On the Group of Transvections of ADE-Diagrams

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On the Group of Transvections of ADE-Diagrams

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Submitted in Partial Fulfillment of the Requirements for the Degree of Master of Science in Mathematics
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2014

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Abstract

In this thesis we examine symplectic spaces with forms generated by the ADE-diagrams. Specifically, we determine the generators of the group of transvections for each space under the standard basis, $\mathcal{S}$, of $K^n$ (where $K$ is a field with characteristic 0) and the hyperbolic basis, $\mathcal{H}$, we get from the classification theorem of symplectic spaces. Further, we examine how the generators of these groups are related via $g : G_{f,\mathcal{S}} \rightarrow SL(Z)_n$ where $g(X) = P^{-1}XP$ where $P$ is the change of basis matrix for $\mathcal{S}$ to $\mathcal{H}$. 
# Table of Contents

**Abstract** ......................................................... ii

**Chapter 1 Introduction** ......................................... 1

**Chapter 2 Preliminaries** ......................................... 3
  2.1 Skew-Symmetric Forms ........................................ 3
  2.2 Transvections of Hyperbolic Space .......................... 6
  2.3 ADE-Diagrams .................................................. 10

**Chapter 3 Transvections of ADE-diagrams** ....................... 17
  3.1 Transvections of $A_n$ ........................................ 17
  3.2 Transvections of $D_n$ ........................................ 27
  3.3 Transvections of $E_6, E_7, E_8$ ............................ 33

**Chapter 4 Future Questions** .................................... 38

**Bibliography** .................................................... 40
Historically, the ADE-diagrams come from the study of reflection groups. Coxeter proved that every reflection group is a Coxeter group [3]; a Coxeter group is a group with the presentation \( \langle r_1, r_2, \ldots, r_n | (r_i r_j)^{m_{ij}} = 1 \rangle \) where \( m_{ii} = 1 \) and \( m_{ij} \geq 2 \) for \( i \neq j \). He went on to show that the finite Coxeter groups could be classified by Coxeter-Dynkin diagrams [4]. In this thesis, we will study five of these diagrams: \( A_n, D_n, E_6, E_7, \) and \( E_8 \). Further, each of the diagrams we will consider the associated Coxeter group is a Weyl group. In Lie algebra, a Weyl group is the reflection group generated by roots.

In this thesis we will study the ADE-diagrams from the skew-symmetric perspective rather than the traditional symmetric. Specifically, we will study symplectic spaces with forms that are derived from an ADE-diagram. We will look at how the group of transvections in terms of the standard basis relate to the group of transvections in terms of a hyperbolic basis. We will construct an explicit hyperbolic basis for each ADE-diagram by using the following theorem:

**Theorem 1.1.** A non-singular symplectic space \( V \) is an orthogonal sum of hyperbolic planes; in other words \( V \) is a hyperbolic space. Further the dimension of \( V \) is always even.

While this theorem can be used to find a hyperbolic basis for spaces with each diagram, the hyperbolic basis we will use for our computations is not unique. However, regardless of our choice of hyperbolic basis we will have the following result:
Theorem 1.2. *Let* $H$ *be a hyperbolic space with dimension* $2s$. *The group of transvections of* $H$ *is the direct product of* $s$ *copies of* $SL_2(\mathbb{Z})$.

It is worth noting that for two distinct hyperbolic bases may give rise to two distinct (but isomorphic) group of transvections that are subgroups of $SL_n(\mathbb{Z})$. Moreover, we will see in Chapter 3 that many of the generators of the group of transvections for the ADE-diagrams with the standard basis will map to the generators of the group of transvections, $G_{f,S}$ for the ADE-diagrams with our particular choice of hyperbolic basis via $f : G_{f,S} \rightarrow SL_n(\mathbb{Z})$.

An outline of this thesis is as follows. In Chapter 2 we will provide all of the relevant preliminary materials. In Section 2.1 we will give a discussion of symplectic spaces with the goal of proving that symplectic spaces can be expressed in terms of a hyperbolic space. In Section 2.2 we will define transvections and determine the group of transvection of a symplectic space in terms of the hyperbolic basis. In Section 2.3 we will introduce the ADE-diagrams and symplectic spaces with forms associated to the diagrams. In Chapter 3 we will determine how the group of transvections of the symplectic space associated to the ADE-diagrams under the standard basis compares to the groups we determined in Section 2.3. In Chapter 4 we will discuss some future questions that can be explored concerning this thesis.
Chapter 2
Preliminaries

Throughout this chapter we will use the following conventions: $K$ is a field of characteristic 0, and all vectors are column vectors. Our treatment of symplectic spaces and transvections come from [1] and [5].

2.1 Skew-Symmetric Forms

Definition 2.1. A skew-symmetric form $f$ over a finite dimensional vector space $V$ is an alternating bilinear form that satisfies $f(n, n) = 0$ for all $n \in V$.

We note that Definition 2.1 is equivalent to $f(n, m) = -f(m, n)$ for all $n, m \in V$. In addition, if we choose a basis $B$ on $V$, then we have a matrix representation $A$ of $f$ in terms of $B$. If we suppose that $B = \{e_1, \cdots, e_t\}$ where $t$ is the dimension of $V$, then we have $A = (a_{i,j})$ where $a_{i,j} = f(e_i, e_j)$. This representation $A$ of $f$ satisfies $A^t = -A$. This condition is also equivalent to Definition 2.1. Moreover since $A$ is the matrix associated to $f$, we have that $f(n, m) = n^t Am$ for $n, m \in V$. For the rest of this thesis we will denote $n^t Am$ by $A(n, m)$; we may write $A_B(n, m)$ when we wish to emphasize the basis.

Definition 2.2. A symplectic vector space $V$ is a finite dimensional vector space over $K$ with a skew-symmetric form $f$.

We will note that a skew-symmetric form $f$ is non-degenerate provided that if $f(u, n) = 0$ for $u \in V$ fixed and all $n \in V$ then $u = 0$. Further, a symplectic space $V$
is non-singular if \( f \) is non-degenerate. We also note that \( f \) being non-degenerate is equivalent to the \( \det(A) = 0 \).

**Definition 2.3.** If \( V \) is a symplectic space and \( f(n, m) = 0 \) for two vectors \( n, m \in V \) then we say that \( n \) is orthogonal to \( m \).

Let \( V^\perp = \{ n \in V : f(n, m) = 0 \text{ for all } m \in V \} \). If we have that \( f \) non-degenerate then we have that the image of \( f(x, -) \) over \( V^\perp \) is the set \( \{0\} \).

**Definition 2.4.** The subspace \( V^\perp \) of \( V \) is called the radical of \( V \), and is denoted by \( \text{rad}(V) \).

In practice we can determine \( \text{rad}(V) \) by taking an arbitrary vector \( x \in V \) and solving \( x^t A = 0 \).

**Lemma 2.5.** Let \( V \) be a non-singular symplectic space over \( K \) with skew-symmetric form \( f \), and let \( n \in V \) be a non-zero vector. Then there exists \( m \in V \) so that \( f(n, m) = 1 \).

**Proof.** Let \( n \) be a non-zero vector in \( V \). Since \( \text{rad}(V) = 0 \), then there exists \( r \in V \) so that \( f(n, r) \neq 0 \). Let \( a := f(n, r) \). We get the desired result by taking \( m := \frac{1}{a} \cdot r \). \( \square \)

**Definition 2.6.** If the vector space \( V \) can be expressed as a direct sum \( V = U_1 \oplus \cdots \oplus U_r \) of subspace which are mutually orthogonal, then we say \( V \) is the orthogonal sum of \( U_i \) and denote it as \( V = U_1 \perp \cdots \perp U_r \).

**Definition 2.7.** If \( f \) is a bilinear form on a vector space \( V \), then a non-zero vector \( n \in V \) is called isotropic if \( f(n, n) = 0 \).

Observe that we could have defined a symplectic vector space \( V \) by: given a bilinear form \( f \) on a vector space \( V \) such that every vector is isotropic.
Definition 2.8. Let a bilinear form $f$. A non-singular plane $P$ that contains an isotropic vector is called a hyperbolic plane. Also, if $V = P_1 \perp \cdots \perp P_s$ for hyperbolic planes $P_i$ then we call $V$ a hyperbolic space.

Note that a hyperbolic plane $P$ is spanned by $n, m$ with $f(n, n) = 0 = f(m, m)$ and $f(n, m) = 1$. We will also note that a plane $P$ in a symplectic space always contains an isotropic vector.

Lemma 2.9. Let $H = P_1 \perp \cdots \perp P_s$ for hyperbolic planes $P_1, \cdots, P_s$. Let non-zero $n \in P_i$ then $n \notin P_j$ for $i \neq j$.

Proof. Let $n \in P_i$ be non-zero. Suppose that $n \in P_j$ for some $i \neq j$. Since $n \in P_i$, there exists non-zero $m \in P_i$ so that $f(n, m) = 1$. Since $n \in P_j$, $m \in P_i$ and $P_i \perp P_j$, $f(n, m) = 0$. This is impossible and thus $n \notin P_j$. \qed

Lemma 2.10. If $H$ is a hyperbolic space, then $H$ has even dimension.

