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Hadronic Lorentz Violation in Chiral Perturbation Theory

Rasha Kamand
University of South Carolina

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by

Rasha Kamand

Bachelor of Science
American University of Beirut 2008

Master of Science
University of South Carolina 2013

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Accepted by:

Matthias R. Schindler, Major Professor

Brett D. Altschul, Committee Member

Vladimir Gudkov, Committee Member

Vladimir Temlyakov, Committee Member

Cheryl L. Addy, Vice Provost and Dean of the Graduate School

Abstract

The leading-order Lorentz-violating hadronic Lagrange densities are constructed using chiral perturbation theory. This is done for both pions and nucleons starting from a two-flavor quark-level Lagrangian that consists of dimension-four Lorentz-violation operators. The effective Lagrangians are first constructed in the absence of external fields. The formalism is then extended to include interactions with external fields. The presence of Lorentz violation modifies the transformation behavior of external fields under the chiral group $SU(2)_L \times SU(2)_R$. This in turn leads to modified pion and nucleon covariant derivatives. By expanding parts of both mesonic and baryonic Lagragians in terms of physical pion and nucleon fields, new approximate bounds on the effective pion Lorentz-violation coefficients are placed using experimental observations from the proton and neutron sectors. The resulting constraints on four pion parameters are at the 10^{-23} level.

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Chapter 1

Introduction and Motivation

Lorentz symmetry (LS) is at the foundation of what physicists think is their best description so far of the fundamental particles and forces observed in nature. It is a symmetry of space and time (space-time), named after Dutch physicist Hendrik Antoon Lorentz, which represents the invariance of the laws of physics for different observers and has two components: rotational symmetry and boost (change of velocity) symmetry. The framework of relativistic quantum field theory (RQFT) is the result of integrating Lorentz symmetry into quantum mechanics. Elementary particles and their interactions, excluding gravitation, are currently described by what is known as the Standard Model (SM) of particle physics, which itself is a RQFT.

The SM Lagrangian which controls the kinematics and dynamics of the theory is constructed from fundamental particles, namely quarks, leptons (both classified as fermions), gauge bosons and the Higgs boson. Quarks and leptons are the matter particles, while the gauge bosons are the force carriers. The Higgs boson interacts with fermions to give them mass. Each kind of particle is represented by a dynamical field pervading space-time. The SM Lagrangian is invariant under the Poincaré group which is the full symmetry of special relativity. It is generated by translations and Lorentz transformations. In addition to this space-time symmetry, the SM is defined by an internal symmetry, $SU(3) \times SU(2) \times U(1)$ local gauge symmetry.

Despite its many successes in describing and providing an explanation of almost

all experimental results as well as accurately predicting a wide range of observed phenomena, the SM is still incomplete. It not only fails to account for the fundamental force of gravity, but also leaves a variety of important questions unanswered, such as those related to the nature of dark matter and dark energy, matter-antimatter asymmetry, neutrino masses and other issues. Therefore, the SM is thought to be only a part of a broader picture that hides within it new physics.
The last two decades have witnessed a growing interest in the possibility that the fundamental Lorentz and CPT symmetries might actually be violated in nature. CPT symmetry is the product of charge conjugation (C)—replacing particles with anti-particles and vice versa—parity (P)—space inversion; reversal of the spatial coordinates—and time reversal (T)—replacing t by -t. The CPT theorem [1] states that a local quantum field theory which is Lorentz invariant and has a Hermitian Hamiltonian must have CPT symmetry. If CPT is violated in field theory, then LS must also be broken.

This rapid rush of interest in these two symmetries being violated has arisen because their violation might be part of a fundamental theory of quantum gravity; however, no LS violation has yet been detected experimentally. Indeed, if it is ever discovered, this will be proof of new fundamental physics beyond the SM and general relativity, providing substantial information about the nature of the new physics.

It is important to distinguish between *observer* and *particle* Lorentz transformations in order to understand Lorentz violation (LV). LS violation signifies that there is a noticeable difference between two systems related only by a particle Lorentz transformation. An observer Lorentz transformation is a rotation, a boost (or a combination of both) of the observer frame that does not affect the laws of physics. Measurements made in frames of reference with differing velocities and orientations are related by observer Lorentz transformations. So, all observers will agree on the laws of physics since this transformation is simply a change of coordinates. On the other hand, a particle Lorentz transformation is a rotation or boost (or a combination of both) of the particle or physical system under consideration. In this case, the same inertial observer studies identical experiments that are rotated or boosted relative to each other whereby the same experimental setup is physically transformed into a new configuration.

In vacuum, these two types of Lorentz transformation (usually known as *passive* and *active* Lorentz transformations, respectively) are inversely related; however, this relation no longer holds true in Lorentz-violating theories because of the presence of fixed background fields extending over all space and time. These background fields, which are tensor-like quantities, create preferred directions and boost-dependent effects and are thus the source of the symmetry breaking. When LS is violated, physical laws remain invariant under observer (or coordinate) transformations because a change of coordinates cannot affect the physics. So, observer LS is expected for all theories including the Lorentz-violating ones, nevertheless, effects that are measurable arise when the physical system which is sensitive to one of the background fields gets rotated or boosted with the background fields remaining unchanged. The invariance of the laws of physics under observer Lorentz transformations is implemented in field theories by writing a scalar Lagrangian in which space-time indices are properly contracted.

As an illustration of the above, consider a particle with magnetic moment $\vec{\mu}$. In vacuum, either the coordinate system used to describe the particle (*coordinate/observer* transformation) or the particle itself (*particle* transformation) can be rotated, and these two transformations are inversely related. Invariance under particle Lorentz transformations signifies the *physical symmetry* of the system [see Figure 1.1]. However,

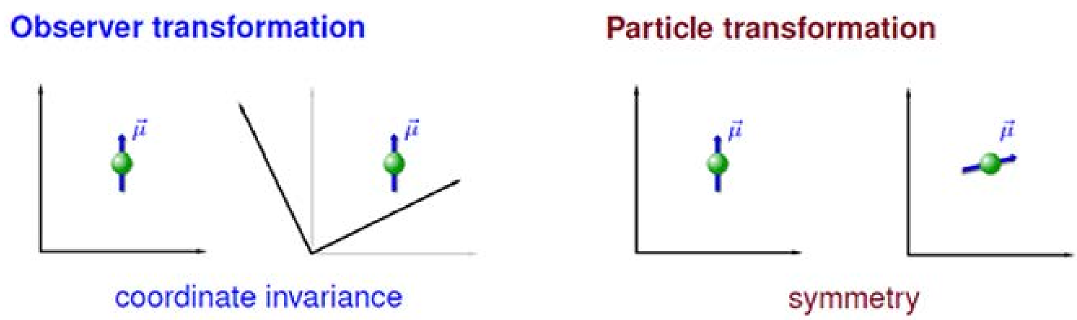

Figure 1.1 Magnetic moment $\vec{\mu}$ in vacuum

Credit:J.S. Diaz [2] (used with permission)

the existence of a background field, such as a magnetic field $\vec{B}$, breaks the physical symmetry of the system under particle transformations (the system can be physically distinguished from its transformed version), while it retains invariance under transformations of the observer's frame [see Figure 1.2].

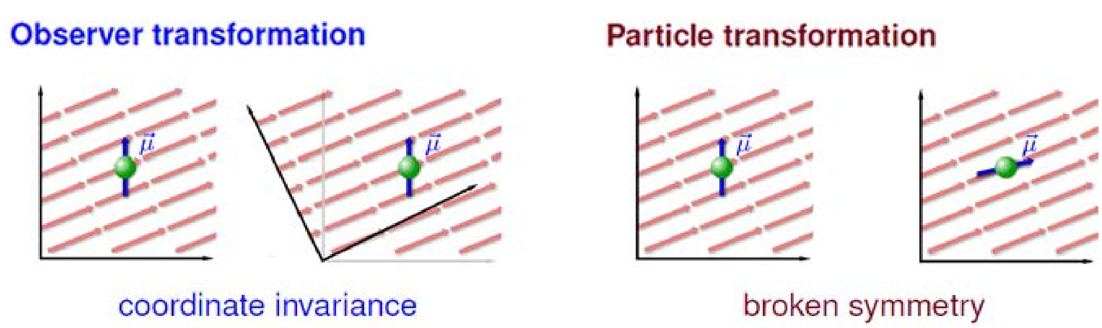

Figure 1.2 Magnetic moment $\vec{\mu}$ in a background magnetic field $\vec{B}$

Credit:J.S. Diaz [2] (used with permission)

This example shows rotation invariance as the broken symmetry, but the same idea can as well be extended to Lorentz invariance, which constitutes invariance under *both* rotations and boosts.

More recently, interest and research in LS violation has grown because of the development of an effective field theory (EFT), known as the Standard Model Extension (SME), that can be used to describe all forms of LV that may exist in a quantum field theory built around the standard model fields [3, 4]. The SME is quite general, and even the minimal SME (mSME)—which is a subset of the full SME restricted to contain only gauge invariant, power-counting renormalizable operators in its action—includes many more forms of LV than previous studies had ever looked at.

However, understanding of the SME is still far from complete. The SME is formulated as a relativistic field theory, in terms of the fundamental quark, lepton, gauge, and Higgs fields of the SM, and the relationships between the parameters in the fundamental SME Lagrangian and experimental observables can be complicated. The most important outstanding issue in this area arises from the fact that at low energies, the SM's strongly interacting degrees of freedom are composite hadrons. There are many extremely precise constraints on effective LV coefficients for protons and neutrons, as well as weaker constraints for other hadrons. However, it has not been possible to systematically translate these constraints into bounds on the more basic parameters appearing in the SME action. Non-perturbative QCD calculations that include LV are yet unavailable.

As mentioned above, in the low-energy regime of quantum chromo-dynamics (QCD), due to confinement, hadrons are the "new" fundamental degrees of freedom rather than quarks and gluons. Chiral perturbation theory (χPT) [5, 6] provides a systematic approach to transition from the quark and gluon level to that of hadrons. It is the low-energy EFT of the strong interactions (QCD). Using χPT, we shall examine the relationships between quark- and hadron-level parameterizations of LV. This will be done by first constructing the χPT action in two-flavor QCD for both pions and nucleons, up to appropriate orders thus revealing how Lorentz-violating parameters in the quark sector contribute to particular operators built out of the hadron fields. Moreover, we shall find that even at the lowest chiral orders, there are terms in the hadronic theory that have previously not been studied. Coupling to external fields will also be considered in both pion and nucleon sectors. This will necessarily lead to modified pion and nucleon covariant derivatives. Finally, we shall look at how bounds on the Lorentz-violating behavior of one variety of hadron may be used to place constraints on different phenomena involving different species of particles.

Chapter 2

Standard Model Extension

It was mentioned in the previous chapter that some theories of quantum gravity give rise to the violation of the fundamental Lorentz and CPT symmetries. So, given this theoretical impetus, the SME was formulated to help facilitate experimental research in the area of Lorentz and CPT violation. According to the pioneers of the SME, it is a general theoretical framework that includes all possible Lorentz and CPT violation effects. More specifically, it is developed as an extension of the SM aimed at treating spontaneous LV occuring at the level of some more fundamental theory. This treatment is done in a low energy EFT context where specific terms can be added that seem to break LS explicitly [3, 4]. In constructing the SME Lagrangian, two important points had to be taken into account. The first is invariance under observer Lorentz transformations remains intact, while the presence of a fixed background field affects only invariance under particle Lorentz transformations. The second point is related to the fact that the SM has shown wide experimental success which means that any LV must be comparatively small. The SME coefficients are assumed to be heavily suppressed by inverse powers of the Planck mass; however, it is not possible to assign definite values to these coefficients. The alternative way is to follow a phenomenological approach by regarding these coefficients as quantities to be bounded in experiments.

Following these two requirements, the full $\mathcal{L}_{\text{SME}}$ is constructed from all possible Lorentz-violating terms that remain invariant under observer Lorentz transformations. The generic form of a Lorentz-violating term is composed of two parts, one that acts as a coupling coefficient representing a fixed background field and another part built from basic SM fields. The SME coefficients are constrained by the requirement that the Lagrangian be Hermitian. The field part may contain covariant derivatives, and in case of fermions, it will contain gamma matrices too. The space-time indices carried by the two parts are contracted such that the whole Lorentz-violating term is a singlet under observer Lorentz transformations.

2.1 Minimal Standard Model Extension

The full SME consists of an infinite number of terms, so as a starting point, a subset of the full SME was constructed that preserves gauge invariance—which requires that the field part be a singlet under $SU(3) \times SU(2) \times U(1)$—and power-counting renormalizability. The reasoning is that since these two features are at the foundation of our current understanding in particle physics, it made sense to construct a subset theory that maintains them. It is known as the minimal SME (mSME) and contains a finite number of terms constructed from operators with mass dimension four or less. The minimal $\mathcal{L}_{\mathrm{SME}}$ describes leading-order effects of LV, but one must keep in mind that certain types of LV might only occur at sub-leading orders. The mSME consists of a fermion sector, a Yukawa sector, a gauge fields sector, and a Higgs sector.

The fermion sector is built of four sets of terms classified according to whether they involve quarks or leptons and whether CPT is odd or even. The Yukawa sector contains CPT-even as well as CPT-odd terms; it describes the coupling between fermions and the Higgs field. The two remaining sectors are composed of both CPT-even and CPT-odd terms. The SME coefficients in these sector can be dimensionless or have dimensions of mass. Some are traceless and anti-symmetric in $(\mu\nu)$ indices. An important point to note is that some of the mSME terms can be eliminated by field redefinitions, such as position-dependent phase redefinitons or field-normalization redefinitions [3, 4, 7, 8]. In such cases, the explicit LV in the theory has no physical effects due to the theory being equivalent to a Lorentz-invariant one through field redefinitions.

After its construction, the mSME has been employed widely by theorists and experimentalists to search for leading-order signals of LV. Many sensitive tests of Lorentz and CPT symmetry are performed in high-precision atomic and particle experiments that involve photons and charged particles. These tests are examined in a generalized QED framework that allows for possible Lorentz and CPT violations [4]. The ability of the SME to parameterize a wide array of Lorentz-violating phenomena has led to a tremendous expansion in experimental tests of Lorentz symmetry—in practically all sectors of the theory. An up-to-date summary of the results of these tests may be found in [9].

2.2 Lorentz Violation at the Quark Level

The portion of the mSME Lagrange density relevant to our present work is given by [4]

$$\begin{aligned} \mathcal{L}_{\text{quark}}^{\text{CPT-even}} &= \frac{i}{2}(c_Q)_{\mu\nu AB}\bar{Q}_A\gamma^\mu \overleftrightarrow{D}^\nu Q_B \\ &+ \frac{i}{2}(c_U)_{\mu\nu AB}\bar{U}_A\gamma^\mu \overleftrightarrow{D}^\nu U_B \\ &+ \frac{i}{2}(c_D)_{\mu\nu AB}\bar{D}_A\gamma^\mu \overleftrightarrow{D}^\nu D_B, \end{aligned} \tag{2.1}$$

where the left- and right-handed quark multiplets are denoted by

$$Q_A = \begin{pmatrix} u_A \\ d_A \end{pmatrix}_L , \quad U_A = (u_A)_R\,, \quad D_A = (d_A)_R\,, \tag{2.2}$$

and $A, B = 1, 2, 3$ label the quark generations, with $u_A = (u, c, t)$, and $d_A = (d, s, t)$. The $c_{\mu\nu}$ parameters in Eq. (2.1) are dimensionless coupling coefficients that are Hermitian in quark generation space spanned by A and B, while μ and ν are space-time indices.

Restricting the Lagrange density of Eq. (2.1) to up (u) and down (d) quarks, we rewrite it as[1]

$$\mathcal{L}^{\text{CPT-even}}_{\text{light quarks}} = i\bar{Q}_L C_{L_{\mu\nu}} \gamma^\mu D^\nu Q_L + i\bar{Q}_R C_{R_{\mu\nu}} \gamma^\mu D^\nu Q_R, \tag{2.3}$$

where now $Q_{L/R} = (u,d)^T_{L/R}$ and the couplings are collected in the matrices

$$C^{\mu\nu}_{L/R} = \begin{pmatrix} c^{\mu\nu}_{u_{L/R}} & 0 \\ 0 & c^{\mu\nu}_{d_{L/R}} \end{pmatrix}. \tag{2.4}$$

Note that this formalism allows for there to be different $c^{\mu\nu}$ coefficients for the left-handed u and d quarks. Physically, the $SU(2)_L$ gauge invariance of the mSME requires that the coefficients for these two chiral fermion species be identical. Moreover, the separate coefficients for left- and right-handed chiral fermions are not typically what are observed in experiments with baryons. Experimental constraints are typically placed on the combinations $c^{\mu\nu} = \frac{1}{2}(c^{\mu\nu}_L + c^{\mu\nu}_R)$ and $d^{\mu\nu} = \frac{1}{2}(c^{\mu\nu}_L - c^{\mu\nu}_R)$.
It may also be convenient to split the coefficients into isosinglet and isotriplet pieces. These are ${}^1C^{\mu\nu}_{L/R} = \mathrm{Tr}(C^{\mu\nu}_{L/R})$ and ${}^3C^{\mu\nu}_{L/R} = C^{\mu\nu}_{L/R} - (\mathbb{1}/2)\,{}^1C^{\mu\nu}_{L/R}$ (where $\mathbb{1}$ is the identity in flavor space).

It will frequently be important that the portions of these Lorentz-violating two-index tensors that are antisymmetric in their Lorentz indices cannot be observed at linear order. Only at second order in the Lorentz violation do these antisymmetric combinations have physical effects. This is a consequence of the fact that field redefinitions such as $Q'_L = [1 - (i/2)(C^{\mu\nu}_L - C^{\nu\mu}_L)\sigma_{\mu\nu}]Q_L$ can actually eliminate the

[1]For a more complete analysis, one should explicitly integrate out the heavy degrees of freedom via the renormalization group.

antisymmetric terms from the Lagrange density at first order [8]. In addition, as discussed below, in the absence of external gauge fields the antisymmetric terms do not contribute to the effective hadronic Lagrange density at leading order in the chiral power counting. We will thus assume $C^{\mu\nu}_{L/R}$ to be symmetric in the following.

Using the Lagrangian of Eq. (2.3), we will construct the effective LV hadronic Lagrangians consisting of pion and nucleon degrees of freedom. This will be done by applying the framework of χPT. Before we show the details that go into constructing the above-mentioned effective Lagrangians, we will use the next chapter to introduce the EFT formalism. We will also explain the framework of χPT and demonstrate how it is applied in the construction of effective Lagrangians.

Chapter 3

Effective Field Theory

The physical universe ranges from the microscopic world at the Planck scale ($\approx 1.5 \times 10^{-35}$ m) to the macroscopic world of galaxies at length scales of hundreds of thousands of light-years. Nevertheless, the dynamics of physical processes at low energies (long distances) do not depend on the details of what is taking place at high energies (short distances). This basic tenet allows physicists to find a suitable description and explanation, a theory, of the important and relevant dynamics in each energy range. Such a theory of the relevant physics at a particular energy (length) scale is called an *effective* theory. So, an EFT is a field theory framework appropriate for the description of "low-energy" physical phenomena, low with respect to some energy scale or *cutoff* Λ below which the EFT is valid.

The idea behind an effective theory is setting the parameters that are negligible in comparison to the physical quantities we are interested in to zero, and those which are very large to infinity. Then the effects of these parameters can be treated as small perturbations. Thus, an effective theory is only applicable in a limited energy domain and is therefore a systematic approximation to a more "fundamental" theory, which is valid across a larger range of energy.

It is certainly not essential or mandatory to use an effective theory approach in cases where the more fundamental theory is known and established, but it is actually more convenient and simpler to do so in many cases. One example that comes to mind is Newtonian classical mechanics. Newtonian mechanics is an effective theory valid in the range of small velocities compared to the speed of light c. The fundamental theory, in this case, is Einstein's theory of special relativity. However, there are situations where the structure of the fundamental theory is not understood. It could also be the case that the more basic theory is very difficult to solve. In these cases, it is therefore necessary to formulate an effective theory and employ it in order to find a useful and simple picture of the important and relevant physics. For instance, theoretical particle physicists have not yet been able to solve quantum chromo-dynamics (QCD), which is the theory of the strong interactions, at low energies; however, the framework of χPT was developed to deal with QCD at low energies. Thus, χPT is the low-energy effective theory of QCD.

3.1 Effective Field Theory Techniques

The basic idea behind an EFT is to treat the light particles whose mass $m \ll \Lambda$ as the only relevant degrees of freedom (dof). The heavy particle dof with mass $M \gg \Lambda$ are integrated out resulting in non-local interactions. These get replaced by a collection of local interactions in the effective theory constructed in such a way as to reproduce the same physics at low energies. However, it should be noted that the distinction between light and heavy dof is not *always* possible because the labeling of "light" and "heavy" is not clear or obvious and can be misleading in some cases. In these situations, the light dof in the underlying theory might not be those one actually observes, so it then becomes necessary to construct a framework using dof that can be detected in experiments. For example, χPT is constructed from pions and nucleons—which are part of the hadron spectrum—as the *relevant* dof while the QCD Lagrangian is written in terms of quarks and gluons. χPT will be discussed in more details in the subsequent section.

