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## On the electrovacuum solutions to the Einstein-Maxwell system in General Relativity

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ON THE ELECTROVACUUM SOLUTIONS TO THE EINSTEIN-MAXWELL SYSTEM IN  
GENERAL RELATIVITY

by

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Bachelor of Science  
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for the Degree of Master of Science in  
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Frank Avignone III, Director of Thesis

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## DEDICATION

*To the memory of my father, Nelson Posada Pedraza.*

## ACKNOWLEDGMENTS

*To the Universe, for its blessings.*

*To my mother Martha Aguirre, for her love and prayers.*

*To the Department of Physics and Astronomy at USC, for its support during these years.*

*To Prof. Frank Avignone III for his infinite patience and consideration.*

*To my friends, those who are close and those who are far, for being friends still.*

## ABSTRACT

In this masters thesis, we will present the analysis of the solution to the Einstein field equation, known as *electrovac universe*. In this model, an electric charge is located somewhere in an empty universe. An important result from this scenario, is that there is a functional relation between the electrostatic potential and the metric components, assuming a conformastat metric. On the other hand, the Einstein's equations implies that the metric function satisfy the Laplace equation. We extended this model considering the Einstein's equation with cosmological constant, which increased the complexity of the equations. The metric function satisfies now a non-linear second-order, partial differential equation. We offered solutions to this equation in 1 and 2 dimensions.

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## INTRODUCTION

In 1916 Albert Einstein published a paper, titled “The foundation of the General Theory of Relativity”[1], which summarizes his investigations of a new theory of gravitation. This magnificent achievement, inspired by the idea of reconcile his Special Theory of Relativity with the gravitational effects, was developed between 1912 to 1916, and it was surrounded by a dramatic time for Einstein, when he had to learn new techniques and push the physics ideas beyond anyone else’s at that time. After almost 100 years of the birth of general relativity, it’s still considered the Einstein’s “magnum opus”. Einstein’s theory has produced a dramatic paradigm shift in our view of the universe, which is still far from being completely understood.

In contrast to the other natural forces found in nature, as the electromagnetic, weak and strong, in general relativity’s heart lies the simple idea of considering gravity as the curvature of *spacetime*. The newtonian picture of a gravitational force given by  $\mathbf{F} = \frac{GmM}{r^2}\mathbf{e}_{(\mathbf{r})}$ , is now replaced by a geometric effect of the spacetime fabric, determined by the energy-momentum content. About this idea, J. Wheeler wrote: “*space acts on matter, telling it how to move. In turn, matter reacts back on space, telling it how to curve*”[2]. Inspired by this idea, Einstein proposed a new set of field equations, in analogy to the classical Poisson’s equation  $\nabla^2\phi = 4\pi G\rho$ , to determine the field given the sources. The Einstein’s equation in its final form reads:

$$R_{\mu\nu} - \frac{1}{2}(R - 2\Lambda)g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

The left side of this equation, describes the geometry of the spacetime which is ‘encoded’ in the metric tensor  $g_{\mu\nu}$ . The right side of the equation involves the *energy-*

*momentum tensor*  $T_{\mu\nu}$  which describes the content of matter and momentum. The Einstein equation represents in a beautiful way the relationship between gravitational field (geometry) and matter; it's in the same spirit as the Maxwell equations which portray electric and magnetic fields, in terms of charge densities and currents.

A few months after GR was published, K. Schwarzschild found a solution to the Einstein equation, which describes the gravitational field in the neighborhood of a spherical mass. In 1918, G. Nordström, and H. Reissner (independently) found a class of exact solutions to the Einstein equation for the gravitational field of a spherical charged mass. Other important solutions to Einstein's equation are: the Kerr metric which describes the gravitational field of a rotating mass, and the Friedmann-Lemaitre-Robertson-Walker metric which portrays an homogeneous and isotropic universe. One interesting class of solutions are known as *Statical Universes* or *comformastat metric*[6]. I will devote the first part of Chapter 4 to discuss this type of solution. One can study the most general case of a comformastat metric, considering a charged mass distribution, which Synge called *electrovac universes*[6, 7, 8, 9]. We will see that in this model, the potential satisfies the Laplace equation. I will show the extension of these results considering the cosmological constant. We will find that the complexity increases formidably, and the new equation for the potential turns out to be a non-homogeneous differential equation. I offer some solutions to this new equation.[10]

The general overview of this masters thesis is as follows: In Chapter 1, I will summarize conceptually and historically the most important aspects of general relativity. In Chapters 2 and 3, I will introduce the mathematical machinery and its application to the Einstein's theory. In the last Chapter, I will introduce the formalism of the *comformastat metric* and the *electrovac universes*. Finally, the extension of this model considering the  $\Lambda$  term will be discussed.

# CHAPTER 1

## A VERY BRIEF HISTORY OF GRAVITY

*One day in the year 1666 Newton had gone to the country,  
and seeing the fall of an apple, as his niece told me, let himself  
be led into a deep meditation on the cause which thus  
draws every object along a line whose extension would pass  
almost through the center of the Earth*

Voltaire (1738)

The major conceptual and historical aspects of the gravitational theory are introduced. The idea that gravity is just the curvature of the *spacetime* fabric is discussed. Some major equations are briefly discussed, but the technical discussion will be given in Chapters 2 and 3 where the formalism is introduced.

### 1.1 THE MARRIAGE BETWEEN INERTIA AND GRAVITATION

In 1687 Isaac Newton published his *Mathematical Principles of Natural Philosophy* (See Fig. 1.1), which can be considered as the most important and influential work in the history of Physics. In this masterpiece, Newton established the laws of motion for physical bodies. At the end of the *Principia*, Newton described gravity as a force that acts on rocks, the sun and planets, in the following way: “*according to the quantity of solid matter which they contain and propagates on all sides to immense distances, decreasing always as the inverse square distances*”[11]. In mathematical terms, the Newton’s gravitational theory can be written as

$$\mathbf{F} = \frac{GmM}{r^2}\mathbf{e}_{(r)} \quad (1.1)$$

where  $r$  is the separation distance between the masses, and  $G = 6.67 \times 10^{-11} m^3/kg s^2$ . As far as is known, most of the physical phenomena, can be represented in terms of the 4 fundamental forces: *strong and weak nuclear interactions, electromagnetic and gravitational*. Of these forces, the gravitational is the weakest (The ratio  $Gm^2/e^2$  between gravitational and electric forces for two electrons is around  $10^{-40}$ ). Despite the fact of being the weakest, *gravity* shapes the large scale structure of the universe as we see it today. The nuclear forces are of short range ( $\sim 10^{-13}cm$ ), and although electromagnetism is a long range interaction (mathematically it has the same form as equation (1.1)), the balance between repulsive and attractive electrical forces is predominant in celestial bodies[12]. The last sentence, is just a manner to say that celestial bodies like planets, stars, etc., are neutral electrically.

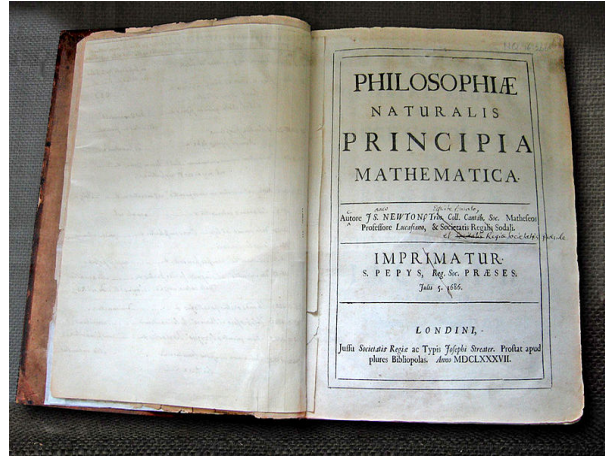


Figure 1.1 Sir Isaac Newton's own first copy of his *Principia*. Photograph: Andrew Dunn (2004)

One peculiar characteristic of the gravitational force (1.1) is its appearance as always being attractive (This is in contrast with the electric force, which can be repulsive or attractive). According to Newton's second law, this force will produce an acceleration given by

$$\mathbf{F} = m\mathbf{a} \tag{1.2}$$

where  $m$  is associated with the *inertial* mass of the body. Instead of working in terms of forces, we can express the gravity Newton's law in terms of the gravitational potential  $\phi$ , following the Poisson's equation

$$\nabla^2\phi = 4\pi G\rho \tag{1.3}$$

where  $\rho$  is the mass density distribution. Despite the fact that it was Newton who gave the final form to the gravitational theory, it was Galileo Galilei who previously realized that bodies of different mass in “free fall”, take the same time to cover the same distance given the same initial conditions. About this Galileo wrote[2]:

*The variation of speed in air between balls of gold, lead, copper, porphyry, and other heavy materials is so slight that in a fall of 100 cubits[about 46 meters] a ball of gold would surely not outstrip one of copper by as much as four fingers. Having observed this, I came to the conclusion that in a medium totally void of resistance all bodies would fall with the same speed”*

However, we can ask ourselves: Is the *inertial* mass entering in the second Newton's law, the same as the *gravitational* mass that appears in the gravitational law?. Newton thought that these masses were not likely the same. If we write the gravitational law as

$$\mathbf{F}_g = m\mathbf{g} \tag{1.4}$$

where  $\mathbf{g}$  is the gravitational field that depends on the position and the mass of the particle. Comparing (1.4) with (1.2) we have

$$\mathbf{a} = \left( \frac{m_g}{m_i} \right) \mathbf{g} \quad (1.5)$$

which states that the acceleration at some point, will be different depending of the ratio  $(m_g/m_i)$ . Newton developed several experiments with pendulums of the same length but different materials, but he did not find any significant difference in their periods[11]. It wasn't until 1889, that a more authoritative experimental test, proposed and successfully performed by Eötvös, removed any shadow of doubt about the equivalence between inertial and gravitational mass. In the following lines, the general ideas of the experiment will be discussed.<sup>1</sup>.

In his experiment, Eötvös placed two masses A and B, hanging of a very thin wire at the center of a 40cm bar (See Fig. 1.2). In the equilibrium, the system satisfies

$$l_A(m_{gA}g - m_{iA}g'_z) = l_B(m_{gB}g - m_{iB}g'_z) \quad (1.6)$$

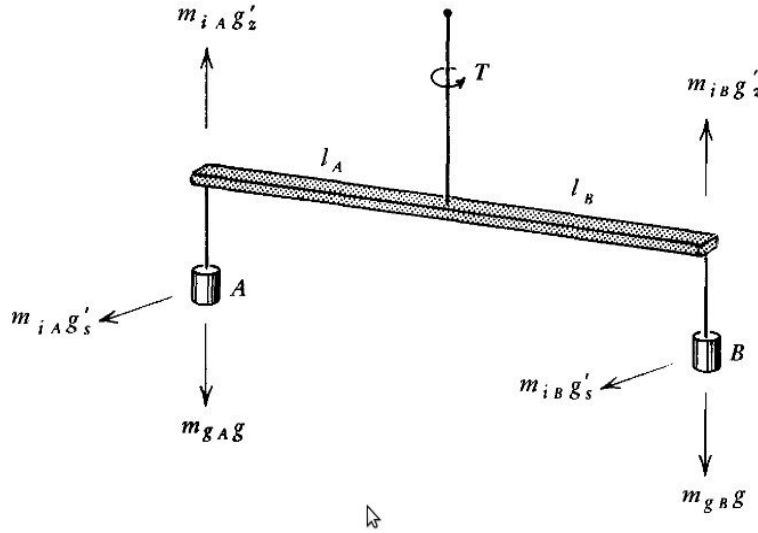


Figure 1.2 Schematic view of the Eötvös experiment[11]

where  $g$  is the local gravitational acceleration. The lab is a rotating frame (it's rotating with the Earth), therefore we will have a centripetal acceleration, where  $g'_z$

---

<sup>1</sup>I am following the very neat explanation given by Weinberg[11]

corresponds to its vertical component. Clearly, in the north hemisphere, the centripetal acceleration will have an appreciable horizontal component, denoted by  $g'_s$ . The total torque around the thin wire (vertical axis) is

$$\tau_y = l_A m_{iA} g'_s - l_B m_{iB} g'_s \quad (1.7)$$

Using the equilibrium condition (1.6) in (1.7) we have

$$\tau_y = l_A m_{iA} g'_s \left[ 1 - \left( \frac{m_{gA}}{m_{iA}} g - g'_z \right) \left( \frac{m_{Bg}}{m_{iB}} g - g'_z \right)^{-1} \right] \quad (1.8)$$

expanding the second term in the bracket, under the condition  $g'_z \ll g$  we have

$$\left( \frac{m_{Bg}}{m_{iB}} g - g'_z \right)^{-1} \approx \frac{m_{iB}}{m_{gB}} \left( \frac{1}{g} + \frac{g'_z m_{iB}}{g^2 m_{gB}} \right)$$

using the last result in (1.8), and simplifying we have

$$\tau = l_A m_{iA} g'_s \left[ \frac{m_{iA}}{m_{gA}} - \frac{m_{iB}}{m_{gB}} \right] \quad (1.9)$$

Any difference in the ratio  $m_i/m_g$  should produce a torque and therefore a twisting in the wire. Eötvös found no twist, and his results showed that the difference for  $m_i/m_g$  for wood and platinum was less than  $10^{-9}$ .

More recent experiments, based on Eötvös idea<sup>2</sup>, aimed to put stringent limits to the average quantity

$$\eta \equiv \frac{\left( \frac{m_{gA}}{m_{iA}} - \frac{m_{gB}}{m_{iB}} \right)}{\frac{1}{2} \left( \frac{m_{gA}}{m_{iA}} + \frac{m_{gB}}{m_{iB}} \right)} \quad (1.10)$$

The more stringent limits were published by Su et.al.[13]. In their experiment, using masses of Beryllium and Copper, they bounded  $\eta$  in the value

$$\eta = (-0.2 \pm 2.8) \times 10^{-12}$$

---

<sup>2</sup>Such experiments are called Eötvös experiments

This authoritative result, stands the equality of gravitational and inertial mass, as one of the fundamental principles of Physics. Taking as his flag the very accurate results by Eötvös about the equality of inertial and gravitational mass, Einstein realized a deep connection between *inertia* and *gravitation*. About this he wrote[18]:

*The assumption of the complete physical equivalence of the systems of co-ordinates,  $K$  (inertial system) and  $K'$  (uniformly accelerated respect to  $K$ ), we call the “principle of equivalence”; this principle is evidently intimately connected with the theorem of the equality between the inert and the gravitational mass, and signifies an extension of the principle of relativity to co-ordinate systems which are in non-uniform motion relatively to each other. In fact, through this conception we arrive at the unity of the nature of inertia and gravitation*

With these ideas in mind, Einstein was ready to jump formally into the Equivalence principle and the beginning of a new theory of gravitation. Before embedding us in that discussion, let us make a brief review of the principle of relativity and the Mach’s ideas about space, which influenced (partially) the Einstein’s theory.

## 1.2 NEWTON VS MACH, AND A ROTATING BUCKET

In Newtonian mechanics we study the motion of particles defining *inertial reference frames*, which can be defined as: *reference systems where  $\mathbf{F} = \frac{d\mathbf{p}}{dt}$  is valid*. The relations between inertial reference frames is determined by the *Galileo Group*

$$\begin{aligned}\mathbf{x}' &= R\mathbf{x} + \mathbf{v}t + \mathbf{d} \\ t' &= t + \tau\end{aligned}\tag{1.11}$$

This is a group of 10 parameters: 3 Euler angles, 3 components for  $\mathbf{v}$ , 3 components for  $\mathbf{d}$ , and the time  $t$ . The invariance of the laws of motion under transformation of



the Galileo group, is called the *Principle of Galilean Relativity*. For example, the Newton's gravitational law (1.1) is invariant under these transformations.

Let us analyze the following situation: an observer on the Earth's surface, might say that she is at rest. She makes experiments and finds that Newton's laws are satisfied. She finds that the Sun, the moon and the stars are moving relative to her. She is the center of the universe!. However, an observer standing on the Sun (if something like that was possible), will notice that us, the rest of the planets, and the distant stars are moving relative to her. She says: I am the one who is at rest. Now we put a third observer in the center of the galaxy (assuming that she has not crossed the event horizon of the central black hole), and she will notice that all the stars, planets and components of the galaxy are going around her. She is convinced that she is the only one at rest. But it comes out that our galaxy is also moving around the great attractor in the galaxy cluster, and everything is going farther away because the universe is expanding.

