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Subdivision of Measures of Squares

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by

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Abstract

The primary goal of our work is to establish a method to relate simple measures to a given set of moments. We calculate the moments of squares via linear polynomial weight measures and straight line cuts and use this to calculate the centre of mass of the square. The one-to-one correspondence that is found is needed to represent surfaces with gaps, which can estimate arbitrary measures on squares. From this, a subdivision scheme is developed, which successively quadrisects squares and uses the relation to estimate the new measures in order to provide a good representation of the original surface. One application of this work is for processing point clouds and their related surfaces.

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Chapter 1

Preliminaries

1.1 Introduction

Given a square of constant density, it is straightforward to calculate its centre of mass. For a square represented by $R^* = [-1, 1]^2$, the centre of mass is $(0, 0)$. Given a straight line cut through the square, two distinct regions are produced. Given a region, any calculus textbook [4] will tell you that its centre of mass is given by $\overline{x} = \dfrac{M_y}{m}$ and $\overline{y} = \dfrac{M_x}{m}$, where m is the mass of the object, M_x is the moment about the x-axis and M_y is the moment about the y-axis.

$$m = \int_{R^*} 1 \, d\zeta \tag{1.1}$$

$$M_y = \int_{R^*} x \, d\zeta \tag{1.2}$$

$$M_x = \int_{R^*} y \, d\zeta \tag{1.3}$$

In these equations, $d\zeta = \chi_{\text{cut}} dA$.

Given these equations, one can calculate the centre of mass on one side of an arbitrary cut. In Chapter 2, we will be calculating the centre of mass of both parts of the square, and using the centre of mass to find the unique equation of the corresponding cut. In Chapter 3, we will be using this one-to-one correspondence to develop a subdivision scheme to estimate the measures of the successively smaller squares.

1.2 Subdivision

Subdivision schemes are used nearly universally in the field of computer graphics. Computer-animated movies, 3D modelling, and video games all use subdivision schemes to create smoother objects from more jagged ones. Various subdivision schemes can be both easy to implement and computationally efficient [6], without requiring a significant increase in data or computation time.

Common subdivision schemes use Bézier curves or B-splines to generate smoother curves in one dimension. In higher dimensions, the same principles are applied, to create notable subdivision schemes [2] as Catmull-Clark, Doo-Sabin, Loop, and 4-8 subdivision.

Our goal is to establish a method to create a one-to-one correspondence between simple measures and a given set of moments. Specifically, a relation is found between parameters of a cut in a square and the moments, which can be used to find the related centre of mass. This is needed in subdivision to represent surfaces with holes or gaps. Ignoring the fact that the data does not cover the entire region can result in bad approximation, while directly identifying the missing area is difficult and would eventually lead to a more complicated representation. By using a subdivision scheme based on the partial measures, we can resolve these issues and have an effective strategy. Since the moments of smaller sets can be defined by the moments of bigger ones, we can take a square with known moments, then cut it in quadrants and use techniques developed in Chapter 2 to calculate the new moments. By repeating this process, the squares become finer and finer. This subdivision scheme will be explained in Chapter 3. One particular application of this scheme is for processing point clouds and relating surfaces to them.

Chapter 2

Basic Calculations

2.1 Moments

The points in a region R can be difficult to work with. By calculating aggregate values of these points, we can simplify calculations, while retaining important data about the set. These aggregate quantities of a region R will be called $Q(R)$. The moments are useful quantities, because they enable us to calculate values such as average values, variance, and covariance [2]. Additionally, the moments are additive, while the averages and variance are not, which enables fewer computations later on.

Definition 2.1. The i-th moments of a region R are given by:

$$M_i(R) = \int_R x^i \, dA. \tag{2.1}$$

Definition 2.2. The moments for $\mathbf{x} = (x, y)$ over a region R is given by:

$$M_{ij}(R) = \int_R x^i y^j \, dA. \tag{2.2}$$

Given the moments of an unknown measure ρ, we wish to find a known measure μ such that $Q(R, \rho) = Q(R, \mu)$, and we say that μ represents ρ. Then we approximate the quantities of the children with respect to ρ by the direct calculation of the quantities with respect to the known measure μ.

Our quantities will be defined as follows:

$$Q(R, \mu) = (M_{00}(R),\ M_{10}(R),\ M_{01}(R)) \tag{2.3}$$

with

$$\begin{aligned} M_{00}(R) &= \int_R 1 \, d\mu \\ M_{10}(R) &= \int_R x \, d\mu \\ M_{01}(R) &= \int_R y \, d\mu. \end{aligned} \tag{2.4}$$

If the region is not a lamina of constant density, but rather a collection of points $\{(x_i, y_i)\}_{i=1}^{m}$, then we can approximate the moments by summing over the points as follows:

$$Q(R, \mu) = (m, \ \sum_{i=1}^{m} x_i, \ \sum_{i=1}^{m} y_i). \tag{2.5}$$

Given a measure μ over R^*, we can see that $M_{00}(R^*, \mu)$ gives us the mass of the region.

2.2 Approximation of Quantities

Our standard region will be $R^* = [-1, 1]^2$. We chose this so that we can easily subdivide into four smaller squares later on. One reason to calculate the moments is so they can be used in this scheme to approximate the moments of subdivided regions of our squares.

Definition 2.3. Define for a measure μ over R^*,

$$\begin{aligned} u(\mu) &= \frac{M_{10}(R^*, \mu)}{M_{00}(R^*, \mu)} \\ v(\mu) &= \frac{M_{01}(R^*, \mu)}{M_{00}(R^*, \mu)} \\ q(\mu) &= (u(\mu), v(\mu)). \end{aligned} \tag{2.6}$$

From equations (1.1) to (1.3), we can see that u and v represent the average x and y values respectively, and q represents the centre of mass of the region. Naturally, it will lie within the convex hull of R^*.

We develop two methods based on the types of measures we wish to reproduce in order to calculate the quantities $Q(R)$ of a region. The measures we consider are of the form $d\mu = \omega(\mathbf{x})\, d\mathbf{x}$ with:

- **Linear Polynomial Weight Measures:**

$$\omega(x, y) = \alpha x + \beta y + \gamma \tag{2.7}$$

  The parameters are α and β, which we will optimize so that $\omega(x) \geq 0$ on R.

- **Subdomain Measures:**

$$\omega(\mathbf{x}) = \chi_{\text{cut}} \quad \text{or} \quad \omega(\mathbf{x}) = 1 - \chi_{\text{cut}} \tag{2.8}$$

  The parameters are two of a, b, c and d, representing the coordinates of the intersection of our cut with our standard region R^*.

We describe our schemes over the standard region $R^* = [-1, 1]^2$ as there is an affine transformation to and from a general square (or rectangle) and R^* [1][3]. Thus, the parent region can be divided into four children, representing the top left, top right, bottom left, and bottom right quadrants, numbered 1, 2, 3, and 4, respectively.

Any element from a general square with bottom left coordinate (x_1, y_1) and top right coordinate (x_2, y_2) can be transformed to the standard region R^* by the affine transformation T, described as follows:

$$T = \begin{bmatrix} \frac{2}{x_2 - x_1} & 0 & 0 \\ 0 & \frac{2}{y_2 - y_1} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & \frac{x_1 + x_2}{-2} \\ 0 & 1 & \frac{y_1 + y_2}{-2} \\ 0 & 0 & 1 \end{bmatrix}. \tag{2.9}$$

Then

$$T = \begin{bmatrix} \frac{2}{x_2 - x_1} & 0 & \frac{x_1 + x_2}{x_1 - x_2} \\ 0 & \frac{2}{y_2 - y_1} & \frac{y_1 + y_2}{y_1 - y_2} \\ 0 & 0 & 1 \end{bmatrix} \tag{2.10}$$

so that any point (x, y) in a square can be transformed to (x^*, y^*) in the reference square via the transformation

$$T \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} = \begin{bmatrix} x^* \\ y^* \\ 1 \end{bmatrix}. \tag{2.11}$$

Similarly, any point (x^*, y^*) in the reference square R^* can be transformed to (x, y) in a general square by the affine transformation

$$T^{-1} \begin{bmatrix} x^* \\ y^* \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ 1 \end{bmatrix}. \tag{2.12}$$

2.3 Linear Polynomial Weight Measures

Consider the probability measures μ over the reference square R^* such that $d\mu = C\omega(\mathbf{x})\,d\mathbf{x}$ where $\omega(\mathbf{x}) = \alpha x + \beta y + \gamma$. We define the weight functions so that:

$$\int_{R^*} \omega(\mathbf{x})\,d\mathbf{x} = 1 \tag{2.13}$$

$$\omega(\mathbf{x}) \geq 0 \text{ for } \mathbf{x} \in R^*. \tag{2.14}$$

Equation (2.13) implies

$$\int_{R^*} \omega(\mathbf{x})\,d\mathbf{x} = \int_{-1}^{1}\int_{-1}^{1} \alpha x + \beta y + \gamma\,dy\,dx = 4\gamma \tag{2.15}$$

so that $\gamma = \dfrac{1}{4}$.

