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Explorations in Elementary and Analytic Number Theory

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Abstract

In this dissertation, we investigate two distinct questions in number theory. Each question is dedicated its own chapter.

First, we consider arithmetic progressions in the polygonal numbers with a fixed number of sides. We will show that four-term arithmetic progressions cannot exist. We then describe explicitly how to find all three-term arithmetic progressions. Additionally we show that there are infinitely many three-term arithmetic progressions starting with an arbitrary polygonal number. This is joint work with K. Brown and J. Harrington and appears in [4].

Second, we will show certain irreducibility criteria for polynomials. Let $f(x)$ be a polynomial with non-negative integer coefficients such that $f(b)$ is prime for some integer $2 \leq b \leq 20$. A. Cohn's criteria states that if $b = 10$ and each coefficient is ≤ 9, then $f(x)$ is irreducible. In 1988, M. Filaseta showed in [8] that the bound 9 can be replaced by 10^{30}. We will look at work that was done to further increase this bound and then generalize this for an arbitrary base b. Along the way, we will also establish additional irreducibility criteria.

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Chapter 1

Arithmetic Progressions in the Polygonal Numbers

1.1 Introduction

We recall that an arithmetic progression with a common difference d is a sequence of numbers, finite or infinite, such that the difference of any two consecutive terms is a constant d. Throughout this chapter, let s be a fixed integer with $s \geq 3$. We will use the notation $P_s(n)$ to represent the n-th s-gonal number – that is, the number of points that are needed to create a regular s-gon with each side being of length $n - 1$. See Figure 1.1. This number is given by $P_s(n) = (s/2 - 1)n^2 - (s/2 - 2)n$.

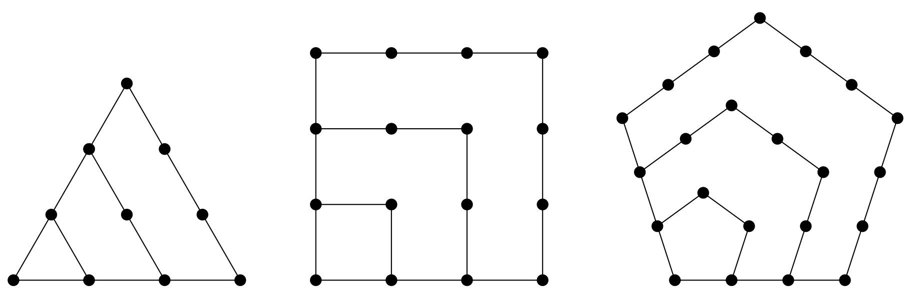

Figure 1.1 Examples of Polygonal Numbers for $s = 3, 4, 5$ and $n = 1, 2, 3, 4$

In this chapter, we will show that four-term arithmetic progressions with common difference $d \neq 0$ in the polygonal numbers do not exist. We then show that not only are there infinitely many three-term arithmetic progressions with common difference $d > 0$, but that there are infinitely many such progressions starting with an arbitrary polygonal number. Finally, we describe explicitly how to find all such three-term arithmetic progressions with common difference $d > 0$.

A natural question that arises from the results in this chapter is to consider arithmetic progressions in the polyhedral numbers and then to generalize this to analyzing arithmetic progressions in the figurative numbers.

For a more general problem one could ask about arithmetic progressions in the sequence $f(n)$ for positive integers n and an arbitrary integer polynomial $f(x)$. This appears to be far from trivial, however. In particular, we note that if $f(x) = x^3$, then finding three-term arithmetic progressions with common difference $d \neq 0$ in $\{f(n) : n \in \mathbb{Z}^+\}$ would amount to solving the Diophantine equation $A^3 + C^3 = 2B^3$ in positive integers $A < B < C$. That there are no such three-term arithmetic progressions for third-powers follows then from [13, Theorem 3, p. 126]. On the other hand, if instead $f(x) = x^3 - x$, then the numbers $f(1) = 0$, $f(4) = 60$ and $f(5) = 120$ form a three-term arithmetic progression with common difference $d = 60$.

We note that everything in this chapter is joint work with Kenny Brown and Josh Harrington and appears in [4].

1.2 Four-Term Arithmetic Progressions

We first show that four-term arithmetic progressions with a common difference $d \neq 0$ cannot occur in the polygonal numbers. To do this, we will reference the following result from [13, pages 21–22] and [16, page 75]:

Theorem (Mordell 1969; Sierpiński 1964)**.** *There cannot be four squares in arithmetic progression with common difference $d \neq 0$.*

Using this, we have

Theorem 1.1. *Let s be a fixed integer with $s \geq 3$ Then there cannot be four s-gonal numbers in arithmetic progression with common difference $d \neq 0$.*

Proof. Let s be a fixed integer with $s \geq 3$. By way of contradiction, suppose that there is a four-term arithmetic progression with common difference $d \neq 0$ in the s-gonal numbers. Then there exists positive integers n, a, b, and c satisfying

$$P_s(a) - P_s(n) = P_s(b) - P_s(a) = P_s(c) - P_s(b) = d \neq 0.$$

First, we consider any two adjacent terms in the arithmetic progression, say $P_s(n)$ and $P_s(a)$. We have that

$$P_s(a) = (s/2 - 1)a^2 - (s/2 - 2)a \quad \text{and} \quad P_s(n) = (s/2 - 1)n^2 - (s/2 - 2)n.$$

By assumption, we also have that $d = P_s(a) - P_s(n)$, so that

$$2d = 2P_s(a) - 2P_s(n) = a^2(s-2) - a(s-4) - n^2(s-2) + n(s-4).$$

We now consider $(2a(s-2)-(s-4))^2$ and $(2n(s-2)-(s-4))^2$. We see that

$$\begin{aligned}(2a(s-2)-(s-4))^2 &- (2n(s-2)-(s-4))^2 \\ &= 4(s-2)\left(a^2(s-2) - a(s-4) - n^2(s-2) + n(s-4)\right) \\ &= 8(s-2)d.\end{aligned}$$

Similarly, we have that

$$(2b(s-2)-(s-4))^2 - (2a(s-2)-(s-4))^2 = 8(s-2)d$$

and

$$(2c(s-2)-(s-4))^2 - (2b(s-2)-(s-4))^2 = 8(s-2)d.$$

This contradicts Sierpiński's and Mordell's theorem, completing the proof. □

1.3 Three-Term Arithmetic Progressions

In order to examine three-term arithmetic progressions with common difference d in the polygonal numbers, we first prove a short lemma. To simplify the proof of the lemma, we will momentarily consider P_s to be a continuous function from $\mathbb{C}$ into $\mathbb{C}$.

Lemma 1.2. *Let s be a fixed integer with $s \geq 3$ and $n, a,$ and b be complex numbers. Then $P_s(n)$, $P_s(a)$, and $P_s(b)$ satisfy $P_s(a) - P_s(n) = P_s(b) - P_s(a)$ if and only if*

$$N = 2(s-2)n - (s-4), \quad A = 2(s-2)a - (s-4), \quad \textit{and} \quad B = 2(s-2)b - (s-4)$$

satisfy the equation $B^2 - 2A^2 = -N^2$.

Proof. Let s be a fixed integer with $s \geq 3$. Suppose that n, a, and b are complex numbers such that $P_s(a) - P_s(n) = P_s(b) - P_s(a)$. It follows that

$$\left(\frac{s}{2}-1\right)a^2 - \left(\frac{s}{2}-2\right)a - \left(\frac{s}{2}-1\right)n^2 + \left(\frac{s}{2}-2\right)n \\ = \left(\frac{s}{2}-1\right)b^2 - \left(\frac{s}{2}-2\right)b - \left(\frac{s}{2}-1\right)a^2 + \left(\frac{s}{2}-2\right)a. \quad (1.3.1)$$

Multiplying both sides of (1.3.1) by $8(s-2)$ and rearranging, we obtain that (1.3.1) is equivalent to $B^2 - 2A^2 = -N^2$, where N, A, and B are as defined in the statement of the lemma.

Since these steps work in reverse, the converse is immediate. This proves the lemma. □

We get an immediate consequence of Lemma 1.2 if we revert to viewing P_s as a function from $\mathbb{N}$ into $\mathbb{N}$. If $P_s(n)$, $P_s(a)$, and $P_s(b)$ form a three-term arithmetic progression for positive integers n, a, and b with $n \leq a \leq b$, then $B^2 - 2A^2 = -N^2$ is satisfied for N, A, and B as given in Lemma 1.2. Conversely, every positive integer solution N, A, and B to $B^2 - 2A^2 = -N^2$ where

$$n = \frac{N + (s-4)}{2(s-2)}, \quad a = \frac{A + (s-4)}{2(s-2)}, \quad \text{and} \quad b = \frac{B + (s-4)}{2(s-2)}$$

are positive integers with $n \leq a \leq b$ gives us that $P_s(n)$, $P_s(a)$, and $P_s(b)$ form a three-term arithmetic progression in the s-gonal numbers.

We now show that there are infinitely many three-term arithmetic progressions with common difference $d > 0$ starting at a given polygonal number, which is our second theorem. The proof of this theorem uses some basic algebraic number theory as detailed in [11] or [12].

Theorem 1.3. *Let s be a fixed integer with $s \geq 3$. Let n be an arbitrary positive integer. Then there exist infinitely many integers $d > 0$ such that there is a three-term arithmetic progressions with a common difference d in the s-gonal numbers beginning with $P_s(n)$.*

Proof. Let s be a fixed integer with $s \geq 3$. Let n be an arbitrary positive integer. Let $N = 2(s-2)n - (s-4)$ as in Lemma 1.2.

Suppose that X and Y are positive integers satisfying

$$X^2 - 2Y^2 = -1, \quad X \equiv 1 \pmod{2(s-2)}, \quad \text{and} \quad Y \equiv 1 \pmod{2(s-2)}. \quad (1.3.2)$$

Notice that $X = 1$ and $Y = 1$ satisfy (1.3.2), so such X and Y exist.

Observe that by multiplying the equation in (1.3.2) by N^2 we have

$$(NX)^2 - 2(NY)^2 = -N^2.$$

Our goal is to apply Lemma 1.2.

Now let

$$a = \frac{NY + (s-4)}{2(s-2)} = nY + \frac{(1-Y)(s-4)}{2(s-2)} \quad (1.3.3)$$

and

$$b = \frac{NX + (s-4)}{2(s-2)} = nX + \frac{(1-X)(s-4)}{2(s-2)}. \quad (1.3.4)$$

Since $s \geq 3$ and $Y \equiv 1 \pmod{2(s-2)}$, we may write $Y = 1 + 2(s-2)k$ for some integer $k \geq 0$. Thus, from (1.3.3),

$$a = n(1 + 2(s-2)k) - (s-4)k = n + (2n(s-2) - (s-4))k \geq n + 2k > 0.$$

Hence, a is a positive integer.

Similarly from (1.3.4), b is also a positive integer.

As already noted, we have $(NX)^2 - 2(NY)^2 = -N^2$. Observe that if $X > Y > 1$ then $n < a < b$. Thus, by the comments after Lemma 1.2, $P_s(n)$, $P_s(a)$, and $P_s(b)$ would form a three-term arithmetic progression with common difference $d > 0$ in the s-gonal numbers. Therefore it suffices to show that there are infinitely many positive integers X and Y satisfying (1.3.2) with $X > Y > 1$.

The solutions to $X^2 - 2Y^2 = -1$ with X and Y being positive integers are given by $X + Y\sqrt{2} = \left(1 + \sqrt{2}\right)^m$ where m is an odd positive integer. We know that

$$G = \left((\mathbb{Z}/(2(s-2))\mathbb{Z})\left[\sqrt{2}\right]\right)^{\times}$$

is a finite group and $1+\sqrt{2}$ is an element of G. Letting m be 1 plus any even multiple of the order of $1+\sqrt{2}$ in G gives us that $\left(1+\sqrt{2}\right)^m$ is equivalent to $1+\sqrt{2}$ in G. Thus, for any such m, $X + Y\sqrt{2} = \left(1+\sqrt{2}\right)^m$ satisfies $X \equiv 1 \pmod{2(s-2)}$ and $Y \equiv 1 \pmod{2(s-2)}$. With the exception of $X = Y = 1$, we have that $X > Y > 1$. This guarantees that $P_s(a) - P_s(n) = P_s(b) - P_s(a) = d$ with $d > 0$.

This completes the proof of the theorem. □

1.4 Remarks

We conclude with a few comments on Theorem 1.3. The special cases of $s = 3$ and $s = 4$, corresponding to the triangular numbers and squares respectively, provide interesting examples. For $s = 3$, integers X and Y satisfying the equation in (1.3.2) give solutions a and b to (1.3.1) given by (1.3.3) and (1.3.4). In other words,

$$a = nY + \frac{Y-1}{2} \quad \text{and} \quad b = nX + \frac{X-1}{2}.$$

It is easy to show, however, that for every integral solution X and Y to the equation in (1.3.2), both X and Y are odd. Thus every integral solution to the equation in (1.3.2) gives us integral solutions a and b to (1.3.1).

With $s = 4$, the integral solutions X and Y to (1.3.2) give solutions a and b to (1.3.1) by $a = nY$ and $b = nX$. Again, we have integral solutions a and b to (1.3.1) for every integral solution X and Y to (1.3.2).

For both $s = 3$ and $s = 4$, every integral solution to the equation in (1.3.2) gives an integral solution to (1.3.1). However, for each $s \geq 5$ this is no longer the case. Take $s = 5$ and $n = 1$, for example, and consider arithmetic progressions with common difference d in the pentagonal numbers starting with $P_5(1) = 1$. Here $X = 7$ and $Y = 5$ is the first non-trivial solution to the equation in (1.3.2), but this does not give an integral solution to (1.3.1). In fact, the first non-trivial solution to the equation in (1.3.2) that does give an integral solution to (1.3.1) is $X = 1393$ and $Y = 985$, which gives us the three-term arithmetic progression $P_5(1) = 1$, $P_5(821) = 1010651$, and $P_5(1161) = 2021301$ with the common difference of 1010650.

Furthermore, not every integral solution to (1.3.1) is given by a solution to (1.3.2). Take the case of $s = 3$ and $n = 3$. Here $P_3(3) = 6$, $P_3(8) = 36$, and $P_3(11) = 66$ is an arithmetic progression with common difference of 30, but $a = 8$ and $b = 11$ are not given by a solution to (1.3.2). This choice of a and b do arise, however, from the discussion after the proof of Lemma 1.2 with $A = 17$ and $B = 23$, as illustrated next.

We wish to find all three-term arithmetic progressions beginning with $P_3(3)$ as discussed after Lemma 1.2. Thus we want to find all positive solutions A and B to the Pell equation $B^2 - 2A^2 = -N^2$, where $N = 7$.

For every divisor δ of $N = 7$, we have the associated Pell equation

$$X^2 - 2Y^2 = -\left(\frac{N}{\delta}\right)^2,$$

where $B = \delta X$, $A = \delta Y$, and we wish for X and Y to be relatively prime. In our case $N = 7$, so we consider the two equations

$$X^2 - 2Y^2 = -1 \qquad (1.4.1)$$

and

$$X^2 - 2Y^2 = -49. \qquad (1.4.2)$$

In the case of (1.4.1), we have $\delta = 7$; and in the case of (1.4.2), we have $\delta = 1$.

Equation (1.4.1) as previously noted has the solution $X = 1$ and $Y = 1$. All other solutions $X + Y\sqrt{2}$ are given by $\left(1 + \sqrt{2}\right)^m$ for any odd positive integer m. Again, we note that the solution $X = 1$ and $Y = 1$ gives the trivial arithmetic progression with common difference $d = 0$.

The solutions X and Y, with X and Y relatively prime, for equation (1.4.2) are given by $X = |U|$ and $Y = |V|$ where $U + V\sqrt{2} = (1 + 5\sqrt{2}\,)(1 + \sqrt{2}\,)^r$, where r is an even integer, possibly negative. Since $\delta = 1$ in this case, if we set $r = 2$, we obtain the values $A = 17$ and $B = 23$ as previously mentioned.

We note that in general there may not be any relatively prime solutions to a Pell equation, as in the case of $X^2 - 2Y^2 = -9$.

We can explicitly describe in a finite number of steps all three-term arithmetic progressions in the s-gonal numbers for any fixed $s \geq 3$ beginning with $P_s(n)$ for any positive integer n. One method for achieving this is through the use of continued fractions, as presented in [6, pages 423-527]. Of special importance to note is that the algorithm that is presented in [6] terminates in a finite number of steps, giving a description of all solutions to a Pell equation in terms of certain constructed general solutions to the Pell equation.

Chapter 2

Some Irreducibility Criteria for Polynomials with Non-Negative Integer Coefficients

2.1 Introduction

Given a prime number p, we can take the decimal representation $p = d_n d_{n-1} \cdots d_0$, where $0 \leq d_j \leq 9$ for $j \leq n$, and let $f(x) = \sum_{j=0}^{n} d_j x^j$. We have that $f(10) = p$. Pólya and Szegö in [14] attribute to A. Cohn that $f(x)$ is irreducible over the integers. In [3], Billhart, Filaseta, and Odlyzko extended this result for all bases $b \geq 2$. Additionally, they showed that for $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ where each $a_j \geq 0$ and $f(10)$ is prime, if each $a_j \leq 167$, then $f(x)$ is irreducible over the integers.

Filaseta in [8] extended this in two ways. First he showed that if the degree of $f(x)$ is $n \leq 31$ with no upper bound on the a_j's, then $f(x)$ is irreducible over the integers. Secondly, he showed that if the degree of $f(x)$ is $n \geq 32$ and each $a_j \leq a_n 10^{30}$, then $f(x)$ is irreducible over the integers.

In 2012, Filaseta and Gross extended this even further in [9] and [10]. They showed that if the degree of $f(x)$ is ≥ 32 and each $a_j \leq 49598666989151226098104244512918$, then $f(x)$ is irreducible over the integers. Furthermore, they showed that if each $a_j \leq 8592444743529135815769545955936773$, then $f(x)$ is either irreducible over the integers or is divisible by $x^2 - 20x + 101$.

Cole in [7] extended the results of Filaseta and Gross to bases b with $11 \leq b \leq 20$ and gave partial results for bases $b = 8$ and $b = 9$.