Proof. Since $H$ is a hyperbolic space, $H = P_1 \perp \cdots \perp P_s$ for some $s \in \mathbb{N}$ and $P_i$’s hyperbolic planes. We note that each $P_i$ is a subspace of $H$. Also by Lemma 2.9 we have that $P_i \cap P_j = \{0\}$ for $i \neq j$. Furthermore, by the additivity of dimension we have

$$\dim(H) = \dim(P_1 \perp \cdots \perp P_s)$$

$$= \sum_{i=1}^{s} \dim(P_i)$$

$$= 2s.$$

\qed

Theorem 2.11. A non-singular symplectic space $V$ is an orthogonal sum of hyperbolic planes; in other words $V$ is a hyperbolic space. Further the dimension of $V$ is always even.
Proof. Let \( n \in V \) be a non-zero vector. By Lemma 2.5 we have that \( m \) exists so that \( f(n, m) = 1 \). We will also note that \( f(n, n) = 0 = f(m, m) \). Hence we have that \( \langle n, m \rangle \) is a hyperbolic plane in \( V \). If \( V \neq P \) then we have \( V = P \perp P^\perp \) with \( P^\perp \neq 0 \). Since \( P^\perp \neq 0 \), we have a non-zero \( w \in P^\perp \). Since \( V \) is non-singular, there exists \( v \in V \) so that \( f(w, v) \neq 0 \). Moreover, we can write \( v = v_1 + v_2 \) for \( v_1 \in P \) and \( v_2 \in P^\perp \). Hence \( f(w, v_2) \neq 0 \). This is true for every non-zero \( w \in P^\perp \). Hence we have that \( P^\perp \) is non-singular. Thus we can apply the previous case to \( P^\perp \).

Since \( V \) is finite dimensional, this process has terminates. Hence we have \( V = P_1 \perp \cdots \perp P_s \). Since each \( P_i = \langle n_i, m_i \rangle \), we conclude by Lemma 2.10 that \( V \) has even dimension. 

Suppose that we remove the non-singular condition from the hypotheses in the previous theorem. We note that \( \text{rad}(V) \neq 0 \). Hence we can express \( V = \text{rad}(V) \perp W \) for \( W \) a non-singular subspace of \( V \). Hence we can apply the theorem to \( W \) to obtain \( V = \text{rad}(V) \perp P_1 \perp \cdots \perp P_s \) for hyperbolic planes \( P_1, \cdots, P_s \).

2.2 Transvections of Hyperbolic Space

In this section we will introduce a special type of linear transformation and some of its properties that will be of interest for the remainder of this thesis.

**Proposition 2.12.** Let \( A \) be a skew-symmetric form on the finite vector space \( V \). If \( v \in V \), then we can define a linear transformation \( T_v : V \to V \) defined by \( T_v(v') = v' + A(v', v) \cdot v \). Then \( T_v \) satisfies the following:

1. \( T_v \) fixes \( (v)^\perp \).

2. If \( \langle v, v' \rangle \) is a hyperbolic plane, then \( T_v(v') = v' - v \).

3. \( T_v \) preserves the the pairing of two vectors; \( A(T_v(v'_1), T_v(v'_2)) = A(v'_1, v'_2) \).
Proof. We will begin by noting that $T_v(-)$ is a linear transformation because it is a sum of two linear maps.

1. Let $a$ be a scalar and $v' \in (v)^\perp$. We have $T_v(av') = av' + aA(v', v) \cdot v = av'$.

2. Let $\langle v, v' \rangle$ be a hyperbolic plane. We have $T_v(v') = v' - A(v', v) \cdot v = v' - v$.

3. Let $v_1', v_2'$ be vectors in $V$. We have that

$$A(T_v(v_1'), T_v(v_2')) = A(v_1' + A(v_1', v) \cdot v, v_2' + A(v_2', v) \cdot v)$$

$$= A(v_1', v_2') + A(v_1', A(v_2', v) \cdot v) + A(A(v_1', v) \cdot v, v_2')$$

$$+ A(A(v_1', v) \cdot v, A(v_2', v) \cdot v)$$

$$= A(v_1', v_2') + A(v_1', A(v_2', v) \cdot v) + A(A(v_1', v) \cdot v, v_2')$$

$$= A(v_1', v_2') + A(v_2', v)A(v_1', v) + A(v_1', v)A(v_2', v)$$

$$= A(v_1', v_2') - A(v, v_2')A(v_1', v) + A(v, v_2')A(v_1', v)$$

$$= A(v_1', v_2').$$

The transformation $T_v(-)$ from the previous proposition is called a transvection. Also, we will note that (1) implies that $T_v$ fixes the line spanned by $v$.

Let $V$ be a symplectic space with form $f$ and basis $\mathcal{B}$. Since each transvection $T_v$ is a linear transformation, we can determine the matrix representation $M_v$ in terms of $\mathcal{B}$. Further, the set of all such $M_v$’s generate a group under matrix multiplication. We will call this group the group of transvections, and denote this group by $G_{f,\mathcal{B}}$.

We will now determine the group of transvections of a hyperbolic space in terms of its hyperbolic basis. We will denote the hyperbolic basis by $\mathcal{H}$.

**Definition 2.13.** $L_1 := \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and $L_2 := \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$. 

7
Lemma 2.14. \( SL_2(\mathbb{Z}) \) is generated by \( L_1 \) and \( L_2 \).

Proof. We begin by noting that \( \langle L_1, L_2 \rangle \subset SL_2(\mathbb{Z}) \). Observe that

\[
L_1^n = \begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix}, \quad \text{and} \quad L_2^n = \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix}.
\]

Also

\[
L_1^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a - cn & b - dn \\ c & d \end{pmatrix} \quad \text{and} \quad L_2^n \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ an + c & bn + d \end{pmatrix}.
\]

Now consider \( \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \). If \( |a| \geq |c| \), then we have that \( a = q_1 c + r_1 \) for \( |c| > r_1 \geq 0 \). Then we have that

\[
L_1^{q_1} \gamma = \begin{pmatrix} r_1 & b - dq_1 \\ c & d \end{pmatrix}.
\]

We will note that if \( |c| > |a| \) then we could take \( q_1 = 0 \) and continue with the present case to attain the desired result. Furthermore since \( |c| > r_1 \geq 0 \), we have \( c = q_2 r_1 + r_2 \) for \( r_1 > r_2 \geq 0 \), then we have

\[
L_2^{-q_2} L_1^{q_1} \gamma = \begin{pmatrix} r_1 & b - dq_1 \\ r_2 & bq_2 - dq_1 q_2 + d \end{pmatrix}.
\]

Since we are performing division over \( \mathbb{Z} \) we have

\[
\left( \prod_{i=1}^{s} L_2^{-q_2_i} L_1^{q_2_{i-1}} \right) \gamma = \begin{pmatrix} 1 & x \\ 0 & y \end{pmatrix}
\]

for some \( s \in \mathbb{N} \) and \( x, y \in \mathbb{Z} \). Moreover, since the right hand side is in \( SL_2(\mathbb{Z}) \), we have that \( y = 1 \). Hence \( \left( \prod_{i=1}^{s} L_2^{-q_2_i} L_1^{q_2_{i-1}} \right) \gamma = L_1^{-x} \). Also, since \( SL_2(\mathbb{Z}) \) is a group, we also have that \( \gamma = \left( \prod_{i=1}^{s} L_2^{-q_2_i} L_1^{q_2_{i-1}} \right)^{-1} L_1^{-x} \). Therefore \( SL_2(\mathbb{Z}) \subset \langle L_1, L_2 \rangle \). Hence by set equality we get the desired result. \( \Box \)
Lemma 2.15. Let $P = \langle n, m \rangle$ be a hyperbolic plane, then group of transvections of $P$ is $SL_2(\mathbb{Z})$.

Proof. We note that $T_n(m) = m - n$ and $T_n(n) = n$, and that $T_m(n) = n + m$ and $T_m(m) = m$. These mappings give rise to the linear transformations $L_1$ and $L_2$. The result follows from applying Lemma 2.14. \hfill \Box

We will note that under the standard basis, the group of transvections for the plane may be different then we found in Lemma 2.15. In fact, the difference in the group of transvections with different bases will be discussed later in this thesis.

If we have $V = P \perp P^\perp$ for $P$ as in the hypothesis of Lemma 2.15 we get that $T_n(v) = v = T_m(v)$ for all $v \in P^\perp$. Hence we can conclude that $T_m|_{P^\perp} = T_n|_{P^\perp} = id_{P^\perp}$. From this we obtain the generators for the group of transvections of a hyperbolic space $H = P_1 \perp \cdots \perp P_s$ are

\[
\{\text{Diag}(I_{2i}, L_1, I_{2s-2i-2}) : 0 \leq i < s\} \cup \\
\{\text{Diag}(I_{2i}, L_2, I_{2s-2i-2}) : 0 \leq i < s\}.
\]

Theorem 2.16. Let $H$ be a hyperbolic space with dimension $2s$. The group of transvections of $H$ is the direct product of $s$ copies of $SL_2(\mathbb{Z})$.