We rise the question: Can we find an *absolute* reference frame, which allows us to define other inertial frames?. The first one in giving an answer to this question was Newton, who said that there is an absolute space, and respect to this, all reference frames can be determined. In his own words[14]:

*Absolute space, in its own nature and with regard to anything external, always remain similar and unmovable. Relative space is some movable dimension or measure of absolute space, which our senses determine by its position with respect to other bodies, and is commonly taken for absolute space.*

In order to prove his concept, Newton analyzed the following experiment. Let us have a bucket hanging by a long rope. We start twisting the bucket until the rope is strongly twisted. Now, we fill the bucket with water. We release the bucket

and the rope starts to unwrap producing a rotation. There are 3 main stages of this experiment (See Fig. 1.2):

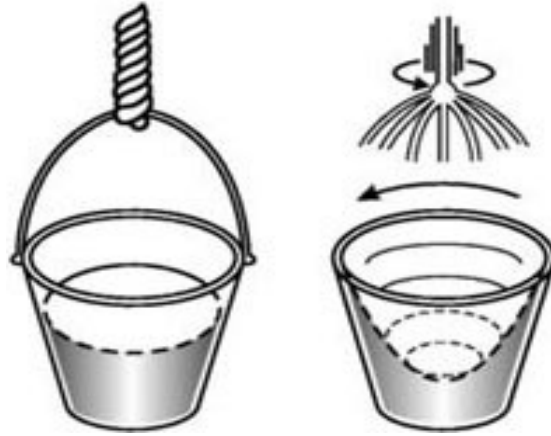


Figure 1.3 Newton's bucket experiment[15]

1. Initially, the bucket is rotating but the level of the water keeps its original level.
2. Gradually the bucket transmits its motion to the water, which starts to lift up by the walls of the bucket.
3. The water increases its rotation, receding from the axis and forming a concave shape.

How can this experiment show the existence of an absolute space?. About this Newton wrote[14]:

*At first, when the relative motion of the water in the vessel was greatest, that motion produced no tendency whatever of recession from the axis, the water made no endeavor to move upwards towards the circumference by rising at the sides of the vessel, but remained level, and for that reason its true circular motion had not yet begun. But afterwards, when the relative motion of the water had decreased, the rising of the water at the sides of*

*the vessel indicated an endeavor to recede from the axis; and this endeavor reveals the real circular motion of the water, continually increasing till it had reached its greatest point, when relatively the water was at rest in the vessel...*

This idea of absolute space, was strongly rejected by G. Leibniz, who argued that the idea of space only makes sense in terms of relative motion between bodies. This problem created a debate between the finest thinkers of the time, like Euler, Kant and Berkeley. However, it was until 1880 when E. Mach gave a serious critic of the Newton's space conception. In his book *Die Mechanik in ihrer Entwicklung*[16] Mach wrote:

*Newton's experiment with the rotating vessel of water simply informs us, that the relative rotation of the water with respect to the sides of the vessel produces no noticeable centrifugal forces, but that such forces are produced by its relative motion with respect to the mass of the Earth and the other celestial bodies. No one is competent to say how the experiment would turn out if the sides of the vessel increased in thickness and mass until they were several leagues thick.*

The postulate that the inertial properties of a body are determined by the mass distribution in the universe, is called the *Mach's Principle*. Einstein was very impressed with Mach's ideas, and he tried to incorporate them in his theory of gravitation. However, it turned out to be, that Mach's principle did not get a full representation in the general theory of relativity. Although the spacetime geometry is affected by the mass content, there are no *boundary conditions* well established which would allow to introduce the Mach's ideas. Let us suppose we have an experimenter in a small lab, and we have removed all matter from the universe. The lab is small, such that we can neglect its effect on the spacetime, therefore we can approximate the

situation as a Lorentzian reference frame. Whatever experiment she does, the physics laws will have the special relativity form. Now, she open the window and stars firing a bazooka tangentially. According to general relativity, a gyroscope inside the lab would be pointing relatively fixed towards the receding bullet. It seems to be that the “small” distant bullet affects more importantly the dynamics of the gyroscope, than the walls and mass of the lab and the experimenter. This conception looks more like an absolute space in the Newton spirit, than a relative space *a la* Mach.

Another limitation of the general relativity incorporating the Mach’s principle is related to the motion of a particle in a spherically symmetric gravitational field. As will be discussed in Chapter 3, this field is described by the Schwarzschild metric. However, the dynamics of a test particle moving under this field, is only determined by the mass that is producing the field, but the effect of the rest of masses in the universe is not considered. In 1961, Brans and Dicke[17] formulated a theory of gravitation that incorporated the Mach’s principle (partially). Despite the fact that their theory was conceptually more consistent with Mach’s ideas, it did not get support of experimental results.

### 1.3 EQUIVALENCE PRINCIPLE & THE EINSTEIN’S “GLÜCKLICHSTE GEDANKE”

In Section 1.1 we discussed the intimate relation between *inertia* and *gravitation*, as a consequence of the equality between gravitational and inertial mass:

$$m_i = m_g$$

The transition to establish the *equivalence principle* was immediately realized by Einstein, who wrote[19]:

*There then occurred to me the ‘glücklichste Gedanke meines Lebens’, the happiest thought of my life, in the following form. The gravitational field*

*has only a relative existence.... Because for an observer falling freely from the roof of a house there exists-at least in his immediate surroundings-no gravitational field. Indeed, if the observer drops some bodies then these remain relative to him in a state of rest or uniform motion, independent of their particular chemical or physical nature (in this consideration air resistance is, of course, ignored). The observer has the right to interpret his state as ‘at rest’.*

This means that we can “turn off” the gravitational effects *locally*, using a suitable accelerated frame of reference. Let us discuss a “gerdanke” experiment in the way as Einstein taught us. Suppose there is an astronaut (let’s call her Alice) inside a space ship in orbit around the earth. Alice is “weightless”. Tools, cups, books, remain at rest or moving in uniform motion with respect to them and the walls of the ship<sup>3</sup>. In principle, Alice can’t say if she is falling freely in a *uniform* gravitational field, or whether she is at rest in a *local* region far from any gravitational field. In this situation, Alice has *locally* “removed” the gravitational field (See Fig. 1.4).

But the equality of inertial and gravitational mass, provides more consequences. Suppose we bring back Alice (and her ship) to Earth. We ask her to develop some experiments to measure the local gravitational field  $\mathbf{g}$ . Alice finds that if she drops a book and a cup, they will fall to the floor of the ship with the same acceleration  $\mathbf{g}$ . Now, let’s return Alice to her initial orbit position. Let’s suppose now that we stick a hook in the top of the ship. Then we come in a bigger ship with a rope which we hang to the hook and we start to accelerate *upwards* at acceleration  $\mathbf{g}$ . Now we ask Alice to develop the same experiments again, and she finds that the book and the cup fall towards the ship floor with the same acceleration  $\mathbf{g}$ .

With this “thought experiment” in our mind, we can express the *Weak Equivalence Principle* (WEP) in the following form: *The motion of freely-falling particles are*

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<sup>3</sup>I ignore the air resistance as Einstein did



Figure 1.4 Crew inside the Zero G plane. The plane is descending to  $45^\circ$  low noise, providing the “no gravity” environment for a short time (around 15 seconds). Following Einstein’s idea, they have removed the gravitational field *locally*. They stay at rest relative to each other and the walls of the plane. Photograph: Joe McNally

*the same in a gravitational field and a uniformly accelerated frame, in local regions of spacetime*[5]. By *local* regions we mean, regions which are small enough such that deviations in the uniformity of the gravitational field can’t be detected. In our experiment, if we set a very large ship and we let it fall freely, Alice will find that the gravitational field varies depending on the position, and bodies will follow the line connecting their positions to the center of the Earth, which in different positions will be different.

But not everything in the universe is gravity<sup>4</sup>. What happens to the Electromagnetism laws, the hydrodynamic equations, the nuclear interactions and the rest of the laws of physics near to a gravitational field?. Well, the answer is given by the *Einstein’s Equivalence Principle* (EEP): *In any and every local Lorentz frame, anywhere and anytime in the universe, all the (nongravitational) laws of Physics must take on their familiar special relativistic forms.*[2] In other words, there is no way to

---

<sup>4</sup>Einstein and Rosen[20] proposed the idea of expressing an atomistic and electromagnetic theory of matter, using only the spacetime geometry determined by  $g_{\mu\nu}$  and the electromagnetic potential  $\phi$ . These ideas led to the concept Einstein-Rosen bridge. Despite the great interest that this model awakes, I leave it out of my discussion.

make a distinction between a Lorentz local frame in some infinitesimal region, from a different Lorentz local frame in any other region of spacetime. This is the *strongest* form of the Einstein's equivalence principle, and over it, rests the foundations of the general relativity.

The power of the EEP allows us to generalize any equation valid in a Lorentz flat spacetime, to a *curved* spacetime just by the rule: *comma (partial derivative, flat spacetime gradient) goes to semi-colon (covariant derivative, curved spacetime gradient)*[2]. For example, a particle moving in absence of external forces will move, according to the inertia principle, uniformly in a *straight* line<sup>5</sup>. Einstein realized that the simplest extension of the equation of motion for the general relativity spacetime, is the *geodesic* equation. In Einstein's words[18]:

*The natural, that is, the simplest, generalization of the straight line which is plausible in the system of concepts of Riemann's general theory of invariants is that of the straightest, or geodesic, line.*

Following the EEP, the motion of a particle is governed by the equation

$$\boxed{\frac{d^2 x_\mu}{ds^2} + \Gamma_{\alpha\beta}^\mu \frac{dx_\alpha}{ds} \frac{dx_\beta}{ds} = 0} \quad (1.12)$$

where  $\Gamma_{\alpha\beta}^\mu$  is called the *Christoffel symbol* which depends of first-order derivatives of the components of the metric tensor  $g_{\mu\nu}$ . If  $\Gamma_{\alpha\beta}^\mu = 0$  (a metric tensor constant for example), (1.12) reduces to the Newtonian equation of motion

$$\frac{d^2 x_\mu}{ds^2} = 0 \quad (1.13)$$

Note in this example the beautiful connection provided by the EEP through the rule “comma goes to semi-colon”. To go from (1.13) (flat spacetime) to (1.12) (curved

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<sup>5</sup>In the Lorentzian spacetime of special relativity, it means an euclidean straight line.

spacetime) we just exchange the standard derivative by the *covariant* derivative, which involves the  $\Gamma_{\alpha\beta}^{\mu}$ . That’s why (1.12) has the form that it has.

There is a distinction between *gravitational* and *non-gravitational* effects. The EEP excludes the gravitational interaction, therefore we can go just one-step further, and establish the *Strong Equivalence Principle* to include gravitational and any other interaction. The EEP implies that gravity can’t be “screened” globally, it will be always there. Is worth to emphasize here that the “weightless” condition, or “remove” gravity, is valid only in a local Lorentz frame. However, there is no a universal frame that can “remove” the Earth’s gravitational field, everywhere and everytime. This is what makes the gravitational interaction so special. In electromagnetism for instance we can screen the electromagnetic fields (we can use a Faraday cage for example), and the nuclear forces are only of short range (we don’t feel them, but we are still made of atoms).

The implications of this are astonishing. Thinking about acceleration due to gravity is non-sense in the context of general relativity. It makes more sense to think that a “freely falling” particle is *unaccelerated*. For instance, Alice sitting on a chair over the top of the Empire State building, is more “accelerated” than Bob who decided to jump from the top in “free fall”. The Einstein’s genius moment (not the only one of course) was to embrace this idea and realize that gravitation can’t be described as a force in the newtonian sense, but just in terms of the *spacetime geometry*.

#### 1.4 FROM NEWTON’S APPLE TO EINSTEIN’S CURVED SPACETIME

The set of equations (1.12) represents the marriage between inertia and gravitation. Despite the fact that each member separately can’t be considered as a tensor quantity, the whole expression transforms as a tensor. Establishing the analogy with the Newtonian picture, the first term can be associated to the inertia and the second one can be associated to gravitation[18].



The next step in his road towards a formal theory of gravitation, Einstein attempted to find the set of equations that govern the gravitational interaction. He was inspired by the Newtonian gravity in terms of the Poisson's equation

$$\nabla^2\phi = 4\pi G\rho \quad (1.14)$$

where  $\rho$  is the matter density. The left side of (1.14) gives information about the gravitational field, the right side describes the mass distribution. This equation expresses the idea that the matter density  $\rho$  produces the gravitational field. In order to generalize this relation in his new theory of gravitation, Einstein wrote[18]:

*We must next attempt to find the laws of the gravitational field. For this purpose, Poisson's equation  $\Delta\phi = 4\pi k\rho$  of the Newtonian theory must serve as a model. This equation has its foundation in the idea that gravitational field arises from the density  $\rho$  of ponderable matter. It must also be so in the general theory of relativity.*

In his quest, Einstein realized that he needed *tensor* equations, such that the covariance principle was satisfied. As will be discussed in Chapter 3, the generalization of the mass density  $\rho$  is the energy-momentum tensor  $T_{\mu\nu}$  of second rank, which will be known provisionally. In special relativity, this tensor must satisfy the divergenceless condition  $\partial_\mu T_{\mu\nu} = 0$ . In general relativity the co-variance of the equation must be accepted. If we denote by  $\tau_{\mu\nu}$  the mixed tensor density, our generalization takes the form[18]

$$0 = \frac{\partial \tau_\sigma^\alpha}{\partial x_\alpha} - \Gamma_{\sigma\beta}^\alpha \tau_\alpha^\beta \quad (1.15)$$

In general relativity it is not correct to discuss energy-momentum conservation for matter only. There is also an energy density for the gravitational field, which is expressed in the second term of (1.15). In Einstein's words: "*the gravitational field*

*transfers energy and momentum to the matter, in that it exerts forces upon it and gives it energy.*[18]

So we have the analogue to the right side of the Poisson's equation, but what about the left side?. Einstein realized that this must be a tensor equation for the metric tensor  $g_{\mu\nu}$ , which describes the geometry of the spacetime. Einstein imposed 3 conditions that this tensor must satisfy[18]:

1. It should not involve second order coefficients of the  $g_{\mu\nu}$ .
2. It must be linear and homogeneous in the second derivatives of the  $g_{\mu\nu}$ .
3. Its covariant derivative must vanish identically.

Note that the first two conditions are just a consequence of the analogy with Poisson's equation (1.14). Following these requirements, Einstein proposed the following equation for the gravitational field:

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = -kT_{\mu\nu}} \quad (1.16)$$

where  $R_{\mu\nu}$  is the Ricci tensor,  $R$  is the Ricci scalar (or curvature scalar),  $g_{\mu\nu}$  is the metric tensor,  $T_{\mu\nu}$  is the energy-momentum tensor and  $k = \frac{8\pi G}{c^4}$ . Equation (1.16) is called the *Einstein equation*, and it can be considered as the most beautiful equation in Physics<sup>6</sup>. In the Einstein equation are condensed the revolutionary ideas of space-time of general relativity, which according to P.A.M. Dirac can be considered as '*the greatest scientific discovery that ever was made*'. [21]

Note that this equation is a *postulate* based on physics arguments and the correspondence with the Newtonian mechanics. This is the trend of thought in physics as I see it. The physicists propose some fundamental equation to describe some phenomena; then the validity of the theory will be determined by the concordance with

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<sup>6</sup>This is my personal point of view.

*experimental results.* Again, it should be said that the Einstein equation is a postulate, as it is the Schrödinger equation and Dirac equation. This equation is not expected to be “derived” from some first principles as was suggested and elaborated by D. Hilbert. There is a controversy related to some historical issue, about who was the first in writing the field equations. Before discussing that historical problem, let us discuss another more immediate one.

After writing the gravitational equations in *vacuum*:  $R_{\mu\nu} = 0$ , Einstein realized that this equation is not the more general that satisfies the 3 requirements discussed above. About this Einstein wrote[3]:

*Properly speaking, this (divergenceless condition) can be affirmed only of the tensor:  $G_{\mu\nu} + \lambda g_{\mu\nu} g^{\alpha\beta} G_{\alpha\beta}$  where  $\lambda$  is a constant. If, however, we set this tensor = 0, we come back again to the equation  $G_{\mu\nu} = 0$ ”*

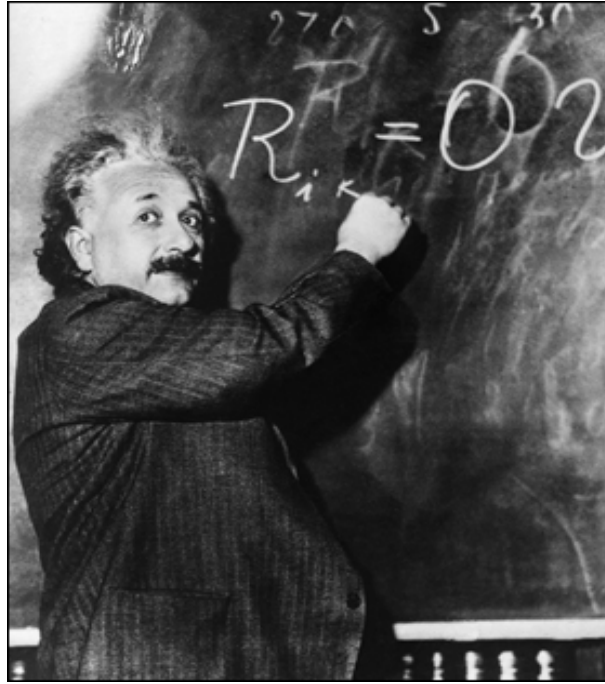


Figure 1.5 A. Einstein writing the field equation for vacuum[24]

where  $G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$  is the Einstein tensor. Clearly the addition of this constant  $\lambda$  preserves the requirements of the theory. In fact, written in this way the

equation represents the most general set of equations for the gravitational field. Is quite interesting to note that Einstein mentioned this constant  $\lambda$ , before discussing any idea about cosmology. However, later on, Einstein introduced this constant again in order to keep a static universe as was “suggested” by the still primitive observations in that time which were in contradiction with his equations which *predicted* an expanding universe. Sadly the  $\lambda$ -term<sup>7</sup> has suffered an unfair discrimination by the relativistic community, as a consequence of the unfortunate quote by Einstein as his “greatest blunder”. However, clearly the term is a *natural* addition to the most general field equation. A fascinating discussion of this historical problem, can be found in reference[4]. More recently, after the discovery of the *accelerated* expansion of the universe[22],  $\Lambda$  resurrected[23] as the term associated to the mysterious “dark energy”, which is driving this accelerated expansion. However, the nature of  $\Lambda$  is still a mystery.