We will modify these methods to give complete results for bases b with $4 \leq b \leq 9$ and also give partial results for base $b = 3$ and base $b = 2$. Our goal will be to prove the following Theorem 2.1. We will let $\Phi_n(x)$ denote the n-th cyclotomic polynomial throughout this chapter.

Theorem 2.1. *Fix an integer b such that $b \geq 2$ and let $M_1(b)$ and $M_2(b)$ be as given in Table 2.1 and Table 2.2 respectively. Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ be such that $a_j \geq 0$ for each j and $f(b)$ is prime. If each $a_j \leq M_1(b)$, then $f(x)$ is irreducible. Also, for bases $3 \leq b \leq 5$, if each $a_j \leq M_2(b)$ and $f(x)$ is reducible, then $f(x)$ is* *divisible by* $\Phi_3(x-b)$*. Similarly, for bases* $6 \leq b \leq 20$*, if each* $a_j \leq M_2(b)$ *and* $f(x)$ *is reducible, then* $f(x)$ *is divisible by* $\Phi_4(x-b)$*.*

We will show that for bases $3 \leq b \leq 20$ that the bound $M_1(b)$ is sharp. For bases $4 \leq b \leq 20$ we will likewise show that the bound $M_2(b)$ is sharp.

Also of note is the bound $M_1(2) = 7$ as given in Table 2.1, which we suspect is not sharp. The previous bound on the coefficients in this case was 4 as shown by [1]. Consider the example

$$f(x) = x^{15} + 9x^{10} + 9x^9 + 9x^8 + 9x^7 + 9x^6 + 8x^5 + 10x^4 + 7x^3 + 10x^2 + 9x + 3.$$

Here $f(2) = 51157$ is prime, the largest coefficient of $f(x)$ is 10, and $f(x)$ is divisible by $x^2 - 3x + 3$. This example shows that the largest possible value of $M_1(2)$ is ≤ 9. Therefore this largest possible value is 7, 8, or 9.

Table 2.1 $M_1(b)$ for Bases $2 \leq b \leq 20$

b	$M_1(b)$
2	7
3	3795
4	8925840
5	56446139763
6	568059199631352
7	4114789794835622912
8	75005556404194608192050
9	1744054672674891153663590400
10	49598666989151226098104244512918
11	1754638089240473418053140582402752512
12	770402337502343186973808858801675881457 22
13	4163976197614743889240641877839816882986680320
14	274327682731486702351640132483696971555362645663790
15	532378204096072367538873751706765373387566379879922401 28
16	826743902509790173824819141451861039372680293578372832721 3632
17	126851405272079175658294461380208517509620085899496335987327578931 2
18	21007537854400487219032582960683605163219237120221608166828460963749 9040
19	3862536865580805292769435930162027257682225220024725436969612854940863037 4400
20	796509781584164390068427657717403682160575603517386313338062798297971858847052 8880

Table 2.2 $M_2(b)$ for Bases $2 \leq b \leq 20$

b	$M_2(b)$
2	–
3	–
4	48391200
5	125096244608
6	618804424079121
7	20721057406576714163
8	945987466487208056191224
9	55940538191331708311472104400
10	8592444743529135815769545955936773
11	1105373397761828143241737786386991708671
12	265147852448848502098555773338261457838146021
13	113377707741342790682562542077632396490643820979692
14	24009263205154407934683568810167126075855812416879485120
15	22547247502066821801492753280147763291252392992548016988539633
16	19350424243438912354196828588241701700337532166126769432980017078701
17	9771327410580082069204544811203201727273697038452545098276035319668495967
18	18439243120912559342277005462816793883105685612493543792760301014308216264410886
19	22643757580438427563497442159186765674826769157538919581661674785897250981739624957239
20	29644302367525205637719953585031678840057791870868847598894287680701297351967464608428822343

2.2 Preliminary Results

We begin with an instructive lemma adapted from [3] that will be of immense use to us later. Throughout this paper, whenever we refer to reducibility, we are referring to reducibility over the integers.

Lemma 2.2. *Fix an integer b such that $b \geq 2$. Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ be such that each $a_j \geq 0$ and $f(b)$ is prime. If $f(x)$ is reducible, then $f(x)$ has a non-real root in the disc $\mathcal{D}_b = \{z \in \mathbb{C} : |b - z| \leq 1\}$.*

Proof. Let b be an integer ≥ 2 and let $f(x)$ be as above. We do note that there is no upper bound on the size of the coefficients a_j. Assume that $f(x)$ is reducible. Then we may write $f(x) = g(x)h(x)$ for some $g(x)$ and $h(x)$ with integer coefficients, positive leading coefficients, $g(x) \not\equiv \pm 1$, and $h(x) \not\equiv \pm 1$. Since $f(b)$ is prime, one of $g(b)$ or $h(b)$ is ± 1. Without loss of generality, we may assume that $g(b) = \pm 1$. Since $g(x) \not\equiv \pm 1$, we know that $g(x)$ has positive degree.

Let c be the leading coefficient of $g(x)$ and $\beta_1, \beta_2, \ldots, \beta_r$ be the roots of $g(x)$ including multiplicities. Thus the degree of $g(x)$ is r and we have

$$1 = |g(b)| = |c| \prod_{j=1}^{r} |b - \beta_j| \geq \prod_{j=1}^{r} |b - \beta_j| \,.$$

Therefore at least one root of $g(x)$ is in the disc $\mathcal{D}_b = \{z \in \mathbb{C} : |b - z| \leq 1\}$.

We complete the lemma by noting that since $f(x)$ has non-negative coefficients, $f(x)$ has no positive real roots, and therefore neither does $g(x)$. □

Another useful lemma deals with the degree of $f(x)$ and is adapted from [8].

Lemma 2.3. *Fix an integer b such that $b \geq 2$ and let $D = D(b)$ as given in Table 2.3. Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ such that each $a_j \geq 0$ and $f(b)$ is prime. If the degree of $f(x)$ is $n \leq D$, then $f(x)$ is irreducible.*

Proof. Let b be an integer ≥ 2, let $D = D(b)$ be the integer given in Table 2.3, and let $f(x)$ be defined as above with $0 < \deg f \leq D$. By way of contradiction, suppose that $f(x)$ is reducible. Then $f(x)$ has a non-real root $\alpha \in \mathcal{D}_b = \{z \in \mathbb{C} : |b - z| \leq 1\}$ by Lemma 2.2. Since the complex conjugate of α is also a root of $f(x)$, we may assume that α has a positive imaginary part.

We write $\alpha = re^{i\theta}$, where $r \geq b-1$ and $0 < \theta \leq \arcsin(1/b)$. A few easy computations show that for each $k \in \{1, 2, \ldots, D\}$, we have that $0 < k\theta \leq D \arcsin(1/b) < \pi$. This gives us that

$$\Im\left(\alpha^k\right) = r^k \sin(k\theta) > 0 \text{ for } 1 \leq k \leq D.$$

Our polynomial $f(x)$ has non-negative coefficients and $\deg f = n$ with $1 \leq n \leq D$, so

$$\Im(f(\alpha)) \geq \Im(\alpha^n) > 0,$$

but this contradicts the fact that α is a root of $f(x)$. Thus $f(x)$ is irreducible. □

Table 2.3 Maximum Degree based on Base b

Base b	2	3	4	5	6	7	8	9	10	11
Degree $D = D(b)$	5	9	12	15	18	21	25	28	31	34

Base b	12	13	14	15	16	17	18	19	20	
Degree $D = D(b)$	37	40	43	47	50	53	56	59	62	

For the most part, the bounds $D(b)$ given in Table 2.3 are sharp. Take for example base $b = 4$. We see that

$$f(x) = x^{13} + x^3 + 235835x + 16576651$$

is of degree 13, $f(4) = 84628919$ is prime, each coefficient is ≤ 16576651, and $f(x)$ is divisible by $\Phi_3(x-4) = x^2 - 7x + 13$. Thus $D(4)$ in Table 2.3 is sharp. In Section 2.4, we will give sharp bounds $D(b)$ for the other bases b. Additionally, although not the focus of this paper, we will give sharp bounds on the size of the coefficients when $f(x)$ is reducible and of degree $D(b) + 1$.

2.3 A Root Bounding Function

The goal of our research is, for a given $b \in \{2, 3, \ldots, 20\}$, to find sharp upper bounds for $M_1(b)$ and $M_2(b)$ in Theorem 2.1. To this end, we will utilize three main methods as in [9] and [10]. First, we will introduce certain rational functions that will give us information on the location of possible roots of $f(x)$. Second, we use the ideas in [1] to get an initial value for $M_1(b)$ and $M_2(b)$. Finally, we use information gained from recursive relations on the possible factors of $f(x)$, as outlined in [9] and [10], to establish sharp values of $M_1(b)$ for $b \geq 3$ and sharp values of $M_2(b)$ for $b \geq 4$. In this section, we focus on the first of these ideas.

We recall that $\Phi_n(x)$ denotes the n-th cyclotomic polynomial and ζ_n is the n-th root of unity $e^{2\pi i/n}$. Fix an integer b with $2 \leq b \leq 20$. Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ have each $a_j \geq 0$ and $f(b)$ be prime. Although our results will restrict to $2 \leq b \leq 20$, these methods can be used to obtain similar results for any integer b with $b \geq 2$.

Now, as in the proof of Lemma 2.2, we suppose that $f(b)$ is prime, $f(x)$ is reducible with $f(x) = g(x)h(x)$, where each of $g(x)$ and $h(x)$ are polynomials with integer coefficients, are not identically ± 1, and have positive leading coefficients. Without loss of generality, we may take $g(b) = \pm 1$. Lemma 2.2 implies that $g(x)$ has a non-real root in the disc $\mathcal{D}_b = \{z \in \mathbb{C} : |b - z| \leq 1\}$. Using the ideas of [9] and [10], we wish to show that either $g(x)$ has a root in common with

$$\Phi_3(x - b) = x^2 - (2b - 1)\, x + b^2 - b + 1,$$

$$\Phi_4(x - b) = x^2 - 2bx + b^2 + 1,$$

and

$$\Phi_6(x - b) = x^2 - (2b + 1)\, x + b^2 + b + 1,$$

or $g(x)$ has roots in the region $\mathcal{R}_b$, which we define shortly.

Taking inspiration from [9] and [10], we define

$$F_b(z) = \frac{N_b(z)}{D_b(z)}, \tag{2.3.1}$$

where

$$N_b(z) = |b-1-z|^{2e_1} \left(|b+\zeta_3 - z| \left|b+\overline{\zeta_3} - z\right|\right)^{2e_3} \cdot \left(|b+i-z|\,|b-i-z|\right)^{2e_4} \left(|b+\zeta_6 - z| \left|b+\overline{\zeta_6} - z\right|\right)^{2e_6},$$

$$D_b(z) = |b-z|^{4(e_3+e_4+e_6)+2(e_1+d+1)},$$

and $e_1 = e_1(b), e_3 = e_3(b), e_4 = e_4(b), e_6 = e_6(b)$ and $d = d(b)$ are all non-negative integers. For Theorem 2.1, the numbers e_1, e_3, e_4, e_6 and d for a given base b are given in Table 2.4.

Table 2.4 Numbers Used in $F_b(z)$ for Base b

Base b	2	3	4	5	6	7	8	9	10	$11 \le b \le 20$
$e_1 = e_1(b)$	20	0	0	0	0	0	0	0	0	0
$e_3 = e_3(b)$	4	15	9	6	4	4	4	4	4	4
$e_4 = e_4(b)$	0	2	2	2	2	2	2	2	2	2
$e_6 = e_6(b)$	0	0	3	3	3	3	3	3	3	3
$d = d(b)$	0	3	3	3	3	3	3	3	3	3

We pause to note that these are not the only choices for $e_1(b), e_3(b), e_4(b), e_6(b)$, and $d(b)$ that can serve our purposes. For example, the choice of $e_1(10) = 0$, $e_3(10) = 3$, $e_4(10) = 2$, $e_6(10) = 3$, and $d(10) = 3$ are the numbers for base $b = 10$ that were used in [9] and [10]. These same numbers were also used for bases $8 \le b \le 20$ in [7]. Our choices for these numbers in Table 2.4 will give us improvements later.

Taking $z = x + iy$, direct computations give us that

$$\begin{aligned} |b-1-z|^2 &= |b-1-x-iy|^2 \\ &= x^2 + y^2 + (2-2b)\,x + b^2 - 2b + 1, \end{aligned}$$

$$
\begin{aligned}
\left(|b+\zeta_3-z|\left|b+\overline{\zeta_3}-z\right|\right)^2 &= \left(|b+\zeta_3-x-iy|\left|b+\overline{\zeta_3}-x-iy\right|\right)^2 \\
&= x^4+(2-4b)x^3+2y^2x^2+\left(3-6b+6b^2\right)x^2 \\
&\quad +(2-4b)\,xy^2-\left(4b^3-6b^2+6b-2\right)x+y^4 \\
&\quad -\left(1+2b-2b^2\right)y^2+b^4-2b^3+3b^2-2b+1,
\end{aligned}
$$

$$
\begin{aligned}
(|b+i-z|\,|b-i-z|)^2 &= (|b+i-x-iy|\,|b-i-x-iy|)^2 \\
&= x^4-4bx^3+2x^2y^2+\left(2+6b^2\right)x^2-4bxy^2 \\
&\quad -\left(4b+4b^3\right)x+y^4-\left(2-2b^2\right)y^2+b^4+2b^2+1,
\end{aligned}
$$

$$
\begin{aligned}
\left(|b+\zeta_6-z|\left|b+\overline{\zeta_6}-z\right|\right)^2 &= \left(|b+\zeta_6-x-iy|\left|b+\overline{\zeta_6}-iy\right|\right)^2 \\
&= x^4-(2+4b)x^3+2x^2y^2+\left(3+6b+6b^2\right)x^2 \\
&\quad -(2+4b)xy^2-\left(2+6b+6b^2+4b^3\right)x+y^4 \\
&\quad -\left(1-2b-2b^2\right)y^2+b^4+2b^3+3b^2+2b+1,
\end{aligned}
$$

and

$$
\begin{aligned}
|b-z|^2 &= |b-x-iy|^2 \\
&= x^2+y^2-2bx+b^2.
\end{aligned}
$$

Therefore $N_b(z)$ and $D_b(z)$ are polynomials in $\mathbb{Z}[b,x,y]$, so $F_b(z)$ is a rational function in b, x and y.

The motivation for $F_b(z)$ is the same as in [9] and [10], which we reproduce here with the obvious changes. We write $g(x)$ in the form

$$
g(x)=c\prod_{j=1}^{r}(x-\beta_j),
$$

where c is the leading coefficient of $g(x)$, and $\beta_1,\ldots,\beta_r$ are the roots of $g(x)$, and therefore also roots of $f(x)$.

For ease of notation, we define

$$
\widetilde{g_b}(n)=g\left(b+\zeta_n\right)g\left(b+\overline{\zeta_n}\right).
$$

This gives us that the two expressions

$$\frac{|g(b-1)|^{2e_1}\,|\widetilde{g}_b(3)|^{2e_3}\,|\widetilde{g}_b(4)|^{2e_4}\,|\widetilde{g}_b(6)|^{2e_6}}{|g\,(b)|^{4(e_3+e_4+e_6)+2(e_1+d+1)}}$$

and

$$\frac{1}{c^{2(d+1)}}\prod_{j=1}^{r}F_b(\beta_j)$$

are equal. We denote these common values by V.

Now, each of $\widetilde{g}_b(3)$, $\widetilde{g}_b(4)$, and $\widetilde{g}_b(6)$ is a symmetric polynomial, with integer coefficients, in the roots of an irreducible monic quadratic in $\mathbb{Z}[x]$. Hence, each of these expressions is an integer. Clearly $g(b-1)$ is an integer. Thus, the numerator of the first expression for V above is an integer. Since $g(b)=\pm 1$ and $V \geq 0$, we know that either $V=0$ or $V \in \mathbb{Z}^+$.

We recall that $f(x)$ is a polynomial with non-negative integer coefficients. Thus $f(x)$ cannot have a positive real root, and neither can $g(x)$ which is a factor of $f(x)$. Therefore $g(b-1) \neq 0$. Now, the definition of V also implies that $V=0$ only if at least one of $\Phi_3(x-b)$, $\Phi_4(x-b)$ and $\Phi_6(x-b)$ is a factor of $g(x)$. If none of these quadratics is a factor of $g(x)$, we necessarily have that $V \in \mathbb{Z}^+$. In this case, the product in the second expression for V above must be a positive integer. Since $F_b(z)$ is a non-negative real number for all $z \in \mathbb{C}$, we know that $F_b(\beta_j) \geq 1$ for at least one value of $j \in \{1, 2, \ldots, r\}$. In other words, there is a root β of $g(x)$, and consequently of $f(x)$, satisfying $F_b(\beta) \geq 1$.

We pause a moment to summarize the importance that $F_b(z)$ plays. Given only that $g(x) \in \mathbb{Z}[x]$, $g(b-1) \neq 0$, $g(x) \not\equiv \pm 1$, and $g(b) = \pm 1$, we have shown that either $g(x)$ has at least one of the factors $\Phi_3(x-b)$, $\Phi_4(x-b)$, and $\Phi_6(x-b)$, or $g(x)$ has a root β in the region $\mathcal{R}_b$ defined as

$$\mathcal{R}_b = \{z \in \mathbb{C} : F_b(z) \geq 1\}\,. \tag{2.3.2}$$

In the latter case, we use an analysis of the region $\mathcal{R}_b$ in the complex plane to obtain important information about the location of β.

The following graphs depict regions $\mathcal{R}_b$ for various choices of $e_1(b), e_3(b), e_4(b), e_6(b)$ and $d(b)$ as given in Table 2.4. The circle imposed on the graph is the unit circle centered at b, and is placed there for reference only.

