Newman multiple by giving an explicit result for the case that $f(z)$ is equal to the first polynomial entry in Table 1.3. We show that not only is there no multiple of $f(z)$ in $\mathbb{Z}[z]$ that is Newman, but further there is no multiple of $f(z)$ in $\mathbb{R}[z]$ having all nonnegative coefficients bounded above by $1.5713809\ldots$. We actually show more, namely that there is a root β of $f(z)$ which cannot be a root of a polynomial having all nonnegative real coefficients bounded by $1.5713809\ldots$.

CHAPTER 2

ALGORITHMS

2.1 DETERMINING WHEN A POLYNOMIAL DOES NOT HAVE A NEWMAN MULTIPLE

Hare and Mossinghoff (2014) outline an algorithm for determining whether a real negative Pisot number $\beta \in (-\phi, -1)$ is a root of a Newman polynomial. Let $f(z)$ be the minimal polynomial of β. Then $f(z)$ divides a Newman polynomial $F(z)$ if and only if $F(\beta) = 0$. If such a Newman polynomial exists, we can construct it by adding powers of β to 1 and checking when this value is 0. Let $\mathscr{N}_0$ denote the set of non-zero $0,1$-polynomials. For a nonnegative integer d, set

$$\mathscr{N}'(\beta, d) = \{F(\beta) : F \in \mathscr{N}_0 \text{ and } \deg F(z) \leq d\}.$$

Note that $\mathscr{N}'(\beta, 0) = \{1\}$, and note that with each iteration of d, we have

$$\begin{aligned}\mathscr{N}'(\beta, d+1) = \mathscr{N}'(\beta, d) \cup \{\beta\alpha : \alpha \in \mathscr{N}'(\beta, d)\} \\ \cup \{\beta\alpha + 1 : \alpha \in \mathscr{N}'(\beta, d)\}.\end{aligned}$$

Our goal is either to find an element of $\mathscr{N}'(\beta, d+1)$ that is equal to 0 or to prove that β is not a root of any Newman polynomial. Unfortunately, after d iterations, the size of $\mathscr{N}'(\beta, d)$ can be as large as 2^d which becomes cumbersome to compute. We describe a method next that Hare and Mossinghoff (2014) use to cut down the search space.

Suppose $\beta \in (-\phi, -1)$ is root of a Newman polynomial

$$F(z) = \sum_{j=0}^{n} \varepsilon_j z^j,$$

with $\varepsilon_0 = \varepsilon_n = 1$. Then $F(\beta) = 0$, and $0 \in \mathcal{N}'(\beta, n)$. For $d \in \{0, \ldots, n\}$, define

$$F_d(z) = \left(\sum_{j=n-d}^{n} \varepsilon_j z^j \right) / z^{n-d}.$$

Note that $F_0(z) = 1$, $F_n(z) = F(z)$, and $F_d(z) \in \mathcal{N}'(\beta, d)$. Evaluating these polynomials at β, we see that for $d \in \{0, \ldots, n-1\}$, we have $F_{d+1}(\beta) = \beta F_d(\beta)$ or $F_{d+1}(\beta) = \beta F_d(\beta) + 1$. Since $F(\beta) = 0$, we deduce

$$\beta^{n-d} F_d(\beta) = \sum_{j=n-d}^{n} \varepsilon_j \beta^j = -\sum_{j=0}^{n-d-1} \varepsilon_j \beta^j.$$

Dividing by β^{n-d} and recalling that $\beta < 0$, we obtain

$$\begin{aligned} F_d(\beta) = -\sum_{j=0}^{n-d-1} \varepsilon_j \beta^{j-n+d} &= -\varepsilon_{n-d-1} \frac{1}{\beta} - \varepsilon_{n-d-2} \frac{1}{\beta^2} - \cdots - \varepsilon_0 \frac{1}{\beta^{n-d}} \\ &> -\frac{1}{\beta^2} - \frac{1}{\beta^4} - \cdots = -\frac{1}{\beta^2} \sum_{j=0}^{\infty} \left(\frac{1}{\beta^2} \right)^j = \frac{-1}{\beta^2 - 1}. \end{aligned}$$

Similarly,

$$\begin{aligned} F_d(\beta) &= -\varepsilon_{n-d-1} \frac{1}{\beta} - \varepsilon_{n-d-2} \frac{1}{\beta^2} - \cdots - \varepsilon_0 \frac{1}{\beta^{n-d}} \\ &< -\frac{1}{\beta} - \frac{1}{\beta^3} - \frac{1}{\beta^5} - \cdots = -\frac{1}{\beta} \sum_{j=0}^{\infty} \left(\frac{1}{\beta^2} \right)^j = \frac{-\beta}{\beta^2 - 1}. \end{aligned}$$

Hence, if β is a root of a Newman polynomial $F(z)$, then for $0 \le d \le n$, the value $F_d(\beta) \in \mathcal{N}'(\beta, d)$ must lie in the interval

$$\mathcal{I}(\beta) = \left(\frac{-1}{\beta^2 - 1}, \frac{-\beta}{\beta^2 - 1} \right).$$

For each integer $d \ge 0$, we define the smaller set

$$\mathcal{N}(\beta, d) = \mathcal{N}'(\beta, d) \cap \mathcal{I}(\beta).$$

Thus, for $0 \le d \le n$, we have $F_d(\beta) \in \mathcal{N}(\beta, d)$. We also note that the recursive formula

$$\begin{aligned} \mathcal{N}(\beta, d+1) = \mathcal{N}(\beta, d) &\cup (\{\beta\alpha : \alpha \in \mathcal{N}(\beta, d)\} \cap \mathcal{I}(\beta)) \\ &\cup (\{\beta\alpha + 1 : \alpha \in \mathcal{N}(\beta, d)\} \cap \mathcal{I}(\beta)) \end{aligned}$$

holds for $d \geq 0$ and provides a means to construct $\mathscr{N}(\beta, d+1)$ from $\mathscr{N}(\beta, d)$. Observe that $\mathscr{N}(\beta,0) \subseteq \mathscr{N}(\beta,1) \subseteq \mathscr{N}(\beta,2) \subseteq \cdots$.

The algorithm of Hare and Mossinghoff (2014) is to construct $\mathscr{N}(\beta,0)$, $\mathscr{N}(\beta,1)$, $\mathscr{N}(\beta,2),\ldots$ using the recursive formula above until we find the first positive integer d for which $\mathscr{N}(\beta,d)$ satisfies one of two possibilities: (i) $0 \in \mathscr{N}(\beta,d)$, or (ii) $\mathscr{N}(\beta,d) = \mathscr{N}(\beta,d-1)$. Note that $\mathscr{N}(\beta,d)$ is the set of all $0,1$-polynomials of degree d, evaluated at β, restricted to values which lie in $\mathscr{I}(\beta)$. Thus, in the case of (i), we have that β is a root of a Newman polynomial. In the case of (ii), each $\beta\alpha$ or $\beta\alpha+1$, with $\alpha \in \mathscr{N}(\beta,d-1)$, used in constructing $\mathscr{N}(\beta,d)$ either does not lie in $\mathscr{I}(\beta)$ or is in $\mathscr{N}(\beta,d-1)$. Observe that if we have for some d that $\mathscr{N}(\beta,d-1) = \mathscr{N}(\beta,d)$, then

$$\mathscr{N}(\beta,d-1) = \mathscr{N}(\beta,d) = \mathscr{N}(\beta,d+1) = \cdots = \bigcup_{d' \geq 0} \mathscr{N}(\beta,d').$$

Thus, in the case of (ii) above, where $0 \notin \mathscr{N}(\beta,d-1)$, we see that $0 \notin \mathscr{N}(\beta,d)$ for all positive integers d. Hence, in this case, β is not a root of a Newman polynomial. We address next why this algorithm must terminate.

Define

$$\mathscr{N}(\beta) = \bigcup_{d \geq 0} \mathscr{N}(\beta,d).$$

One can see that $\mathscr{N}(\beta,d) = \mathscr{N}(\beta,d+1)$ for some $d \geq 0$ by a result of A.M. Garsia (1962). Garsia showed that there exists a constant $C := C(\beta)$, independent of d, such that for any two values $x,y \in \mathscr{N}(\beta)$, either $x=y$ or $|x-y| > C > 0$. Since the elements are inside a bounded set $\mathscr{I}(\beta)$, the number of distinct elements of $\mathscr{N}(\beta)$ is finite. As $\mathscr{N}(\beta,0) \subseteq \mathscr{N}(\beta,1) \subseteq \cdots$, we deduce that $\mathscr{N}(\beta,d) = \mathscr{N}(\beta,d+1)$ for some $d \geq 0$. This justifies then that the algorithm of Hare and Mossinghoff (2014) terminates for β a negative Pisot number. This algorithm can also be generalized for any negative algebraic integer β, but as Hare and Mossinghoff (2014) note the algorithm is only known to terminate when β is a negative Pisot number.

We introduce now a variation of the previous algorithm for a complex number β. Suppose $\beta \in A_\phi \setminus \mathbb{R}$ and let $|\beta| > 1$. Then, we define

$$\mathscr{I}_{\mathbb{C}}(\beta) = \left\{ z \in \mathbb{C} : |z| < \frac{|\beta|}{|\beta|-1} \right\}$$

and

$$\mathscr{N}_{\mathbb{C}}(\beta,d) = \{F(\beta) : F \in \mathscr{N}_0 \text{ and } \deg(F) \le d\} \cap \mathscr{I}_{\mathbb{C}}(\beta).$$

If $F \in \mathscr{N}_0$ and $|F(\beta)| \ge \dfrac{|\beta|}{|\beta|-1}$, then

$$|\beta F(\beta)| \ge \frac{|\beta|^2}{|\beta|-1} \ge \frac{|\beta|}{|\beta|-1},$$

and

$$\begin{aligned} |\beta F(\beta)+1| \ge |\beta F(\beta)| - 1 \ge \frac{|\beta|^2}{|\beta|-1} - 1 &= \frac{|\beta|^2 - |\beta| + 1}{|\beta|-1} \\ &= \frac{|\beta|^2 - 2|\beta| + 1 + |\beta|}{|\beta|-1} = \frac{(|\beta|-1)^2 + |\beta|}{|\beta|-1} \ge \frac{|\beta|}{|\beta|-1}. \end{aligned}$$

Hence,if $F(\beta) \notin \mathscr{I}_{\mathbb{C}}(\beta)$, then $\beta F(\beta) \notin \mathscr{I}_{\mathbb{C}}(\beta)$, and $\beta F(\beta)+1 \notin \mathscr{I}_{\mathbb{C}}(\beta)$.

As before, we begin with $d = 0$ and $\mathscr{N}'_{\mathbb{C}}(\beta,0) = \{1\}$, and with each iteration of d, we generate all $0,1$-polynomials of degree d evaluated at β. This is done by taking each element $\alpha \in \mathscr{N}_{\mathbb{C}}(\beta, d-1)$ and applying a simple construction to obtain the new values, $\beta\alpha$ and $\beta\alpha+1$. For example, the first three iterations of d may appear as follows:

$$\begin{array}{ll} d=0: & 1 \\ d=1: & \beta, \quad \beta+1 \\ d=2: & \beta^2, \quad \beta^2+\beta, \quad \beta^2+1, \quad \beta^2+\beta+1. \end{array}$$

Note that doing this for every $\alpha \in \mathscr{N}'_{\mathbb{C}}(\beta, d-1)$ will yield all possible values $F(\beta)$ for all $F \in \mathscr{N}_0$ of degree $\le d$. The values generated which lie in the interval $\mathscr{I}_{\mathbb{C}}(\beta)$ we denote by $\mathscr{N}_{\mathbb{C}}(\beta,d)$. Recalling that if $F(\beta) \notin \mathscr{I}_{\mathbb{C}}(\beta)$, then $\beta F(\beta) \notin \mathscr{I}_{\mathbb{C}}(\beta)$ and $\beta F(\beta)+1 \notin \mathscr{I}_{\mathbb{C}}(\beta)$, we deduce

$$\begin{aligned} \mathscr{N}_{\mathbb{C}}(\beta,d) = \mathscr{N}_{\mathbb{C}}(\beta,d-1) &\cup (\{\beta\alpha : \alpha \in \mathscr{N}_{\mathbb{C}}(\beta,d-1)\} \cap \mathscr{I}_{\mathbb{C}}(\beta)) \\ &\cup (\{\beta\alpha+1 : \alpha \in \mathscr{N}_{\mathbb{C}}(\beta,d-1)\} \cap \mathscr{I}_{\mathbb{C}}(\beta)). \end{aligned}$$

We use the above to obtain now an algorithm as before that recursively constructs $\mathscr{N}_{\mathbb{C}}(\beta,d)$. The algorithm terminates if either (i$'$) $0 \in \mathscr{N}_{\mathbb{C}}(\beta,d)$, or (ii$'$) $\mathscr{N}_{\mathbb{C}}(\beta,d) = \mathscr{N}_{\mathbb{C}}(\beta,d-1)$. Observe that in the case of (i$'$), we have $F(\beta) = 0$ for some $F \in \mathscr{N}$ of degree d. In other words, in this case, β is the root of some Newman polynomial of degree d. If (ii$'$) occurs, then $\mathscr{N}_{\mathbb{C}}(\beta,d-1) = \mathscr{N}_{\mathbb{C}}(\beta,d) = \mathscr{N}_{\mathbb{C}}(\beta,d+1) = \cdots$, and β is not a root of a Newman polynomial.

The above algorithm for complex β is also described in Hare and Mossinghoff (2014). Unlike the first algorithm for negative real β, it is unknown whether this algorithm will always terminate with (i$'$) or (ii$'$) occurring. On the other hand, for both algorithms, one must handle the situation where the sets $\mathscr{N}(\beta,d)$ or $\mathscr{N}_{\mathbb{C}}(\beta,d)$ grow in size to a point where computationally it becomes infeasible to continue constructing these sets for larger d. One can set up bounds for the size of these sets or degree bounds to force the algorithms to terminate, but one then is left without resolving whether or not β is a root of a Newman polynomial. This situation led us to search for other methods for determining whether a given β is a root of a Newman polynomial. One method is to check for Newman multiples by multiplying the minimal polynomial for β by various products of cyclotomic polynomials. This often led to a quick determination that the minimal polynomials we were considering had Newman multiples. Yet another approach we used is described in Chapter 5 of this thesis.

2.2 A new class of polynomials

In regards to Problem 1, Theorem 1.1 suggests the possibility that σ exists and that perhaps one can take $\sigma = \phi$. Having considered the problem restricted to the class of polynomials with negative real Pisot numbers as roots, we will introduce a new related class of polynomials.

Let $f(z) \in \mathbb{Z}[z]$ be a monic, irreducible polynomial with roots $\{\beta_1, \beta_2, \ldots, \beta_n\} \in A_\phi$ with $\beta_1 = \beta$ and $\beta_2 = \overline{\beta}$. Suppose that $\beta_1 \neq \beta_2$, $|\beta| = |\overline{\beta}| > 1$, and $|\beta_k| < 1$ for $3 \leq k \leq n$.

Then we call β and $\overline{\beta}$ *complex Pisot numbers*. Note that if $f(z)$ is the minimal polynomial of any negative Pisot number β, then $f(z^2)$ is the minimal polynomial of $\pm\sqrt{|\beta|}\,i$. The fact that $f(z^2)$ is irreducible in this situation is due to the observations that $\pm\sqrt{|\beta|}\,i$ are two imaginary conjugates which must be roots of the same minimal polynomial and the product of the remaining roots of $f(z^2)$ has absolute value < 1. Since $f(z^2)$ is the minimal polynomial of $\pm\sqrt{|\beta|}\,i$ in this case, we see that if β is a negative Pisot number, then $\pm\sqrt{|\beta|}\,i$ are two complex Pisot numbers.

One would hope that we are able to classify a number of complex Pisot numbers as was done for negative real Pisot numbers in the work of Bertin, Decomps-Guilloux, Grandet-Hugot, Pathiaux-Delefosse, and Schreiber (1992). As a result of an ICERM summer project by Z. Blumenstein, A. Lamarche, Mossinghoff, and S. Saunders (2014), a few such polynomial families were found, and we discuss these next.