The main point is that in constructing an EFT, one relies on the *symmetries* of the underlying "fundamental" theory and starts by writing down an effective Lagrangian $\mathcal{L}_{\rm eff}$ using the *relevant* dof and that contains *all* the terms allowed by the symmetries of the underlying theory [5, 10]

$$\mathcal{L}_{\rm eff} = \sum_i c_i\, O_i. \tag{3.1}$$

The above sum is an infinite expansion in powers of $\frac{q}{\Lambda}$. Here, q stands for energy, mass, or momentum of the relevant dof such that $q \ll \Lambda$. As mentioned previously, Λ is the energy scale that determines the range of applicability of the EFT. The O_i's are local operators constructed from the relevant particles and are consistent with the symmetries of the high-energy theory. The low-energy constants (LECs), c_i's, carry the information on the high-energy interactions.

Even though $\mathcal{L}_{\rm eff}$ contains an infinite number of terms, the reason behind the EFT approach being useful is that in calculating physical observables, one only requires a certain *finite* accuracy of results. For a given accuracy, only a finite number of terms have to be taken into account. These terms can be identified using *power counting*. Power counting is an organization scheme. $\mathcal{L}_{\rm eff}$ is arranged in increasing powers of $\frac{q}{\Lambda}$. Feynman rules can be derived from the Lagrangian which then allows one to construct Feynman diagrams for the observable of interest. The observables are also to be expanded in the ratio of scales. Power counting tells us which diagrams have to be considered to calculate a given observable up to a particular power in this expansion. So in order to achieve a given accuracy, one has to expand up to a given order in $\frac{q}{\Lambda}$ and thus works with a finite number of terms in the effective Lagrangian (see e.g. [11]).

In quantum field theories (QFTs) and therefore in EFTs, when calculating the values of observable quantities, one encounters integrals that diverge at high momenta. These integrals come from quantum loop corrections (for more details, see e.g. [12]). Physical observables can be measured in experiments and of course are not infinite. These infinite integrals are to be dealt with so one calculates finite values for observable quantities that in turn are to be compared to experimental measurements. The ability to handle these infinities saves the theory from losing its predictive power.

Taking care of the divergences is a two-step process starting with *regularization* then *renormalization*. I will briefly explain the ideas behind each (see e.g. [13]). Regularization is basically a mathematical trick whereby the divergences are concealed and the integral looks finite. This is done by using a regulator of some kind. For example, the regulator can be a fictitious mass or a cut-off scale; after performing the integration the fictitious mass is set equal to zero or in the case of cut-off scale regularization, the resulting expression is then taken in the limit of the cut-off becoming infinite. There are other methods to regularize divergent integrals such as Pauli-Villars or dimensional regularization to name a couple (see e.g. [12] and [13]). Following regularization is the renormalization procedure which removes the divergences. The basic general idea behind this is the following. One starts with a Lagrangian that contains a certain number of *parameters*, sometimes known as "bare couplings". Upon encountering these divergent integrals, the first step, as mentioned above, is to regularize the integral. So, an extra parameter, the regulator, is introduced to the calculation. In the next step (known as renormalization), the parameters appearing in the Lagrangian are redefined appropriately (according to a particular scheme) in such a way that after this redefinition accompanied with the introduction of the regulator, the divergences are removed. After this is done, one is left with finite values of the couplings now called "renormalized couplings". It should be mentioned that there are different renormalization schemes one can employ, and for a thorough examination of these, one can consult references such as [13].

Before ending this discussion, it is important to address renormalization in the context of EFTs. Traditionally, a QFT is said to be renormalizable if there are a finite number of parameters in the Lagrangian that need to be adjusted or redefined to reproduce physical observables. For instance, in quantum electrodynamics (QED), these two parameters are the electron mass and its charge. Once they are adjusted, every other physical quantity can be calculated, and all divergences get removed. An EFT has an infinite number of these parameters in the effective Lagrangian and is therefore not renormalizable in the traditional sense. Nevertheless, this does not strip an EFT from its usefulness and predictive power. Thanks to the *power counting* tool and the fact that physical observables are calculated to a finite accuracy, one needs to expand the effective Lagrangian up to a given finite order in powers of $\frac{q}{\Lambda}$. This means that only a finite number of parameters need to be adjusted up to the given order in power counting. Therefore, in this sense EFTs are not different from "traditionally renormalizable" QFTs (see e.g. [14]).

3.2 Chiral Perturbation Theory

QCD is the QFT that describes the strong interactions. The QCD Lagrangian is written down in terms of quarks and gluons—quarks are the matter fields and gluons are the gauge bosons—both of which carry *color* charges and interact with a coupling strength g. QCD's renormalized coupling, g, is momentum-dependent and decreases with the increase in the momentum scale Q. This is known as *running* of the strong coupling constant. Due to the decrease in the value of g at high momenta, the quarks become quasi-free and QCD is said to be an *asymptotically free* theory. So perturbation theory in $\alpha_s(Q) = \frac{g^2(Q)}{4\pi}$ is valid for large Q. However, when Q is decreased, α_s increases, becoming sufficiently large, so that perturbation theory fails to converge and thus cannot be used to describe the physics at low momentum scales. Some methods have been devised to deal with the non-perturbative regime such as lattice QCD, which provides numerical solutions to QCD, and χPT, the low-energy EFT of the strong interactions.

The idea behind χPT is based on observing hadrons at low energy. In this domain, quarks are not free; they are rather bound and confined in hadrons (*baryons* such as the protons and neutrons and *mesons* like pions, kaons, etc.). Thus in the low-energy regime, it is sensible to make use of hadrons as they are the relevant *effective* dof. The QCD Lagrangian then needs to be replaced by an effective Lagrangian which is constructed from hadronic fields and contains all terms allowed by the symmetries of QCD.

3.2.1 Chiral Symmetry and Construction of the χPT Leading-order (LO) Pionic Effective Lagrangian

The QCD Lagrangian is [1]

$$\mathscr{L}_{\text{QCD}} = \sum_{\substack{f=u,d,s,\\ c,b,t}} \bar{q}_f(i\gamma^\mu D_\mu - m_f)q_f - \frac{1}{2}\text{Tr}(G^{\mu\nu}G_{\mu\nu}), \tag{3.2}$$

where f denotes the quark flavor, and each flavor comes in 3 different colors (red, green and blue), m_f is the quark mass parameter, γ^μ are the gamma matrices, $D_\mu q = \partial_\mu q - igG_\mu q$ is the co-variant derivative with G_μ being the gluon fields, and $G_{\mu\nu} = \partial_\mu G_\nu - \partial_\nu G_\mu - ig[G_\mu, G_\nu]$ is the gluon field strength tensor. The trace Tr is taken in color space.

In compact notation,

$$\mathscr{L}_{\text{QCD}} = \bar{q}(i\gamma^\mu D_\mu - m_q)q - \frac{1}{2}\text{Tr}(G^{\mu\nu}G_{\mu\nu}) \tag{3.3}$$

where q is a column vector consisting of the six quark flavors and m_q is a 6×6 quark mass matrix.

[1] We use $\mathscr{L}$ to denote Lorentz-conserving and $\mathcal{L}$ for Lorentz-violating Lagrange densities.

Table 3.1 Quark Masses [15]

Flavor	Mass (GeV)
u up	0.005
d down	0.01
s strange	0.2
c charm	1.5
b bottom	4.7
t top	180

The six quark flavors can be grouped according to their current-quark masses into two categories of light (u, d, s) and heavy (c, b, t) quarks. Compared to the typical scale of hadron masses $\approx$ 1 GeV, the c, b, and t quarks are relatively very heavy and thus are to be integrated out leaving only the light quarks whose current masses range from a few MeV to 100 MeV. Compared to the up and down quarks, the strange quark's mass is at least 16 times larger than that of u and d. So, the s quark can also be integrated out, leaving only the up and down quarks as the relevant dof. Hence, the limit $m_u = m_d = 0$ known as the *chiral limit* is a good point to start constructing the low-energy effective theory of QCD.

Let us denote the QCD Lagrangian in the chiral limit by $\mathscr{L}^0_{\text{QCD}}$.

$$\mathscr{L}^0_{\text{QCD}} = \bar{q} i \gamma^\mu D_\mu q - \frac{1}{2}\text{Tr}(G^{\mu\nu} G_{\mu\nu}), \tag{3.4}$$

where q is now a column vector consisting of only the two light quarks (u,d) with $m_u = m_d = 0$. We introduce the projection operators

$$P_R = \frac{1}{2}(\mathbf{1} + \gamma_5), \quad P_L = \frac{1}{2}(\mathbf{1} - \gamma_5) \tag{3.5}$$

where the subscripts R and L denote right- and left-handed, respectively, and $\gamma_5 = \gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$ is called the chirality matrix. Using the above two operators, we can project the chiral components q_R and q_L of the Dirac field q such that

$$q_R = P_R q, \quad q_L = P_L q. \tag{3.6}$$

These are the right- and left-handed quark fields, respectively. Then, using the chiral quark fields, the Lagrangian of Eq.(3.4) can be re-written as

$$\mathscr{L}^0_{\mathrm{QCD}} = \bar{q}_R i\gamma_\mu D^\mu q_R + \bar{q}_L i\gamma_\mu D^\mu q_L - \frac{1}{2}\mathrm{Tr}(G^{\mu\nu}G_{\mu\nu}). \tag{3.7}$$

It can now be observed that under independent transformations of the right- and left-handed quark fields

$$q_R \to Rq_R, \quad q_L \to Lq_L \tag{3.8}$$

with $R \in U(2)_R$ and $L \in U(2)_L$, where $U(2)$ denotes the group of all unitary 2×2 matrices, $\mathscr{L}^0_{\mathrm{QCD}}$ is invariant owing to the flavor-independence of the covariant derivative D^μ. The $U(2)_L \times U(2)_R$ transformations can be decomposed into $SU(2)_L \times SU(2)_R \times U(1)_L \times U(1)_R$, where $SU(2)$ is the group of all unitary 2×2 matrices with unit determinant. The $U(1)_L$ and $U(1)_R$ transformations can be combined into $U(1)_V$ and $U(1)_A$, where V and A denote vector and axial, respectively. Then, the group $U(2)_L \times U(2)_R$ can be decomposed into $SU(2)_L \times SU(2)_R \times U(1)_V \times U(1)_A$. However, the $U(1)_A$ transformation where $q \to e^{i\alpha\gamma_5}q$ is not a symmetry of the quantum theory. It is only conserved at the classical level, and so the $U(1)_A$ symmetry is broken by quantum effects leading to an anomaly [16, 17]. We are then left with a $SU(2)_L \times SU(2)_R \times U(1)_V$ symmetry. The $U(1)_V$ symmetry is that under which both left- and right-handed quarks of all flavors pick up a common phase. It results in baryon number (B) conservation and leads to a classification of hadrons into mesons with $B = 0$ and baryons with $B = 1$. The invariance of $\mathscr{L}^0_{\mathrm{QCD}}$ under the remaining $SU(2)_L \times SU(2)_R$, also known as the chiral group, is referred to as *chiral symmetry* of $SU(2)$ massless QCD.

According to Noether's theorem, a conserved current J_μ with $\partial_\mu J^\mu = 0$ is associated with each continuous symmetry of a Lagrangian. The corresponding charge

$$Q(t) = \int d^3x \; J_0(t, \mathbf{x}) \tag{3.9}$$

is time-independent, $\frac{dQ}{dt} = 0$, and commutes with the Hamiltonian. For example, invariance of the Lagrangian under temporal and spatial translations as well as under rotations, imply conservation of energy, linear momentum and angular momentum, respectively.

The conserved currents of chiral symmetry, in the chiral limit of QCD, are

$$R_\mu^a = \bar{q}_R \gamma_\mu \frac{\tau^a}{2} q_R, \quad L_\mu^a = \bar{q}_L \gamma_\mu \frac{\tau^a}{2} q_L \tag{3.10}$$

with the corresponding invariant charges

$$Q_R^a = \int d^3x \, R_0^a(t, \mathbf{x}), \quad Q_L^a = \int d^3x \, L_0^a(t, \mathbf{x}), \tag{3.11}$$

respectively, where τ^a are the Pauli matrices, and a = 1,2,3.

The linear combinations

$$Q_V^a \equiv Q_R^a + Q_L^a, \text{ and } \quad Q_A^a \equiv Q_R^a - Q_L^a \tag{3.12}$$

commute with the QCD Hamiltonian in the chiral limit, H^0_{QCD}, and have opposite parity,

$$Q_V^a \xrightarrow{P} Q_V^a, \quad Q_A^a \xrightarrow{P} -Q_A^a. \tag{3.13}$$

It is therefore expected to observe mass degenerate states of opposite parity; however, this pattern is not verified in the particle spectrum [18]. As an example, the scalar ($J^P = 0^+$) mesons, have a significantly higher mass than the light pseudo-scalar ($J^P = 0^-$) mesons.

This seeming paradox is resolved by the Nambu-Goldstone realization of chiral symmetry [19]. It is assumed that the QCD vacuum, $|0\rangle$, is not invariant under the action of the axial charges

$$Q_V^a \, |0\rangle = 0, \qquad Q_A^a \, |0\rangle \neq 0. \tag{3.14}$$

So, it is said that the chiral $SU(2)_L \times SU(2)_R$ symmetry of the QCD Hamiltonian is spontaneously broken down to $SU(2)_V$. The spontaneous breakdown of a symmetry happens if the vacuum state does not share the full symmetry group of the Hamiltonian.

Goldstone's theorem states that the consequence of a spontaneously broken continuous symmetry is the appearance of massless Goldstone bosons (GB). In the case of massless two-flavor QCD, the axial charges Q_A^a create states $|\phi^a\rangle = Q_A^a |0\rangle$ which are energetically degenerate with the vacuum $|0\rangle$ since $[Q_A^a, H^0_{\text{QCD}}] = 0$. These created states are three massless GBs. The three pseudo-scalar mesons (π^+, π^- and π^0) are considered to be the GBs of spontaneously broken two-flavor chiral symmetry. In reality, these three pseudo-scalars are not exactly massless owing to the non-zero masses of the light quarks—only up and down quarks in our case—which cause the *explicit* breaking of chiral symmetry.

We can now outline the general principles for constructing the chiral SU(2) effective Lagrangian ($\mathscr{L}_\pi$). The quarks and gluons of the QCD Lagrangian get replaced by pions in the chiral SU(2) $\mathscr{L}_\pi$. The pion fields (π^+, π^-, π^0) get collected in a matrix, $U(x) \in \text{SU}(2)$, which has the following transformation behavior under the SU(2) chiral group [20],[21]

$$U(x) \to U'(x) = RU(x)L^\dagger, \tag{3.15}$$

where $(L, R) \in SU(2)_L \times SU(2)_R$.

$U(x)$ can be represented in different ways; for our purpose, the exponential representation is a convenient choice [20, 21]

$$U(x) = \exp\Big[\frac{i}{F_0}\tau^a\phi^a(x)\Big], \quad \Phi(x) = \sum_a \tau_a\phi_a = \begin{pmatrix} \phi_3 & \phi_1 - i\phi_2 \\ \phi_1 + i\phi_2 & -\phi_3 \end{pmatrix}, \tag{3.16}$$

where ϕ_a are the Cartesian pion fields, τ_a are the Pauli isospin matrices, and $F_0 \approx 92.4\,\mathrm{MeV}$ is the pion decay constant in the chiral limit.

As mentioned earlier, $\mathscr{L}_\pi$ must have the same symmetries as the underlying theory, which is QCD in this case. These are charge conjugation (C), parity (P), time reversal (T), Lorentz invariance and chiral $SU(2)_L \times SU(2)_R$ symmetry. At low energies the χPT power counting dictates that derivatives acting on the pion fields are suppressed, and $\mathscr{L}_\pi$ is then expanded in increasing powers of derivatives [5]. This ordering of the effective Lagrangian is referred to as *chiral power counting* or *chiral ordering*. The derivative on a pion field corresponds to the pion momentum, q, which is the small quantity used in the expansion. Therefore, in the counting scheme of χPT, the building blocks count as $U(x) = \mathcal{O}(q^0)$ and $\partial^\mu U(x) = \mathcal{O}(q)$. Since the Lagrangian is a Lorentz scalar, all space-time indices must be contracted. This means that terms with only an even number of derivatives of the pion fields are allowed. Moreover, at each order $\mathscr{L}_\pi$ must be invariant under chiral transformations. At zeroth chiral order, there is only one term that satisfies this requirement and that is $\mathrm{Tr}(UU^\dagger) = \mathrm{Tr}(\mathbf{1}) = 2$. Being a constant, $\mathscr{L}_\pi^{(0)}$ can be dropped.
At LO, i.e. second chiral order,

$$\mathscr{L}_\pi^{(2)} = c_1 \mathrm{Tr}[\partial^\mu U^\dagger \partial_\mu U] + c_2 \mathrm{Tr}[U^\dagger \partial^\mu \partial_\mu U], \tag{3.17}$$

where c_1 and c_2 are called low-energy constants (LECs), and they are related to the pion decay constant F_0. Note, however, that the second term in Eq.(3.17) can be re-expressed as

$$\mathrm{Tr}[U^\dagger \partial^\mu \partial_\mu U] = \partial^\mu[\mathrm{Tr}(U^\dagger \partial_\mu U)] - \mathrm{Tr}[\partial^\mu U^\dagger \partial_\mu U]. \tag{3.18}$$

Up to a total derivative, it can be reduced to the first term in that equation. We know that adding a total derivative to the Lagrangian does not change the equations of motion. Therefore, the total derivative can be dropped. So, the LO chiral effective

Lagrangian is

$$\mathscr{L}_\pi^{\mathrm{LO}} = c\mathrm{Tr}[\partial^\mu U^\dagger \partial_\mu U] \tag{3.19}$$

where c can be chosen to be $\frac{F_0^2}{4}$.

Before concluding the discussion in this section, it is important to introduce the mass term—in reality, the up and down quarks have finite masses—which explicitly breaks chiral symmetry and to show how this is dealt with in χPT framework while constructing the effective Lagrangian. The pattern that breaks chiral symmetry at the quark level must be reproduced at the effective Lagrangian level. The quark-mass term of the QCD Lagrangian is

$$\mathscr{L}_{\mathcal{M}} = -\bar{q}_R \mathcal{M} q_L - \bar{q}_L \mathcal{M}^\dagger q_R \tag{3.20}$$

where the quark-mass matrix is $\mathcal{M} = \mathrm{diag}(m_u, m_d)$. In order to include this term in the LO chiral $\mathscr{L}_\pi$, a so-called "spurion" field is introduced. The idea behind this is that although $\mathcal{M}$ is a constant matrix and does not transform along with the quark fields, it is promoted to a hypothetical dynamical field that transforms in such a way as to maintain the invariance of the total QCD Lagrangian under chiral transformations. So, if $\mathcal{M}$ transformed as

$$\mathcal{M} \to R\mathcal{M}L^\dagger, \tag{3.21}$$

then $\mathscr{L}_{\mathrm{QCD}}$ (with the mass term included) would be invariant under chiral transformations. The matrix $\mathcal{M}$ with the assumed transformation is used as a building block for $\mathscr{L}_\pi$ to construct invariant terms. The chiral counting rule is $\mathcal{M} = \mathcal{O}(q^2)$, which means that the quark masses are of second chiral order. In this way, the pattern of symmetry breaking is matched from the quark level to the hadronic level. It is now possible to construct a symmetry-breaking mass term and add it to the effective Lagrangian. At lowest order in the symmetry-breaking parameter, $\mathcal{M}$, $\mathscr{L}_{\mathrm{s.b.}}$ is constructed to look like [2]

$$\mathscr{L}_{\text{s.b.}} = c'\text{Tr}[\mathcal{M}U^\dagger + U\mathcal{M}^\dagger], \tag{3.22}$$

with c' equal to $\frac{F_0^2 B_0}{2}$ where B_0 is related to the scalar quark condensate [10]. So, the QCD LO SU(2) chiral effective Lagrangian is

$$\mathscr{L}_\pi^{\text{LO}} = \frac{F_0^2}{4}\text{Tr}[\partial^\mu U^\dagger \partial_\mu U] + \frac{F_0^2 B_0}{2}\text{Tr}[\mathcal{M}U^\dagger + U\mathcal{M}^\dagger]. \tag{3.23}$$

Covariant Derivative and Local Chiral Invariance

We have derived the LO pion effective Lagrangian for a *global* $SU(2)_L \times SU(2)_R$ symmetry. We want to extend the χPT formalism to include external fields. To this end, the QCD Lagrangian in the SU(2) chiral limit is to be extended in the presence of external fields [10]

$$\mathscr{L}_{\text{QCD}} = \mathscr{L}^0_{\text{QCD}} + \mathscr{L}_{\text{ext}}, \tag{3.24}$$

where

$$\begin{aligned}\mathscr{L}_{\text{ext}} &= \sum_{a=1}^{3} \nu_a^\mu \bar{q}\gamma_\mu \frac{\tau_a}{2} q + \frac{1}{3}\nu^\mu_{(s)}\bar{q}\gamma_\mu q + \sum_{a=1}^{3} a_a^\mu \bar{q}\gamma_\mu\gamma_5 \frac{\tau_a}{2} q \\ &\quad - \sum_{a=0}^{3} s_a \bar{q}\tau_a q + \sum_{a=0}^{3} p_a i\bar{q}\gamma_5\tau_a q \\ &= \bar{q}\gamma_\mu\Big(\nu^\mu + \frac{1}{3}\nu^\mu_{(s)} + \gamma_5 a^\mu\Big)q - \bar{q}(s - i\gamma_5 p)q.\end{aligned} \tag{3.25}$$

The external fields ν^μ, a^μ, s, p, and $\nu^\mu_{(s)}$ denote the vector, axial-vector, scalar, pseudo-scalar and singlet vector fields, respectively.