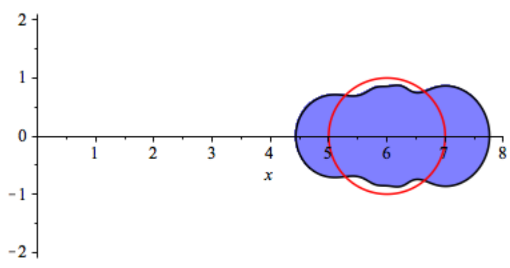

Figure 2.1 R_6 to Scale

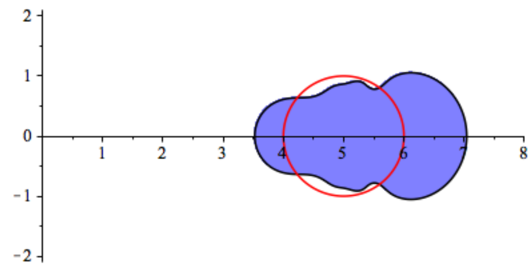

Figure 2.2 R_5 to Scale

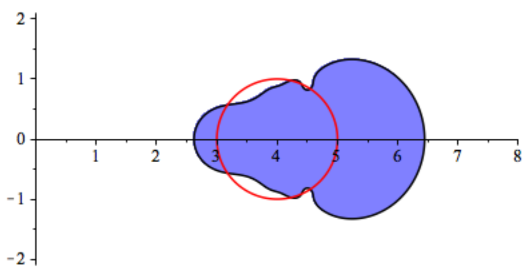

Figure 2.3 R_4 to Scale

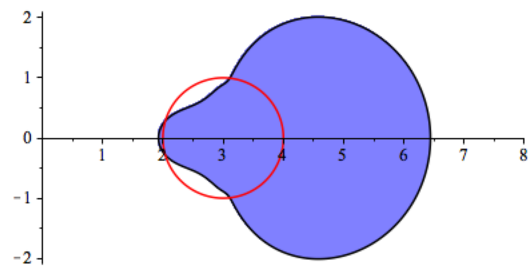

Figure 2.4 R_3 to Scale

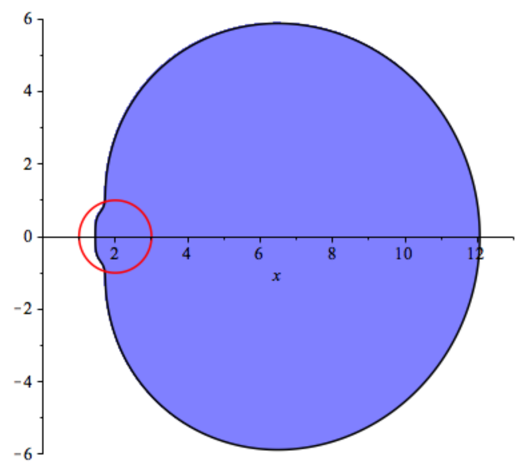

Figure 2.5 R_2 to Scale

We also pause to compare our choice of $e_1(b), e_3(b), e_4(b), e_6(b)$ and $d(b)$ for base $b = 10$ with those that were used in [9], [10], and [7]. Figure 2.6 shows our choice of $e_1(10) = 0, e_3(10) = 4, e_4(10) = 2, e_6(10) = 3$ and $d(10) = 3$ while Figure 2.7 shows their choice of $e_1(10) = 0, e_3(10) = 3, e_4(10) = 2, e_6(10) = 3$ and $d(10) = 3$. Both Figure 2.6 and Figure 2.7 have the y-axis placed at $x = 8$ for ease of viewing. Although subtle, Figure 2.7 is symmetric about the vertical line $x = 10$, while Figure 2.6 is slightly narrower at the front of the region.

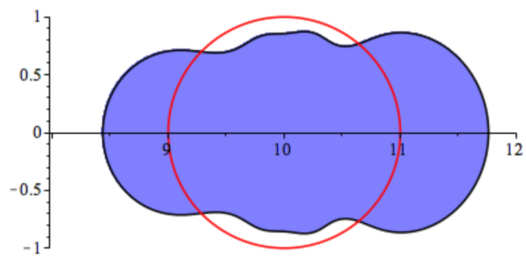

Figure 2.6 New R_{10}

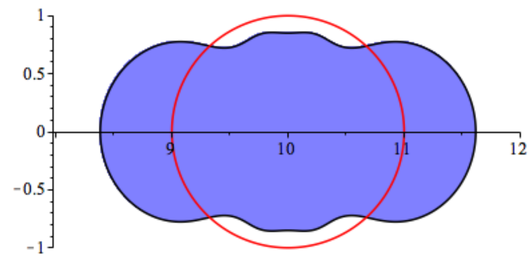

Figure 2.7 Old R_{10}

Although these graphs are only based on numerical approximations, they do help to motivate the subsequent arguments that give us precise information about β that we require. We will sometimes refer to points (x, y) being in $\mathcal{R}_b$, and this is to be interpreted as the point $z = x + iy$ in the complex plane being in $\mathcal{R}_b$. For example, taking base $b = 6$, we will see later that all the points $(x, y) \in \mathcal{R}_6$ lie below the line $y = \tan(\pi/21)x$. This then means that any point $z = x + iy \in \mathcal{R}_6$ satisfies $y \le \tan(\pi/21)x$.

To further help us analyze the region R_b in the complex plane, we define

$$P_b(x, y) = D_b(x + iy) - N_b(x + iy). \tag{2.3.3}$$

With direct computations, which are given in Appendix A.1, we can write

$$P_b(x, y) = \sum_{j=0}^{r} a_j(b, x) y^{2j}, \tag{2.3.4}$$

where $r = 2(e_3 + e_4 + e_6) + e_1 + d + 1$ and each $a_j(b, x)$ is an integer polynomial in b and x. Furthermore, the definition of $D_b(z)$ implies that $D_b(z) > 0$ for all $z \in \mathbb{C}$

with $z \neq b$. Thus,

$$F_b(x+iy) \geq 1 \quad \text{and} \quad P_b(x,y) \leq 0$$

are equivalent for $z \neq b$. Also, we have that the equation $F_b(x+iy) = 1$ and $P_b(x,y) = 0$ are equivalent for $z \neq b$. Therefore, the $z \in \mathbb{C}$ such that $F_b(x+iy) = 1$ correspond exactly to the numbers where $P_b(x,y) = 0$.

We introduce the following technical Lemma:

Lemma 2.4. *Fix an integer $2 \leq b \leq 20$. Then there exists real numbers $a_0 = a_0(b)$, $a_1 = a_1(b)$, and a function $\rho_b(x)$ defined on an interval $I_b = [b - a_0, b + a_1]$ such that the following conditions hold:*

1. *For any given $x \notin I_b$, $P_b(x,y) = 0$ has no real roots in y.*
2. *$\rho_b(b - a_0) = 0$ and $\rho_b(b + a_1) = 0$.*
3. *$P_b(x, \rho_b(x)) = 0$ for all $x \in I_b$.*
4. *$\rho_b(x)$ is a continuously differentiable function on the interior of I_b and continuous on I_b.*
5. *If x and y are real numbers for which $P_b(x,y) \leq 0$, then $x \in I_b$ and $|y| \leq \rho_b(x)$.*

Given the above lemma, complex numbers of the form $x + i\rho_b(x)$ are boundary points of $\mathcal{R}_b$ which are on or above the real axis. Since $P_b(x,y)$ is a polynomial in y^2 with coefficients in $\mathbb{Z}[b,x]$, our region $\mathcal{R}_b$ is symmetric about the real axis. Thus the points $x - i\rho_b(x)$ are boundary points of $\mathcal{R}_b$ which are on or below the real axis. The points $b - a_0$ and $b + a_1$ are boundary points on the real axis.

To prove Lemma 2.4, we will use the oft-cited Implicit Function Theorem, which can be found in [5], [15], [17], or even in [9] and [10], and is stated here.

Lemma 2.5. *Let $\mathfrak{D}$ be an open set in $\mathbb{R}^2$ and let $W : \mathfrak{D} \to \mathbb{R}$. Suppose W has continuous partial derivatives W_x and W_y on $\mathfrak{D}$. Let $(x_0, y_0) \in \mathfrak{D}$ be such that*

$$W(x_0, y_0) = 0 \text{ and } W_y(x_0, y_0) \neq 0.$$

Then there is an open interval $\mathfrak{I} \in \mathbb{R}$ and a real valued, continuously differentiable function ϕ defined on $\mathfrak{I}$ such that $x_0 \in \mathfrak{I}$, $\phi(x_0) = y_0$, $(x, \phi(x)) \in \mathfrak{D}$ for all $x \in \mathfrak{I}$, and $W(x, \phi(x)) = 0$ for all $x \in \mathfrak{I}$.

Our proof of Lemma 2.4 is a variation of the proof of the corresponding lemma in [9].

Proof of Lemma 2.4. First we fix an integer $2 \leq b \leq 20$ and let $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$ and $d = d(b)$ be as in Table 2.4. Using this choice of numbers, we set $r = 2(e_3 + e_4 + e_6) + e_1 + d + 1$ and let $P_b(x, y)$ be as shown in (2.3.4). Now, for $0 \leq j \leq r$, define $p_j(b, x) = a_j(b, x + b)$, and set

$$\overleftarrow{P_b}(x, y) = \sum_{j=0}^{r} p_j(b, x) y^j = \sum_{j=0}^{r} a_j(b, x + b) y^j.$$

Thus,

$$\overleftarrow{P_b}\left(x, y^2\right) = P_b(x + b, y). \tag{2.3.5}$$

We comment that the notation $p_j(b, x)$ is slightly misleading since each p_j is a polynomial with integer coefficients in only the variable x and depends on b only for the choice of $e_1(b)$, $e_3(b)$, $e_4(b)$, $e_6(b)$, and $d(b)$ used above. Since the same choice of $e_1(b)$, $e_3(b)$, $e_4(b)$, $e_6(b)$, and $d(b)$ are used for bases $6 \leq b \leq 20$, we have only five unique sets of these numbers to consider. The exact values of each $p_j(b, x)$ for various choices of $e_1(b)$, $e_3(b)$, $e_4(b)$, $e_6(b)$, and $d(b)$ have been computed and are explicitly shown in Appendix A.2.

To simplify our notation and avoid confusion, we use $\overleftarrow{P_b}(y)$ for $\overleftarrow{P_b}(x, y)$ when we are viewing $\overleftarrow{P_b}(x, y)$ as a polynomial in y whose coefficients are polynomials in x.

Also, the degree of $\overleftarrow{P_b}(y)$ is r. Table 2.5 lists r for the various bases.

Table 2.5 Degree r of $\overleftarrow{P_b}(y)$ for Base b

Base b	r
$b = 2$	$r = 29$
$b = 3$	$r = 38$
$b = 4$	$r = 32$
$b = 5$	$r = 26$
$6 \leq b \leq 20$	$r = 22$

Now, using a Sturm sequence, we verify that $p_0(b, x)$ has exactly two distinct real roots. One checks that $p_0(b, x) = 0$ has a negative root, which, in absolute value, we call a_0, and a positive root, which we will call a_1. Computations give us the values of a_0 and a_1 for various bases b, accurate to the digits shown in Table 2.6. We show that a_0 and a_1 have the properties stated in Lemma 2.4.

Table 2.6 Values of a_0 and a_1 for Base b

Base b	a_0	a_1	$\hat{a_0}$	$\hat{a_1}$
$b = 2$	$0.5523770847\ldots$	$10.0651310946\ldots$	0.5523	10.06
$b = 3$	$1.0721963435\ldots$	$3.4397713145\ldots$	1.07	3.43
$b = 4$	$1.3782037799\ldots$	$2.4446162254\ldots$	1.37	2.44
$b = 5$	$1.4754544841\ldots$	$2.0416766993\ldots$	1.47	2.04
$6 \leq b \leq 20$	$1.5638035689\ldots$	$1.7605007116\ldots$	1.56	1.76

Let J_b denote the interval $[-a_0, a_1]$. Using Sturm sequences, one can verify that for each $j \in \{1, 2, \ldots, r\}$, the polynomial $p_j(b, x)$ has all of its real roots in the interval $[-\hat{a_0}, \hat{a_1}] \subset J_b$, where $\hat{a_0}$ and $\hat{a_1}$ are given in Table 2.6. These computations are shown in Appendix A.3.

Recalling equation (2.3.5), we see that to prove part (1), we need only show that for any given $x_0 \notin J_b$, the real roots of $\overleftarrow{P_b}(x_0, y)$ are all negative. A simple calculation shows that $p_j(b, \pm 11) > 0$ for all $j \in \{0, 1, \ldots, r\}$. Since none of the $p_j(b, x)$ have real roots outside of J_b, we deduce that $p_j(b, x_0) > 0$ for each j. From Descartes' rule of signs, we obtain that $\overleftarrow{P_b}(x_0, y)$ has no positive real roots. Since $\overleftarrow{P_b}(x_0, 0) = p_0(b, x_0) \neq 0$, part (1) now follows.

For the purposes of part (5), we note that this implies that $P_b(x, y) > 0$ for all $x \notin I_b$.

We turn to the remaining parts of Lemma 2.4. For a given $x \in I_b$, we define $\rho_b(x)$ to be the largest real root of $P_b(x, y)$. We will need to show that such a root exists and is non-negative. From (2.3.5), we see that for $x \in J_b$, we want $(\rho_b(x + b))^2$ to be a root of $\overleftarrow{P_b}(y)$. Further, showing $P_b(x, y)$ has a non-negative root for each $x \in I_b$ is equivalent to showing $\overleftarrow{P_b}(y)$ has a non-negative root for each $x \in J_b$.

A direct computation gives that $p_0(b, 0) = -1$ and $p_r(b, x) = 1$. Since $p_0(b, x)$ has only the two real roots $-a_0$ and a_1, it follows that $p_0(b, x_0) < 0$ for all $x_0 \in (-a_0, a_1)$. Since $\overleftarrow{P_b}(y)$ is monic and of degree $r > 0$, it follows that $\overleftarrow{P_b}(x_o, y) = 0$ has a positive real root in y for all $x_0 \in (-a_0, a_1)$.

We now consider the case that $x_0 = -a_0$ or $x_0 = a_1$. As noted earlier, for each $j \in \{1, 2, \ldots, r\}$, the polynomial $p_j(b, x)$ has its roots in the interval $[-\hat{a_0}, \hat{a_1}]$ and $p_j(b, \pm 11) > 0$. Since each of $-a_0$, a_1, and ± 11 are not in $[-\hat{a_0}, \hat{a_1}]$ while $x_0 = -a_0$ or $x_0 = a_1$, it follows that $p_j(b, x_0) > 0$ for each such j. From Descartes' rule of signs, we deduce that $\overleftarrow{P_b}(x_0, y)$ has no positive real roots. Thus, $\overleftarrow{P_b}(x_0, y)$ has 0 as its largest real root.

For each $x \in J_b$, define

$$\psi_b(x) = \max\left\{y \in \mathbb{R} : \overleftarrow{P_b}(y) = 0\right\}.$$

Since $\overleftarrow{P_b}(y)$ has real roots for any given $x \in J_b$, then $\psi_b(x)$ is well defined. Moreover, we have now seen that $\psi_b(x) > 0$ for all $x \in (-a_0, a_1)$, and $\psi_b(-a_0) = 0$, $\psi_b(a_1) = 0$. Parts (2) and (3) now follow by observing that $\rho_b(x) = \sqrt{\psi_b(x - b)}$ for each $x \in I_b$.

Next, we establish part (4). To prove $\rho_b(x)$ is a continuously differentiable function on $(b - a_0, b + a_1)$, it is sufficient to show that, given any $x_0 \in (-a_0, a_1)$, there exists an open interval $J' \subseteq (-a_0, a_1)$ containing x_0 such that $\psi_b(x)$ is a continuously differentiable function on J'. To prove that $\rho_b(x)$ is a continuous function on

$[b-a_0, b+a_1]$, we will also need to show that

$$\lim_{x\to -a_0^+} \psi_b(x) = 0 \quad \text{ and } \quad \lim_{x\to a_1^-} \psi_b(x) = 0.$$

Fix $x_0 \in (-a_0, a_1)$, and let $y_0 = \psi_b(x_0)$. We make use of Lemma 2.5 with $W(x,y) = \overleftarrow{P_b}(x,y)$. Since then $W(x,y)$ is a polynomial, both W_x and W_y are continuous on all of $\mathbb{R}^2$. The definition of y_0 implies $W(x_0, y_0) = 0$.

We also need to show that $W_y(x_0, y_0) \neq 0$. In the case that $b \neq 2$, we calculate the discriminant $\Delta_b(x)$ of $\overleftarrow{P_b}(x,y)$. A Sturm sequence computation shows that $\Delta_b(x) \neq 0$ for all $x \in \mathbb{R}$. Therefore, in the case that $b \neq 2$, we have that $\overleftarrow{P_b}(x_0, y)$ has no repeated roots, so $W_y(x_0, y_0) \neq 0$. In the case that $b = 2$, a Sturm sequence computation shows that $\Delta_2(x)$ is non-zero on J_2 when $x \neq -1/2$. Thus, we have that $\overleftarrow{P_2}(x,y)$ has a repeated root for $x \in J_b$ only when $x = -1/2$. By factoring $\overleftarrow{P_b}(-1/2, y)$, one sees that the only repeated root of $\overleftarrow{P_b}(-1/2, y)$ is $y = -1/4$. Therefore, in our case that $y_0 \geq 0$, $W_y(x_0, y_0) \neq 0$.

Now define $\mathfrak{D} = \{(x,y) \in \mathbb{R}^2 : -a_0 < x < a_1 \text{ and } y > 0\}$. From Lemma 2.5, there exists an open interval $J'' \subseteq (-a_0, a_1)$ containing x_0 and a continuously differentiable function $\phi(x)$ defined on J'' such that $\phi(x_0) = y_0$ and also that $\overleftarrow{P_b}(x, \phi(x)) = 0$ for all $x \in J''$. By the definition of $\psi_b(x)$, we know that $\phi(x) \leq \psi_b(x)$ for all $x \in J''$. We will show that there exists an open interval $J' \subseteq J''$ containing x_0 such that $\psi_b(x) = \phi(x)$ for all $x \in J'$.