Equation (2.14) implies $\omega(\mathbf{x})$ will be minimized when $\mathbf{x} = (\pm 1, \pm 1)$, resulting in four equations:

$$\begin{aligned} \frac{1}{4} + \alpha + \beta \geq 0 \\ \frac{1}{4} - \alpha + \beta \geq 0 \\ \frac{1}{4} - \alpha - \beta \geq 0 \\ \frac{1}{4} + \alpha - \beta \geq 0 \end{aligned} \tag{2.16}$$

or equivalently

$$\begin{aligned} -\frac{1}{4} \leq \alpha + \beta \leq \frac{1}{4} \\ -\frac{1}{4} \leq \alpha - \beta \leq \frac{1}{4}. \end{aligned} \tag{2.17}$$

This is a rhombus in the $\alpha\beta$-plane with vertices at $(1/4, 0)$, $(0, 1/4)$, $(-1/4, 0)$, and $(0, -1/4)$. Let $S = \left\{(\alpha, \beta) : -\frac{1}{4} \leq \alpha + \beta \leq \frac{1}{4} \text{ and } -\frac{1}{4} \leq \alpha - \beta \leq \frac{1}{4}\right\}$.

Then the moments are:

$$M_{00}(R^*, \mu) = \int_{R^*} \omega(\mathbf{x})\, d\mathbf{x} = \int_{-1}^{1} \int_{-1}^{1} \alpha x + \beta y + \frac{1}{4}\, dy\, dx = 1 \tag{2.18}$$

$$M_{10}(R^*, \mu) = \int_{R^*} x\omega(\mathbf{x})\, d\mathbf{x} = \int_{-1}^{1} \int_{-1}^{1} x \left(\alpha x + \beta y + \frac{1}{4}\right) dy\, dx = \frac{4\alpha}{3} \tag{2.19}$$

$$M_{01}(R^*, \mu) = \int_{R^*} y\omega(\mathbf{x})\, d\mathbf{x} = \int_{-1}^{1} \int_{-1}^{1} y \left(\alpha x + \beta y + \frac{1}{4}\right) dy\, dx = \frac{4\beta}{3}. \tag{2.20}$$

In the uv-plane, we have that

$$u = \frac{M_{10}(R^*, \mu)}{M_{00}(R^*, \mu)} = \frac{4\alpha}{3} \quad \text{and} \quad v = \frac{M_{01}(R^*, \mu)}{M_{00}(R^*, \mu)} = \frac{4\beta}{3}. \tag{2.21}$$

By plugging in extreme values of α and β, we can determine that the image of S in the uv-plane is also a rhombus T with vertices at $(1/3, 0)$, $(0, 1/3)$, $(-1/3, 0)$, and $(0, -1/3)$. Then $T = \left\{(u, v) : -\frac{1}{3} \leq u + v \leq \frac{1}{3} \text{ and } -\frac{1}{3} \leq u - v \leq \frac{1}{3}\right\}$.

The correspondence between (α, β) and (u, v) is one-to-one and the inverse is given by

$$\alpha = \frac{3u}{4} \quad \text{and} \quad \beta = \frac{3v}{4}. \tag{2.22}$$

Lemma 2.4. *Let ρ be a measure on R^*. If $q(\rho)$ is in T, then we construct a polynomial weight measure η of the form $d\eta = C\omega(\mathbf{x})\, d\mathbf{x}$ such that $Q(R^*, \rho) = Q(R^*, \eta)$.*

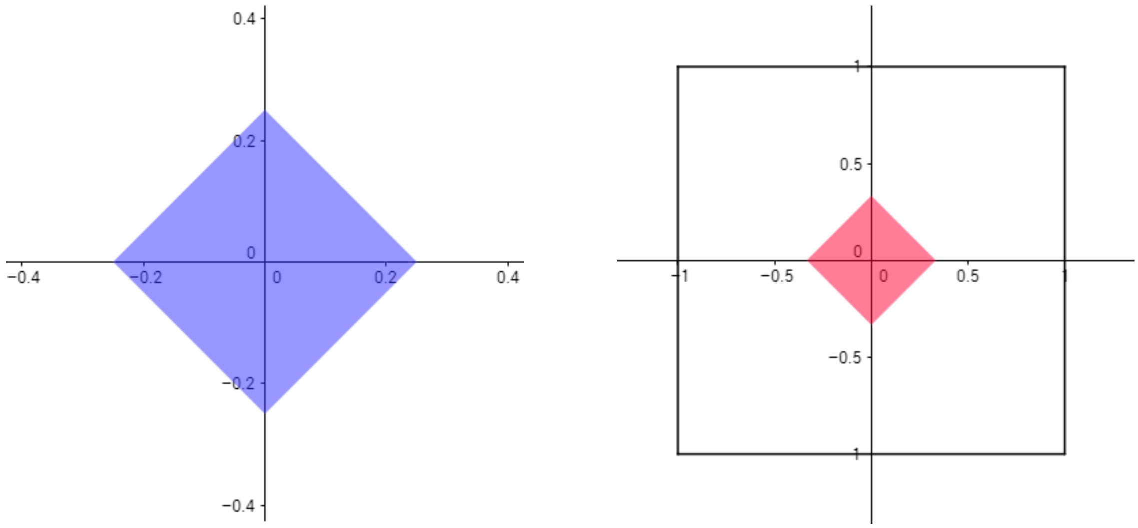

Figure 2.1: On the left we have the set S in the $\alpha\beta$-plane. On the right is the corresponding region T in the uv-plane.

Proof. The formulas in (2.22) give us values of α and β. Then we have

$$u(\rho) = \frac{\int_{R^*} x \, d\rho}{\int_{R^*} 1 \, d\rho} = \int_{R^*} x\omega(\mathbf{x}) \, d\mathbf{x} \tag{2.23}$$

$$v(\rho) = \frac{\int_{R^*} y \, d\rho}{\int_{R^*} 1 \, d\rho} = \int_{R^*} y\omega(\mathbf{x}) \, d\mathbf{x} \tag{2.24}$$

Therefore, we let $C = \int_{R^*} 1 \, d\rho$, so that

$$M_{00}(R^*, \rho) = C = \int_{R^*} 1 \, d\eta = C \int_{R^*} \omega(\mathbf{x}) \, d\mathbf{x} = M_{00}(R^*, \eta). \tag{2.25}$$

Additionally,

$$M_{10}(R^*, \rho) = u(\rho) M_{00}(R^*, \rho) = Cu(\rho) = C \int_{R^*} x\omega(\mathbf{x}) \, d\mathbf{x} = M_{10}(R^*, \eta). \tag{2.26}$$

Finally,

$$M_{01}(R^*, \rho) = v(\rho) M_{00}(R^*, \rho) = Cv(\rho) = C \int_{R^*} y\omega(\mathbf{x}) \, d\mathbf{x} = M_{01}(R^*, \eta). \tag{2.27}$$

Therefore, $Q(R^*, \rho) = Q(R^*, \eta)$.

□

Given a measure ρ with $q(\rho)$ in T, we determine the representative measure η. With the representative measure η known, in order to subdivide the square, we can simply calculate the quantities for the children.

However, because the acceptable region T is not equal to R^*, some measures will not have a linear polynomial weight representation. Thus we need to develop an additional scheme.

2.4 Subdomain Measures

In this section, we consider the probability measures μ over the reference square R^* such that $d\mu = C\omega(\mathbf{x})\,d\mathbf{x}$ where $\omega(\mathbf{x}) = \chi_{\text{cut}}$ or $\omega(\mathbf{x}) = 1 - \chi_{\text{cut}}$. We will refer to measures of the first form as primary measures, and the second as complementary measures. Here, *cut* represents the area on the smaller side of a single straight cut dividing the reference square into two disjoint sets, whose union is R^*. Specifically, the primary measure represents the area which does not contain the origin $(0, 0)$.