The Lagrangian of Eq.(3.24) can be re-written in terms of the right- and left-handed quark fields as follows

$$\begin{aligned}\mathscr{L}_{\text{QCD}} =& \mathscr{L}^0_{\text{QCD}} + \bar{q}_L\gamma^\mu\Big(l_\mu + \frac{1}{3}\nu^{(s)}_\mu\Big)q_L + \bar{q}_R\gamma^\mu\Big(r_\mu + \frac{1}{3}\nu^{(s)}_\mu\Big)q_R \\ &- \bar{q}_R(s+ip)q_L - \bar{q}_L(s-ip)q_R,\end{aligned} \tag{3.26}$$

[2]The subscript"s.b." refers to symmetry breaking.

where $r_\mu = \nu_\mu + a_\mu$ and $l_\mu = \nu_\mu - a_\mu$.

By promoting global chiral SU(2) symmetry to a local one, interactions can be introduced in the effective Lagrangian. The Lagrangian of Eq.(3.26) remains invariant under local chiral $SU(2)_L \times SU(2)_R \times U(1)_V$ transformations, where $V_R(x)$ and $V_L(x)$ are space-time dependent SU(2) matrices

$$\begin{aligned} q_R &\longrightarrow \exp\left(-i\frac{\Theta\,(x)}{3}\right) V_R(x) q_R, \\ q_L &\longrightarrow \exp\left(-i\frac{\Theta\,(x)}{3}\right) V_L(x) q_L, \end{aligned} \tag{3.27}$$

provided the external fields are subject to the transformations

$$\begin{aligned} r_\mu &\to V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger, \\ l_\mu &\to V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger, \\ \nu_\mu^{(s)} &\to \nu_\mu^{(s)} - \partial_\mu \Theta, \\ s + ip &\to V_R (s + ip) V_L^\dagger, \\ s - ip &\to V_L (s - ip) V_R^\dagger. \end{aligned} \tag{3.28}$$

Local chiral symmetry must also be maintained at the effective pion Lagrangian level in the presence of external fields. This requirement is satisfied by introducing a pion covariant derivative, $D_\mu U$, acting on the pion matrix $U(x)$ and which transforms in the same manner as $U(x)$,

$$U(x) \longrightarrow V_R(x) U(x) V_L^\dagger(x). \tag{3.29}$$

So, the normal partial derivative gets promoted to a covariant derivative

$$\partial_\mu U \longrightarrow D_\mu U \equiv \partial_\mu U - i r_\mu U + i U l_\mu, \tag{3.30}$$

which transforms under chiral transformations as

$$D_\mu U \longrightarrow V_R (D_\mu U) V_L^\dagger. \tag{3.31}$$

In the power counting scheme of χPT, $D_\mu U$ is of first chiral order like the normal derivative, $\partial_\mu U$. Since U is of chiral order zero, the external fields, r_μ and l_μ count

as chiral order one to match the chiral power of $\partial_\mu U$. The linear combination $\chi = 2B_0(s+ip)$ is also introduced as a building block of the locally invariant chiral effective Lagrangian; it counts as second chiral order in the power counting scheme [22].
Then, the LO most general *locally* invariant chiral pion effective Lagrangian is [23]

$$\mathscr{L}_\pi^{\mathrm{LO}} = \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U (D^\mu U)^\dagger] + \frac{F_0^2}{4}\mathrm{Tr}[\chi U^\dagger + U\chi^\dagger]. \tag{3.32}$$

Beyond LO effective Lagrangians have been considered. Mesonic χPT has been employed to construct the most general $SU(3)_L \times SU(3)_R$-invariant Lagrangian at $\mathcal{O}(q^4)$ [10]. Higher orders (up to q^6) have also been investigated [24].

3.2.2 LO Baryonic Effective Lagrangian

In this section we aim to construct the LO pion-nucleon (πN) effective Lagrangian $\mathscr{L}_{\pi N}^{(1)}$ in χPT. The form of the effective Lagrangian is the same in both cases of *global* and *local* $SU(2)_L \times SU(2)_R$ symmetries. The only difference is at the levels of the covariant derivative and the so-called chiral vielbein, a building block of the baryonic chiral effective Lagrangian.
As a shorthand notation, let the chiral group $SU(2)_L \times SU(2)_R$ be denoted by G and the nucleon doublet by

$$\Psi = \begin{pmatrix} p \\ n \end{pmatrix}, \tag{3.33}$$

where p and n are four-component Dirac fields for the proton and neutron, respectively. We also recall the $SU(2)$ matrix $U(x)$ that contains the pion fields and its transformation behavior under the chiral group G in both the global and local cases.
Under global G, U transforms as

$$U(x) \longrightarrow R U(x) L^\dagger, \tag{3.34}$$

where the independent matrices L and R are space-time-independent such that $(L, R) \in$ global G. In the local case, the transformation of U looks similar

$$U(x) \longrightarrow V_R(x)U(x)V_L(x)^{\dagger}, \tag{3.35}$$

where V_L and V_R are independent space-time-dependent matrices that belong to local G.

The unitary square root of U is denoted by u such that $u^2(x) = U(x)$. The $SU(2)$-valued function $K(L, R, U)$ is defined by [22]

$$u(x) \longrightarrow u'(x) = \sqrt{RUL^{\dagger}} \equiv RuK^{\dagger}(L, R, U) = KuL^{\dagger}, \tag{3.36}$$

where it should be understood that when we are dealing with the local case, wherever the matrices L and R appear, they must be replaced by their local counter-parts, $V_L(x)$ and $V_R(x)$.

The nucleon doublet Ψ transforms as

$$\Psi(x) \rightarrow K[L, R, U(x)]\Psi(x) \qquad \text{or} \qquad \Psi(x) \rightarrow K[V_L(x), V_R(x), U(x)]\Psi(x) \tag{3.37}$$

in the global/local case, respectively.

Before we discuss the nucleon covariant derivative, we will modify our symmetry group G to include the $U(1)_V$ transformation. So, the full symmetry group under consideration is $\tilde{G} = SU(2)_L \times SU(2)_R \times U(1)_V$. The $U(1)_V$ group only affects the nucleon field transformation by introducing a phase factor $\exp[-i\Theta(x)]$ in the following manner

$$\Psi(x) \rightarrow \exp[-i\Theta(x)]K[V_L(x), V_R(x), U(x)]\Psi(x). \tag{3.38}$$

Since K depends not only on L (or V_L) and R (or V_R) but also on U, the covariant derivative of the nucleon field, $D_\mu\Psi$, is expected to contain u via the so-called chiral connection Γ_μ, where

$$\begin{aligned} \Gamma_\mu^g &= \frac{1}{2}(u^{\dagger}\partial_\mu u + u\partial_\mu u^{\dagger}) \qquad \text{(global case)} \\ \Gamma_\mu^l &= \frac{1}{2}[u^{\dagger}(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^{\dagger}] \qquad \text{(local case)} \end{aligned} \tag{3.39}$$

where r_μ and l_μ are the right- and left-handed external fields appearing in Eq.(3.26). $D_\mu\Psi$ has the usual property of transforming in the same way as Ψ. It has the following form:

$$D^g_\mu\Psi = (\partial_\mu + \Gamma^g_\mu)\Psi \qquad \text{(global case)} \tag{3.40}$$

$$D^l_\mu\Psi = (\partial_\mu + \Gamma^l_\mu - i\nu^{(s)}_\mu)\Psi \qquad \text{(local case)} \tag{3.41}$$

where $\nu^{(s)}_\mu$ is the singlet-vector field also appearing in Eq.(3.26).

At LO, another Hermitian building block exists, u_μ, the chiral vielbein,

$$u^g_\mu \equiv i(u^\dagger\partial_\mu u - u\partial_\mu u^\dagger) \qquad \text{(global case)} \tag{3.42}$$

$$u^l_\mu \equiv i[u^\dagger(\partial_\mu - ir_\mu)u - u(\partial_\mu - il_\mu)u^\dagger] \qquad \text{(local case)} \tag{3.43}$$

which transforms under $\tilde{G}$ as

$$u_\mu \to K u_\mu K^\dagger \tag{3.44}$$

where K depends on L (or V_L), R (or V_R) and $U(x)$.

Keeping in mind that the effective pion-nucleon Lagrangian must have $SU(2)_L \times SU(2)_R \times U(1)_V$ *global* or *local* symmetry, therefore it should be of the general form $\bar{\Psi}\hat{O}\Psi$, where $\hat{O}$ is an operator in Dirac and iso-spin space which transforms as $K\hat{O}K^\dagger$ under $\tilde{G}$. Just like the pion effective Lagrangian, the baryonic Lagrangian must be a Hermitian Lorentz scalar, even under C, P, and T. So, the most general LO baryonic effective Lagrangian with the least number of derivatives is [25]

$$\mathscr{L}^{(1)}_{\pi N} = \bar{\Psi}(i\not{D} - m + \frac{g_A}{2}\gamma^\mu\gamma_5 u_\mu)\Psi. \tag{3.45}$$

There are two LECs, m, the nucleon mass in the chiral limit, and g_A, the axial-vector coupling constant in the chiral limit. Both of these parameters are not determined by chiral symmetry.

Unlike the pion mass, the nucleon mass, m_N, is a new, heavy mass scale that does not vanish in the chiral limit. This means that $\partial^0\Psi$ is not a "small" quantity, so there are different power counting rules in the baryonic sector of χPT which are summarized as follows [26]:

$$\begin{aligned}&\Psi, \bar{\Psi}, D_\mu\Psi = \mathcal{O}(q^0), (i\not{D} - m) = \mathcal{O}(q), \Gamma_\mu, u_\mu = \mathcal{O}(q),\\ &\mathbb{1}, \gamma_\mu, \gamma_5\gamma_\mu, \sigma_{\mu\nu} = \mathcal{O}(q^0), \gamma_5 = \mathcal{O}(q),\end{aligned} \tag{3.46}$$

where the order given is the minimal one. As an example, $\gamma_5\gamma_\mu$ has two pieces of different orders, $\gamma_5\gamma_0$ of $\mathcal{O}(q^0)$ and $\gamma_5\gamma_i$ of $\mathcal{O}(q)$. It should be mentioned that higher-order baryonic Lagrangians have been constructed, e.g. Fettes et al., to order q^4 [27].

Chapter 4

Lorentz-Violating Hadronic Lagrangians

In the previous chapter, we have shown how χPT is used to construct the LO pion and pion-nucleon effective Lagrangians in the absence of LV. In order to construct the effective Lagrangian including LV in terms of hadronic dof, we have to match symmetry properties of the quark-level Lagrangian from Eq.(2.3),

$$\mathcal{L}^{\text{CPT-even}}_{\text{light quarks}} = i\bar{Q}_L C_{L_{\mu\nu}} \gamma^\mu D^\nu Q_L + i\bar{Q}_R C_{R_{\mu\nu}} \gamma^\mu D^\nu Q_R,$$

onto the hadronic level. Under global chiral transformations, $Q_R \to RQ_R$, $Q_L \to LQ_R$, the above Lagrangian transforms as

$$\mathcal{L}^{\text{CPT-even}}_{\text{light quarks}} \to i\bar{Q}_L L^\dagger C_{L_{\mu\nu}} L \gamma^\mu D^\nu Q_L + i\bar{Q}_R R^\dagger C_{R_{\mu\nu}} R \gamma^\mu D^\nu Q_R. \tag{4.1}$$

The matrices $C^{\mu\nu}_{L/R}$ are constant, and chiral symmetry is broken by the terms in Eq.(2.3). Following the method for the quark-mass term in the QCD Lagrangian, Eq.(3.20), which was described in the previous chapter, we note that the Lorentz-violating action *would be* invariant under global chiral transformations *if* $C^{\mu\nu}_{L/R}$ transformed as

$$C^{\mu\nu}_L \to L C^{\mu\nu}_L L^\dagger, \quad C^{\mu\nu}_R \to R C^{\mu\nu}_R R^\dagger. \tag{4.2}$$

Because of the cyclic property of the trace, this implies for the isosinglet and isotriplet components

$$\begin{aligned} {}^1C^{\mu\nu}_L &\to {}^1C^{\mu\nu}_L, \quad {}^3C^{\mu\nu}_L \to L\,{}^3C^{\mu\nu}_L L^\dagger, \\ {}^1C^{\mu\nu}_R &\to {}^1C^{\mu\nu}_R, \quad {}^3C^{\mu\nu}_R \to R\,{}^3C^{\mu\nu}_R R^\dagger. \end{aligned} \tag{4.3}$$

Using this transformation behavior to construct a Lagrangian that is invariant under global chiral transformations, the pattern of symmetry breaking in the quark-level action is matched onto the hadronic Lagrangian.

We recall the $SU(2)$ pion matrix and the nucleon doublet

$$\begin{aligned} U(x) &= \exp\left[i\frac{\Phi(x)}{F_0}\right], \\ \Psi(x) &= \begin{pmatrix} p(x) \\ n(x) \end{pmatrix}, \end{aligned} \tag{4.4}$$

and their transformation behavior under global chiral transformations

$$\begin{aligned} U(x) &\to U'(x) = RU(x)L^\dagger, \\ \Psi(x) &\to K(L, R, U)\Psi(x). \end{aligned} \tag{4.5}$$

With these basic building blocks, we can construct the chirally invariant, Lorentz-violating LO effective Lagrangians for the pure pion sector and for pion-nucleon interactions.

4.1 Leading-order Lorentz-violating Mesonic Lagrangian

We start by writing down all possible LO expressions that are invariant under global chiral transformations. The number of indices on $C^{L/R}_{\mu\nu}$ requires us to have at least two pion derivatives which count as $\mathcal{O}(q^2)$ in the pion χPT scheme. These terms are the following:

$$\begin{aligned} &(a)\ \ \mathrm{Tr}(C^L_{\mu\nu})\mathrm{Tr}[(\partial^\mu U)^\dagger \partial^\nu U] \\ &(b)\ \ \mathrm{Tr}[\partial^\mu U C^L_{\mu\nu}(\partial^\nu U)^\dagger] \\ &(c)\ \ \mathrm{Tr}[UC^L_{\mu\nu}U^\dagger \partial^\mu U(\partial^\nu U)^\dagger] \\ &(d)\ \ \mathrm{Tr}[UC^L_{\mu\nu}\partial^\mu\partial^\nu U^\dagger] + \mathrm{Tr}[\partial^\nu\partial^\mu U C^L_{\mu\nu}U^\dagger] \\ &(e)\ \ \mathrm{Tr}[UC^L_{\mu\nu}(\partial^\mu U)^\dagger U(\partial^\nu U)^\dagger] \end{aligned} \tag{4.6}$$

Listed above are the left-handed terms with the $C^L_{\mu\nu}$ building block; however, similar matching right-handed terms can also be written with $C^R_{\mu\nu}$. The upcoming discussion applies to both the left- and right-handed terms.

Terms (c), (d) and (e) in Eq.(4.6) are not independent and can be shown to be equal or proportional to (b). We will make use of the following two identities:

$$\begin{aligned} \partial^\mu U^\dagger \, U &= -U^\dagger \, \partial^\mu U \\ (\partial^\mu\partial^\nu U^\dagger)U + U^\dagger(\partial^\mu\partial^\nu U) &= -\partial^\mu U^\dagger \partial^\nu U - \partial^\nu U^\dagger \partial^\mu U, \end{aligned} \tag{4.7}$$

which can easily be derived using $\partial_\mu(U^\dagger U) = 0$ and $\partial_\mu\partial_\nu(U^\dagger U) = 0$, respectively. Term (d) can be written in the following way

$$\mathrm{Tr}[UC^L_{\mu\nu}\partial^\mu\partial^\nu U^\dagger] + \mathrm{Tr}[\partial^\nu\partial^\mu U C^L_{\mu\nu} U^\dagger] = \mathrm{Tr}[C^L_{\mu\nu}(\partial^\mu\partial^\nu U^\dagger \, U + U^\dagger \, \partial^\nu\partial^\mu U)], \tag{4.8}$$

where we have used the cyclic property of the trace. Employing the second identity of Eq.(4.7), we get

$$\begin{aligned} \mathrm{Tr}[UC^L_{\mu\nu}\partial^\mu\partial^\nu U^\dagger] + \mathrm{Tr}[\partial^\nu\partial^\mu U C^L_{\mu\nu} U^\dagger] &= -\mathrm{Tr}[\partial^\nu U C^L_{\mu\nu}\partial^\mu U^\dagger + \partial^\mu U C^L_{\mu\nu}\partial^\nu U^\dagger] \\ &= -2\mathrm{Tr}[\partial^\mu U C^L_{\mu\nu}\partial^\nu U^\dagger], \end{aligned} \tag{4.9}$$

since the $C_{\mu\nu}$'s are symmetric in $(\mu\nu)$ at LO.

Terms (c) and (e) are equal after using the first identity of Eq.(4.7). As for (c),

$$\begin{aligned} \mathrm{Tr}[UC^L_{\mu\nu}U^\dagger\partial^\mu U(\partial^\nu U)^\dagger] &= \mathrm{Tr}[C^L_{\mu\nu}U^\dagger\partial^\mu U(\partial^\nu U)^\dagger U] \\ &= \mathrm{Tr}[C^L_{\mu\nu}(\partial^\mu U)^\dagger U U^\dagger \partial^\nu U] \\ &= \mathrm{Tr}[C^L_{\mu\nu}(\partial^\mu U)^\dagger \partial^\nu U] \\ &= \mathrm{Tr}[\partial^\nu U C^L_{\mu\nu}(\partial^\mu U)^\dagger] \\ &= \mathrm{Tr}[\partial^\mu U C^L_{\mu\nu}(\partial^\nu U)^\dagger], \end{aligned} \tag{4.10}$$

where we have used the first identity of Eq.(4.7) twice going from line 1 to line 2 of the above equation.

So, as can be seen, there are only two terms from the above list that survive and are independent. These terms are (a) $\mathrm{Tr}(C^L_{\mu\nu})\mathrm{Tr}[(\partial^\mu U)^\dagger \partial^\nu U]$ and (b) $\mathrm{Tr}[\partial^\mu U C^L_{\mu\nu}(\partial^\nu U)^\dagger]$. In a similar manner, for the right-handed terms, we also find that only two independent terms exist which are $\mathrm{Tr}(C^R_{\mu\nu})\mathrm{Tr}[(\partial^\mu U)^\dagger \partial^\nu U]$ and $\mathrm{Tr}[(\partial^\mu U)^\dagger C^R_{\mu\nu}\partial^\nu U]$.

The LO Lorentz-violating pion effective Lagrangian that can be constructed from these terms has the following structure

$$\begin{aligned}\mathcal{L}_\pi^{\mathrm{LO}} =&[\alpha\frac{F_0^2}{4}\mathrm{Tr}(C_{R_{\mu\nu}}) + \alpha'\frac{F_0^2}{4}\mathrm{Tr}(C_{L_{\mu\nu}})]\mathrm{Tr}[(\partial^\mu U)^\dagger \partial^\nu U] \\ &+ \mathrm{Tr}[\beta\frac{F_0^2}{4}(\partial^\mu U)^\dagger C_{R_{\mu\nu}}\partial^\nu U + \beta'\frac{F_0^2}{4}\partial^\mu U C_{L_{\mu\nu}}(\partial^\nu U)^\dagger],\end{aligned} \tag{4.11}$$

where α, α', β, and β' are dimensionless low-energy couplings (LECs). The factors of $F_0^2/4$ are present for dimensional reasons and to mirror their appearance in the standard pion Lagrange density.

Moreover, the transformation properties under P and C of the Lagrange density in Eq.(4.11) will also produce relationships among the right- and left-handed terms[1]. The Lorentz-violating terms in the quark-level Lagrange density are the only sources of C, P, and T violations in this theory. So, at LO, the terms in the pion Lagrange density need to have the same discrete symmetries as the terms in the underlying quark density that are multiplied by the same $C^{\mu\nu}_{L/R}$ coefficients. This forces the coefficients for left- and right-handed quark fields to enter the pion Lagrange density multiplied by the same numerical LECs, drastically reducing the number of independent terms.