Proof. Let $H = P_1 \perp \cdots \perp P_s$ for hyperbolic planes $P_i = \langle n_i, m_i \rangle$. We will take $\mathcal{H} := \{n_1, m_1, \ldots, n_s, m_s\}$ to be our basis of $H$. From the previous lemma, we have that the group of transvections of each $P_i$ is $SL_2(\mathbb{Z})$. Let

\[
R_k = \begin{pmatrix} c_{11} & \ldots & c_{1n} \\ \vdots & \ddots & \vdots \\ c_{n1} & \ldots & c_{nn} \end{pmatrix} \quad \text{and} \\
S_k = \begin{pmatrix} d_{11} & \ldots & d_{1n} \\ \vdots & \ddots & \vdots \\ d_{n1} & \ldots & d_{nn} \end{pmatrix}
\]
\[ c_{ij} = d_{ij} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \text{ for } i \neq j, \]
\[ c_{ii} = \begin{cases} L_1 & : i = k \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & : i \neq k \end{cases}, \quad \text{and} \]
\[ d_{ii} = \begin{cases} L_2 & : i = k \\ \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} & : i \neq k \end{cases}. \]

The group \( \langle R_i, S_i \rangle \cong \text{SL}_2(\mathbb{Z}) \) is the subgroup of transvections associated to the hyperbolic plane \( P_i \). Further by matrix multiplication we have that \( R_i \) commutes with \( R_j \) and \( S_j \) for \( i \neq j \), and that \( S_i \) commutes with \( S_j \) for \( i \neq j \). Hence the group of transvections of \( H \) is the external direct sum of \( s \) copies of \( \text{SL}_2(\mathbb{Z}) \).

**Remark.** If we have a degenerate symplectic space \( V \), then \( V = H_2s \perp \text{rad}(V) \). Since vectors in \( \text{rad}(V) \) pairs with everything to be 0, we have that the only transvection generated by \( \text{rad}(V) \) is the identity map. Thus the group of transvections of \( V \) with basis \( \mathcal{H} \) is isomorphic to the group of transvections on \( H_2s \).

The group of transvections of \( V \) under a different basis may not be the direct sum of several copies of \( \text{SL}_2(\mathbb{Z}) \). In fact, how the group of transvections is affected by different choices of bases will be our main interest in Chapter 3.

### 2.3 ADE-Diagrams

In this section we will introduce the ADE-diagrams which will be our main focus for the remainder of this thesis.

**Definition 2.17.** The following graphs are the ADE-diagrams:

\[ A_n: \quad \bullet \rightarrow \bullet \rightarrow \cdots \rightarrow \bullet \]
If we impose a direction to each edge, $v_i \to v_j$ for $i < j$, then we can generate a skew-symmetric form from the ADE-diagrams. Specifically, a skew-symmetric form can be defined by the following pairing:

$$v_i \cdot v_j = 0 \text{ if no edge connects } v_i \text{ to } v_j,$$

$$v_i \cdot v_j = 1 \text{ if there is a directed edge connecting } v_i \text{ to } v_j, \text{ and}$$

$$v_i \cdot v_j = -v_j \cdot v_i.$$

Each vertex $v_i$ can be associated to the corresponding $i$th basis element in the standard basis of $K^n$. We will presently determine the determinant for the matrix representation of the skew-symmetric form generated from the ADE-diagrams under this basis.

**Lemma 2.18.** The matrix representation of $A_{2n}$ has determinant 1.
Proof. Note that $A_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$ and $\det(A_2) = 1$. Suppose that the matrix representation of $A_{2k}$ has determinant 1. We will now consider $A_{2(k+1)}$. Note that

$$A_{2(k+1)} = \begin{pmatrix} 0 & 1 & O_{1 \times 2k} \\ -1 & 0 & e_1^t \\ O_{2k \times 1} & -e_1 & A_{2k} \end{pmatrix}$$

where $e_1 = (1, 0 \cdots, 0)$. Now,

$$\det(A_{2k}) = -\det\left( \begin{pmatrix} -1 & e_1^t \\ O_{2k \times 1} & A_{2k} \end{pmatrix} \right)$$

$$= \det(A_{2k})$$

$$= 1.$$

Note that we used the fact that the determinant of a matrix with a column of all 0’s is 0. Therefore $\det(A_{2n}) = 1$ for all $n \in \mathbb{N}$. \hfill \Box

**Theorem 2.19.** The vector space $V$ with skew-symmetric form generated by $A_{2n}$ can be expressed as

$$V_{A_{2n}} = \bigoplus_{i=1}^{n} \langle N_i, M_i \rangle$$

for $N_i = (n_j)$ where $n_{2r+1} = 1$ for $0 \leq r < i$ and $n_j = 0$ otherwise, and $M_i = (m_j)$ where $m_{2i} = 1$ and $m_j = 0$ for $j \neq 2i$. Also, if $V$ is paired with the skew-symmetric form generated by $A_{2n+1}$ then $V$ can be expressed as

$$V_{A_{2n+1}} = V_{A_{2n}} \perp \langle N_{2n+1} \rangle.$$

Proof. Consider the matrix representation of $A_2$, say $B_2$. Let $N^t = (1, 0)$, and $M^t = (0, 1)$. Hence we have $B_2(N, M) = 1$. We can conclude that $H_{A_2} = \langle N, M \rangle$.

Suppose that the theorem is true for $A_{2k}$ for some $2 \leq k \in \mathbb{N}$. We will denote the matrix representation of $A_{2(k+1)}$ as $B_{2(k+1)}$. Since $N_i, M_i$ for $i \leq k$ work for $A_{2k}$,
they work for $A_{2(k+1)}$ because embedded of subspaces. Note that $M_{k+1}^t B_{2(k+1)} = (a_i)^t$ where $a_{2(k+1)} = -1$ and otherwise $a_j = 0$. Further, $(a_i)^t N_{k+1} = -1$. Thus $B_{2(k+1)}(N_{k+1}, M_{k+1}) = 1$ and $\langle N_{k+1}, M_{k+1} \rangle$ is a hyperbolic plane. Finally,

$$B_{2(k+1)}(N_{k+1}, M_i) = B_{2(k+1)}(N_{k+1}, N_i) = B_{2(k+1)}(M_{k+1}, M_i) = 0$$

for all $i \neq k + 1$. Hence $V_{A_{2(k+1)}} = \perp_{i=1}^n \langle N_i, M_i \rangle$. Therefore the first statement in the theorem is true for all $n$.

Now we will consider $A_{2n+1}$. Since that $N_{2n+1}^t A_{2n+1} = 0$, and $V_{A_{2n}} \subset V_{A_{2n+1}}$ is a subspace then the desired result follows.

Lemma 2.20. For $n \geq 4$, the matrix representation of $D_n$ has determinant 0.

Proof. The $(n-1)$th and $n$th rows of the matrix representation are both of the form $(0, \cdots, 0, -1, 0)$. Hence we can conclude that $\det(D_n) = 0$.

The graph of first $n - 1$ vertices in the graph of $D_n$ is the graph of $A_{n-1}$. So it is reasonable to expect that the symplectic space of $A_{n-1}$ is a subspace of the symplectic space of $D_n$.

Theorem 2.21. If $V$ is associated with the skew-symmetric form generated by $D_{2n}$ then

$$V_{D_{2n}} = V_{A_{2n-1}} \perp \langle N'_n \rangle$$

for $N'_n = (n_j)$ where $n_{2r+1} = 1$ for $0 \leq r < n - 1$, $n_{2n} = 1$, and $n_j = 0$ otherwise.

Also, if $V$ is associated with the skew-symmetric form generated by $D_{2n+1}$ then

$$V_{D_{2n+1}} = V_{A_{2n}} \perp \langle K_n \rangle$$

for $K = (k_j)$ for $k_{2n} = -1, k_{2n+1} = 1$ and $k_j = 0$ otherwise.
Proof. Since there are two distinct $A_{2n-1}$ is as subgraph of $D_{2n}$ which differs only by the last vertex the result follows. If we consider $D_{2n+1}$ then we have that $A_{2n}$ is a subgraph of $D_{2n+1}$ which gives us: $V_{A_{2n}}$ is a subspace of $V_{D_{2n}}$. Further, $K^tB_{2n+1} = O_{1 \times 2n+1}$ where $B_{2n+1}$ is the matrix representation of $D_{2n+1}$. 

The matrix representations of $E_6$, $E_7$, and $E_8$ have determinant 1, 0 and 1 respectively. We will now determine the hyperbolic basis of $E_6$, $E_7$, and $E_8$.