There are several types of cuts that we have to account for, but this reduces to only two due to symmetry. Define four points on the boundary of R^* as follows:

- $\mathbf{a} = (a, -1) \quad \text{where } -1 \leq a \leq 1,$
- $\mathbf{b} = (-1, b) \quad \text{where } -1 \leq b \leq 1,$
- $\mathbf{c} = (c, 1) \quad \text{where } -1 \leq c \leq 1,$
- $\mathbf{d} = (1, d) \quad \text{where } -1 \leq d \leq 1.$

From here, we can envision eight different cuts, falling into two general categories:

- Type 1: Corner (Figure 2.2)

  Cuts of Type 1 isolate a single corner of R^*. They may be triangles consisting of $(\mathbf{a}, \mathbf{b}, (-1, -1))$, $(\mathbf{b}, \mathbf{c}, (-1, 1))$, $(\mathbf{c}, \mathbf{d}, (1, 1))$, or $(\mathbf{d}, \mathbf{a}, (1, -1))$.

- Type 2: Edge (Figure 2.3)

  Cuts of Type 2 isolate a two corners of R^*. They may be trapezoids consisting of $(\mathbf{b}, \mathbf{d}, (1,-1), (-1,-1)$, $(\mathbf{c}, \mathbf{a}, (-1,-1), (-1,1)$, $(\mathbf{d}, \mathbf{b}, (-1,1), (1,1))$, or $(\mathbf{a}, \mathbf{c}, (1,1), (1,-1))$.

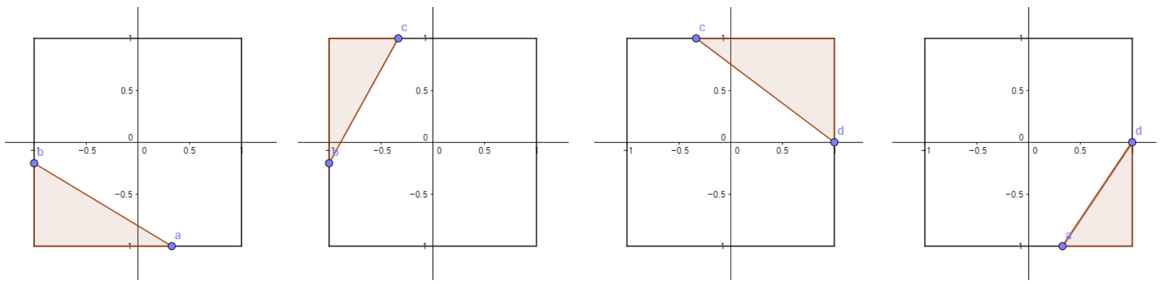

Figure 2.2: Four possible representations of cuts of Type 1, that isolate a single corner.

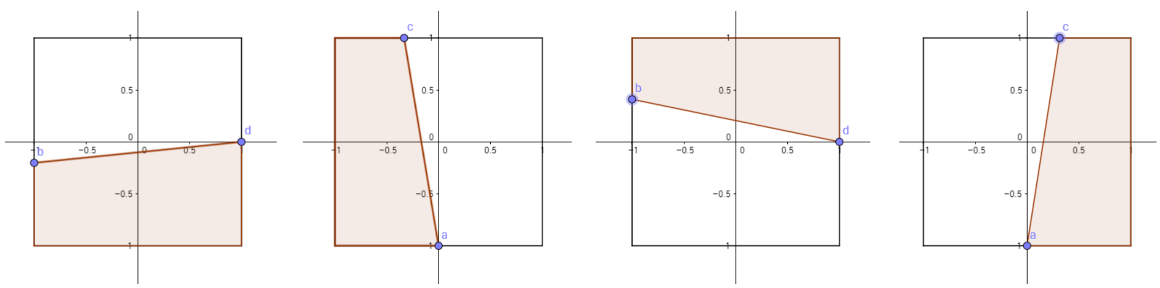

Figure 2.3: Four possible representations of cuts of Type 2, that isolate a single edge.

We will only calculate the moments for one measure each of Type 1 and Type 2, and use symmetry to derive the rest.

2.4.1 Primary Measures

For measures of Type 1, the cut itself can have several different forms, depending on which representation is being used. That is, if points **a** and **b** are connected with a straight line, the general form of the line will be different than if **c** and **d** are connected, although both can still be considered Type 1 measures. Due to symmetry, we only need to calculate one of these.

For this example, we will connect **b** and **c**, as in the second image of Figure 2.2. This is a line with slope $\frac{1-b}{c+1}$ and equation $y = \frac{1-b}{c+1}(x+1) + b$, called L_{bc}.

Then the moments are:

$$\begin{aligned} M_{00}(R^*, \mu) &= \int_{R^*} 1\, d\mu = \int_{\text{cut}} C\, d\mathbf{x} = \int_{-1}^{c} \int_{L_{bc}}^{1} C\, dy\, dx \\ &= C\frac{(c+1)(1-b)}{2} \end{aligned} \tag{2.28}$$

$$\begin{aligned} M_{10}(R^*, \mu) &= \int_{R^*} x\, d\mu = \int_{\text{cut}} Cx\, d\mathbf{x} = \int_{-1}^{c} \int_{L_{bc}}^{1} Cx\, dy\, dx \\ &= C\frac{(c-2)(c+1)(1-b)}{6} \end{aligned} \tag{2.29}$$

$$\begin{aligned} M_{01}(R^*, \mu) &= \int_{R^*} y\, d\mu = \int_{\text{cut}} Cy\, d\mathbf{x} = \int_{-1}^{c} \int_{L_{bc}}^{1} Cy\, dy\, dx \\ &= C\frac{(b+2)(c+1)(1-b)}{6}. \end{aligned} \tag{2.30}$$

Therefore, the transformation from (b, c) to (u, v) is

$$u = \frac{M_{10}(R^*, \mu)}{M_{00}(R^*, \mu)} = \frac{c-2}{3} \quad \text{and} \quad v = \frac{M_{01}(R^*, \mu)}{M_{00}(R^*, \mu)} = \frac{b+2}{3}. \tag{2.31}$$

Since the transformation is linear, the inverse is easily calculated as

$$c = 3u + 2 \quad \text{and} \quad b = 3v - 2. \tag{2.32}$$

Since $-1 \le b \le 1$ and $-1 \le c \le 1$, one can see that all the points $q = (u(b,c), v(b,c))$ will fall in a square A with vertices $(-1, 1/3)$, $(-1, 1)$, $(-1/3, 1)$, and $(-1/3, 1/3)$ (Figure 2.4).

Similar computations for the other Type 1 Primary Measures result in Table 2.1.

Therefore, measures corresponding to a vector $q \in \{A,\ C,\ E,\ \text{or}\ G\}$ can be modeled by a primary measure of Type 1.

For measures of Type 2, the cut can have also several different forms, depending on which representation is being used. That is, if points **a** and **c** are connected with

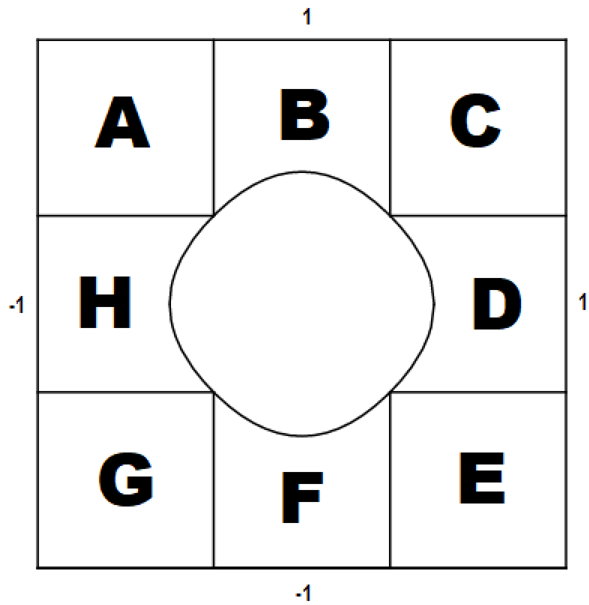

Figure 2.4: Each labeled region corresponds to a Primary Measure of Type 1 (ACEG) or Type 2 (BDFH).