In order to show in more explicit details how α is related to α' and β to β', the transformation behavior of the quark-level Lagrange density, Eq.(2.3), under charge

[1] T does not give any additional constraint due to the quark-level Lagrange density being CPT invariant.

conjugation C will be examined more closely,

$$
\begin{aligned}
\mathcal{L}^{\text{CPT-even}}_{\text{light quark}} &= i\bar{Q}_L C^L_{\mu\nu}\gamma^\mu D^\nu Q_L + i\bar{Q}_R C^R_{\mu\nu}\gamma^\mu D^\nu Q_R \\
&= i(c^L_u)_{\mu\nu}\bar{u}_L\gamma^\mu D^\nu u_L + i(c^L_d)_{\mu\nu}\bar{d}_L\gamma^\mu D^\nu d_L \\
&+ i(c^R_u)_{\mu\nu}\bar{u}_R\gamma^\mu D^\nu u_R + i(c^R_d)_{\mu\nu}\bar{d}_R\gamma^\mu D^\nu d_R,
\end{aligned}
\tag{4.12}
$$

where $(q_{L/R})_f = P_{L/R}q_f = \frac{1}{2}(\mathbf{1} \mp \gamma_5)q_f$, in which $P_{L/R}$ is the left/right-handed projection operator, and f refers to quark flavor, which in our case, can only be up (u) or down (d). Under C, the quark fields transform as

$$
q_{\alpha,f} \to C_{\alpha\beta}\ \bar{q}_{\beta,f} \tag{4.13}
$$

$$
\bar{q}_{\alpha,f} \to -q_{\beta,f}\ C^{-1}_{\beta\alpha}, \tag{4.14}
$$

where $C = i\gamma^2\gamma^0 = -C^{-1} = -C^\dagger = -C^T$ is the charge conjugation matrix, and the subscripts α and β are Dirac-spinor indices. Then, the right- and left-handed quark fields transform as

$$
\begin{aligned}
(q_{R/L})_{\alpha,f} &= \frac{1}{2}(\mathbf{1} \pm \gamma_5)_{\alpha\beta}\ q_{\beta,f} \\
&\xrightarrow{C} \frac{1}{2}(\mathbf{1} \pm \gamma_5)_{\alpha\beta} C_{\beta\gamma}\bar{q}_{\gamma,f} \\
(\bar{q}_{R/L})_{\alpha,f} &= \bar{q_{\beta,f}}\ \frac{1}{2}(\mathbf{1} \mp \gamma_5)_{\beta\alpha} \\
&\xrightarrow{C} -q_{\gamma,f}\ C^{-1}_{\gamma\beta}\ \frac{1}{2}(\mathbf{1} \mp \gamma_5)_{\beta\alpha}.
\end{aligned}
\tag{4.15}
$$

For quark bilinears, this implies

$$
\begin{aligned}
(\bar{q}_{R/L})_f\Gamma(q_{R/L})_f &= (\bar{q}_{R/L})_{\alpha,f}\ \Gamma_{\alpha\beta}\ (q_{R/L})_{\beta,f} \\
&\xrightarrow{C}\ -q_{\lambda,f}\ C^{-1}_{\lambda\mu}\ \frac{1}{2}(\mathbf{1} \mp \gamma_5)_{\mu\alpha}\ \Gamma_{\alpha\beta}\ \frac{1}{2}(\mathbf{1} \pm \gamma_5)_{\beta\rho}\ C_{\rho\sigma}\ \bar{q}_{\sigma,f} \\
&= -\bar{q_{\sigma,f}}\ C_{\lambda\mu}\ \frac{1}{2}(\mathbf{1} \mp \gamma_5)_{\mu\alpha}\ \Gamma_{\alpha\beta}\ \frac{1}{2}(\mathbf{1} \pm \gamma_5)_{\beta\rho}\ C_{\rho\sigma}\ q_{\lambda,f} \\
&= -\bar{q}_{\sigma,f}[CP_{L/R}\Gamma P_{R/L}C]^T_{\sigma\lambda}\ q_{\lambda,f} \\
&= -\bar{q}_{\sigma,f}(CP^T_{R/L}\Gamma^T P^T_{L/R}C)_{\sigma\lambda}\ q_{\lambda,f}
\end{aligned}
\tag{4.16}
$$

where Γ denotes one of the sixteen 4×4 matrices. In our case, $\Gamma = \gamma^\mu$.

Using $\gamma_5^T = \gamma_5$, we find that $P_{R/L}^T = P_{R/L}$. We also find that $CP_{R/L} = P_{R/L}C$. So,

$$
\begin{aligned}
(\bar{q}_{R/L})_f \Gamma (q_{R/L})_f \xrightarrow{C} & -\bar{q}_f P_{R/L} C \Gamma^T C P_{L/R} q_f \\
&= (\bar{q}_{L/R})_f (-C\Gamma^T C)(q_{L/R})_f,
\end{aligned}
\tag{4.17}
$$

with $[-C(\gamma^\mu)^T C] = -\gamma^\mu$. We note that there is a derivative on $(q_{R/L})_f$, which means that

$$
(\bar{q}_{R/L})_f \gamma^\mu D^\mu (q_{R/L})_f \xrightarrow{C} D^\mu (\bar{q}_{L/R})_f (-\gamma^\mu)(q_{L/R})_f
\tag{4.18}
$$

In order to move the derivative back onto q, not $\bar{q}$, we integrate by parts and this gives an additional factor of (-1).

Therefore,

$$
\begin{aligned}
\mathcal{L}_{\text{light quark}}^{\text{CPT-even}} \xrightarrow{C} \mathcal{L}'^{\text{CPT-even}}_{\text{light quark}} &= i(c_u^L)_{\mu\nu} \bar{u}_R \gamma^\mu D^\nu u_R + i(c_d^L)_{\mu\nu} \bar{d}_R \gamma^\mu D^\nu d_R \\
&+ i(c_u^R)_{\mu\nu} \bar{u}_L \gamma^\mu D^\nu u_L + i(c_d^R)_{\mu\nu} \bar{d}_L \gamma^\mu D^\nu d_L \\
&= i\bar{Q}_R C_{\mu\nu}^L \gamma^\mu D^\nu Q_R + i\bar{Q}_L C_{\mu\nu}^R \gamma^\mu D^\nu Q_L.
\end{aligned}
\tag{4.19}
$$

We notice that for the general case ($C_{\mu\nu}^L \neq C_{\mu\nu}^R$), the LV quark-level Lagrange density is no longer invariant under charge conjugation. However, if $C_{\mu\nu}^L = C_{\mu\nu}^R$, $\mathcal{L}_{\text{light quark}}^{\text{CPT-even}}$ is even under C, whereas it is odd (i.e. $\mathcal{L}'^{\text{CPT-even}}_{\text{light quark}} = -\mathcal{L}_{\text{light quark}}^{\text{CPT-even}}$) if $C_{\mu\nu}^L = -C_{\mu\nu}^R$.

Next, we need to look at the transformation behavior of the LV pion effective Lagrange density, Eq.(4.11), under charge conjugation. First let us consider the following part,

$$
\begin{aligned}
\mathcal{L}_{\pi,\beta,\beta'}^{\text{LO}} &= \text{Tr}[\beta \frac{F_0^2}{4} (\partial^\mu U)^\dagger C_{\mu\nu}^R \partial^\nu U + \beta' \frac{F_0^2}{4} \partial^\mu U C_{\mu\nu}^L (\partial^\nu U)^\dagger] \\
&= \beta \frac{F_0^2}{4} \text{Tr}[(\partial^\mu U)^\dagger C_{\mu\nu}^R \partial^\nu U] + \beta' \frac{F_0^2}{4} \text{Tr}[\partial^\mu U C_{\mu\nu}^L (\partial^\nu U)^\dagger].
\end{aligned}
\tag{4.20}
$$

Under C, $U \xrightarrow{C} U^{\text{T}}$, then $\mathcal{L}_{\pi,\beta,\beta'}^{\text{LO}}$ transforms into

$$
\begin{aligned}
\mathcal{L}'^{\text{LO}}_{\pi,\beta,\beta'} &= \beta \frac{F_0^2}{4} \text{Tr}[(\partial^\mu U^\dagger)^{\text{T}} C_{\mu\nu}^R (\partial^\nu U)^{\text{T}}] + \beta' \frac{F_0^2}{4} \text{Tr}[(\partial^\mu U)^{\text{T}} C_{\mu\nu}^L (\partial^\nu U^\dagger)^{\text{T}}] \\
&= \beta \frac{F_0^2}{4} \text{Tr}\{[\partial^\nu U (C_{\mu\nu}^R)^{\text{T}} \partial^\mu U^\dagger]^{\text{T}}\} + \beta' \frac{F_0^2}{4} \text{Tr}\{[\partial^\nu U^\dagger (C_{\mu\nu}^L)^{\text{T}} \partial^\mu U]^{\text{T}}\}.
\end{aligned}
\tag{4.21}
$$

Using the property, $\mathrm{Tr}(A) = \mathrm{Tr}(A^{\mathrm{T}})$, Eq.(4.21) yields the following result

$$\mathcal{L'}^{\mathrm{LO}}_{\pi,\beta,\beta'} = \beta\frac{F_0^2}{4}\mathrm{Tr}[\partial^\mu U C^R_{\mu\nu}(\partial^\nu U)^\dagger] + \beta'\frac{F_0^2}{4}\mathrm{Tr}[(\partial^\mu U)^\dagger C^L_{\mu\nu}\partial^\nu U]. \tag{4.22}$$

In arriving at Eq.(4.22), we made use of $C^{L/R}_{\mu\nu} = (C^{L/R}_{\mu\nu})^{\mathrm{T}}$ as well as $C^{L/R}_{\mu\nu} = C^{L/R}_{\nu\mu}$ since $C^{L/R}_{\mu\nu}$ are diagonal in SU(2) space and symmetric in μ, ν at LO in LV.

Recall that the LV quark-level Lagrange density is invariant under C if $C^L_{\mu\nu} = C^R_{\mu\nu}$. This special case should also hold at the level of the effective Lagrange density $\mathcal{L}^{\mathrm{LO}}_{\pi,\beta,\beta'}$. Noting that in general $\mathrm{Tr}(ABC) \neq \mathrm{Tr}(CBA)$, this gives the requirement $\beta = \beta'$ for $\mathcal{L}^{\mathrm{LO}}_{\pi,\beta,\beta'}$ to also be invariant.

The remaining part of the Lagrange density in Eq.(4.11) is

$$\mathcal{L}^{\mathrm{LO}}_{\pi,\alpha,\alpha'} = [\alpha\frac{F_0^2}{4}\mathrm{Tr}(C_{R_{\mu\nu}}) + \alpha'\frac{F_0^2}{4}\mathrm{Tr}(C_{L_{\mu\nu}})]\mathrm{Tr}[(\partial^\mu U)^\dagger\partial^\nu U], \tag{4.23}$$

which transforms under C into itself, so

$$\mathcal{L'}^{\mathrm{LO}}_{\pi,\alpha,\alpha'} = [\alpha\frac{F_0^2}{4}\mathrm{Tr}(C_{R_{\mu\nu}}) + \alpha'\frac{F_0^2}{4}\mathrm{Tr}(C_{L_{\mu\nu}})]\mathrm{Tr}[(\partial^\mu U)^\dagger\partial^\nu U]. \tag{4.24}$$

In order to find the relation between α and α', we choose the special case $C^L_{\mu\nu} = -C^R_{\mu\nu}$ which results in the quark-level Lagrange density being odd under C. This must also hold at the level of $\mathcal{L}^{\mathrm{LO}}_{\pi,\alpha,\alpha'}$. Setting $C^L_{\mu\nu} = -C^R_{\mu\nu}$ in Eqs.(4.23) and (4.24) and demanding that $\mathcal{L}^{\mathrm{LO}}_{\pi,\alpha,\alpha'} = -\mathcal{L'}^{\mathrm{LO}}_{\pi,\alpha,\alpha'}$, we arrive at the relation $\alpha = \alpha'$.

So, Eq.(4.11) becomes

$$\begin{aligned}\mathcal{L}^{\mathrm{LO}}_{\pi} = \alpha\frac{F_0^2}{4}&[\mathrm{Tr}(C_{R_{\mu\nu}}) + \mathrm{Tr}(C_{L_{\mu\nu}})]\mathrm{Tr}[(\partial^\mu U)^\dagger\partial^\nu U] \\ &+ \beta\frac{F_0^2}{4}\mathrm{Tr}[(\partial^\mu U)^\dagger C_{R_{\mu\nu}}\partial^\nu U + \partial^\mu U C_{L_{\mu\nu}}(\partial^\nu U)^\dagger],\end{aligned} \tag{4.25}$$

which can be re-written in terms of ${}^1C^{\mu\nu}_{L/R}$ and ${}^3C^{\mu\nu}_{L/R}$ in the following manner

$$\begin{aligned}\mathcal{L}^{\mathrm{LO}}_{\pi} \;=\; &\beta^{(1)}\frac{F_0^2}{4}\left({}^1C_{R\mu\nu} + {}^1C_{L\mu\nu}\right)\mathrm{Tr}[(\partial^\mu U)^\dagger\partial^\nu U] \qquad (4.26)\\ &+\beta^{(2)}\frac{F_0^2}{4}\mathrm{Tr}[(\partial^\mu U)^\dagger\,{}^3C_{R\mu\nu}\partial^\nu U + \partial^\mu U\,{}^3C_{L\mu\nu}(\partial^\nu U)^\dagger],\end{aligned}$$

where $\beta^{(1)}$ and $\beta^{(2)}$ are dimensionless LECs that encode short-distance physics and that cannot be determined from symmetry arguments. In principle, they can be determined from nonperturbative QCD calculations, which however are currently not available. The factor of F_0^2 in eq. (4.26) is also chosen such that based on naive dimensional analysis [28] the $\beta^{(i)}$ are expected to be of natural size, i.e. $\mathcal{O}(1)$.

The isotriplet part, the term with $\beta^{(2)}$, of the pion Lagrange density of Eq.(4.26) does not contribute in the pion sector as we are currently considering it (leading order and no external fields). There exists another parametrization of the matrix $U(x)$, which turns out to be more convenient to use in showing this. It is given by

$$U(x) = \frac{1}{F_0}\Big[\sigma(x)\mathbb{1} + i\vec{\Pi}(x)\cdot\vec{\tau}\Big], \tag{4.27}$$

where $\sigma(x) = \sqrt{F_0^2 - \vec{\Pi}^2(x)}$. Our isotriplet Lagrangian is proportional to (and analogously for U and $U^\dagger$ interchanged)

$$\begin{aligned}\mathcal{L} &\sim \mathrm{Tr}(\partial_\mu U\tau_3\partial_\nu U^\dagger)\\ &\sim \partial_\mu\sigma\partial_\nu\sigma\mathrm{Tr}(\tau_3) + i\partial_\mu\Pi^a\partial_\nu\sigma\mathrm{Tr}(\tau_a\tau_3)\\ &\quad - i\partial_\mu\sigma\partial_\nu\Pi^a\mathrm{Tr}(\tau_3\tau_a) + \partial_\mu\Pi^a\partial_\nu\Pi^b\mathrm{Tr}(\tau_a\tau_3\tau_b).\end{aligned} \tag{4.28}$$

The first term is identically zero, while the remaining terms become

$$2i[\partial_\mu\Pi^3\partial_\nu\sigma - \partial_\mu\sigma\partial_\nu\Pi^3] + 2i(\partial_\mu\vec{\Pi}\times\partial_\nu\vec{\Pi})_3, \tag{4.29}$$

which vanish when contracted with a tensor symmetric in $\mu\nu$, with this being the case with ${}^3C^{L/R}_{\mu\nu}$ at LO.

In principle there is a second, nearly-identical-looking copy of the Lagrange density of Eq. (4.26) contracted with the antisymmetric parts of the $C^{\mu\nu}_{L/R}$. These terms would be accompanied by an independent set of LECs. However, all the terms involved can be shown to be total derivatives, so they may be dropped; and thus only the symmetric part of the $C^{\mu\nu}_{L/R}$ contributes at LO. Therefore, the LO minimal mesonic Lorentz-violating Lagrange density is given by

$$\mathcal{L}_\pi^{\mathrm{LO}} = \beta^{(1)}\frac{F_0^2}{4}\left({}^1C_{R\mu\nu} + {}^1C_{L\mu\nu}\right)\mathrm{Tr}[(\partial^\mu U)^\dagger\partial^\nu U]. \tag{4.30}$$

Expanding $U(x)$ in terms of the pion fields shows that the Lagrange density in Eq. (4.30) not only contains corrections to the pion propagator, but also induces new multi-pion interactions. The two-pion portion of the Lagrange density is

$$\mathcal{L}_{\pi}^{\mathrm{LO},2\phi} = \frac{\beta^{(1)}}{2}(c_{u_L}^{\mu\nu} + c_{d_L}^{\mu\nu} + c_{u_R}^{\mu\nu} + c_{d_L}^{\mu\nu})\partial_\mu\phi_a\partial_\nu\phi_a. \tag{4.31}$$

It would have been interesting to have a three-pion term due to its completely novel-looking structure. It does not have a Lorentz-invariant analogue as it depends on C-odd forms of LV. Unfortunately, all three-pion vertices vanish when the symmetric parts of the $c_{L/R}^{\mu\nu}$ are involved.

The four-pion vertex takes the form

$$\mathcal{L}_{\pi}^{\mathrm{LO},4\phi} = \frac{\beta^{(1)}}{6F^2}(c_{u_L}^{\mu\nu} + c_{d_L}^{\mu\nu} + c_{u_R}^{\mu\nu} + c_{d_L}^{\mu\nu})(\phi_a\phi_b\partial_\mu\phi_a\partial_\nu\phi_b - \phi_b\phi_b\partial_\mu\phi_a\partial_\nu\phi_a). \tag{4.32}$$

This term is a straightforward Lorentz-violating generalization of the usual four-pion vertex. Many Lorentz-violating operators in the SME Lagrange density are structurally similar to operators found in the usual standard model. For example, the quark kinetic terms from Eq. (2.3) resemble standard kinetic terms, but instead of the indices on γ^μ and D^ν being contracted with the metric tensor $g_{\mu\nu}$, they are contracted with the Lorentz-violating backgrounds. The four-pion vertex can be similarly viewed as a deformation of the standard model four-pion vertex. Vertices with more pion fields can similarly be derived. It is important however to point out that the coefficient of the Lorentz-violating four-pion vertex is fixed; it is the same as that of the two-pion. This constraint comes from chiral symmetry.

4.1.1 Local Chiral Transformations and The Modified Pion Covariant Derivative

So far, we have dealt with global chiral symmetry and the absence of external fields. We constructed the LO Lorentz-violating effective pion Lagrange density invariant under global chiral transformations. We shall now extend our formalism to include external fields in the presence of LV. To this end, global chiral symmetry needs to be promoted to a local symmetry. In the previous chapter, we have shown how in the absence of LV, the introduction of external fields on the level of the quark Lagrangian leads to introducing the pion covariant derivative

$$\partial_\mu U \longrightarrow D_\mu U \equiv \partial_\mu U - i r_\mu U + i U l_\mu, \tag{4.33}$$

where r_μ and l_μ are the right-and left-handed external fields in Eq.(3.26).
Under local chiral transformations, and in the absence of any LV

$$\begin{aligned} D_\mu U &\longrightarrow V_R (D_\mu U) V_L^\dagger, \\ r_\mu &\longrightarrow V_R r_\mu V_R^\dagger + i V_R \partial_\mu V_R^\dagger, \\ l_\mu &\longrightarrow V_L l_\mu V_L^\dagger + i V_L \partial_\mu V_L^\dagger. \end{aligned} \tag{4.34}$$

where $r_\mu = \nu_\mu + a_\mu$ and $l_\mu = \nu_\mu - a_\mu$.

Let us now consider our Lorentz-violating quark-level Lagrange density and study its transformation behavior under local chiral $SU(2)_L \times SU(2)_R$ transformations

$$\begin{aligned} q_R &\longrightarrow V_R(x) q_R, \\ q_L &\longrightarrow V_L(x) q_L, \end{aligned} \tag{4.35}$$

Recall the Lorentz-violating quark-level Lagrange density

$$\mathcal{L}^{\text{CPT-even}}_{\text{light quarks}} = \mathcal{L}_{\text{LV}} = \frac{i}{2}\bar{Q}_L C_{L_{\mu\nu}} \gamma^\mu \overleftrightarrow{D}^\nu Q_L + \frac{i}{2}\bar{Q}_R C_{R_{\mu\nu}} \gamma^\mu \overleftrightarrow{D}^\nu Q_R, \tag{4.36}$$

where $A \overleftrightarrow{\partial}_\mu B \equiv A\partial_\mu B - (\partial_\mu A)B$ and $D^\nu Q_{L/R} = (\partial^\nu + i g_3 \mathcal{A}^\nu) Q_{L/R}$, with g_3 and $\mathcal{A}^\nu$ being the strong coupling constant and the gluon fields, respectively.

From Eq.(4.36), we get

$$\begin{aligned}\mathcal{L}_{\text{LV}} =& \frac{i}{2}\bar{Q}_L C_{L_{\mu\nu}}\gamma^\mu \partial^\nu Q_L - \frac{i}{2}\partial^\nu \bar{Q}_L C_{L_{\mu\nu}}\gamma^\mu Q_L \\ &+ i\bar{Q}_L C_{L_{\mu\nu}}\gamma^\mu (ig_3\mathcal{A}^\nu) Q_L + (R \leftrightarrow L).\end{aligned} \tag{4.37}$$

Under the transformations of Eq.(4.35),

$$\begin{aligned}\partial^\nu Q_{L/R} &\longrightarrow \partial^\nu V_{L/R} Q_{L/R} + V_{L/R}\partial^\nu Q_{L/R} \\ \partial^\nu \bar{Q}_{L/R} &\longrightarrow \bar{Q}_{L/R}\partial^\nu V^\dagger_{L/R} + \partial^\nu \bar{Q}_{L/R} V^\dagger_{L/R}.\end{aligned} \tag{4.38}$$

Then, with $C^{\mu\nu}_{L/R} \longrightarrow V_{L/R}\, C^{\mu\nu}_{L/R}\, V^\dagger_{L/R}$ under local chiral transformations,

$$\begin{aligned}\mathcal{L}_{\text{LV}} \longrightarrow& \frac{i}{2}\bar{Q}_L C_{L_{\mu\nu}}\gamma^\mu V^\dagger_L \partial^\nu V_L Q_L - \frac{i}{2}\bar{Q}_L \partial^\nu V^\dagger_L\, V_L C_{L_{\mu\nu}}\gamma^\mu Q_L \\ &+ \frac{i}{2}\bar{Q}_L C_{L_{\mu\nu}}\gamma^\mu \partial^\nu Q_L - \frac{i}{2}\partial^\nu \bar{Q}_L C_{L_{\mu\nu}}\gamma^\mu Q_L \\ &+ i\bar{Q}_L C_{L_{\mu\nu}}\gamma^\mu (ig_3\mathcal{A}^\nu) Q_L + (R \leftrightarrow L).\end{aligned} \tag{4.39}$$

We notice that upon transformation, additional terms get generated that were not present in the original Lorentz-violating quark-level Lagrange density. These extra terms are

$$\begin{aligned}&(1)\ \ \frac{i}{2}\bar{Q}_{L/R} C_{L/R_{\mu\nu}}\gamma^\mu V^\dagger_{L/R}\partial^\nu V_{L/R} Q_{L/R} \\ &(2)\ -\frac{i}{2}\bar{Q}_{L/R}\partial^\nu V^\dagger_{L/R}\, V_{L/R} C_{L/R_{\mu\nu}}\gamma^\mu Q_{L/R}.\end{aligned} \tag{4.40}$$

Since these terms got generated by promoting the global chiral symmetry to a local one, they need to be absorbed by the transformations of the external fields, namely, l_μ and r_μ. Upon taking this step, we will be able to render $\mathcal{L}_{\text{LV}}$ invariant under local chiral transformations.