Using the families of polynomials from the work of Bertin, Decomps-Guilloux, Grandet-Hugot, Pathiaux-Delefosse, and Schreiber (1992) (see the discussion before Lemma 1.2 in this thesis), we replace z with z^2 to obtain

$$\begin{aligned}
-p_{2n}(-z^2) &= z^{4n+2} - z^{4n-2} + z^{4n-4} - \cdots - z^2 + 1,\\
-q_{2n+1}(-z^2) &= z^{4n+2} + z^{4n} + z^{4n-4} + \cdots + z^4 + 1,\\
r_{2n}(-z^2) &= z^{4n}(z^4 + z^2 - 1) + z^4 - 1,\\
-r_{2n+1}(-z^2) &= z^{4n+2}(z^4 + z^2 - 1) - z^4 + 1,\\
g(-z^2) &= z^{12} + 2z^{10} + z^8 - z^4 - z^2 - 1.
\end{aligned}$$

Defining

$$\begin{aligned}
P_n(z) &= z^n(z^4 + z^2 - 1) + 1,\\
Q_n(z) &= z^n(z^4 + z^2 - 1) - 1,\\
R_n(z) &= z^n(z^4 + z^2 - 1) + z^4 - 1,\\
S_n(z) &= z^n(z^4 + z^2 - 1) - z^4 + 1,\\
G(z) &= z^{12} + 2z^{10} + z^8 - z^4 - z^2 - 1,
\end{aligned} \tag{2.1}$$

one can check that

$$-p_{2n}(-z^2) = \frac{P_{4n}(z)}{z^2+1},$$
$$-q_{2n+1}(-z^2) = \frac{Q_{4n+2}(z)}{z^4-1},$$
$$r_{2n}(-z^2) = R_{4n}(z),$$
$$-r_{2n+1}(-z^2) = S_{4n+2}(z),$$
$$g(-z^2) = G(z).$$

As each negative Pisot number in $(-\phi,-1)$ is a root of one of $-p_{2n}(-z)$, $-q_{2n+1}(-z)$, $r_{2n}(-z)$, $-r_{2n+1}(-z)$, and $g(-z)$, the above discussion leads to complex Pisot numbers β among the roots of $P_{4n}(z)$, $Q_{4n+2}(z)$, $R_{4n}(z)$, $S_{4n+2}(z)$, and $G(z)$ which satisfy $1 < |\beta| < \sqrt{\phi}$. The idea is to consider instead the larger families $P_n(z)$, $Q_n(z)$, $R_n(z)$, $S_n(z)$, and $G_n(z)$ and to show that these generate even more complex Pisot numbers. There is no reason to believe that these larger families represent all complex Pisot numbers with modulus less than ϕ or $\sqrt{\phi}$, but the discussion above implies that every complex Pisot number of the form $\pm\sqrt{|\beta|}\,i$ where β is a negative real Pisot number in $(-\phi,-1)$ is a root of one of the polynomials $P_n(z)$, $Q_n(z)$, $R_n(z)$, $S_n(z)$, and $G_n(z)$ for some n.

We make another observation. Recall that each negative real Pisot number in $(-\phi,-1)$, having no positive real conjugates, is a root of some $F(z) \in \mathscr{N}$ by Theorem 1.1. Hence, the complex Pisot numbers arising as roots of the polynomials $-p_{2n}(-z^2)$, $-q_{2n+1}(-z^2)$, $r_{2n}(-z^2)$, $-r_{2n+1}(-z^2)$, and $g(-z^2)$ discussed above, provided they have no positive real conjugates, will also be roots of $F(z^2)$ for some $F(z) \in \mathscr{N}$. Thus, these complex Pisot numbers are roots of Newman polynomials. We do not however yet know if the same is true of all complex Pisot numbers that are roots of the larger families $P_n(z)$, $Q_n(z)$, $R_n(z)$, $S_n(z)$, and $G_n(z)$.

As an indication of the work achieved by Blumenstein, Lamarche, Mossinghoff, and Saunders (2014), we state and prove the following result obtained by them.

Theorem 2.1. *For every odd $n \geq 3$, each of $P_n(z)$, $Q_n(z)$, $R_n(z)$, $S_n(z)$, and $G(z)$ has exactly two roots outside $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$ with both roots non-real.*

Proof. We begin with $P_n(z)$. The reciprocal of $P_n(z)$ is $\widetilde{P}_n(z) = 1 + z^2 - z^4 + z^{n+4}$. It suffices to show that $\widetilde{P}_n(z)$ contains exactly two non-real roots inside of $\mathscr{C}$, that is with absolute value < 1, for every odd $n \geq 3$. To show this, we will apply Rouché's theorem. For odd $n \geq 3$, observe that $P_n(z)$ has a root at $z = -1$. Thus, we will not be able to use $\mathscr{C}$ as the boundary of our region when applying Rouché's theorem. We will instead use Rouché's theorem over a contour which contains part of the unit circle not including $z = -1$. To help motivate the contour we use, we show first that $P_n(z)$ has no roots on $\mathscr{C}$ besides $z = -1$. We do this by showing that $z + 1$ is the only irreducible reciprocal factor of $P_n(z)$.

Suppose, by way of contradiction, that $P_n(z)$ has an irreducible reciprocal factor $w(z)$ with $w(z) \neq z + 1$. Then, we must have that $w(z) \mid \widetilde{P}_n(z)$, and hence, $w(z) \mid \big(\widetilde{P}_n(z) - P_n(z)\big)$. Observe that

$$\widetilde{P}_n(z) - P_n(z) = z^2 - z^4 + z^n - z^{n+2} = z^2(1 - z^2 + z^{n-2} - z^n) = z^2(1 - z^2)(1 + z^{n-2}).$$

We discard the z^2 factor since $z \nmid P_n(z)$. Hence, $w(z) \mid (1 - z^2)(1 + z^{n-2})$. Again, we discard the factor $1 - z^2$ since $w(z) \neq z + 1$ and $P_n(1) \neq 0$. Thus, $w(z) \mid (1 + z^{n-2})$. This implies that $w(z)$ divides

$$\widetilde{P}_n(z) - z^6(1 + z^{n-2}) = 1 + z^2 - z^4 - z^6 = (1 + z^2)(1 - z^4).$$

As $w(z) \nmid (1 - z^2)$, we have that $w(z) = 1 + z^2$. One checks that $P_n(i) \neq 0$ since n is odd, so we obtain a contradiction. Hence, $P_n(z)$ has no irreducible reciprocal factors other than $z + 1$.

We apply Rouché's theorem to the contour $\mathscr{C}'$, an arc C and chord ℓ as shown in Figure 2.2. Here, we view $\varepsilon > 0$ as some sufficiently small number. First, consider the arc C. Let $f(z) = -1 - z^2 + z^4$, and note that $f(z)$ has exactly two roots inside of the unit circle $\mathscr{C}$. On the arc C, we have that

$$|f(z) + \widetilde{P}_n(z)| = |z^{n+4}| = 1.$$

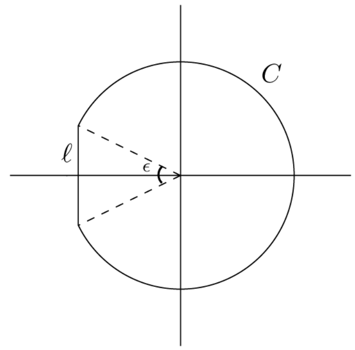

Figure 2.1 Contour $\mathscr{C}'$

A direct computation shows that $f(e^{i\theta}) = u + iv$, where

$$u = 8\cos^4(\theta) - 10\cos^2(\theta) + 1 \quad \text{and} \quad v = 8\cos^3(\theta)\sin(\theta) - 6\cos(\theta)\sin(\theta).$$

Another direct computation gives

$$|f(e^{i\theta})|^2 = u^2 + v^2 = -16\cos^4(\theta) + 16\cos^2(\theta) + 1.$$

Since $|\cos\theta| \leq 1$ for all θ, we have that $\cos^2(\theta) \geq \cos^4(\theta)$. We deduce that for z on the arc C, we have $|f(z)| \geq 1$. Since $P_n(z)$, and hence $\widetilde{P}_n(z)$, has no root on $\mathscr{C}$ besides $z = -1$, we further see that for z on the arc C, we have $|\widetilde{P}_n(z)| > 0$. We deduce that

$$|f(z) + \widetilde{P}_n(z)| = 1 < |f(z)| + |\widetilde{P}_n(z)|.$$

Now, consider the chord ℓ where again $\varepsilon > 0$ is sufficiently small. For z on ℓ, we have

$$|f(z) + \widetilde{P}_n(z)| = |z^{n+4}| \leq 1.$$

We wish to show that $|f(z)| > 1$ for such z. For z on ℓ, we see that the real part of z is $-\cos\varepsilon$ and the imaginary part is $t\sin\varepsilon$ for some $t \in [-1, 1]$. We therefore write

$$z = -\cos(\varepsilon) + it\sin(\varepsilon) \quad \text{with } -1 \leq t \leq 1.$$

A computation here, shows that the derivative of $f(z)\overline{f(z)}$ with respect to t can be expressed in the form $4t(\cos(\varepsilon) - 1)(\cos(\varepsilon) + 1)g(\varepsilon, t)$, where $g(\varepsilon, t)$ is a polynomial of degree 6 in t

with the property that the coefficients of t^j for $j \in \{1,2,\ldots,6\}$ approach 0 as ε approaches 0 and $g(\varepsilon,0)$ approaches -7 as ε approaches 0. As $|t| \le 1$, we see that since ε is sufficiently small, we have $g(\varepsilon,t) \ne 0$ for $|t| \le 1$. Furthermore, for such t, the derivative of $f(z)\overline{f(z)}$ with respect to t equals 0 only when $t = 0$ and is negative for $t < 0$ and positive for $t > 0$. Thus, $|f(z)|^2 = f(z)\overline{f(z)}$ has a minimum at $t = 0$. At $t = 0$, we obtain

$$f(z)\overline{f(z)} = f(-\cos\varepsilon)^2 = (\cos^4\varepsilon - \cos^2\varepsilon - 1)^2.$$

Since $\varepsilon > 0$ is small, we have $0 < \cos\varepsilon < 1$ so that $\cos^4\varepsilon - \cos^2\varepsilon < 0$. Hence, we see that $|f(z)| > 1$. Thus, for z on ℓ, we obtain

$$|f(z) + \widetilde{P}_n(z)| \le 1 < |f(z)| \le |f(z)| + |\widetilde{P}_n(z)|.$$

We now have that $|f(z) + \widetilde{P}_n(z)| < |f(z)| + |\widetilde{P}_n(z)|$ for all $z \in \mathscr{C}'$. Therefore, by Rouché's theorem, we deduce that $\widetilde{P}_n(z)$ has two roots inside $\mathscr{C}'$. As we let ε tend to 0, we obtain that $\widetilde{P}_n(z)$ has exactly two roots inside the unit circle $\mathscr{C}$.

Finally, we must show that these two roots are non-real. For $z \in (-1,1)$, we have $z^2 \ge z^4$ so that $\widetilde{P}_n(z) = 1 + z^2 - z^4 + z^{n+4} \ge 1 + z^{n+4} > 0$. We deduce that $\widetilde{P}_n(z)$ has no roots in the interval $(-1,1)$. In other words, the two roots of $\widetilde{P}_n(z)$ inside $\mathscr{C}$ cannot be real. Hence, we can now deduce that $P_n(z)$ has exactly two roots outside $\mathscr{C}$, both of which are non-real.

Next, we consider $Q_n(z)$. Since we are interested in odd n, we observe that for such n, we have

$$P_n(-z) = -z^n(z^4 + z^2 - 1) + 1 = -Q_n(z).$$

Thus, the roots of $Q_n(z)$ are precisely $-\beta$ where β runs through the roots of $P_n(z)$. We deduce from the above then that $Q_n(z)$ has exactly two roots outside $\mathscr{C}$, both of which are non-real.

Next, we consider the family $R_n(z)$. Here, we can proceed like we did in Lemma 1.2. Replacing z with $-z$ in (1.1), we obtain that

$$|z^2 + z - 1| > |z^2 - 1| \quad \text{for all } z \in \mathscr{C}.$$

Now, replacing z with z^2 gives

$$|z^4+z^2-1| > |z^4-1| \quad \text{for all } z \in \mathscr{C}.$$

Taking here $\rho_n(z) = z^n(z^4+z^2-1)$, we see that for $z \in \mathscr{C}$, we have

$$|R_n(z) - \rho_n(z)| = |z^4-1| < |z^4+z^2-1| = |\rho_n(z)|.$$

From Rouché's Theorem, we see that $R_n(z)$ and $\rho_n(z)$ have the same number of roots, counted to their multiplicity, inside $\mathscr{C}$. As z^4+z^2-1 has exactly two roots in $\mathscr{C}$, we deduce $R_n(z)$ has exactly $n+2$ roots inside $\mathscr{C}$ and exactly two roots outside $\mathscr{C}$. To see that $R_n(z)$ does not have real roots outside of $\mathscr{C}$, suppose that $z \in \mathbb{R}$ and $|z| > 1$. Then

$$\begin{aligned} |R_n(z)| &= \left|z^n(z^4+z^2-1)+z^4-1\right| \geq |z|^{n+4}+|z|^{n+2}-|z|^n-\left|z^4-1\right| \\ &= |z|^{n+4}+|z|^{n+2}-|z|^n-z^4+1 = \left(|z|^{n+2}-|z|^n\right)+\left(|z|^{n+4}-z^4\right)+1 > 1 > 0. \end{aligned}$$

Thus, $R_n(z)$ has no real roots outside of $\mathscr{C}$. Hence, $R_n(z)$ has exactly two roots outside $\mathscr{C}$, both of which are non-real.

Next, we consider $S_n(z)$. Since we are interested in odd n, we observe that for such n, we have

$$R_n(-z) = -z^n(z^4+z^2-1)+z^4-1 = -S_n(z).$$

Thus, the roots of $S_n(z)$ are precisely $-\beta$ where β runs through the roots of $R_n(z)$. We deduce that $S_n(z)$ has exactly two roots outside $\mathscr{C}$, both of which are non-real.

To finish the proof of the theorem, it remains to note that with a simple computation one can deduce that $G(z)$ has exactly two complex roots lying outside the unit circle $\mathscr{C}$, both of which are not only non-real but are in fact purely imaginary. □

Now we show how these polynomials factor for all odd n.

Theorem 2.2. *For odd $n \geq 3$, the polynomial $P_n(z)$ is $z+1$ times an irreducible polynomial, the polynomial $Q_n(z)$ is $z-1$ times an irreducible polynomial, and each of the polynomials $R_n(z)$, $S_n(z)$, and $G(z)$ is irreducible.*

Proof. One checks directly that $G(z)$ is irreducible. So we need now only consider the polynomials $P_n(z)$, $Q_n(z)$, $R_n(z)$, and $S_n(z)$.

In the proof of Theroem 2.1, we showed that the only root of $P_n(z)$ on the unit circle $\mathscr{C}$ is -1. In fact, with n odd, we have $P_n(-1) = 0$ and

$$P_n'(-1) = (n+4)(-1)^{n+3} + (n+2)(-1)^{n+1} - n(-1)^{n-1} = n+6 \neq 0.$$

Thus, $P_n(z)/(z+1)$ is a monic polynomial with no roots on $\mathscr{C}$. In the proof of Theroem 2.1, we also showed that $|R_n(z) - \rho_n(z)| < |\rho_n(z)|$ for all $z \in \mathscr{C}$. Observe that this implies $R_n(z) \neq 0$ for all $z \in \mathscr{C}$. Thus, $R_n(z)$ is a monic polynomial with no roots on $\mathscr{C}$. Further, we showed in the proof of Theroem 2.1 that $P_n(-z) = -Q_n(z)$ and $R_n(-z) = -S_n(z)$ from which we deduce that $Q_n(z)/(z-1)$ and $S_n(z)$ are monic polynomials with no roots on $\mathscr{C}$.

Let $f(z)$ be one of the polynomials $P_n(z)/(z+1)$, $Q_n(z)/(z-1)$, $R_n(z)$, and $S_n(z)$, where $n \geq 3$ is odd. Then $f(z)$ is monic in $\mathbb{Z}[x]$ and has exactly two roots outside the unit circle $\mathscr{C}$, both of which are non-real and hence they are complex conjugates. Also, as seen above, $f(z)$ has no roots on the unit circle $\mathscr{C}$. Now, assume $f(z)$ is reducible. Then $f(z) = u(z)v(z)$ for some monic $u(z)$ and $v(z)$ in $\mathbb{Z}[x]$ with positive degrees and non-zero constant terms. As the two roots of $f(z)$ outside $\mathscr{C}$ are non-real complex conjugate roots, they must both be roots of $u(z)$ or both be roots of $v(z)$. Without loss of generality, we may suppose they are roots of $u(z)$. As the remaining roots of $f(z)$ all have modulus less than 1, the roots of $v(z)$ all have modulus less than 1. But this is a contradiction since the absolute value of the product of the roots of $v(z)$ is $|v(0)| \geq 1$. Therefore, $f(z)$ is irreducible, establishing the theorem. □

2.3 Generating Infinitely Many Polynomials of Small Measure

The polynomials considered in Theorem 2.1 motivated the direction taken in the next chapter. Taking $\phi(z) = z^4 + z^2 - 1$, we see that

$$P_n(z) = z^n \phi(z) + 1,$$

$$Q_n(z) = z^n \phi(z) - 1,$$

$$R_n(z) = z^n \phi(z) + z^4 - 1,$$

$$S_n(z) = z^n \phi(z) - z^4 + 1.$$

Computationally, these polynomials appear to have Mahler measure very close to the golden ratio ϕ as n tends to infinity. In fact, we have

$$\phi = 1.61803398874989\ldots$$

$$M(P_{100}(z)) = 1.61803398873400\ldots$$

$$M(Q_{100}(z)) = 1.61803398876578\ldots$$

$$M(R_{100}(z)) = 1.61803398872418\ldots$$

$$M(S_{100}(z)) = 1.61803398877560\ldots.$$

This observation suggests that maybe by considering more general polynomials of the form $u(z)z^n + v(z)$, one might be able to find some interesting limiting values of Mahler measures and relate them to our investigations of polynomials which have small Mahler measure and no Newman multiples.