Table 2.1: Transformation and Inverses of Type 1 Primary Measures

Region	**Line**	**Transformation**	**Inverse**
A	L_{bc}	$u = \dfrac{c-2}{3},\ v = \dfrac{b+2}{3}$	$c = 3u+2,\ b = 3v-2$
C	L_{cd}	$u = \dfrac{c+2}{3},\ v = \dfrac{d+2}{3}$	$c = 3u-2,\ d = 3v-2$
E	L_{ad}	$u = \dfrac{a+2}{3},\ v = \dfrac{d-2}{3}$	$a = 3u-2,\ d = 3v+2$
G	L_{ab}	$u = \dfrac{a-2}{3},\ v = \dfrac{b-2}{3}$	$a = 3u+2,\ b = 3v+2$

a straight line, the general form of the line will be different than if **b** and **d** are connected, although both can still be considered Type 2 measures. Due to symmetry, we only need to calculate one of these.

For this example, we will connect **a** and **c**, as in the second image of Figure 2.3. This is a line with slope $\dfrac{2}{c-a}$ and equation $y = \dfrac{2}{c-a}(x-c)+1$, called L_{ac}. However, there are now two different areas we must consider: the region to the left and to the right of this line. The one that contains the origin will be the complementary measure; in this section we will examine the primary measure. We will assume $a + c < 0$ so that the primary measure is to the left of the line.

Then the moments are:

$$\begin{aligned} M_{00}(R^*, \mu) &= \int_{R^*} 1\, d\mu = \int_{\text{cut}} C\, d\mathbf{x} = \int_{-1}^{1} \int_{-1}^{L_{ac}} C\, dy\, dx \\ &= C(a + c + 2) \end{aligned} \tag{2.33}$$

$$\begin{aligned} M_{10}(R^*, \mu) &= \int_{R^*} x\, d\mu = \int_{\text{cut}} Cx\, d\mathbf{x} = \int_{-1}^{1} \int_{L_{ac}}^{1} Cx\, dy\, dx \\ &= \frac{C}{3}(a^2 + ac + c^2 - 3) \end{aligned} \tag{2.34}$$

$$\begin{aligned} M_{01}(R^*, \mu) &= \int_{R^*} y\, d\mu = \int_{\text{cut}} Cy\, d\mathbf{x} = \int_{-1}^{1} \int_{L_{ac}}^{1} Cy\, dy\, dx \\ &= \frac{C}{3}(c - a). \end{aligned} \tag{2.35}$$

Therefore, the transformation from (a, c) to (u, v) is

$$u = \frac{M_{10}(R^*, \mu)}{M_{00}(R^*, \mu)} = \frac{a^2 + ac + c^2 - 3}{3a + 3c + 6} \quad \text{and} \quad v = \frac{M_{01}(R^*, \mu)}{M_{00}(R^*, \mu)} = \frac{c - a}{3a + 3c + 6}. \tag{2.36}$$

This transformation is not linear, however the inverse can still be calculated as

$$a = \frac{-3v^2 - 6uv + 2u - 6y + 1}{3v^2 + 1} \quad \text{and} \quad c = \frac{-3v^2 + 6uv + 2u + 6v + 1}{3v^2 + 1}. \tag{2.37}$$

Since $-1 \leq a \leq 1$ and $-1 \leq c \leq 1$, one can see that all the points $q = (u(a, c), v(a, c))$ will fall in a region H defined by $-1/3 \leq v \leq 1/3$, $-1 \leq u \leq 3/2\, v^2 - 1/2$ (Figure 2.4).

Exploiting the properties of symmetry, computations for the other Type 2 Primary measures result in Table 2.2.

Therefore, measures corresponding to a vector $q \in \{B,\ D,\ F, \text{ or } H\}$ can be modeled by a primary measure of Type 2.

For each type of primary measure, we can calculate the parameters of the measure given $q = (u, v)$ of the measure over the parent square. These parameters uniquely

Table 2.2: Transformation and Inverses of Type 2 Primary Measures

Region	Line	Transformation	Inverse
B	L_{bd}	$u = \dfrac{d-b}{3b+3d-6}$, $v = \dfrac{b^2+bd+d^2-3}{3b+3d-6}$	$b = \dfrac{3u^2-6uv+6u+2v-1}{3u^2+1}$, $d = \dfrac{3u^2+6uv-6u+2v-1}{3u^2+1}$
D	L_{ca}	$u = \dfrac{a^2+ac+c^2-3}{3a+3c-6}$, $v = \dfrac{c-a}{3a+3c-6}$	$a = \dfrac{3v^2-6uv+2u+6v-1}{3v^2+1}$, $c = \dfrac{3v^2+6uv+2u-6v-1}{3v^2+1}$
F	L_{db}	$u = \dfrac{d-b}{3b+3d+6}$, $v = \dfrac{b^2+bd+d^2-3}{3b+3d+6}$	$b = \dfrac{-3u^2-6uv-6u+2v+1}{3u^2+1}$, $d = \dfrac{-3u^2+6uv+6u+2v+1}{3u^2+1}$
H	L_{ac}	$u = \dfrac{a^2+ac+c^2-3}{3a+3c+6}$, $v = \dfrac{c-a}{3a+3c+6}$	$a = \dfrac{-3v^2-6uv+2u-6v+1}{3v^2+1}$, $c = \dfrac{-3v^2+6uv+2u+6v+1}{3v^2+1}$

determine the measure. In the case of polynomial weight measure, the constant C was simply the zeroth moment of the parent. In the case of subdomain measures, it is not as simple.

Lemma 2.5. *Let ρ be a measure on R^*. If $q(\rho)$ is in A, then there exists a unique subdomain measure μ of the form $d\mu = C\chi_{cut}\,d\mathbf{x}$ where the cut is of the form L_{bc}, such that $Q(R^*,\rho) = Q(R^*,\mu)$.*

Proof. The formulas in (2.31) give us values of

$$u = \frac{c-2}{3} \quad \text{and} \quad v = \frac{b+2}{3}. \tag{2.38}$$

Then we let

$$C = \frac{2M_{00}(R^*,\rho)}{(c+1)(1-b)}. \tag{2.39}$$

Define the cut to be the line L_{bc}, so that the cut is the triangle with vertices $(\mathbf{b}, \mathbf{c}, (-1,1))$ and μ to be the measure such that $d\mu = C\chi_{\text{cut}}\,d\mathbf{x}$. Then equations

(2.28) to (2.30) give us that

$$M_{00}(R^*,\mu) = C\frac{(c+1)(1-b)}{2} = \frac{M_{00}(R^*,\rho)(c+1)(1-b)}{(c+1)(b-1)}$$
$$= M_{00}(R^*,\rho) \tag{2.40}$$

$$M_{10}(R^*,\mu) = C\frac{(c-2)(c+1)(1-b)}{6} = \frac{M_{00}(R^*,\rho)(c-2)}{3} = u(\rho)M_{00}(R^*,\rho)$$
$$= M_{10}(R^*,\rho) \tag{2.41}$$

$$M_{01}(R^*,\mu) = C\frac{(b+2)(c+1)(1-b)}{6} = \frac{M_{00}(R^*,\rho)(b+2)}{3} = v(\rho)M_{00}(R^*,\rho)$$
$$= M_{01}(R^*,\rho). \tag{2.42}$$

Therefore, $Q(R^*,\mu) = Q(R^*,\rho)$. □

Lemma 2.6. *As above, let ρ be a measure on R^*. If $q(\rho)$ is in H, then there exists a unique subdomain measure μ of the form $d\mu = C\chi_{cut}\,d\mathbf{x}$ where the cut is of the form L_{ac}, such that $Q(R^*,\rho) = Q(R^*,\mu)$.*

Proof. The formulas in (2.36) give us values of

$$u = \frac{a^2+ac+c^2-3}{3a+3c+6} \quad \text{and} \quad v = \frac{c-a}{3a+3c+6}. \tag{2.43}$$

Then we let

$$C = \frac{M_{00}(R^*,\rho)}{a+c+2}. \tag{2.44}$$

Define the cut to be the line L_{ac}, so that the cut is the trapezoid with vertices $(\mathbf{a}, \mathbf{c}, (-1,1), (-1,-1))$ and μ to be the measure such that $d\mu = C\chi_{\text{cut}}\,d\mathbf{x}$. Then equations (2.33) to (2.35) give us that

$$M_{00}(R^*,\mu) = C(a+c+2) = \frac{M_{00}(R^*,\rho)(a+c+2)}{(a+c+2)}$$
$$= M_{00}(R^*,\rho) \tag{2.45}$$

$$\begin{aligned} M_{10}(R^*,\mu) = C\frac{a^2+ac+c^2-3}{3} &= \frac{M_{00}(R^*,\rho)(a^2+ac+c^2-3)}{3a+3c+6} = u(\rho)M_{00}(R^*,\rho) \\ &= M_{10}(R^*,\rho) \end{aligned} \tag{2.46}$$

$$\begin{aligned} M_{01}(R^*,\mu) = C\frac{(c-a)}{3} &= \frac{M_{00}(R^*,\rho)(c-a)}{3a+3c+6} = v(\rho)M_{00}(R^*,\rho) \\ &= M_{01}(R^*,\rho). \end{aligned} \tag{2.47}$$

Therefore, $Q(R^*,\mu) = Q(R^*,\rho)$.