By observing that the vertices $v_1, v_2$, and $v_5, v_6$ for the only two subdiagrams of $A_2$ (with exactly one edge exiting the subdiagram) we suspect that the following are hyperbolic planes in $V_{E_6}$:

$$\langle (1,0,0,0,0,0), (0,1,0,0,0,0) \rangle,$$
$$\langle (0,0,0,0,1,0), (0,0,0,0,0,1) \rangle.$$

These planes can be easily verified to be orthogonal hyperbolic planes. Further, we could have view $A_4$ as a subdiagram with vertices $v_1, v_2, v_3, v_4$. By Theorem 2.19 we expect that the following are hyperbolic planes in $V_{E_6}$:

$$\langle (1,0,0,0,0,0), (0,1,0,0,0,0) \rangle,$$
$$\langle (1,0,1,0,0,0), (0,0,0,1,0) \rangle.$$

These are orthogonal hyperbolic planes, however $(1,0,1,0,0,0)$ is not orthogonal to $(0,0,0,1,0)$. By replacing $(1,0,1,0,0,0)$ with $(1,0,1,0,0,1)$ we get the desired result. Hence $V_{E_6}$ can be expressed as a hyperbolic space as

$$V_{E_6} = \langle (1,0,0,0,0,0,0), (0,1,0,0,0,0,0) \rangle$$
$$\perp \langle (0,0,0,0,1,0), (0,0,0,0,0,1) \rangle$$
$$\perp \langle (1,0,1,0,0,1), (0,0,0,1,0) \rangle.$$

Since $V_{E_6}$ is a subspace of $V_{E_7}$ and $V_{E_8}$ we have that the basis elements of $V_{E_6}$ are basis elements of $V_{E_7}$ and $V_{E_8}$.
As described earlier we can determine the \( \text{rad}(V_{E_7}) \) by solving \( X^t E_7 = 0 \) for the arbitrary vector \( X \). Hence we have that \( V_{E_7} \) can be expressed as

\[
V_{E_7} = V_{E_6} \perp \langle (0, 0, 0, 1, -1, 0, -1) \rangle.
\]

Finally, we have that \( (0, 0, 0, 1, -1, 0, -1, 0)\!^t E_8 = (0, 0, 0, 0, 0, 0, 0, -1) \). Since \( V_{E_7} \) is a subspace of \( V_{E_8} \) we have that \( V_{E_8} \) is the hyperbolic space

\[
V_{E_8} = V_{E_6} \perp \langle (0, 0, 0, 1, -1, 0, -1, 0), (0, 0, 0, 0, 0, 0, 0, -1) \rangle.
\]

We can compute the rank of a skew-symmetric form by \( \text{rank}(f) = \dim(V) - \text{rad}(V) \).

From earlier work we get the following table:

**Table 2.1 Diagram and Rank**

<table>
<thead>
<tr>
<th>Diagram</th>
<th>Dimension</th>
<th>( \dim(\text{rad}(V)) )</th>
<th>Rank</th>
</tr>
</thead>
<tbody>
<tr>
<td>( A_{2n} )</td>
<td>( 2n )</td>
<td>( 0 )</td>
<td>( 2n )</td>
</tr>
<tr>
<td>( A_{2n+1} )</td>
<td>( 2n + 1 )</td>
<td>( 1 )</td>
<td>( 2n )</td>
</tr>
<tr>
<td>( D_{2n} )</td>
<td>( 2n )</td>
<td>( 2 )</td>
<td>( 2(n-1) )</td>
</tr>
<tr>
<td>( D_{2n+1} )</td>
<td>( 2n + 1 )</td>
<td>( 1 )</td>
<td>( 2n )</td>
</tr>
<tr>
<td>( E_6 )</td>
<td>( 6 )</td>
<td>( 0 )</td>
<td>( 6 )</td>
</tr>
<tr>
<td>( E_7 )</td>
<td>( 7 )</td>
<td>( 1 )</td>
<td>( 6 )</td>
</tr>
<tr>
<td>( E_8 )</td>
<td>( 8 )</td>
<td>( 0 )</td>
<td>( 8 )</td>
</tr>
</tbody>
</table>

Since symplectic spaces are isomorphic up to their dimension and dimension of their radical, we can observe the following relations between symplectic spaces associated by some of the ADE-diagrams as follows:

**Theorem 2.22.** If we let \( V_{A_n} \) and \( V_{E_n} \) denote the symplectic space generated by the appropriate graph, then we have the following relations:

\[
V_{A_6} \cong V_{E_6},
\]

\[
V_{A_7} \cong V_{E_7} \text{ and}
\]

\[
V_{A_8} \cong V_{E_8}.
\]
We do not attain such an isomorphism with involving $V_{D_n}$ because $\dim(V_{D_n}) = \dim(V_{A_n}) + 1$.

Finally, we are able to give a description of the group of transvections of the ADE-diagrams in terms of the classification theorem of symplectic spaces and Theorem 2.16.

**Corollary 2.23.** The group of transvections of $V_{A_{2n}}$ and $V_{A_{2n+1}}$ is the direct product of $n$ copies of $SL_2(\mathbb{Z})$.

*Proof.* This follows from Theorems 2.19, 2.16. □

**Corollary 2.24.** The group of transvections of $V_{D_{2n+1}}$ and $V_{D_{2(n+1)}}$ is the direct product of $n$ copies of $SL_2(\mathbb{Z})$.

*Proof.* This follows from Theorems 2.21, 2.16, and Corollary 2.23. □

**Corollary 2.25.** In terms of the classification theorem, the group of transvections for $V_{E_6}$ and $V_{E_7}$ is $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$, and the group of transvections of $V_{E_8}$ is $SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z}) \times SL_2(\mathbb{Z})$.

*Proof.* This follows from the computations done earlier in this section and Theorem 2.16. □
CHAPTER 3

TRANSVECTIONS OF ADE-DIAGRAMS

In this chapter we will determine the group of transvections for the ADE-diagrams in terms of the standard basis of \( K^n \), denoted \( S \). For convenience we will denote the group of transvections of symplectic space \( V \) with basis \( B \) by \( G_{f,B} \) where \( f \) is the associated skew-symmetric form. In this chapter we will also use the convention that all vectors are column vectors unless otherwise stated.

3.1 Transvections of \( A_n \)

In this section we will determine the \( G_{f_{A_n},S} \) and how it relates to \( G_{f_{A_n},H} \) where \( f_{A_n} \) is the form associated to the graph \( A_n \).

By examining the diagram of \( A_n \) we see that there three types of vertices: \( v_1, v_n \), and \( v_i \) for \( 1 < i < n \). By considering each of vertices as separate cases we will be able to determine the generators of \( G_{f_{A_n},S} \).

\[
\begin{array}{c}
\bullet \\
v_1 \\
\rightarrow \\
v_2 \\
\ldots \\
v_{n-1} \\
\rightarrow \\
v_n \\
\end{array}
\]

We will determine \( T_{v_1} \) presently. Since there is no edge connecting \( v_1 \) to \( v_i \) for \( i \neq 2 \), we have that \( T_{v_1}(v_i) = v_i \) for all \( i \neq 2 \). Also, we can compute that \( T_{v_1}(v_2) = v_2 - v_1 \). Thus we have that

\[
T_{v_1} = \begin{pmatrix}
L_1 & O_{2 \times (n-2)} \\
O_{(n-2) \times 2} & I_{n-2}
\end{pmatrix}.
\]
In a similar manner, we obtain that

\[
T_{v_n} = \begin{pmatrix}
I_{n-2} & O_{2 \times (n-2)} \\
O_{2 \times (n-2)} & L_2
\end{pmatrix}.
\]

The transvections for \( v_i \) with \( 1 < i < n \) will be more complicated since there are two edges connecting \( v_i \) to the rest of the graph. If we consider \( T_{v_i} \) we get that

\[
T_{v_i}(v_{i-1}) = v_{i-1} + v_i
\]

\[
T_{v_i}(v_{i+1}) = v_{i+1} - v_{i+1}, \quad \text{and}
\]

\[
T_{v_i}(v_j) = v_j \text{ for } j \neq i \pm 1.
\]

Hence we have that

\[
T_{v_i} = \begin{pmatrix}
1 & \cdots & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\
0 & \cdots & 1 & 1 & -1 & \cdots & 0 \\
0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \cdots & \cdots & \cdots & \cdots & 1
\end{pmatrix}.
\]

Throughout the rest of this thesis we will use

\[
L_3 := \begin{pmatrix}
1 & 0 & 0 \\
1 & 1 & -1 \\
0 & 0 & 1
\end{pmatrix}
\]

to simplify our expression of \( T_{v_i} \) as

\[
T_{v_i} = \begin{pmatrix}
I_{i-2} & O_{(i-2) \times 3} & O_{(i-2) \times (n-i-1)} \\
O_{3 \times (i-2)} & L_3 & O_{3 \times (n-i-1)} \\
O_{(n-i-1) \times (i-2)} & O_{(n-i-1) \times 3} & I_{(n-i-1) \times (n-i-1)}
\end{pmatrix}.
\]
Now that we have the generators for $G_{f_{A_n}, S}$ and $G_{f_{A_n}, H}$ we will shift our attention to determining how the groups are related. Let $V$ be the symplectic space with skew-symmetric form $f_{A_n}$. Since we have two bases for $V$ there is a change of basis transformation $P_{A_n}$ that maps the $S$ into $H$. We can use $P_{A_n}$ to construct a homomorphism $g : G_{f_{A_n}, S} \to \text{SL}_n(\mathbb{Z})$ defined by $g(X) = P_{A_n}^{-1} XP_{A_n}$. It is worth noting that $g(X)$ is an isomorphism with inverse given by conjugation by $P^{-1}$. Further, if we recall from Theorem 2.19 we have the basis $H = \{N_1, M_1, \cdots, N_n, M_n\}$ for $V_{A_{2n}}$ and $H = \{N_1, M_1, \cdots, N_n, M_n, N_{n+1}\}$ for $V_{A_{2n+1}}$, then we have that the columns of $P_{A_{2n}}$ and $P_{A_{2n+1}}$ are the vectors in the appropriate $H$. In general, if we have two arbitrary bases $B_1, B_2$, then we can find the change of basis transformation by solving $[B_1] = [B_2]P$ for $P \in \text{GL}_n(K)$ [2].