Let $f(z)$ be a monic irreducible polynomial in $\mathbb{Z}[z]$ such that $f(z) \neq \widetilde{f}(z)$, where $\widetilde{f}(z)$ denotes the reciprocal of $f(z)$ defined in (1.2). In the next chapter, we consider

$$F_n(z) = z^n f(z) + \widetilde{f}(z), \tag{2.2}$$

and show that it yields an infinite number of distinct polynomials with Mahler measures that approach $M(f)$ as n becomes large. By choosing $f(z)$ appropriately, we are led then to finding an infinite number of complex Pisot numbers which have distinct Mahler measures $< 1.56 < \phi$ and which are not roots of any Newman polynomial.

We are also able to obtain some computational results by testing other variations of this construction. For instance, we also consider $F_n(z) = z^n f(z) - \widetilde{f}(z)$, $F_n(z) = z^n f(z^k) \pm \widetilde{f}(z^k)$ for integers $k \geq 2$, $F_n(z) = z^n f(-z) \pm \widetilde{f}(-z)$, and $F_n(z) = z^n \widetilde{f}(z) \pm f(z)$. We will explore these computational results in Chapter 4. In Chapter 5, we use a polynomial discovered in the investigations in Chapter 4 to explain another approach to showing that a given polynomial does not have a Newman multiple.

CHAPTER 3

GENERATING INFINITELY MANY POLYNOMIALS WITHOUT NEWMAN MULTIPLES

In this chapter, we expound on the construction $F_n(z) = z^n f(z) + \widetilde{f}(z)$, outlined in the last chapter. Many of the results here were provided by M. Filaseta.

Lemma 3.1. *Let k and ℓ be nonnegative integers. Let $u(z)$ be a polynomial in $\mathbb{R}[z]$ having exactly k distinct roots which lie outside the circle $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$, and let $v(z)$ be a polynomial in $\mathbb{R}[z]$ having exactly ℓ distinct roots which lie inside $\mathscr{C}$. Let these roots be denoted $\alpha_1, \dots, \alpha_k$ and $\beta_1, \dots, \beta_\ell$, respectively. Let e_j denote the multiplicity of the roots α_j in $u(z)$, and let e'_j denote the multiplicity of the roots β_j in $v(z)$. Thus,*

$$u(z) = (z-\alpha_1)^{e_1} \cdots (z-\alpha_k)^{e_k} w_1(z) \quad \textit{and} \quad v(z) = (z-\beta_1)^{e'_1} \cdots (z-\beta_k)^{e'_\ell} w_2(z),$$

where $w_1(z)$ and $w_2(z)$ necessarily have real coefficients, with the roots of $w_1(z)$ on or inside $\mathscr{C}$, and with the roots of $w_2(z)$ on or outside $\mathscr{C}$. For n a positive integer, define $f_n(z) = u(z)z^n + v(z)$. Then for every $\varepsilon > 0$ sufficiently small and every n sufficiently large ($n \geq N_0(\varepsilon, u(z), v(z))$), each of the following holds:

(i) For each $j \in \{1, 2, \dots, k\}$, the disk $\{z \in \mathbb{C} : |z - \alpha_j| < \varepsilon\}$ has exactly e_j roots of $f_n(z)$.

(ii) For each $j \in \{1, 2, \dots, \ell\}$, the disk $\{z \in \mathbb{C} : |z - \beta_j| < \varepsilon\}$ has exactly e'_j roots of $f_n(z)$.

(iii) The remaining roots of $f_n(z)$ all lie in the annulus $\{z \in \mathbb{C} : 1 - \varepsilon < |z| < 1 + \varepsilon\}$.

Proof. If $v(0) = 0$, then it is not difficult to see that, for n sufficiently large, 0 is a root of $f_n(z)$ and with multiplicity equal to the multiplicity of 0 as a root of $v(z)$. By factoring out the appropriate power of z, we may therefore consider now only the case that $v(0) \neq 0$ and do so.

Fix $\varepsilon > 0$ sufficiently small, in particular so that each closed disk centered at a root of $u(z)$ of radius ε contains only that root of $u(z)$, so that each closed disk centered at a root of $v(z)$ of radius ε contains only that root of $v(z)$, and so that each such disk centered at one of $\alpha_1, \ldots, \alpha_k$ or one of $\beta_1, \ldots, \beta_\ell$ does not intersect the annulus in (iii). Let n be sufficiently large. For $j \in \{1, 2, \ldots, k\}$, set

$$\mathscr{C}_j = \{z \in \mathbb{C} : |z - \alpha_j| = \varepsilon\} \quad \text{and} \quad \mathscr{D}_j = \{z \in \mathbb{C} : |z - \alpha_j| \leq \varepsilon\}.$$

Observe that there is a $t_1 > 0$ such that $|v(z)| \leq t_1$ for every z in each $\mathscr{D}_j$. Recall that $u(z) = 0$ for $z \in \mathscr{D}_j$ if and only if $z = \alpha_j$. Hence, there is a $t_2 > 0$ such that $|u(z)| \geq t_2$ for every z on each circle $\mathscr{C}_j$. Since each $\mathscr{D}_j$ lies outside $\mathscr{C}$, there is also a $t_3 > 0$ such that if $z \in D = \mathscr{D}_1 \cup \cdots \cup \mathscr{D}_k$, then $|z| > 1 + t_3$. We deduce that, for n sufficiently large and $z \in \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_k$, we have

$$|v(z)| \leq t_1 < t_2(1+t_3)^n \leq |u(z)z^n|.$$

Set $g_n(z) = -u(z)z^n$. For $z \in \mathscr{C}_1 \cup \cdots \cup \mathscr{C}_k$ and n sufficiently large, we have

$$|f_n(z) + g(z)| = |v(z)| < |g(z)|.$$

Thus, by Rouché's Theorem, we deduce $f_n(z)$ has exactly e_j roots in $\mathscr{D}_j$ for each $j \in \{1, 2, \ldots, k\}$, establishing (i).

By applying (i) to the reciprocal polynomial $\tilde{f}_n(z) = \tilde{v}(z)z^{n+\deg u - \deg v} + \tilde{u}(z)$, one can see that (ii) holds as well.

Now, suppose n is sufficiently large and $f_n(z_0) = 0$ and z_0 is not within ε of one of the α_j or β_j. We show first that $|z_0|$ is bounded above by

$$B = 2 + \max_{1 \leq j \leq k} \{|\alpha_j|\} + \max\{|\beta| : \beta \in \mathbb{C}, v(\beta) = 0\}.$$

Let a denote the leading coefficient of $u(z)$ and b denote the leading coefficient of $v(z)$. If $|z_0| > B$, then the distance from z_0 to each root of $u(z)$ or $v(z)$ is at least 2 and no more than $2|z_0|$. Hence,

$$|u(z_0)| \geq 2^{\deg u} a \quad \text{and} \quad |v(z_0)| \leq (2|z_0|)^{\deg v} b.$$

For n sufficiently large, we have

$$\begin{aligned} |f_n(z_0)| = |u(z_0)z_0^n + v(z_0)| &\geq 2^{\deg u} a |z_0|^n - (2|z_0|)^{\deg v} b \\ &\geq |z_0|^{\deg v} (2^{n+\deg u - \deg v} a - 2^{\deg v} b) > 0, \end{aligned}$$

contradicting that $f_n(z_0) = 0$. Thus, $|z_0| \leq B$.

Assume now that $|z_0| \geq 1 + \varepsilon$. Since $z_0 \notin D$ and all the roots of $u(z)$ other than the α_j are on or inside $\mathscr{C}$, there is a $t_4 > 0$, not depending on z_0, such that $|u(z_0)| > t_4$. Since $|z_0| \leq B$, there is a $t_5 > 0$ such that $|v(z_0)| \leq t_5$. Therefore,

$$|f_n(z_0)| = |u(z_0)z_0^n + v(z_0)| \geq t_4(1+\varepsilon)^n - t_5.$$

Since n is sufficiently large, we obtain a contradiction. Hence, we must have $|z_0| < 1 + \varepsilon$. By again considering the reciprocal polynomial $\tilde{f}_n(z) = \tilde{v}(z) z^{n+\deg u - \deg v} + \tilde{u}(z)$, we deduce that for n sufficiently large, we must also have $|z_0| > 1 - \varepsilon$. □

We will want to have some idea of how close the roots of $f_n(z)$ are to the roots of $u(z)$ in Lemma 3.1. Observe that in the argument for Lemma 3.1, if we decrease the size of ε, the values of t_1 and t_3 can remain constant. Given the factorization of $u(z)$ in the statement of Lemma 3.1, we have for $z \in \mathscr{C}_j$ that

$$|u(z)| \geq t_6 |z - \alpha_j|^{e_j} = t_6 \varepsilon^{e_j},$$

where t_6 is a constant depending only on $u(z)$ and $\varepsilon > 0$ being sufficiently small. The inequality $|v(z)| < |u(z)z^n|$ then follows provided that

$$t_6 (1+t_3)^n \varepsilon^{e_j} > t_1.$$

Therefore, for the purposes of (i), we can take

$$\varepsilon = t_7/(1+t_3)^{n/e_j},$$

for some t_7 depending only on $u(z)$ and $v(z)$. Rouché's Theorem applies, and we deduce that $f_n(z)$ has a root within $t_7/(1+t_3)^{n/e_j}$ of each root α_j in Lemma 3.1. A similar argument applies to the roots β_j and (ii). Thus, we have the following.

Corollary 3.2. *In the set-up of Lemma 3.1, let*

$$E = \max\{e_1, \ldots, e_k, e'_1, \ldots, e'_\ell\}.$$

For $n \in \mathbb{Z}^+$, $z_0 \in \mathbb{C}$ and positive real numbers A and B, set

$$\mathscr{D}_n(z_0) = \mathscr{D}_n(z_0, A, B) = \{z \in \mathbb{C} : |z - z_0| < A/(1+B)^{n/E}\}.$$

Then there exist positive constants $A = A(u(z), v(z))$ and $B = B(u(z), v(z))$ such that each of the following holds for n sufficiently large:

(i) For each $j \in \{1, 2, \ldots, k\}$, the disk $\mathscr{D}_n(\alpha_j)$ has exactly e_j roots of $f_n(z)$.

(ii) For each $j \in \{1, 2, \ldots, \ell\}$, the disk $\mathscr{D}_n(\beta_j)$ has exactly e'_j roots of $f_n(z)$.

Next, we turn to obtaining a result for the case where $v(z)$ is the reciprocal of $u(z)$. Our next lemma will help us to show that, in this case, for n sufficiently large, the roots described by (iii) of Lemma 3.1 are not only close to $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$ but actually all lie on $\mathscr{C}$.

Lemma 3.3. *Fix $c \in \mathbb{R}$ with $c \geq 1$. Let I be an open interval containing c, and let J be a closed interval. Suppose $H(z,t)$ is a real valued function having continuous first order partial derivatives for $z \in I$ and $t \in J$. Suppose further that $H(c,t) > 0$ for all $t \in J$. For $n \in \mathbb{Z}^+$, define $F_n(z,t) = z^{2n} H(z,t)$. Then there exists an $n_0(H,c)$ such that*

$$\left.\frac{\partial F_n(z,t)}{\partial z}\right|_{z=c} \geq 1 \quad \text{for all } n \geq n_0(H,c) \text{ and all } t \in J.$$

Proof. First, observe that since J is compact and $H(c,t)$ is a continuous function of t in J, the function $H(c,t)$ obtains its minimum in J. As $H(c,t) > 0$ for all $t \in J$, there is an $\varepsilon > 0$ such that $H(c,t) > \varepsilon$ for all $t \in J$. Similarly, there is an $M > 0$ such that

$$\left| \frac{\partial H(z,t)}{\partial z}\bigg|_{z=c} \right| \leq M \quad \text{for all } t \in J. \tag{3.1}$$

Since

$$\frac{\partial F_n(z,t)}{\partial z}\bigg|_{z=c} = 2nc^{2n-1}H(c,t) + c^{2n}\frac{\partial H(z,t)}{\partial z}\bigg|_{z=c} = c^{2n-1}\left(2nH(c,t) + c\frac{\partial H(z,t)}{\partial z}\bigg|_{z=c}\right),$$

the conditions $c \geq 1$, $H(c,t) > \varepsilon$ for all $t \in J$ and (3.1) imply

$$\frac{\partial F_n(z,t)}{\partial z}\bigg|_{z=c} \geq 2n_0\,\varepsilon - cM \geq 1$$

for $n \geq n_0$ and $n_0 = n_0(H,c)$ sufficiently large. Hence, the lemma follows. □

Lemma 3.4. *Let $f(z)$ be a polynomial in $\mathbb{R}[z]$ which has exactly k roots outside the unit circle $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$ and no roots on $\mathscr{C}$. For n sufficiently large, the polynomial $f(z)z^n + \tilde{f}(z)$ has exactly k roots outside $\mathscr{C}$.*

Proof. Observe that $f(z)$ having no roots on $\mathscr{C}$ implies $\tilde{f}(z)$ has no roots on $\mathscr{C}$. Let $z \in \mathscr{C}$. Then z^k and z^{-k} are complex conjugates for every positive integer k. Therefore, $f(z)$ and $f(1/z)$ are complex conjugates and, hence, have the same absolute value. We deduce that

$$\left|\frac{f(z)}{\tilde{f}(z)}\right| = \left|\frac{f(z)}{z^{\deg f} f(1/z)}\right| = \frac{|f(z)|}{|z|^{\deg f}|f(1/z)|} = 1 \quad \text{for all } z \in \mathscr{C}. \tag{3.2}$$

Now, set $z = re^{i\theta}$, where $r \geq 0$ and $\theta \in [0, 2\pi)$. Define

$$H(r,\theta) = \frac{|f(z)|^2}{|\tilde{f}(z)|^2} = \frac{f(r\cos\theta + ir\sin\theta)f(r\cos\theta - ir\sin\theta)}{\tilde{f}(r\cos\theta + ir\sin\theta)\tilde{f}(r\cos\theta - ir\sin\theta)} \quad \text{and} \quad F_n(r,\theta) = r^{2n}H(r,\theta).$$

Then $H(r,\theta)$ and $F_n(r,\theta)$ are real-valued rational functions in r with coefficients that are polynomials in $\cos\theta$ and $\sin\theta$. These functions are continuous in r and θ away from points where the denominator is 0; in particular, since $f(z)$ and $\tilde{f}(z)$ do not have zeroes on $\mathscr{C}$, there is an open annulus $\mathscr{A}$ containing $\mathscr{C}$ such that $H(r,\theta)$ and $F_n(r,\theta)$ are continuous in r and θ for $re^{i\theta}$ in $\mathscr{A}$.

From (3.2), we deduce

$$H(1,\theta) = 1 \quad \text{for all } \theta \in [0,2\pi].$$

From Lemma 3.3, there is an N such that $n \geq N$ implies

$$\left.\frac{\partial F_n(r,t)}{\partial r}\right|_{r=1} \geq 1 \quad \text{for all } \theta \in [0,2\pi].$$

Fix $\theta_0 \in [0,2\pi)$ and $n \geq N$. By the continuity of $\partial F_n(r,\theta)/\partial r$ in r and θ around $(r,\theta) = (1,\theta_0)$, there is an $\varepsilon(\theta_0) > 0$ such that for $z = re^{i\theta}$ in the open disk $\mathscr{D}(\theta_0) = \{z \in \mathbb{C} : |z - e^{i\theta_0}| < \varepsilon(\theta_0)\}$, we have $\partial F_n(r,\theta)/\partial r \geq 1/2$.