As I mentioned before, there was a controversy about who was the first in writing the field equation. The notable German mathematician D. Hilbert, during the first World War was completely absorbed in the physics problems of the time[21]. By 1914, Hilbert was fascinated by the ideas of Einstein and G. Mie (Mie was working in a theory of gravitation and electromagnetism). Hilbert invited Einstein to Göttingen to present his ideas about the theory of relativity. In a letter to A. Sommerfeld, dated 15 July 1915, Einstein wrote[21]:

*I had the great joy of seeing in Göttingen that everything (about the theory of relativity) is understood to the last detail. With Hilbert I am just enraptured. An important man!*

On 4, 11, 18 and 25 November 1915, Einstein presented a series of communications on general relativity to the Prussian Academy. By the same time, on 20 November

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<sup>7</sup>In more recent literature, the capital letter  $\Lambda$  is used

1915, Hilbert presented a derivation of Einstein equation to the Royal Academy of Sciences in Göttingen. Hilbert's communications appeared on the third issue of the Proceedings of the Göttingen Academy for 1915, and in his publication he referred to all communications of November 1915 by Einstein. The Hilbert's approach was based on variational principles, starting from the action:

$$S_H = \int \sqrt{g}(R + L)d^4x \quad (1.17)$$

where  $R = g^{\mu\nu}R_{\mu\nu}$  and  $L$  is a function of the metric tensor  $g^{\mu\nu}$  and the generalized coordinates  $q_s, q_{sk}$ [25]. From his communication of November 20, there is not a shadow of doubt that Hilbert found the same equation independently. In fact, only until November 25, Einstein gave the final form of the gravitational equation. In that sense, Hilbert was ahead of Einstein by 5 days!. About this Hilbert wrote[21]:

*It seems to me that the differential equations of gravitation so realized (by me) are in agreement with the beautiful theory of general relativity proposed by Einstein in his later (25 November 1915) memoir.*

Hilbert's approach was more a formal derivation, in contrast with Einstein's who wrote the field equation as a *postulate* inspired in the equivalence principle, the covariance of the theory and the analogy with the Newtonian mechanics. Despite the fact that Hilbert's approach could be considered more "elegant" from the mathematical point of view, were the Einstein's ideas about spacetime, and his insight, which gave the physics and philosophical foundation to the theory. Far away from the "controversy" (see reference[25] for example), the Einstein's name is the one associated with the general theory of relativity.

The consequences of general relativity have been astonishing: black holes, gravitational waves, gravitational Doppler redshift, neutron stars, among others. One

example is the prediction of the bending of light due to the presence of a mass (See Fig. 1.6).

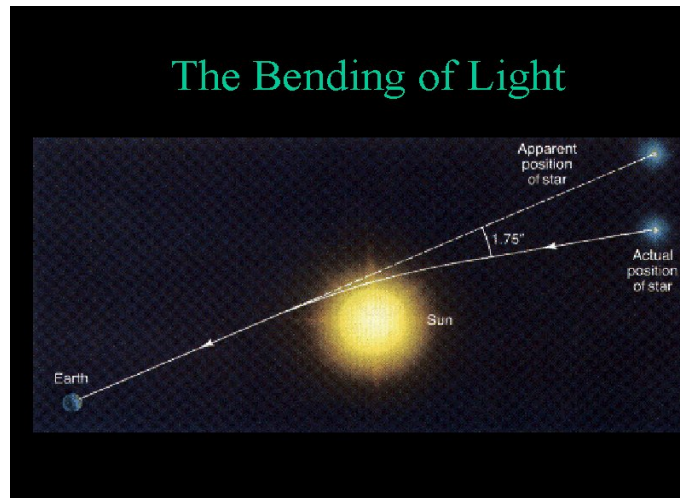


Figure 1.6 A schematic diagram of the bending of light. The spacetime in the vicinity of the sun is curved, causing the light to bend. This bending produces an apparent position of the star, different to the actual one.[27]

The key idea in general relativity is that mass *curves* spacetime. So we can imagine that a ray of light emitted from a distant star, when passing near the neighborhood of a mass like our sun, it will find a curved spacetime (the bigger the mass, the bigger the curvature) and it will follow the geodesic line in that region, which in this case deviates of an Euclidean straight line. In fact, before the full theory was completed, Einstein derived this result and found a formula to calculate this deviation, using only the equivalence principle. However, his prediction was off by  $(1/2)$ . With the field equation in its final form, the calculation was corrected.

In 1919 the British astronomer sir A. Eddington, lead an expedition to observe a solar eclipse in Africa to corroborate the general relativity prediction which was  $1.75''$  for the sun[26]. When Eddington confirmed the prediction of the theory, Einstein became instantly in a celebrity, with the media showing news about the new theory of gravitation and the Einstein's genius. Since then, Einstein became not only an icon, but also the most important scientist of XX century.

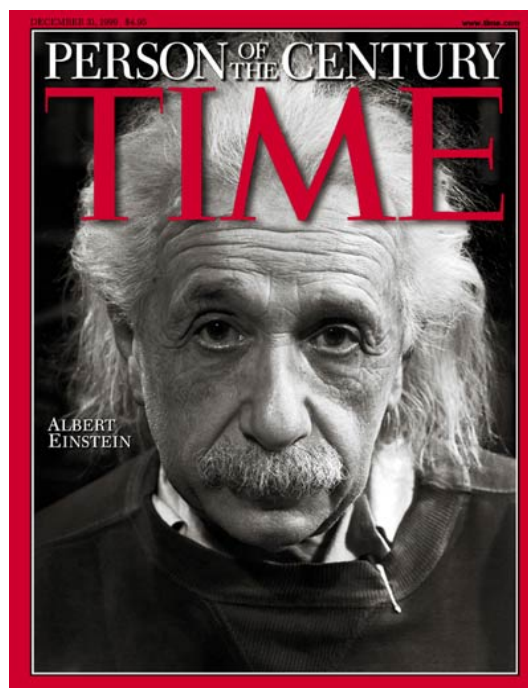


Figure 1.7 Time magazine cover. December 31, 1999. Cover Credit: Philippe Halsman

## CHAPTER 2

### THE MATH MACHINERY: TENSOR ANALYSIS IN FLAT SPACETIME

*Therefore I chance to think that all Nature and the  
graceful sky are symbolized in the art of geometry.... Now as  
God the maker play'd He taught the game to Nature whom  
He created in His image; taught her the self-same game  
which He played to her.*

Johannes Kepler, *Tertius Interveniens*.

The math of relativity are presented. The concepts of vectors, one-forms and metric tensor are discussed. The operational machinery involving these objects are introduced. In our approach we focus more on the geometrical nature of tensors, instead of their transformation properties. The concepts here introduced, will be applied in the context of general relativity in Chapter 3.

**Notation:** We will follow the notation by MTW[2] and Schutz[28], where Greek index run from  $\{0, \dots, 3\}$  and Latin index (space index) run from  $\{1, \dots, 3\}$ . We also use geometrized units with  $c = 1$ . Bold letters indicate vectors and basis vectors as usual. We use a metric with signature  $\{-1, 1, 1, 1\}$ .



## 2.1 VECTORS AND TETRADS

We can borrow most of the tools used in Special Relativity, concerning to 4-vectors and its transformations rules. As we know, the Lorentz transformation of a 4-vector  $\mathbf{A} \rightarrow (A^0, A^1, A^2, A^3) = \{A^\alpha\}$  is given by

$$A^{\alpha'} = \Lambda^{\alpha'}_{\beta} A^{\beta} \quad (2.1)$$

where I am using the Einstein summation convention. In my procedures, I shall try to emphasize the *geometrical* character of a vector, which exists independent of the coordinates system. This will be highly important when we generalize this methods to tensors and finally its application to general relativity.

We can introduce a set of basis vectors denoted by  $\mathbf{e}_\mu$ <sup>1</sup>, such that the vector  $\mathbf{A}$  can be written as

$$\mathbf{A} = A^\alpha \mathbf{e}_\alpha \quad (2.2)$$

where the basis vector written in components takes the form

$$\begin{aligned} \mathbf{e}_0 &\rightarrow (1, 0, 0, 0) \\ \mathbf{e}_1 &\rightarrow (0, 1, 0, 0) \\ \mathbf{e}_2 &\rightarrow (0, 0, 1, 0) \\ \mathbf{e}_3 &\rightarrow (0, 0, 0, 1) \end{aligned} \quad (2.3)$$

This basis satisfies the relation  $(\mathbf{e}_\alpha)^\beta = \delta_\alpha^\beta$ . These transformation rules applies also to any other frame, it means

$$\mathbf{A} = A^{\alpha'} \mathbf{e}_{\alpha'} \quad (2.2)$$

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<sup>1</sup>Also known as tetrad

Note that in general, these “primed” components and basis, are *not* the same as the ones in (2.2). They are two different frames. However, after the sum is done, the total vector will be the same

$$A^\alpha \mathbf{e}_\alpha = A^{\alpha'} \mathbf{e}_{\alpha'} \quad (2.3)$$

This relation is important, because from here we can find the transformation rules for the basis vectors. Using (2.1) for  $A^{\alpha'}$  and replacing in (2.3) we have

$$\Lambda^{\alpha'}_{\beta} A^{\beta} \mathbf{e}_{\alpha'} = A^{\alpha} \mathbf{e}_{\alpha}$$

$$A^{\beta} \Lambda^{\alpha'}_{\beta} \mathbf{e}_{\alpha'} = A^{\alpha} \mathbf{e}_{\alpha}$$

Note that  $\alpha$  and  $\beta$  are dummy index, so we can exchange them

$$A^{\alpha} \left( \Lambda^{\beta'}_{\alpha} \mathbf{e}_{\beta'} - \mathbf{e}_{\alpha} \right) = 0$$

which reduces to

$$\boxed{\mathbf{e}_{\alpha} = \Lambda^{\beta'}_{\alpha} \mathbf{e}_{\beta'}} \quad (2.4)$$

More than a components transformation, (2.4) gives the linear transformation between frames  $O$  and  $O'$ . Note that this is different of (2.1). An important example in physics is the 4-momentum vector which is defined as  $\mathbf{P} = m\mathbf{U}$ , which has the components;  $\mathbf{P} \rightarrow (E, p^1, p^2, p^3)$ . In analogy with the interval in Minkowski geometry

$$\Delta s^2 = -(\Delta t)^2 + (\Delta x)^2 + (\Delta y)^2 + (\Delta z)^2$$

we define the magnitude of a vector like

$$\mathbf{A}^2 = -(A^0)^2 + (A^1)^2 + (A^2)^2 + (A^3)^2 \quad (2.5)$$

The magnitude is obviously a scalar (frame independent quantity), therefore it's a Lorentz invariant. The minus sign is not just a signal of the difference with the euclidean geometry (Newtonian mechanics), but also, it informs that the magnitude is not only defined positive. In particular, we have three different cases

$$\mathbf{A}^2 \begin{cases} < 0, & \text{Timelike} \\ > 0, & \text{Spacelike} \\ = 0, & \text{Null} \end{cases}$$

Care must be taken about the Null condition. It does not mean that all components are zero, it means that the sum (2.5) is zero. The scalar product between two vectors is determined by

$$\mathbf{A} \cdot \mathbf{B} = -A^0 B^0 + A^1 B^1 + A^2 B^2 + A^3 B^3 \quad (2.6)$$

If the dot product vanishes  $\mathbf{A} \cdot \mathbf{B} = 0$ , it means the vectors are orthogonal. Note that the minus sign (again) indicates that  $\mathbf{A}$  and  $\mathbf{B}$  does not form right angles in a spacetime diagram. What?, yes, they are orthogonal in the Minkowski spacetime!. For example, the basis vectors  $e_\mu$  form an *orthogonal* vector basis (or orthonormal tetrad), which satisfies

$$\mathbf{e}_0 \cdot \mathbf{e}_0 = -1,$$

$$\mathbf{e}_1 \cdot \mathbf{e}_1 = \mathbf{e}_2 \cdot \mathbf{e}_2 = \mathbf{e}_3 \cdot \mathbf{e}_3 = +1$$

$$\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = 0, \quad \alpha \neq \beta$$

We can summarize the results above as

$$\boxed{\mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \eta_{\alpha\beta}} \quad (2.7)$$

where  $\eta_{\alpha\beta}$  is the *metric tensor*, which in matrix form can be written as

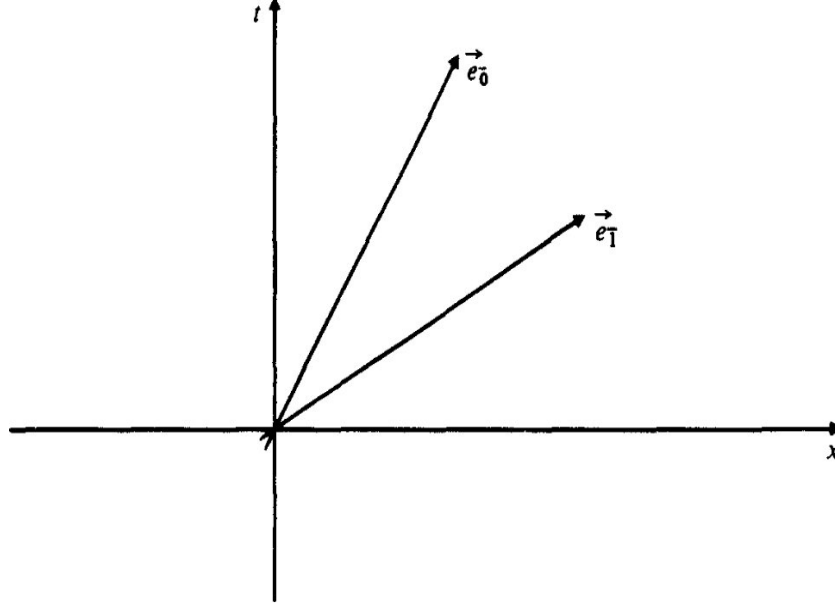


Figure 2.1 Basis vectors  $O'$  are not ‘perpendicular’ in the euclidean way, when they are drawn in the frame  $O$ . They are orthogonal in the Minkowski spacetime. (Figure adapted of [28])

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (2.8)$$

This particular form of metric tensor is valid only for the Minkowski spacetime (special relativity). We will see that in general relativity, the spacetime is *curved*, and the complexity of the metric tensor will increase. This looks a little bit “naive” definition of a metric tensor, a more rigorous definition will be discussed in the following.

## 2.2 INTRODUCING TENSORS: VECTORS AND ONE-FORMS

Let’s consider two vectors in the representation of some basis  $\mathbf{e}_\mu$  in some frame  $O$

$$\mathbf{A} = A^\alpha \mathbf{e}_\alpha ; \quad \mathbf{B} = B^\beta \mathbf{e}_\beta \quad (2.9)$$

Taking the dot product of (2.9)

$$\mathbf{A} \cdot \mathbf{B} = (A^\alpha \mathbf{e}_\alpha) \cdot (B^\beta \mathbf{e}_\beta) = A^\alpha B^\beta (\mathbf{e}_\alpha \cdot \mathbf{e}_\beta)$$

using (2.7) we have

$$\boxed{\mathbf{A} \cdot \mathbf{B} = A^\alpha B^\beta \eta_{\alpha\beta}} \quad (2.10)$$

where  $\eta_{\alpha\beta}$  corresponds to the components of the metric tensor. So, what is a tensor?.

We follow this definition[28]

*A tensor of type  $\binom{0}{N}$  is a function of  $N$  vectors into the real numbers, which is linear in each of its arguments.*

The symbol  $\binom{0}{N}$  is not indicating the binomial coefficient, it is related to the number of vectors and one-forms that the tensor composes. I will discuss more about that later. For the moment, let's analyze that definition for the case we discussed in (2.10). The rule says, a tensor  $\binom{0}{2}$  is a function that takes two vectors, and it gives a real number. Comparing this analysis with (2.10) we see that the relation is satisfied. On the other hand, linearity condition means

$$(\alpha \mathbf{A}) \cdot \mathbf{B} = \alpha (\mathbf{A} \cdot \mathbf{B})$$

$$(\mathbf{A} + \mathbf{B}) \cdot \mathbf{C} = \mathbf{A} \cdot \mathbf{C} + \mathbf{B} \cdot \mathbf{C}$$

In more concrete terms, we define the metric tensor  $\mathbf{g}$  as

$$\boxed{\mathbf{g}(\mathbf{A}, \mathbf{B}) \equiv \mathbf{A} \cdot \mathbf{B}} \quad (2.11)$$

Note that this definition of tensor is *independent* of the coordinates, we did not mention components here. The power of the tensor concept, is that it must give the same real number (scalar), in any reference frame. Here we can start to visualize the relation with the equivalence principle in general relativity, but let's stop there, we need to do more math before going there.