□

By using the properties of reflection or rotation, we can use the same method for the remaining regions B, C, D, E, F, and G by considering $q'(\rho) = (\pm u, \pm v)$ and letting μ be the measure such that $d\mu = C\chi_{\text{cut}}\, d(\pm x)\, d(\pm y)$, so that $Q(R^*,\mu) = Q(R^*,\rho)$ for all $q(\rho)$ of primary measures. However, from Figure 2.4, we can see that this does not cover all of R^*. To deal with the area that remains, we have to deal with complementary measures.

2.4.2 Complementary Measures

Complementary measures are measures μ of the form $d\mu = C(1-\chi\text{cut})\, d\mathbf{x}$, where the *cut* is the same as the previous section: the smaller side of a single straight line dividing the reference square into two disjoint sets, whose union is R^*. As before, the primary measure represents the area which does not contain the origin $(0,0)$. Since this is a complementary measure, the measure represents the *larger* of the two sides; the one containing the origin.

Again, there are two types of cuts, Type 1 and Type 2, which are identical to the previous definition. The only difference is the values of **a**, **b**, **c**, and **d** are such that the pentagon (Type 1) or trapezoid (Type 2) contains the origin. Again, due to symmetry, we only need to calculate one of each.

For this example, we will connect **b** and **d**. This is a line with slope $\frac{d-b}{2}$ and equation $y = \frac{d-b}{2}(x-1)+d$, called L_{bd}.

Then the moments are:

$$\begin{aligned} M_{00}(R^*, \mu) &= \int_{R^*} 1\, d\mu = \int_{-1}^{1} \int_{L_{bd}}^{1} C\, dy\, dx \\ &= C(-b-d+2) \end{aligned} \tag{2.48}$$

$$\begin{aligned} M_{10}(R^*, \mu) &= \int_{R^*} x\, d\mu = \int_{-1}^{1} \int_{L_{bd}}^{1} Cx\, dy\, dx \\ &= C\frac{b-d}{3} \end{aligned} \tag{2.49}$$

$$\begin{aligned} M_{01}(R^*, \mu) &= \int_{R^*} y\, d\mu = \int_{-1}^{c} \int_{L_{bc}}^{1} Cy\, dy\, dx \\ &= C\frac{-b^2-bd-d^2+3}{3}. \end{aligned} \tag{2.50}$$

Therefore, the transformation from (b, d) to (u, v) is

$$u = \frac{M_{10}(R^*, \mu)}{M_{00}(R^*, \mu)} = \frac{d-b}{3b+3d-6} \quad \text{and} \quad v = \frac{M_{01}(R^*, \mu)}{M_{00}(R^*, \mu)} = \frac{b^2+bd+d^2-3}{3b+3d-6}. \tag{2.51}$$

This transformation is not linear, however the inverse can still be calculated as

$$b = \frac{3x^2-6xy+6x+2y-1}{3x^2+1} \quad \text{and} \quad d = \frac{3x^2+6xy-6x+2y-1}{2x^2+1}. \tag{2.52}$$

Since $-1 \leq b \leq 1$ and $-1 \leq d \leq 1$, when $b+d < 0$, one can see that all the points $q = (u(b,d), v(b,d))$ will fall in region f bounded by $v = -\frac{3}{2}u^2 + \frac{1}{2}$, $v = -\frac{3u(u-1)}{3u+1}$ and $v = \frac{3u(u+1)}{3u-1}$, with vertices at $(1/3, 1/3)$, $(-1/3, 1/3)$, and $(0, 0)$ (Figure 2.5).

A sharp-eyed reader may notice that these formulas are the same as for region B of the primary measures. Indeed, they are derived the exact same way. The only difference is that the measure for B does not include the origin, while for f it does. Similar computations for the other Type 2 complementary measures result in the familiar Table 2.3.

Therefore, measures corresponding to a vector $q \in \{b,\ d,\ f,\ \text{or}\ h\}$ can be modeled by a complementary measure of Type 2. Interestingly, this is not the only way to

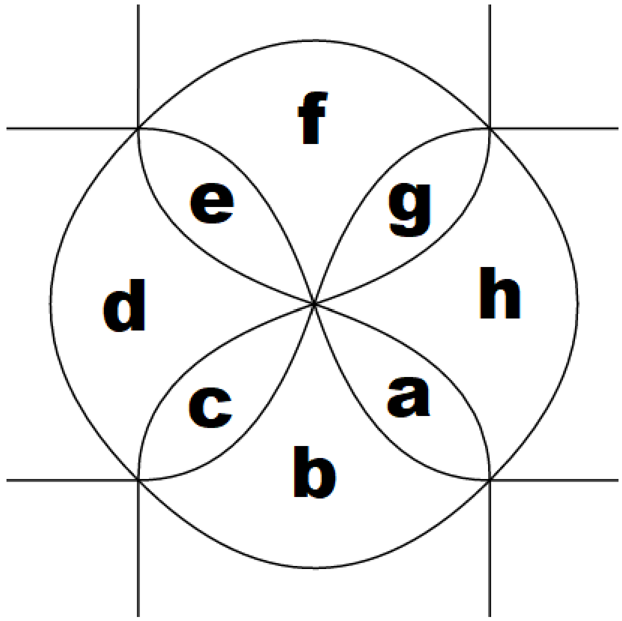

Figure 2.5: Each labeled region corresponds to a Complementary Measure of Type 1 (aceg) or Type 2 (bdfh).

calculate these parameters. Using the fact that moments are additive, we can deduce that the sum of the moments of χ_{cut} and $1-\chi_{\text{cut}}$ are the same as the moments of R^*. That is, as long as disjoint regions $J \cup K = R^*$,

$$\begin{aligned} M_{00}(J) + M_{00}(K) &= M_{00}(R^*) \\ M_{10}(J) + M_{10}(K) &= M_{10}(R^*) \\ M_{01}(J) + M_{01}(K) &= M_{01}(R^*). \end{aligned} \tag{2.53}$$

Then, the moments of Lebesgue measure of the entire reference square are

$$\begin{aligned} M_{00}(R^*, \mu) &= \int_{-1}^{1}\int_{-1}^{1} C\,dy\,dx = 4C \\ M_{10}(R^*, \mu) &= \int_{-1}^{1}\int_{-1}^{1} Cx\,dy\,dx = 0 \\ M_{01}(R^*, \mu) &= \int_{-1}^{1}\int_{-1}^{1} Cy\,dy\,dx = 0. \end{aligned} \tag{2.54}$$

Therefore, the Lebesgue measure over the entire reference square gives

$$u = \frac{M_{10}(R^*)}{M_{00}(R^*)} = 0 \quad \text{and} \quad v = \frac{M_{01}(R^*)}{M_{00}(R^*)} = 0. \tag{2.55}$$

Table 2.3: Transformation and Inverses of Type 2 Complementary Measures

Region	**Line**	**Transformation**	**Inverse**
b	L_{bd}	$u = \dfrac{d-b}{3b+3d+6}$ $v = \dfrac{b^2+bd+d^2-3}{3b+3d+6}$	$b = \dfrac{-3u^2-6uv-6u+2v+1}{3u^2+1}$ $d = \dfrac{-3u^2+6uv+6u+2v+1}{3u^2+1}$
d	L_{ca}	$u = \dfrac{a^2+ac+c^2-3}{3a+3c+6}$ $v = \dfrac{c-a}{3a+3c+6}$	$a = \dfrac{-3v^2-6uv+2u-6v+1}{3v^2+1}$ $c = \dfrac{-3v^2+6uv+2u+6v+1}{3v^2+1}$
f	L_{db}	$u = \dfrac{d-b}{3b+3d-6}$ $v = \dfrac{b^2+bd+d^2-3}{3b+3d-6}$	$b = \dfrac{3u^2-6uv+6u+2v-1}{3u^2+1}$ $d = \dfrac{3u^2+6uv-6u+2v-1}{3u^2+1}$
h	L_{ac}	$u = \dfrac{a^2+ac+c^2-3}{3a+3c-6}$ $v = \dfrac{c-a}{3a+3c-6}$	$a = \dfrac{3v^2-6uv+2u+6v-1}{3v^2+1}$ $c = \dfrac{3v^2+6uv+2u-6v-1}{3v^2+1}$

For cuts of Type 1, using the traditional methods of integrating in order to find the moments can lead to a number of problems. While you can find the moments, it is difficult to find the inverse. Thus, a new technique is used: using additivity of the moments, as above. Given a cut of Type 1, instead of directly computing the moments, we will calculate them based on the known moments of the primary measure given by the cut, and develop a new scheme to find the inverse. Due to symmetry, we only need to calculate one of these.