By way of contradiction, assume that no such interval J' exists. Then there exists a sequence $\{x_n\}_{n=1}^\infty$ satisfying $\lim\limits_{n\to\infty} x_n = x_0$ and having the property that, for all $n \geq 1$, $\psi_b(x_n) > \phi(x_n)$. Since $x_0 \in J''$, we suppose further as we may that each $x_n \in J''$. Define $y_n = \psi_b(x_n)$. In particular, $\overleftarrow{P_b}(x_n, y_n) = 0$. We justify that $\{y_n\}_{n=1}^\infty$ is a bounded sequence. In fact, we show that there is an absolute constant M such that for $x' \in J_b$ and $z \in \mathbb{C}$ satisfying $\overleftarrow{P_b}(x', z) = 0$, we have $|z| \leq M$. Since each $p_j(b,x)$ is continuous on J_b and J_b is compact, there exists an absolute constant $A \geq 0$ such that $|p_j(b,x)| \leq A$ for all $j \in \{0, \ldots, r\}$ and $x \in J_b$. Recall $p_r(b,x) = 1$. Since

$x' \in J_b$ and $\overleftarrow{P_b}\,(x', z) = 0$, we deduce

$$0 = \left| \sum_{j=0}^{r} p_j\,(b, x')\, z^j \right| \geq |z|^r - \sum_{j=0}^{r-1} |p_j\,(b, x')|\, |z|^j \geq |z|^r - A \sum_{j=0}^{r-1} |z|^j\,.$$

Thus, $|z|$ is less than or equal to the positive real root M of the polynomial

$$x^r - Ax^{r-1} - Ax^{r-2} - \cdots - Ax - A.$$

We deduce that $\{y_n\}_{n=1}^{\infty}$ is a sequence with $|y_n| \leq M$ for all n. Hence, the sequence $\{y_n\}_{n=1}^{\infty}$ has a convergent subsequence $\left\{y_{n_j}\right\}_{j=1}^{\infty}$. Let $L = \lim_{j\to\infty} y_{n_j}$. The continuity of $\overleftarrow{P_b}\,(x, y)$ implies

$$\overleftarrow{P_b}\,(x_0, L) = \lim_{j\to\infty} \overleftarrow{P_b}\left(x_{n_j}, y_{n_j}\right) = 0.$$

Since

$$y_0 = \psi_b(x_0) = \max\{y \in \mathbb{R} : \overleftarrow{P_b}\,(x_0, y) = 0\},$$

we deduce that $L \leq y_0$. Since $\phi(x)$ is continuous on J'' and $\phi\left(x_{n_j}\right) \leq \psi_b\left(x_{n_j}\right) = y_{n_j}$ for all $j \geq 1$, we also have that

$$L = \lim_{j\to\infty} y_{n_j} = \lim_{j\to\infty} \psi_b\left(x_{n_j}\right) \geq \lim_{j\to\infty} \phi\left(x_{n_j}\right) = \phi\left(\lim_{j\to\infty} x_{n_j}\right) = \phi\left(x_0\right) = y_0.$$

Thus, $L = y_0$. In particular,

$$\lim_{j\to\infty} \psi_b\left(x_{n_j}\right) = y_0 = \lim_{j\to\infty} \phi\left(x_{n_j}\right). \tag{2.3.6}$$

We show that this implies a contradiction.

Consider

$$\left|W\left(x_{n_j}, \psi_b\left(x_{n_j}\right)\right) - W\left(x_{n_j}, \phi\left(x_{n_j}\right)\right)\right| = 0.$$

By the Mean Value Theorem, we have that

$$\left|\psi_b\left(x_{n_j}\right) - \phi\left(x_{n_j}\right)\right| \left|W_y\left(x_{n_j}, \xi_j\right)\right| = 0 \tag{2.3.7}$$

for some $\xi_j \in \left[\phi\left(x_{n_j}\right), \psi_b\left(x_{n_j}\right)\right]$. Since $\psi_b\left(x_{n_j}\right) > \phi\left(x_{n_j}\right)$, we have from (2.3.7) that

$$|W_y\,(x_0, \xi_j)| = 0.$$

Taking the limit as $j \to \infty$, we have by (2.3.6) that $\lim_{j\to\infty} \xi_j = y_0$ so that

$$|W_y(x_0, y_0)| = 0.$$

But this contradicts the fact that $W_y(x_0, y_0) \neq 0$. Therefore, there exists an open interval $J' \subseteq J''$ containing x_0 such that $\psi_b(x) = \phi(x)$ for all $x \in J'$.

To finish the proof of part (4), we need only to show that $\psi_b(x)$ is continuous at the endpoints of J_b. Let $\{x_n\}_{n=1}^{\infty} \subset J_b$ be a sequence that converges to one of the endpoints of J_b, say a_1. Take $y_n = \psi_b(x_n)$. With M as before, we have that $|y_n| \leq M$. To show that

$$\lim_{n\to\infty} \psi_b(x_n) = 0 = \psi_b(a_1),$$

it suffices to prove that every convergent subsequence of y_n converges to 0.

Suppose that $\left\{y_{n_j}\right\}$ is such that $\lim_{j\to\infty} y_{n_j} = L$ for some $L \in \mathbb{R}$. Since we know that $y_{n_j} = \psi_b\left(x_{n_j}\right) \geq 0$, we deduce $0 \leq L \leq M$. Now

$$\overleftarrow{P}_b(a_1, L) = \lim_{j\to\infty} \overleftarrow{P}_b\left(x_{n_j}, y_{n_j}\right) = \lim_{j\to\infty} \overleftarrow{P}_b\left(x_{n_j}, \psi_b\left(x_{n_j}\right)\right) = 0.$$

Therefore, $L \leq \psi_b(a_1) = 0$. Hence, $L = 0$. A similar argument holds for the endpoint $-a_0$, completing the proof of part (4).

To establish part (5), we first observe that the definition of $\rho_b(x)$ implies that if $x \in I_b$ and $y \in \mathbb{R}$ are such that $P_b(x, y) = 0$, then $|y| \leq \rho_b(x)$. Part (1) also implies if $P_b(x, y) = 0$ for some real numbers x and y, then $x \in I_b$. Now, consider real numbers x_0 and y_0 for which $P_b(x_0, y_0) < 0$. One checks that $P_b(0, 0) > 0$. Since $P_b(x, y)$ is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$, we deduce that along any path from $(0, 0)$ to (x_0, y_0) in $\mathbb{R}^2$, there must be a point (x, y) satisfying $P_b(x, y) = 0$. We use again that for any $x \in J_b$, the number M is a bound on the absolute value of the roots of $\overleftarrow{P_b}(y)$. We deduce from (2.3.5) that $\rho_b(x) \leq \sqrt{M}$ for all $x \in I_b$. If $x_0 \notin I_b = [b - a_0, b + a_1]$ or if $x_0 \in I_b$ and $y_0 > \rho_b(x_0)$, one can consider the path consisting of line segments from $(0, 0)$ to $\left(0, 1 + \sqrt{M}\right)$, from $\left(0, 1 + \sqrt{M}\right)$ to $\left(x_0, 1 + \sqrt{M}\right)$ and from $\left(x_0, 1 + \sqrt{M}\right)$

to (x_0, y_0) to obtain a contradiction. If $x_0 \in I_b$ and $y_0 < -\rho_b(x_0)$, one can consider a similar path but from $(0,0)$ to $\left(0, -1-\sqrt{M}\right)$ to $\left(x_0, -1-\sqrt{M}\right)$ to (x_0, y_0) to obtain a contradiction. Therefore, we must have $x_0 \in I_b$ and $|y_0| \leq \rho_b(x_0)$. This establishes part (5), completing the proof. $\square$

Now that we have proven Lemma 2.4, we will use it in the next sections to prove irreducibility criteria based on the degree of $f(x)$ and on the size of the coefficients of $f(x)$.

2.4 Irreducibility Criteria Based on Degree

We take a moment to consider the region $\mathcal{R}_b$ as defined in the previous section. First we fix a base b with $2 \leq b \leq 20$ and then let $f(x) \in \mathbb{Z}[x]$ have non-negative coefficients. Furthermore, we assume that $f(b)$ is prime. In Section 2.2 with Lemma 2.3 we gave irreducibility criteria based on the degree of $f(x)$, although these bounds were not necessarily sharp. We now use the region $\mathcal{R}_b$ to make these bounds sharp.

Take for example base $b = 6$. Lemma 2.3 and Table 2.3 gives us that if $f(6)$ is prime and the degree of $f(x)$ is ≤ 18, then $f(x)$ is irreducible. We now show that if $f(6)$ is prime and the degree of $f(x)$ is ≤ 19, then $f(x)$ is irreducible. Furthermore, we give an example to show that this bound is sharp. The argument we give is modified from the one that appears in [9].

First, we state a short lemma:

Lemma 2.6. *Let n be a positive integer. A complex number $\alpha = re^{i\theta}$, such that $0 < \theta < \pi/n$, cannot be a root of a non-zero polynomial with non-negative integer coefficients and degree $\leq n$.*

The proof of this lemma is immediate from the proof of Lemma 2.3 in Section 2.1.

Now, we can show the following:

Theorem 2.7 (Improvement to Lemma 2.3)**.** *Fix an integer b such that $2 \leq b \leq 20$ and let $D = D(b)$, $D_1 = D_1(b)$, and $D_2 = D_2(b)$ be as in Table 2.7. Consider $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $a_j \geq 0$ for each j and $f(b)$ is prime. If the degree of $f(x)$ is $\leq D$, then $f(x)$ is irreducible. Additionally, if the degree of $f(x)$ is $\leq D_1$ and $f(x)$ is reducible, then $f(x)$ is divisible by $\Phi_4(x-b)$ and not divisible by $\Phi_3(x-b)$. Furthermore, if the degree of $f(x)$ is $\leq D_2$ and $f(x)$ is reducible, then $f(x)$ is divisible by either $\Phi_4(x-b)$ or $\Phi_3(x-b)$.*

Table 2.7 Values of $D(b)$, $D_1(b)$, $D_2(b)$, $\vartheta(B)$, and $m(b)$ for Bases $2 \leq b \leq 20$.

Base b	$D(b)$	$D_1(b)$	$D_2(b)$	$\vartheta(b)$	$m(b)$
$b=2$	6	–	7	$\pi/7$	13/27
$b=3$	9	–	10	$\pi/10$	12/37
$b=4$	12	–	14	$\pi/14$	5/22
$b=5$	15	16	18	$\pi/18$	70/397
$b=6$	19	20	21	$\pi/21$	3/20
$b=7$	22	23	25	$\pi/25$	1/8
$b=8$	25	27	29	$\pi/29$	5/46
$b=9$	28	30	32	$\pi/32$	6/61
$b=10$	31	34	36	$\pi/36$	2/23
$b=11$	34	38	40	$\pi/40$	7/89
$b=12$	37	41	43	$\pi/43$	3/41
$b=13$	40	45	47	$\pi/47$	1/15
$b=14$	44	49	50	$\pi/50$	1/16
$b=15$	47	52	54	$\pi/54$	5/86
$b=16$	50	56	58	$\pi/58$	2/37
$b=17$	53	59	61	$\pi/61$	2/39
$b=18$	56	63	65	$\pi/65$	43/889
$b=19$	59	67	68	$\pi/68$	3/65
$b=20$	62	70	72	$\pi/72$	1/23

Proof. The proof of this theorem is similar to the proof of [9, Corollary 2.4]. Fix a base $2 \leq b \leq 20$ and let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ with $a_j \geq 0$ for each j. Suppose further that $f(b)$ is prime.

For bases $b = 2$ and $b = 3$, let $F_b(z)$ be as defined in (2.3.1) and $P_b(x, y)$ be as defined in (2.3.3), using the numbers $e_1(b) = 2$, $e_3(b) = 7$, $e_4(b) = 2$, $e_6(b) = 3$, and $d(b) = 3$. For base $b = 4$, let $F_b(z)$ be as defined in (2.3.1) and $P_b(x, y)$ be as defined in (2.3.3), using $e_1(b) = 2$, $e_3(b) = 4$, $e_4(b) = 2$, $e_6(b) = 3$, and $d(b) = 3$. For bases $b \geq 5$, let $F_b(z)$ be as defined in (2.3.1) and $P_b(x, y)$ be as defined in (2.3.3), using the numbers $e_1(b) = 0$, $e_3(b) = 4$, $e_4(b) = 2$, $e_6(b) = 3$, and $d(b) = 3$. Let $\mathcal{R}_b$ be as defined in (2.3.2). Finally, take $D = D(b)$, $D_1 = D_1(b)$, $D_2 = D_2(b)$, $\vartheta = \vartheta(b)$ and $m = m(b)$ as given in Table 2.7. We note that m is a rational number. Furthermore, the proof of Lemma 2.4 did not explicitly consider these number sets $e_1(b)$, $e_3(b)$, $e_4(b)$, $e_6(b)$, and $d(b)$ for bases $2 \leq b \leq 4$. The proof of Lemma 2.4 is identical with these number sets, however.

We consider the line $y = \tan(\vartheta)\, x$ or equivalently the points $x + i \tan(\vartheta)\, x$ in the complex plane. A simple computation gives us that $\tan(\vartheta) > m$. So the line $y = mx$ lies strictly below the line $y = \tan(\vartheta)\, x$. Applying Lemma 2.4, we know that $\rho_b(b - a_0) = 0$ and that $\rho_b(x)$ is continuous. We use a Sturm sequence to verify that $P_b(x, mx)$ has no real roots. Since the coefficients of $P_b(x, mx)$ are rational, this computation is exact. Using Lemma 2.4 part 3, we can deduce that $\mathcal{R}_b$ does not intersect the line $y = mx$. Therefore then entire region $\mathcal{R}_b$ lies below the line $y = mx$.

Now, from Section 2.3, we assumed that $f(x)$ was reducible and could write $f(x) = g(x)h(x)$, where both $g(x)$ and $h(x)$ are in $\mathbb{Z}[x]$, $g(x) \not\equiv \pm 1$, $h(x) \not\equiv \pm 1$, and both $g(x)$ and $h(x)$ have positive leading coefficients. Furthermore, without loss of generality, we assumed that $g(b) = 1$ and showed that either $g(x)$ has a root in common with $\Phi_3(x - b)$, $\Phi_4(x - b)$, and $\Phi_6(x - b)$, or $g(x)$ has a root $\beta \in \mathcal{R}_b$. Since $f(x)$ has non-negative coefficients, and the real numbers in $\mathcal{R}_b$ are positive, we know that $\beta \notin \mathbb{R}$. We note that $b + \zeta_6$ also lies below the line $y = mx$. Therefore, we conclude that either $g(x)$ has a root in common with $\Phi_3(x - b)$ and $\Phi_4(x - b)$, or $g(x)$ has a root $\beta = \sigma + it$ such that $0 < t < \tan(\vartheta)\, \sigma$.

For bases $b \geq 3$, a computation gives $\pi/\arg(b+\zeta_3) > D$ and $\pi/\arg(b+\zeta_4) > D$. Thus, by Lemma 2.6, we have that $f(x)$ is irreducible. This is illustrated in Figure 2.8 for base $b = 6$, where the straight line passes through the origin and its slope is $\tan(\pi/19)$. Although difficult to tell from the graph, the point $6+\zeta_4$ lies strictly below this line.

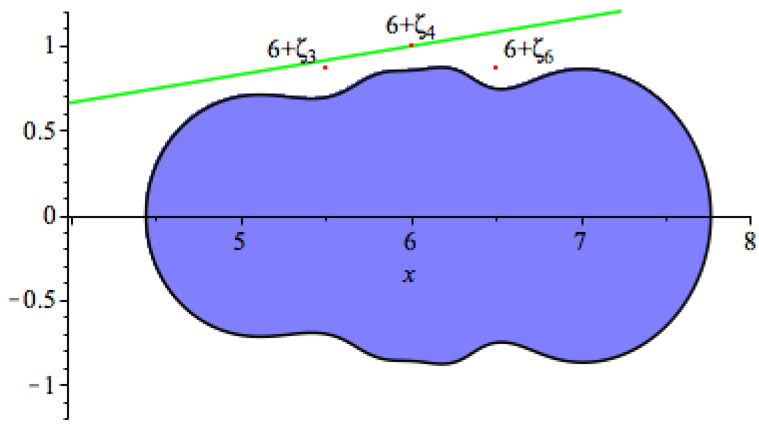

Figure 2.8 R_6 Illustrating $D(6)$

In the case of base $b = 2$, we have that $\pi/\arg(2+\zeta_4) > D$ but $\pi/\arg(2+\zeta_3) = D$. We quickly show that in this case, if $\deg f(x) = 6$ and $f(x)$ is divisible by $\Phi_3(x-2)$, then $f(2)$ is necessarily composite, contradicting our original assumption.

Since we want $\Phi_3(x-2) = x^2 - 3x + 3$ to be a factor of $f(x)$, and $\deg f(x) = 6$, we know that the other factor of $f(x)$ is $u_1x^4 + u_2x^3 + u_3x^2 + u_4x + u_5$, where $u_1, u_2, u_3, u_4, u_5 \in \mathbb{Z}$ and $u_1 \geq 1$. This gives us that

$$\begin{aligned} f(x) &= \left(x^2 - 3x + 3\right)\left(u_1x^4 + u_2x^3 + u_3x^2 + u_4x + u_5\right) \\ &= u_1x^6 + (u_2 - 3\,u_1)\,x^5 + (3\,u_1 - 3u_2 + u_3)\,x^4 + (3u_2 - 3\,u_3 + u_4)\,x^3 \\ &\quad + (3\,u_3 - 3\,u_4 + u_5)\,x^2 + (3\,u_4 - 3\,u_5)\,x + 3\,u_5. \end{aligned}$$

Observe that $2+\zeta_3$ is a root of $f(x)$ and each coefficient of $f(x)$ is non-negative. If one of the coefficients of x, x^2, x^3, x^4, or x^5 in $f(x)$ is > 0, then $\Im\left(f\left(2+\zeta_3\right)\right) > 0$, contradicting the fact that $2+\zeta_3$ is a root of $f(x)$. Thus, we have that $u_2 - 3u_1 = 0$,

$3u_1 - 3u_2 + u_3 = 0$, $3u_2 - 3u_3 + u_4 = 0$, $3u_3 - 3u_4 + u_5 = 0$, and $3u_4 - 3u_5 = 0$. Solving for u_2, u_3, u_4 and u_5, we have that $u_2 = 3u_1$, $u_3 = 6u_1$, $u_4 = 9u_1$, and $u_5 = 9u_1$. This gives us that $f(x) = u_1x^6 + 27u_1$. Clearly if $u_1 > 1$, we have that $f(2)$ is composite. In the case that $u_1 = 1$, we have that $f(x) = x^6 + 27$. In this case $f(2) = 91 = 7 \times 13$, so again $f(2)$ is composite. This gives us our desired result for base $b = 2$.