Lemma 3.1. Given the graph $A_{2n}$, $P_{A_{2n}}^{-1} = [T_1, \cdots, T_{2n}]^t$ where $T_{2i}$ consists of all 0’s except for 1 in the $(2i)$th position, and $T_{2i+1}$ consists of all 0’s except for 1 in $(2i + 1)$th position and -1 in $(2i + 3)$th position for $0 \leq i \leq n$. Further, for the graph $A_{2n+1}$ then $P_{A_{2n+1}}^{-1} = [T_1, \cdots, T_{2n+1}]^t$.

Proof. Before we compute $P_{A_n}^{-1} P_{A_n}$ we will compute all possible dot products of the rows of $P_{A_n}^{-1}$ with the columns of $P_{A_n}$:

$$T_{2i+1} \cdot N_j = n_{2i+1} - n_{2i+3} = \begin{cases} 1 & j = i \\ 0 & j \neq i \end{cases},$$

$$T_{2i+1} \cdot M_j = m_{2i+1} - m_{2i+3} = 0,$$

$$T_{2i} \cdot M_j = m_{2j} = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases},$$

and

$$T_{2i} \cdot N_j = n_{2i} = 0.$$
If we consider the graph $A_{2n}$ then we have that

$$P_{A_{2n}}^{-1} P_{A_{2n}} = \begin{pmatrix} T_1 \\ \vdots \\ T_{2n} \end{pmatrix} \begin{pmatrix} N_1 M_1 & \cdots & N_n M_n \end{pmatrix}$$

$$= \begin{pmatrix} T_1 N_1 & \cdots & T_1 M_n \\ \vdots & \ddots & \vdots \\ T_{2n} N_1 & \cdots & T_{2n} M_n \end{pmatrix}$$

$$= I_{2n \times 2n}.$$ 

The result for $A_{2n+1}$ follows by the same argument. □

Later it will be beneficial to express both $P_{A_n}$ and $P_{A_n}^{-1}$ as block matrices:

$$P_{A_{n}} = \begin{pmatrix} I_2 & S_{n-2} \\ O_{(n-2) \times 2} & P_{A_{n-2}} \end{pmatrix} = \begin{pmatrix} P_{A_i} & W_i & U_{i \times (n-i-3)} \\ O_{3 \times i} & X_{3 \times 3} & Z_{3 \times (n-i-3)} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & \cdots \end{pmatrix},$$

where

$$S_{n-2} = \begin{pmatrix} \end{pmatrix}.$$
\[ W_i = \begin{cases} 
\begin{pmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots 
\end{pmatrix} & : i \equiv 0 \pmod{2} \\
\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots 
\end{pmatrix} & : i \equiv 1 \pmod{2} 
\end{cases} \]

\[ U_{i \times (n-i-3)} = \begin{cases} 
\begin{pmatrix} 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots 
\end{pmatrix} & : i \equiv 0 \pmod{2} \\
\begin{pmatrix} \vdots \vdots \vdots \vdots \vdots \\
\end{pmatrix} & : i \equiv 1 \pmod{2} 
\end{cases} \]

\[ X_3 = \begin{cases} 
P_{A_3} & : i \equiv 0 \pmod{2} \\
I_3 & : i \equiv 1 \pmod{2} 
\end{cases}, \]

\[ Z_{3 \times (n-i-3)} = \begin{cases} 
\begin{pmatrix} 0 & 1 & 0 & 1 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 1 & 0 & 1 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots \\ 1 & 0 & 1 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots 
\end{pmatrix} & : i \equiv 1 \pmod{2} 
\end{cases} \]
\[
Y_{(n-i-3)\times(n-i-3)} = \begin{cases}
\begin{pmatrix}
1 & O_{1 \times (n-i-3)} \\
O_{(n-i-3) \times 1} & P_{A_{n-i-3}}
\end{pmatrix} : i \equiv 0 \pmod{2}, & \text{and} \\
P_{A_{(n-i-3)\times(n-i-3)}} : i \equiv 1 \pmod{2}
\end{cases}
\]

\[
P_{A_n}^{-1} = \begin{pmatrix}
I_2 & S'_{n-2} \\
O_{(n-2) \times 2} & P_{A_{n-2}}^{-1}
\end{pmatrix}
= \begin{pmatrix}
P_{A_i}^{-1} & W'_i & O_{1 \times (n-i-3)} \\
O_{3 \times i} & X'_3 & Z'_3 \times (n-i-3)
\end{pmatrix}
\]
for

\[
S'_{n-2} = \begin{pmatrix}
-1 & 0 & \cdots \\
0 & 0 & \cdots
\end{pmatrix},
\]

\[
W'_i = \begin{cases}
\begin{pmatrix}
0 & 0 & 0 \\
: & : & : \\
-1 & 0 & 0
\end{pmatrix} : i \equiv 0 \pmod{2}, \\
\begin{pmatrix}
0 & 0 & 0 \\
: & : & : \\
0 & -1 & 0
\end{pmatrix} : i \equiv 1 \pmod{2}
\end{cases}
\]

\[
X'_3 = \begin{cases}
P_{A_3}^{-1} : i \equiv 0 \pmod{2}, \\
I_3 : i \equiv 1 \pmod{2}
\end{cases}
\]

\[
Z'_{3 \times (n-i-3)} = \begin{pmatrix}
0 & 0 & \cdots \\
0 & 0 & \cdots \\
0 & -1 & 0 & \cdots \\
0 & 0 & \cdots \\
-1 & 0 & \cdots \\
0 & 0 & \cdots
\end{pmatrix} : i \equiv 1 \pmod{2}
\]
Observe that we have $G_{f_{A_2},S} = G_{f_{A_2},H}$ since the $H = S$. However this is no longer the case for $n > 2$. Furthermore, since $G_{f_{A_3},H} = G_{f_{A_2},H}$ and $G_{f_{A_2},S} \subset G_{f_{A_3},S}$ then we have that $G_{f_{A_3},H} \subset G_{f_{A_3},S}$. However for $n > 3$ the relationship between the two groups is no longer clear.

We will now determine compute $f(T_{vi})$ for all $1 \leq i \leq n$ for $A_n$ with $n > 2$.

**Theorem 3.2.** Given the diagram $A_n$, we have the transvections $\{T_{v_1}, \cdots, T_{v_n}\}$ which generate the group of transvection under the standard basis. Then

$$g(T_{v_1}) = T_{N_1},$$

$$g(T_{v_{2i}}) = T_{M_i} \text{ for } 2i \leq n, \text{ and}$$

$$g(T_{v_{2i+1}}) = \begin{pmatrix} I_{2i-1} & C_{(2i-1) \times 3} & O_{(2i-1) \times (n-2i-2)} \\ O_{3 \times (2i-1)} & L_3 & O_{3 \times (n-2i-2)} \\ O_{(n-2i-2) \times (2i-1)} & O_{(n-2i-2) \times 3} & I_{n-2i-2} \end{pmatrix}$$

for $C_{k \times 3} = \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ -1 & 0 & 1 \end{pmatrix}$.