In order to do this, the transformations of l_μ and r_μ need to be modified to include a Lorentz-violating part. So, we let

$$\begin{aligned}l_\mu &\longrightarrow l'_\mu + X^l_\mu \\ r_\mu &\longrightarrow r'_\mu + X^r_\mu\end{aligned} \tag{4.41}$$

where $l'_\mu = V_L l_\mu V_L^\dagger + iV_L\partial_\mu V_L^\dagger$ and $r'_\mu = V_R r_\mu V_R^\dagger + iV_R\partial_\mu V_R^\dagger$. We now consider the terms of

$$\begin{aligned}\mathcal{L}_{\text{ext}} =& \bar{q}_L\gamma^\mu\Big(l_\mu + \frac{1}{3}\nu_\mu^{(s)}\Big)q_L + \bar{q}_R\gamma^\mu\Big(r_\mu + \frac{1}{3}\nu_\mu^{(s)}\Big)q_R \\ &- \bar{q}_R(s+ip)q_L - \bar{q}_L(s-ip)q_R,\end{aligned} \tag{4.42}$$

that have the l_μ and r_μ external fields in them. Under local chiral transformations in the presence of LV,

$$\bar{Q_L}\gamma^\mu l_\mu Q_L \longrightarrow \bar{Q_L}V_L^\dagger\gamma^\mu l'_\mu V_L Q_L + \bar{Q_L}V_L^\dagger\gamma^\mu X_\mu^l V_L Q_L. \tag{4.43}$$

A similar expression holds for the right-handed term. The first term of Eq.(4.43) along with its right-handed counter part takes care of the extra terms generated from locally transforming

$$\mathcal{L}^0_{\text{QCD}} = \bar{q}_R i\gamma_\mu D^\mu q_R + \bar{q}_L i\gamma_\mu D^\mu q_L - \frac{1}{2}\text{Tr}(G^{\mu\nu}G_{\mu\nu}). \tag{4.44}$$

The second term of Eq.(4.43) along with its right-handed counter part takes care of the extra terms generated from locally transforming the Lorentz-violating quark-level Lagrangian. Therefore, we require

$$\frac{i}{2}[C_{L/R_{\mu\nu}}V_{L/R}^\dagger\partial^\nu V_{L/R} - \partial^\nu V_{L/R}^\dagger\, V_{L/R}C_{L/R_{\mu\nu}}] = -V_{L/R}^\dagger X_\mu^{l/r}V_{L/R}, \tag{4.45}$$

and find

$$X_\mu^{l/r} = \frac{i}{2}[V_{L/R}C_{L/R_{\mu\nu}}\partial^\nu V_{L/R}^\dagger - \partial^\nu V_{L/R}C_{L/R_{\mu\nu}}V_{L/R}^\dagger]. \tag{4.46}$$

So the transformations of l_μ and r_μ get modified in the presence of LV to take the form

$$\begin{aligned} l_\mu &\longrightarrow V_L l_\mu V_L^\dagger + iV_L\partial_\mu V_L^\dagger + \frac{i}{2}[V_L C_{L_{\mu\nu}}\partial^\nu V_L^\dagger - \partial^\nu V_L C_{L_{\mu\nu}}V_L^\dagger], \\ r_\mu &\longrightarrow V_R r_\mu V_R^\dagger + iV_R\partial_\mu V_R^\dagger + \frac{i}{2}[V_R C_{R_{\mu\nu}}\partial^\nu V_R^\dagger - \partial^\nu V_R C_{R_{\mu\nu}}V_R^\dagger].\end{aligned} \tag{4.47}$$

The modification of the transformation behavior of the external left- and right-handed fields due to the presence of LV necessarily leads to the modification of the pion covariant derivative. Upon transforming $D_\mu U$ under chiral transformations in the presence of LV, we get

$$\begin{aligned} D_\mu U \longrightarrow\ & V_R(D_\mu U)V_L^\dagger \\ & - i[\frac{i}{2}(V_R C_{R_{\mu\nu}}\partial^\nu V_R^\dagger - \partial^\nu V_R C_{R_{\mu\nu}} V_R^\dagger)](V_R U V_L^\dagger) \\ & + i(V_R U V_L^\dagger)[\frac{i}{2}(V_L C_{L_{\mu\nu}}\partial^\nu V_L^\dagger - \partial^\nu V_L C_{L_{\mu\nu}} V_L^\dagger)], \end{aligned} \tag{4.48}$$

where $U \longrightarrow V_R U V_L^\dagger$ under local chiral transformations, and $D_\mu U = \partial_\mu U - ir_\mu U + iUl_\mu$. The extra terms appearing in Eq.(4.48) can be removed by adding certain terms to $D_\mu U$ that transform as $V_R(...)V_L^\dagger$ in addition to generating opposite terms that cancel the extra ones. The terms that need to be added are $\frac{i}{2}[C^R_{\mu\nu}r^\nu U - UC^L_{\mu\nu}l^\nu]$ and $-\frac{i}{2}[Ul^\nu C^L_{\mu\nu} - r^\nu C^R_{\mu\nu}U]$. So, the modified pion covariant derivative in the presence of LV, which is denoted by $D_\mu^{\text{LV}}U$, will take the form

$$D_\mu^{\text{LV}}U = D_\mu U + \frac{i}{2}\{C^R_{\mu\nu}, r^\nu\}U - \frac{i}{2}U\{C^L_{\mu\nu}, l^\nu\}, \tag{4.49}$$

where $\{C^{L/R}_{\mu\nu}, l^\nu/r^\nu\}$ is the anti-commutator of $C^{L/R}_{\mu\nu}$ with l^ν/r^ν.

Having arrived at the form of the modified pion covariant derivative, we go back to the list of terms in Eq.(4.6) and replace the partial derivative $\partial_\mu U$ with $D_\mu^{\text{LV}}U$ and expand it as $D_\mu U + \frac{i}{2}\{C^R_{\mu\nu}, r^\nu\}U - \frac{i}{2}U\{C^L_{\mu\nu}, l^\nu\}$ retaining only terms with a single $C^{L/R}_{\mu\nu}$. The reason for this is that we are only working up to $\mathcal{O}(C^{L/R}_{\mu\nu})$. We find that we get the same list as in Eq.(4.6) with the regular pion covariant derivative $D_\mu U$ replacing $\partial_\mu U$. These terms exhaust all the possibilities at LO; however, they are not all independent. Here again the same dependent terms can be eliminated following exactly what was done in the beginning of section 4.1. The identities in Eq.(4.7) still hold with $D_\mu U$ taking the place of $\partial_\mu U$,

$$\begin{gathered} (D^\mu U)^\dagger U = -U^\dagger(D^\mu U) \\ (D^\mu D^\nu U^\dagger)U + U^\dagger(D^\mu D^\nu U) = -(D^\mu U)^\dagger(D^\nu U) - (D^\nu U)^\dagger(D^\mu U). \end{gathered} \tag{4.50}$$

Before continuing our main discussion, it is worthwhile to prove the above two identities as they are not as obvious like the ones in Eq.(4.7) where there is the regular partial derivative acting on the pion field matrix.

The first line appearing in Eq.(4.50) can be proven as follows,

$$\begin{aligned}
(D^\mu U)^\dagger U &= \partial^\mu U^\dagger U + iU^\dagger r^\mu U - il^\mu U^\dagger U \\
&= -U^\dagger \partial^\mu U + iU^\dagger r^\mu U - il^\mu \\
-U^\dagger (D^\mu U) &= -U^\dagger \partial^\mu U - U^\dagger(-ir_\mu U) - iU^\dagger U l_\mu \\
&= -U^\dagger \partial^\mu U + iU^\dagger r^\mu U - il^\mu \\
&= (D^\mu U)^\dagger U,
\end{aligned} \tag{4.51}$$

where r^μ and l^μ are Hermitian.

As for the second identity in Eq.(4.50), we will refer to the paper by Fearing and Scherer [24] in proving it. In this paper, the authors show that one can define a product chain rule for covariant derivatives analogous to ordinary derivatives. This is done by first defining the covariant derivatives of certain objects that transform in the following way under local chiral transformations [24]:

$$\begin{aligned}
&A \to A; && D_\mu A \equiv \partial_\mu A, \\
&B \to V_L B V_L^\dagger; && D_\mu B \equiv \partial_\mu B - il_\mu B + iBl_\mu, \\
&C \to V_R C V_R^\dagger; && D_\mu C \equiv \partial_\mu C - ir_\mu C + iCr_\mu, \\
&D \to V_L D V_R^\dagger; && D_\mu D \equiv \partial_\mu D + iDr_\mu - il_\mu D, \\
&E \to V_R E V_L^\dagger; && D_\mu E \equiv \partial_\mu E - ir_\mu E + iEl_\mu.
\end{aligned} \tag{4.52}$$

Given the product $V = MN$ such that M, N, and V all transform like any of the objects in Eq.(4.52), one can still apply the well-known product chain rule

$$D_\mu V = D_\mu(MN) = (D_\mu M)N + M(D_\mu N). \tag{4.53}$$

To prove the second identity, we start with

$$\begin{aligned}
0 &= D_\mu D_\nu (U^\dagger U) \\
&= D_\mu[(D_\nu U^\dagger)U + U^\dagger(D_\nu U)] \\
&= (D_\mu D_\nu U^\dagger)U + (D_\nu U^\dagger)(D_\mu U) + (D_\mu U^\dagger)(D_\nu U) + U^\dagger(D_\mu D_\nu U).
\end{aligned} \tag{4.54}$$

This then gives the desired result

$$(D_\mu D_\nu U^\dagger)U + U^\dagger(D_\mu D_\nu U) = -(D_\mu U^\dagger)(D_\nu U) - (D_\nu U^\dagger)(D_\mu U). \tag{4.55}$$

Going back to our main discussion, we find that after eliminating the dependent terms, the ones that remain are the following

$$\begin{aligned}
&\mathrm{Tr}(C^L_{\mu\nu})\mathrm{Tr}[(D^\mu U)^\dagger D^\nu U]\\
&\mathrm{Tr}[D^\mu U C^L_{\mu\nu}(D^\nu U)^\dagger]\\
&\mathrm{Tr}(C^R_{\mu\nu})\mathrm{Tr}[(D^\mu U)^\dagger D^\nu U]\\
&\mathrm{Tr}[(D^\mu U)^\dagger C^R_{\mu\nu} D^\nu U].
\end{aligned} \tag{4.56}$$

Hence, the constructed LO Lorentz-violating pion effective Lagrangian in the presence of external fields has the form

$$\begin{aligned}
\mathcal{L}^{\mathrm{LO}}_{\pi,\mathrm{r}_\mu,\mathrm{l}_\mu} =& [\alpha\frac{F_0^2}{4}\mathrm{Tr}(C_{R_{\mu\nu}}) + \alpha'\frac{F_0^2}{4}\mathrm{Tr}(C_{L_{\mu\nu}})]\mathrm{Tr}[(D^\mu U)^\dagger D^\nu U]\\
&+ \mathrm{Tr}[\beta\frac{F_0^2}{4}(D^\mu U)^\dagger C_{R_{\mu\nu}} D^\nu U + \beta'\frac{F_0^2}{4}D^\mu U C_{L_{\mu\nu}}(D^\nu U)^\dagger],
\end{aligned} \tag{4.57}$$

where α, α', β, and β' are the dimensionless LECs appearing in Eq.(4.11).

The above Lagrange density has exactly the same transformation behavior under C as that in Eq.(4.11). Due to $\alpha = \alpha'$ and $\beta = \beta'$, Eq.(4.57) becomes

$$\begin{aligned}
\mathcal{L}^{\mathrm{LO}}_{\pi,\mathrm{r}_\mu,\mathrm{l}_\mu} =& \alpha\frac{F_0^2}{4}[\mathrm{Tr}(C_{R_{\mu\nu}}) + \mathrm{Tr}(C_{L_{\mu\nu}})]\mathrm{Tr}[(D^\mu U)^\dagger D^\nu U]\\
&+ \beta\frac{F_0^2}{4}\mathrm{Tr}[(D^\mu U)^\dagger C_{R_{\mu\nu}} D^\nu U + D^\mu U C_{L_{\mu\nu}}(D^\nu U)^\dagger].
\end{aligned} \tag{4.58}$$

We can again re-write the above Lagrange density in terms of ${}^1C^{\mu\nu}_{L/R}$ and ${}^3C^{\mu\nu}_{L/R}$ in the following manner

$$\begin{aligned}
\mathcal{L}^{\mathrm{LO}}_{\pi,r_\mu,l_\mu} =& \beta^{(1)}\frac{F_0^2}{4}\left({}^1C_{R\mu\nu} + {}^1C_{L\mu\nu}\right)\mathrm{Tr}[(D^\mu U)^\dagger D^\nu U]\\
&+ \beta^{(2)}\frac{F_0^2}{4}\mathrm{Tr}[(D^\mu U)^\dagger\, {}^3C_{R\mu\nu} D^\nu U + D^\mu U\, {}^3C_{L\mu\nu}(D^\nu U)^\dagger],
\end{aligned} \tag{4.59}$$

where $\beta^{(1)}$ and $\beta^{(2)}$ are the same dimensionless LECs in Eq.(4.26).

In this case too, that is, in the presence of external fields accompanied with LV, there is in principle a second, nearly-identical-looking copy of the Lagrange density of Eq. (4.59) contracted with the antisymmetric parts of the $C^{\mu\nu}_{L/R}$. These terms would be accompanied by an independent set of LECs. However, all the terms involved can be shown to vanish; and thus even in the presence of external fields, we find that only the symmetric part of the $C^{\mu\nu}_{L/R}$ contributes at LO.

The Lagrange density in Eq.(4.59) is an exact copy of the one in Eq.(4.26) with the covariant derivative substituting the partial derivative. However, taking the standard LO effective pion Lagrangian

$$\mathscr{L}_{\pi}^{\mathrm{LO}} = \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger], \tag{4.60}$$

and replacing $D_\mu U$ by $D_\mu^{\mathrm{LV}}U = D_\mu U + \frac{i}{2}\{C^R_{\mu\nu}, r^\nu\}U - \frac{i}{2}U\{C^L_{\mu\nu}, l^\nu\}$, then expanding up to $\mathcal{O}(C^{L/R}_{\mu\nu})$, we find two new terms in addition to recovering the standard pion Lagrangian term,

$$\frac{F_0^2}{4}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger] \longrightarrow \frac{F_0^2}{4}\mathrm{Tr}[D_\mu^{\mathrm{LV}}U(D^{\mathrm{LV}^\mu}U)^\dagger], \tag{4.61}$$

where

$$\begin{aligned}\frac{F_0^2}{4}\mathrm{Tr}[D_\mu^{\mathrm{LV}}U(D^{\mathrm{LV}^\mu}U)^\dagger] = {} & \frac{F_0^2}{4}\mathrm{Tr}[D_\mu U(D^\mu U)^\dagger] \\ & + i\frac{F_0^2}{4}\Big[\mathrm{Tr}\Big(D^\mu U^\dagger\{C^R_{\mu\nu}, r^\nu\}U\Big) + \mathrm{Tr}\Big(D^\mu U\{C^L_{\mu\nu}, l^\nu\}U^\dagger\Big)\Big].\end{aligned} \tag{4.62}$$

It is reassuring that the last two terms in Eq.(4.62) above have the expected transformation behavior under C, where

$$\begin{aligned}i\mathrm{Tr}\Big(D^\mu U^\dagger\{C^R_{\mu\nu}, r^\nu\}U\Big) + i\mathrm{Tr}\Big(D^\mu U\{C^L_{\mu\nu}, l^\nu\}U^\dagger\Big) \to {} & i\mathrm{Tr}\Big(D^\mu U\{C^R_{\mu\nu}, l^\nu\}U^\dagger\Big) \\ & + i\mathrm{Tr}\Big(D^\mu U^\dagger\{C^L_{\mu\nu}, r^\nu\}U\Big).\end{aligned} \tag{4.63}$$

Therefore, the total LO Lorentz-violating pion effective Lagrange density in the presence of external fields is

$$\begin{aligned}\mathcal{L}^{\mathrm{LO}}_{\pi,\mathrm{r}_\mu,\mathrm{l}_\mu} = {} & \alpha\frac{F_0^2}{4}[\mathrm{Tr}(C_{R_{\mu\nu}}) + \mathrm{Tr}(C_{L_{\mu\nu}})]\mathrm{Tr}[(D^\mu U)^\dagger D^\nu U] \\ & + \beta\frac{F_0^2}{4}\mathrm{Tr}[(D^\mu U)^\dagger C_{R_{\mu\nu}} D^\nu U + D^\mu U C_{L_{\mu\nu}}(D^\nu U)^\dagger] \\ & + i\frac{F_0^2}{4}\Big[\mathrm{Tr}\Big(D^\mu U^\dagger\{C^R_{\mu\nu}, r^\nu\}U\Big) + \mathrm{Tr}\Big(D^\mu U\{C^L_{\mu\nu}, l^\nu\}U^\dagger\Big)\Big].\end{aligned} \tag{4.64}$$

Application: The Electromagnetic Interaction

At the quark-level, if one considers only the two-flavor version of QCD, i.e. only u and d quarks (which is the case in this dissertation), the coupling of quarks to an external electromagnetic four-potential $\mathcal{A}_\mu$ is given by setting [22, 29]

$$r_\mu = l_\mu = -\frac{e}{2}\mathcal{A}_\mu \tau_3, \quad \nu_\mu^{(s)} = -\frac{e}{2}\mathcal{A}_\mu, \tag{4.65}$$

where $\nu_\mu^{(s)}$ is the $U(1)_V$ gauge field. Instead, one can replace the traceless fields r_μ and l_μ by non-traceless ones of the form [29]

$$\begin{aligned} \tilde{r}_\mu &= r_\mu + \frac{1}{3}\nu_\mu^{(s)}\mathbb{1}_{2\times 2} = -\frac{e}{2}\mathcal{A}_\mu(\tau_3 + \frac{1}{3}\mathbb{1}_{2\times 2}), \\ \tilde{l}_\mu &= l_\mu + \frac{1}{3}\nu_\mu^{(s)}\mathbb{1}_{2\times 2} = -\frac{e}{2}\mathcal{A}_\mu(\tau_3 + \frac{1}{3}\mathbb{1}_{2\times 2}). \end{aligned} \tag{4.66}$$

As an application, we consider the last term of Eq.(4.64) and substitute $r^\nu = l^\nu = -\frac{e}{2}\mathcal{A}^\nu(\tau_3 + \frac{1}{3}\mathbb{1}_{2\times 2})$ where $e > 0$ is the elementary charge. We write

$$C_{\mu\nu}^{R/L} = \frac{1}{2}C_{+\mu\nu}^{R/L}\mathbb{1} + \frac{1}{2}C_{-\mu\nu}^{R/L}\tau_3, \tag{4.67}$$

where $C_{+\mu\nu}^{R/L} = c_{u_{\mu\nu}}^{R/L} + c_{d_{\mu\nu}}^{R/L}$ and $C_{-\mu\nu}^{R/L} = c_{u_{\mu\nu}}^{R/L} - c_{d_{\mu\nu}}^{R/L}$, and expand the term

$$i\frac{F_0^2}{4}\Big[\mathrm{Tr}\Big(D^\mu U^\dagger\{C_{\mu\nu}^R, r^\nu\}U\Big) + \mathrm{Tr}\Big(D^\mu U\{C_{\mu\nu}^L, l^\nu\}U^\dagger\Big)\Big] \tag{4.68}$$

with

$$U = \exp\Big(\frac{i\phi_a\tau_a}{F_0}\Big) = \mathbb{1} + \frac{i}{F_0}\phi_a\tau_a + ... \tag{4.69}$$

Up to one pion field, we get

$$\begin{aligned} \mathrm{Tr}\Big(D^\mu U^\dagger\{C_{\mu\nu}^R, r^\nu\}U\Big) &= \mathrm{Tr}\Big\{(-\frac{i}{F_0}\tau_a\partial^\mu\phi_a)(-\frac{e}{2}\mathcal{A}^\nu) \\ &\qquad [(\frac{1}{3}C_{+\mu\nu}^R + C_{-\mu\nu}^R)\mathbb{1} + (\frac{1}{3}C_{-\mu\nu}^R + C_{+\mu\nu}^R)\tau_3](\mathbb{1})\Big\} \\ &= \frac{ie}{2F_0}(\frac{1}{3}C_{-\mu\nu}^R + C_{+\mu\nu}^R)\mathcal{A}^\nu\partial^\mu\phi_a\mathrm{Tr}(\tau_a\tau_3) \\ &= \frac{ie}{F_0}(\frac{1}{3}C_{-\mu\nu}^R + C_{+\mu\nu}^R)\mathcal{A}^\nu\partial^\mu\phi_3. \end{aligned} \tag{4.70}$$

Similarly, we find that up to a single pion field,

$$\mathrm{Tr}\Big(D^\mu U\{C_{\mu\nu}^L, l^\nu\}U^\dagger\Big) = -\frac{ie}{F_0}(\frac{1}{3}C_{-\mu\nu}^L + C_{+\mu\nu}^L)\mathcal{A}^\nu\partial^\mu\phi_3. \tag{4.71}$$

Therefore, the final result of expanding Eq.(4.68) to a single pion field is

$$-e\frac{F_0}{4}\Big[\frac{1}{3}(C^R_{-\mu\nu} - C^L_{-\mu\nu}) + (C^R_{+\mu\nu} - C^L_{+\mu\nu})\Big]\mathcal{A}^\nu\partial^\mu\phi_3. \tag{4.72}$$

In physical pion fields, $\phi_3 = \pi^0$, so Eq.(4.72) represents a Lorentz-violating process whereby a photon, via interacting with the fixed background field, produces a neutral pion. Looking at the quantum numbers of the particles involved, we find that both carry zero charge and have negative parity. While the spin of the neutral pion is $s = 0$, that of the photon is $s = 1$. This means that in SM physics, an incoming photon cannot produce a π^0 due to non-conservation of angular momentum. In cases where rotation invariance is not an exact symmetry, the conversion of a photon to a neutral pion is not forbidden by angular momentum conservation. In a Lorentz-violating process, angular momentum need no longer be conserved owing to the presence of a background field. Thus, in the presence of LV, the process of an incoming photon interacting with the background and producing a neutral pion becomes admissible.