With $n \geq N$ still fixed, we observe that the open disks $\mathscr{D}(\theta)$ for all $\theta \in [0,2\pi)$ form an open covering of the compact unit circle $\mathscr{C}$. Hence, there is a finite subcovering of the unit circle using say $\mathscr{D}(\theta_1),\ldots,\mathscr{D}(\theta_s)$ for some $s \in \mathbb{Z}^+$ and $\theta_j \in [0,2\pi)$. By considering the intersection points of the boundaries of overlapping disks and the minimum distance of these intersection points to the unit circle, we deduce that there is an $\varepsilon > 0$ such that

$$\frac{\partial F_n(r,\theta)}{\partial r} \geq 1/2 \quad \text{for all } re^{i\theta} \in |\{z \in \mathbb{C} : 1-\varepsilon \leq |z| \leq 1+\varepsilon\}|. \tag{3.3}$$

The significance of (3.3) is the following. For a fixed $\theta \in [0,2\pi)$, we have that the function $F_n(r,\theta)$ is strictly increasing as a function of $r \in [1,1+\varepsilon]$. Further, $F_n(1,\theta) = 1$ from (3.2). Hence, $F_n(1+\varepsilon,\theta) > 1$. We deduce that

$$|z|^n|f(z)| > |\tilde{f}(z)| \quad \text{for all } z \in \mathscr{C}_\varepsilon = \{z \in \mathbb{C} : |z| = 1+\varepsilon\}.$$

Let $g(z) = -f(z)z^n$. Then

$$\big|(f(z)z^n + \tilde{f}(z)) + g(z)\big| = |\tilde{f}(z)| < |z|^n|f(z)| = |g(z)| \quad \text{for all } z \in \mathscr{C}_\varepsilon. \tag{3.4}$$

Observe that (3.4) holds for each $n \geq N$ where $\varepsilon = \varepsilon(n)$. However, in the above, one may take $\varepsilon > 0$ arbitrarily small. We obtain from Rouché's Theorem and (3.4) that, for $n \geq N$, the polynomials $f(z)z^n + \tilde{f}(z)$ and $g(z)$ have the same number of roots, counted to their multiplicity, on the closed unit disk $\mathscr{D} = \{z \in \mathbb{C} : |z| \leq 1\}$. As $g(z) = -f(z)z^n$ has $n + \deg f - k$ such roots, the same is true of $f(z)z^n + \tilde{f}(z)$. The lemma now follows. □

Corollary 3.5. *Let $f(z)$ be a polynomial in $\mathbb{R}[z]$ which has exactly k roots outside the unit circle $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$ and no roots on $\mathscr{C}$. For n sufficiently large, the polynomial $f(z)z^n + \tilde{f}(z)$ has exactly $n + \deg f - 2k$ roots on $\mathscr{C}$.*

Proof. Apply Lemma 3.4 to see that, for n large, $f(z)z^n + \tilde{f}(z)$ has exactly k roots outside of $\mathscr{C}$. Since $f(z)z^n + \tilde{f}(z)$ is reciprocal, we deduce $f(z)z^n + \tilde{f}(z)$ has exactly k roots inside $\mathscr{C}$. The corollary follows. □

The next lemma follows, for example, from Proposition 11.2.4 in Rahman and Schmeisser (2002). We give a proof here.

Lemma 3.6. *Let $f(z) = \sum_{j=0}^{r} a_j z^{d_j} \in \mathbb{Z}[z]$ with $r \geq 1$, $a_0 \neq 0$, and $d_0 = 0$. Then each root of $f(z)$ has multiplicity at most r.*

Proof. Assume α is a root of $f(z)$ with multiplicity $k \geq r+1$. Then $f^{(u)}(\alpha) = 0$ for $0 \leq u \leq r$ so that

$$\sum_{j=0}^{r} a_j d_j (d_j - 1) \cdots (d_j - u + 1) \alpha^{d_j - u} = 0 \quad \text{for } 0 \leq u \leq r.$$

We claim that this is only possible if each a_j equals 0, contradicting $a_0 \neq 0$. As the above $r+1$ equations in the $r+1$ numbers a_j form linear equations in the a_j's, it suffices to show that $\det M \neq 0$, where $M = (m_{ij})$ is the $(r+1) \times (r+1)$ matrix with

$$m_{ij} = \begin{cases} \alpha^{d_j} & \text{if } i = 0, \\ d_j (d_j - 1) \cdots (d_j - i + 1) \alpha^{d_j - i} & \text{if } 0 < i \leq r, \end{cases}$$

where for convenience we use subscripts for i and j from $\{0, 1, \ldots, r\}$ so that the first row corresponds to $i = 0$ and the first column corresponds to $j = 0$. Note that $\alpha \neq 0$ since $a_0 \neq 0$ and $d_0 = 0$. We can remove a factor of α^{d_j} from the $(j+1)$st column and then a factor of α^{-i} from the $(i+1)$st row to obtain a new $(r+1) \times (r+1)$ matrix $M' = (m'_{ij})$ where

$$m'_{ij} = \begin{cases} 1 & \text{if } i = 0, \\ d_j (d_j - 1) \cdots (d_j - i + 1) & \text{if } 0 < i \leq r. \end{cases}$$

The matrix M' has the property that $\det M \neq 0$ if and only if $\det M' \neq 0$. We can further multiply the $(i+1)$st row by $1/i!$ to obtain an $(r+1)\times(r+1)$ matrix $M'' = (m''_{ij})$ where

$$m''_{ij} = \binom{d_j}{i}.$$

The matrix M'' similarly satisfies the property that $\det M \neq 0$ if and only if $\det M'' \neq 0$. The matrix M'' (or the matrix M') can be computed by connecting it to a Vandermonde matrix Pólya and Szegő (1976) (see Part V, Problem 96 Solution). We obtain

$$\det M'' = \frac{\prod_{0\le i<j\le r}(d_j - d_i)}{\prod_{0\le i<j\le r}(j-i)} \neq 0.$$

The lemma follows. $\square$

For the results that follow, we will require $f(z) \in \mathbb{Z}[z]$ have degree ≥ 1 and satisfy $\gcd(f(z), \tilde{f}(z)) = 1$. We note that for such $f(z) \in \mathbb{Z}[z]$, we have as a consequence that $f(z)$ has no roots on the unit circle $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$, as any such root would be a root of both $f(z)$ and $\tilde{f}(z)$.

Lemma 3.7. *Let $f(z) = \sum_{j=0}^{r} a_j z^j \in \mathbb{Z}[z]$ with $r \geq 1$, $a_r = 1$, $a_0 \neq 0$ and $\gcd(f(z), \tilde{f}(z)) = 1$. Then there exists a positive integer M such that for all positive integers n and m satisfying $\Phi_m(z)$ divides $f(z)z^n + \tilde{f}(z)$, we have $m \leq M$.*

Proof. For n a positive integer, set $F_n(z) = f(z)z^n + \tilde{f}(z)$. Suppose m and n are positive integers for which $\Phi_m(z)$ divides $F_n(z)$. Let $\zeta = \zeta_m = e^{2\pi i/m}$. Then $F_n(\zeta) = 0$, and we obtain

$$0 = f(\zeta)\zeta^n + \tilde{f}(\zeta) = f(\zeta)\zeta^n + \zeta^r f(1/\zeta) \implies \zeta^{n-r} = -\frac{f(1/\zeta)}{f(\zeta)}. \tag{3.5}$$

Since ζ and $1/\zeta$ are complex conjugates, we deduce $f(\zeta)$ and $f(1/\zeta)$ are complex conjugates. Hence, $|f(1/\zeta)/f(\zeta)| = 1$. The basic idea now is to use that ζ is very close to 1 for large m so that the right-hand side of the last equality in (3.5) is close to -1 and then to show that the left-hand side of this equation can only be close to -1 for m from a finite list of possibilities.

The right-hand side of the last equality in (3.5) is

$$-\frac{f(1/\zeta)}{f(\zeta)} = -1+\eta, \quad \text{where } \eta = \frac{f(\zeta)-f(1/\zeta)}{f(\zeta)}.$$

Denote the roots of $f(z)$, up to their multiplicities, by $\alpha_1, \ldots, \alpha_r$. By the conditions in the statement of the lemma,

$$\tau = \min\left\{\left|z-\alpha_j\right| : 1 \le j \le r, z \in \mathscr{C}\right\} = \min_{1\le j\le r}\left\{\left|1-|\alpha_j|\right|\right\} > 0.$$

We deduce the inequalities

$$|f(\zeta)-f(1/\zeta)| = \left|\sum_{j=0}^{r} a_j(\zeta^j-\overline{\zeta}^j)\right| \le |\zeta-\overline{\zeta}|\sum_{j=0}^{r}|a_j|\left|\sum_{k=0}^{j-1}\zeta^k(\overline{\zeta})^{j-1-k}\right| \le |\zeta-\overline{\zeta}|\sum_{j=0}^{r} j|a_j|,$$

$$|\zeta-\overline{\zeta}| = 2|\sin(2\pi/m)| \le \frac{4\pi}{m},$$

and

$$|f(\zeta)| = |a_r|\prod_{j=1}^{r}|\zeta-\alpha_j| \ge |a_r|\tau^r.$$

Thus,

$$|\eta| \le \frac{C_1}{m}, \quad \text{where } C_1 = \frac{4\pi}{|a_r|\tau^r}\sum_{j=0}^{r} j|a_j|.$$

We now turn to the left-hand side of the last equality in (3.5). Let t be an integer in $[0, m-1]$ such that $n-r \equiv t \pmod{m}$. Then $\zeta^{n-r} = \zeta^t$. We treat $t \in (m/4, 3m/4)$ separately from other t. For $t \in (m/4, 3m/4)$, we set $t' = t-(m/2) \in (-m/4, m/4)$. Observe that

$$\zeta^t = \zeta_{2m}^{2t} = \zeta_{2m}^{m}\zeta_{2m}^{2t'} = -\zeta_{2m}^{2t'} \quad \text{and} \quad |2\pi t'/m| < \pi/2.$$

Hence,

$$\left|1+\zeta^t\right| = \left|1-\zeta_{2m}^{2t'}\right| \ge \left|\operatorname{Im}(1-\zeta_{2m}^{2t'})\right| = |\sin(2\pi t'/m)| \ge \frac{2}{\pi}\cdot\frac{2\pi|t'|}{m} = \frac{4|t'|}{m}.$$

We deduce that

$$\zeta^{n-r} = \zeta^t = -1+\eta', \quad \text{where } |\eta'| \ge (4|t'|)/m.$$

From (3.5), we obtain

$$\frac{4|t'|}{m} \leq \frac{C_1}{m} \implies |t'| \leq C_2 = \frac{\pi}{|a_r|\tau^r} \sum_{j=0}^{r} j|a_j|.$$

Note that t' is not necessarily an integer, but the definition of t' implies that $2t' = t''$ for some $t'' \in \mathbb{Z}$. Thus,

$$\zeta^{n-r} = \zeta^t = -\zeta_{2m}^{2t'} = -\zeta_{2m}^{t''},$$

where $|t''| \leq 2C_2$. Fix $t'' \in \mathbb{Z}$ satisfying $|t''| \leq 2C_2$. Since $F_n(\zeta) = 0$ and $\zeta = \zeta_{2m}^2$, we obtain

$$f(\zeta)\zeta^n + \tilde{f}(\zeta) = 0 \implies f(\zeta)\zeta^{n-r} + f(1/\zeta) = 0 \implies -f(\zeta_{2m}^2)\zeta_{2m}^{t''} + f(1/\zeta_{2m}^2) = 0.$$

Clearing denominators in this last equation by multiplying through by $\zeta_{2m}^{2r+\max\{-t'',0\}}$, we obtain a polynomial in ζ_{2m} with a non-zero leading coefficient. More precisely, the polynomial

$$-f(z^2)z^{t''+2r+\max\{-t'',0\}} + z^{\max\{-t'',0\}}\tilde{f}(z^2) \tag{3.6}$$

has ζ_{2m} as a root and has leading coefficient c, where

$$c = \begin{cases} -a_r & \text{if } t'' > -2r \\ a_0 & \text{if } t'' < -2r \\ a_{r-k} - a_k & \text{if } t'' = -2r, \text{ where } k = \max\{j : 0 \leq j \leq r/2, a_j \neq a_{r-j}\}. \end{cases}$$

Note that such a k exists since $\gcd(f(z), \tilde{f}(z)) = 1$. As the polynomial in (3.6) only depends on $f(z)$ and t'' and $|t''| \leq 2C_2$, we see that as t'' varies, there are only finitely many possibilities for the roots of these polynomials and, hence, for the value of $2m$. Thus, if $t \in (m/4, 3m/4)$, the lemma follows.

In the case that $t \in [0, m-1]$ with $n - r \equiv t \pmod{m}$ and $t \notin (m/4, 3m/4)$, we proceed as follows. First, we observe here that $\zeta^t = e^{2\pi i t/m} = e^{\theta_m i}$ for some $\theta_m \in [-\pi/2, \pi/2]$. Hence,

$$\left|1 + \zeta^t\right| \geq \text{Re}(1 + \zeta^t) = 1 + \cos(\theta_m) \geq 1.$$

In other words, $\zeta^t = -1 + \eta'$, where $|\eta'| \geq 1$. From (3.5), we obtain

$$1 \leq |\eta'| = |\eta| \leq \frac{C_1}{m} \implies m \leq C_1 = \frac{4\pi}{|a_r|\tau^r} \sum_{j=0}^{r} j|a_j|.$$

Thus, in this case, we also get that m is bounded as in the lemma, finishing the proof. $\square$

For computational purposes, we made the constants C_1 and C_2 explicit above. The proof can therefore be used then to obtain an explicit M as in the statement of the lemma. We comment, however, that the expression $|a_r|\tau^r$ in C_1 and C_2 creates a less than optimal bound for $|\eta|$ and $|t'|$, and one can replace it with a value closer to $|f(1)|$.

For the next result, we recall the notation $M(f)$ for the Mahler measure of the polynomial $f(z)$.

Theorem 3.8. *Let $f(z) = \sum_{j=0}^{r} a_j z^j \in \mathbb{Z}[z]$ with $r \geq 1$, $a_r = 1$, $a_0 \neq 0$ and $\gcd(f(z), \tilde{f}(z)) = 1$. Suppose $f(z)$ has no positive real roots. Then for every $\varepsilon > 0$, there exist infinitely many polynomials $h(z)$ having no positive real roots and no cyclotomic factors such that*

$$\big|M(f) - M(h)\big| < \varepsilon. \tag{3.7}$$

Furthermore, these $h(z)$ can be chosen so that the number of roots of $h(z)$ outside $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$ is equal to the number of roots of $f(z)$ outside $\mathscr{C}$.

Proof. For n a positive integer, set $F_n(z) = f(z)z^n + \tilde{f}(z)$. We take n sufficiently large so that the following arguments hold.

Let k denote the number of roots of $f(z)$, counted to their multiplicity, outside of $\mathscr{C}$. By Lemma 3.4, the polynomial $F_n(z)$ has exactly k roots, counted to their multiplicity, outside of $\mathscr{C}$. Furthermore, since n is sufficiently large, Lemma 3.1 implies that these k roots are close to the k roots of $f(z)$ outside of $\mathscr{C}$ in such a way that (3.7) holds with $h(z) = F_n(z)$.

Let $h_n(z)$ denote the product of the monic irreducible factors of $F_n(z)$ having roots outside $\mathscr{C}$. The polynomial $F_n(z)/h_n(z)$ has only roots on or inside $\mathscr{C}$. The product of the roots of $F_n(z)/h_n(z)$ must be a non-zero integer since $a_0 \neq 0$. We deduce, therefore, that $F_n(z)/h_n(z)$ has only roots on $\mathscr{C}$. As cyclotomic polynomials are the only monic irreducible polynomials containing every root on $\mathscr{C}$, we deduce that $F_n(z)/h_n(z)$ is a product of cyclotomic polynomials.

From Lemma 3.6 and Lemma 3.7, there are positive integers R and M independent of n such that $F_n(z)/h_n(z)$ divides

$$\prod_{m\le M}\Phi_m(z)^R.$$

As n tends to infinity, this expression is a polynomial of bounded degree and $\deg F_n$ tends to infinity, so we deduce that $\deg h_n$ tends to infinity. As $M(h_n)=M(F_n)$, we deduce that there are infinitely many h as in (3.7). Furthermore, since $f(z)$ and hence $\tilde{f}(z)$ has no positive real roots, Lemma 3.1 and Corollary 3.5 imply $h_n(z)$ has no positive real roots off of $\mathscr{C}$. Since $h_n(z)$ has no cyclotomic factors, we also have $h_n(1)\neq 0$, so $h_n(z)$ has no positive real roots, completing the proof. □

There is a much simpler proof that polynomial $h(z)$ as in (3.7) exist by making use of $f(z^n)$ instead of $h_n(z)$. However, we will want to take advantage of the fact that, for n large, the polynomials $h_n(z)$ in the proof of Theorem 3.8 each has the same number of roots outside $\mathscr{C}$ as $f(z)$ and that these roots of $h_n(z)$ are close to the roots of $f(z)$.

We briefly review and partially revise the algorithms discussed in the previous chapter. Let β be an algebraic integer in $(-\tau,-1)$ where $\tau=(1+\sqrt{5})/2$. Let

$$I(\beta)=\left[\frac{-1}{\beta^2-1},\frac{-\beta}{\beta^2-1}\right].$$

Observe that this differs from $\mathscr{I}(\beta)$ only in that we have made this a closed interval for our purposes here. Set $\mathscr{N}(\beta,0)=\{1\}$ and, for d a positive integer, define

$$\mathscr{N}(\beta,d)=\big(\{\beta\omega:\omega\in\mathscr{N}(\beta,d-1)\}\cup\{\beta\omega+1:\omega\in\mathscr{N}(\beta,d-1)\}\big)\cap I(\beta).$$

This differs from our previous definitions discussed earlier in that we do not necessarily include the elements of $\mathscr{N}(\beta,d-1)$ in $\mathscr{N}(\beta,d)$. However, it is not difficult to check that the arguments in the previous chapter imply that β is not a root of a Newman polynomial if and only if there is a positive integer $d_0 = d_0(\beta)$ such that either $\mathcal{N}(\beta, d_0) = \emptyset$ or both $\mathcal{N}(\beta, d_0) = \mathcal{N}(\beta, d_0 - d)$ for some positive integer $d \le d_0$ and $0 \notin \mathcal{N}(\beta, d_0)$. Note that, in the case $\mathcal{N}(\beta, d_0) = \emptyset$, one necessarily has $0 \notin \mathcal{N}(\beta, d)$ for all $d \le d_0$.