This proves the first part of our theorem. We now consider the statements concerning D_1 and D_2

For bases $b \geq 5$, we have that $\pi/\arg(b + \zeta_3) > D_1$, $\pi/\arg(b + \zeta_4) < D_1$, and $D_1 > D$. Thus, by Lemma 2.6, we have that if $f(x)$ is reducible and $\deg f(x) \leq D_1$, then $f(x)$ is divisible by $\Phi_4(x - b)$. This is illustrated in Figure 2.9 for base $b = 6$, where the straight line passes through the origin and its slope is $\tan(\pi/20)$. Although difficult to tell from the graph, the point $6 + \zeta_3$ lies strictly below this line.

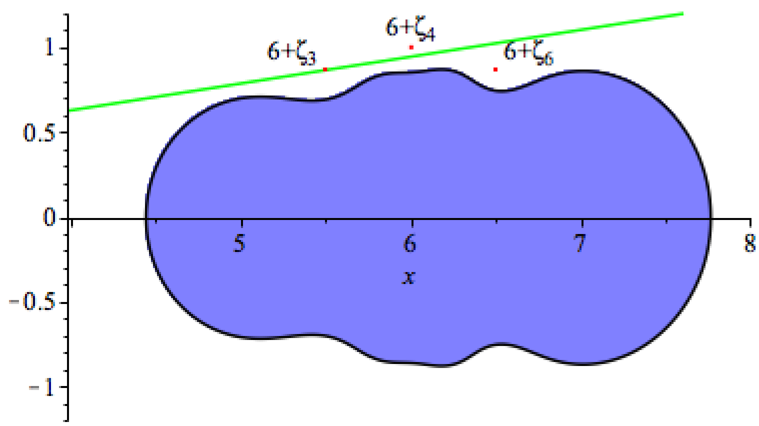

Figure 2.9 R_6 Illustrating $D_1(6)$

Similarly, for bases $2 \leq b \leq 20$, we have $\pi/\arg(b + \zeta_3) < D_2$, $\pi/\arg(b + \zeta_4) < D_2$, and $D_2 > D$. Thus, by Lemma 2.6, we have that if $f(x)$ is reducible and the degree of $f(x)$ is $\leq D_2$, then $f(x)$ is divisible by $\Phi_3(x - b)$ or $\Phi_4(x - b)$. This is illustrated in Figure 2.10 for base $b = 6$, where the straight line passes through the origin and its slope is $\tan(\pi/21)$.

For bases $b = 2$, $b = 3$, and $b = 4$, we note that there is no value for D_1 since

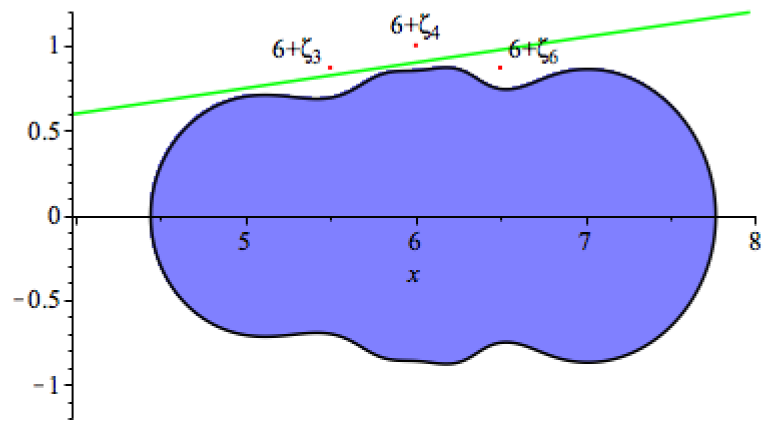

Figure 2.10 R_6 Illustrating $D_2(6)$

$\lfloor \pi/\arg(b+\zeta_4) \rfloor = \lfloor \pi/\arg(b+\zeta_3) \rfloor$.

This completes the proof. □

Table 2.8, Table 2.9, Table 2.10, and Table 2.11 give explicit examples that show that the bounds $D(b)$ and $D_1(b)$ are sharp. For example, take base $b = 6$, where we see that $D(6)$ has increased from 18 in Lemma 2.3 to 19 in Theorem 2.7. The polynomial

$$f(x) = x^{20} + 2x^3 + 13519269991320x^2 + 610418402115746x + 610418402115527$$

is of degree 20, $f(6) = 8415780974560931$ is prime, each coefficient of $f(x)$ is at most 610418402115746, and $f(x)$ is divisible by $\Phi_3(x-6) = x^2 - 12x + 37$. Although not our ultimate goal in this chapter, we will prove later in Section 2.8 that this polynomial is optimal in another sense, as stated in the following theorem.

Theorem 2.8. *Fix an integer b such that $b \geq 2$, let $D = D(b)$, $D_1 = D_1(b)$ and $D_2 = D_2(b)$ be as given in Table 2.3, let $N_1 = N_1(b)$ be as given in Table 2.12, and let $N_2 = N_2(b)$ be as given in Table 2.13. Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ be such that $a_j \geq 0$ for each j and $f(b)$ is prime. If $\deg f(x) = D+1$ and each $a_j \leq N_1$, then $f(x)$ is irreducible. In the case that $5 \leq b \leq 20$, if $\deg f(x) = D_1 + 1$ and each $a_j \leq N_2$, then $f(x)$ is not divisible by $\Phi_3(x-b)$.*

Explicit examples illustrating Theorem 2.7 and Theorem 2.8 are given in Table 2.8, Table 2.9, Table 2.10, and Table 2.11. Please note that the examples for base $b = 10$ are necessarily the same as the example given in [9].

The proof of Theorem 2.8 will be given in Section 2.8.

Table 2.8 Examples Illustrating that $D(b)$ is Sharp for Bases $2 \leq b \leq 14$

b	$f(x)$
2	$x^7 + x^3 + 4\,x^2 + 9\,x + 21$
3	$x^{10} + x^3 + 3\,x^2 + 3692\,x + 7238$
4	$x^{13} + x^3 + 235835\,x + 16576651$
5	$x^{16} + 14\,x^2 + 3494829940\,x + 191323668588$
6	$x^{20} + 2x^3 + 13519269991320x^2 + 610418402115746x + 610418402115527$
7	$x^{23} + 8\,x^2 + 4203172930619131792\,x + 48476922112812036000$
8	$x^{26} + 5\,x^3 + 33832763767045412768629\,x + 975072233254529906548650$
9	$x^{29} + 4\,x^3 + 20\,x^2 + 379368440324759571320027856\,x$ $+2200192048605247301544844664$
10	$x^{32} + 4\,x^3 + 10\,x^2 + 560328675401014156716157263892 4\,x$ $+6109104104761309555986010605548 9$
11	$x^{35} + 28\,x^2 + 1022309986895297769091585369570011 12\,x$ $+2119463830567700564381021297555803480$
12	$x^{38} + 4\,x^3 + 6\,x^2 + 2068989162106320012180039042071801322988\,x$ $+9156421224413095255016580698872377281093 5$
13	$x^{41} + 2\,x^2 + 32085320557683790183457220835906490863714252\,x$ $+48819031282379755942821318567777163455705910 60$
14	$x^{45} + 9x^3 + 190427015436250536820510121014683293286454259749x^2$ $+278336811480425292328491552981955444943583501062424x$ $+278336811480425292328491552981955444943583501059837$

Table 2.9 Examples Illustrating that $D(b)$ is Sharp for Bases $15 \le b \le 20$

b	$f(x)$
15	$x^{48} + 9\,x^3 + 9\,x^2$ $+16912197798051084357861219245523514058313814877373720384\,x$ $+61074859962290535565373333952146687505375635458305881678$
16	$x^{51} + 4\,x^3 + 11\,x^2$ $+1185763498642604673111010510892742232929342931151849710352293\,x$ $+9401468271903366135972500856333110049503488294231938850008747$
17	$x^{54} + 12\,x^2$ $+95308787665469452744336500106288544910208974417523586171630780008\,x$ $+1431397180112955678634451120632703115867308362290036476121595252120$
18	$x^{57} + 2\,x^3 + 41\,x^2$ $+8463281062202111533878239172104686314687322333773655853967026651221621\,x$ $+235429303540695115385709981455936954415388002209380091524801717697233925$
19	$x^{60} + 12\,x^2$ $+772784749771101496631976160205347792570129801187057002485017083402361241144\,x$ $+43022718318161585107154947899035503608645093219967711021015380107341305221368$
20	$x^{63} + 5\,x^3 + 33\,x^2$ $+57735712055255563324819612927610828614857142493350013713356499456113585161611486\,x$ $+8823216088819058575067389247090576700176541906366627393606717738052119209880430673$

Table 2.10 Examples Illustrating $D_1(b)$ is Sharp for Bases $5 \leq b \leq 13$

b	$f(x)$
5	$x^{17} + 2\,x^3 + 10\,x^2 + 18096425331\,x + 91182226359$
6	$x^{21} + 4662361342700\,x^2 + 674230217165570\,x + 674230217165581$
7	$x^{24} + 6\,x^3 + 17\,x^2 + 1721492003084292463\,x$ $+2874211188654189792 4$
8	$x^{28} + x^3 + 14\,x^2 + 340829997121795439379717\,x$ $+1253983385808624632627229$
9	$x^{31} + 6\,x^3 + 13\,x^2 + 5266425187874612926923438 90\,x$ $+7164314540293359134627129 9995$
10	$x^{35} + 3\,x^3 + 3\,x^2 + 891572422312872968547877442943784\,x$ $+1071112974878289533198669427384 4451$
11	$x^{39} + 21\,x^2 + 598715106212835649790162712234228135069\,x$ $+1348312606061131031866295541636879998622$
12	$x^{42} + x^3 + 25\,x^2 + 705845596192484534231861889953672659300 4509\,x$ $+3176996790603319890009722329188177828150822 47$
13	$x^{46} + x^3 + 37\,x^2 + 14904215247993281188655041013866062095454855150432\,x$ $+13383684297286329426437833914427282894008330748 2002$

Table 2.11 Examples Illustrating $D_1(b)$ is Sharp for Bases $14 \le b \le 20$

b	$f(x)$
14	$x^{50} + 2\,x^3$ $+5423718181968966282264555835956879354006170863939 6236\,x^2$ $+24387207020849741198805521258225261909442987625989599229\,x$ $+24387207020849741198805521258225261909442987625989599493$
15	$x^{53} + 6\,x^3 + 18\,x^2$ $+911236263215448210017499389419205223064882811352158858229529\,x$ $+2599709957888578907166650788038895111723636569086137477917 6373$
16	$x^{57} + x^3 + 37\,x^2$ $+35492715345001171443399569805244757735921654842855782400832168 10053\,x$ $+2210166939653449230976983739225710952503007228453341911539423753 2819$
17	$x^{60} + 3\,x^3 + x^2$ $+660353921591460289424682734939820739509020666370172961577846964258 00719\,x$ $+11068765075055445663455770678250929757451117392105995069831359511491 726051$
18	$x^{64} + 5\,x^3 + 36\,x^2$ $+102959926115844106587935265979880581375772139016096795484660744149412514 7435685\,x$ $+207357056341397645350880610882225484326499834543425565728110344739656497 91911451$
19	$x^{68} + 4\,x^3 + 28\,x^2$ $+71928559586201009731978729543967457031294429581274130333461353975447519480048 42591571\,x$ $+25299051628958894639347305083391076959171276290019053473973793001833084973083 685273179$
20	$x^{71} + x^3 + 33\,x^2$ $+4630607494244746811585316950445042966234816184172240231349141799857228053284532785 31636154\,x$ $+3292851079308193395910000675188650040251317406064440505883009797768861309358485135 8050701813$

Table 2.12 $N_1(b)$ for Bases $2 \le b \le 20$

b	$N_1(b)$
2	20
3	7237
4	16576650
5	191323668587
6	610418402115745
7	4847692211281203599
8	97507223325452990654864
9	2200192048605247301544844663
10	61091041047613095559860106055488
11	2119463830567700564381021297555803479
12	91564212244130952550165806988723772810934
13	4881903128237975594282131856777716345570591059
14	278336811480425292328491552981955444943583501059836
15	61074859962290535565373333952146687505375635458305881677
16	9401468271903366135972500856333110049503488294231938850008746
17	1431397180112955678634451120632703115867308362290036476121595252119
18	235429303540695115385709981455936954415388002209380091524801717697233924
19	43022718318161585107154947899035503608645093219967711021015380107341305221367
20	8823216088819058575067389247090576700176541906366627393606717738052119209880430672

Table 2.13 $N_2(b)$ for Bases $2 \leq b \leq 20$

b	$N_2(b)$
5	91182226358
6	674230217165580
7	2874211886541897923
8	12539833858086246326272228
9	716431454029335913462712999994
10	10711129748782895331986694273844450
11	134831260606113103186629554163687999 8621
12	317699679060331989000972232918817782815082246
13	1338368429728632942643783391442728289400833074820 01
14	24387207020849741198805521258225261909442987625989599492
15	2599709957888578907166650788038895111723636569086137477917 6372
16	22101669396534492309769837392257109525030072284533419115394237532818
17	110687650750554456634557706782509297574511173921059950698313595114917260 50
18	207357056341397645350880610882225484326499834543425565728110344739656497919 11450
19	25299051628958894639347305083391076959171276290019053473973793001833084973083685273178
20	32928510793081933959100006751886500402513174060644405058830097977688613093584851358050701812

2.5 A First Bound on the Coefficients

We pause briefly to summarize the previous sections and set the goal for this section.

We have fixed an integer b with $2 \le b \le 20$, and considered a polynomial $f(x)$ such that each coefficient of $f(x)$ is non-negative and $f(b)$ is prime. If $f(x)$ is reducible, with $f(x) = g(x)h(x)$, we have used the fact that $f(b)$ is prime to take $g(b) = \pm 1$. We then showed that either $g(x)$, and thus $f(x)$, has a factor of at least one of $\Phi_3(b-x)$, $\Phi_4(b-x)$, and $\Phi_6(b-x)$, or $g(x)$ has a root $\beta \in \mathcal{R}_b$. Throughout this section, $\mathcal{R}_b$ is as defined in (2.3.2), with $F_b(z)$ given by (2.3.1) and $P_b(x,y)$ given by (2.3.3). The numbers $e_1(b)$, $e_3(b)$, $e_4(b)$, $e_6(b)$, and $d(b)$ are as given in Table 2.4.

Now we consider the latter case, that $g(x)$, and thus $f(x)$, has a root $\beta \in \mathcal{R}_b$ to obtain a lower bound on the coefficients of $f(x)$. The method outlined here is based on [9] and [10]. We will rely heavily on the following lemma.

Lemma 2.9. *Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$, where $a_j \ge 0$ for $j \in \{0, 1, \ldots, n\}$. Suppose $\alpha = re^{i\theta}$ is a root of $f(x)$ with $0 < \theta < \pi/2$ and $r > 1$. Let*

$$B = \max_{\pi/(2\theta) < k < \pi/\theta} \left\{ \frac{r^k(r-1)}{1 + \cot(\pi - k\theta)} \right\},$$

where the maximum is over $k \in \mathbb{Z}$. Then there is some $j \in \{0, 1, \ldots, n-1\}$ such that $a_j > Ba_n$.

The proof of this lemma is similar to the proof of [**Theorem 5**, 8], and is reproduced from [9] in Appendix A.4.

Our use of Lemma 2.9 is to prove the following Corollary:

Corollary 2.10. *Fix an integer b with $b \ge 2$. Let $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ be such that $a_j \ge 0$ for each j and $f(b)$ is prime. If*

$$0 \le a_j \le B_b a_n \quad \text{for} \quad 0 \le j \le n-1 \quad \text{and} \quad B_b \text{ as in Table 2.14},$$

then either $f(x)$ is irreducible or $f(x)$ is divisible by at least one of $\Phi_3(x-b)$, $\Phi_4(x-b)$ and $\Phi_6(x-b)$.

Table 2.14 Values of B_b

Base b	2	3	4	5
B_b	7	4712	5.8802×10^{7}	4.149×10^{11}
Base b	6	7	8	9
B_b	6.616×10^{14}	8.762×10^{19}	1.401×10^{25}	1.412×10^{30}
Base b	10	11	12	13
B_b	2.749×10^{35}	5.203×10^{40}	1.159×10^{46}	6.969×10^{51}
Base b	14	15	16	17
B_b	2.689×10^{57}	1.598×10^{63}	1.869×10^{69}	1.269×10^{75}
Base b	18	19	20	
B_b	2.075×10^{81}	1.245×10^{87}	3.942×10^{93}	

For a fixed integer base $2 \leq b \leq 20$, let θ and θ' be real numbers such that $0 \leq \theta < \theta' \leq R_b$, where R_b is given in Table 2.15. We are interested in the set of points $\mathcal{R}_b(\theta, \theta')$ that are in $\mathcal{R}_b$ between the line passing through the origin making an angle θ with the positive x-axis and the line passing through the origin making an angle θ' with the positive x-axis. Explicitly, we define this as

$$\mathcal{R}_b(\theta, \theta') = \{(x, y) \in \mathcal{R}_b : \tan\theta \leq y/x < \tan\theta'\}.$$

We are still considering the case that $g(x)$ has a root $\beta \in \mathcal{R}_b$. We write $\beta = x_0 + iy_0$ for some $(x_0, y_0) \in \mathcal{R}_b$, where we may take $y_0 > 0$.

Along the lines of the proof of Theorem 2.7, we use a Sturm sequence to show that the line $y = R_b x$ does not intersect the region $\mathcal{R}_b$. In other words, the region $\mathcal{R}_b$ lies completely under the line $y = R_b x$.