4.2 Leading-order Lorentz-violating Baryonic Lagrangian

The LV effective pion-nucleon Lagrangian, $\mathcal{L}^{\mathrm{LO}}_{\pi N}$, that we will construct is of zeroth chiral order, i.e. $O(q^0)$, and it contains up to two derivatives on the nucleon field. The $O(q^0)$ elements that form its building blocks are the following [22, 27]:

$$\Psi,\ \bar{\Psi},\ D_\mu\Psi,\ u,\ \gamma_\mu,\ \gamma_\mu\gamma_5,\ \sigma_{\mu\nu},\ g_{\mu\nu},\ \text{and}\ \varepsilon_{\lambda\mu\nu\rho}, \tag{4.73}$$

where γ_μ are the so-called gamma or Dirac matrices satisfying the anti-commutation relation $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}\mathbb{1}_{4\times4}$ with $g_{\mu\nu}$ being the space-time metric tensor. The anti-symmetric combination of two gamma matrices is denoted by $\sigma_{\mu\nu} = -\sigma_{\nu\mu} = \frac{1}{2}[\gamma_\mu, \gamma_\nu] = \frac{1}{2}(\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)$ [30]. Finally, $\varepsilon_{\lambda\mu\nu\rho}$ is the totally anti-symmetric tensor in four dimensions $(\lambda, ...\rho = 0, 1, 2, 3)$.

Recall that Ψ is the nucleon field doublet, $D_\mu \Psi = (\partial_\mu + \Gamma_\mu)\Psi$ is the nucleon covariant derivative, where $\Gamma_\mu = \frac{1}{2}(u^\dagger \partial_\mu u + u \partial_\mu u^\dagger)$ is the chiral conection, and $u^2 = U$ such that under the global chiral group $\mathrm{G} = SU(2)_L \times SU(2)_R$ [see section 3.2.2]

$$\begin{pmatrix} u \\ \Psi \\ D_\mu \Psi \end{pmatrix} \longrightarrow \begin{pmatrix} u' \\ \Psi' \\ (D_\mu \Psi)' \end{pmatrix} = \begin{pmatrix} RuK^\dagger = KuL^\dagger \\ K[L,R,U]\Psi \\ K[L,R,U](D_\mu \Psi) \end{pmatrix} \tag{4.74}$$

Since $\mathcal{L}_{\pi N}^{\mathrm{LO}}$ must be chirally invariant, its terms should have the general form $\bar{\Psi}\hat{O}\Psi$, where $\hat{O}$ is an operator in Dirac and isospin space which transforms as $K\hat{O}K^\dagger$ under chiral transformations. Bearing in mind all of the above, we can now write down all of the following chirally invariant $O(q^0)$ terms where $a_{R/L}^{(n)}$'s are LECs:

1. $\bar{\Psi}[(a_R^{(1)} u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(1)} u\, C_{L_{\mu\nu}} u^\dagger)(\gamma^\nu D^\mu + \gamma^\mu D^\nu)]\Psi$

2. $[a_R^{(2)}\, \mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(2)}\, \mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}(\gamma^\nu D^\mu + \gamma^\mu D^\nu)]\Psi$

3. $\bar{\Psi}[(a_R^{(3)} u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(3)} u\, C_{L_{\mu\nu}} u^\dagger)(\gamma^\nu \gamma^5 D^\mu + \gamma^\mu \gamma^5 D^\nu)]\Psi$

4. $[a_R^{(4)}\, \mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(4)}\, \mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}(\gamma^\nu \gamma^5 D^\mu + \gamma^\mu \gamma^5 D^\nu)\Psi$

5. $\bar{\Psi}(a_R^{(5)} u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(5)} u\, C_{L_{\mu\nu}} u^\dagger)\sigma^{\mu\nu}\Psi$

6. $[a_R^{(6)}\, \mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(6)}\, \mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}\sigma^{\mu\nu}\Psi$

7. $\bar{\Psi}(a_R^{(7)} u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(7)} u\, C_{L_{\mu\nu}} u^\dagger)\epsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda}\Psi$

8. $[a_R^{(8)}\, \mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(8)}\, \mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}\epsilon^{\mu\nu\rho\lambda}\sigma_{\rho\lambda}\Psi$

9. $\bar{\Psi}(a_R^{(9)} u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(9)} u\, C_{L_{\mu\nu}} u^\dagger)\epsilon^{\mu\nu\rho\lambda}\gamma_\rho D_\lambda \Psi$

10. $[a_R^{(10)}\, \mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(10)}\, \mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}\epsilon^{\mu\nu\rho\lambda}\gamma_\rho D_\lambda \Psi$

11. $\bar{\Psi}(a_R^{(11)} u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(11)} u\, C_{L_{\mu\nu}} u^\dagger)\epsilon^{\mu\nu\rho\lambda}\gamma_\rho \gamma_5 D_\lambda \Psi$

12. $[a_R^{(12)}\, \mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(12)}\, \mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}\epsilon^{\mu\nu\rho\lambda}\gamma_\rho \gamma_5 D_\lambda \Psi$

13. $\bar{\Psi}[(a_R^{(13)}u^\dagger\, C^\mu_{R_\mu} u + a_L^{(13)} u\, C^\mu_{L_\mu} u^\dagger)\sigma^{\rho\lambda}(D_\rho D_\lambda + D_\lambda D_\rho)]\Psi$

14. $[a_R^{(14)}\,\mathrm{Tr}(C^\mu_{R_\mu}) + a_L^{(14)}\,\mathrm{Tr}(C^\mu_{L_\mu})]\bar{\Psi}\sigma^{\rho\lambda}(D_\rho D_\lambda + D_\lambda D_\rho)\Psi$

15. $\bar{\Psi}[(a_R^{(15)}u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(15)} u\, C_{L_{\mu\nu}} u^\dagger)\epsilon^{\mu\nu\rho\lambda}(D_\rho D_\lambda + D_\lambda D_\rho)]\Psi$

16. $[a_R^{(16)}\,\mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(16)}\,\mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}\epsilon^{\mu\nu\rho\lambda}(D_\rho D_\lambda + D_\lambda D_\rho)\Psi$

17. $\bar{\Psi}[(a_R^{(17)}u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(17)} u\, C_{L_{\mu\nu}} u^\dagger)(D^\mu D^\nu + D^\nu D^\mu)]\Psi$

18. $[a_R^{(18)}\,\mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(18)}\,\mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}(D^\mu D^\nu + D^\nu D^\mu)\Psi$

19. $\bar{\Psi}[(a_R^{(19)}u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(19)} u\, C_{L_{\mu\nu}} u^\dagger)\sigma^{\nu\rho}(D^\mu D_\rho + D_\rho D^\mu)]\Psi$

20. $[a_R^{(20)}\,\mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(20)}\,\mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}\sigma^{\nu\rho}(D^\mu D_\rho + D_\rho D^\mu)\Psi$

21. $\bar{\Psi}(a_R^{(21)}u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(21)} u\, C_{L_{\mu\nu}} u^\dagger)g^{\mu\nu}\Psi$

22. $[a_R^{(22)}\,\mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(22)}\,\mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}g^{\mu\nu}\Psi$

The majority of these terms vanish. All terms in which $C^{L/R}_{\mu\nu}$ are contracted with an anti-symmetric tensor such as $\sigma^{\mu\nu}$ or $\epsilon^{\mu\nu\rho\lambda}$ or a combination of these two result in a zero contribution to the pion-nucleon Lagrange density. The reason behind this is that in the fermion sector of the mSME, the anti-symmetric portions of $C_{\mu\nu}$ cannot be observed at linear order in LV [8]. So, at LO, $C^{L/R}_{\mu\nu}$ are taken to be symmetric, therefore, terms no. 5 through no. 12 are identically zero.

Due to $C^{L/R}_{\mu\nu}$ being symmetric, only the operators that are symmetric in their Lorentz indices will contribute at this order. This avoids the presence of terms such as those containing $[D_\mu, D_\nu]$. In addition, the anti-symmetric combination of two nucleon covariant derivatives is of higher order in the χPT power counting as well [27]. Moreover, every term in which $\sigma^{\mu\nu}$ or $\epsilon^{\mu\nu\rho\lambda}$ is contracted with the symmetric combination of two nucleon covariant derivative vanishes. These terms are no. 13 through no. 16.

Terms no. 17 and 18 where $C_{\mu\nu}$ is contracted with the symmetric combination $(D^\mu D^\nu + D^\nu D^\mu)$ are reduced to terms no. 1 and 2, respectively, up to higher-order chiral corrections. This is done by using the equations of motion:

$$
\begin{aligned}
i\gamma_\mu D^\mu \Psi = m_N \Psi + h.o. \Rightarrow \not{D}\Psi = -im_N\Psi, \\
-i\bar{\Psi}\overleftarrow{\not{D}} = m_N\bar{\Psi} + h.o. \Rightarrow \bar{\Psi}\overleftarrow{\not{D}} = im_N\bar{\Psi},
\end{aligned}
\tag{4.75}
$$

where $h.o.$ stands for higher-order chiral corrections. The equation of motion in the first line of Eq.(4.75) can be re-written as

$$
D_\nu\Psi = \sigma_{\rho\nu}D^\rho\Psi - im_N\gamma_\nu\Psi, \tag{4.76}
$$

then,

$$
D^\mu D^\nu\Psi = -im_N\gamma^\nu D^\mu\Psi + \sigma^{\rho\nu}D^\mu D_\rho\Psi. \tag{4.77}
$$

Here, m_N is the nucleon mass. After inserting the last equation above in no. 17 and 18, one gets terms which are proportional to no. 1 and 2 up to another term of the form $\sigma^{\rho\nu}D^\mu D_\rho\Psi$. The second term, $\sigma^{\rho\nu}D^\mu D_\rho\Psi$, appears in no. 19 and 20 which will be shown to vanish next. So, at LO, terms no. 17 and 18 are proportional to no. 1 and 2, respectively. Thus, they are not independent.

As for terms no. 19 and 20 with the combination $\sigma^{\nu\rho}(D^\mu D_\rho + D_\rho D^\mu)$, they vanish at LO. This can be seen in the following manner.

For term no. 20,

$$
\begin{aligned}
\bar{\Psi}\sigma^{\nu\rho}(D^\mu D_\rho + D_\rho D^\mu)\Psi &= \bar{\Psi}(\gamma^\nu\gamma^\rho - g^{\nu\rho})(D^\mu D_\rho + D_\rho D^\mu)\Psi \\
&= \bar{\Psi}\Big(\gamma^\nu D^\mu\not{D} - \not{D}\gamma^\nu D^\mu + [D^\nu, D^\mu]\Big)\Psi \\
&= \bar{\Psi}\gamma^\nu D^\mu\not{D}\Psi + (\bar{\Psi}\overleftarrow{\not{D}})\gamma^\nu D^\mu\Psi + \bar{\Psi}[D^\nu, D^\mu]\Psi,
\end{aligned}
\tag{4.78}
$$

where we have applied integration by parts to the second term of line 2 and neglected the total derivative which resulted from this partial integration. Using the LO equation of motion and its Hermitian conjugate appearing in Eq.(4.75), the first two terms of line 3 in the above equation cancel each other. The remaining term containing $[D^\nu, D^\mu]$ is of higher order in the χPT power counting and is neglected. When applying integration by parts for term no. 19, one has to make sure to include the structure $C^{L/R}_{\mu\nu}$ sandwiched in between the pion fields u and $u^\dagger$.

Let us denote the combination of the pion fields and the Lorentz-violating coefficients, namely, $u^\dagger\, C_{R_{\mu\nu}} u$ and $u\, C_{L_{\mu\nu}} u^\dagger$ by X, so we have

$$\begin{aligned}\bar\Psi X \sigma^{\nu\rho}(D^\mu D_\rho + D_\rho D^\mu)\Psi &= \bar\Psi X(\gamma^\nu\gamma^\rho - g^{\nu\rho})(D^\mu D_\rho + D_\rho D^\mu)\Psi \\ &= \bar\Psi X\Big(\gamma^\nu D^\mu \not{D} - \not{D}\gamma^\nu D^\mu + [D^\nu, D^\mu]\Big)\Psi \\ &= \bar\Psi X\gamma^\nu D^\mu \not{D}\Psi - \bar\Psi X\not{D}\gamma^\nu D^\mu\Psi + \bar\Psi X[D^\nu, D^\mu]\Psi,\end{aligned} \tag{4.79}$$

where the last term of the last line of the above equation is of higher order in chiral power counting and can thus be ignored. Again using the LO equation of motion $i\gamma_\mu D^\mu\Psi = m_N\Psi$, the first term $\bar\Psi X\gamma^\nu D^\mu\not{D}\Psi$ is reduced to $-im\bar\Psi X\gamma^\nu D^\mu\Psi$.

As for the second term, we have

$$\begin{aligned}\bar\Psi X\not{D}\gamma^\nu D^\mu\Psi &= \bar\Psi X\gamma^\rho\gamma^\nu D_\rho D^\mu\Psi \\ &= \bar\Psi X\gamma^\rho\gamma^\nu(\partial_\rho + \Gamma_\rho)D^\mu\Psi \\ &= \bar\Psi X\gamma^\rho\gamma^\nu\partial_\rho(D^\mu\Psi) + \bar\Psi X\gamma^\rho\gamma^\nu\Gamma_\rho D^\mu\Psi \\ &= \text{total derivative} - \partial_\rho(\bar\Psi X)\gamma^\rho\gamma^\nu D^\mu\Psi + \bar\Psi X\gamma^\rho\gamma^\nu\Gamma_\rho D^\mu\Psi.\end{aligned} \tag{4.80}$$

Neglecting the total derivative, we find

$$\begin{aligned}\bar\Psi X\not{D}\gamma^\nu D^\mu\Psi &= -(\partial_\rho\bar\Psi)X\gamma^\rho\gamma^\nu D^\mu\Psi - \bar\Psi\partial_\rho X\gamma^\rho\gamma^\nu D^\mu\Psi + \bar\Psi X\gamma^\rho\gamma^\nu\Gamma_\rho D^\mu\Psi \\ &= -(\partial_\rho\bar\Psi)X\gamma^\rho\gamma^\nu D^\mu\Psi + \bar\Psi\Gamma_\rho X\gamma^\rho\gamma^\nu D^\mu\Psi - \bar\Psi\Gamma_\rho X\gamma^\rho\gamma^\nu D^\mu\Psi \\ &\quad + \bar\Psi X\gamma^\rho\gamma^\nu\Gamma_\rho D^\mu\Psi - \bar\Psi\partial_\rho X\gamma^\rho\gamma^\nu D^\mu\Psi \\ &= -(\bar\Psi\overleftarrow{\not{D}})X\gamma^\nu D^\mu\Psi - \bar\Psi(\Gamma_\rho X - X\Gamma_\rho)\gamma^\rho\gamma^\nu D^\mu\Psi \\ &\quad - \bar\Psi\partial_\rho X\gamma^\rho\gamma^\nu D^\mu\Psi \\ &= -(\bar\Psi\overleftarrow{\not{D}})X\gamma^\nu D^\mu\Psi - \bar\Psi[\Gamma_\rho, X]\gamma^\rho\gamma^\nu D^\mu\Psi - \bar\Psi\partial_\rho X\gamma^\rho\gamma^\nu D^\mu\Psi \\ &= -(\bar\Psi\overleftarrow{\not{D}})X\gamma^\nu D^\mu\Psi - \bar\Psi\Big(\partial_\rho X + [\Gamma_\rho, X]\Big)\gamma^\rho\gamma^\nu D^\mu\Psi.\end{aligned} \tag{4.81}$$

We can then define a covariant derivative of X as $D_\rho X = \partial_\rho X + [\Gamma_\rho, X]$, so that up to a total derivative,

$$\bar{\Psi} X \not{D} \gamma^\nu D^\mu \Psi = -(\bar{\Psi} \overleftarrow{\not{D}}) X \gamma^\nu D^\mu \Psi - \bar{\Psi} D_\rho X \gamma^\rho \gamma^\nu D^\mu \Psi. \tag{4.82}$$

Note that $D_\rho X$ is of higher chiral order, and the second term in the equation above will thus be dropped, resulting in

$$\begin{aligned} \bar{\Psi} X \sigma^{\nu\rho}(D^\mu D_\rho + D_\rho D^\mu)\Psi &= -im\bar{\Psi} X \gamma^\nu D^\mu \Psi + (\bar{\Psi}\overleftarrow{\not{D}}) X \gamma^\nu D^\mu \Psi + h.o. \\ &= -im\bar{\Psi} X \gamma^\nu D^\mu \Psi + im\bar{\Psi} X \gamma^\nu D^\mu \Psi \\ &= 0. \end{aligned} \tag{4.83}$$

The last two terms that vanish are the ones where $C_{\mu\nu}$ is contracted with the metric tensor, $g^{\mu\nu}$. These are terms no. 21 and 22. The reason behind their vanishing is that the combination of $C^{L/R}_{\mu\nu}$ and $g^{\mu\nu}$ results in $C^{(L/R)\mu}_{\mu}$. This is the trace of $C^{L/R}_{\mu\nu}$ in the space of Lorentz indices which can be taken to be zero [4].

We finally find that the only terms that remain from the long list of nucleon terms are no. 1 through no. 4. The operators in these terms are not the only structures with two free Lorentz indices that may be constructed out of Dirac matrices and nucleon covariant derivatives. As an example, one can add to these operators $D^\mu D_\mu$ or $\gamma^\mu D_\mu$ terms sandwiched between $\bar{\Psi}$ and Ψ. However, these terms can be eliminated using the equations of motion, Eq.(4.75). Therefore, at LO, any operator with an extra $D^\mu D_\mu$ or $\gamma^\mu D_\mu$ can be absorbed into the terms no. 1 through 4.

So, the LO Lorentz-violating independent pion-nucleon terms are

$$\begin{aligned} &\bar{\Psi}[(a_R^{(1)} u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(1)} u\, C_{L_{\mu\nu}} u^\dagger)(\gamma^\nu i D^\mu + \gamma^\mu i D^\nu)]\Psi + h.c. \\ &[a_R^{(2)}\, \mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(2)}\, \mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}(\gamma^\nu i D^\mu + \gamma^\mu i D^\nu)]\Psi + h.c. \\ &\bar{\Psi}[(a_R^{(3)} u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(3)} u\, C_{L_{\mu\nu}} u^\dagger)(\gamma^\nu\gamma^5 i D^\mu + \gamma^\mu\gamma^5 i D^\nu)]\Psi + h.c. \\ &[a_R^{(4)}\, \mathrm{Tr}(C_{R_{\mu\nu}}) + a_L^{(4)}\, \mathrm{Tr}(C_{L_{\mu\nu}})]\bar{\Psi}(\gamma^\nu\gamma^5 i D^\mu + \gamma^\mu\gamma^5 i D^\nu)\Psi + h.c. \end{aligned} \tag{4.84}$$

where $h.c.$ stands for Hermitian conjugate.