Given the moments of a primary measure, you can calculate the moments of the complementary measure, and vice versa. Specifically,

$$\begin{aligned} M_{00}(R^*, \mu^{\complement}) &= 4C - M_{00}(R^*, \mu) \\ M_{10}(R^*, \mu^{\complement}) &= -M_{10}(R^*, \mu) \\ M_{01}(R^*, \mu^{\complement}) &= -M_{01}(R^*, \mu). \end{aligned} \tag{2.56}$$

For example, if we were to find the moments of the complementary measure of the cut connecting **b** and **c**, we would get the same line with slope $\dfrac{1-b}{c+1}$ and equation $y = \dfrac{1-b}{c+1}(x+1) + b$, again called L_{bc}.

This time, the moments are:

$$\begin{aligned} M_{00}(R^*, \mu^{\complement}) &= \int_{R^*} 1\, d\mu^{\complement} = \int_{R^*} C\, d\mathbf{x} - \int_{\text{cut}} C\, d\mathbf{x} \\ &= 4C - C\frac{(c+1)(1-b)}{2} = \frac{C}{2}(7 + b - c + bc) \end{aligned} \tag{2.57}$$

$$\begin{aligned} M_{10}(R^*, \mu^{\complement}) &= \int_{R^*} x\, d\mu^{\complement} = \int_{R^*} Cx\, d\mathbf{x} - \int_{\text{cut}} Cx\, d\mathbf{x} \\ &= -\frac{C}{6}(c-2)(c+1)(1-b) \end{aligned} \tag{2.58}$$

$$\begin{aligned} M_{01}(R^*, \mu^{\complement}) &= \int_{R^*} y\, d\mu^{\complement} = \int_{R^*} Cy\, d\mathbf{x} - \int_{\text{cut}} Cy\, d\mathbf{x} \\ &= -\frac{C}{6}(b+2)(c+1)(1-b) \end{aligned} \tag{2.59}$$

Therefore, the transformation from (b, c) to (u, v) is

$$u = \frac{M_{10}(R^*, \mu^{\complement})}{M_{00}(R^*, \mu^{\complement})} = \frac{(c-2)(c+1)(b-1)}{3(bc + b - c + 7)} \tag{2.60}$$

$$v = \frac{M_{01}(R^*, \mu^{\complement})}{M_{00}(R^*, \mu^{\complement})} = \frac{(b+2)(c+1)(b-1)}{3(bc + b - c + 7)}. \tag{2.61}$$

In order to determine the shape of the region in the uv-plane, we analyze the extrema of b and c. Since $-1 \leq b \leq 1$ and $-1 \leq c \leq 1$, one can see that all the points $q = (u(b, c), v(b, c))$ will fall in region a bounded by $v = \dfrac{3u(u-1)}{3u+1}$ and its reflection over $v = -u$, $u = \dfrac{3v(v+1)}{3v-1}$, with vertices at $(1/3, -1/3)$, and $(0, 0)$ (Figure 2.5).

Of course, we have not yet found the inverse transformation over this region, due to the difficulty in making this calculation. Even computational algebra systems such as WolframAlpha struggle here [5]. Thus, we find a new method.

Since we already have a way to transform and invert primary measures, we can develop a method to find primary parameters from their complementary counterparts. That is, given a complementary measure, we can find the corresponding primary measure, and use the formulas from Tables 2.1 and 2.2.

From equation (2.28) and Table 2.1, we have

$$M_{00}(R^*,\mu) = C\frac{(c+1)(1-b)}{2} \tag{2.62}$$

$$c = 3u+2 \quad \text{and} \quad b = 3v-2. \tag{2.63}$$

Substituting the values for (2.63) into (2.62), we get

$$M_{00}(R^*,\mu) = C\frac{(3u+3)(3-3v)}{2} = \frac{9C}{2}(u+1)(1-v). \tag{2.64}$$

Now, from equations (2.56), we have

$$M_{10}(R^*,\mu) = -M_{10}(R^*,\mu^{\mathsf{C}}) \quad \text{and} \quad M_{01}(R^*,\mu) = -M_{01}(R^*,\mu^{\mathsf{C}}) \tag{2.65}$$

or by equations (2.41) and (2.42)

$$\begin{aligned} M_{00}(R^*,\mu)u(\mu) &= -M_{00}(R^*,\mu^{\mathsf{C}})u(\mu^{\mathsf{C}}) \\ M_{00}(R^*,\mu)v(\mu) &= -M_{00}(R^*,\mu^{\mathsf{C}})v(\mu^{\mathsf{C}}). \end{aligned} \tag{2.66}$$

Define

$$\gamma = -\frac{M_{00}(R^*,\mu^{\mathsf{C}})}{M_{00}(R^*,\mu)} \tag{2.67}$$

so that

$$u(\mu) = \gamma u(\mu^{\mathsf{C}}) \quad \text{and} \quad v(\mu) = \gamma v(\mu^{\mathsf{C}}). \tag{2.68}$$

By (2.56),

$$\gamma = \frac{M_{00}(R^*,\mu) - 4C}{M_{00}(R^*,\mu)} = 1 - \frac{4C}{M_{00}(R^*,\mu)} \tag{2.69}$$

so that

$$M_{00}(R^*,\mu) = \frac{4C}{1-\gamma}. \tag{2.70}$$

Finally, from equation (2.64), we have

$$\frac{4C}{1-\gamma} = C\frac{(3u+3)(3-3v)}{2} = \frac{9C}{2}(u+1)(1-v) = \frac{9C}{2}(\gamma u(\mu^{\mathsf{C}})+1)(1-\gamma v(\mu^{\mathsf{C}})) \tag{2.71}$$

from equation (2.68).

Rearranging gives

$$\frac{8}{9} = (\gamma u(\mu^{\complement}) + 1)(1 - \gamma v(\mu^{\complement}))(1 - \gamma) \tag{2.72}$$

so that

$$P_3(\gamma) = (\gamma u(\mu^{\complement}) + 1)(1 - \gamma v(\mu^{\complement}))(1 - \gamma) - \frac{8}{9} \tag{2.73}$$

which is a polynomial of degree 3 in terms of γ. Solving $P_3(\gamma) = 0$ for γ either exactly or numerically lets us apply equation (2.68) to solve for u and v, where we can use Table 2.1 to solve for b and c. $P_3(\gamma)$ is guaranteed to have at least one real solution, although it may have up to 3. In that case, we reject the solutions that would imply b or c are outside of their bounds, and take the smallest negative root.

Doing the same calculations for the complementary regions of C, E, and G results in Table 2.4.