To utilize Lemma 2.9, we specify a set $\Theta_b = \{\theta_0, \theta_1, \cdots, \theta_{m-1}, \theta_m\}$ where

$$0 = \theta_0 < \theta_1 < \ldots < \theta_{m-1} < \theta_m = R_b,$$

and $\theta_1 = 1/1000$. Thus, we have that

$$(x_0, y_0) \in \bigcup_{l=0}^{m-1} \mathcal{R}_b(\theta_l, \theta_{l+1}).$$

Table 2.15 Values of R_b

Base b	2	3	4	5	6
R_b	5/3	13/27	4/15	9/50	3/20
Base b	7	8	9	10	11
R_b	1/8	11/100	12/125	2/23	7/89
Base b	12	13	14	15	16
R_b	3/41	1/15	1/16	5/86	2/37
Base b	17	18	19	20	
R_b	2/39	43/889	3/65	1/23	

Next, for each $l \in \{0, 1, \ldots, m-1\}$, we use Lemma 2.9 to find a bound $B'_b(\theta_l, \theta_{l+1})$ so that for all $(x_0, y_0) \in \mathcal{R}_b(\theta_l, \theta_{l+1})$, there is a $j \in \{0, 1, \cdots, n-1\}$ for which $a_j > B'_b(\theta_l, \theta_{l+1})\, a_n$. We can then deduce that some coefficient of $f(x)$ must exceed

$$\min_{0 \le l \le m-1} \{B'_b(\theta_l, \theta_{l+1})\} \cdot a_n. \tag{2.5.1}$$

We very carefully choose each $\theta_l \in \Theta_b$ so that each $B'_b(\theta_l, \theta_{l+1}) > B_b$, where B_b is listed in Table 2.14. Corollary 2.10 then follows immediately.

We begin by considering the first sector $\mathcal{R}_b(\theta_0, \theta_1)$, where we have already stated that $\theta_0 = 0$ and $\theta_1 = 1/1000$, regardless of the base b.

Take $k = k(\theta) = \left\lfloor \dfrac{25\pi}{26\theta} \right\rfloor$ where $0 \le \theta \le \arctan\left(\dfrac{1}{1000}\right)$. We note that

$$k \in \left(\frac{\pi}{2\theta}, \frac{\pi}{\theta}\right)$$

since

$$k\theta \le \frac{25\pi}{26} < \pi$$

and

$$k\theta > \left(\frac{25\pi}{26\theta} - 1\right)\theta = \frac{25\pi}{26} - \theta \ge \frac{25\pi}{26} - \arctan\left(\frac{1}{1000}\right) > \frac{\pi}{2}.$$

A fact that we will use later is that $\dfrac{\pi}{2} \ge \pi - k\theta \ge \pi - \dfrac{25\pi}{26} = \dfrac{\pi}{26}$, which gives us that $\cot(\pi - k\theta) \le \cot\left(\dfrac{\pi}{26}\right)$.

From our definition of k and range of θ above, we have that

$$k = \left\lfloor \frac{25\pi}{26\theta} \right\rfloor \geq \left\lfloor \frac{\frac{25\pi}{26}}{\arctan\left(\frac{1}{1000}\right)} \right\rfloor = 3020.$$

We recall that for each $z \in \mathcal{R}_b$, regardless of the base b we are using, we have that the $\Re(z) \geq 1.447$, as seen in Table 2.6. Thus, for each $z = re^{i\theta} \in \mathcal{R}_b$, we have that $r = |z| \geq 1.447$. For each such z, we have that

$$\frac{r^k(r-1)}{1+\cot(\pi - k\theta)} \geq \frac{1.447^{3020}(1.447-1)}{1+\cot\left(\pi/26\right)} > 1.99 \times 10^{483}.$$

From Lemma 2.9, with $\theta_0 = 0$ and $\theta_1 = \arctan\left(\frac{1}{1000}\right)$, we see that we may take

$$B_b'\left(\theta_0, \theta_1\right) = B_b'\left(0, \arctan\left(\frac{1}{1000}\right)\right) = 1.99 \times 10^{483}. \tag{2.5.2}$$

Our choice of $\theta_1 = 1/1000$ was chosen so that $B_b'\left(\theta_0, \theta_1\right) = 1.99 \times 10^{483}$ would be larger than any B_b for any base $2 \leq b \leq 20$.

This leaves us much freedom in choosing the values of θ_l for each base. As we have just seen, we want to know where the line $y = (\tan\theta_l)\,x$ intersects $\mathcal{R}_b$. Since the boundary of $\mathcal{R}_b$ consists of the points (x, y) such that $P_b(x, y) = 0$, we need the real numbers x such that $P\left(x, (\tan\theta_l)\,x\right) = 0$. We must, however, avoid basing our approximations to the real roots of a polynomial whose coefficients are themselves just approximations. To this end, we take $r_l = \tan\theta_l$, where r_l is a rational number. We then find a close rational approximation x_l to the minimum real root of $P_b\left(x, r_l x\right) = 0$. Then, using a Sturm sequence, we can verify, with exact arithmetic, that $P_b\left(x, r_l x\right)$ has no roots in the interval $[0, x_l]$.

The values of $r_l = \tan\theta_l$ for each base $b \geq 3$ are given in Appendix A.5. The values for base $b = 2$ are shown later in Table 2.16.

For a fixed $l \in \{1, 2, \ldots, m-1\}$, we will now show how to obtain a value for $B_b'\left(\theta_l, \theta_{l+1}\right)$. We have already shown how to find a verifiable lower bound x_l for the left-most point (x, y) on the intersection of the line $y = \tan\left(\theta_l\right)x$ and $\mathcal{R}_b$. This was done using a Sturm sequence with rational numbers, so these calculations are exact.

Now take

$$\alpha = x_0 + iy_0 = re^{i\theta} \text{ where } (x_0, y_0) \in \mathcal{R}_b(\theta_l, \theta_{l+1}). \tag{2.5.3}$$

We will show that both $x_0 \geq x_l$ and $y_0 \geq \tan(\theta_l) x_l$. We begin with the former.

By way of contradiction, suppose that $x_0 < x_l$. Let (x_1, y_1) be the point where $y = \tan(\theta) x$ intersects $\mathcal{R}_b$ with x_1 being minimal. Therefore (x_1, y_1) lies on the boundary of $\mathcal{R}_b$, and, by Lemma 2.4, we have that $y_1 = \rho_b(x_1)$.

Also, $x_1 \leq x_0 < x_l$ and, by Lemma 2.4 part (1), $x_1 \geq b - a_0$ where a_0 is given in Table 2.6. By Lemma 2.4 part (4), the function $\rho_0(x) = \rho_b(x) - r_l x$ is a continuous function such that $\rho_0(b - a_0) < 0$.

However, since $(x_1, y_1) \in R_b(\theta_l, \theta_{l+1})$, it lies above the line $y = \tan(\theta_l) x$. This gives us that

$$\rho_b(x_1) = y_1 = \tan(\theta) x_1 \geq \tan(\theta_l) x_1 = r_l x_1,$$

so $\rho_0(x_1) \geq 0$. By the Intermediate Value Theorem, there exists a $u \in [b - a_0, x_1]$ such that $\rho_0(u) = 0$. Thus $\rho_b(u) = r_l u$, which gives us that $P_b(u, r_l u) = 0$. Putting this all together, we see that

$$u \leq x_1 \leq x_0 < x_l,$$

but this contradicts the definition of x_l. Therefore $x_0 \geq x_l$.

To show that $y_0 \geq \tan(\theta_l) x_0$, we notice that

$$y_0 = \tan(\theta) x_0 \geq \tan(\theta_l) x_0 \geq \tan(\theta_l) x_l.$$

To get a value for $B_b'(\theta_l, \theta_{l+1})$, we will let R_l be a lower bound approximation of $\sec(\theta_l) x_l$ so that, for any $\alpha = re^{i\theta}$ as in (2.5.3), we have that

$$r = \sqrt{x_0^2 + y_0^2} \geq \sqrt{1 + \tan^2(\theta_l)} x_l \geq R_l.$$

Now, for any $l \in \{1, 2, \ldots, m-1\}$, we let k_1 be the largest integer $\leq \dfrac{\pi}{\theta_{l+1}}$ and let $k_2 = k_1 - 1$. Using 100 digit approximations in Maple 17, we check that both the inequalities

$$\frac{\pi}{2\theta_l} + 10^{-10} \le k_2 \quad \text{and} \quad k_1 \le \frac{\pi}{\theta_{l+1}} - 10^{-10}$$

hold. If the first inequality does not hold, we verify that

$$\frac{\pi}{2\theta_l} + 10^{-10} \le k_1$$

and set $k_2 = k_1$. Thus, for any $\theta \in [\theta_l, \theta_{l+1}]$, we have that

$$\frac{\pi}{2\theta} \le \frac{\pi}{2\theta_l} < k_2 \le k_1 < \frac{\pi}{\theta_{l+1}} \le \frac{\pi}{\theta}.$$

So $k_1, k_2 \in \left(\frac{\pi}{2\theta}, \frac{\pi}{\theta}\right)$.

Now for each such θ, we compute, again using 100 digit approximations in Maple 17, $c(k_1)$ and $c(k_2)$ such that

$$\cot(\pi - k_j\theta) \le \cot(\pi - k_j\theta_{l+1}) \le c(k_j) - 10^{-10} \text{ for } j \in \{1,2\}.$$

Lemma 2.9 now allows us to take

$$B_b'(\theta_l, \theta_{l+1}) = \max\left\{\frac{R_l^{k_1}(R_l - 1)}{1 + c(k_1)}, \frac{R_l^{k_2}(R_l - 1)}{1 + c(k_2)}\right\}.$$

Each one of these bounds, combined with (2.5.2) and (2.5.1), gives us the lower bound of $B_b a_n$ for a least one of the coefficients of $f(x)$, where B_b is listed in Table 2.14.

For example, we consider base $b = 2$. Using the numbers $e_1(2)$, $e_3(2)$, $e_4(2)$, $e_6(2)$, and $d(2)$ from Table 2.4, we have a region $\mathcal{R}_2$, which we divide into 49 sectors. Table 2.16 shows the sectors and the associated $B_2'(\theta_l, \theta_{l+1})$, accurate to the digits shown. The Maple 17 code that was used for these calculations is given in Appendix A.6 as the function GetBound.

From this, we have that $\min\limits_{0 \le l < m} \{B_2'(\theta_l, \theta_{l+1})\} \ge B_2 = 7$, as desired.

We make a few comments about our choices for r_l. As noted earlier, we do have much freedom. We did, however, explicitly avoid having an $r_l = 1$. More specifically, we avoided the choice of $\theta = \frac{\pi}{4}$ and $k_1 = 4$, where $\cot(\pi - k_1\theta)$ is undefined. This explains the choice of r_{47} and r_{48}.

Table 2.16 Values of $B_2'(\theta_l, \theta_{l+1})$

l	$r_l = \tan(\theta_l)$	$B_2'(\theta_l, \theta_{l+1})$	l	$r_l = \tan(\theta_l)$	$B_2'(\theta_l, \theta_{l+1})$
0	$0 = 0$	1.99×10^{483}	25	$\frac{3}{10} = 0.3$	8.48120
1	$\frac{1}{1000} = 0.001$	1.67316×10^{333}	26	$\frac{31}{100} = 0.31$	7.68540
2	$\frac{3}{2000} = 0.0015$	1.88152×10^{249}	27	$\frac{8}{25} = 0.32$	7.86165
3	$\frac{1}{500} = 0.002$	1.78851×10^{165}	28	$\frac{33}{100} = 0.33$	7.61940
4	$\frac{3}{1000} = 0.003$	2.25395×10^{123}	29	$\frac{17}{50} = 0.34$	7.31188
5	$\frac{1}{250} = 0.004$	1.59285×10^{98}	30	$\frac{7}{20} = 0.35$	7.41486
6	$\frac{1}{200} = 0.005$	3.13071×10^{81}	31	$\frac{71}{200} = 0.355$	7.20197
7	$\frac{3}{500} = 0.006$	3.66576×10^{69}	32	$\frac{9}{25} = 0.36$	7.22629
8	$\frac{7}{1000} = 0.007$	3.99316×10^{60}	33	$\frac{37}{100} = 0.37$	7.28552
9	$\frac{1}{125} = 0.008$	3.51475×10^{53}	34	$\frac{19}{50} = 0.38$	7.34184
10	$\frac{9}{1000} = 0.009$	1.01194×10^{48}	35	$\frac{39}{100} = 0.39$	7.38453
11	$\frac{1}{100} = 0.01$	2.52294×10^{31}	36	$\frac{2}{5} = 0.4$	7.39514
12	$\frac{3}{200} = 0.015$	1.13455×10^{23}	37	$\frac{41}{100} = 0.41$	7.74498
13	$\frac{1}{50} = 0.02$	8.071030×10^{14}	38	$\frac{21}{50} = 0.42$	7.72610
14	$\frac{3}{100} = 0.03$	6.506270×10^{10}	39	$\frac{11}{25} = 0.44$	7.95266
15	$\frac{1}{25} = 0.04$	2.576910×10^{8}	40	$\frac{47}{100} = 0.47$	8.65642
16	$\frac{1}{20} = 0.05$	5.92576×10^{6}	41	$\frac{1}{2} = 0.5$	8.64546
17	$\frac{3}{50} = 0.06$	479437	42	$\frac{11}{20} = 0.55$	8.47305
18	$\frac{7}{100} = 0.07$	62346.5	43	$\frac{3}{5} = 0.6$	7.34988
19	$\frac{2}{25} = 0.08$	12234.5	44	$\frac{7}{10} = 0.7$	8.44235
20	$\frac{9}{100} = 0.09$	4547.64	45	$\frac{3}{4} = 0.75$	8.10185
21	$\frac{1}{10} = 0.1$	118.104	46	$\frac{4}{5} = 0.8$	7.69225
22	$\frac{3}{20} = 0.15$	28.2727	47	$\frac{9}{10} = 0.9$	7.46715
23	$\frac{1}{5} = 0.2$	11.9817	48	$\frac{11}{10} = 1.1$	7.72974
24	$\frac{1}{4} = 0.25$	7.41419	49	$\frac{17}{10} = 1.7$	–

Additionally, our choices of r_l might not be optimal in the sense that it might be possible to choose a fewer number of sectors to analyze. However, these choices for r_l are a reasonable number that do give the desired result.

Furthermore, these choices for r_l appear to give us the best value of B_2 with our choice of numbers $e_1(2)$, $e_3(2)$, $e_4(2)$, $e_6(2)$, and $d(2)$. By carefully analyzing the choice of $r_l = .36$ (we choose to refine this sector into 4 sub-sectors from .35999999999999999999 to .36), we see that the best we can achieve is 7.51877930829. Thus, we cannot do better than $B_2 = 7$.

2.6 Bounds Based on Recursive Relations

We will now examine another method to bound the coefficients of $f(x)$ that is motivated by Corollary 2.10. In the case that $f(x)$ is divisible by one of the quadratics $\Phi_3(b-x)$, $\Phi_4(b-x)$, or $\Phi_6(b-x)$, we wish to find sharp lower bounds for the maximum coefficient of $f(x)$. The bound that we find will depend on our choice of base b and quadratic.

The arguments in the section are in essence identical to those introduced in [9] and [10] for base $b = 10$. We reproduce their arguments here, with the obvious minor changes to accommodate the various bases $b \geq 2$.

Fix positive integers A and B. Let b_j be integers such that

$$\Big(b_0 x^s + b_1 x^{s-1} + \cdots + b_{s-1} x + b_s\Big)\Big(x^2 - Ax + B\Big) \tag{2.6.1}$$

is a polynomial of degree $s+2$ with non-negative coefficients. In the setting of this chapter, we will want A and B chosen so that the quadratic on the right is one of $\Phi_3(x-b)$, $\Phi_4(x-b)$, and $\Phi_6(x-b)$. With $f(x) = g(x)h(x)$ as before and $g(x)$ being the quadratic, we view the polynomial factor on the left in (2.6.1) as $h(x)$ and further $n = \deg f(x) = s+2$. The choice of b_j as the coefficient of x^{s-j} will help us view the b_j as forming a sequence and be more appropriate for the arguments that follow. If (2.6.1) is expanded, we obtain $f(x)$ so that the resulting coefficients are all non-negative.

For convenience, we define $b_j = 0$ for all $j < 0$ and all $j > s$. Since the coefficients of $f(x)$ are all non-negative, we deduce that

$$b_0 \geq 1 \quad \text{and} \quad b_j \geq Ab_{j-1} - Bb_{j-2} \ \text{ for all } j \in \mathbb{Z}. \tag{2.6.2}$$

Define

$$\beta_j = \begin{cases} 0 & \text{if } j < 0 \\ 1 & \text{if } j = 0 \\ A\beta_{j-1} - B\beta_{j-2} & \text{if } j \geq 1, \end{cases}$$

so the β_j satisfy a recursive relation for $j \geq 0$. In particular, $\beta_1 = A$ and $\beta_2 = A^2 - B$. Of some significance to our problem, the values of β_j may vary in sign and will do so for the choices of A and B that interest us. Let J be a positive integer for which

$$\beta_j \geq 0 \quad \text{for } j \leq J. \tag{2.6.3}$$

Then for $1 \leq j \leq J+1$, we have

$$\begin{aligned} b_j &\geq Ab_{j-1} - Bb_{j-2} \geq A\left(Ab_{j-2} - Bb_{j-3}\right) - Bb_{j-2} \\ &\geq \beta_2 b_{j-2} - B\beta_1 b_{j-3} \geq \beta_2\left(Ab_{j-3} - Bb_{j-4}\right) - B\beta_1 b_{j-3} \\ &\geq \beta_3 b_{j-3} - B\beta_2 b_{j-4} \geq \beta_3\left(Ab_{j-4} - Bb_{j-5}\right) - B\beta_2 b_{j-4} \\ &\geq \beta_4 b_{j-4} - B\beta_3 b_{j-5} \geq \cdots \geq \beta_{j-1} b_1 - B\beta_{j-2} b_0 \geq \beta_j b_0. \end{aligned} \tag{2.6.4}$$

We deduce the inequality

$$b_j \geq \beta_j b_0 \quad \text{for all integers } j \leq J+1. \tag{2.6.5}$$

Let

$$U = \max_{j \geq 0} \{b_j\} \quad \text{and} \quad L = \min_{j \geq 0} \{b_j\}.$$

Since $b_j = 0$ for $j > s$, we have the trivial bound $L \leq 0$. We can obtain rather precise information about U and L for specific A and B by making use of (2.6.5), and we do so now. Table 2.17 and Table 2.18 show the A, B, J, and $\beta_{\max}$ for various bases b that are of interest to us, where

$$\beta_{\max} = \max_{0 \le j \le J} \{\beta_j\}.$$

The value of J we used in (2.6.5) corresponds to the least positive J for which $\beta_{J+1} \le 0$. Thus, (2.6.3) holds. From (2.6.5), we obtain $\beta_{\max} b_0$ as a lower bound for U.