In some 'oldies' texts, the metric tensor is defined in terms of components, as an object that transforms according to the rule (see for instance [6, 11, 29])

$$g'_{\mu\nu} = \frac{\partial x^\alpha}{\partial x'^\mu} \frac{\partial x^\beta}{\partial x'^\nu} g_{\alpha\beta}$$

Although this definition is operationally correct, it does not allow us to have a wider view of a tensor as a *geometric* object independent of the coordinates<sup>2</sup>. Of course, when we need to solve particular problems (finding exact solutions to Einstein equation for instance) we need to choose some coordinate system and a reference frame, and then do the calculations with components. Again, this is more an 'operational' definition of a tensor.

The components of a tensor corresponds to the *values that the function takes when its arguments are the basis vectors*. Let's recall that the arguments of the metric tensor are the *vectors* themselves, not the components. In terms of components, the metric tensor can be written as

$$\boxed{\mathbf{g}(\mathbf{e}_\alpha, \mathbf{e}_\beta) = \mathbf{e}_\alpha \cdot \mathbf{e}_\beta = \eta_{\alpha\beta}} \quad (2.12)$$

which corresponds to the metric tensor of special relativity, or *Lorentz metric*. A tensor of type  $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$  is called a *covariant vector* (covector), or in modern terminology: *one-form*. In our notation, we will use a tilde  $\tilde{p}$  to denote one-forms. Given an

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<sup>2</sup>This method of working with tensors in components, was the one that used Einstein in his original papers[3]. However, the tools of the modern differential geometry were not completely developed by that time.

arbitrary one-form  $\tilde{p}$ , when a vector is given to it as an argument, we obtain a scalar:  $\tilde{p}(\mathbf{A}) = k$ , where  $k$  is a real number. Additionally, the one-forms satisfies the same properties as a space vector

- $\tilde{s} = \tilde{p} + \tilde{q}$
- $\tilde{r} = \alpha\tilde{p}$
- $\tilde{s}(\mathbf{A}) = \tilde{p}(\mathbf{A}) + \tilde{q}(\mathbf{A})$
- $\tilde{r}(\mathbf{A}) = \alpha\tilde{p}(\mathbf{A})$

The one-forms space is called the *dual space vector*. In terms of components, a one-form is written as

$$p_\alpha \equiv \tilde{p}(\mathbf{e}_\alpha) \quad (2.13)$$

When components are denoted with single lower index, by convention, these corresponds to one-forms. Upper index corresponds to vectors. This difference in the index notation is important because it determines the rules of one-forms acting on vectors, as follows

$$\tilde{p}(\mathbf{A}) = \tilde{p}(A^\alpha \mathbf{e}_\alpha) = A^\alpha \tilde{p}(\mathbf{e}_\alpha)$$

$$\boxed{\tilde{p}(\mathbf{A}) = A^\alpha p_\alpha} \quad (2.14)$$

which gives a real number. In principle, this is a more fundamental operation than the dot product between vectors, because (2.14) does not require another tensor to operate.<sup>3</sup> In analogy as we proceed with vectors, we can find the transformation rules

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<sup>3</sup>Let's remember that in the case of dot product we need a metric tensor.

of one-forms in the basis of tetrads  $\mathbf{e}_\beta$

$$\begin{aligned} p_{\beta'} &\equiv \tilde{p}(\mathbf{e}_{\beta'}) = \tilde{p}(\Lambda^\alpha_{\beta'} \mathbf{e}_\alpha) \\ &= \Lambda^\alpha_{\beta'} \tilde{p}(\mathbf{e}_\alpha) = \Lambda^\alpha_{\beta'} p_\alpha \end{aligned}$$

$$\mathbf{e}_{\beta'} = \Lambda^\alpha_{\beta'} \mathbf{e}_\alpha \quad (2.15)$$

comparing to (2.4) we see that components of one-forms transforms in *opposite* (I mean with the inverse transformation) way, compared to how vectors components transforms. In analogy as we did for vectors, we can define a ‘one-forms’ basis, considering that one-forms satisfies the properties of vector space. In my notation convention, I will use  $\{\tilde{\omega}^\alpha\}$  where  $\alpha = 0, \dots, 3$ . In this basis, a one-form reads as

$$\tilde{p} = p_\alpha \tilde{\omega}^\alpha \quad (2.16)$$

Let us act  $\tilde{p}$  on a vector  $\mathbf{A}$ . From (2.16) we have then

$$\tilde{p}(\mathbf{A}) = p_\alpha \tilde{\omega}^\alpha(\mathbf{A}) \quad (2.17)$$

substituting (2.9) in (2.17) we have

$$\begin{aligned} \tilde{p}(\mathbf{A}) &= p_\alpha \tilde{\omega}^\alpha(A^\beta \mathbf{e}_\beta) \\ &= p_\alpha A^\beta \tilde{\omega}^\alpha(\mathbf{e}_\beta) \end{aligned}$$

the last line can be only the invariant  $p_\alpha A^\beta$  if

$$\boxed{\tilde{\omega}^\alpha(\mathbf{e}_\beta) = \delta^\alpha_\beta} \quad (2.18)$$

which defines the one-forms basis in terms of the vector basis. In components, (2.18) is given by



$$\begin{aligned}
\tilde{\omega}^0 &\rightarrow (1, 0, 0, 0), \\
\tilde{\omega}^1 &\rightarrow (0, 1, 0, 0), \\
\tilde{\omega}^2 &\rightarrow (0, 0, 1, 0), \\
\tilde{\omega}^3 &\rightarrow (0, 0, 0, 1).
\end{aligned}
\tag{2.19}$$

Although we can describe vector and one-forms basis in terms of 4 numbers, their *geometrical* significance is different. We are familiar with the representation of vectors as arrows, but what about one-forms?. Warning: a one-form is *not* an arrow. From (2.14) we found that a one-form *acts* on a vector to produce a real number<sup>4</sup>. And also, we discussed that this action does not need an additional tensor to operate, in contrast with the dot product which needs a metric tensor to operate. A visual representation used in mathematics to describe one-forms, consists of a set of surfaces where the spacing between them, determines the magnitude of the one-form: “larger the space, smaller the magnitude” (see Fig. 2.2).[28]

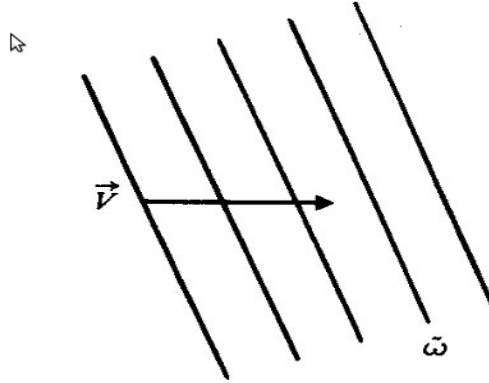


Figure 2.2 Visual representations of a vector and a one-form. The vector corresponds to the ‘standard’ arrow. The one-form can be seen as a set of surfaces, where the spacing determines its magnitude. A one-form ‘acting’ on a vector, gives a scalar which corresponds to the number of surfaces that the arrow ‘crosses’. In this case  $\tilde{\omega}(\mathbf{V}) = 2.5$ . (Figure adapted of [28])

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<sup>4</sup>In more technical terms, it *maps* a vector into a real number.

Note that in analogy to vectors, which are represented by ‘straight’ arrows, one-forms corresponds to surfaces straight and parallel. It is possible, because we are working with one-forms at a point (‘tangent’ one-forms as tangent vector).

**Comment on notation for derivatives:** I will use the following notation to indicate partial derivative

$$\phi_{,x} \equiv \frac{\partial \phi}{\partial x} \quad ; \quad \phi_{,\alpha} \equiv \frac{\partial \phi}{\partial x^\alpha} \quad ; \quad x^\alpha_{,\beta} = \delta^\alpha_\beta \quad (2.20)$$

In Chapter 3 we will introduce the *covariant derivative* which we will denote as semi-colon  $T_{\mu\nu;\nu}$ , but let us discuss that later.

### 2.3 MORE TENSORS: $\binom{0}{2}$ TENSORS AND THE METRIC TENSOR

Basically  $\binom{0}{2}$  type tensors corresponds to tensors that have two arguments. An example of this, is the metric tensor which was discussed in the previous section. We said that the dot product between two vectors, demands a metric tensor to produce a real number. Another important example of tensors  $\binom{0}{2}$  corresponds to the product of two one-forms. The rule is as follows: given two one-forms  $\tilde{p}$  and  $\tilde{q}$ , then  $\tilde{p} \otimes \tilde{q}$  is the tensor  $\binom{0}{2}$  which when acts on vectors  $\mathbf{A}$  and  $\mathbf{B}$  gives the number  $\tilde{p}(\mathbf{A}) \cdot \tilde{q}(\mathbf{B})$ . Is worth to mention that the ‘outer product’  $\otimes$  is not commutative, so  $\tilde{p} \otimes \tilde{q} = \tilde{p}(\mathbf{A}) \cdot \tilde{q}(\mathbf{B})$  but  $\tilde{q} \otimes \tilde{p} = \tilde{q}(\mathbf{A}) \cdot \tilde{p}(\mathbf{B})$ . In general we can write a  $\binom{0}{2}$  tensor in a basis  $\mathbf{e}_\mu$  as

$$f_{\alpha\beta} = \mathbf{f}(\mathbf{e}_\alpha, \mathbf{e}_\beta) \quad (2.21)$$

Acting on the vectors  $\mathbf{A}$  and  $\mathbf{B}$ , we can write (2.21) in components

$$\mathbf{f}(\mathbf{A}, \mathbf{B}) = \mathbf{f}(A^\alpha \mathbf{e}_\alpha, B^\beta \mathbf{e}_\beta) = A^\alpha B^\beta \mathbf{f}(\mathbf{e}_\alpha, \mathbf{e}_\beta)$$

therefore:

$$\boxed{\mathbf{f}(\mathbf{A}, \mathbf{B}) = A^\alpha B^\beta \mathbf{f}_{\alpha\beta}} \quad (2.22)$$

where in the last line we used (2.21). Note that (2.22) is written in terms of vectors, and for instance, its basis. However, we can also find a basis  $\tilde{\omega}^{\alpha\beta}$  for tensors  $\binom{0}{2}$ , in analogy to (2.16). What we are looking for, is something like

$$\mathbf{f} = f_{\alpha\beta} \tilde{\omega}^{\alpha\beta} \quad (2.23)$$

in that case, using (2.21) we demand

$$f_{\mu\nu} = \mathbf{f}(\mathbf{e}_\mu, \mathbf{e}_\nu) = f_{\alpha\beta} \tilde{\omega}^{\alpha\beta}(\mathbf{e}_\mu, \mathbf{e}_\nu)$$

we demand that

$$\tilde{\omega}^{\alpha\beta}(\mathbf{e}_\mu, \mathbf{e}_\nu) = \delta_\mu^\alpha \delta_\nu^\beta \quad (2.23)$$

but from (2.18) we know that  $\delta_\mu^\alpha$  corresponds to the value of  $\tilde{\omega}^\alpha$  acting on  $\mathbf{e}_\mu$ , so this implies

$$\tilde{\omega}^{\alpha\beta} = \tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta$$

and we conclude that

$$\boxed{\mathbf{f} = f_{\alpha\beta} (\tilde{\omega}^\alpha \otimes \tilde{\omega}^\beta)} \quad (2.24)$$

The ordering of the arguments of  $\binom{0}{2}$  tensors is an important characteristic which will have relevant consequences when we introduce the physical ideas in Chapter 3. We can consider *symmetric* tensors which satisfy

$$\mathbf{f}(\mathbf{A}, \mathbf{B}) = \mathbf{f}(\mathbf{B}, \mathbf{A}) \quad (2.25)$$

which implies

$$f_{\alpha\beta} = f_{\beta\alpha} \quad (2.26)$$

An example of this is the metric tensor  $g_{\mu\nu}$ , which is the fingerprint of Riemannian geometries<sup>5</sup>. As before, we can introduce the symmetrization

$$h_{(s)\alpha\beta} \equiv h_{(\alpha\beta)} = \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha}) \quad (2.27)$$

On the other hand, a tensor is called *antisymmetric* if it satisfies

$$\mathbf{f}(\mathbf{A}, \mathbf{B}) = -\mathbf{f}(\mathbf{B}, \mathbf{A}) \quad (2.28)$$

which implies

$$f_{\alpha\beta} = -f_{\beta\alpha} \quad (2.29)$$

Similarly, we can construct an antisymmetric tensor by

$$h_{(A)\alpha\beta} \equiv h_{[\alpha\beta]} = \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha}) \quad (2.30)$$

The important characteristic of symmetrization and antisymmetrization, is that we can write any  $\binom{0}{2}$  tensor in terms of its symmetric and skew-symmetric parts like this

$$h_{\alpha\beta} = \frac{1}{2}(h_{\alpha\beta} + h_{\beta\alpha}) + \frac{1}{2}(h_{\alpha\beta} - h_{\beta\alpha}) = h_{(\alpha\beta)} + h_{[\alpha\beta]} \quad (2.31)$$

With these tools in our hands, we can introduce more formally the *metric tensor*, which will be of great importance in general relativity. The basic role of the metric tensor, is to act as a *mapping* between vectors and one-forms to produce a real number. We define that ‘acting’ as

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<sup>5</sup>There are alternative models where a skew-symmetric tensor is considered, which are called non-symmetric gravitational theories. See for instance[30].

$$\tilde{V}(\mathbf{A}) \equiv \mathbf{g}(\mathbf{V}, \mathbf{A}) = \mathbf{V} \cdot \mathbf{A} \quad (2.32)$$

This is an important idea to point out. The metric tensor acts as a machine, where once we supply with a vector and a one-forms, it gives a real number. Note that the idea of metric is implicitly given when we operate a dot product. Let us take a look now to  $\tilde{V}$  in components

$$\begin{aligned} V_\alpha &\equiv \tilde{V}(\mathbf{e}_\alpha) = \mathbf{V} \cdot \mathbf{e}_\alpha = \mathbf{e}_\alpha \cdot \mathbf{V} \\ &= \mathbf{e}_\alpha \cdot (V^\beta \mathbf{e}_\beta) \\ &= (\mathbf{e}_\alpha \cdot \mathbf{e}_\beta) V^\beta \end{aligned}$$

finally

$$\boxed{V_\alpha = \eta_{\alpha\beta} V^\beta} \quad (2.33)$$

Note that equation (2.33) shows the idea discussed above: the metric tensor acts as a mapping between vectors (in this case  $V^\beta$ ) and one-forms ( $V_\alpha$ ). Clearly the index position is important in this formalism. In operational terms, the metric tensor can be understood as the “machine” that allows us to raise and low indexes. We can go backwards, and find the components of the one-form given the vector. Considering that  $\eta_{\alpha\beta}$  is non-singular we have

$$V^\alpha = \eta^{\alpha\beta} V_\beta \quad (2.34)$$

We conclude that the mapping produced by the metric tensor  $\mathbf{g}$  is one-to-one and invertible. For instance, the flat spacetime of special relativity is determined by the *Lorentz metric* which takes the form (2.8)

$$\eta_{\alpha\beta} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

When we introduce the ideas of general relativity in this context, we will see that the content of mass and momentum disturbs the spacetime producing *curvature*, which makes the metric more complicated.

The application of this formalism of tensor algebra, vectors and one-forms, is not only limited to the Einstein's relativity. In quantum mechanics, using the Dirac formalism we learned that the state of a system is represented by a 'ket'  $|\psi\rangle$ . Then, the Hilbert space is introduced where we can define all these vectors (kets) and the operations between them. The Hilbert space is a *space vector*. Similarly, we define the dual to a ket, or 'bra'  $\langle\phi|$ , where it's defined in a dual vector space. Analogously to the behavior of vectors and one-forms, when we 'act' a bra on a ket, or in better terms we construct a 'braket', we obtain the number  $\langle\phi|\psi\rangle$ .

So far, we have found differences between vectors and one-forms. However, when we calculate magnitudes both of them gives the same result

$$\mathbf{p}^2 = \tilde{p}^2 = \eta_{\alpha\beta} p^\alpha p^\beta \quad (2.35)$$

using (2.34) we have

$$\tilde{p}^2 = \eta_{\alpha\beta} (\eta^{\alpha\mu} p_\mu) (\eta^{\beta\nu} p_\nu)$$

the sum over  $\beta$  drops, which implies

$$\eta_{\alpha\beta} \eta^{\beta\nu} = \delta_\alpha^\nu \quad (2.36)$$

then we have finally

$$\tilde{p}^2 = \eta^{\alpha\mu} p_\mu p_\alpha \quad (2.37)$$

In geometric terms (I mean visual terms, all we have done is geometry!) we consider a *normal* vector to a surface, if its associated one-form is a normal one-form. It is the general case of the well known euclidean space, where a normal vector is defined as the vector orthogonal to the space of tangent vectors. The powerful characteristic of our tensor formalism, is that we now recognize the ‘normal’ as a one-form independent of the coordinates, therefore we don’t need to specify a metric.