Similarly, for β an algebraic integer in $\mathbb{C}$ with $|\beta| > 1$, we define

$$I'(\beta) = \left\{ z \in \mathbb{C} : |z| \le \frac{|\beta|}{|\beta| - 1} \right\},$$

$$\mathcal{N}'(\beta, 0) = \{1\},$$

and

$$\mathcal{N}'(\beta, d) = \big(\{\beta\omega : \omega \in \mathcal{N}'(\beta, d)\} \cup \{\beta\omega + 1 : \omega \in \mathcal{N}'(\beta, d)\} \big) \cap I'(\beta), \;\text{ for } d \ge 1.$$

We have here that if there is a positive integer $d_0' = d_0'(\beta)$ such that $\mathcal{N}(\beta, d_0') = \emptyset$, then β is not a root of a Newman polynomial. As above, in this case, one necessarily has $0 \notin \mathcal{N}(\beta, d)$ for all $d \le d_0'$. Note that the existence of $d_0'(\beta)$ is not a necessary condition for a $\beta \in \mathbb{C}$ with $|\beta| > 1$ to avoid being a root of a Newman polynomial.

Theorem 3.9. *Let $f(z) = \sum_{j=0}^{r} a_j z^j \in \mathbb{Z}[z]$ with $r \ge 1$, $a_r = 1$, $a_0 \ne 0$ and $\gcd(f(z), \tilde{f}(z)) = 1$. Suppose $f(z)$ has no positive real roots. Suppose that $f(z)$ has a root $\beta \in \mathbb{C}$ with $|\beta| > 1$ and that the second algorithm described above establishes that β is not a root of a Newman polynomial. Then for every $\varepsilon > 0$, there exist infinitely many polynomials $h(z)$ having no positive real roots and no cyclotomic factors such that both*

$$\big| M(f) - M(h) \big| < \varepsilon \tag{3.8}$$

and no multiple of $h(z)$ in $\mathbb{Z}[z]$ is a Newman polynomial. Furthermore, these $h(z)$ can be chosen so that the number of roots of $h(z)$ outside $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$ is equal to the number of roots of $f(z)$ outside $\mathscr{C}$.

Before turning to the proof, we note that one can take $f(z)$ in Theorem 3.9 to be $f_1(z) = z^6 - z^5 - z^3 + z^2 + 1$ and to be $f_2(z) = z^{10} - z^8 - z^5 + z + 1$. Here, $M(f_1) = 1.556\ldots$ and

$M(f_2) = 1.419\ldots$. Thus, there are infinitely many polynomials with no Newman multiples and Mahler measure as close as we want to the Mahler measure of $f_1(z)$, and similarly for $f_2(z)$.

Proof. Recall that taking $h = h_n(z)$, where $h_n(z)$ is described in the proof of Theorem 3.8, and n sufficiently large, one obtains (3.8). Further, the set of such $h_n(z)$ is infinite and the number of roots of $h_n(z)$ outside $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$ is equal to the number of roots of $f(z)$ outside $\mathscr{C}$. To establish the theorem, we need only now show that for n sufficiently large, the polynomial $h_n(z)$ has no Newman multiple.

Let β_n' be a root of $h_n(z)$ that is as close to β as possible. As shown in the proof of Lemma 3.1, the difference $|\beta_n' - \beta|$ tends to 0 as n tends to infinity. Let $d_0' = d_0'(\beta)$ be as defined before the statement of the theorem. Define

$$\mathscr{N}'(\beta) = \bigcup_{0 \le d \le d_0} \mathscr{N}'(\beta, d).$$

One may view $\mathscr{N}'(\beta)$ as the expressions

$$\varepsilon_{d_0}\beta^{d_0} + \varepsilon_{d_0-1}\beta^{d_0-1} + \cdots + \varepsilon_2\beta^2 + \varepsilon_1\beta + \varepsilon_0, \qquad \text{with each } \varepsilon_j \in \{0,1\}, \tag{3.9}$$

which lie in $I'(\beta)$. Let $\overline{\mathscr{N}'}(\beta)$ denote the expressions in (3.9) which do not lie in $I'(\beta)$. As this is a finite set disjoint from the closed set $I'(\beta)$, there is an $\varepsilon > 0$ such that each element of $\overline{\mathscr{N}'}(\beta)$ is a distance of at least $\varepsilon/2$ from $I'(\beta)$. As β is not a root of a Newman polynomial, we may also choose ε so that each expression in (3.9) is a distance of at least $\varepsilon/2$ from 0. Observe that the endpoints of $I'(s)$ are continuous functions of $s \in \mathbb{C}$ in a neighborhood of β. Thus, there is a $\delta > 0$ such that

$$I'(\gamma) = \left\{z \in \mathbb{C} : |z| \le \frac{|\gamma|}{|\gamma| - 1}\right\} \subseteq \left\{z \in \mathbb{C} : |z| \le \frac{|\beta|}{|\beta| - 1} + \frac{\varepsilon}{4}\right\} \quad \text{for } |\gamma - \beta| < \delta.$$

In addition, we may take δ so that $|\gamma - \beta| < \delta$ implies

$$\left|\sum_{j=0}^{d_0} \varepsilon_j \gamma^j - \sum_{j=0}^{d_0} \varepsilon_j \beta^j\right| < \frac{\varepsilon}{4}$$

and, consequently,

$$\left|\sum_{j=0}^{d_0} \varepsilon_j \gamma^j\right| > \frac{\varepsilon}{4},$$

for every choice of $\varepsilon_j \in \{0,1\}$.

Observe that, still with $|\gamma - \beta| < \delta$, if the ε_j's are chosen so that

$$z_1 = \varepsilon_{d_0}\gamma^{d_0} + \varepsilon_{d_0-1}\gamma^{d_0-1} + \cdots + \varepsilon_2\gamma^2 + \varepsilon_1\gamma + \varepsilon_0 \in I'(\gamma),$$

then

$$z_2 = \varepsilon_{d_0}\beta^{d_0} + \varepsilon_{d_0-1}\beta^{d_0-1} + \cdots + \varepsilon_2\beta^2 + \varepsilon_1\beta + \varepsilon_0$$

satisfies

$$|z_2| \le |z_2 - z_1| + |z_1| < \frac{\varepsilon}{4} + \frac{|\beta|}{|\beta|-1} + \frac{\varepsilon}{4} = \frac{|\beta|}{|\beta|-1} + \frac{\varepsilon}{2}.$$

Thus, $|z_2|$ is a distance $< \varepsilon/2$ from $I'(\beta)$. By our choice of ε, we deduce $z_2 \in I'(\beta)$. Thus, if

$$\varepsilon_{d_0}\gamma^{d_0} + \varepsilon_{d_0-1}\gamma^{d_0-1} + \cdots + \varepsilon_2\gamma^2 + \varepsilon_1\gamma + \varepsilon_0$$

is in $I'(\gamma)$ for some choice of $\varepsilon_j \in \{0,1\}$, then it is non-zero and the corresponding element from (3.9) is in $I'(\beta)$. As a consequence,

$$\left\{(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d) : \sum_{j=0}^{d} \varepsilon_j \gamma^j \in \mathscr{N}'(\gamma, d), \varepsilon_j \in \{0,1\}\, \forall j\right\}$$
$$\subseteq \left\{(\varepsilon_0, \varepsilon_1, \ldots, \varepsilon_d) : \sum_{j=0}^{d} \varepsilon_j \beta^j \in \mathscr{N}'(\beta, d), \varepsilon_j \in \{0,1\}\, \forall j\right\}$$

for each positive integer $d \le d_0$. In particular, $N'(\gamma, d_0) = \emptyset$ for such γ.

Recall the definition of β_n'. For n sufficiently large, we obtain $|\beta_n' - \beta| < \delta$. Thus, for n sufficiently large, the second algorithm described earlier establishes that $h_n(z)$ does not have a Newman multiple, completing the proof of the theorem. □

Theorem 3.10. *Let $f(z) \in \mathbb{Z}[z]$ be monic and non-reciprocal. Suppose that $f(z)$ has no positive real roots and exactly two roots outside $\mathscr{C}$, both non-real and with multiplicity one. Suppose further that* $\gcd(f(z), \tilde{f}(z)) = 1$. *For n a positive integer, define $h_n(z) \in \mathbb{Z}[z]$ as* *the largest degree monic factor of $f(z)z^n + \tilde{f}(z)$ not divisible by a cyclotomic polynomial. Then the polynomials $h_n(z)$ include infinitely many distinct irreducible polynomials with distinct Mahler measures approaching the Mahler measure of $f(z)$ as n tends to infinity and such that $h_n(z)$ has exactly two roots outside $\mathscr{C}$. Furthermore, under the conditions of a root β of $f(z)$ in Theorem 3.9, we may deduce that for n sufficiently large, no multiple of $h_n(z)$ in $\mathbb{Z}[z]$ is a Newman polynomial.*

To illustrate Theorem 3.10, we consider again $f_1(z) = z^6 - z^5 - z^3 + z^2 + 1$ and $f_2(z) = z^{10} - z^8 - z^5 + z + 1$. The condition of having exactly two non-real roots outside the unit circle $\mathscr{C}$ is satisfied by $f_1(z)$ but not by $f_2(z)$. We can conclude that $M(f_1) = 1.556\ldots$ is a limiting value for the Mahler measures of polynomials with no Newman multiples. On the other hand, Theorem 3.9 implies that there are infinitely many polynomials $h(z)$ with no Newman multiples and with Mahler measure arbitrarily close $M(f_2) = 1.419\ldots$. The distinction here, however, is that we have not established that the Mahler measures of these $h(z)$ differ from $M(f_2) = 1.419\ldots$, so we have not eliminated the possibility that all of the polynomials $h(z)$ given by Theorem 3.9 have the same Mahler measure, though it is likely they do not. It is certainly reasonable to conjecture, therefore, that $M(f_2) = 1.419\ldots$ is a limiting value for Mahler measures of polynomials with no Newman multiples, though we have not established this.

Proof of Theorem 3.10. From Theorem 3.8 and the proof of Theorem 3.9, it suffices to show that there are infinitely many n for which $h_n(z)$ is irreducible and has Mahler measure different from $M(f)$. Indeed, since $M(h_n)$ approaches $M(f)$ as n tends to infinity, an infinite sequence of n for which $M(h_n) \neq M(f)$ must have a subsequence of distinct $M(h_n)$ approaching $M(f)$.

We begin by showing that $h_n(z)$ is irreducible for all sufficiently large n. By Lemma 3.1 and Lemma 3.4, for n large, the polynomial $h_n(z)$ has exactly two non-real complex conjugate roots outside of $\mathscr{C}$. There is a monic irreducible factor $w_n(z)$ of $h_n(z)$ that has these two complex conjugate numbers as roots. Since any remaining monic irreducible factor will have all its roots on the unit circle and since $h_n(z)$ has no cyclotomic factors, we deduce that $h_n(z)/w_n(z) = 1$. Thus, $h_n(z) = w_n(z)$ is irreducible.

Assume that for all n sufficiently large, we have $M(h_n) = M(f)$. By our above remarks, the proof of the theorem will be complete if we can obtain a contradiction to this assumption. With this in mind, we take n sufficiently large now so that $M(h_n) = M(f)$. Observe that the conditions in the theorem imply that $f(z)$ has a root $\beta = re^{i\theta}$ where $0 < \theta < \pi$ and $r > 1$. Also, $h_n(z)$ has a root $\beta_n = r_n e^{i\theta_n}$ where $0 < \theta_n < \pi$ and $r_n > 1$. Furthermore, the only roots of $f(z)$ outside $\mathscr{C}$ are β and $\overline{\beta}$, and the only roots of $h_n(z)$ outside $\mathscr{C}$ are β_n and $\overline{\beta_n}$. Also, β_n approaches β as n tends to infinity.

We claim that $\beta_n \neq \beta_m$ for any distinct positive integers n and m. Assume $\beta_n = \beta_m$ for some $n > m$. Since $h_j(z)$ is a factor of $f(z)z^j + \tilde{f}(z)$ for each j, we have

$$f(\beta_n)\beta_n^n + \tilde{f}(\beta_n) = f(\beta_n)\beta_n^m + \tilde{f}(\beta_n) = 0,$$

where in the second expression we have used that $\beta_m = \beta_n$. Taking a difference, we obtain $f(\beta_n)\beta_n^m(\beta_n^{n-m} - 1) = 0$. Since $|\beta_n| > 1$, we deduce $f(\beta_n) = 0$. But then $f(\beta_n)\beta_n^n + \tilde{f}(\beta_n) = 0$ implies $\tilde{f}(\beta_n) = 0$, contradicting that $\gcd(f(z), \tilde{f}(z)) = 1$. Hence, the β_n are distinct as n varies.

Viewing n again as sufficiently large, we have

$$r^2 = \beta\overline{\beta} = M(f) = M(h_n) = \beta_n\overline{\beta_n} = r_n^2,$$

and we can conclude $r_n = r$. Since the β_n are distinct, we necessarily have distinct θ_n for distinct n. Also, θ_n approaches θ as n tends to infinity.

Since $h_n(z)$ is a factor of $f(z)z^n + \tilde{f}(z)$, we deduce that

$$f(re^{i\theta_n})r^n e^{in\theta_n} + \tilde{f}(re^{i\theta_n}) = 0. \tag{3.10}$$

Let $s = \deg f$. Then

$$f(z+h) = f(z) + f'(z)h + \frac{f''(z)}{2!}h^2 + \cdots + \frac{f^{(s)}(z)}{s!}h^s,$$

so that

$$f(\beta_n) = f(\beta) + (\beta_n - \beta) f'(\beta) + \frac{f''(\beta)}{2!}(\beta_n - \beta)^2 + \cdots + \frac{f^{(s)}(\beta)}{s!}(\beta_n - \beta)^s.$$

As $f(z)$ has β as a root with multiplicity one, we deduce that $f'(\beta) \neq 0$ and

$$\left| f(\beta_n) - (\beta_n - \beta) f'(\beta) \right| \leq C_f \left| \beta_n - \beta \right|^2,$$

where C_f is a constant depending only on $f(z)$. Similarly, we have

$$\beta_n - \beta = r\left(e^{i\theta_n} - e^{i\theta}\right) = (\theta_n - \theta)\tau_n, \tag{3.11}$$

where τ_n approaches $ire^{i\theta}$ as n tends to infinity. From (3.10), we deduce that

$$e^{in\theta_n} = -\frac{\tilde{f}(re^{i\theta_n})}{f(re^{i\theta_n})r^n} = -\frac{\tilde{f}(re^{i\theta_n})}{r^n\big((\theta_n - \theta)\tau_n f'(\beta) + \mu_n\big)}, \tag{3.12}$$

where

$$|\mu_n| \leq C_f(\beta_n - \beta)^2 \leq C_f \tau_n^2 (\theta_n - \theta)^2.$$

As n tends to infinity, the value of $\tilde{f}(re^{i\theta_n})$ approaches $\tilde{f}(re^{i\theta})$ which is non-zero since $f(re^{i\theta}) = 0$ and $\gcd(f(z), \tilde{f}(z)) = 1$. Also, the expression $\tau_n f'(\beta)$ approaches $ire^{i\theta} f'(\beta) \neq 0$. We deduce that

$$\lim_{n \to \infty} \frac{|\mu_n|}{\left|(\theta_n - \theta)\tau_n f'(\beta)\right|} = 0.$$

We let $\arg(z)$ denote the argument of a complex number z, with $0 \leq \arg(z) < 2\pi$ and equate the endpoints by referring to $\arg(z) \bmod 2\pi$. We write the right-hand side of (3.12) in the form $\rho_n e^{i\phi_n}$ with $\rho_n > 0$ and $0 \leq \phi_n < 2\pi$. As n tends to infinity through values of n for which $\theta_n > \theta$, assuming infinitely many such n exist, we obtain

$$\phi_n \bmod 2\pi \ \to \ \arg\left(-\tilde{f}(re^{i\theta})/(ie^{i\theta} f'(re^{i\theta}))\right) \bmod 2\pi.$$

If instead n tends to infinity through values of n for which $\theta_n < \theta$, assuming infinitely many such n exist, we obtain

$$\phi_n \bmod 2\pi \ \to \ \arg\left(\tilde{f}(re^{i\theta})/(ie^{i\theta} f'(re^{i\theta}))\right) \bmod 2\pi.$$

We deduce then that $\phi_n \bmod 2\pi$ approaches (has as a limit point) at most two distinct values in $[0, 2\pi)$.