Recall that we are interested in considering A and B such that $f(x)$ is divisible by $x^2 - Ax + B$. We consider an $f(x)$ with non-negative coefficients as before but with the largest coefficient as small as possible. Let $M = M(A, B)$ denote the maximum coefficient for such an $f(x)$. In the way of an important example, we consider the polynomial

$$\begin{aligned} f(x) = x^{24} &+ 9130158x^{12} + 48391200x^{11} + 48391200x^{10} \\ &+ 48391200x^9 + 48391200x^8 + 48391200x^7 + 48391200x^6 \\ &+ 48391200x^5 + 48391201x^4 + 48391200x^3 + 48391191x^2 \\ &+ 48391032x + 39261687, \end{aligned}$$

for which $f(4) = 705276789296711$ is prime, but

$$\begin{aligned} f(x) = \left(x^2 - 8\,x + 17\right) &\left(x^{22} + 8\,x^{21} + 47\,x^{20} + 240\,x^{19} + 1121\,x^{18} + 4888\,x^{17}\right. \\ &+ 20047\,x^{16} + 77280\,x^{15} + 277441\,x^{14} + 905768\,x^{13} + 2529647\,x^{12} + 4839120\,x^{11} \\ &+ 4839119\,x^{10} + 4839112\,x^9 + 4839073\,x^8 + 4838880\,x^7 + 4837999\,x^6 \\ &\left.+4834232\,x^5 + 4819073\,x^4 + 4761840\,x^3 + 4561680\,x^2 + 3933360\,x + 2309511\right). \end{aligned}$$

Thus $f(4)$ is prime, but $f(x)$ is divisible by $\Phi_4(x - b) = x^2 - 8\,x + 17$. We may conclude that

$$M(8, 17) \le 48391201. \tag{2.6.6}$$

Table 2.17 Values of $\beta_{\max}$ for bases $2 \leq b \leq 12$

b	A	B	J	$\beta_{\max}$
2	3	3	4	9
2	4	5	5	44
2	5	7	8	1265
3	5	7	8	1265
3	6	10	8	7696
3	7	13	11	1275120
4	7	13	11	1275120
4	8	17	11	4839120
4	9	21	15	4342010751
5	9	21	15	4342010751
5	10	26	14	7358602624
5	11	31	19	29466877337101
6	11	31	19	29466877337101
6	12	37	18	21848430755052
6	13	43	22	668421206663764973
7	13	43	22	668421206663764973
7	14	50	21	111210534995557376
7	15	57	26	2199970852295832688 8168
8	15	57	26	21999708522958326888168
8	16	65	24	15001111280838921638 41
8	17	73	29	98141295072511768967 4949200
9	17	73	29	981412950725117689674949200
9	18	82	27	268316103488444792871 32160
9	19	91	33	117704722514097750900952684327901
10	19	91	33	117704722514097750900952684327901
10	20	101	30	604861792550624708513466396499
10	21	111	37	1214696041496514443122788776249441 4381
11	21	111	37	12146960414965144431227887762494414381
11	22	122	33	17372654348915578396565748340621312
11	23	133	40	238871939143106758647347543547983295349 6811
12	23	133	40	2388719391431067586473475435479832953496811
12	24	145	36	63147732582159277620804004819809498 4801
12	25	157	44	852463967980020982575658211110018018726645270524

Table 2.18 Values of β_{max} for bases $13 \leq b \leq 20$

b	A	B	J	β_{max}
13	25	157	44	852463967980020982575658211110018018726645270524
13	26	170	39	28717077224929268201659599157515978503356416
13	27	183	48	152925243344932534615818909618898892202903263801780160
14	27	183	48	152925243344932534615818909618898892202903263801780160
14	28	197	43	1613692251361686484421412544021746891502133209787
14	29	211	51	123209002743534545363348378580042422356570453511191349664151
15	29	211	51	123209002743534545363348378580042422356570453511191349664151
15	30	226	46	270242743195975821085722716602418971262724050700468224
15	31	241	55	91708171769852665185766960133846927489751337280221656080474014591
16	31	241	55	91708171769852665185766960133846927489751337280221656080474014591
16	32	257	49	36581588606627883797558369090790311476667269627361629766432
16	33	273	58	40544927014855112320350808345241500943044386051670311611103880994475087
17	33	273	58	40544927014855112320350808345241500943044386051670311611103880994475087
17	34	290	52	4935852345217088547015348691836907296094166766517367159039983616
17	35	307	62	67543015094917799788560459570757486751302877701441552354433337048748044924582
18	35	307	62	67543015094917799788560459570757486751302877701441552354433337048748044924582
18	36	325	55	724397857048292662725261481402882936662732314490400281614774515991376
18	37	343	66	73758168014457418773607303450119757898458531457781497008669950442662055315112784877
19	37	343	66	73758168014457418773607303450119757898458531457781497008669950442662055315112784877
19	38	362	58	118847288171717085931367259389600838697914622154606936522141933998180401152
19	39	381	69	86426537514650745299475083338284777959352162888830459471995007815455677410983861832154001
20	39	381	69	86426537514650745299475083338284777959352162888830459471995007815455677410983861832154001
20	40	401	61	22003032640446530112387504356834355860789381312634981031438198848010272343841240
20	41	421	73	$250714312379800306559196007794041584507088620364 \times 10^{48}$ $+ 503305220058795561622034471001070474059605249481$

Let $k \geq 0$ and $\ell \geq 1$ be integers. We take a weighted average of ℓ consecutive coefficients of $f(x)$. More precisely, define $\tilde{a}_j = b_j - Ab_{j-1} + Bb_{j-2}$ for all integers j so that $\tilde{a}_j$ is the coefficient of x^{s+2-j} in $f(x)$ for $0 \leq j \leq s+2$. Suppose $b_k \neq 0$, and define t_j by

$$b_{k+j} = t_j b_k \quad \text{for } j \in \mathbb{Z}. \tag{2.6.7}$$

Thus,

$$\tilde{a}_{k+j+2} = (t_{j+2} - At_{j+1} + Bt_j)\, b_k \quad \text{for } j \in \mathbb{Z}.$$

We consider the weighted average of $\tilde{a}_j$ given by

$$W(k,\ell) = \sum_{j=0}^{\ell-1} \mu_j \tilde{a}_{k+j+2}, \quad \text{where } 0 \leq \mu_j \leq 1 \text{ for } 0 \leq j \leq \ell-1 \text{ and } \sum_{j=0}^{\ell-1} \mu_j = 1.$$

Observe that $W(k,\ell) = W_0(k,\ell) b_k$, where

$$\begin{aligned} W_0(k,\ell) &= \sum_{j=0}^{\ell-1} \mu_j \,(t_{j+2} - At_{j+1} + Bt_j) \\ &= \mu_0 B t_0 + (-\,\mu_0 A + \mu_1 B)\, t_1 + \sum_{j=2}^{\ell-1} (\mu_{j-2} - \mu_{j-1} A + \mu_j B) t_j \\ &\qquad + (\mu_{\ell-2} - \mu_{\ell-1} A)\, t_\ell + \mu_{\ell-1} t_{\ell+1}. \end{aligned}$$

The idea is to choose the μ_j so that the coefficients of $t_1, t_2, \ldots, t_{\ell-1}$ are all zero above. In other words, we want to choose the μ_j so that the matrix equation

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & \cdots & 1 & 1 & 1 & 1 \\ -A & B & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 1 & -A & B & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 1 & -A & B & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -A & B & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & \cdots & B & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & -A & B & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 1 & -A & B & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & -A & B \end{pmatrix} \begin{pmatrix} \mu_0 \\ \mu_1 \\ \mu_2 \\ \mu_3 \\ \mu_4 \\ \vdots \\ \mu_{\ell-4} \\ \mu_{\ell-3} \\ \mu_{\ell-2} \\ \mu_{\ell-1} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

holds. Note that the above corresponds to a system of ℓ equations in the ℓ unknowns μ_j where $0 \leq j \leq \ell - 1$. The system of equations depends only on A, B and ℓ, and not on k. The first equation in this system guarantees that the condition $\sum_{j=0}^{\ell-1} \mu_j = 1$ holds. It is not the case that the solution of this system necessarily satisfies $0 \leq \mu_j \leq 1$ for every $j \in \{0, 1, \ldots, \ell - 1\}$. There is no reason to concern ourselves with this or whether even a solution exists; we will specify A, B and ℓ shortly and be able to explicitly solve the system of equations and obtain the information we need.

Let the μ_j be a fixed solution to the above system of equations. Since A and B are integers, we deduce that there are computable rational numbers u, v and w, independent of k, for which

$$W_0(k, \ell) = ut_0 + vt_\ell + wt_{\ell+1}.$$

These u, v and w will be positive for our choices of A, B and ℓ. Recall that the definition of t_j in (2.6.7) depends on k. We consider first the case where k is chosen so that $b_k = U$. Since the maximum coefficient of $f(x)$ is M, we deduce from (2.6.7) that

$$\begin{aligned} M \geq W(k, \ell) &= W_0(k, \ell) b_k = ut_0 b_k + vt_\ell b_k + wt_{\ell+1} b_k \\ &= ub_k + vb_{k+\ell} + wb_{k+\ell+1} \geq uU + vL + wL. \end{aligned} \tag{2.6.8}$$

Next, consider the case where k is chosen so that $b_k = L$. Since each coefficient of $f(x)$ is ≥ 0, we deduce similarly that

$$0 \leq W(k, \ell) = W_0(k, \ell) b_k = ub_k + vb_{k+\ell} + wb_{k+\ell+1} \leq uL + vU + wU. \tag{2.6.9}$$

Multiplying through (2.6.8) by u and through (2.6.9) by $-(v + w)$ and adding, we obtain

$$uM \geq (u^2 - (v + w)^2) U. \tag{2.6.10}$$

Multiplying through (2.6.8) by $v + w$ and through (2.6.9) by $-u$ and adding, we obtain

$$(v + w) M \geq (u^2 - (v + w)^2)(-L), \tag{2.6.11}$$

where we have written the above with $-L$ to emphasize that $L \leq 0$.

Now, we are ready to apply the above information to the cases that are of interest to us. For $A = 8$ and $B = 17$, we take $\ell = 12$. We calculate the values of $\mu_0, \mu_1, \ldots, \mu_{11}$ from the system of equations above and verify directly that each μ_j is in $[0, 1]$. We solve for u, v and w and find

$$u = \frac{582622237229761}{58262224635992}$$
$$v = \frac{4291039}{58262224635992}$$

and

$$w = \frac{46530}{560213698423}.$$

The value of $u^2 - (v + w)^2$ is positive. From our upper bound for $M = M(8, 17)$ in (2.6.6) and from (2.6.10), we deduce

$$U \leq \frac{u}{u^2 - (v + w)^2} \cdot M(8, 17) \leq 4839120.17583\ldots.$$

Observe that the lower bound $\beta_{\max} b_0$ for U given by Table 2.17 now implies $b_0 = 1$ (i.e., the polynomial $h(x)$ is monic) and

$$U = U(8, 17) = 4839120.$$

From (2.6.11), we similarly obtain

$$-L \leq \frac{v + w}{u^2 - (v + w)^2} \cdot M(8, 17) \leq 0.07583\ldots.$$

Since $L \leq 0$, we deduce $L = 0$.

For base $b = 4$ and $A = 7$ and $B = 13$, we want similar information. To get an estimate for $M(7, 13)$, we make use of the following example of a polynomial $f(x)$ with non-negative coefficients, divisible by $x^2 - 7x + 13$, and satisfying $f(4)$ is prime:

$$x^{25} + 1510991\, x^{13} + 8925839\, x^{12} + 8925840\, x^{11} + 8925840\, x^{10} + 8925840\, x^9$$
$$+8925840\, x^8 + 8925840\, x^7 + 8925840\, x^6 + 8925840\, x^5 + 8925841\, x^4$$
$$+8925839\, x^3 + 8925833\, x^2 + 8925764\, x + 7415135.$$

The coefficient of x^4 is the largest coefficient above, so we can conclude

$$M(7,13) \le 8925841. \tag{2.6.12}$$

We take $\ell = 12$ in this case, which is purely coincidental. Following an analogous analysis to the case $A = 8$ and $B = 17$ above, we obtain here that

$$U = U(7,13) = 1275120.$$

and $L = 0$ in this case.

We do not need to do as much in the case of base $b = 4$, $A = 9$, and $B = 21$. We take $\ell = 16$, check that the μ_j are in $[0,1]$ and compute u, v, and w as before. Table 2.17 gives us a lower bound for $U = U(9,21)$. Using (2.6.10), we see that

$$M = M(9,21) \ge \frac{u^2 - (v+w)^2}{u} U \ge 4.34201 \times 10^9.$$

This implies that any polynomial $f(x)$ with non-negative coefficients divisible by $x^2 - 9x + 21$ must have a coefficient as large as $4.34201 \cdot 10^9$. We easily deduce as a consequence of Corollary 2.10 that if $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}$ is such that $f(4)$ is prime and

$$0 \le a_j \le 5.8802 \cdot 10^7 \quad \text{for } 0 \le j \le n,$$

then either $f(x)$ is irreducible or $f(x)$ is divisible by $\Phi_3(x-4) = x^2 - 7x + 13$ or $\Phi_4(x-4) = x^2 - 8x + 17$.

Along similar lines, we can use (2.6.10) to obtain a lower bound for $M = M(A,B)$ in the case that $f(x)$ is divisible by $x^2 - Ax + B$ where $(A,B) \in \{(7,13),(8,17)\}$ by making use of the precise values just obtained for $U(A,B)$. We deduce that

$$M(7,13) \ge 8925840$$

and

$$M(8,17) \ge 48391200.$$

To clarify, we rounded up in these estimates since M and U being integers in (2.6.10) implies that

$$M \geq \left\lceil \frac{u^2 - (v+w)^2}{u} \cdot U \right\rceil. \tag{2.6.13}$$

Note that these lower bounds for $M(A, B)$ are each 1 less than the upper bounds obtained in (2.6.6) and (2.6.12). Combining the above information with Corollary 2.10, we see that to complete a proof of Theorem 2.1 it suffices to show that the upper bounds obtained for $M(A, B)$ are the actual values of $M(A, B)$.

In Section 2.7, we find explicit examples showing that

$$M(A, B) \leq (1 - A + B)\beta_{\max} + 1,$$

for the choices of A and B we need to consider. Repeating the above calculations for bases $2 \leq b \leq 20$, with the aid of Corollary 2.10, one can show that the only quadratics that need to be considered for $g(x)$ are $\Phi_3(x - b)$ and $\Phi_4(x - b)$. Furthermore, one can show that for each such quadratic, the corresponding $h(x)$ must be monic, that all of its coefficients must be non-negative (i.e., $L = 0$) and that its largest coefficient corresponds to the value of $\beta_{\max}$ in Table 2.17 and Table 2.18. Table 2.19 and Table 2.20 lists b, A, B, and a lower bound for $M(A, B)$ obtained from our computations.

Table 2.19 Lower Bound on $M(A,B)$ for bases $3 \leq b \leq 8$

b	A	B	**Lower Bound on** $M(A,B)$ **from** (2.6.13)
3	5	7	3078
3	6	10	38480
4	7	13	8925840
4	8	17	48391200
5	9	21	56446139763
5	10	26	125096244608
6	11	31	358827010528371
6	12	37	77441641171102
7	13	43	207210574065576714162
7	14	50	3587292236348090368
8	15	57	945987466487208056191223
8	16	65	75005556404194608192049

Table 2.20 Lower Bound on $M(A,B)$ for bases $9 \le b \le 20$

b	A	B	**Lower Bound on** $M(A,B)$ **from** (2.6.13)
9	17	73	55940538191331708311472104399
9	18	82	1744054672674891153663590399
10	19	91	8592444743529135815769545955936771
10	20	101	49598666989151226098104244512916
11	21	111	1105373397761828143241737786386991708670
11	22	122	1754638089240473418053140582402752510
12	23	133	265147852448848502098555773338261457838146019
12	24	145	77040233750234318697380885880167588145720
13	25	157	113377707741342790682562542077632396490643820979690
13	26	170	4163976197614743889240641877839816882986680319
14	27	183	12407120390228992708497471586407623566027519706895477117
14	28	197	207816534034072432577282639868785188569775847780400
15	29	211	22547247502066821801492753280147763291252392992548016988539630
15	30	226	53237820409607236753887375170676537338756637987992240126
16	31	241	19350424243438912354196828588241701700337532166126769432980017078699
16	32	257	8267439025097901738248191414518610393726802935783728327213629
17	33	273	9771327410580082069204544811203201727273697038452545098276035319668495966
17	34	290	1268514052720791756582944613802085175096200858994963359873275789309
18	35	307	18439243120912559342277005462816793883105685612493543792760301014308216264410882
18	36	325	210075378544004872190325829606836051632192371202216081668284609637499036
19	37	343	22643757580438427563497442159186765674826769157538919581661674785897250981739624957237
19	38	362	38625368655808052927694359301620272576822252200247254369696128549408630374397
20	39	381	29644302367525205637719953585031678840057791870868847598894287680701297351967464608428822340
20	40	401	7965097815841643900684276577174036821605756035173863133380627982979718588470528878

2.7 A Sharp Bound for $M(A, B)$

We are now ready to finish the proof of Theorem 2.1. The arguments in this section are essentially the same as in [9] and [10], with a notable improvement that was first shown in [7]. These are reproduced here, with the obvious minor changes.