## 2.4 A LITTLE MORE OF TENSORS: $\binom{M}{N}$ TENSORS AND DIFFERENTIATION

We have defined vectors and one-forms and tensors type  $\binom{0}{2}$ . As a particular case of this last one, we introduced the metric tensor. My next step is to generalize these concepts a little further.[28]

A  $\binom{M}{0}$  tensor is a linear function of M one-forms into the real numbers.

In analogy to the case of  $\binom{2}{0}$  tensors, the components of a tensor  $\binom{M}{0}$  corresponds to the basis one-forms  $\tilde{\omega}^\alpha$ . Finally, the most general definition that we will discuss is a  $\binom{M}{N}$  tensor

A  $\binom{M}{N}$  tensor is a linear function of M one-forms and N vectors, into the real numbers.

In components, a  $\binom{M}{N}$  tensor is characterized by M superscripts and N subscripts. For example, doing a boost to another inertial reference frame we have

$$\begin{aligned} R^{\alpha'}_{\beta'} &= \mathbf{R}(\tilde{\omega}^{\alpha'}, \mathbf{e}_{\beta'}) \\ &= \mathbf{R}(\Lambda^{\alpha'}_{\mu} \tilde{\omega}^{\mu}; \Lambda^{\nu}_{\beta'} \tilde{\mathbf{e}}_{\nu}) \end{aligned}$$

therefore

$$R^{\alpha'}_{\beta'} = \Lambda^{\alpha'}_{\mu} \Lambda^{\nu}_{\beta'} R^{\mu}_{\nu} \quad (2.38)$$

In the ‘old fashion’ notation, a  $\binom{M}{N}$  tensor has M contravariant components and N covariant components. For example, let’s suppose  $T^{\alpha\mu}_{\gamma}$  corresponds to the components of a tensor  $\binom{2}{1}$ . We can construct a  $\binom{1}{2}$  tensor by

$$T^{\alpha}_{\beta\gamma} = \eta_{\beta\mu} T^{\alpha\mu}_{\gamma} \quad (2.39)$$

note that the metric tensor acts as the ‘machine’ to raise and low indexes. On the other hand, a different tensor  $\binom{1}{2}$  can be formed by

$$T^{\beta}_{\alpha\gamma} = \eta_{\alpha\mu} T^{\mu\beta}_{\gamma} \quad (2.40)$$

in summary, these operations are called *lowering* and *raising*. Another important property of the metric tensor is

$$\eta^{\alpha}_{\beta} \equiv \eta^{\alpha\mu} \eta_{\mu\beta} = \delta^{\alpha}_{\beta} \quad (2.41)$$

we can conclude that  $\eta^{\alpha\beta}$  corresponds to the components of the tensor  $\binom{2}{0}$  which is ‘mapped’ of the tensor  $\binom{0}{2}$ .

As the last stage in our description of the tensor algebra, we will introduce the differentiation of tensors. In general, when we take derivatives of a tensor it will produce a tensor of higher rank. For example, a scalar function  $f$  is considered a  $\binom{0}{0}$  tensor. Its gradient  $\nabla f$  is a one-form, or tensor  $\binom{0}{1}$ . Let’s suppose we have a  $\binom{1}{1}$  tensor whose components are

$$\mathbf{T} = T^{\alpha}_{\beta} \tilde{\omega}^{\beta} \otimes \mathbf{e}_{\alpha} \quad (2.42)$$



Now, let's suppose we move along a line parametrized with the parameter  $\tau$ , the proper time. Considering that vector basis remains constant in flat spacetime we have

$$\frac{d\mathbf{T}}{d\tau} = \left( \frac{dT^\alpha_\beta}{d\tau} \right) \tilde{\omega}^\beta \otimes \mathbf{e}_\alpha \quad (2.43)$$

where  $\frac{dT^\alpha_\beta}{d\tau} = T^\alpha_{\beta,\gamma} U^\gamma$  is a  $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$  tensor. In general, for any vector  $\mathbf{U}$  we have

$$\frac{d\mathbf{T}}{d\tau} = \left( T^\alpha_{\beta,\gamma} \tilde{\omega}^\beta \otimes \mathbf{e}_\alpha \right) U^\gamma \quad (2.44)$$

from (2.44) we can define the *gradient* of  $\mathbf{T}$

$$\boxed{\nabla \mathbf{T} \equiv T^\alpha_{\beta,\gamma} \tilde{\omega}^\beta \otimes \tilde{\omega}^\gamma \otimes \mathbf{e}_\alpha} \quad (2.45)$$

Is worth to remark that this definition was possible just because the basis vectors remains constant in the flat spacetime of special relativity. In the next chapter, we will see that we must modify this definition, once we put all this mathematical 'machinery' to the service of the principles of the general theory of relativity.

## CHAPTER 3

### GENERAL RELATIVITY IN A NUTSHELL

*Einstein's theory of gravitation was inspired by  
and based on the "principle of equivalence", which  
states that when gravity is present, as when it is absent,  
free particles move along extremal (geodesic) lines of spacetime-  
spacetime now being curved, not flat.*

Ya. B. Zel'dovich (1971)

The formalism of tensors, which was reviewed in Chapter 2, is now applied to the general theory of relativity. The idea that gravitation is just a manifestation of the *curvature* of the spacetime, is introduced. The concept of manifold is discussed. The Riemann, Ricci and energy-momentum tensors are introduced. The Einstein equation is written and the idea that matter determines geometry is presented. Some solutions to Einstein equation are discussed (but not derived), which will be cited in Chapter 4.

#### 3.1 WHAT IS A MANIFOLD?

To go further in our description of the geometry of the spacetime, we would like to introduce a sort of mathematical structure, which looks locally flat (special relativity spacetime) but its curvature grows in complexity once we cover extended regions. The mathematical object that embraces these ideas, is a *manifold*. The notion of manifold corresponds to a space which *locally* looks like  $\mathbb{R}^n$  (Euclidean)[5]. When

we say “looks like” it does not mean the metric is the same. What it means, is that functions and coordinates work in a similar fashion.

An important feature of a manifold, is that it can be *parametrized*, where the number of parameters corresponds to the *dimension* of the manifold. Is not hard to see that these parameters corresponds to the *coordinates* of the manifold. For example, in special relativity we deal with a manifold of dimension 4, which corresponds to the three spatial coordinates  $x^i$  and the time coordinate  $x^0$ . Newtonian physics for instance, works in a manifold of dimension 3, considering that time is not a coordinate.

The definition of manifold given above might look a little “vague”. We can give a more rigorous definition, following[5]: *a  $C^\infty$   $n$ -dimensional manifold (or  $n$ -manifold) is simply a set  $M$  provided with a maximal atlas, which contains every possible compatible chart.* An atlas must be understood as an indexed collection of *charts* which satisfies:

- The union of  $U_\alpha$  is equal to  $M$ , it means, that the charts covers all the manifold  $M$ .
- The charts must be “sewn” smoothly together. If two charts overlap,  $U_\alpha \cap U_\beta \neq \emptyset$ , then the map  $(\phi_\alpha \circ \phi_\beta^{-1})$  take points in  $\phi_\beta(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$  onto an open set  $\phi_\alpha(U_\alpha \cap U_\beta) \subset \mathbb{R}^n$ , and all these maps must be  $C^\infty$  where they are defined.

where we understand a *chart* or *coordinate system* as a subset  $U \subset M$ , along with a one-to-one map  $\phi : U \rightarrow \mathbb{R}^n$ . In Fig. 3.1 is shown the idea of overlapped charts.

Note that we have not introduced the metric in our development. Sometimes we don’t need to introduce the idea of metric into the manifold, it must be understood as a geometrical object independent of the coordinates. However, in general relativity the metric is indispensable, because it carries the information about the clocks rates and distances between points, just like the Lorentz metric does for special relativity.

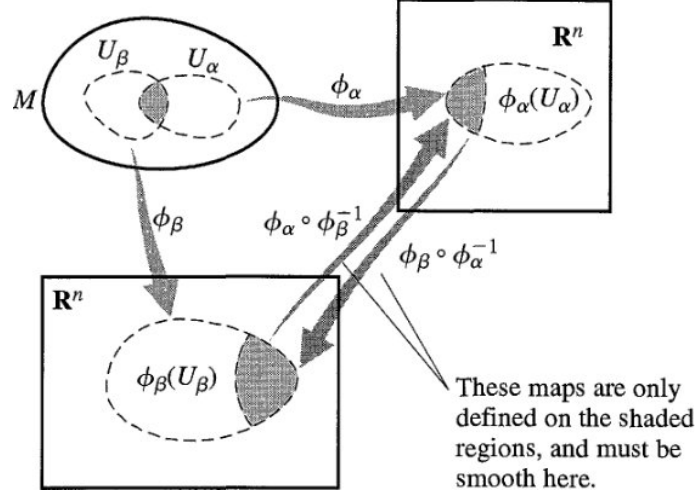


Figure 3.1 Overlapping coordinate charts. (Figure adapted of [5])

One important example is a *Riemannian manifold*, which corresponds to a  $C^\infty$  manifold which has been provided with a tensor  $\binom{0}{2} \mathbf{g}$  field<sup>1</sup>. We will see that once we provide the manifold with a metric tensor, it will define completely the geometry of the spacetime. We will discuss some important metrics in general relativity later. Before going there, we need to understand how we are going to take derivatives of vectors in curves manifolds, which will take us to the notion of covariant derivative and geodesic equation.

### 3.2 CURVATURE: COVARIANT DERIVATIVE, PARALLEL TRANSPORT AND GEODESICS

In flat space geometry, the derivative of a vector field corresponds to the difference between vectors at two different points (in the limit when the separation between vectors goes to zero). However, when we extend our study to *curved* spaces, the notion of vectors in two near points must be analyzed carefully. In principle, we should expect some “correction” in the partial derivative term, due to the fact that

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<sup>1</sup>More formally, a Riemannian manifold is characterized by the condition  $\mathbf{g}(\mathbf{V}, \mathbf{V}) > 0$  for all  $\mathbf{V} \neq 0$ . From our discussion in Chapter 2, we found that for the Lorentz metric,  $\mathbf{g}(\mathbf{V}, \mathbf{V})$  can be positive, negative or null. The Lorentz metric is called *pseudo-Riemannian*. That will be also the case in general relativity.

the basis vectors are changing in this curved space. The concept of Riemannian manifold is of great help in this matter, because as we discussed in the previous section, locally this manifold looks like  $\mathbb{R}^n$ , so it's expected that locally the derivative reduces to the standard partial derivative.

In general, the *covariant derivative* of a vector  $V^\nu$  is given by the partial derivative  $\partial_\mu$  plus some correction which is related to the change in the vector basis

$$\boxed{V^\nu{}_{;\mu} = V^\nu{}_{,\mu} + \Gamma^\nu{}_{\alpha\mu} V^\alpha} \quad (3.1)$$

where we use the semi-colon to denote the covariant derivative, and the comma to denote the partial derivative  $V_{,\mu} = \frac{\partial V}{\partial x^\mu}$ . The “gamma” terms  $\Gamma^\nu{}_{\alpha\mu}$  are called the *connection coefficients* or *Christoffel symbols* and are given by

$$\Gamma^\alpha{}_{\mu\nu} = \frac{1}{2} g^{\alpha\beta} (g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta}) \quad (3.2)$$

Note that in a local inertial frame, we recover the flat spacetime of SR. In that case  $V^\nu{}_{;\mu} = V^\nu{}_{,\mu}$  at some point  $P$  in that frame. Equation (3.1) is valid for any tensor, including the metric

$$g_{\alpha\beta;\gamma} = g_{\alpha\beta,\gamma} = 0 \quad \text{at } P \quad (3.3)$$

Note that equation (3.3) is a tensor equation, therefore it is valid in any basis. This is an important result in general relativity, which we can summarize like this: *the covariant derivative of the metric tensor is zero in any basis*. It's worth to mention that the Christoffel symbols are *not* tensors. That's not a problem, because what we want is that the whole expression (3.1) transforms as a tensor, which it does<sup>2</sup>. We have told that  $\Gamma^\alpha{}_{\mu\nu} = 0$  in any local inertial frame. However, this is not going to be true in general, because the  $\Gamma$ 's involves partial derivatives of the metric tensor.

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<sup>2</sup>To see a proof of this result see[31].

Although we can find local inertial frames where the connection coefficients vanish, is not possible to find a *global* basis where this holds true. In flat spacetime for instance,  $\Gamma^\alpha_{\mu\nu} = 0$  *everywhere*.

Let's summarize some formulas for covariant derivatives of one-forms and  $\binom{2}{0}$  tensors

$$\begin{aligned}\omega_{\nu;\mu} &= \omega_{\nu,\mu} - \Gamma^\alpha_{\nu\mu} \omega_\alpha \\ T^{\alpha\beta}_{;\gamma} &= T^{\alpha\beta}_{,\gamma} + \Gamma^\alpha_{\mu\gamma} T^{\mu\beta} + \Gamma^\beta_{\mu\gamma} T^{\alpha\mu}\end{aligned}\tag{3.4}$$

We are in position now to apply the previous concepts to the notion of *parallel transport*. Let us define a vector field  $\mathbf{V}$  at every point along a curve, which is parametrized by  $\lambda$  (See Fig. 3.2). If the vectors  $\mathbf{V}$  at infinitesimally neighbor points of the curve, are parallel to each other, we say that  $\mathbf{V}$  has been parallel-transported along the curve.

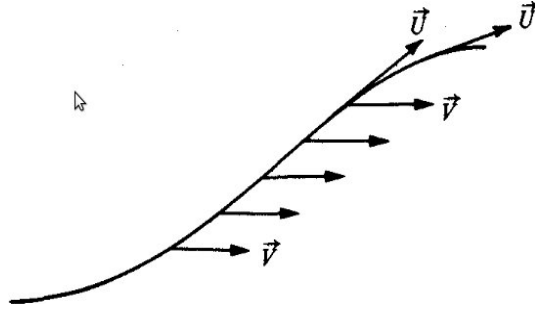


Figure 3.2 Parallel transport of  $\mathbf{V}$  along  $\mathbf{U}$ . (Figure adapted of [28])

Let  $\mathbf{U} = \frac{dx}{d\lambda}$  be a tangent vector to the curve. We know that the main characteristic of a manifold, is that it looks locally like  $\mathbb{R}^n$ . Therefore, in a locally inertial coordinates system at point  $P$ , the components of  $\mathbf{V}$  must be constant along the curve at  $P$ :

$$\frac{dV^\alpha}{d\lambda} = 0 \quad \text{at } P\tag{3.5}$$

we can write (3.5) as

$$\frac{dV^\alpha}{d\lambda} = \frac{dV^\alpha}{dx^\beta} \frac{dx^\beta}{d\lambda} = U^\beta V^\alpha_{;\beta} = U^\beta V^\alpha_{;\beta} = 0 \quad (3.6)$$

at point  $P$ . Note that the last equality in (3.6) is because the Christoffel symbols vanishes at  $P$  (locally flat). The last term involves a covariant derivative, and this is a tensor. Therefore this expression holds in any basis, so we have a general definition of *parallel transport* of a vector  $\mathbf{V}$  along  $\mathbf{U}$ :

$$U^\beta V^\alpha_{;\beta} = 0 \Rightarrow \frac{d}{d\lambda} \mathbf{V} = \nabla_{\mathbf{U}} \mathbf{V} = 0 \quad (3.7)$$

where the notation  $\nabla_{\mathbf{U}} \mathbf{V} \rightarrow \{V^\alpha_{\beta,\gamma} U^\gamma\}$ . How can we relate this to geodesics?. Well, we know from Euclidean geometry, that two parallel lines will keep being parallel, no matter how far we extend them. More precisely, the tangent to the curve at some point, is parallel to the tangent at a previous neighbor point. In terms of our definition (3.7), a straight line in Euclidean space, is the only one that parallel-transport its own tangent vector. This is not the case on a curved surface, like a sphere for example. On a sphere the space is *curved* and a vector initially pointing along the equator, it will be pointing towards the south hemisphere after being parallel transported until its initial position (See Fig. 3.3).