The left-hand side of (3.12) has argument $n\theta_n \bmod 2\pi$. In order to examine the left-hand side further, we will want information on how close θ_n is to θ. Observe that in (3.11), we have $|\tau_n|$ approaches r as n increases so that $|\tau_n| \geq r/2$ for n large. We make use of Corollary 3.2 with $u(z) = f(z)$ and $v(z) = \tilde{f}(z)$. Since $f(z)$ only has two roots outside $\mathscr{C}$, both with multiplicity one, we deduce $\tilde{f}(z)$ only has two roots inside $\mathscr{C}$, both with multiplicity one. Hence, in Corollary 3.2, we have $E = 1$. Letting $A = A(f, \tilde{f}) > 0$ and $B = B(f, \tilde{f}) > 0$ be as defined there, we deduce from Corollary 3.2 and (3.11) that

$$|\beta_n - \beta| < \frac{A}{(1+B)^n} \implies |\theta_n - \theta| < \frac{2A}{r(1+B)^n}.$$

In particular, for n sufficiently large, we have

$$|n\theta_n - n\theta| < \frac{2nA}{r(1+B)^n} < \frac{\theta}{n}. \tag{3.13}$$

We consider the two cases $\theta/\pi \in \mathbb{Q}$ and $\theta/\pi \notin \mathbb{Q}$ separately. In the case that $\theta/\pi \in \mathbb{Q}$, since $\theta \in (0, \pi)$, we have that $\theta \notin \{0, \pi\}$ so that $\theta = a\pi/b$ where $b \geq 2$ and $\gcd(a, b) = 1$. Taking an appropriate $m \in \mathbb{Z}^+$ relatively prime to b, we see that $m\theta \bmod 2\pi$ takes on the value π/b. Letting n now be multiples of m, we obtain that $n\theta \bmod 2\pi$ takes on $2b$ distinct values, each infinitely often, as n tends to infinity. In the case that $\theta/\pi \notin \mathbb{Q}$, a classical result of H. Weyl (1916) implies that $n\theta/(2\pi)$ is equidistributed modulo 1 so that $n\theta \bmod 2\pi$ is arbitrarily close to each number in $[0, 2\pi)$ for infinitely many n. In either case, whether $\theta/\pi \in \mathbb{Q}$ or not, we see that $n\theta_n \bmod 2\pi$ is arbitrarily close to at least 4 different numbers in $[0, 2\pi)$. As we have just seen that the corresponding argument of the right-hand side of (3.12) approaches at most two different values as n tends to infinity, we obtain a contradiction, completing the proof. □

For some $f(z)$ as in Theorem 3.10, it is possible to obtain a little more information on the polynomials $h_n(z)$. We illustrate this with $f(z) = f_1(z) = z^6 - z^5 - z^3 + z^2 + 1$ used for previous examples here. Our goal is to show that there is a sequence of irreducible polynomials having each coefficient in $\{-1,0,1\}$ and with Mahler measures approaching the Mahler measure of $f_1(z)$, that is

$$M(f_1) = 1.556014485\ldots.$$

Given Theorem 3.10, the main idea is to show that $h_n(z) = f(z)z^n + \tilde{f}(z)$ for many n. This requires having then more information on the cyclotomic factors of $f(z)z^n + \tilde{f}(z)$, which we explore next.

The next lemma is a consequence of Corollary 1 in Filaseta and Schinzel (2004).

Lemma 3.11. *Let $f(z)$ be a polynomial with r terms. If $f(z)$ is divisible by a cyclotomic polynomial, then there is an $m \in \mathbb{Z}^+$ such that every prime divisor of m is $\le r$ and $\Phi_m(z)$ divides $f(z)$.*

We set

$$f_1(z) = z^6 - z^5 - z^3 + z^2 + 1 \quad \text{so that} \quad \tilde{f}_1(z) = \tilde{u}(z) = z^6 + z^4 - z^3 - z + 1.$$

We show that often $f_1(z)z^n + \tilde{f}_1(z)$ does not have cyclotomic roots.

Lemma 3.12. *The lower asymptotic density of positive integers n for which $f_1(z)z^n + \tilde{f}_1(z)$ has no cyclotomic factors is at least $1/20$. In other words, if*

$$\mathscr{S} = \{n \in \mathbb{Z}^+ : f_1(z)z^n + \tilde{f}_1(z) \textit{ is not divisible by a cyclotomic polynomial}\},$$

then

$$\liminf_{z\to\infty} \frac{|\{n \in \mathscr{S} : n \le z\}|}{z} \ge \frac{1}{20}.$$

Proof. Let m be a positive integer. The polynomial $f_1(z)$ itself is easily checked to have no roots on $\mathscr{C}$ and, hence, no cyclotomic factors. We deduce that $\Phi_m(z)$ divides $f_1(z)z^n + \tilde{f}_1(z)$ if and only if $\Phi_m(z)$ divides $f_1(z)z^{n+m} + \tilde{f}_1(z)$, and further that there is at most one $n \in [0, m-1]$ such that $\Phi_m(z)$ divides $f_1(z)z^n + \tilde{f}_1(z)$. It follows that for a fixed $m \in \mathbb{Z}^+$, we can determine the n for which $\Phi_m(z)$ divides $f_1(z)z^n + \tilde{f}_1(z)$ by a direct computation using $n \in [0, m-1]$. In particular, one checks that

$$\Phi_2(z) \text{ divides } f_1(z)z^n + \tilde{f}_1(z) \iff n \equiv 1 \pmod{2}$$

$$\Phi_4(z) \text{ divides } f_1(z)z^n + \tilde{f}_1(z) \iff n \equiv 0 \pmod{4}$$

$$\Phi_8(z) \text{ divides } f_1(z)z^n + \tilde{f}_1(z) \iff n \equiv 2 \pmod{8}$$

$$\Phi_{10}(z) \text{ divides } f_1(z)z^n + \tilde{f}_1(z) \iff n \equiv 2 \pmod{10}$$

$$\Phi_{14}(z) \text{ divides } f_1(z)z^n + \tilde{f}_1(z) \iff n \equiv 1 \pmod{14}$$

$$\Phi_{18}(z) \text{ divides } f_1(z)z^n + \tilde{f}_1(z) \iff n \equiv 17 \pmod{18}$$

$$\Phi_{30}(z) \text{ divides } f_1(z)z^n + \tilde{f}_1(z) \iff n \equiv 22 \pmod{30},$$

and that, for all positive integers n, $\Phi_m(z)$ does not divide $f_1(z)z^n + \tilde{f}_1(z)$ for any other positive integers $m \leq 2050$. One checks that the positive integers n that do not satisfy any of the congruences on the right above are the n which are $6 \pmod{8}$ and not $2 \pmod{5}$. The density of such positive integers is

$$\frac{1}{8} - \frac{1}{40} = \frac{1}{10}.$$

Next, we apply Lemma 3.11. As $f_1(z)z^n + \tilde{f}_1(z)$ has at most 10 terms, Lemma 3.11 implies that $f_1(z)z^n + \tilde{f}_1(z)$ will not be divisible by a cyclotomic polynomial if it is not divisible by $\Phi_m(z)$ for every $m \in \mathbb{Z}^+$ with all its prime factors < 10. Observe that we have verified asymptotically $1/10$ of the positive integers n satisfy that $\Phi_m(z)$ does not divide $f_1(z)z^n + \tilde{f}_1(z)$ for every $m \leq 2050$. On the other hand, for each $m > 2050$, we have that $\Phi_m(z)$ divides $f_1(z)z^n + \tilde{f}_1(z)$ for at most a density of $1/m$ positive integers n. As we need only consider those m having each prime factor < 10, we obtain that the density of positive integers n which are not divisible by $\Phi_m(z)$ for $m \in \{2, 4, 8, 10, 14, 18, 30\}$ and which are divisible by some $\Phi_m(z)$ for some $m > 2050$ is at most

$$\left(1 - \frac{1}{2}\right)^{-1} \left(1 - \frac{1}{3}\right)^{-1} \left(1 - \frac{1}{5}\right)^{-1} \left(1 - \frac{1}{7}\right)^{-1} - \sum_{m=1}^{2050}{}^{*} \frac{1}{m},$$

where the $*$ indicates that the sum is over those m having largest prime factor ≤ 7. A computation gives that the difference above is

$$\frac{35}{8} - \frac{461502875167}{106686720000} < \frac{1}{20}.$$

It follows that the asymptotic density of the positive integers m for which $f_1(z)z^n + \tilde{f}_1(z)$ is divisible by some $\Phi_m(z)$ is

$$> \frac{1}{10} - \frac{1}{20} = \frac{1}{20},$$

completing the proof. □

Theorem 3.13. *There is a set $\mathscr{T} \subseteq \mathscr{S}$ such that the sequence $F_n(z)$ with $n \in \mathscr{T}$ consists of monic irreducible polynomials which have distinct Mahler measures that approach $M(f_1) = 1.556014485\ldots$. Furthermore, $\mathscr{T}$ can be chosen in such a way that each $F_n(z)$ with $n \in \mathscr{T}$ does not have a Newman multiple.*

Proof. Most of the result is a direct consequence of Theorem 3.10 and the definition of $\mathscr{S}$. However, Theorem 3.10 involves a subsequence of $h_n(z)$ which we want to show includes $F_n(z)$ with n from a subset of $\mathscr{S}$.

We refer to the proof of Theorem 3.10 using the notation in the statement of Theorem 3.13. In the proof of Theorem 3.10, we considered n sufficiently large and chose a root $\beta = re^{i\theta}$ of $f_1(z)$ and a root $\beta_n = r_n e^{i\theta_n}$ of $F_n(z)$ with both β and β_n outside $\mathscr{C}$ and with both θ and θ_n in $(0, \pi)$. If $M(F_n) = M(f_1)$, then $r_n^2 = r^2$ so that $r_n = r$. Further, as seen there, $\theta_n \neq \theta$. We showed then that (3.12) holds, and that the argument of the right-hand side of (3.12) has two limit points, say L_1 and L_2. We then analyzed the argument $n\theta_n$ of the left-hand side of (3.12), taking advantage of (3.13). Here, we want to analyze the use of (3.13) with more precision.

As $\beta = re^{i\theta}$ is the root of $f_1(z) = z^6 - z^5 - z^3 + z^2 + 1$ which has absolute value > 1 and lies in the upper half-plane, we can compute that

$$\frac{\theta}{\pi} = 0.0961214629596479892115 71\ldots.$$

A direct computation gives

$$\min\left\{\left|\frac{\theta}{\pi}-\frac{a}{b}\right| : 1 \le b \le 100, 0 \le a \le b\right\} > 0.00003.$$

In particular, as $\theta/\pi \in (0,1)$, we can deduce that if $\theta/\pi \in \mathbb{Q}$, then $\theta/\pi = a/b$ for some positive relatively prime integers a and b with $b > 100$.

We now follow the approach at the end of the proof of Theorem 3.10. In the case that $\theta/\pi \in \mathbb{Q}$, we deduce that as n varies, $n\theta \bmod 2\pi$ takes on over 200 distinct values, each for the same positive density of integers. This positive density is $< 1/200$, and hence the density of n for which $n\theta$ is in a small neighborhood of either L_1 or L_2 modulo 2π is $< 1/100$. In this case, since the lower asymptotic density of n in $\mathscr{S}$ is at least $1/20$, we see that there are infinitely many $n \in \mathscr{S}$ for which (3.12) fails to hold. For such n, we must have $M(F_n) \neq M(f_1)$. As $M(F_n)$ approaches $M(f_1)$ as n tends to infinity, we deduce that some infinite subsequence $n \in \mathscr{S}$ gives distinct values for $M(F_n)$ which approach $M(f_1)$.

The case that $\theta/\pi \notin \mathbb{Q}$ is handled similarly. Here, the theorem of H. Weyl (1916) implies that the positive integers n for which $n\theta$ is within $1/400$ modulo 2π of either L_1 or L_2 is $\le 1/100$ (where equality can be shown to hold since $L_1 \bmod 2\pi$ equals $L_2 + \pi \bmod 2\pi$). As in the case $\theta/\pi \in \mathbb{Q}$ above, we deduce that some infinite subsequence $n \in \mathscr{S}$ gives distinct values for $M(F_n)$ which approach $M(f_1)$. This completes the proof of the theorem. □

CHAPTER 4

MINIMIZING MAHLER MEASURE

In this chapter, we explore the results given by the methods described in the previous chapter. So far, in the search for an upper bound for σ in Problem 1, the smallest upper bound to date was given by Drungilas, Jankauskas, and Šiurys (2016) as $1.4366322261\ldots$ given by showing the polynomial $z^9+z^8+z^7-z^5-z^4-z^3+1$ has Mahler measure $1.4366322261\ldots$, no positive real roots, and no Newman multiple. Before this, Hare and Mossinghoff (2014) showed that the polynomial $z^6-z^5-z^3+z^2+1$, has Mahler measure $1.55601\ldots$, no positive real roots, and no Newman multiple. We recall in Table 4.1 the data from Table 1.3 in Chapter 1. In this chapter, we explain some background associated with the discovery of the polynomials in these tables.

4.1 PRODUCING POLYNOMIALS WITH SMALL MEASURES HAVING NO NEWMAN MULTIPLES

Our goal in this section is to use the construction $F_n(z) = z^n f(z) + \widetilde{f}(z)$ with a monic irreducible non-reciprocal polynomial $f(z)$ in $\mathbb{Z}[z]$ with small Mahler measure, no positive real roots, and no Newman multiple, to generate other polynomials with small Mahler measure, no positive real roots, and no Newman multiple. Our main examples elaborated on in Table 1.3 and Table 4.1 involve polynomials with Mahler measure < 1.31, and so it is worth noting that a result of Smyth (1971) implies that any polynomial in $\mathbb{Z}[z]$ with Mahler measure $< 1.324717957\ldots$ is necessarily reciprocal.

As in the last chapter, we set

$$f_1(z) = z^6 - z^5 - z^3 + z^2 + 1 \quad \text{and} \quad f_2(z) = z^{10} - z^8 - z^5 + z + 1.$$

Table 4.1 Some polynomials of small measure with no Newman multiple

Polynomial	Mahler measure
$z^{44}-z^{42}+z^{40}-z^{38}-z^{33}-z^{32}+z^{31}+z^{30}-2z^{29}-z^{28}$ $+2z^{27}+z^{26}-z^{25}+z^{23}+z^{22}+z^{21}-z^{19}+z^{18}+2z^{17}-z^{16}$ $-2z^{15}+z^{14}+z^{13}-z^{12}-z^{11}-z^{6}+z^{4}-z^{2}+1$	$1.263095875\ldots$
$z^{26}-z^{23}-z^{21}+z^{15}+z^{13}+z^{11}-z^{5}-z^{3}+1$	$1.272019269\ldots$
$z^{50}-z^{49}+z^{48}-z^{47}-z^{40}+z^{39}-z^{38}+z^{37}-z^{36}+z^{35}$ $-z^{34}+z^{33}+z^{30}-z^{29}+z^{28}-z^{27}+z^{26}-z^{25}+z^{24}-z^{23}$ $+z^{22}-z^{21}+z^{20}+z^{17}-z^{16}+z^{15}-z^{14}+z^{13}-z^{12}+z^{11}$ $-z^{10}-z^{3}+z^{2}-z+1$	$1.273464959\ldots$
$z^{48}-z^{47}+z^{46}-z^{45}-z^{38}+z^{37}-z^{36}+z^{35}-z^{34}+z^{33}$ $-z^{32}+z^{31}+z^{28}-z^{27}+z^{26}-z^{25}+z^{24}-z^{23}+z^{22}-z^{21}$ $+z^{20}+z^{17}-z^{16}+z^{15}-z^{14}+z^{13}-z^{12}+z^{11}-z^{10}-z^{3}$ $+z^{2}-z+1$	$1.279464310\ldots$
$z^{48}-z^{47}+z^{46}-z^{45}+z^{44}-z^{43}-z^{40}+z^{39}-2z^{38}+2z^{37}$ $-2z^{36}+2z^{35}-z^{34}+z^{33}+z^{30}-z^{29}+z^{28}-z^{27}+z^{26}-z^{25}$ $+z^{24}-z^{23}+z^{22}-z^{21}+z^{20}-z^{19}+z^{18}+z^{15}-z^{14}+2z^{13}$ $-2z^{12}+2z^{11}-2z^{10}+z^{9}-z^{8}-z^{5}+z^{4}-z^{3}+z^{2}-z+1$	$1.279702474\ldots$
$z^{30}-z^{29}-z^{23}+z^{22}+z^{16}-z^{15}+z^{14}+z^{8}-z^{7}-z+1$	$1.299764321\ldots$
$z^{28}-z^{26}-z^{25}+z^{22}+z^{21}-z^{19}+z^{14}-z^{9}+z^{7}+z^{6}-z^{3}-z^{2}+1$	$1.309200435\ldots$

We also let

$$f_3(z) = z^9 + z^8 + z^7 - z^5 - z^4 - z^3 + 1.$$

The polynomial $f_1(z)$ is the polynomial given by Hare and Mossinghoff (2014) mentioned above, $f_2(z)$ was found by experimentation in Maple 2015, and $f_3(z)$ is the polynomial found by Drungilas, Jankauskas, and Šiurys (2016) stated above. The polynomial $f_1(z)$ is a non-reciprocal polynomial $f(z)$ satisfying (i) $f(z)$ has exactly two roots outside $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$, (ii) $f(z)$ has no positive real roots, (iii) $f(z)$ has no Newman multiple. Of all polynomials $f(z) \in \mathbb{Z}[z]$, the polynomial $f_1(z)$ is one of smallest Mahler measure which has been found to date. The polynomial $f_2(z)$ is a non-reciprocal polynomial $f(z) \in \mathbb{Z}[z]$ satisfying (ii) and (iii) of smallest Mahler measure which has been found to date. The polynomial $f_3(z)$ was prior to this work a non-reciprocal $f(z) \in \mathbb{Z}[z]$ satisfying (ii) and (iii) of smallest Mahler measure which has been found to date. Note that $f_1(z)$ and $f_2(z)$ were used for our examples in the last chapter. Their Mahler measures of the $f_k(x)$ are

$$1.556014485\ldots, \quad 1.419404632\ldots, \quad \text{and} \quad 1.436632260\ldots,$$

for $k = 1$, 2, and 3, respectively. The number of roots outside $\mathscr{C}$ are 2, 4, and 5, respectively.