**Proof.** Consider $A_n$. To make the multiplication more clear we will use block matrices.

Recall that

$$T_{v_1} = \begin{pmatrix} L_1 & O_{2 \times (n-2)} \\ O_{(n-2) \times 2} & I_{n-2} \end{pmatrix}$$

By explicit computation we have

$$g(T_{v_1}) = P_{A_n}^{-1}T_{v_1}P_{A_n}$$

$$= P_{A_n}^{-1} \begin{pmatrix} L_1 & O_{2 \times (n-2)} \\ O_{(n-2) \times 2} & I_{n-2} \end{pmatrix} \begin{pmatrix} I_2 & S_{n-2} \\ O_{(n-2) \times 2} & P_{A_{n-2}} \end{pmatrix}$$
Now we will compute \( g(T_{v_i}) \) for \( 1 < i \leq n \), also for convenience we will use \( k = i - 2 \).

\[
g(T_{v_i}) = P_{A_n}^{-1} T_{v_i} P_{A_n}
\]

\[
= P_{A_n}^{-1} \begin{pmatrix}
I_k & O_{k \times 3} & O_{k \times (n-k-3)} \\
O_{3 \times k} & L_3 & O_{3 \times (n-k-3)} \\
O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & I_{n-k-3}
\end{pmatrix}
\]

\[
= P_{A_n}^{-1} \begin{pmatrix}
P_{A_k} & W_k & U_{k \times (n-k-3)} \\
O_{3 \times k} & L_3 & O_{3 \times (n-k-3)} \\
O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & I_{n-k-3}
\end{pmatrix}
\]

\[
= P_{A_n}^{-1} \begin{pmatrix}
P_{A_k} & W'_{k} & O_{k \times (n-k-3)} \\
O_{3 \times k} & X_3' & Z_{3 \times (n-k-3)} \\
O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & Y'_{n-k-3}
\end{pmatrix}
\]

\[
= \begin{pmatrix}
P_{A_n}^{-1} & W''_{k} & O_{k \times (n-k-3)} \\
O_{3 \times k} & X_3'' & Z_{3 \times (n-k-3)} \\
O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & Y'_{n-k-3}
\end{pmatrix}
\]

\[
= T_{N_1}.
\]
From earlier work we have two cases: $k$ even and $k$ odd. We will begin by considering $k$ even. Note that this corresponds to $i = 2l$ for some $l$. Now we can finish computing $g(T_{v_i})$:

$$g(T_{v_i}) = g(T_{v_{2l}}) = \begin{pmatrix} I_k & P_{A_k}^{-1}W_k + W'kL_3X_3 & P_{A_k}^{-1}U_{k \times (n-k-3)} + W'kL_3Z_{3 \times (n-k-3)} \\ O_{3 \times k} & X_3' L_3 X_3 & X_3' L_3 Z_{3 \times (n-k-3)} + Z_{3 \times (n-k-3)}' Y_{n-k-3} \\ O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & Y_{n-k-3}' Y_{n-k-3} \end{pmatrix}$$

$$= \begin{pmatrix} I_k & P_{A_k}^{-1}W_k + W'kL_4 & P_{A_k}^{-1}U_{k \times (n-k-3)} + W'kL_3Z_{3 \times (n-k-3)} \\ O_{3 \times k} & L_2' & L_5 Z_{3 \times (n-k-3)} + Z_{3 \times (n-k-3)}' Y_{n-k-3} \\ O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & Y_{n-k-3}' Y_{n-k-3} \end{pmatrix}$$

$$= \begin{pmatrix} I_k & O_{k \times 3} & O_{k \times (n-k-3)} \\ O_{3 \times k} & L_2' & O_{3 \times (n-k-3)} \\ O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & I_{n-k-3} \end{pmatrix}$$

$$= \begin{pmatrix} I_k & O_{k \times 2} & O_{k \times (n-k-2)} \\ O_{2 \times k} & L_2 & O_{k \times 2} \\ O_{(n-k-2) \times k} & O_{(n-k-2) \times 2} & I_{n-k-2} \end{pmatrix}$$

$$= T_{M_i}$$

We will now consider $k$ odd. Note that this corresponds to $i = 2l + 1$ for some $l$. 

Observe that

\[
P_{A_3}^{-1}L_3P_{A_3} = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =: L_2',
\]

\[
L_3P_{A_3} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =: L_4 \text{ and}
\]

\[
P_{A_3}^{-1}L_3 = \begin{pmatrix} 1 & 0 & -1 \\ 1 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} =: L_5.
\]

\[
g(T_{v_i}) = \begin{pmatrix} I_k & P_{A_k}^{-1}W_k + W_k' L_3 X_3 & P_{A_k}^{-1}U_k \times (n-k-3) + W_k' L_3 Z_3 \times (n-k-3) \\ O_{3 \times k} & X_3' L_3 X_3 & X_3' L_3 Z_3 \times (n-k-3) + Z_3' \times (n-k-3) Y_{n-k-3} \\ O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & Y_{n-k-3}' Y_{n-k-3} \end{pmatrix}
\]

\[
= \begin{pmatrix} I_k & P_{A_k}^{-1}W_k + W_k' L_3 & P_{A_k}^{-1}U_k \times (n-k-3) + W_k' L_3 Z_3 \times (n-k-3) \\ O_{3 \times k} & L_3 & L_3 Z_3 \times (n-k-3) + Z_3' \times (n-k-3) Y_{n-k-3} \\ O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & Y_{n-k-3}' Y_{n-k-3} \end{pmatrix}
\]

\[
= \begin{pmatrix} I_k & C_{k \times 3} & O_{k \times (n-k-3)} \\ O_{3 \times k} & L_3 & O_{3 \times (n-k-3)} \\ O_{(n-k-3) \times k} & O_{(n-k-3) \times 3} & I_{n-k-3} \end{pmatrix}
\]

for

\[
C_{k \times 3} = \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ -1 & 0 & 1 \end{pmatrix}
\]
3.2 Transvections of $D_n$

In this section we will determine how the $G_{fD_n,S}$ compares to $G_{fD_n,H}$. To do this we will not only perform similar computations that we did in the previous section for $A_n$, but will also build on them.

We observe that since $A_{n-1}$ is a subgraph of $D_n$, we have that the transvections $T_{v_1}, \ldots, T_{v_{n-3}}$ are the same as we saw for $A_n$. Due to this similarity, we will find it useful to write $T'_{v_i}$ to refer to the transvection defined by vertex $v_i$ in $A_{n-1}$. The transvections defined by $v_{n-2}$, $v_{n-1}$ and $v_n$ are

\[
T_{v_{n-2}} = \begin{pmatrix}
I_{n-3} & O(n-3) \times 3 \\
0 \ldots 1 & 1 \ 1 \ 1 \\
0 \ldots 0 & 0 \ 1 \ 0 \\
0 \ldots 0 & 0 \ 0 \ 1
\end{pmatrix}, \\
T_{v_{n-1}} = \begin{pmatrix}
I_{n-2} & O(n-2) \times 2 \\
0 \ldots 1 & 1 \ 0 \\
0 \ldots 0 & 0 \ 1
\end{pmatrix}, \text{ and} \\
T_{v_n} = \begin{pmatrix}
I_{n-1} & O(n-1) \times 1 \\
0 \ldots 1 & 0 \ 1
\end{pmatrix}.
\]

Recall that $H$ for $V$ with $f_{D_n}$ depends on the parity of $n$; specifically we have

\[
V_{D_{2n}} = V_{A_{2n-1}} \perp \langle N'_n \rangle \ \text{and} \\
V_{D_{2n+1}} = V_{A_{2n}} \perp \langle K'_n \rangle.
\]
for \( N'_n = (n_j) \) where \( n_{2r+1} = 1 \) for \( 0 \leq r < n - 1 \), \( n_{2n} = 1 \), and \( n_j = 0 \) otherwise, and for \( K = (k_j) \) for \( k_{2n} = -1 \), \( k_{2n+1} = 1 \) and \( k_j = 0 \) otherwise. The difference in bases is significant enough that we will have to consider the parity of \( n \) in separate cases.

We will begin by considering \( n \) even. If \( P_{D_{2n}} \) is the change of basis transformation of \( V \) from \( S \) to \( H \), then the vectors in \( H \) are the columns of \( P_{D_{2n}} \). Hence we have that

\[
P_{D_{2n}} = \begin{pmatrix} P_{A_{2n-1}} & X \\ O_{1 \times (2n-1)} & 1 \end{pmatrix} \quad \text{and} \quad P_{D_{2n}}^{-1} = \begin{pmatrix} P_{A_{2n-1}}^{-1} & Y \\ O_{1 \times (2n-1)} & 1 \end{pmatrix}
\]

for \( X' = (1, 0, 1, 0, \ldots, 1, 0, 0) \) and \( Y' = (0, \ldots, 0, -1, 0, 0) \).

In the following theorem we will consider the homomorphism \( g : G_{D_{2n}, S} \to \text{SL}_{2n}(\mathbb{Z}) \) defined by \( g(X) = P_{D_{2n}}^{-1} X P_{D_{2n}} \).