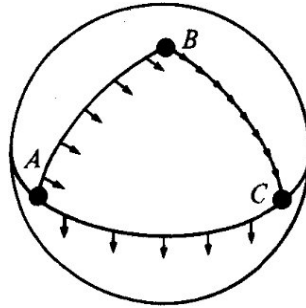


Figure 3.3 Parallel transport along a spherical triangle. (Figure adapted of [28])

On curved spaces, we find that the analogue to an ‘Euclidean straight line’, is a *geodesic*

$$\nabla_{\mathbf{U}} \mathbf{U} = 0 \quad (3.8)$$

which in components can be written as

$$U^\beta U^\alpha_{;\beta} = U^\beta U^\alpha_{,\beta} + \Gamma^\alpha_{\mu\beta} U^\mu U^\beta = 0$$

letting  $\lambda$  be the parameter of the curve, we have:  $U^\alpha = \frac{dx^\alpha}{d\lambda}$  and  $U^\beta \frac{\partial}{\partial x^\beta} = \frac{d}{d\lambda}$ .

Therefore we have

$$U^\beta \frac{d}{dx^\beta} \left( \frac{dx^\alpha}{d\lambda} \right) + \Gamma^\alpha_{\mu\beta} \frac{dx^\mu}{d\lambda} \frac{dx^\beta}{d\lambda} = 0$$

which gives

$$\boxed{\frac{d^2 x^\alpha}{d\lambda^2} + \Gamma^\alpha_{\mu\beta} \frac{dx^\mu}{d\lambda} \frac{dx^\beta}{d\lambda} = 0} \quad (3.9)$$

This is the geodesic equation, which determines the equations of motion of material particles *under the action only of inertia and gravitation*[18]. For Euclidean space, we know that the  $\Gamma$  connections vanishes, so (3.9) reduces to  $\frac{d^2 x^\alpha}{d\lambda^2} = 0$  which corresponds to a straight line.

### 3.3 MORE TENSORS: RIEMANN AND EINSTEIN

With the notion of parallel-transport discussed in the previous section, we are now in position to build more formally the idea of curvature of a *manifold*. Following a conceptual reasoning, we should expect that the curvature will depend on the covariant derivative of a vector and the affine connections. It turns out to be that curvature is quantified by the *Riemann tensor*, which is derived of the connection. We are not going to discuss the formal derivation of the Riemann tensor (see [5] for instance), but we can say that it will come of parallel-transport of a vector. Let's recall that when a vector is parallel-transported, it's transformed. This transformation



depends on the curvature of the manifold. In mathematical terms, the Riemann tensor is given by

$$\boxed{R^\alpha_{\beta\mu\nu} \equiv \Gamma^\alpha_{\beta\nu,\mu} - \Gamma^\alpha_{\beta\mu,\nu} + \Gamma^\alpha_{\sigma\mu}\Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu}\Gamma^\sigma_{\beta\mu}} \quad (3.10)$$

Some important properties of this tensor are the following

$$R_{\alpha\beta\mu\nu} = -R_{\beta\alpha\mu\nu} = -R_{\alpha\beta\nu\mu} = R_{\mu\nu\alpha\beta} \quad (3.11)$$

$$R_{\alpha\beta\mu\nu} + R_{\alpha\nu\beta\mu} + R_{\alpha\mu\nu\beta} = 0 \quad (3.12)$$

Note the antisymmetry of  $R_{\alpha\beta\mu\nu}$  in the first pair and on the second pair of indexes, and the symmetry on exchange of two pairs. Is worth to recall that these are tensor equations, therefore they are valid in any basis. An important result that can be obtained using properties (3.11) and (3.12) in (3.10), is that the number of independent components of the Riemann tensor are 20, in 4 dimensions<sup>3</sup>.

A flat manifold corresponds to one where the Riemann tensor vanishes  $R_{\alpha\beta\mu\nu} = 0$ . We can also obtain this result in a local reference frame. Let's remember that by definition, a manifold is a structure that locally looks like  $\mathbb{R}^n$ , which implies that  $\Gamma^\alpha_{\beta\mu,\nu} = 0$ , so from (3.2) we have

$$\Gamma^\alpha_{\mu\nu,\sigma} = \frac{1}{2}g^{\alpha\beta}(g_{\beta\mu,\nu\sigma} + g_{\beta\nu,\mu\sigma} - g_{\mu\nu,\beta\sigma}) \quad (3.13)$$

but second derivatives of the metric tensor does not vanish, then we have from (3.10)

$$R^\alpha_{\beta\mu\nu} = \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\beta,\nu\mu} + g_{\sigma\nu,\beta\mu} - g_{\beta\nu,\sigma\mu} - g_{\sigma\beta,\mu\nu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu})$$

considering that partial derivatives always commute, we have

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<sup>3</sup>The independent components of the Riemann tensor can be reduced even further imposing symmetry conditions. An important example is the Schwarzschild solution which is built under spherical symmetry. We will discuss this later.

$$R^\alpha_{\beta\mu\nu} = \frac{1}{2}g^{\alpha\sigma}(g_{\sigma\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\sigma\mu}) \quad (3.14)$$

finally, we can lower the index  $\alpha$  using the metric tensor to obtain finally

$$R_{\alpha\beta\mu\nu} \equiv g_{\alpha\lambda}R^\lambda_{\beta\mu\nu} = \frac{1}{2}(g_{\alpha\nu,\beta\mu} - g_{\sigma\mu,\beta\nu} + g_{\beta\mu,\sigma\nu} - g_{\beta\nu,\alpha\mu}) \quad (3.15)$$

So, (3.15) corresponds to the Riemann tensor in a locally inertial reference frame.

Let's differentiate (3.15) with respect to  $x^\lambda$

$$R_{\alpha\beta\mu\nu,\lambda} = \frac{1}{2}(g_{\alpha\nu,\beta\mu\lambda} - g_{\alpha\mu,\beta\nu\lambda} + g_{\beta\mu,\alpha\nu\lambda} - g_{\beta\nu,\alpha\mu\lambda})$$

from the symmetry condition  $g_{\alpha\beta} = g_{\beta\alpha}$  and that partial derivatives commute, we have

$$R_{\alpha\beta\mu\nu,\lambda} + R_{\alpha\beta\lambda\mu,\nu} + R_{\alpha\beta\nu\lambda,\mu} = 0 \quad (3.16)$$

but this is a tensorial equation, we can apply the rule “comma goes to semi-colon” and then write

$$\boxed{R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu} = 0} \quad (3.17)$$

This is an important result known as the *Bianchi identities*. In the following we will explore the consequences of these identities. Before going there, is useful to define the contracted Riemann tensor or *Ricci tensor*

$$R_{\alpha\beta} \equiv R^\mu_{\alpha\mu\beta} = R_{\beta\alpha} \quad (3.18)$$

From the Ricci tensor we can define the *Ricci scalar* or curvature scalar

$$R \equiv g^{\alpha\beta}R_{\alpha\beta} = g^{\mu\nu}g^{\alpha\beta}R_{\alpha\mu\beta\nu} \quad (3.19)$$

Let us contract indexes in the Bianchi identities (3.17)

$$g^{\alpha\mu} [R_{\alpha\beta\mu\nu;\lambda} + R_{\alpha\beta\lambda\mu;\nu} + R_{\alpha\beta\nu\lambda;\mu}] = 0 \quad (3.20)$$

using (3.3) and the fact that

$$g^{\alpha\mu} R_{\alpha\beta\lambda\mu;\nu} = -g^{\alpha\mu} R_{\alpha\beta\mu\lambda;\nu} = -R_{\beta\lambda;\nu}$$

we have from (3.20)

$$R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^\mu_{\beta\nu\lambda;\mu} = 0$$

contracting one more time in  $\beta$  and  $\nu$  we have

$$g^{\beta\nu} [R_{\beta\nu;\lambda} - R_{\beta\lambda;\nu} + R^\mu_{\beta\nu\lambda;\mu}] = 0$$

which reduces to

$$R_{;\lambda} - R^\mu_{\lambda;\mu} - R^\mu_{\lambda;\mu} = 0$$

last equation can be written as

$$(2R^\mu_{\lambda} - \delta^\mu_{\lambda} R)_{;\mu} = 0 \quad (3.21)$$

We can define the *Einstein symmetric tensor* as

$$\boxed{G^{\alpha\beta} \equiv R^{\alpha\beta} - \frac{1}{2} R g^{\alpha\beta}} \quad (3.22)$$

which from (3.21) satisfies

$$G^{\alpha\beta}_{;\beta} = 0 \quad (3.23)$$

Conclusion: the Einstein tensor is just a consequence of the Riemann tensor and the metric tensor, and it's *divergenceless*. This result will be highly important for

the discussion of the next section, where we will write the field equations of general relativity.

### 3.4 GRAVITY IS GEOMETRY: THE EINSTEIN'S EQUATION

The table is set, and all our tensors ‘army’ has been organized. We are ready now to embrace the physics ideas that lies in the core of the general relativity. The discussion will go in two aspects: how the gravitational field determines the inertia of test bodies, and how the matter determines the gravitational field. As we discussed in Chapter 1, Einstein found inspiration in the Newtonian theory, particularly in the Poisson’s equation (Eq. 1.3) which relates matter density with gravitational field

$$\nabla^2\phi = 4\pi G\rho$$

As a first guess, Einstein tried the equation

$$R_{\mu\nu} = kT_{\mu\nu} \tag{3.24}$$

but this equation shows problems with the energy conservation law  $T_{\mu\nu;\nu} = 0$ . However from (3.23) we know the Einstein tensor satisfies the divergenceless condition  $G^{\alpha\beta}_{;\beta} = 0$ , which is in concordance with the energy conservation. Therefore, we can present finally the *Einstein’s field equation*

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi GT_{\mu\nu}} \tag{3.25}$$

where  $G$  is the Newton’s constant of gravitation. Equation (3.25) informs us how the geometry of the spacetime (left side) will be determined by the content of energy and momentum (right side). A worth point to remark here, is that any type of energy that we can write in the energy-momentum tensor will *curve* the spacetime. For example, a charged particle will produce an electric field in its neighborhood, which

will affect the geometry of the spacetime. This scenario, called *electrovac universe*, will be discussed in the next chapter.

In summary, Einstein postulated his equation based on the following principles:

- Resemblance with the Poisson's equation: matter as the source of gravitational field.
- General covariance principle (no preference for any coordinate system).
- Local conservation of energy-momentum for any  $g_{\mu\nu}$ .

Einstein's equation corresponds to a set of second-order differential equations for the metric  $g_{\mu\nu}$ . Due to the symmetry in the two-index tensors, it reduces to ten independent equations. Going further, the Bianchi identities provides 4 constraints on the Ricci tensor, so at the end we have 6 independent equations in (3.25). The complexity of Einstein's equation is formidable, the non-linearity of the theory makes a very hard task to find exact solutions to the equations. However, few months after Einstein published his theory, Karl Schwarzschild found a solution to Einstein equation which describes the external gravitational field due to a spherical mass[5]. In polar coordinates  $(t, r, \theta, \phi)$  the Schwarzschild metric reads

$$ds^2 = -\left(1 - \frac{2GM}{r}\right) dt^2 + \left(1 - \frac{2GM}{r}\right)^{-1} dr^2 + r^2(d\theta^2 + \sin^2\theta d\phi^2) \quad (3.26)$$

where  $M$  is interpreted as the mass of the object producing the gravitational field<sup>4</sup>. This is a very important result which allows to study some of the experimental tests in GR, namely: deflection of light due to a mass and the precession of Mercury perihelia. Even further, the Schwarzschild metric *predicts* the existence of 'black holes'<sup>5</sup>. Note

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<sup>4</sup>A clear derivation of this metric is found in [5].

<sup>5</sup>The term 'black hole' was introduced by J.A. Wheeler in the 60's.

that when  $r = 2GM$  (called the Schwarzschild radius and denoted by  $r_s$ ), the radial component  $g_{rr}$  metric diverges while the time component  $g_{tt}$  vanishes. Once the geodesic motion in this spacetime is studied, it turns out to be that particles moving along timelike and null geodesics, cannot scape from the inner region to  $r_s$ .

The surface  $r = 2GM$ , although locally is regular, globally behaves as a ‘non-return point’. Once a particle (even a photon!) crosses the surface, it can never scape<sup>6</sup>. The radius  $r = 2GM$  forms what is called an *event horizon*. Once the matter crosses the event horizon, it collapses to a singular point in  $r = 0$ , called a *singularity*. In the 70’s Hawking[12] and Penrose showed some theorems related to the properties of these singularities in general relativity. A deeper study of singularities and black holes is beyond the scope of this thesis, but the interested reader should consult the references[2, 12, 31]<sup>7</sup>.

### 3.5 THE EINSTEIN EQUATION PLUS THE COSMOLOGICAL CONSTANT

After the culmination of the general relativity (Nov. 1915) and the publication of the Schwarzschild solution; physicists started to apply the Einstein’s equation to describe the whole universe. Einstein was ‘guided’ (in fact he was misguided as we will see) by the still ‘primitive’ observations at the time, which indicated that the velocity of the distant stars is negligible. This observation suggested that the universe was *static*. However, in 1922 the Soviet mathematician A. Friedmann found a set of solutions to Einstein’s equation that describe the dynamics of an *expanding* universe. In his model, Friedmann assumed the universe to be *isotropic* and *homogeneous*, which can be described by the following energy-momentum tensor

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<sup>6</sup>Even light will be trapped inside the Schwarzschild radius, so it’s not possible for us to see inside. That’s why the term ‘black hole’.

<sup>7</sup>An alternative model to ‘black hole’, which alleviates most of its problems, was proposed by Mazur and Mottola which is called a *gravastar*[32]. A detailed discussion of this model is out of the scope of this thesis.

$$T_{\mu\nu} = \begin{pmatrix} \rho & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ 0 & 0 & 0 & p \end{pmatrix} \quad (3.27)$$

When (3.27) is applied to the Einstein's equation, assuming a Robertson-Walker metric, one finds the *Friedmann equations*

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} \quad (3.28)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) \quad (3.29)$$

where  $a(t) = \frac{R(t)}{R_0}$  is called the *scale factor*, which measures the universal expansion rate. The scale factor is a function of time only, and it tells us how physical separations grows in time. Note that these equations show an *evolution* of the scale factor  $a$ , implying an evolving universe. In summary, general relativity predicts an *expanding universe*. In 1927, Georges Lemaitre who was a Belgium priest, astronomer and physics professor at the Université catholique de Louvain, proposed a model of an expanding universe where the universe had a beginning at the ‘big bang’<sup>8</sup>.

In order to conciliate his new theory of gravitation with a static universe, Einstein introduced the *cosmological constant*  $\Lambda$  in his equations

$$\boxed{R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} + \Lambda g_{\mu\nu} = 8\pi GT_{\mu\nu}} \quad (3.30)$$

the lambda-term affects the dynamical Friedmann equations like[33]

$$\left(\frac{\dot{a}}{a}\right)^2 = \frac{8\pi G}{3}\rho - \frac{k}{a^2} + \frac{\Lambda}{3} \quad (3.31)$$

$$\frac{\ddot{a}}{a} = -\frac{4\pi G}{3}(\rho + 3p) + \frac{\Lambda}{3} \quad (3.32)$$

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<sup>8</sup>The cosmological ideas of Lemaitre were driven as well by his religious beliefs. He thought this ‘big bang’ corresponded to the moment of creation of the universe by God.

Note that a sufficient large positive value of  $\Lambda$ , compensates the gravitational attraction represented by the first term to the right in (3.32). However in 1924, the American astronomer Edwin Hubble concluded of his observations, that the distant galaxies are ‘receding’ from us, validating the Friedmann-Lemaitre model. This led Einstein to consider the inclusion of  $\Lambda$  as his “biggest blunder” in all his very prolific career. Once the observations by Hubble supported the original predictions of his theory, Einstein was determined to eliminate  $\Lambda$  of his equations. In a letter to H. Weyl, Einstein wrote[19]: *“If there is no quasi-static world, then away with the cosmological term”*.

However, once the rabbit is out of the hat, is not easy to put it back again. Eddington was one of the detractors of the idea of eliminating  $\Lambda$ , which he considered was a natural addition to the equations. In fact, Einstein himself was aware of this term before discussing any idea about cosmology (see §1.4). Einstein realized that the more general  $\binom{0}{2}$  tensor that satisfies the divergenceless condition is:  $G_{\mu\nu} + \lambda g_{\mu\nu}$ . However, he neglected this additional term, because it ‘removed the beauty of the theory’. But as we see, he recalled this term again once he found his theory was not in concordance with a static universe.

How it is possible that the man who had the courage to change the Newtonian ideas of space and time which were reigning in physics during almost two centuries, was not brave enough to trust in his theory and *predict* an expanding universe and to push the astronomers to improve his observations?. Instead of that he just added a term, in a completely *ad-hoc* manner, just to match his theory with primitive “observations”. I think what Einstein called his “biggest blunder” was the fact of not being confident enough in his theory. Even a great genius, can be a ‘fool’ sometimes.