For a fixed $k \in \{1,2,3\}$, define $F_n(z) = F_{n,k}(z) = f_k(z)z^n + \widetilde{f}_k(z)$, and let $h_n(z) = h_{n,k}(z)$ denote the largest degree monic factor of $F_n(z)$ in $\mathbb{Z}[z]$ that is not divisible by a cyclotomic polynomial. We note that each of the polynomials $f_1(z)$ and $f_2(z)$ can be shown to have no Newman multiples by making use of the algorithm described in Chapter 2 taken from Hare and Mossinghoff (2014) for complex β. Here, $f_2(z)$ was found by experimentation in Maple 2015. The polynomial $f_3(z)$ was shown not to have a Newman multiple by Drungilas, Jankauskas, and Šiurys (2016).

Based on the material from the last chapter, we know that for each $k \in \{1,2,3\}$, the polynomial $F_{n,k}(z)$ will have Mahler measure approaching $M(f_k)$ as n tends to infinity, and furthermore $F_{n,k}(z)$ and the corresponding $h_{n,k}(z)$ will have no Newman multiple for n sufficiently large. Experimentally, the Mahler measures of the polynomials $F_{n,k}(z)$ varied a bit for smaller values of n, as did the property of whether $F_{n,k}(z)$ or $h_{n,k}(z)$ has a Newman multiple. We tabulate here some of the data we obtained, the data that led to our more interesting examples.

All of the tables below list the leading digits (with no rounding) of the Mahler measure of $h_n(z) = h_{n,k}(z)$, the number of roots of $h_n(z)$ outside the unit circle $\mathscr{C}$, the number of those roots outside $\mathscr{C}$ which are real, the number of positive real roots of the factor $h_n(z)$, and whether or not $h_n(z)$ has a Newman multiple for $n \leq 35$. Each table records only the data from polynomials satisfying $M(h_n(z)) < M(f_k(z))$ for each k.

We note that each polynomial represented in Table 4.2 has a smaller Mahler measure than $f_1(z)$, the same number of roots outside of $\mathscr{C}$, and no Newman multiple. Table 4.2 also illustrates the content of Theorem 3.13, where we know that the Mahler measure of $F_n(z)$ approaches $M(f_1) = 1.556014485\ldots$ as n tends to infinity, $h_n(z) = F_n(z)$ for positive integers n from some infinite set $\mathscr{T}$, and the Mahler measures of $F_n(z)$ as n varies in $\mathscr{T}$ are distinct.

There are 8 polynomials represented in Table 4.3 that have no Newman multiple, some of which are of small measure, the smallest being the non-cyclotomic factor of $F_{31}(z) = f_2(z)z^{31} + \widetilde{f_2}(z)$ with Mahler measure $1.29976\ldots$.

For Tables 4.5 and 4.6, we revise the definition of $h_n(z)$ so that it is the product of the largest degree monic factor of $F_n(z) = f_k(z^2)z^n + \widetilde{f_k}(z^2)$ in $\mathbb{Z}[z]$ that is not divisible by a cyclotomic polynomial. Similarly, for each of the remaining tables given, the value of $F_n(z)$ is indicated and $h_n(z)$ denotes the product of the largest degree monic factor of $F_n(z)$ in $\mathbb{Z}[z]$ that is not divisible by a cyclotomic polynomial.

We observe that Table 4.6 gives $F_{29}(z) = f_2(z^2)z^{29} + \widetilde{f_2}(z^2)$, which produces a polynomial $h_{29}(z)$ of Mahler measure $1.27946431096\ldots$, and $F_{31}(z) = f_2(z^2)z^{31} + \widetilde{f_2}(z^2)$, which produces a polynomial $h_{31}(z)$ of Mahler measure $1.27346495964\ldots$. Neither of these polynomials has a positive real root or a Newman multiple. Another interesting example is given in Table 4.8 where the value of $h_{31}(z)$ for $F_{31}(z) = \widetilde{f_3}(z^2)z^{31} + f_3(z^2)$ has Mahler measure $1.27970247401\ldots$, no positive real root, and no Newman multiple. Also, the example $F_{28}(z) = f_2(z)z^{28} - \widetilde{f_2}(z)$ from Table 4.9 produces an $h_{28}(z)$ with Mahler measure $1.30920043575\ldots$, no positive real root, and no Newman multiple. The polynomial giving the second entry listed in Table 1.3 and again here in Table 4.1 is not listed in the other tables in this chapter. This polynomial is the factor $h_{19}(z)$ of $f_4(z^2)z^{19} + \widetilde{f_4}(z^2)$ having Mahler measure $1.27201926934\ldots$, where $f_4(z) = z^9 - z^7 - z^5 + z^3 + z + 1$. The polynomial $f_4(z)$ is an example from Drungilas, Jankauskas, and Šiurys (2016) with no Newman multiple, no positive real roots, and Mahler measure $1.48958132144\ldots$. The polynomial producing the first entry in Table 1.3 and Table 4.1 is discussed in the next section.

The notations ✓and ✗ in the last column serve as a “yes" and “no," respectively, to the question of whether $h_n(z)$ has a Newman multiple. The notation — indicates that we were not able to determine the answer to this question. We tested many more polynomials than those listed in these tables, and it should be noted that we could not successfully use our methods to determine whether or not Newman multiples existed for a number of them. The situation with the number of — indicated in Table 4.4 was not uncommon. A superscript $*$ next to an entry on the right-most column of a table means that some approach other than searching for cyclotomic multipliers to find a Newman multiple or using a direct application of the algorithms of Hare and Mossinghoff (2014) was used to verify the entry. We comment on these next.

For Table 4.5, the entries for $n \in \{30, 32, 34\}$, were dealt with as follows. In each case, the polynomial $h_n(z)$ is of the form $w_n(z^2)$ where $w_n(z)$ is the monic polynomial dividing $f_1(z)z^n + \widetilde{f_1}(z)$ of largest possible degree which has no cyclotomic factors. Observe that if $h_n(z)u(z)$ is in $\mathscr{N}$ for some $u(z) \in \mathbb{Z}[z]$, then we can write $u(z) = a(z^2) + zb(z^2)$ where $a(z)$ and $b(z)$ are in $\mathbb{Z}[z]$. Then

$$h_n(z)u(z) = w_n(z^2)a(z^2) + zw_n(z^2)b(z^2) \in \mathscr{N},$$

where $w_n(z^2)a(z^2)$ corresponds to the terms in $h_n(z)u(z)$ of even degree. In particular, this means that $w_n(z^2)a(z^2) \in \mathscr{N}$ so that $w_n(z)a(z) \in \mathscr{N}$. On the other hand, we already observed in Table 4.2 that $w_n(z)$ does not have a Newman multiple. Hence, $h_n(z)$ cannot have a Newman multiple.

For Table 4.8 with $n = 15$, taking the root

$$\beta_1 = (1.0928857359\ldots) + i(0.1254857623\ldots)$$

of $h_{15}(z)$, the method of Chapter 5 was used to verify that if β_1 is a root of an $F(z) \in \mathscr{N}$ of degree m, then $m > 15$ and

$$F(z) = z^m + \varepsilon_{15}z^{m-15} + \varepsilon_{16}z^{m-16} + \ldots + \varepsilon_m, \quad \text{where each } \varepsilon_j \in \{0,1\}.$$

Taking the real root $\beta_2 = -1.1248126357\ldots$ of $h_{15}(z)$, then the approach of Hare and Mossinghoff (2014) for real roots can be used. In this case, this amounts to the observation

$$\left|\frac{F(\beta_2)}{\beta_2^m}\right| > 1 - \frac{1}{|\beta_2|^{15}} - \frac{1}{|\beta_2|^{17}} - \cdots = 1 - \frac{1}{|\beta_2|^{15} - |\beta_2|^{13}} > 0,$$

so it is impossible for $F(\beta_2)$ to be 0. We note that β_2 is the root of $h_{15}(z)$ with maximum absolute value. For Table 4.8 with $n = 15$, the method of Hare and Mossinghoff (2014) was used with the root

$$\beta = (1.0949971342\ldots) + i(0.1042054033\ldots)$$

of $h_{19}(z)$. What made this worth noting is that β is not the root of $h_{19}(z)$ with largest absolute value. The root with largest absolute value is the real root $-1.1009452505\ldots$.

Table 4.2 Data for $F_n(z) = f_1(z)z^n + \widetilde{f_1}(z)$ with $M(F_n) < M(f_1) = 1.556\ldots$

n	$M(h_n(z))$	Number of roots outside $\mathscr{C}$	Number of real roots outside $\mathscr{C}$	Number of positive real roots	$h_n(z)$ has a Newman multiple?
15	1.5369179477682034	2	0	0	✗
16	1.5229957493128481	2	0	0	✗
17	1.5180589114389942	2	0	0	✗
18	1.5216200155288215	2	0	0	✗
19	1.5296307833019841	2	0	0	✗
20	1.5381835217766994	2	0	0	✗
21	1.5454469961759745	2	0	0	✗
22	1.5509638159120815	2	0	0	✗
23	1.5548344890951955	2	0	0	✗
35	1.5558158008316200	2	0	0	✗

Table 4.3 Data for $F_n(z) = f_2(z)z^n + \widetilde{f_2}(z)$ with $M(F_n) < M(f_2) = 1.419\ldots$

n	$M(h_n(z))$	Number of roots outside $\mathscr{C}$	Number of real roots outside $\mathscr{C}$	Number of positive real roots	$h_n(z)$ has a Newman multiple?
3	1.3019549434966640	2	0	0	✓
5	1.3001931433967972	2	0	0	✓
6	1.3993912890938539	2	0	0	✓
13	1.4155418842084992	2	0	0	✗
18	1.3696117008585243	2	0	0	✗
19	1.3615292809044729	2	0	0	✗
24	1.3275068021121254	2	0	0	✗
25	1.3220077539604978	2	0	0	✗
30	1.4031648828031895	4	0	0	✗
31	1.2997643210059706	2	0	0	✗
32	1.4095325205272812	4	0	0	✗

Table 4.4 Data for $F_n(z) = f_3(z)z^n + \widetilde{f_3}(z)$ with $M(F_n) < M(f_3) = 1.436\ldots$

n	$M(h_n(z))$	Number of roots outside $\mathscr{C}$	Number of real roots outside $\mathscr{C}$	Number of positive real roots	$h_n(z)$ has a Newman multiple?
3	1.3509803377162373	1	1	0	✓
4	1.2728183650834955	2	0	0	✓
6	1.3979993139693446	2	0	0	✓
7	1.4052124163112895	3	1	0	✓
8	1.2528286630316362	2	0	0	✓
10	1.2277855586945986	2	0	0	✓
11	1.2800820372203617	3	1	0	✓
16	1.2868840708651366	2	0	0	✓
18	1.3030748928169405	5	1	0	—
21	1.2194468759409303	3	1	0	✓
24	1.3195081637317398	3	1	0	—
27	1.3916311259038360	3	1	0	—
28	1.4297660580707468	5	1	0	—
29	1.3960141705798196	3	1	0	—
32	1.3593194660549923	3	1	0	—
34	1.4365773212651716	5	1	0	—
35	1.4067905794817218	5	1	0	—

Table 4.5 Data for $F_n(z) = f_1(z^2)z^n + \widetilde{f}_1(z^2)$ with $M(F_n) < M(f_1) = 1.556\ldots$

n	$M(h_n(z))$	Number of roots outside $\mathscr{C}$	Number of real roots outside $\mathscr{C}$	Number of positive real roots	$h_n(z)$ has a Newman multiple?
3	1.3776747893793782	1	1	0	✓
11	1.4681273540397895	3	1	0	✗
13	1.3510990619153372	2	0	0	✗
15	1.3349325020779348	2	0	0	✗
17	1.4628464414409443	4	0	0	✗
19	1.5442402396411085	4	0	0	✗
30	1.5369179477682034	4	0	0	✗*
32	1.5229957493128481	4	0	0	✗*
33	1.5531039916596791	4	0	0	—
34	1.5180589114389942	4	0	0	✗*
35	1.5512910936184789	4	0	0	—

Table 4.6 Data for $F_n(z) = f_2(z^2)z^n + \widetilde{f}_2(z^2)$ with $M(F_n) < M(f_2) = 1.419\ldots$

n	$M(h_n(z))$	Number of roots outside $\mathscr{C}$	Number of real roots outside $\mathscr{C}$	Number of positive real roots	$h_n(z)$ has a Newman multiple?
3	1.3320736964915977	1	1	0	✓
6	1.3019549434966640	4	0	0	✓
7	1.2000265239873915	1	1	0	✓
9	1.3736135811120419	5	1	0	✓
10	1.3001931433967972	4	0	0	✓
12	1.3993912890938539	4	0	0	✓
21	1.3767579101491908	5	1	0	✗
23	1.3913427635879796	5	1	0	✗
25	1.3592808747592152	5	1	0	✗
26	1.4155418842084992	4	0	0	✗
27	1.3584604307655911	7	1	0	✗
29	1.2794643109583782	4	0	0	✗
31	1.2734649596362572	4	0	0	✗
33	1.3646840825337791	8	0	0	✗
35	1.3580771338262559	6	0	0	✗

Table 4.7 Data for $F_n(z) = \widetilde{f_3}(z)z^n + f_3(z)$ with $M(F_n) < M(f_3) = 1.436\ldots$

n	$M(h_n(z))$	Number of roots outside $\mathscr{C}$	Number of real roots outside $\mathscr{C}$	Number of positive real roots	$h_n(z)$ has a Newman multiple?
2	1.3984561816152523	2	0	0	✓
5	1.3434948980986468	2	0	0	✓
13	1.4331345249485368	2	0	0	✗
14	1.4234311159715063	2	0	0	✗
18	1.3878827700420173	2	0	0	✗
23	1.3534163551018263	2	0	0	✗
27	1.3339979896237377	2	0	0	✗
28	1.4269214465986548	4	0	0	✗
32	1.3205703863583893	2	0	0	✗

Table 4.8 Data for $F_n(z) = \widetilde{f_3}(z^2)z^n + f_3(z^2)$ with $M(F_n) < M(f_3) = 1.436\ldots$

n	$M(h_n(z))$	Number of roots outside $\mathscr{C}$	Number of real roots outside $\mathscr{C}$	Number of positive real roots	$h_n(z)$ has a Newman multiple?
4	1.3984561816154015	4	0	0	✓
5	1.2527759374101137	1	1	0	✓
7	1.3484400894061053	3	1	0	✓
10	1.3434948980986485	4	0	0	✓
15	1.3916821984841463	5	1	0	✗*
19	1.4322043293234774	5	1	0	✗*
21	1.4288045592816294	5	1	0	✗
23	1.4074563000697322	5	1	0	✗
25	1.369606470305066	5	1	0	✗
26	1.4331345249485399	4	0	0	✗
27	1.3628156629302184	5	1	0	✗
28	1.4234311159714992	4	0	0	✗
29	1.3636483746277421	7	1	0	✗
31	1.2797024740084288	4	0	0	✗
33	1.3351370446158135	8	0	0	✗
35	1.3616825994601507	6	0	0	✗

Table 4.9 Data for $F_n(z) = f_2(z)z^n - \widetilde{f_2}(z)$ with $M(F_n) < M(f_2) = 1.419\ldots$

n	$M(h_n(z))$	Number of roots outside $\mathscr{C}$	Number of real roots outside $\mathscr{C}$	Number of positive real roots	$h_n(z)$ has a Newman multiple?
4	1.4012683679398549	1	2	2	✗
7	1.2612309611371388	1	2	2	✗
8	1.2303914344072247	1	2	2	✗
9	1.2026167436886042	1	2	2	✗
10	1.1762808182599175	1	2	2	✗
11	1.3516891084166915	3	2	2	✗
12	1.3357332210166238	3	2	2	✗
13	1.2486111656859293	3	2	2	✗
17	1.2883596645367590	4	0	0	✓
18	1.3765014052915571	4	0	0	✓
19	1.4014602019654860	4	0	0	✓
20	1.3730577151000097	4	0	0	✓
21	1.2648330803003662	2	0	0	✓
22	1.2775721230452175	2	0	0	✓
27	1.3854250650019093	4	0	0	✗
28	1.3092004357501738	2	0	0	✗
34	1.3123232555392172	2	0	0	✗
35	1.4151721596850971	4	0	0	✗

4.2 ANOTHER EXAMPLE OF A POLYNOMIAL WITH SMALL MEASURE HAVING NO NEWMAN MULTIPLE

The computations demonstrated in part in the previous section motivated a search of known examples of polynomials with small Mahler measure using the data listed by Mossinghoff (2011). In particular, based on the success of showing no Newman multiples existed for polynomials which had a root with large positive real part and a relatively small imaginary part (i.e., polynomials with a root close to the positive real axis), we looked for such polynomials among those given by this data. We found only one new and interesting one to report, namely

$$\begin{aligned} f(z) = z^{44} - z^{42} + z^{40} - z^{38} - z^{33} - z^{32} + z^{31} + z^{30} - 2z^{29} - z^{28} + 2z^{27} \\ + z^{26} - z^{25} + z^{23} + z^{22} + z^{21} - z^{19} + z^{18} + 2z^{17} - z^{16} \\ - 2z^{15} + z^{14} + z^{13} - z^{12} - z^{11} - z^{6} + z^{4} - z^{2} + 1. \end{aligned} \tag{4.1}$$

This polynomial has Mahler measure $M(f) = 1.263095875\ldots$, has no positive real root and has no Newman multiple. Thus, $f(z)$ provides us with the best upper bound thus far on σ (assuming it exists) in Problem 1, and we can report

$$\sigma \leq 1.263095875\ldots.$$

The value

$$\beta = (1.079315910\ldots) + i(0.752389188\ldots)$$

is a root of $f(z)$ with the largest absolute value, and the algorithm for complex roots given by Hare and Mossinghoff (2014) provides a proof that $f(z)$ has no Newman multiple. In the next chapter, we show an alternative way of showing this polynomial has no Newman multiple that also allows us to obtain some additional information in the case that the multiplier is in $\mathbb{R}[z]$.