**Theorem 3.3.** Given the diagram \( D_{2n} \), we have the transvections \( T_{v_1}, \ldots, T_{v_{2n}} \) which generate the group \( G_{f_{D_{2n}}, S} \). Then

\[
g(T_{v_1}) = T_{N_1},
\]

\[
g(T_{v_{2i}}) = T_{M_i} \text{ for } 2i \leq 2n - 1,
\]

\[
g(T_{v_{2i+1}}) = \begin{pmatrix} g_{A_{2n-1}}(T'_{2i+1}) & O_{(2n-1) \times 1} \\ O_{1 \times (2n-1)} & 1 \end{pmatrix},
\]

\[
g(T_{v_{2n}}) = \begin{pmatrix} I_{2n-3} & C'_{(2n-3) \times 3} \\ O_{3 \times (2n-3)} & L'_3 \end{pmatrix}
\]

for

\[
L'_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 0 & -1 \end{pmatrix}
\]

\[
C'_{(2n-3) \times 3} = \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ -1 & 0 & 0 \end{pmatrix}
\]

**Proof.** We will now compute how the generators of \( G_{g_{D_{2n}}, S} \) maps via \( g(X) = P^{-1} X P \).
By explicit computation we have

\[ g(T_{v_1}) = \left. P_{D_{2n}}^{-1} T_{v_1} P_{D_{2n}} \right| \]

\[ = P_{D_{2n}}^{-1} \begin{pmatrix} T'_{v_1} & O_{(2n-1)\times 1} \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \begin{pmatrix} P_{A_{2n-1}} & X \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \]

\[ = P_{D_{2n}}^{-1} \begin{pmatrix} T'_{v_1} P_{A_{2n-1}} & T'_{v_1} X \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} P_{A_{2n-1}}^{-1} Y \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \begin{pmatrix} T'_{v_1} P_{A_{2n-1}} & T'_{v_1} X \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \]

\[ = P_{A_{2n-1}}^{-1} \begin{pmatrix} T'_{v_1} P_{A_{2n-1}} & P_{A_{2n-1}}^{-1} T'_{v_1} X + Y \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} T'_{N_1} & O_{(2n-1)\times 1} \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \]

\[ = T_{N_1} \]

We will now compute \( g(T_{v_k}) \) for \( 1 < k \neq 2n - 2 \) and \( k \neq 2n \).

\[ g(T_{v_k}) = \left. P_{D_{2n}}^{-1} T_{v_k} P_{D_{2n}} \right| \]

\[ = P_{D_{2n}}^{-1} \begin{pmatrix} T'_{v_k} & O_{(2n-1)\times 1} \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \begin{pmatrix} P_{A_{2n-1}} & X \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \]

\[ = P_{D_{2n}}^{-1} \begin{pmatrix} T'_{v_k} P_{A_{2n-1}} & T'_{v_k} X \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \]

\[ = \begin{pmatrix} P_{A_{2n-1}}^{-1} Y \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \begin{pmatrix} T'_{v_k} P_{A_{2n-1}} & T'_{v_k} X \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \]

\[ = P_{A_{2n-1}}^{-1} \begin{pmatrix} T'_{v_k} P_{A_{2n-1}} & P_{A_{2n-1}}^{-1} T'_{v_k} X + Y \\ O_{1\times(2n-1)} & 1 \end{pmatrix} \]

If we consider \( k \) even, then \( k = 2i \). Hence we have that

\[ g(T_{v_k}) = g(T_{v_{2i}}) \]
\[
\begin{pmatrix}
P_{A_{2n-1}}^{-1} T'_{v_{2i}} P_{A_{2n-1}} & P_{A_{2n-1}}^{-1} T'_{v_{2i}} X + Y \\
O_{1 \times (2n-1)} & 1
\end{pmatrix}
\begin{pmatrix}
T'_{M_i} & O_{(2n-1) \times 1} \\
O_{1 \times (2n-1)} & 1
\end{pmatrix}
= T_{M_i}.
\]

If we consider \( k \) odd, then \( k = 2i + 1 \). Hence we have that
\[
g(T_{v_k}) = g(T_{v_{2i+1}})
\]
\[
= \begin{pmatrix}
P_{A_{2n-1}}^{-1} T'_{v_{2i+1}} P_{A_{2n-1}} & P_{A_{2n-1}}^{-1} T'_{v_{2i+1}} X + Y \\
O_{1 \times (2n-1)} & 1
\end{pmatrix}
= \begin{pmatrix}
g_{A_{2n-1}}(T'_{2i+1}) & O_{(2n-1) \times 1} \\
O_{1 \times (2n-1)} & 1
\end{pmatrix}.
\]

Observe that the order \( N_n \) and \( N'_{n-1} \) is arbitrary. Hence if we swap the corresponding row and columns of \( T_{v_{2n-1}} \) we will get \( T_{v_{2n}} \). Thus we have that
\[
g(T_{v_{2n}}) = \begin{pmatrix}
I_{2n-3} & C'_{(2n-3) \times 3} \\
O_{3 \times (2n-3)} & L'_{3}
\end{pmatrix}
\]
for
\[
L'_{3} = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
1 & 0 & -1
\end{pmatrix}
\]
and
\[
C'_{(2n-3) \times 3} = \begin{pmatrix}
0 & 0 & 0 \\
\vdots & \vdots & \vdots \\
-1 & 0 & 0
\end{pmatrix}.
\]

If we consider \( n \) odd, then we have that
\[
P_{D_{2n+1}} = \begin{pmatrix}
P_{A_{2n}} X' \\
O_{1 \times 2n} & 1
\end{pmatrix}
\]
and
\[
P_{D_{2n+1}}^{-1} = \begin{pmatrix}
P_{A_{2n}}^{-1} -X' \\
O_{1 \times 2n} & 1
\end{pmatrix}.
\]

\( \square \)
for $X'' = (0, \cdots, 0, -1)$.

In the following theorem we will consider the homomorphism $g : G_{D_{2n+1}, S} \rightarrow \text{SL}_{2n+1}(\mathbb{Z})$ defined by $g(X) = P^{-1}_{D_{2n+1}, X}P_{D_{2n+1}}$.

**Theorem 3.4.** Given the diagram $D_{2n+1}$, we have the transvections $T_{v_1}, \cdots, T_{v_{2n+1}}$ which generate the group $G_{D_{2n+1}, S}$. Then

$$g(T_{v_i}) = \begin{pmatrix} g_{A_{2n}}(T''_{v_i}) & O_{2n \times 1} \\ O_{1 \times 2n} & 1 \end{pmatrix}$$

for $i \neq 2n + 1$, and

$$g(T_{v_{2n+1}}) = \begin{pmatrix} g_{A_{2n}}(T_{v_{2n+1}}) & O_{2n \times 1} \\ K & 1 \end{pmatrix}$$

for $K = (0, \cdots, 1, 0)$.

**Proof.** Observe that

$$g(T_{v_i}) = P^{-1}_{D_{2n+1}, T_{v_i}}P_{D_{2n+1}}$$

$$= \begin{pmatrix} P^{-1}_{A_{2n}, T_{v_i}} & -X' \\ O_{1 \times 2n} & 1 \end{pmatrix} \begin{pmatrix} T'_{v_i} & O_{2n \times 1} \\ O_{1 \times 2n} & 1 \end{pmatrix} \begin{pmatrix} P_{A_{2n}, X'} \\ O_{1 \times 2n} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} P^{-1}_{A_{2n}, T_{v_i}} & -X' \\ O_{1 \times 2n} & 1 \end{pmatrix} \begin{pmatrix} T'_{v_i} P_{A_{2n}, T_{v_i}} & T''_{v_i} X' \\ O_{1 \times 2n} & 1 \end{pmatrix}$$

$$= \begin{pmatrix} P^{-1}_{A_{2n}, T_{v_i}} & -X' \\ O_{1 \times 2n} & 1 \end{pmatrix} \begin{pmatrix} P_{A_{2n}, X'} \\ O_{1 \times 2n} & 1 \end{pmatrix}$$

for $i \neq 2n + 1$ and $i \neq 2n - 1$.

We will now compute $g(T_{v_{2n+1}})$. Let $C'' = (0, \cdots, -1, 0)$. We have

$$g(T_{v_{2n+1}}) = P^{-1}_{D_{2n+1}, T_{v_{2n+1}}}P_{D_{2n+1}}$$

$$= \begin{pmatrix} P^{-1}_{A_{2n}, T_{v_{2n+1}}} & -X' \\ O_{1 \times 2n} & 1 \end{pmatrix} \begin{pmatrix} T''_{v_{2n+1}} C' \\ O_{1 \times 2n} & 1 \end{pmatrix} \begin{pmatrix} P_{A_{2n}, X'} \\ O_{1 \times 2n} & 1 \end{pmatrix}$$
\[
\begin{pmatrix}
P_{A_{2n}}^{-1} & -X' \\
O_{1\times2n} & 1
\end{pmatrix}
\begin{pmatrix}
T'_{v2n-1}P_{A_{2n}} & T'_{v2n-1}X' + C \\
O_{1\times2n} & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
P_{A_{2n}}^{-1}T'_{v2n-1}P_{A_{2n}} & P_{A_{2n}}^{-1}(T'_{v2n-1}X' + C) - X' \\
O_{1\times2n} & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
g_{A_{2n}}(T'_{v2n-1}) & P_{A_{2n}}^{-1}(X' + C) - X' \\
O_{1\times2n} & 1
\end{pmatrix}
\]

Now if we consider \(i = 2n + 1\). Let \(K := (0, \cdots, 1, 0)\). We have

\[
g(T_{v2n+1}) = P_{A_{2n}}^{-1}T_{v2n+1}P_{D_{2n+1}}
\]

\[
= \begin{pmatrix}
P_{A_{2n}}^{-1} & -X' \\
O_{1\times2n} & 1
\end{pmatrix}
\begin{pmatrix}
I_{2n} & O_{2n\times1} \\
K & 1
\end{pmatrix}
\begin{pmatrix}
P_{A_{2n}} & X' \\
O_{1\times2n} & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
P_{A_{2n}}^{-1} & -X' \\
O_{1\times2n} & 1
\end{pmatrix}
\begin{pmatrix}
P_{A_{2n}} & X' \\
KP_{A_{2n}} & 1
\end{pmatrix}
\]