In order to find the relation between the LECs $a_R^{(n)}$ and $a_L^{(n)}$, again we need to consider the transformation behavior of the pion-nucleon terms in Eq.(4.84) under charge conjugation. In the previous section, we worked out the transformation behavior of $\mathcal{L}^{\text{CPT-even}}_{\text{light quark}}$ under C. Please refer to section 4.1 for details. As discussed in that section, for the general case ($C^L_{\mu\nu} \neq C^R_{\mu\nu}$), the LV quark-level Lagrange density is not invariant under charge conjugation. However, if $C^L_{\mu\nu} = C^R_{\mu\nu}$, then $\mathcal{L}^{\text{CPT-even}}_{\text{light quark}}$ is even under C, whereas if $C^L_{\mu\nu} = -C^R_{\mu\nu}$, it is odd (i.e. $\mathcal{L}'^{\text{CPT-even}}_{\text{light quark}} = -\mathcal{L}^{\text{CPT-even}}_{\text{light quark}}$).
The first Lagrangian term of Eq.(4.84) above will be used as an example and shown how it transforms under C. The same method applies to the remaining terms.
First, let us denote by $\mathcal{L}_1^{\mathcal{R}}$ the term $\bar{\Psi}[(a_R^{(1)} u^\dagger\, C_{R_{\mu\nu}} u)(\gamma^\mu i D^\nu + \gamma^\nu i D^\mu)]\Psi$, and find its Hermitian conjugate, so that

$$
\begin{aligned}
\mathcal{L}_1^{\mathcal{R}} &= \bar{\Psi}[(a_R^{(1)} u^\dagger\, C_{R_{\mu\nu}} u)(\gamma^\mu i D^\nu + \gamma^\nu i D^\mu)]\Psi \\
&= \bar{\Psi} a_R^{(1)} (u^\dagger\, C_{R_{\mu\nu}} u)[\gamma^\mu i(\partial^\nu + \Gamma^\nu) + (\mu \leftrightarrow \nu)]\Psi,
\end{aligned}
\tag{4.85}
$$

and[2]

$$
(\mathcal{L}_1^{\mathcal{R}})^{(h.c.)} = -i a_R^{(1)} \bar{\Psi}(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu) u^\dagger C_{R_{\mu\nu}} u \Psi,
\tag{4.86}
$$

where $\bar{\Psi}\overleftarrow{D^\nu} \equiv \partial^\nu \bar{\Psi} - \bar{\Psi}\Gamma^\nu$. The same method applies to the left-handed part of this term, and so

$$
\mathcal{L}_1{}^{h.c.} = -i\bar{\Psi}[(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu)(a_R^{(1)} u^\dagger C_{R_{\mu\nu}} u + a_L^{(1)} u C_{L_{\mu\nu}} u^\dagger)]\Psi
\tag{4.87}
$$

is the Hermitian conjugate of

$$
\mathcal{L}_1 = i\bar{\Psi}[(a_R^{(1)} u^\dagger\, C_{R_{\mu\nu}} u + a_L^{(1)} u\, C_{L_{\mu\nu}} u^\dagger)(\gamma^\nu D^\mu + \gamma^\mu D^\nu)]\Psi.
\tag{4.88}
$$

The next step is to find out how these two terms transform under charge conjugation. Here again the right-handed part of both terms will be presented knowing that the

[2]See appendix A for details.

left-handed part is similar.

Under charge conjugation, we get[3]

$$(\mathcal{L}_1^{\mathcal{R}})^{(C)} = -ia_R^{(1)}\bar{\Psi}(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu)uC^R_{\mu\nu}u^\dagger\Psi, \tag{4.89}$$

so that

$$\mathcal{L}_1^{(C)} = -i\bar{\Psi}[(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu)(a_R^{(1)}uC^R_{\mu\nu}u^\dagger + a_L^{(1)}u^\dagger C_{L_{\mu\nu}}u)]\Psi. \tag{4.90}$$

In exactly the same way, we can show that under charge conjugation, $\mathcal{L}_1^{h.c.}$ transforms into

$$(\mathcal{L}_1^{h.c.})^{(C)} = i\bar{\Psi}[(a_R^{(1)}uC^R_{\mu\nu}u^\dagger + a_L^{(1)}u^\dagger C_{L_{\mu\nu}}u)(\gamma^\nu D^\mu + \gamma^\mu D^\nu)]\Psi. \tag{4.91}$$

Finally, we find that under C,

$$\begin{aligned} \mathcal{L}_1^{tot.} &= \mathcal{L}_1 + \mathcal{L}_1^{h.c.} \\ &= i\bar{\Psi}[(a_R^{(1)}u^\dagger\, C_{R_{\mu\nu}}u + a_L^{(1)}u\, C_{L_{\mu\nu}}u^\dagger)(\gamma^\nu D^\mu + \gamma^\mu D^\nu)]\Psi \\ &\quad - i\bar{\Psi}[(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu)(a_R^{(1)}u^\dagger C_{R_{\mu\nu}}u + a_L^{(1)}uC_{L_{\mu\nu}}u^\dagger)]\Psi \end{aligned} \tag{4.92}$$

transforms as

$$\begin{aligned} (\mathcal{L}_1^{tot.})^{(C)} &= \mathcal{L}_1^{(C)} + (\mathcal{L}_1^{h.c.})^{(C)} \\ &= -i\bar{\Psi}[(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu)(a_R^{(1)}uC^R_{\mu\nu}u^\dagger + a_L^{(1)}u^\dagger C_{L_{\mu\nu}}u)]\Psi \\ &\quad + i\bar{\Psi}[(a_R^{(1)}uC^R_{\mu\nu}u^\dagger + a_L^{(1)}u^\dagger C_{L_{\mu\nu}}u)(\gamma^\nu D^\mu + \gamma^\mu D^\nu)]\Psi. \end{aligned} \tag{4.93}$$

For the special case of $C^L_{\mu\nu} = C^R_{\mu\nu}$, $\mathcal{L}^{\text{CPT-even}}_{\text{light quark}}$ is even under C. This must also hold at the level of the effective Lagrange density. Therefore, we see that the requirement of $\mathcal{L}_1^{tot.} = \mathcal{L}_1^{tot.(C)}$ for $C^L_{\mu\nu} = C^R_{\mu\nu}$ forces $a_R^{(1)}$ to be equal to $a_L^{(1)}$.

The same method and arguments apply for the remaining pion-nucleon terms of

[3]See appendix A for details.

Eq.(4.84), and we obtain the following relations between the left and right LECs:

$$
\begin{aligned}
a_L^{(1)} &= a_R^{(1)} \\
a_L^{(2)} &= a_R^{(2)} \\
a_L^{(3)} &= -a_R^{(3)} \\
a_L^{(4)} &= -a_R^{(4)}.
\end{aligned}
\tag{4.94}
$$

Written in terms of the iso-singlet and iso-triplet parts of $C_{\mu\nu}^{L/R}$, the minimal LO Lorentz-violating baryonic Lagrange density is

$$
\begin{aligned}
\mathcal{L}_{\pi N}^{\mathrm{LO}} \; = \; & \Big\{ \alpha^{(1)} \bar{\Psi}[(u^\dagger \, {}^3C_R^{\mu\nu} u + u \, {}^3C_L^{\mu\nu} u^\dagger)(\gamma_\nu i D_\mu + \gamma_\mu i D_\nu)]\Psi \\
& + \alpha^{(2)} \left({}^1C_R^{\mu\nu} + \, {}^1C_L^{\mu\nu} \right) \bar{\Psi}(\gamma_\nu i D_\mu + \gamma_\mu i D_\nu)]\Psi \\
& + \alpha^{(3)} \bar{\Psi}[(u^\dagger \, {}^3C_R^{\mu\nu} u - u \, {}^3C_L^{\mu\nu} u^\dagger)(\gamma_\nu \gamma^5 i D_\mu + \gamma_\mu \gamma^5 i D_\nu)]\Psi \\
& + \alpha^{(4)} \left({}^1C_R^{\mu\nu} - \, {}^1C_L^{\mu\nu} \right) \bar{\Psi}(\gamma_\nu \gamma^5 i D_\mu + \gamma_\mu \gamma^5 i D_\nu)\Psi \Big\} + h.c.,
\end{aligned}
\tag{4.95}
$$

where the $\alpha^{(n)}$'s are dimensionless LECs that by naive dimensional analysis are expected to be $\mathcal{O}(1)$.

4.2.1 Local Chiral Transformations and The Modified Nucleon Covariant Derivative

Analogous to the pion case, extending the χPT formalism in the presence of LV to include interactions of nucleons with external fields forces global chiral symmetry to be promoted to a local one. This is also true in the absence of any LV. We recall the transformation behavior of the building blocks of the baryonic effective Lagrangian under the local $SU(2)_L \times SU(2)_R$ chiral group, denoted by G, in the absence of LV.

$$
\begin{pmatrix} u \\ \Psi \\ D_\mu \Psi \end{pmatrix} \longrightarrow \begin{pmatrix} u' \\ \Psi' \\ (D_\mu \Psi)' \end{pmatrix} = \begin{pmatrix} V_R u K^\dagger = K u V_L^\dagger \\ K[V_L, V_R, U]\Psi \\ K[V_L, V_R, U](D_\mu \Psi) \end{pmatrix}
\tag{4.96}
$$

where $V_L(x)$ and $V_R(x)$ are independent space-time-dependent matrices that belong to the local group G. The nucleon covariant derivative in the presence of external fields and absence of LV is

$$\begin{aligned} D_\mu\Psi &= (\partial_\mu + \Gamma_\mu)\Psi, \\ \Gamma_\mu &= \frac{1}{2}[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger]. \end{aligned} \tag{4.97}$$

We have seen that the transformations of the external fields l_μ and r_μ get modified in the presence of LV,

$$\begin{aligned} l_\mu &\longrightarrow V_L l_\mu V_L^\dagger + iV_L\partial_\mu V_L^\dagger + \frac{i}{2}[V_L C_{L_{\mu\nu}}\partial^\nu V_L^\dagger - \partial^\nu V_L C_{L_{\mu\nu}} V_L^\dagger] \\ r_\mu &\longrightarrow V_R r_\mu V_R^\dagger + iV_R\partial_\mu V_R^\dagger + \frac{i}{2}[V_R C_{R_{\mu\nu}}\partial^\nu V_R^\dagger - \partial^\nu V_R C_{R_{\mu\nu}} V_R^\dagger]. \end{aligned} \tag{4.98}$$

Since l_μ and r_μ enter the nucleon covariant derivative via the chiral connection Γ_μ, $D_\mu\Psi$ needs to be modified in the presence of LV in order to retain its transformation behavior under local transformations.

With LV and under local chiral transformations, Γ_μ transforms as

$$\Gamma_\mu \longrightarrow \Gamma'_\mu + \tilde{\Gamma}_\mu, \tag{4.99}$$

where Γ'_μ is the transform of Γ_μ in the absence of LV, and

$$\begin{aligned} \tilde{\Gamma}_\mu &= -\frac{i}{2}(Ku^\dagger V_R^\dagger)\Big[\frac{i}{2}V_R C_{R_{\mu\nu}}\partial^\nu V_R^\dagger - \frac{i}{2}\partial^\nu V_R C_{R_{\mu\nu}} V_R^\dagger\Big](V_R u K^\dagger) \\ &\quad - \frac{i}{2}(KuV_L^\dagger)\Big[\frac{i}{2}V_L C_{L_{\mu\nu}}\partial^\nu V_L^\dagger - \frac{i}{2}\partial^\nu V_L C_{L_{\mu\nu}} V_L^\dagger\Big](V_L u^\dagger K^\dagger) \\ &= \frac{1}{4}Ku^\dagger C_{R_{\mu\nu}}\partial^\nu V_R^\dagger V_R u K^\dagger - \frac{1}{4}Ku^\dagger V_R^\dagger \partial^\nu V_R C_{R_{\mu\nu}} u K^\dagger \\ &\quad + \frac{1}{4}KuC_{L_{\mu\nu}}\partial^\nu V_L^\dagger V_L u^\dagger K^\dagger - \frac{1}{4}KuV_L^\dagger\partial^\nu V_L C_{L_{\mu\nu}} u^\dagger K^\dagger. \end{aligned} \tag{4.100}$$

These four extra terms in Eq.(4.100) can be canceled out by adding four terms to Γ_μ which upon being transformed produce terms that have the desired transformation behavior and other terms that cancel the extra ones in the equation above. Let us denote the terms that need to be added to Γ_μ by $\Gamma_\mu^{(C_{\mu\nu})}$, where

$$\Gamma_\mu^{(C_{\mu\nu})} = \frac{i}{4}u^\dagger\{C^R_{\mu\nu}, r^\nu\}u + \frac{i}{4}u\{C^L_{\mu\nu}, l^\nu\}u^\dagger. \tag{4.101}$$

So,

$$\begin{aligned}\Gamma_\mu^{\rm LV} &\equiv \Gamma_\mu + \Gamma_\mu^{(C_{\mu\nu})} \\ &= \frac{1}{2}\Big[u^\dagger(\partial_\mu - ir_\mu + \frac{i}{2}\{C^R_{\mu\nu}, r^\nu\})u + u(\partial_\mu - il_\mu + \frac{i}{2}\{C^L_{\mu\nu}, l^\nu\})u^\dagger\Big],\end{aligned} \tag{4.102}$$

and

$$D_\mu^{\rm LV}\Psi = (\partial_\mu + \Gamma_\mu^{\rm LV})\Psi \tag{4.103}$$

is the modified nucleon covariant derivative that maintains the transformation behavior $K[V_L, V_R, U](D_\mu^{\rm LV}\Psi)$ in the presence of both external fields and LV.

We can then take the standard LO baryonic effective Lagrange density

$$\mathscr{L}^{(1)}_{\pi N} = \bar\Psi(i\not{D} - m + \frac{g_A}{2}\gamma^\mu\gamma_5 u_\mu)\Psi, \tag{4.104}$$

and replace $D_\mu\Psi$ by $D_\mu^{\rm LV}\Psi$. So,

$$\bar\Psi(i\not{D} - m + \frac{g_A}{2}\gamma^\mu\gamma_5 u_\mu)\Psi \to \bar\Psi(i\not{D}^{LV} - m + \frac{g_A}{2}\gamma^\mu\gamma_5 u_\mu)\Psi, \tag{4.105}$$

where

$$\bar\Psi(i\not{D}^{LV} - m + \frac{g_A}{2}\gamma^\mu\gamma_5 u_\mu)\Psi = \mathscr{L}^{(1)}_{\pi N} + \bar\Psi i\not{\Gamma}^{(C_{\mu\nu})}\Psi. \tag{4.106}$$

Therefore, we recover the standard LO baryonic effective Lagrangian, $\mathscr{L}^{(1)}_{\pi N}$, as well as generate a Lorentz-violating term of the form $\bar\Psi i\not{\Gamma}^{(C_{\mu\nu})}\Psi$ which is of the same chiral order, i.e. $\mathcal{O}(q)$.

Application: The Electromagnetic Interaction

As an application, we would like to consider the electromagnetic interaction with baryons in the presence of LV. For this purpose, we will first look at the term $\bar\Psi i\not{\Gamma}^{(C_{\mu\nu})}\Psi$ with $\Gamma_\mu^{(C_{\mu\nu})} = \frac{i}{4}u^\dagger\{C^R_{\mu\nu}, r^\nu\}u + \frac{i}{4}u\{C^L_{\mu\nu}, l^\nu\}u^\dagger$. In the case of baryons, the electromagnetic interaction is dealt with by substituting $r^\nu = l^\nu = -\frac{e}{2}\mathcal{A}^\nu(\mathbb{1} + \tau_3)$ and taking $u = u^\dagger = \mathbb{1}$, considering zero pion fields. Writing the nucleon doublet Ψ as $(p\ n)^T$ and

$$C^{R/L}_{\mu\nu} = \frac{1}{2}C^{R/L}_{+\mu\nu}\mathbb{1} + \frac{1}{2}C^{R/L}_{-\mu\nu}\tau_3, \tag{4.107}$$

with $C^{R/L}_{+\mu\nu} = c^{R/L}_{u_{\mu\nu}} + c^{R/L}_{d_{\mu\nu}}$ and $C^{R/L}_{-\mu\nu} = c^{R/L}_{u_{\mu\nu}} - c^{R/L}_{d_{\mu\nu}}$, gives the result

$$\bar{\Psi} i \Gamma^{(C_{\mu\nu})}_{\mu} \gamma^{\mu} \Psi = \frac{e}{4}(C^{R}_{+\mu\nu} + C^{L}_{+\mu\nu} + C^{R}_{-\mu\nu} + C^{L}_{-\mu\nu})\mathcal{A}^{\nu}\gamma^{\mu}\bar{p}p. \tag{4.108}$$

In our LO Lorentz-violating baryonic Lagrange density, Eq.(4.95), external fields can be accommodated by taking the chiral connection Γ_μ to be

$$\Gamma_\mu = \frac{1}{2}[u^\dagger(\partial_\mu - ir_\mu)u + u(\partial_\mu - il_\mu)u^\dagger], \tag{4.109}$$

with external right- and left-handed fields, r_μ and l_μ, respectively. The portion of the Lagrangian of Eq.(4.95) that contains interactions of baryons with external fields is

$$\begin{aligned} \mathcal{L}^{\mathrm{LO}}_{\mathrm{int.}} \;=\; & \Big\{\alpha^{(1)}\bar{\Psi}[(u^\dagger\, {}^3C^{\mu\nu}_R u + u\, {}^3C^{\mu\nu}_L u^\dagger)(\gamma_\nu i\Gamma_\mu + \gamma_\mu i\Gamma_\nu)]\Psi \\ & + \alpha^{(2)}\left({}^1C^{\mu\nu}_R + {}^1C^{\mu\nu}_L\right)\bar{\Psi}(\gamma_\nu i\Gamma_\mu + \gamma_\mu i\Gamma_\nu)]\Psi \\ & + \alpha^{(3)}\bar{\Psi}[(u^\dagger\, {}^3C^{\mu\nu}_R u - u\, {}^3C^{\mu\nu}_L u^\dagger)(\gamma_\nu\gamma^5 i\Gamma_\mu + \gamma_\mu\gamma^5 i\Gamma_\nu)]\Psi \\ & + \alpha^{(4)}\left({}^1C^{\mu\nu}_R - {}^1C^{\mu\nu}_L\right)\bar{\Psi}(\gamma_\nu\gamma^5 i\Gamma_\mu + \gamma_\mu\gamma^5 i\Gamma_\nu)\Psi\Big\} + h.c., \end{aligned} \tag{4.110}$$

where ${}^3C^{\mu\nu}_{R/L} = \frac{1}{2}C^{(R/L)\mu\nu}_{-}\tau_3$ and ${}^1C^{\mu\nu}_{R/L} = C^{(R/L)\mu\nu}_{+}$.

Here again we substitute $r_\mu = l_\mu = -\frac{e}{2}\mathcal{A}_\mu(\mathbb{1} + \tau_3)$ and take $u = u^\dagger = \mathbb{1}$ in Eq. (4.110). Therefore, the expression for the electromagnetic interaction with baryons in the presence of LV is

$$\begin{aligned} \mathcal{L}^{\mathrm{LO}}_{\mathrm{int.}} = \Big\{ & -\frac{\alpha^{(1)}}{2}e(C^{R\mu\nu}_{-} + C^{L\mu\nu}_{-})\bar{p}(\gamma_\nu\mathcal{A}_\mu + \gamma_\mu\mathcal{A}_\nu)p \\ & - \alpha^{(2)}e(C^{R\mu\nu}_{+} + C^{L\mu\nu}_{+})\bar{p}(\gamma_\nu\mathcal{A}_\mu + \gamma_\mu\mathcal{A}_\nu)p \\ & - \frac{\alpha^{(3)}}{2}e(C^{R\mu\nu}_{-} - C^{L\mu\nu}_{-})\bar{p}(\gamma_\nu\gamma^5\mathcal{A}_\mu + \gamma_\mu\gamma^5\mathcal{A}_\nu)p \\ & - \alpha^{(4)}e(C^{R\mu\nu}_{+} - C^{L\mu\nu}_{+})\bar{p}(\gamma_\nu\gamma^5\mathcal{A}_\mu + \gamma_\mu\gamma^5\mathcal{A}_\nu)p\Big\} + h.c. \end{aligned} \tag{4.111}$$

Chapter 5

Experimental Constraints

Physicists interested in the possibility of LV have widely used the mSME to search for leading-order signals of LV. Many experiments provide tests of Lorentz invariance; however, relatively few of them have the necessary sensitivity to observe possible LV signals. This is because the Lorentz-violating coupling coefficients in the SME are small. Nevertheless, a few high-precision tests were able to place bounds on some of the coefficients for LV in the SME in the different particle sectors. These include neutral-meson oscillation experiments [31, 32, 33, 34, 35, 36], QED tests [37, 38, 39, 40, 41], analyses of high-energy astrophysical processes [42, 43, 44, 45, 46] and others. An extensive summary of the results of the various tests can be found in [9].

5.1 Bounds on Pion Lorentz-violating Coefficients

We will now use the hadronic Lagrange densities constructed in chapter 4 to set new constraints on the effective LV coefficients in the pion sector. This will be done by using experimental observations from the proton/neutron sector. It is important to note that the limited number of underlying Lorentz violations on the quark level determines the effective coefficients for different types of mesons and baryons. Moreover, when a certain combination of the quark coefficients is measured or bounded using one type of hadrons, it is possible to transfer this information to an altogether different kind of particle. Bounds placed in the second particle sector are of limited precision due to the presence of numerous LECs. However, there is still order of magnitude validity assuming the LECs have natural sizes.