Chapter 5

Bounding the coefficients of a multiple over the reals

5.1 Introduction

The polynomial (4.1) is the polynomial with smallest Mahler measure which we found having the property that there is no multiple of $f(z)$ in $\mathbb{Z}[z]$ which is a Newman polynomial. We have here that $M(f) = 1.263095875\ldots$, and $f(z)$ has no positive real root and has exactly 4 complex roots outside $\mathscr{C} = \{z \in \mathbb{C} : |z| = 1\}$. In this chapter, we show an alternative approach to establishing that there is no Newman multiple of $f(z)$ which can easily be turned into an algorithm as well. Like the previous algorithm, we cannot show that the algorithm will successfully show that an arbitrary polynomial with no Newman multiple does not have a Newman multiple. So our emphasis in this chapter will be different. We focus on using the algorithm to establish a bit more about $f(z)$, which we describe next.

As noted above, there is no Newman multiple for $f(z)$ in $\mathbb{Z}[z]$. But is there a Newman multiple for $f(z)$ in $\mathbb{R}[z]$? The answer is that there is not, and we show more, namely the following.

Theorem 5.1. *Let $f(z)$ be as in* (4.1). *Let $g(z) \in \mathbb{R}[z]$ for which $f(z)g(z)$ has non-negative coefficients. Then at least one of the coefficients is greater than* $1.5713809\ldots$.

As one would expect, we give a general method that would give a similar result for the other polynomials we have found earlier in this thesis which do not have Newman multiples, though $1.5713809\ldots$ would need to be replaced by a different value > 1 depending on the polynomial being considered. We also note that we actually show a bit more, that any polynomial with non-negative real coefficients, which has the root $1.0793159\ldots + i0.0752389\ldots$ in common with $f(z)$, has a coefficient greater than $1.5713809\ldots$.

The argument will be based on numeric approximations for complex roots of $f(z)$, and we do not concern ourselves (though we should) on whether these approximations suffice to justify the arguments given. We use Maple 2015 set at 50 digits of precision but do not indicate all the digits below in our argument for Theorem 5.1.

5.2 Proof of Theorem 5.1

Proof. One computationally checks that $f(z)$ has no real roots. Furthermore, two of its roots are

$$\alpha_1 = 1.0793159\ldots + i0.0752389\ldots \quad \text{and} \quad \alpha_2 = 1.0793159\ldots - i0.0752389\ldots.$$

Both α_1 and α_2 are outside the unit circle $\mathscr{C}$. There are 2 additional roots of $f(z)$ outside of $\mathscr{C}$ which will not play a role in this argument.

The idea is to take advantage of an approach from Cole, Dunn, and Filaseta (2016) and Filaseta and Gross (2014) that allowed the authors to show that multiples of certain quadratics of the form $z^2 - Az + B$ cannot have non-negative coefficients unless the maximum coefficient of the multiple exceeds an explicit bound. In Cole, Dunn, and Filaseta (2016) and Filaseta and Gross (2014), A and B are integers, but as we shall see here, the same approach works with real coefficients. For our purposes, we take A and B to be real numbers defined by

$$z^2 - Az + B = (z - \alpha_1)(z - \alpha_2) = z^2 - (2.1586318\ldots)z + 1.1705837\ldots. \tag{5.1}$$

So that the approach can apply to other polynomials besides $f(z)$, we use variables below and, in particular, refer to the quadratic above as $g(z) = z^2 - Az + B$ and allow for the possibility that A and B are not as indicated in (5.1). We require however that A and B

are positive real numbers. Define $b_0, \dots, b_s \in \mathbb{R}$ with $b_0 = 1$ and with the product

$$(b_0 z^s + b_1 z^{s-1} + \cdots + b_{s-1} z + b_s)(z^2 - Az + B) \tag{5.2}$$

equal to a polynomial of degree $s+2$ with non-negative coefficients. Set

$$h(z) = b_0 z^s + b_1 z^{s-1} + \cdots + b_{s-1} z + b_s = z^s + b_1 z^{s-1} + \cdots + b_{s-1} z + b_s$$

and $F(z) = g(z)h(z)$ as given in (5.2). We let M denote the maximal coefficient of $F(z)$ so that the coefficients of $F(z)$ are all in the interval $[0, M]$. In the context of $f(z)$ as in (4.1), among other things, we want to show that $F(z)$ cannot be a Newman polynomial, and we will establish this by showing that necessarily $M > 1$. For the theorem, we want to show $M \geq 1.5713809\ldots$ (still in the case of $f(z)$ as in (4.1)).

We define $b_j = 0$ for $j < 0$ and $j > s$. Since the coefficients of $F(z)$ are ≥ 0, we deduce that

$$b_j \geq A b_{j-1} - B b_{j-2} \quad \text{for all } j \in \mathbb{Z}. \tag{5.3}$$

Since $b_0 = 1$, we deduce $b_1 \geq A$. For each integer j, define

$$\beta_j = \begin{cases} 0 & \text{if } j < 0 \\ 1 & \text{if } j = 0 \\ A\beta_{j-1} - B\beta_{j-2} & \text{if } j \geq 1, \end{cases}$$

so the β_j satisfy a recursive relation for $j \geq 0$. In particular, $\beta_1 = A$ and $\beta_2 = A^2 - B$. For A and B as in (5.1), we have

$$\beta_0 = 1, \quad \beta_1 = 2.1586318\ldots, \quad \beta_2 = 3.4891076\ldots, \quad \beta_3 = 5.0048394\ldots,$$

$$\beta_4 = 6.7193130\ldots, \quad \beta_5 = 8.6459393\ldots, \quad \beta_6 = 10.7978812\ldots,$$

$$\beta_7 = 13.1878541\ldots, \quad \beta_8 = 15.8278975\ldots, \quad \beta_9 = 18.7291158\ldots,$$

$$\beta_{10} = 21.9013860\ldots, \quad \beta_{11} = 25.3530305\ldots, \quad \beta_{12} = 29.0904523\ldots,$$

$$\beta_{13} = 33.1177310\ldots, \quad \text{and} \quad \beta_{14} = 37.4361779\ldots.$$

Let J be minimal such that $\beta_{J+1} < \beta_J$. With A and B as in (5.1), we get $J = 34$, with

$$\beta_{33} = 135.3470045\ldots, \quad \beta_{34} = 135.6907665\ldots,$$

and

$$\beta_{35} = 134.4714052\ldots.$$

Note that, in general, $\beta_j \geq 0$ for $j \leq J$. Also, for $1 \leq j \leq J+1$, we obtain from (5.3) and $A \geq 0$ that

$$\begin{aligned} b_j &\geq Ab_{j-1} - Bb_{j-2} \geq A\big(Ab_{j-2} - Bb_{j-3}\big) - Bb_{j-2} \\ &\geq \beta_2 b_{j-2} - B\beta_1 b_{j-3} \geq \beta_2\big(Ab_{j-3} - Bb_{j-4}\big) - B\beta_1 b_{j-3} \\ &\geq \beta_3 b_{j-3} - B\beta_2 b_{j-4} \geq \beta_3\big(Ab_{j-4} - Bb_{j-5}\big) - B\beta_2 b_{j-4} \\ &\geq \beta_4 b_{j-4} - B\beta_3 b_{j-5} \geq \cdots \geq \beta_{j-1} b_1 - B\beta_{j-2} b_0 \geq \beta_j. \end{aligned} \tag{5.4}$$

We deduce that

$$b_j \geq \beta_j \quad \text{for all integers } j \leq J+1. \tag{5.5}$$

Now, we define

$$U = \max_{j \geq 0}\{b_j\} \quad \text{and} \quad L = \min_{j \geq 0}\{b_j\}.$$

Since $b_j = 0$ for $j > s$, we have $L \leq 0$. In the case of (5.1), we also see that $U \geq b_{34} \geq \beta_{34} = 135.6907665\ldots$. Let $k \geq 0$ and $\ell \geq 1$ be integers. We will want some flexibility on choosing precise values of k and ℓ, and in general some experimentation is helpful in selecting them depending on the choice of A and B.

The idea is to take advantage of a weighted average of ℓ consecutive coefficients of $F(z)$. Define $a_j = b_j - Ab_{j-1} + Bb_{j-2}$ for all integers j so that a_j is the coefficient of z^{s+2-j} in $F(z)$ for $0 \leq j \leq s+2$. Suppose $b_k \neq 0$, and let t_j be given by

$$b_{k+j} = t_j b_k \quad \text{for } j \in \mathbb{Z}. \tag{5.6}$$

Then

$$a_{k+j+2} = b_{k+j+2} - Ab_{k+j+1} + Bb_{k+j} = (t_{j+2} - At_{j+1} + Bt_j)b_k \quad \text{for } j \in \mathbb{Z}.$$

We will be interested in the weighted average of the ℓ numbers $a_{k+2}, a_{k+3}, \dots, a_{k+\ell+1}$ given by

$$W(k,\ell) = \sum_{j=0}^{\ell-1} \mu_j a_{k+j+2}, \qquad \text{where } 0 \le \mu_j \le 1 \text{ for } 0 \le j \le \ell-1 \text{ and } \sum_{j=0}^{\ell-1} \mu_j = 1.$$

Observe that $W(k,\ell) = W_0(k,\ell) b_k$, where

$$\begin{aligned} W_0(k,\ell) &= \sum_{j=0}^{\ell-1} \mu_j \big(t_{j+2} - A t_{j+1} + B t_j\big) \\ &= \mu_0 B t_0 + \big(-\mu_0 A + \mu_1 B\big) t_1 + \sum_{j=2}^{\ell-1} \big(\mu_{j-2} - \mu_{j-1} A + \mu_j B\big) t_j \\ &\qquad + \big(\mu_{\ell-2} - \mu_{\ell-1} A\big) t_\ell + \mu_{\ell-1} t_{\ell+1}. \end{aligned} \tag{5.7}$$

We choose the μ_j so that the coefficients of $t_1, t_2, \dots, t_{\ell-1}$ above are all zero. Keeping in mind that we want the μ_j to sum to 1, the above corresponds to choosing the μ_j so that the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ -A & B & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -A & B & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -A & B & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -A & B & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -A & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -A & B & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -A & B \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \vdots \\ \mu_{\ell-4} \\ \mu_{\ell-3} \\ \mu_{\ell-2} \\ \mu_{\ell-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

is satisfied. Of some interest to us is that the matrix equation depends only on A, B and ℓ, and not on k. The first row corresponds to the equation $\mu_0 + \mu_1 + \cdots + \mu_{\ell-1} = 1$. Recall that we also want $0 \le \mu_j \le 1$ for every $j \in \{0, 1, \dots, \ell-1\}$. Part of the process of choosing ℓ appropriately for given A and B is to ensure that the condition $0 \le \mu_j \le 1$ holds. In other words, the verification that $0 \le \mu_j \le 1$ will be established by solving the matrix equation above and checking directly if the condition holds. If it does not, then a different choice of ℓ needs to be selected.

Suppose now that μ_j is a fixed solution to the above matrix equation. The matrix equation guarantees that the coefficients of $t_1, t_2, \ldots, t_{\ell-1}$ in (5.7) are all zero. Hence, taking

$$a = \mu_0 B, \quad b = \mu_{\ell-2} - \mu_{\ell-1} A \quad \text{and} \quad c = \mu_{\ell-1},$$

we obtain

$$W_0(k,\ell) = at_0 + bt_\ell + ct_{\ell+1}.$$

The values of a, b and c depend on A, B and ℓ. From (5.6), the values of t_j depend on k.

We consider first taking k to satisfy $b_k = U$, which is possible by the definition of U. At this point, we will want knowledge about the signs of a, b and c. For our specific choice in (5.1), we will have that a, b and c are positive. This was also the case in Cole, Dunn, and Filaseta (2016) and Filaseta and Gross (2014). As this is the case of interest to us now as well, we suppose that

$$a > 0, \quad b > 0 \quad \text{and} \quad c > 0$$

but note that modifications can be made to the arguments that follow if for example $a > 0$, $b < 0$ and $c > 0$. Since the maximum coefficient of $F(z)$ is M, we have that (5.6) implies

$$\begin{aligned} M \ge W(k,\ell) &= W_0(k,\ell)b_k = at_0 b_k + bt_\ell b_k + ct_{\ell+1} b_k \\ &= ab_k + bb_{k+\ell} + cb_{k+\ell+1} \ge aU + bL + cL. \end{aligned} \tag{5.8}$$

Next, take k so that $b_k = L$. Since each coefficient of $F(z)$ is ≥ 0, we deduce here that

$$0 \le W(k,\ell) = W_0(k,\ell)b_k = ab_k + bb_{k+\ell} + cb_{k+\ell+1} \le aL + bU + cU. \tag{5.9}$$

Multiplying through (5.8) by a and through (5.9) by $-(b+c)$ and adding, we obtain

$$aM \ge (a^2 - (b+c)^2)U. \tag{5.10}$$

Multiplying through (5.8) by $b+c$ and through (5.9) by $-a$ and adding, we have

$$(b+c)M \geq (a^2-(b+c)^2)(-L), \tag{5.11}$$

where $-L$ is used here to emphasize that $L \leq 0$. We will not make use of (5.11) here, but note that if $F(z)$ is a Newman polynomial, then $M=1$ and (5.11) can be used to give a lower bound on the coefficients of $h(z)$.

Of particular interest to us is (5.10) as it provides a lower bound for M. We return to the case of A and B given by (5.1). We take $\ell = 45$ (arrived at through experimentation). The solution to the matrix equation is given in part by

$$\mu_0 = 0.0099262\ldots, \quad \mu_1 = 0.0183046\ldots, \quad \mu_2 = 0.0252752\ldots,$$

$$\mu_3 = 0.0309719\ldots, \quad \mu_4 = 0.0355222\ldots, \quad \mu_5 = 0.0390468\ldots,$$

$$\mu_6 = 0.0416590\ldots, \quad \mu_7 = 0.0434652\ldots, \mu_8 = 0.0445645\ldots, \quad \ldots,$$

$$\mu_{42} = 0.0007752\ldots, \quad \mu_{43} = 0.0003826\ldots, \quad \text{and} \quad \mu_{44} = 0.0000433\ldots.$$

One checks that $\mu_j \in [0,1]$ for each $j \in \{0,1,\ldots,44\}$. We obtain

$$a = \mu_0 B = 0.0116195\ldots, \quad b = \mu_{\ell-2} - \mu_{\ell-1}A = 0.0002890\ldots$$

$$\text{and} \quad c = \mu_{\ell-1} = 0.0000433\ldots.$$

Recall that

$$U \geq b_{34} \geq \beta_{34} = 135.6907665\ldots.$$

From (5.10), we now obtain that

$$M \geq 1.5713809\ldots.$$

Thus, the maximum coefficient of $F(z)$ must exceed $1.5713809\ldots$, establishing the theorem. □

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