We note that the \((2n - 1)\)th row of \(A_{2n}\) is \(K\). Hence we have

\[
= \begin{pmatrix}
P_{A_{2n}}^{-1} & -X' \\
O_{1\times2n} & 1
\end{pmatrix}
\begin{pmatrix}
P_{A_{2n}} & X' \\
K & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
I_{2n} - X'K & P_{A_{2n}}^{-1}X' - X' \\
K & 1
\end{pmatrix}
\]

\[
= \begin{pmatrix}
g_{A_{2n}}(T_{v2n}) & O_{2n\times1} \\
K & 1
\end{pmatrix}
\]

We note that \(-X'K = \begin{pmatrix}
O_{2n-2} & O_{(2n-2)\times1} & O_{(2n-2)\times1} \\
O_{1\times(2n-2)} & 1 & 0
\end{pmatrix}\).
3.3 Transvections of $E_6$, $E_7$, $E_8$

In this section we will compute how $\text{im}(G_{V,S})$ under $P_{E_i}^{-1}X P_{E_i}$ relates to $G_{E_i,H}$ for the ADE-diagrams $E_6$, $E_7$, and $E_8$.

Recall from earlier that the classification theorem gives us that $V_{E_6}$ can be expressed as

$$V_{E_6} = \langle N_1 := (1,0,0,0,0,0), N_2 := (0,1,0,0,0,0) \rangle$$

$$\perp \langle N_3 := (0,0,0,0,1,0), N_4 := (0,0,0,0,0,1) \rangle$$

$$\perp \langle N_5 := (1,0,1,0,0,0), N_6 := (0,0,0,1,0,0) \rangle.$$

Similarly from earlier if we take these basis elements to be the columns of a matrix $P_{E_6}$ then $P_{E_6}$ is the change of base from the standard basis to the hyperbolic basis. Specifically, we have that

$$P_{E,6} = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \end{pmatrix},$$

and we also have $P_{E,6}^{-1} = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \end{pmatrix}$.

If we consider the homomorphism $g : G_{E_6,S} \to SL_6(\mathbb{Z})$ defined by $f(X) = P_{E_6}^{-1}X P_{E_6}$, then we have the following:

$$g(T_{v_1}) = T_{N_1},$$

$$g(T_{v_2}) = T_{N_2}.$$
\[
g(T_{v_3}) = \begin{pmatrix}
1 & -1 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 1 & 1 & 0 & 1 \\
0 & 1 & -1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix},
\]

\[
g(T_{v_4}) = T_{N_6},
\]

\[
g(T_{v_5}) = T_{N_3} \text{ and}
\]

\[
g(T_{v_6}) = T_{N_4}.
\]

Observe that we have
\[
g(\langle T_{v_1}, T_{v_2} \rangle) = \langle T_{N_1}, T_{N_2} \rangle
\]

\[
g(\langle T_{v_5}, T_{v_6} \rangle) = \langle T_{N_3}, T_{N_4} \rangle.
\]

We note that it appears that \( T_{N_5} \neq g(X) \) for any \( X \in G_{f_{E_6}}.S \). However, by computation we have that
\[
g(T_{v_3}T_{v_1}^{-1}T_{v_6}^{-1}) = \begin{pmatrix}
1 & 0 & 1 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & 1 \\
0 & 1 & -1 & 0 & 1 & -1 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

By block matrix multiplication we can see that \( \langle g(T_{v_3}T_{v_1}^{-1}T_{v_6}^{-1}), g(T_{v_4}) \rangle \cong \langle T_{N_5}, T_{N_6} \rangle \).

Unfortunately \( g(T_{v_3}T_{v_1}^{-1}T_{v_6}^{-1}) \) does not commute with \( T_{N_2} \) and \( T_{N_3} \). Hence it is not clear whether \( g(G_{f_{E_6}}.S) \) contains an isomorphic copy of \( G_{f_{E_6}}.\mathcal{H} \).

If we now consider \( E_7 \), then we have that
\[
V_{E_7} = V_{E_6} \perp \langle N_7' := (0, 0, 0, 1, -1, 0 - 1) \rangle
\]

34
Further, we can also compute that

\[
P_{E,7} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -1 & 0 \\ O_{1\times 6} & -1 \end{pmatrix}
\quad \text{and} \quad
P_{E,7}^{-1} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ O_{1\times 6} & -1 \end{pmatrix}.\]

Hence if we take \( g : G_{f_{E,7}, s} \to \text{SL}_7(\mathbb{Z}) \) defined by \( g(X) = P_{E,7}^{-1}XP_{E,7} \) then we have the following:

\[
g(T_{v_1}) = T_{N_1},
\]

\[
g(T_{v_2}) = T_{N_2},
\]

\[
g(T_{v_3}) = \begin{pmatrix} g_{E_6}(v_3) \\ O_{6 \times 1} \end{pmatrix},
\]

\[
g(T_{v_4}) = T_{N_6},
\]

\[
g(T_{v_5}) = T_{N_3},
\]

\[
g(T_{v_6}) = T_{N_4},
\]

\[
g(T_{v_7}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & -1 & -1 & 0 & 1 \end{pmatrix}.
\]
Similar to $E_6$ we have that $\langle g(T_{v_3}T_{v_1}^{-1}T_{v_6}^{-1}), g(T_{v_4}) \rangle \cong \langle T_{N_7}, T_{N_6} \rangle$. Again it is not clear whether $G_{E_7,H}$ is contained in the image of $G_{E_7,S}$.

Finally, if we consider $E_8$, then we have that

$$V_{E_8} = V_{E_6} \perp \langle N_7 := (0, 0, 0, 1, -1, 0, -1, 0), N_8 := (0, 0, 0, 0, 0, 0, 0, 0, -1) \rangle.$$ 

We can also compute the change of basis matrix $P_{E_8}$ in a similar manner as earlier to obtain

$$P_{E,6} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
1 & 0 & -1 \\
o_{2\times6} & -1 & 0 \\
0 & -1
\end{pmatrix}$$

and $P_{E,8}^{-1} = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
0 & 0 & 1 \\
o_{2\times6} & -1 & 0 \\
0 & -1
\end{pmatrix}$.

Hence if we take $g : G_{E_8,S} \to SL_8(\mathbb{Z})$ defined by $g(X) = P_{E_8}^{-1}X P_{E_8}$ then we have the following:

$$g(T_{v_1}) = T_{N_1},$$

$$g(T_{v_2}) = T_{N_2},$$

$$g(T_{v_3}) = \begin{pmatrix}
g_{E_6}(v_3) & O_{6\times2} \\
o_{2\times6} & I_2
\end{pmatrix},$$

$$g(T_{v_4}) = T_{N_6},$$

36
\[ g(T_{v_0}) = T_{N_3}, \]
\[ g(T_{v_0}) = T_{N_4}. \]
\[ g(T_{v_7}) = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} \]
\[ g(E_7(V_7)) \]
\[ g(T_{v_8}) = T_{N_8}. \]

It again is not clear whether \( G_{f_{E_8}, \mathcal{H}} \) is contained in the image \( G_{f_{E_8}, S} \).
Chapter 4

Future Questions

In this section we will discuss a few directions that this can be taken to build on this thesis.

In Chapter 3 we determine how elements in $S$ maps to in $\text{SL}_n(Z)$ for a variety of diagrams. We saw that for $n \leq 3$ we have $G_{f_{A_n}, H} \subset G_{f_{A_n}, S}$; we had equality for $n = 2$. The relationship became unclear when $n > 3$. For what values of $n > 3$ does $G_{f_{A_n}, H} \cong H \subset G_{f_{A_n}, S}$? Similar questions can be raised for $D_n$, $E_6$, $E_7$, and $E_8$. If there is no such subgroup $H$ for $A_n$ with $n > 3$ then it is reasonable to expect that such containment would fail for the diagrams $D_i$ and $E_i$ that have $A_n$ as a subdiagram.

The results in Chapter 2 and Chapter 3 has been specific for fields of characteristic 0. What would the analog of these results be for fields of characteristic non-zero? Further since we have that skew-symmetric is the same as symmetric in the characteristic 2 case it is reasonable for analog results to not hold. Is it true that the analog results don’t hold for characteristic 2? And why do they fail?

Another natural question is how do these results translate for free $Z$-modules. It is possible to prove an analog of Theorem 2.11 for free $Z$-modules, and to generalize it to free modules over an arbitrary PID. It is worth pointing out that the classification of $Z$-modules is not as nice because Lemma 2.5 is no longer possible because we lose division when working in this case. So how does this decomposition of the modules affect the group of transvection compared to the group of transvections of the ADE-diagrams for the standard basis? Also, how do these groups compare to the results
that we have found for fields?

For both of these questions is it possible to find similar relations that we saw in Theorem 2.22?
BIBLIOGRAPHY