Recently with the discovery of the accelerating expansion of the universe[34], the cosmological constant returns to the game as the most prominent explanation to this effect. In principle the  $\Lambda$ -term is associated to the *vacuum energy*: an energy density



characteristic of empty space[5]. Usually the energy-momentum tensor associated to  $\Lambda$  is required to be Lorentz invariant in a locally inertial frame<sup>9</sup>. Lorentz invariance implies that  $T_{\mu\nu}$  should be proportional to the metric tensor

$$T_{\mu\nu}^{vac} = -\rho_{vac}g_{\mu\nu} \quad (3.33)$$

On the other hand, we know that the energy-momentum tensor of a perfect fluid is given by

$$T_{\mu\nu} = (\rho + p)U_\mu U_\nu + pg_{\mu\nu} \quad (3.34)$$

where  $U_\mu$  is the four-velocity. Comparing (3.33) with (3.34) (assuming a local rest frame such that  $U_\mu = 0$ ) we have

$$p_{vac} = -\rho_{vac} \quad (3.35)$$

which implies that the vacuum energy density behaves as a ‘perfect’ fluid with an isotropic pressure<sup>10</sup>. Using (3.35) we can rewrite the Einstein equation as follows

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G \left( T_{\mu\nu}^{(m)} - \rho_{vac}g_{\mu\nu} \right) \quad (3.36)$$

where  $T_{\mu\nu}^{(m)}$  indicates the energy-momentum tensor of matter (baryonic matter let’s say). Comparing (3.36) with (3.30) we can see a relation between the cosmological constant and  $\rho_{vac}$

$$\boxed{\rho_{vac} = \frac{\Lambda}{8\pi G}} \quad (3.37)$$

In this model, cosmological constant and ‘vacuum energy’ are almost interchangeable. Despite the fact of this ‘neat’ association, the previous interpretation suffers of a

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<sup>9</sup>The vacuum is assumed to be isotropic, it does not pick out a preferred direction.

<sup>10</sup>The minus sign is because we are using a signature  $(-, +, +, +)$ .

terrible trouble. One possible contribution of the vacuum energy, is provided by quantum field theory as the zero-point fluctuations[35]. We know that the lowest energy state has energy  $E_0 = \frac{1}{2}\hbar\omega$ . We can imagine the ‘empty’ space filled with quantum harmonic oscillators<sup>11</sup>. The frequency of each oscillator is given by the dispersion relation  $\omega = \sqrt{m^2 + k^2}$ . Integrating all the contributions of each one of these oscillators we have for the average value of  $\rho_{vac}$

$$\rho_{vac}^{QFT} = \frac{1}{(2\pi)^3} \int_0^\infty d^3k \left( \frac{1}{2}\hbar\omega \right) \quad (3.38)$$

we don’t need to go further to realize that this integral goes to infinity. To bypass this issue, the ‘trick’ is to integrate until a *cut-off* momentum  $k_{max} \gg m$  (ultraviolet momentum) so we can obtain a finite value[36]

$$\rho_{vac}^{QFT} = \frac{1}{(2\pi)^3} \int_0^\infty d^3k \left( \frac{1}{2}\hbar\omega \right) \approx \frac{\hbar}{(2\pi)^3} \int_0^{k_{max}} 4\pi k^2 dk \left( \frac{1}{2}\sqrt{m^2 + k^2} \right) \quad (3.39)$$

we can rewrite (3.39) like

$$\rho_{vac}^{QFT} = \frac{2\pi\hbar}{(2\pi)^3} \int_0^{k_{max}} dk \left( k^3 \sqrt{1 + \frac{m^2}{k^2}} \right)$$

expanding the term in the radical at first order in the ratio  $\frac{m^2}{k^2}$  and integrating, we obtain

$$\rho_{vac}^{QFT} \simeq \frac{\hbar k_{max}^4}{16\pi^2} \quad (3.40)$$

If we believe that we can use QFT up to the Planck scale, where the reduced Planck mass is given by:  $M_P = \frac{1}{\sqrt{8\pi G}} \sim 10^{18} GeV$ [11] we might say that  $\rho_{vac}$  is roughly

$$\rho_{vac}^{QFT} \sim (10^{18} GeV)^4 \sim 10^{112} erg/cm^3 \quad (3.41)$$

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<sup>11</sup>Following the Dr. Creswick’s conjecture: ‘everything is a harmonic oscillator’.

However, the observations of Type Ia Supernova plus observations of anisotropies in the CMB[34] have put limits to the vacuum energy

$$\rho_{vac}^{obs} \sim 10^{-8} erg/cm^3 \sim (10^{-3} eV)^4 \quad (3.42)$$

comparison of (3.41) with (3.42) gives

$$\rho_{vac}^{obs} \sim 10^{-120} \rho_{vac}^{QFT} \quad (3.43)$$

This is the ‘famous’ (I’d rather say infamous) discrepancy of 120 orders of magnitude. This is probably the worst theoretical prediction in all the history of physics. Clearly the cosmological constant suffers of this chronic issue, which is still unsolved. This situation has put the ‘dark energy’ problem as the biggest one in the current physics, and the one that is gaining most of the attention of the community. Some alternative models have been proposed in the literature: quintessence fields, modified gravity, timescape model[37], extensions of general relativity at large scale, among others. The problem is still open, current and future missions (DES, Planck, Euclid) hopefully will provide us with valuable information about the nature of this mysterious cosmological constant.

## CHAPTER 4

### THE EINSTEIN-MAXWELL SYSTEM AND THE ELECTROVAC UNIVERSE

*The physical world is represented as a four-dimensional continuum. If in this I adopt a Riemannian metric, and look for the simplest laws which such a metric can satisfy, I arrive at the relativistic gravitation theory of empty space. If I adopt in this space a vector field, or the antisymmetrical tensor field derived from it, and if I look for the simplest laws which such a field can satisfy, I arrive at the Maxwell equations for free space. ...at any given moment, out of all conceivable constructions, a single one has always proved itself absolutely superior to all the rest...*

Albert Einstein (1934)

In this final chapter a class of exact solutions to the Einstein's equation, known as *electrovac universe*, is discussed. The comformastat metric is introduced and its consequences in the GR context are explored. The comformastat spacetime is applied to analyze the gravitational field due to a charged mass, which produces the Majumdar-Papapetrou solutions. This solution is extended considering the cosmological constant. We offer solutions to the new extended equation.

#### 4.1 STATIC UNIVERSES IN CONFORMASTAT FORM

In the following sections we use geometrized units  $c = G = 1$ . Per definition, a general static universe is represented by the following metric

$$ds^2 = -V^2(x^i)dt^2 + U^2(x^i)dx^i dx^i \quad (4.1)$$

which Synge[6] calls *conformastat*. Note that the metric elements in (4.1) depends only on spatial coordinates, therefore the metric (4.1) is invariant under  $x^0 \rightarrow x^0 + \text{const}$ . In more technical terms, we can find a Killing vector  $\xi^\alpha$  associated with this symmetry, namely

$$\xi \cdot u = g_{tt}\xi^\alpha u^t = -V^2 u^t = \text{const}. \quad (4.2)$$

where  $\xi^\alpha = (1, 0, 0, 0)$  in the basis  $(t, x, y, z)$ . The Christoffel symbols for the metric (4.1) can be readily calculated to be

$$\begin{aligned} \Gamma_{i0}^0 &= V^{-1}V_{,i} \quad ; \quad \Gamma_{00}^i = U^{-2}V V_{,i} \\ \Gamma_{ij}^i &= U^{-1}U_{,j} \quad ; \quad \Gamma_{jj}^i = -U^{-1}U_{,i} \text{ for } i \neq j \end{aligned} \quad (4.3)$$

The spatial part of the Ricci tensor (3.18) is

$$\begin{aligned} R_{ij} &= U^{-1}(U_{,ij} + \delta_{ij}U_{,kk}) - 2U^{-2}U_{,i}U_{,j} + V^{-1}V_{,ij} \\ &\quad - (UV)^{-1}(U_{,i}V_{,j} + U_{,j}V_{,i}) + (UV)^{-1}\delta_{ij}U_{,k}V_{,k} \end{aligned} \quad (4.4)$$

whereas the temporal part takes the form

$$R_{00} = -VU^{-2}(V_{,kk} + U^{-1}U_{,k}V_{,k}) \quad (4.5)$$

This allows us to calculate the Ricci scalar (3.19) as

$$R = 4U^{-3} \left( U_{,kk} - \frac{1}{2}U^{-1}U_{,k}U_{,k} \right) + 2U^{-2}V^{-1}(V_{,kk} + U^{-1}U_{,k}V_{,k}) \quad (4.6)$$

Note that we have not specified any energy-momentum tensor, therefore these results are valid for any metric of the *conformastat* type. Before applying these results to

the Einstein-Maxwell system (where  $T_{\mu\nu}$  is given by the electromagnetic tensor), let us explore what results we obtain if we specialize to the vacuum case i.e.

$$R_{\mu\nu} = 0 \quad (4.7)$$

Equation (4.7) implies

$$R = 0 \quad (4.8)$$

using (4.5) we have

$$V_{,kk} + U^{-1}U_{,k}U_{,k} = 0 \quad (4.9)$$

Therefore equation (4.6) reduces to

$$U_{,kk} - \frac{1}{2}U^{-1}U_{,k}U_{,k} = 0 \quad (4.10)$$

which can be arranged to be written as the Laplace equation

$$\boxed{(\sqrt{U})_{,kk} = 0} \quad (4.11)$$

Therefore, only from the vacuum condition  $R_{\mu\nu} = 0$ , we conclude that all  $\sqrt{U}$  must be *harmonic functions*. Thus, knowing  $U$  we can obtain  $V$  by solving (4.9) which shows the form of the Poisson equation. Let us analyze an example of this general result. A well known solution is, of course, the Schwarzschild spacetime which was discussed in §3.4. In isotropic coordinates, this metric is given by[28]

$$ds^2 = - \left( \frac{1-\xi}{1+\xi} \right)^2 dt^2 + (1+\xi)^4 (d\rho^2 + \rho^2 d\Omega^2) \quad (4.12)$$

where  $\xi = \frac{m}{2\rho}$ . Comparing (4.12) with (4.1) we conclude that

$$U = (1+\xi)^2 \quad ; \quad V = \frac{1-\xi}{1+\xi} \quad (4.13)$$

with  $\rho^2 = x^i x^i$ . Is straightforward to see that  $\sqrt{U}$  is a harmonic function in the isotropic coordinates.

So far, we have specialized on the vacuum case. Let us now introduce the condition  $UV = 1$  such that the metric takes the form

$$ds^2 = -U^{-2}dt^2 + U^2 dx^i dx^j \delta_{ij} \quad (4.14)$$

Equation (4.4) can be written as

$$\begin{aligned} R_{ij} = & U^{-1}(U_{,ij} + \delta_{ij}U_{,kk}) - 2U^{-2}U_{,i}U_{,j} + U(U^{-1})_{,ij} \\ & - U_i(U^{-1})_{,j} - U_j(U^{-1})_{,i} + \delta_{ij}U_{,k}(U^{-1})_{,k} \end{aligned}$$

after simplifications this equation gives

$$R_{ij} = \delta_{ij}U^{-1}(U_{,kk} - U^{-1}U_{,k}U_{,k}) + 2U^{-2}U_{,i}U_{,j} \quad (4.15)$$

where we used  $(U^{-1})_{,ij} = 2U^{-3}U_{,j}U_{,i} - U^{-2}U_{,ij}$ . The corresponding spatial components of the Ricci tensor and the Ricci scalar are:

$$R_{00} = U^{-5}(U_{,kk} - U^{-1}U_{,k}U_{,k}) \quad (4.16)$$

$$R = 2U^{-3}U_{,kk} \quad (4.17)$$

Note that in the last development we have not used the Einstein equation, therefore these results are purely geometrical consequences of the Ricci tensor. We conclude that equation (4.17) represents a general result for the function  $U$  if  $R$  is known (e.g. vacuum case, conformal energy-momentum tensor with  $T^\mu_\mu = 0$ , etc.). We will recall this result when we introduce the energy-momentum tensor and the Einstein's equation.

## 4.2 ELECTROVAC UNIVERSE

Let's suppose we have a static electric charge located somewhere in an empty universe. In the exterior region of the charge, we have only an electric field, but no matter. This scenario was called by Synge the *electrovac universe* [6]. This situation can be

understood as the generalization of the Reissner-Nordström metric [5] who considered the spherically symmetric case. The implications of the Reissner-Nordström metric in a cosmological context, was studied by Posada [40] by using curvature and geodesic coordinates. Let us first choose the metric to be of the form

$$ds^2 = -V^2(x^i)dt^2 + h_{ij}(x^k)dx^i dx^j \quad (4.18)$$

We will first derive general results by using this metric, later on we will specialize to the conformastat case (4.1). We are looking for the solutions of the Einstein equation (here still with  $\Lambda = 0$ )

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi T_{\mu\nu} \quad (4.19)$$

where the source  $T_{\mu\nu}$  is the electromagnetic energy-momentum tensor

$$T_{\mu\nu} = \frac{1}{4}g_{\mu\nu}F_{\alpha\beta}F^{\alpha\beta} - F_{\mu\alpha}F_{\nu}^{\alpha} \quad (4.20)$$

and the Maxwell equations in vacuum

$$F^{\mu\nu}_{;\nu} = F^{\mu\nu}_{;\nu} + \Gamma^{\nu}_{\alpha\nu}F^{\mu\alpha} + \Gamma^{\mu}_{\alpha\nu}F^{\alpha\nu} = 0 \quad (4.21)$$

under the condition that the system is purely electrostatic. This implies that there is only one component of the electromagnetic tensor

$$F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \quad (4.22)$$

which is non-zero<sup>1</sup>:

$$F_{0i} = -A_{0,i} = -\phi_{,i} \quad (4.23)$$

where  $\phi$  is the electric potential. In terms of this potential the spatial components of the electromagnetic tensor can be obtained as

$$T_{ij} = V^{-2} \left( \frac{1}{2}h_{ij}\Delta_1\phi - \phi_{,i}\phi_{,j} \right) \quad (4.24)$$

---

<sup>1</sup>The mathematically possible case  $F_{23} = -F_{32} \neq 0$  would indicate a magnetic monopole which we leave out of our discussion.



where we have defined

$$\Delta_1 \phi \equiv h^{ij} \phi_{,i} \phi_{,j} \quad (4.25)$$

The temporal components are simply

$$T_{00} = -\frac{1}{2} h^{ij} \phi_{,i} \phi_{,j} = -\frac{1}{2} \Delta_1 \phi \quad (4.26)$$

The explicit form of the energy-momentum tensor can be now used to write down the Einstein equation as

$$R_{ij} = 8\pi V^{-2} \left( \frac{1}{2} h_{ij} \Delta_1 \phi - \phi_{,i} \phi_{,j} \right) \quad (4.27)$$

$$R_{00} = 4\pi \Delta_1 \phi = V \Delta_2 V \quad (4.28)$$

In the above we used the traceless condition of the electromagnetic tensor  $T^\mu_\mu = 0$ , such that the Einstein equation takes the form  $R_{\mu\nu} = 8\pi G T_{\mu\nu}$ . Here we introduced a new definition, namely

$$\Delta_2 \phi \equiv h^{ij} \phi_{||ij} \quad (4.29)$$

The double vertical lines indicates the covariant derivative with respect to the spatial metric  $h_{ij}$ . Note that the remaining components of the Einstein equation,  $R_{i0} = 8\pi G T_{i0}$  are identically satisfied. It is clear that the only relevant component of the Maxwell equation is

$$F^{0i}_{;i} = F^{0i}_{;i} + \Gamma^i_{ik} F^{0k} + \Gamma^0_{0i} F^{0i} = 0 \quad (4.30)$$

with

$$F^{0i} = g^{00} h^{ij} F_{0j} = V^{-2} h^{ij} \phi_{,j} \quad (4.31)$$

one obtains easily

$$F^{0i}_{;i} = V^{-2} (-2V^{-1} V_{,i} h^{ij} \phi_{,j} + \phi_{,j} h^{ij}_{;i} + h^{ij} \phi_{,ij}) \quad (4.32)$$

On the other hand we have

$$\Gamma^i_{ik} F^{0k} = \frac{1}{2} V^{-2} h^{lm} (h_{lm,k}) (h^{kj} \phi_{,j}) \quad (4.33)$$

$$\Gamma^0_{0i} F^{0i} = V^{-3} h^{ij} V_{,i} \phi_{,j} \quad (4.34)$$

Using (4.32) and (4.33) equation (4.30) takes the form

$$F^{0i}_{;i} = V(h^{ij} \phi_{,ij} + h^{ij}_{,i} \phi_{,j}) + \frac{1}{2} V [h^{lm} (h_{lm})_{,k} h^{kj} \phi_{,j}] - h^{ij} V_{,i} \phi_{,j} = 0 \quad (4.35)$$

which can be simplified further noticing that  $h^{ij}_{,i} = (h_{ij})^{-1}_{,i} = -h^{-2}_{ij} h_{ij,i}$  and

$$\phi_{||ij} = (\phi_{,i})_{;j} = \phi_{,ij} - \Gamma^\gamma_{ij} \phi_\gamma = \phi_{,ij} - \frac{1}{2} h^{km} h_{km,i} \phi_{,j}$$