Let us now return to the pure pion sector and take a closer look at the pion propagation terms. The free two-pion Lorentz-violating Lagrange density,

$$\mathcal{L}_{\pi}^{\mathrm{LO},2\phi} = \frac{\beta^{(1)}}{2}(c_{u_L}^{\mu\nu} + c_{d_L}^{\mu\nu} + c_{u_R}^{\mu\nu} + c_{d_L}^{\mu\nu})\partial_\mu\phi_a\partial_\nu\phi_a, \tag{5.1}$$

written in terms of the physical pion fields, $\pi^{\pm} = \frac{1}{\sqrt{2}}(\phi_1 \pm i\phi_2)$ and $\pi^0 = \phi_3$, has the form

$$\mathcal{L}_{2\pi}^{\mathrm{LO}} = \frac{\beta^{(1)}}{2}(c_{u_L}^{\mu\nu} + c_{d_L}^{\mu\nu} + c_{u_R}^{\mu\nu} + c_{d_L}^{\mu\nu})(\partial^\mu\pi^+\partial^\nu\pi^- + \partial^\mu\pi^-\partial^\nu\pi^+ + \partial^\mu\pi^0\partial^\nu\pi^0). \tag{5.2}$$

This looks like the standard form for LV involving a spin-0 field. In the general Lagrange density [47]

$$\mathcal{L}_{\text{spin-0}} = \frac{1}{2}\partial^\mu\phi_a\partial_\mu\phi_a + \frac{1}{2}k^{\mu\nu}\partial_\mu\phi_a\partial_\nu\phi_a - \frac{m^2}{2}\phi_a\phi_a, \tag{5.3}$$

the equations of motion, or equivalently, the energy-momentum relation for free propagating particles are modified by the tensor $k^{\mu\nu}$. These propagation modifications lead to observable physical consequences. Modified energy-momentum relations may lead to the presence of upper and lower thresholds for various particle emission and decay processes. As an example, the decay of photons into charged particle-antiparticle pairs (such as $\gamma \to \pi^+ + \pi^-$) may occur, with an appropriate choice of parameters, for sufficiently energetic γ-rays. Another important process is the dominant decay mode for neutral pions ($\pi^0 \to 2\gamma$). However, the presence of LV can lead to some interesting changes in neutral pion decay. For pions with large momenta, the process $\pi^0 \to N + \bar{N}$ may become allowed, where N stands for nucleons. So, above a certain energy threshold, neutral pions can decay into nucleons rather than photons due to the strong pion-nucleon coupling.

It can be a great challenge to make very precise measurements with short-lived particles such as pions. In fact, the tightest constraints on LV in the pion sector come from analyses of high-energy astrophysical data [45, 47, 48]. The reason is that very large distances and very high energies available in extraterrestrial environments can make astrophysical tests of Lorentz symmetry extremely sensitive. Typically, astrophysical observations involving observed quanta at an energy E allow us to have constraints on combinations of $k_\pi^{\mu\nu}$ at the $\sim m_\pi^2/E^2$ level. Currently, the pion bounds are at the 10^{-10}–10^{-13} levels, which are fairly strong [47, 48]. However, these bounds are on combinations of all the $k_\pi^{\mu\nu}$ coefficients, which are determined by the sky coordinates of the sources involved. Moreover, other sectors show much stronger bounds, and there are limited possibilities for improving the direct pion bounds. Major improvements would require observations of substantially more energetic quanta, which can be sparse.

Looking at Eqs. (5.1) and (5.2), we find that common to all the physical pion fields is a single $k_\pi^{\mu\nu}$ tensor having the form

$$k_\pi^{\mu\nu} = \beta^{(1)}(c_{u_L}^{\mu\nu} + c_{u_R}^{\mu\nu} + c_{d_L}^{\mu\nu} + c_{d_R}^{\mu\nu}). \tag{5.4}$$

It is expected that the three pion types share these same LO LV coefficients since in the chiral limit, the pion wave functions all contain equal mixtures of the u and d fields, as well as equal right and left helicities.

We shall now relate the pion LV tensor, $k_\pi^{\mu\nu}$, to the combination $c_p^{\mu\nu} + c_n^{\mu\nu}$ of readily measurable baryon parameters. The mSME Lagrange density for a Dirac fermion is given by [49]

$$\mathcal{L}_{\text{spin}-\frac{1}{2}} = \bar{\psi}\left[i(\gamma^\mu + c^{\nu\mu}\gamma_\nu + d^{\nu\mu}\gamma_5\gamma_\nu)D_\mu - m\right]\psi, \tag{5.5}$$

where the Lorentz-violating kinetic terms in the SME nucleon sector effective Lagrangian are written in terms of four coefficient tensors $c_p^{\mu\nu}$, $c_n^{\mu\nu}$, $d_p^{\mu\nu}$, and $d_n^{\mu\nu}$.

Recall the LO chiral pion-nucleon effective Lagrange density constructed in chapter 4,

$$
\begin{aligned}
\mathcal{L}_{\pi N}^{\text{LO}} &= \Big\{\alpha^{(1)}\bar{\Psi}[(u^{\dagger}\,{}^{3}C_{R}^{\mu\nu}u + u\,{}^{3}C_{L}^{\mu\nu}u^{\dagger})(\gamma_{\nu}iD_{\mu}+\gamma_{\mu}iD_{\nu})]\Psi \\
&\quad +\alpha^{(2)}\left({}^{1}C_{R}^{\mu\nu} + {}^{1}C_{L}^{\mu\nu}\right)\bar{\Psi}(\gamma_{\nu}iD_{\mu}+\gamma_{\mu}iD_{\nu})]\Psi \\
&\quad +\alpha^{(3)}\bar{\Psi}[(u^{\dagger}\,{}^{3}C_{R}^{\mu\nu}u - u\,{}^{3}C_{L}^{\mu\nu}u^{\dagger})(\gamma_{\nu}\gamma^{5}iD_{\mu}+\gamma_{\mu}\gamma^{5}iD_{\nu})]\Psi \\
&\quad +\alpha^{(4)}\left({}^{1}C_{R}^{\mu\nu} - {}^{1}C_{L}^{\mu\nu}\right)\bar{\Psi}(\gamma_{\nu}\gamma^{5}iD_{\mu}+\gamma_{\mu}\gamma^{5}iD_{\nu})\Psi\Big\} + h.c.
\end{aligned}
\tag{5.6}
$$

By looking at the form of the Lagrange densities in Eqs. (5.5) and (5.6), we can see that $c_{p}^{\mu\nu}$ and $c_{n}^{\mu\nu}$ are related to the $\alpha^{(1)}$ and $\alpha^{(2)}$ terms, while $d_{p}^{\mu\nu}$ and $d_{n}^{\mu\nu}$ receive contributions from $\alpha^{(3)}$ and $\alpha^{(4)}$ terms. Writing the nucleon doublet Ψ as $(p\;n)^{T}$ and taking $u = u^{\dagger} = \mathbb{1}$, one gets

$$
\begin{aligned}
\mathcal{L}_{\pi N}^{\text{LO}} &= \Big\{\alpha^{(1)}(\bar{p}\;\bar{n})\Big[\Big(\frac{\tau_{3}}{2}(c_{u_R}^{\mu\nu} - c_{d_R}^{\mu\nu}) + \frac{\tau_{3}}{2}(c_{u_L}^{\mu\nu} - c_{d_L}^{\mu\nu})\Big)(\gamma_{\nu}iD_{\mu}+\gamma_{\mu}iD_{\nu})\Big](p\;n)^{T} \\
&\quad +\alpha^{(2)}(\bar{p}\;\bar{n})\left((c_{u_R}^{\mu\nu} + c_{d_R}^{\mu\nu}) + (c_{u_L}^{\mu\nu} + c_{d_L}^{\mu\nu})\right)(\gamma_{\nu}iD_{\mu}+\gamma_{\mu}iD_{\nu})](p\;n)^{T} \\
&\quad +.....\Big\} + h.c.
\end{aligned}
\tag{5.7}
$$

Expanding the above expression gives the following result,

$$
\begin{aligned}
\mathcal{L}_{\pi N}^{\text{LO}} &= \Big\{\alpha^{(1)}\frac{\bar{p}}{2}\Big[(c_{u_L}^{\mu\nu} + c_{u_R}^{\mu\nu}) - (c_{d_L}^{\mu\nu} + c_{d_R}^{\mu\nu})\Big](\gamma_{\nu}iD_{\mu}+\gamma_{\mu}iD_{\nu})p \\
&\quad +\alpha^{(1)}\frac{\bar{n}}{2}\Big[-(c_{u_L}^{\mu\nu} + c_{u_R}^{\mu\nu}) + (c_{d_L}^{\mu\nu} + c_{d_R}^{\mu\nu})\Big](\gamma_{\nu}iD_{\mu}+\gamma_{\mu}iD_{\nu})n \\
&\quad +\alpha^{(2)}\frac{\bar{p}}{2}\Big[(c_{u_L}^{\mu\nu} + c_{u_R}^{\mu\nu}) + (c_{d_L}^{\mu\nu} + c_{d_R}^{\mu\nu})\Big](\gamma_{\nu}iD_{\mu}+\gamma_{\mu}iD_{\nu})p \\
&\quad +\alpha^{(2)}\frac{\bar{n}}{2}\Big[(c_{u_L}^{\mu\nu} + c_{u_R}^{\mu\nu}) + (c_{d_L}^{\mu\nu} + c_{d_R}^{\mu\nu})\Big](\gamma_{\nu}iD_{\mu}+\gamma_{\mu}iD_{\nu})n \\
&\quad +.....\Big\} + h.c.
\end{aligned}
\tag{5.8}
$$

We can then read off the $c_{p}^{\mu\nu}$ and $c_{n}^{\mu\nu}$ coefficients directly from $\mathcal{L}_{\pi N}^{\text{LO}}$ as

$$
\begin{aligned}
c_{p}^{\mu\nu} &= \left[\frac{1}{2}\alpha^{(1)} + \alpha^{(2)}\right](c_{u_L}^{\mu\nu} + c_{u_R}^{\mu\nu}) + \left[-\frac{1}{2}\alpha^{(1)} + \alpha^{(2)}\right](c_{d_L}^{\mu\nu} + c_{d_R}^{\mu\nu}), \\
c_{n}^{\mu\nu} &= \left[-\frac{1}{2}\alpha^{(1)} + \alpha^{(2)}\right](c_{u_L}^{\mu\nu} + c_{u_R}^{\mu\nu}) + \left[\frac{1}{2}\alpha^{(1)} + \alpha^{(2)}\right](c_{d_L}^{\mu\nu} + c_{d_R}^{\mu\nu}).
\end{aligned}
\tag{5.9}
$$

Table 5.1 Existing constraints on LV (proton and neutron sectors) [9].

Coefficient	Proton Bound	Neutron Bound
$c_Q = c_{XX} + c_{YY} - 2c_{ZZ}$	10^{-21}	10^{-10}
$c_- = c_{XX} - c_{YY}$	10^{-24}	10^{-28}
$c_{(XY)}$	10^{-24}	10^{-29}
$c_{(XZ)}$	10^{-25}	10^{-28}
$c_{(YZ)}$	10^{-25}	10^{-28}
$c_{(TX)}$	10^{-20}	10^{-5}
$c_{(TY)}$	10^{-20}	10^{-5}
$c_{(TZ)}$	10^{-20}	10^{-5}
c_{TT}	10^{-11}	10^{-11}

The sum of these two expressions results in

$$c_p^{\mu\nu} + c_n^{\mu\nu} = 2\alpha^{(2)}(c_{u_L}^{\mu\nu} + c_{u_R}^{\mu\nu} + c_{d_L}^{\mu\nu} + c_{d_R}^{\mu\nu}), \tag{5.10}$$

which is the same combination of quark coefficients occuring in the pion $k_\pi^{\mu\nu}$. This makes it possible to place order-of-magnitude bounds on the $k_\pi^{\mu\nu}$ by combining observations made of the proton and neutron. Most measurements of LV involving protons and neutrons are done non-relativistically, typically using atomic clocks [50, 51, 52, 53, 54, 55]. The best current order of magnitude constraints for the $c_p^{\mu\nu}$ and $c_n^{\mu\nu}$ coefficients are reported in table 5.1. However, new analyses based on more careful nuclear models suggest significant improvements over some of these constraints [56, 57]. Bounds on LV are conventionally expressed in the sun-centered celestial equatorial coordinate system in which X, Y, Z, and T are the coordinates [58]. The Z-axis points along the rotation axis of the Earth, and the X-axis points to the vernal equinox point on the celestial sphere. The Y-direction is determined by the right-hand rule, and time in these coordinates is denoted by T. Table 5.1 shows four types of coefficients for which there are much stronger constraints in both the proton and neutron sectors than in the pion sector. These are the c_-, $c_{(XY)}$, $c_{(XZ)}$ and $c_{(YZ)}$ coefficients. Therefore, in table 5.2, we quote new bounds on four pion parameters. The LECs in the nucleon and meson sectors are unknown but expected to be of $\mathcal{O}(1)$. This educated

Table 5.2 New constraints on pion LV.

Coefficient	Bound
$(k_\pi)_- = (k_\pi)_{XX} - (k_\pi)_{YY}$	10^{-23}
$(k_\pi)_{(XY)}$	10^{-23}
$(k_\pi)_{(XZ)}$	10^{-24}
$(k_\pi)_{(YZ)}$	10^{-24}

assumption allows us to place order-of-magnitude bounds on the corresponding pion coefficients. Thus, we have set the pion constraints to be one order of magnitude weaker than the looser of the contributing proton and neutron bounds. For example, the $c_{(XY)}$ coefficient is bounded at 10^{-24} in the proton sector and at 10^{-29} in the nucleon sector. The bound on the pion parameter $(k_\pi)_{(XY)}$ is set to be one order of magnitude weaker than 10^{-24}. Therefore, we find that the new bound on $(k_\pi)_{(XY)}$ is at the 10^{-23} level.

Better than order-of-magnitude accuracy is not possible because of the presence of unknown LECs in all the hadronic expressions, which at the moment can only be estimated using naive dimensional analysis. Yet we are still able to make improvements of at least ten orders of magnitude over direct astrophysical constraints on the same parameters. Our analysis of the pion bounds is all based on assuming "natural" size of the LECs in the nucleon and meson sectors. In order to make an exact statement, one would need non-perturbative QCD calculations with LV.

Chapter 6

Conclusions and Outlook

Starting from a CPT-even Lorentz-violating quark-level Lagrange density with mass-dimension four operators, we constructed the LO Lorentz-violating effective Lagrangians with pion and nucleon degrees of freedom. This was done using the framework of χPT. The construction of the pion and pion-nucleon Lagrange densities first excluded external fields. In the pure pion sector, χPT provided us with a systematic way to generate multi-pion vertices with definite relations between them imposed by chiral symmetry. The work was then extended to include interactions with external fields in the presence of LV. The inclusion of LV leads to a modification of the transformation behavior of the external fields. This affected the structure of the pion and nucleon covariant derivatives which also got modified in order to retain their usual transformation behavior under the chiral group. We were also able to produce order-of-magnitude bounds on four LV coefficients that affect pion propagation, without looking directly at any pions.

Our χPT analysis was restricted to a subset of the mSME terms that are likely to affect hadrons. There are still more steps that can be taken. We have not considered any LV in the $SU(3)_c$ gauge sector which is an important omission. Pure gauge interactions could make a LO contribution to the two-index tensors such as $k_\pi^{\mu\nu}$. Therefore, a more complete analysis should include the effects of both quark and gauge LV on hadronic fields. There are also other forms of LV that may exist for hadrons. Other quark-level operators such as those with mass-dimension three will contribute in completely different ways to symmetry violations by mesons and baryons. An important future task is to study these operators in details. Moreover, the kind of analysis we applied in constructing the Lorentz-violating pion-nucleon Lagrangian may lead to an understanding of LV for spin-1 and spin-$\frac{3}{2}$ composite particles which have never really been studied in any detail.

One of the most important remaining puzzles in Lorentz-violating effective field theory is how to relate the underlying quark and gluon mSME coefficients to corresponding Lorentz-violating coefficients for hadrons. Using χPT, we were able to look at how quark-level operators get translated into pion, proton, and neutron operators. Our work in addition to other recent work [59] demonstrate the power of the χPT technique, and there remains a lot more to be understood about the use of χPT in the framework of the SME.

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Appendix A

Hermitian and Charge Conjugation Trasnformation Behavior

The Hermitian conjugate of $\mathcal{L}_1$ is calculated as follows.

$$\begin{aligned}
(\mathcal{L}_1^{\mathcal{R}})^{(h.c.)} &= a_R^{(1)}[-i(\partial^\nu\Psi)^\dagger(\gamma^\mu)^\dagger u^\dagger(C_{R_{\mu\nu}})^\dagger u\gamma^0\Psi \\
&\quad + (-i)\Psi^\dagger(\Gamma^\nu)^\dagger(\gamma^\mu)^\dagger u^\dagger(C_{R_{\mu\nu}})^\dagger u\gamma^0\Psi] + (\mu\leftrightarrow\nu) \\
&= -ia_R^{(1)}[(\partial^\nu\Psi)^\dagger(\gamma^\mu)^\dagger u^\dagger C_{R_{\mu\nu}}u\gamma^0\Psi \\
&\quad - \Psi^\dagger(\Gamma^\nu)^\dagger(\gamma^\mu)^\dagger u^\dagger C_{R_{\mu\nu}}u\gamma^0\Psi] + (\mu\leftrightarrow\nu) \\
&= -ia_R^{(1)}[\partial^\nu\bar{\Psi}\gamma^\mu u^\dagger C_{R_{\mu\nu}}u\Psi - \bar{\Psi}\Gamma^\nu\gamma^\mu u^\dagger C_{R_{\mu\nu}}u\Psi] + (\mu\leftrightarrow\nu) \\
&= -i\bar{\Psi}\overleftarrow{D^\nu}\gamma^\mu u^\dagger C_{R_{\mu\nu}}u\Psi + (\mu\leftrightarrow\nu) \\
&= -ia_R^{(1)}\bar{\Psi}(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu)u^\dagger C_{R_{\mu\nu}}u\Psi,
\end{aligned} \tag{A.1}$$

where $\bar{\Psi}\overleftarrow{D^\nu} \equiv \partial^\nu\bar{\Psi} - \bar{\Psi}\Gamma^\nu$. The same method applies to the left-handed part of this term, and so

$$\mathcal{L}_1{}^{h.c.} = -i\bar{\Psi}[(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu)(a_R^{(1)}u^\dagger C_{R_{\mu\nu}}u + a_L^{(1)}uC_{L_{\mu\nu}}u^\dagger)]\Psi \tag{A.2}$$

is the Hermitian conjugate of

$$\mathcal{L}_1 = i\bar{\Psi}[(a_R^{(1)}u^\dagger\, C_{R_{\mu\nu}}u + a_L^{(1)}u\, C_{L_{\mu\nu}}u^\dagger)(\gamma^\nu D^\mu + \gamma^\mu D^\nu)]\Psi. \tag{A.3}$$

Next, we want to find the charge conjugation behavior of $\mathcal{L}_1$. In index notation,

$$\begin{aligned}
\mathcal{L}_1^{\mathcal{R}} &= ia_R^{(1)}\bar{\Psi}_{a_\alpha}(u^\dagger)_{ab}(C_{\mu\nu}^R)_{bc}(u)_{cd}(\gamma^\mu)_{\alpha\beta}(\partial^\nu\Psi)_{d_\beta} \\
&\quad + ia_R^{(1)}\bar{\Psi}_{a_\alpha}(u^\dagger)_{ab}(C_{\mu\nu}^R)_{bc}(u)_{cd}(\gamma^\mu)_{\alpha\beta}(\Gamma^\nu)_{de}\Psi_{e_\beta} + (\mu\leftrightarrow\nu).
\end{aligned} \tag{A.4}$$

Under charge conjugation, we get

$$
\begin{aligned}
(\mathcal{L}_1^{\mathcal{R}})^{(C)} &= -ia_R^{(1)}\Psi_{a_\delta}C^{-1}_{\delta\alpha}[(u^\dagger)^T]_{ab}(C^R_{\mu\nu})_{bc}(u^T)_{cd}(\gamma^\mu)_{\alpha\beta}C_{\beta\lambda}(\partial^\nu\bar{\Psi})_{d_\lambda} \\
&\quad - ia_R^{(1)}\Psi_{a_\delta}C^{-1}_{\delta\alpha}[(u^\dagger)^T]_{ab}(C^R_{\mu\nu})_{bc}(u^T)_{cd}(\gamma^\mu)_{\alpha\beta}C_{\beta\lambda}[-(\Gamma^\nu)^T]_{de}C_{\beta\lambda}\bar{\Psi}_{e_\lambda} + (\mu\leftrightarrow\nu) \\
&= -ia_R^{(1)}(\partial^\nu\bar{\Psi})_{d_\lambda}C_{\lambda\beta}[(\gamma^\mu)^T]_{\beta\alpha}C_{\alpha\delta}u_{dc}(C^R_{\mu\nu})_{cb}(u^\dagger)_{ba}\Psi_{a_\delta} \\
&\quad - ia_R^{(1)}\bar{\Psi}_{e_\lambda}C_{\lambda\beta}[(\gamma^\mu)^T]_{\beta\alpha}C_{\alpha\delta}(-\Gamma^\nu)_{ed}u_{dc}(C^R_{\mu\nu})_{cb}(u^\dagger)_{ba}\Psi_{a_\delta} + (\mu\leftrightarrow\nu) \\
&= ia_R^{(1)}\partial^\nu\bar{\Psi}[-C(\gamma^\mu)^TC]uC^R_{\mu\nu}u^\dagger\Psi - ia_R^{(1)}\bar{\Psi}[-C(\gamma^\mu)^TC]\Gamma^\nu uC^R_{\mu\nu}u^\dagger\Psi + (\mu\leftrightarrow\nu) \\
&= -ia_R^{(1)}\partial^\nu\bar{\Psi}uC^R_{\mu\nu}u^\dagger\gamma^\mu\Psi + ia_R^{(1)}\bar{\Psi}\Gamma^\nu uC^R_{\mu\nu}u^\dagger\gamma^\mu\Psi + (\mu\leftrightarrow\nu) \\
&= -ia_R^{(1)}\bar{\Psi}(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu)uC^R_{\mu\nu}u^\dagger\Psi,
\end{aligned}
\tag{A.5}
$$

so that

$$
\mathcal{L}_1^{(C)} = -i\bar{\Psi}[(\overleftarrow{D^\nu}\gamma^\mu + \overleftarrow{D^\mu}\gamma^\nu)(a_R^{(1)}uC^R_{\mu\nu}u^\dagger + a_L^{(1)}u^\dagger C_{L_{\mu\nu}}u)]\Psi. \tag{A.6}
$$