Table 2.4: Transformation and Inverses of Type 1 Complementary Measures

Region	**Transformation**	$\mathbf{P_3}(\gamma)$
a L_{bc}	$u = \frac{(c-2)(c+1)(b-1)}{3(bc+b-c+7)}$ $v = \frac{(b+2)(c+1)(b-1)}{3(bc+b-c+7)}$	$(\gamma u(\mu^{\complement}) + 1)(1 - \gamma v(\mu^{\complement}))(1 - \gamma) - \frac{8}{9}$
c L_{cd}	$u = \frac{(c+2)(c-1)(d-1)}{3(c+d-cd+7)}$ $v = \frac{(d+2)(c-1)(d-1)}{3(c+d-cd+7)}$	$(\gamma u(\mu^{\complement}) - 1)(\gamma v(\mu^{\complement}) - 1)(1 - \gamma) - \frac{8}{9}$
e L_{ad}	$u = \frac{(a+2)(1-a)(d+1)}{3(ad+a-d+7)}$ $v = \frac{(d-2)(1-a)(d+1)}{3(ad+a-d+7)}$	$(1 - \gamma u(\mu^{\complement}))(\gamma v(\mu^{\complement}) + 1)(1 - \gamma) - \frac{8}{9}$
g L_{ab}	$u = \frac{(2-a)(a+1)(b+1)}{3(ab+a+b-7)}$ $v = \frac{(2-b)(a+1)(b+1)}{3(ab+a+b-7)}$	$(\gamma u(\mu^{\complement}) + 1)(\gamma v(\mu^{\complement}) + 1)(1 - \gamma) - \frac{8}{9}$

Based on the information in Tables 2.1 – 2.4, we are successfully able to represent every point $q = (u, v)$ in R^* as a primary or complementary measure of Type 1 or

Type 2, with the exception of $(0, 0)$, which was already established in equation (2.55). In fact, there are many points along the boundaries of multiple regions. For example $(1/3, 1/3)$ lies on the boundary of B, C, D, f, g, and h. Interestingly enough, for a point exactly on the boundary of two or more regions, the calculations corresponding to either region will work (and be identical). The only issue arises at $(0, 0)$, which is on the boundary of 8 different regions, but has already been addressed.

Chapter 3

The Algorithm

3.1 Blending

Let ρ be a measure with quantities $Q(R^*, \rho)$. Then we use the methods from the previous chapter to find an approximation for the quantities of the subdivided regions using either the linear polynomial weight measure or subdomain measure schemes.

Unfortunately, the linear polynomial weight measures only work on a rhombus T with vertices $(1/3, 0)$, $(0, 1/3)$, $(-1/3, 0)$, and $(0, -1/3)$, which does not cover all of R^*. On the other hand, subdomain measures work everywhere except for $(0, 0)$, but they are highly unstable near the origin. A small change in the moments with $q = (u, v)$ near $(0, 0)$ could result in very large changes in the parameters. In fact, a small perturbation could change the region of $q = (u, v)$, and thus completely change the type of the corresponding cut.

Therefore, we choose to blend the schemes. The quantities of the subdivided regions (hereafter referred to as *children*) will be calculated as a linear combination of the quantities calculated by each measure scheme.

Define

$$\varphi(u, v) = \cos^2\left(\frac{3\pi}{2}(u + v)\right)\cos^2\left(\frac{3\pi}{2}(u - v)\right) \tag{3.1}$$

so that $0 \leq \varphi(T) \leq 1$. Then $\varphi(u, v) = 0$ for (u, v) on the boundary of T, increasing to $\varphi(0, 0) = 1$. Extend $\varphi(u, v)$ to R^* by setting it equal to 0 everywhere outside of T.

Then let $Q(R_i^*, \eta)$ be the quantities calculated from the linear polynomial measure scheme and $Q(R_i^*, \mu)$ be the quantities from the subdomain measure scheme, both for $i = 1, 2, 3, 4$. Then we blend the quantities of the children as follows:

$$\tilde{Q}(R_i^*) = \varphi(u, v)Q(R_i^*, \eta) + (1 - \varphi(u, v))Q(R_i^*, \mu) \tag{3.2}$$

for $i = 1, 2, 3, 4$.

3.2 Subdivision Algorithm

We take this section to present the results of the subdivision algorithm with blending over squares. Theoretically, the algorithm should perform very well for linear functions and regular distributions with gaps similar to the forms above. We begin with a domain $X = [-4, 4]^2$. After 2 levels of quadrisecting, we are left with 16 congruent two-unit squares R_i. At this stage, we use given data to directly calculate the quantities $Q(R_i)$ for each of the squares. This is the only stage in the algorithm that has access to the original data.

The data can be presented in one of two ways. We could define arbitrary uniform regions of the plane by algebraic equations, then calculate the moments either directly, or numerically. For example, we could populate X with three non-parallel lines, and find the quantities for the region of uniform density bounded by all three. Alternatively, we could present the data as a point cloud: a finite set of $d+1$ dimensional data points that correspond to a coordinate (x, y) with a value p at that point [2]. Again, the moments could be calculated directly, from equation (2.5).

With each of M_{00}, M_{10}, and M_{01} calculated, we proceed as follows: First, we transform each region R_i to the standard region R^*, through the affine transformation in equation (2.11), and calculate its centre of mass. This tells us what type of cut the representative measure will have (if any). Second, use the formulas from Chapter 2 to find the corresponding parameters of the cut (two of a, b, c, or d) as well as α and β, if necessary. Third, we calculate the moments the unknown measure ρ, by constructing the known representative measure η such that $Q(R^*, \rho) = Q(R^*, \eta)$.

Fourth, we directly calculate the quantities of the children R_1, R_2, R_3, and R_4 so that $Q(R_i, \rho) \approx Q(R_i, \eta)$ for $i = 1, 2, 3, 4$. Since some measures can be represented as both a linear polynomial weight measure and as a subdomain measure, we have to blend those measures using the formula in equation (3.2). Finally, we continue the subdivision by repeating the whole process from the first step with the new quantities for each subregion. We can continue this subdivision scheme as many levels as we want, until we reach the limits of the machine, but usually until a satisfactory limit surface is reached. Under ideal circumstances, this will be indistinguishable from the original, but can be stored and calculated more efficiently.

We will present an example of the Lebesgue measure over X with a gap between the curves

$$y_1(x) = \frac{3}{8}x + \frac{5}{2} \quad \text{and} \quad y_2(x) = -\frac{3}{8}x - \frac{3}{2}. \tag{3.3}$$

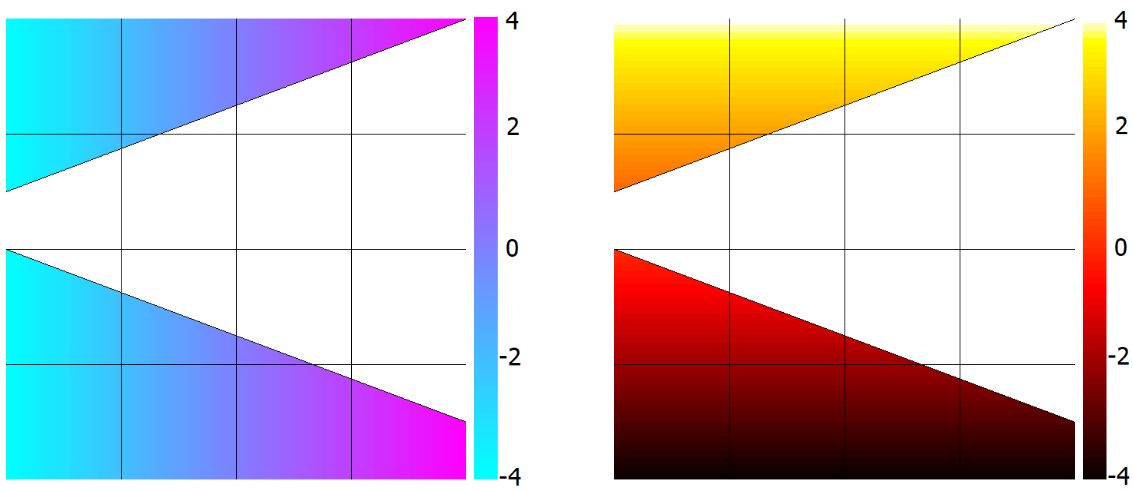

Figure 3.1: The initial x-values (cyan-magenta) and y-values (red-yellow) on the initial grid on X.

The figures in this section are plots of the average x and y-values over the squares. The cyan-magenta figures represent the x-values, which range from -4 to 4. Similarly, the red-yellow figures represent the y-values, which also range from -4 to 4. Note: the colours are only used to better understand the performance of the algorithm;

the shapes are all we care about. Figure 3.1 displays the data being used for the subdivision scheme, which was used to calculate the initial moments. Note that the measures over each square can be approximated well by either the subdomain or complementary measures from Chapter 2.

From the initial data, we will calculate the average x and y-values over their respective regions R_i. In Figure 3.2 we see the initial calculation of the quantities for each of the 16 regions in X. These quantities are represented by the average x-values ($u = M_{10}/M_{00}$) and average y-values ($v = M_{01}/M_{00}$) in all the remaining figures. From there, each square undergoes the subdivision algorithm described earlier, being split into four parts. The moments of each of the children are calculated based on the moments of the parent squares. The first two levels of subdivision of X are shown in Figure 3.2, in which the quantities are calculated from the previous level. After each layer of subdivision, the gap between y_1 and y_2 becomes more apparent. By continuing this subdivision, we can get a better idea of the limiting surface.