Making use of the definition (4.29) we can rewrite equation (4.35) in an elegant form, namely

$$F^{0i}_{;i} = V \Delta_2 \phi - h^{ij} V_{,i} \phi_{,j} = 0 \quad (4.36)$$

In particular, we are interested in solutions where  $V$  and  $\phi$  are functionally related  $V = V(\phi)$  [7, 8]. This condition, allows us to write the following

$$\begin{aligned} V_{||ij} &= V' \phi_{||ij} + V'' \phi_{,i} \phi_{,i} \quad ; \quad V' = \frac{dV}{d\phi} \\ V'' &= \frac{d^2 V}{d\phi^2} \quad ; \quad V_{,i} = V' \phi_{,i} \end{aligned} \quad (4.37)$$

To summarize, we are looking for  $V$ ,  $\phi$  y  $h_{ij}$  such that the Einstein equation (4.27), (4.28) and the Maxwell equation (4.36) are satisfied. Concentrating first on (4.28) and (4.36) this means that we have to solve

$$V \Delta_2 V - 4\pi \Delta_1 \phi = 0 \quad (4.38)$$

and

$$V \Delta_2 \phi - V' \Delta_1 \phi = 0 \quad (4.39)$$

Making explicit use of (4.37) and the definitions (4.25) and (4.29) we have the identities

$$\Delta_1 V = h^{ij} V_{,i} V_{,j} = h^{ij} V'^2 \phi_{,i} \phi_{,j} = V'^2 \Delta_1 \phi \quad (4.40)$$

$$\begin{aligned}\Delta_2 V &= h^{ij} V_{||ij} = h^{ij} (V' \phi_{||ij} + V'' \phi_{,i} \phi_{,j}) \\ &= V' \Delta_2 \phi + V'' \Delta_1 \phi\end{aligned}$$

The above identities are now used to put equation (4.38) in the form

$$VV' \Delta_2 \phi + (VV'' - 4\pi) \Delta_1 \phi = 0 \quad (4.41)$$

This form is in particular useful as multiplying (4.39) by  $(-V')$  and adding the result to (4.41) we arrive at

$$\Delta_1 \phi (VV'' + V'^2 - 4\pi) = 0 \quad (4.42)$$

which is equivalent to

$$VV'' + V'^2 - 4\pi = 0 \quad (4.43)$$

assuming  $\Delta_1 \phi \neq 0$ . Integrated once we obtain

$$VV' - 4\pi\phi - \beta = \frac{1}{2}(V^2)' - 4\pi\phi - \beta = 0 \quad (4.44)$$

where  $\beta$  is an arbitrary integration constant. A second integration yields the desired relation between  $V$  and  $\phi$ , namely

$$\boxed{V^2 = A + B\phi + 4\pi\phi^2} \quad (4.45)$$

where  $A$  y  $B$  are arbitrary constants. This functional relation is part of the Majumdar-Papapetrou solution[7, 8]. Note that we still have not used the Einstein equation (4.27), and we will not do it in the following. Instead we assume that the electrovac universe given by the metric (4.18) takes a particular form of the conformastat type (4.1). This is to say we assume  $h_{ij} + V^{-2}\delta_{ij}$  or

$$ds^2 = -V^2 dt^2 + V^{-2} dx^i dx^j \delta_{ij} \quad (4.46)$$

such that  $V = U^{-1}$ . Recalling that  $T^\mu_\mu = 0$  implies  $R = 0$ , equation (4.17) reduces then to the Laplace equation

$$\boxed{\Delta U \equiv U_{,kk} = 0} \quad (4.47)$$

In vacuum we found that  $\sqrt{U}$  must be a harmonic function (see (4.11)). Now with (4.47) we find that  $U$  satisfies Laplace equation. Therefore we have a simple way to specify an electrovac solution: take any *harmonic* solution  $U$  and define  $V$  by  $UV = 1$ , then use (4.45) to solve for the potential. In order to recover the flat spacetime far from the source, we must choose  $U$  such that  $U^2 \rightarrow 1$  at infinity. As in the vacuum case, the isotropic coordinates play a special role here and it is illuminating to dwell upon their role in the Reissner-Nordström case (which is the spherically symmetric sub-case of the more general one studied above). This metric in Schwarzschild coordinates is given by[5]

$$ds^2 = - \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right) dt^2 + \left(1 - \frac{2m}{r} + \frac{Q^2}{r^2}\right)^{-1} dr^2 + r^2 d\Omega^2 \quad (4.48)$$

The transformation to isotropic coordinate  $\rho$  involves

$$r = \rho \left(1 + \frac{m}{\rho} + \frac{m^2 - Q^2}{4\rho^2}\right) \quad (4.49)$$

and we obtain

$$ds^2 = - \left[ \frac{m^2 - 4\rho^2 - Q^2}{(m + 2\rho)^2 - Q^2} \right]^2 dt^2 + \left(1 + \frac{m}{\rho} + \frac{m^2 - Q^2}{4\rho^2}\right)^2 (d\rho^2 + \rho^2 d\Omega^2) \quad (4.50)$$

Comparing (4.50) with (4.1) we can conclude that the function  $U$  is

$$U = 1 + \frac{m}{\rho} + \frac{m^2 - Q^2}{4\rho^2} \quad (4.51)$$

However, a straightforward calculation tells us that

$$\Delta U = \frac{1}{2\rho^4} (m^2 - Q^2) \quad (4.52)$$

Hence only if  $Q = m$  holds,  $U$  satisfies the Laplace equation. For this extreme case we also have

$$ds^2 = - \left( \frac{1}{1 + \frac{m}{\rho}} \right)^2 dt^2 + \left(1 + \frac{m}{\rho}\right)^2 [d\rho^2 + \rho^2 d\Omega^2] \quad (4.53)$$

and therefore  $V = U^{-1}$ , which shows the consistency of the model.

### 4.3 ELECTROVAC UNIVERSE WITH $\Lambda$

Having reviewed the electrovac universe with vanishing cosmological constant, let us discuss a relative fast derivation of the corresponding situation where  $\Lambda \neq 0$  [10]. Right from the beginning we can specialize to the conformastat case i.e.

$$ds^2 = -f^2 dt^2 + f^{-2} dx^i dx^j \delta_{ij} \quad (4.54)$$

The Einstein equation reads now

$$R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} + \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \quad (4.55)$$

From the traceless condition of the electromagnetic tensor, we have

$$R = 4\Lambda \quad (4.56)$$

This can be used to re-write the Einstein equation in a form which is more suitable for our purposes

$$R_{\mu\nu} - \Lambda g_{\mu\nu} = -8\pi T_{\mu\nu} \quad (4.57)$$

This makes it evident which modification the cosmological constant  $\Lambda$  introduces as compared to the results from the last section. Equation (4.27) becomes

$$R_{ij} - \Lambda h_{ij} = -8\pi f^{-2} \left( \frac{1}{2} h_{ij} \Delta_1 \phi - \phi_{,i} \phi_{,j} \right) \quad (4.58)$$

whereas (4.28) is simply

$$R_{00} + \Lambda f^2 = 4\pi \Delta_1 \phi = f \Delta_2 f \quad (4.59)$$

Reproducing the steps from section 3, we obtain the analogy to (4.42)

$$f \Delta_2 f - \Lambda f^2 - 4\pi \Delta_1 \phi = 0 \quad (4.60)$$

which integrated twice with respect to  $\phi$  gives

$$f^2 = A + B\phi + 4\pi\phi^2 + \Lambda \left[ \frac{1}{(\ln \phi)_{,i}} \right]^2 \quad (4.61)$$

As compared to the algebraic equation (4.45), the above equation (which reduces to (4.45) in the case of  $\Lambda = 0$ ) is a non-linear partial differential equation. Finally, the combination of (4.17) with (4.56) results in

$$\Delta U = 2\Lambda U^3 \quad (4.62)$$

which is the the generalization of (4.47). The linear Laplace equation becomes now in a non-linear partial differential equation for  $U$ . To summarize, if we know  $U$  we can use (4.61) to infer the electric potential  $\phi$  by the relation  $fU = 1$ . We note that the level of mathematical complication introduced by  $\Lambda$  is quite formidable. Note that the right side of (4.62) shows a coupling between the cosmological constant and the function  $U$  which is related to the potential  $\phi$ . This informs us that the electromagnetic phenomena (in our case the electric potential) will be affected by the cosmological constant[10]. In the following section, we will offer some solutions to this equation.

#### 4.4 SOLUTIONS

Before we come to the non-pertubative solution we mention that an iterative one can be found by using the standard technique. In case  $\Lambda$  is small, we can attempt a pertubative solution by the ansatz

$$U = U_0 + \Lambda^1 U_1 + \Lambda^2 U_2 + \dots \quad (4.63)$$

Back into equation (4.62) this ansatz gives first a Laplace equation followed by a series of Poisson equations:

$$\begin{aligned} \Delta U_0 &= 0 \\ \Delta U_1 &= 2U_0^3 \\ \Delta U_2 &= 6U_0^2 U_1 \\ &\dots \end{aligned} \quad (4.64)$$

Next we offer special cases of non-perturbative solutions. Let us first concentrate on the one dimension case. In one dimension equation (4.62) becomes an autonomous second order differential equation, namely

$$\frac{d^2U}{dx^2} = 2\Lambda U^3. \quad (4.65)$$

By means of the substitution  $u(U) = dU/dx$  the above equation can be reduced to the first order ODE

$$\frac{du^2}{dU} = 4\Lambda U^3 \quad (4.66)$$

that can be integrated yielding

$$u^2 = \Lambda U^4 + c_1 \quad (4.67)$$

where  $c_1$  is an integration constant. The last step consists in integrating the ODE

$$\frac{dU}{dx} = \pm \sqrt{\Lambda U^4 + c_1} \quad (4.68)$$

and we obtain

$$c_2 \pm x = \int \frac{dU}{\sqrt{\Lambda U^4 + c_1}}. \quad (4.69)$$

The integration constants  $c_1$  and  $c_2$  should be fixed so that the metric becomes de Sitter in the limit  $x \rightarrow \infty$ . However, the integral can be solved in terms of the elliptic function  $F$  as

$$c_2 \pm x = \frac{1}{\sqrt{i\sqrt{\Lambda c_1}}} F\left(\sqrt{i\sqrt{\Lambda/c_1}}, i\right). \quad (4.70)$$

Independently of the sign of  $c_1$  the above solution will be complex. Hence, the requirement that  $U$  is a real function of the spatial variable  $x$  will imply that  $c_1 = 0$ .

In this case the solution is

$$U(x) = -\frac{1}{\sqrt{\Lambda}(c_2 \pm x)}. \quad (4.71)$$

Now, let us consider the more complicated situation where  $U$  depends on two spatial variables  $x$  and  $y$ . In this case the equation to solve is

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 2\Lambda U^3 \quad (4.72)$$

with  $U = U(x, y)$ . The above equation is a special case of the following more general stationary heat equation with nonlinear source, namely

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = f(U); \quad f(U) = 2\Lambda U^3 \quad (4.73)$$

As in [?] let us suppose that  $U = U(x, y)$  is a solution of our equation. Then, the functions

$$\begin{aligned} U_1 &= U(\pm x + C_1, \pm y + C_2) \\ U_2 &= U(x \cos \beta - y \sin \beta, x \sin \beta + y \cos \beta) \end{aligned} \quad (4.74)$$

where  $C_1, C_2$  and  $\beta$  are arbitrary constants, are also solutions of the original equation. Implicit solutions can be found in the form

$$\begin{aligned} \int \left[ C + \frac{2}{A^2 + B^2} F(U) \right]^{-1/2} &= Ax + By + D \\ F(U) &= \int f(U) dU \end{aligned} \quad (4.75)$$

where  $A, B, C$  and  $D$  are arbitrary constants. Notice that for  $f(U) = 2\Lambda U^3$  the above integral gives rise to a complex elliptic function and again the requirement that  $U$  has to be a real function fixes  $C = 0$  and we obtain

$$U(x, y) = -\sqrt{\frac{A^2 + B^2}{\Lambda}} \frac{1}{Ax + By + D}. \quad (4.76)$$

If we assume a solution with central symmetry about the point  $(-C_1, -C_2)$  with  $U = U(\xi)$  where

$$\xi = \sqrt{(x + C_1)^2 + (y + C_2)^2} \quad (4.77)$$

and  $C_1, C_2$  are arbitrary constants, then the function  $U(\xi)$  is determined by the second order non-linear differential equation

$$\frac{d^2 U}{d\xi^2} + \frac{1}{\xi} \frac{dU}{d\xi} = f(U). \quad (4.78)$$

Since it is a quasi-linear equation it can be reduced to its normal form

$$\frac{d^2 U}{d\omega^2} = 2\Lambda e^{2\omega} U^3 \quad (4.79)$$



by means of the transformation  $\omega = \ln \xi$ . If we set  $2\omega = \tilde{x}$  the previous equation becomes

$$\frac{d^2 U}{d\tilde{x}^2} = \frac{\Lambda}{2} e^{\tilde{x}} U^3 \quad (4.80)$$

which is a particular case of the equation

$$\frac{d^2 y}{dx^2} = A e^x y^m \left( \frac{dy}{dx} \right)^\ell \quad (4.81)$$

given in [38]. Since in our present case  $\ell \neq 1 - m$  we have a particular solution

$$U(\omega) = \frac{1}{\sqrt{2\Lambda}} e^{-\omega}. \quad (4.82)$$

On the other side  $m \neq 0$  and  $\ell \neq 1$  and we can reduce equation (4.80) with the help of the transformation

$$t = \frac{dU}{d\tilde{x}}, \quad w = e^{\tilde{x}} \quad (4.83)$$

to a generalized Emden-Fowler equation with respect to  $w = w(t)$ , namely

$$\frac{d^2 w}{dt^2} = -3 \left( \frac{\Lambda}{2} \right)^{1/3} t w^{-1} \left( \frac{dw}{dt} \right)^{7/3}. \quad (4.84)$$

Unfortunately, the above equation does not match with those listed in [38]. Moreover, equation (4.73) can be seen as a particular case of

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = aU + bU^n. \quad (4.85)$$

For  $a = 0$  there is a self-similar solution of the form [39]

$$U(x, y) = x^{2/(1-n)} F(z), \quad z = \frac{y}{x}. \quad (4.86)$$

In our case for  $b = 2\Lambda$  and  $n = 3$  we shall have

$$U(x, y) = x^{-1} F(z) \quad (4.87)$$

where  $F(z)$  is a solution of the second order nonlinear ODE

$$(1 + z^2) \frac{d^2 F}{dz^2} + 4z \frac{dF}{dz} + 2F = 2\Lambda F^3. \quad (4.88)$$

The solution of the above equation can be expressed in terms of the Jacobi amplitude function  $J_{SN}$  as follows

$$F(z) = \frac{A_2}{\sqrt{(1 - \Lambda + A_2^2 \Lambda)(1 + z^2)}} \times J_{SN} \left( \frac{\sqrt{1 - \Lambda} \arctan(z) + A_1}{\sqrt{1 - \Lambda + A_2^2 \Lambda}}, A_2 \frac{\sqrt{\Lambda(1 - \Lambda)}}{\Lambda - 1} \right) \quad (4.89)$$

Finally, equation (4.62) can be seen as a special case of the more general equation

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = aU^n + bU^{2n-1} \quad (4.90)$$

with  $a = 2\Lambda$ ,  $n = 3$  and  $b = 0$ . For this choice the solutions of the above equation are [39]

$$U(x, y) = \left[ \frac{\Lambda}{2} (x \sin \alpha_1 + y \cos \alpha_1 + \alpha_2) \right]^{-1/2} \quad (4.91)$$

and

$$U(x, y) = \frac{1}{\sqrt{2\Lambda [(x + \alpha_1)^2 + (y + \alpha_2)^2]}} \quad (4.92)$$

where  $\alpha_1$  and  $\alpha_2$  are arbitrary constants. In contrast to the vanishing  $\Lambda$  case, here we must recover the de Sitter spacetime at infinity. Note that (4.92) shows in explicit form the idea discussed previously about the coupling between electromagnetism and cosmology in this theory, i.e., the cosmological constant affects the local electromagnetic phenomena, considering that  $U$  (which now is a function of  $\Lambda$ ) will determine the potential  $\phi$ . Notice that except for the case of a self-similar solution the above results can be easily generalized to the case when  $U$  depends on all three spatial variables[10].

## CONCLUSIONS

- The cosmological constant  $\Lambda$  is a natural addition to the Einstein's equation. There is no physical or mathematical argument, against the introduction of the  $\Lambda$  term. In fact, the current accelerated expansion of the universe, puts  $\Lambda$  in the cosmological scenario as a possible explanation for this phenomena. However, its physical nature is still a mystery.
- The Majumdar-Papapetrou electrovac universe model, provides an intimate relation between metric elements (by using a conformastat spacetime) and the electrostatic potential. The metric elements satisfies the Laplace equation, and these are related to the electrostatic potential by a simple algebraic equation.
- We found that the introduction of the cosmological constant  $\Lambda$  into the electrovac universe, increases the complexity in great manner. The Majumdar-Papapetrou functional relation becomes in a differential equation, and the metric elements now satisfies a non-linear second order differential equation. We investigated families of solutions to this equation in the 1-dimension and 2-dimensions cases, and we found a relation between cosmology and electromagnetism. The more important conceptual result, is that the cosmological constant affects the local electromagnetic phenomena.

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