Note that for each of the x-value figures (left column), there appear to be vertical bands of colour that become more apparent with each layer of subdivision. For each of the y-value figures (right column), horizontal bands appear. The only place this changes is along edges of the gap, where the squares are a slightly different colour. This indicates that the square is only a partial measure, instead of a full measure. Additionally, in the final picture, a few full measures appear to be very slightly miscoloured. This is a product of the blending scheme from Section 3.1 that provides a weighted average for some complementary measures that are mostly full.

After several rounds of subdivision, a limiting surface is achieved. When comparing the algorithm results from Figure 3.3 with Figure 3.1 we see that the limit surface is very close to the initial data in both colour and shape. However, we only care about how close the shape represents the original. As expected, the majority of the error is concentrated along the edge of the gap.

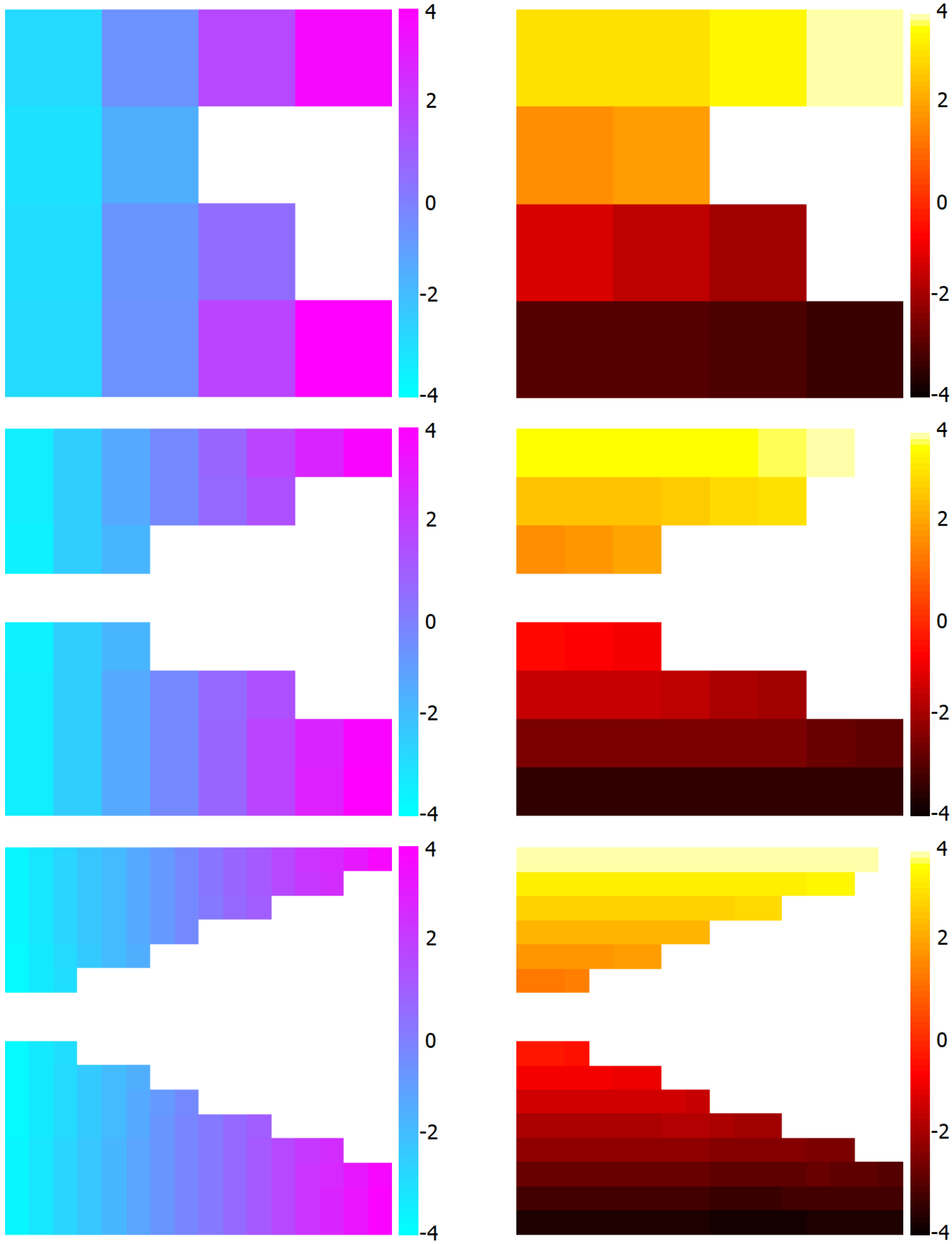

Figure 3.2: The initial average x-values (cyan-magenta) and y-values (red-yellow) over the squares in X, followed by the first two layers of subdivision.

The approximation would be slightly worse had the edges not been straight. Had any measures deviated significantly from the measures described in Chapter 2, the approximation could be made significantly worse, leading to more visible errors.

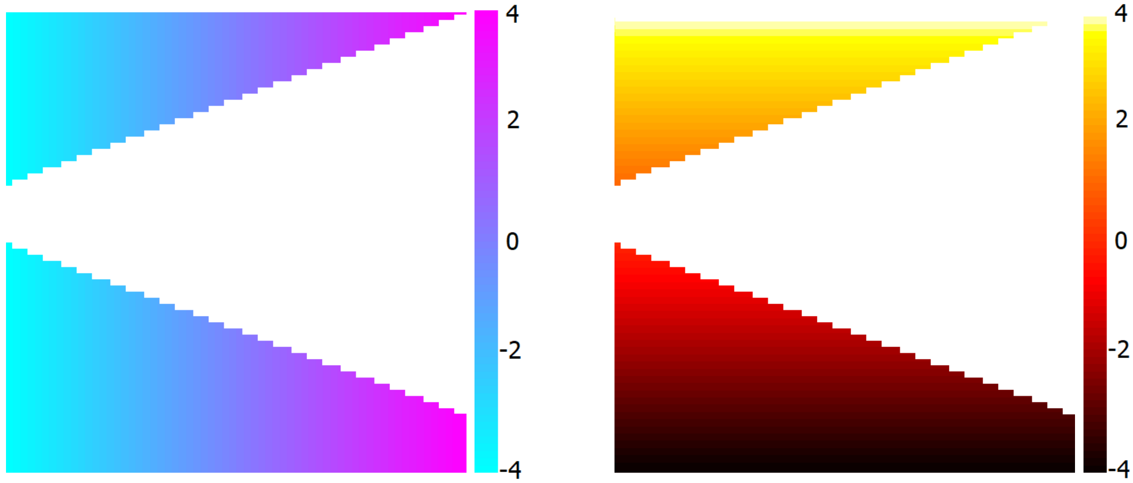

Figure 3.3: The limit surface of successive subdivision steps closely approximates the initial data.

The initial level of subdivision greatly impacts the limiting surface of the subdivision. Since the gap was observed with the initial data, we could even have started the subdivision process one level higher. If the measure of the initial level can be represented well by the subdomain and complementary measures in Chapter 2, then the limiting surface of the algorithm will also be good. On the other hand if the measure is poorly approximated initially, then that error will propagate through the refinement, and be visible in the limit surface. Higher order subdivision schemes could potentially represent more diverse measures.

Chapter 4

Conclusions

In this paper, we have shown that it is possible to represent characteristics of a reference square R^* by just the moments. We developed a one-to-one correspondence between points (u, v) on the square that represent the centre of mass, and parameters α and β or a, b, c, and d that represent coefficients of a linear polynomial or points of a cut. We used these parameters to develop a subdivision algorithm that can successively subdivide a square into quadrants, each of which can be transformed to the reference square.

While standard subdivision schemes are developed to apply for regular distributions, we suggest a new approach based on previous work by Diefenthaler [2] that gives the opportunity to extend this visualization to practically important areas. The algorithms analyze the aggregate quantities from the coarser level, to calculate their counterparts at a finer level. In the future, new subdivision schemes of a higher order could be explored which could contain more diverse representative measures. For example, by extending the main idea of this thesis to calculate higher order moments, a more complex subdivison scheme could be modelled, with better recovery of more diverse measures than the ones presented here. Along those lines, a more sophisticated blending scheme could be implemented in order to minimize errors.

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