Recall that, for a fixed base b with $2 \le b \le 20$, we are interested in the case that $f(x) = g(x)h(x)$ where $g(x) = x^2 - Ax + B$ with $g(x) = \Phi_3(x-b)$ or $g(x) = \Phi_4(x-b)$, and also where $f(x)$ has maximal coefficient equal to $M(A, B)$. In the previous section, we established that $h(x)$ must be monic, that all of its coefficients must be non-negative (i.e., $L = 0$) and that its largest coefficient corresponds to the value of $\beta_{\max}$ indicated in Table 2.17 or Table 2.18. To finish the proof of Theorem 2.1, one checks that it suffices to show $M(A, B) = (1 - A + B) \cdot \beta_{\max} + 1$ for each appropriate choice of (A, B) as shown in Table 2.17 and Table 2.18.

We fix (A, B) and assume to the contrary that

$$M(A, B) \le (1 - A + B) \cdot \beta_{\max}. \tag{2.7.1}$$

We want then to obtain a contradiction for each (A, B) corresponding to $\Phi_3(x - b)$ and $\Phi_4(x - b)$ given in Table 2.17 and Table 2.18.

This argument is independent of our fixed base b and relies only on the choice of A and B (although A and B themselves do depend on the choice of base b).

To obtain a contradiction, we will first want more information about the structure of $h(x)$. As in the Section 2.6, we consider

$$h(x) = b_0 x^s + b_1 x^{s-1} + \cdots + b_{s-1} x + b_s,$$

where we now know $b_0 = 1$. As before, we define $b_j = 0$ if $j < 0$ or $j > s$. We consider J as in Table 2.17 and Table 2.18 so that (2.6.5) holds. We claim now that, with $h(x)$ as above, the inequality in (2.6.5) can be replaced by equality for $j \le J'$, where $J' \le J$ is maximal with $\beta_{J'} = \beta_{\max}$. In other words, since $b_0 = \beta_0 = 1$, we claim that

$$b_j = \beta_j \quad \text{for all integers } j \le J'. \tag{2.7.2}$$

Recall that (2.6.2) holds for the b_j. To justify (2.7.2), it suffices to show that

$$b_j = Ab_{j-1} - Bb_{j-2} \ \text{ for all } j \in \{1, 2, \ldots, J'\}\,.$$

Assume that at least one of these equations for b_j does not hold. Taking $j = J'$ and following the string of inequalities (2.6.4) that led to (2.6.5), we see that $b_{J'} > \beta_{J'}$. This contradicts that the largest coefficient of $h(x)$ is $\beta_{J'}$. Therefore, (2.7.2) holds.

For the coefficients b_j with $j > J'$, we will obtain a different structure. Let t denote the maximal non-negative integer for which

$$b_{J'+1} = b_{J'+2} = \cdots = b_{J'+t} = \beta_{J'}.$$

Thus, $b_{J'+t+1} < \beta_{J'}$. We claim next that

$$b_{J'+t+j+1} = \beta_{J'} - \beta_j \ \text{ for } j \in \{0, 1, \ldots, J'\}\,. \tag{2.7.3}$$

Define

$$\gamma_j = \begin{cases} \beta_{J'} - b_{J'+t+j+1} & \text{for } j \ge 0 \\ 0 & \text{for } j \le -1. \end{cases}$$

Note that

$$\gamma_{-1} = 0 = \beta_{J'} - \beta_{J'} = \beta_{J'} - b_{J'+t}.$$

Since the expansion of (2.6.1) gives us coefficients of $f(x)$, we deduce from our assumption (2.7.1) that, for $j \ge 1$,

$$\begin{aligned}(1 - A + B)\beta_{J'} - &\gamma_j + A\gamma_{j-1} - B\gamma_{j-2} \\ &= b_{J'+t+j+1} - Ab_{J'+t+j} + Bb_{J'+t+j-1} \\ &\le M(A, B) \le (1 - A + B)\beta_{J'}.\end{aligned}$$

Since $b_{J'+t+1} < \beta_{J'}$, it follows that

$$\gamma_0 \geq 1 \quad \text{and} \quad \gamma_j \geq A\gamma_{j-1} - B\gamma_{j-2} \;\; \text{for all } j \in \mathbb{Z}.$$

Observe that this is (2.6.2) with the b_j's replaced by γ_j's. We deduce, as we did there, that (2.6.5) holds but now with the b_j's replaced by γ_j's. So we have

$$\gamma_j \geq \beta_j \gamma_0 \qquad \text{for all integers } j \leq J' + 1.$$

As in the argument for (2.7.2), we either have equality for each $j \leq J'$ or else that $\gamma_{J'} > \beta_{J'}\gamma_0$. This latter inequality is impossible since the b_j are all ≥ 0 which implies

$$\beta_{J'} - \gamma_{J'} = b_{2J'+t+1} \geq 0.$$

Since $\gamma_{J'} \geq \beta_{J'}\gamma_0$, the above inequality also implies $\gamma_0 = 1$. Thus, we have $\gamma_j = \beta_j$ for all $j \leq J'$. This implies (2.7.3).

A direct computation using the values of A, B and J' from Table 2.17 and Table 2.18 shows that $\beta_{J'-1} \leq \beta_{J'}$. From (2.7.3), we deduce

$$b_{2J'+t} \geq 0 \qquad \text{and} \qquad b_{2J'+t+1} = 0.$$

Beginning with $j = -1$ and increasing j, the numbers b_j start at 0, go up to $\beta_{J'}$, possibly remain there for awhile and then come back down to 0. It is possible that there are more non-zero b_j with $j > 2J' + t + 1$. But we get a kind-of carousel effect here, where if there are more non-zero b_j with $j > 2J' + t + 1$, then again they will go up in the same pattern as before to $\beta_{J'}$, possibly linger at $\beta_{J'}$ for awhile and then come back down to 0 again. The increases in the numbers b_j are largely due to the condition that the coefficients of $f(x)$ are ≥ 0; the decreases in the coefficients are largely due to the assumption on the upper bound for the coefficients of $f(x)$ given by (2.7.1). We explain this in some more detail next.

Suppose we know that $b_{k-1} = 0$ and $b_k \neq 0$ for some integer k. In particular, perhaps $k = 2J' + t + 2$. We have already seen that the coefficients of $h(x)$ are ≥ 0.

Hence, $b_k \geq 1$. Define

$$b'_j = \begin{cases} b_{k+j} & \text{for } j \geq 0 \\ 0 & \text{for } j < 0. \end{cases}$$

Since $b_k \geq 1$, we have $b'_0 \geq 1$. From (2.6.2),

$$b'_1 = b_{k+1} \geq Ab_k - Bb_{k-1} = Ab_k = Ab'_0 - Bb'_{-1}.$$

Also, from (2.6.2), we deduce

$$b'_j \geq Ab'_{j-1} - Bb'_{j-2} \quad \text{for all integers } j \geq 2.$$

The definition of b'_j also implies

$$b'_j \geq Ab'_{j-1} - Bb'_{j-2} \quad \text{for all integers } j \leq 0.$$

We deduce that condition (2.6.2) holds with the b_j's replaced by b'_j's. Thus, based on our previous arguments with b_j, we deduce here that

$$b'_j = \begin{cases} \beta_j & \text{for } j \in \{0, 1, \ldots, J'\} \\ \beta_{J'} & \text{for } j \in \{J'+1, J'+2, \ldots, J'+t'\} \\ \beta_{J'} - \beta_{j-J'-t'-1} & \text{for } j \in \{J'+t'+1, J'+t'+2, \ldots, 2J'+t'+1\}, \end{cases}$$

where t' denotes some non-negative integer. Thus, $h(x)$ can be written as a sum over some non-negative integers k of polynomials which are x^k times

$$\begin{aligned} &\left(\beta_0 x^{J'} + \beta_1 x^{J'-1} + \cdots + \beta_{J'}\right) x^{J'+t'} + \left(x^{J'+t'-1} + x^{J'+t'-2} + \cdots + x^{J'}\right) \beta_{J'} \\ &\quad + (\beta_{J'} - \beta_0)\, x^{J'-1} + (\beta_{J'} - \beta_1)\, x^{J'-2} + \cdots + (\beta_{J'} - \beta_{J'-1}), \end{aligned} \tag{2.7.4}$$

where $t' = t'(k)$ is a non-negative integer. We note that the k cannot be arbitrary since we do not want overlapping terms for different k and we want the coefficient of each such x^{k-1} in $h(x)$ to be 0. To finish the proof of Theorem 2.1, we need only to show that $h(b)$ is composite. The approach here differs from that given in [9] and [10], but is identical to that shown in [7]. We reproduce the argument next with minor clarifications.

We refer to the polynomial in (2.7.4) as part of $h(x)$. We begin by showing that with A, B, and J' fixed, but t' arbitrary, each part of $h(x)$ is divisible by

$$h_0(x) = \sum_{j=0}^{J'} \left(\beta_{J'-j} - \beta_{J'-j-1}\right) x^j,$$

where we recall here that $\beta_{-1} = 0$. From this definition of $h_0(x)$, we have

$$\sum_{j=0}^{J'} \beta_{J'-j} x^j \equiv \sum_{j=0}^{J'} \beta_{J'-j-1} x^j \equiv \sum_{j=1}^{J'} \beta_{J'-j} x^{j-1} \pmod{h_0(x)}.$$

We deduce that the polynomial given in (2.7.4) is

$$\begin{aligned}
&\left(\sum_{j=0}^{J'} \beta_{J'-j} x^j\right) x^{J'+t'} + \left(\sum_{j=0}^{J'+t'-1} x^j\right) \beta_{J'} - \sum_{j=1}^{J'} \beta_{J'-j} x^{j-1} \\
&\qquad \equiv \left(\sum_{j=1}^{J'} \beta_{J'-j} x^{j-1}\right) x^{J'+t'} + \left(\sum_{j=0}^{J'+t'-1} x^j\right) \beta_{J'} - \sum_{j=1}^{J'} \beta_{J'-j} x^{j-1} \\
&\qquad \equiv \left(\sum_{j=0}^{J'} \beta_{J'-j} x^j\right) x^{J'+t'-1} + \left(\sum_{j=0}^{J'+t'-2} x^j\right) \beta_{J'} - \sum_{j=1}^{J'} \beta_{J'-j} x^{j-1} \\
&\qquad \equiv \left(\sum_{j=0}^{J'} \beta_{J'-j} x^j\right) x^{J'+t'-2} + \left(\sum_{j=0}^{J'+t'-3} x^j\right) \beta_{J'} - \sum_{j=1}^{J'} \beta_{J'-j} x^{j-1} \\
&\qquad \vdots \\
&\qquad \equiv \sum_{j=0}^{J'} \beta_{J'-j} x^j - \sum_{j=1}^{J'} \beta_{J'-j} x^{j-1} \equiv 0 \pmod{h_0(x)}.
\end{aligned}$$

Thus we obtain that each part of $h(x)$ and, therefore, $h(x)$ itself is divisible by $h_0(x)$. Using that $h(x)$ consists of at least one part as in (2.7.4) with $t' \geq 0$ and $J' \geq 1$, we deduce that

$$h(b) \geq \left(\beta_0 b^{J'} + \beta_1 b^{J'-1} + \cdots + \beta_{J'}\right) b^{J'} > \beta_0 b^{J'} + \beta_1 b^{J'-1} + \cdots + \beta_{J'} > h_0(b) > 1.$$

Hence, $h(b)$ is the integer $h_0(b)$ times an integer that is > 1. Therefore, $h(b)$ is composite. This gives us a contradiction to (2.7.1).

Therefore $M(A, B) \geq (1 - A + B)\beta_{J'} + 1$. By finding explicit examples of $f(x)$ with maximal coefficient equal to $(1 - A + B)\beta_{J'} + 1$, as done for $b = 4$ in Section 2.6, we know that $M(A, B) \leq (1 - A + B)\beta_{J'} + 1$. Thus, $M(A, B) = (1 - A + B)\beta_{J'} + 1$, and our desired result of Theorem 2.1 follows.

To find explicit examples, we fixed a base b with $3 \leq b \leq 20$, choose the appropriate A, B, and J' using Table 2.17 and Table 2.18, and then we took $h_0(x)$ to be as given in (2.7.4) and set t' to 0 (with the exception of base $b = 15$ and $M_1(b)$, where we set $t' = 1$). With some trial and error, we found a quadratic $h_1(x) \in \mathbb{Z}[x]$ such that $h(x) = h_0(x) + h_1(x)$ satisfies the following conditions:

1. $f(x) = g(x)h(x)$ has non-negative coefficients,
2. $f(b)$ is prime,
3. the largest coefficient of $f(x)$ is $(1 - A + B)\beta_{\text{max}} + 1$,

where β_{max} is given in Table 2.17 or Table 2.18.

Table 2.21 below gives our explicit choices of $h_1(x)$ to construct $f(x)$ showing us that the bounds $M_1(b)$ and $M_2(b)$ given in Theorem 2.1 are sharp.

2.8 Final Arguments

We finish by supplying a proof of Theorem 2.8. This will be done in two parts: for bases $2 \leq b \leq 20$ with $b \neq 6$ and $b \neq 14$, and then for the case that $b = 6$ and $b = 14$. We begin with the former, and mimic the proof as in [9].

To help illustrate the proof, we will explicitly use base $b = 7$. We begin by taking $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$ such that $a_j \geq 0$ for each j and $f(7)$ is prime. Let

$$M = N_1(7) = 4847692211281203599$$

as in Table 2.12. Since M is less than the bound

$$M_2(7) = 20721057406576714163$$

Table 2.21 Examples of $h_1(x)$ for $M_1(b)$ and $M_2(b)$

b	$h_1(x)$ for $M_1(b)$	$h_1(x)$ for $M_2(b)$
3	x^2+5x+9	–
4	$x^2+5x+10$	$x^2+8x+40$
5	$x^2+7x+23$	$x^2+10x+44$
6	$x^2+8x+32$	$x^2+11x+48$
5	$x^2+7x+23$	$x^2+10x+44$
7	$x^2+9x+39$	$x^2+13x+46$
8	$x^2+15x+72$	$x^2+15x+106$
9	$x^2+16x+76$	$x^2+17x+115$
10	$x^2+8x+54$	$x^2+11x+66$
11	$x^2+14x+84$	$x^2+21x+133$
12	$x^2+19x+126$	$x^2+23x+135$
13	$x^2+16x+122$	$x^2+13x+83$
14	$x^2+14x+114$	$x^2+23x+164$
15	$x^2+24x+198$	$x^2+15x+123$
16	$x^2+12x+114$	$x^2+31x+565$
17	$x^2+18x+178$	$x^2+19x+176$
18	$x^2+19x+198$	$x^2+35x+742$
19	$x^2+29x+279$	$x^2+27x+272$
20	$x^2+21x+232$	$x^2+39x+522$

from Table 2.2, we can apply Theorem 2.1. Thus, if $f(x)$ is reducible, then it must be divisible by $\Phi_4(x-7) = x^2 - 14\,x + 50$. In this case, $f(x)$ must be of the form given by (2.6.1) with $s = 21$, $A = 14$, and $B = 50$. From (2.6.5), with $J = 21$ as in Table 2.17, we deduce that the constant term of $h(x)$ is

$$b_{21} \geq \beta_{21} b_0 \geq 96953844225624064\, b_0.$$

Since the constant term of $g(x)$ is $B = 50$, we see that the constant term of $f(x)$ must be at least b_0 times

$$96953844225624064 \cdot 50 = 4847692211281203200.$$

Note that this number differs from M by 399, so we still have some work to do but can deduce that $b_0 = 1$.

For $A = 14$ and $B = 50$, the non-zero values of β_j occurring in (2.6.5) begin with

$$\beta_0 = 1, \quad \beta_1 = 14, \quad \beta_2 = 146, \quad \beta_3 = 1344, \quad \text{and} \quad \beta_4 = 11516. \tag{2.8.1}$$

Further, the β_j remain positive for $j \leq 21$, where $\beta_{21} = 96953844225624064$. We define

$$\kappa_j = b_j - Ab_{j-1} + Bb_{j-2}.$$

For u a non-negative integer, we also define

$$\kappa'_u = \sum_{j=0}^{u} \beta_j \kappa_{21-j}.$$

Along the lines of the inequalities in (2.6.4), we deduce the equalities

$$\begin{aligned} b_{21} &= Ab_{20} - Bb_{19} + \kappa_{21} \\ &= A\left(Ab_{19} - Bb_{18} + \kappa_{20}\right) - Bb_{19} + \kappa_{21} \\ &= \beta_2 b_{19} - B\beta_1 b_{18} + \kappa_{21} + \beta_1\kappa_{20} \\ &= \cdots = \beta_{19} b_2 - B\beta_{18} b_1 + \kappa'_{18} \\ &= \beta_{20} b_1 - B\beta_{19} b_0 + \kappa'_{19} \\ &= \beta_{21} b_0 + \kappa'_{20} = \beta_{21} + \kappa'_{20}. \end{aligned}$$

The advantage here is that we have a formulation of how far the constant term b_{21} of $h(x)$ is from β_{21}. Since the constant term of $f(x)$ is $50b_{21}$, we deduce that

$$\beta_{21} + \sum_{j=0}^{20} \beta_j \kappa_{21-j} = b_{21} \leq \frac{M}{50}.$$

Given that $50\beta_{21}$ differs from M by 399 and the values of β_j start as in (2.8.1), we see that

$$\kappa_1 = \kappa_2 = \cdots = \kappa_{19} = 0, \quad \kappa_{20} \leq \left\lfloor \frac{399}{700} \right\rfloor = 0, \quad \text{and} \quad \kappa_{21} \leq \left\lfloor \frac{399}{50} \right\rfloor = 7.$$

We see that by considering the different non-negative integers κ_j with the above restrictions, we are left with ≤ 8 possible $h(x)$ to consider. Of the 8 polynomials $h(x)$ satisfying the conditions on the κ_j above, none satisfies $f(7) = h(7)$ is prime. The example given in Table 2.8 corresponds to $\kappa_{21} = 8$. This justifies the bound $N_1(7)$ from Table 2.12

A similar process can be repeated for $N_2(7)$ from Table 2.13.

For $2 \leq b \leq 20$ with $b \neq 6$ and $b \neq 14$, we are able to apply the same approach as for $N_1(7)$. In each case, we had to consider at most two κ_j's that were non-zero with each such $\kappa_j \leq 279$. In the event that some choices of κ_j's led to $f(b)$ being prime, a quick check of the coefficients of $f(x)$ verified that at least one of them was larger than $N_1(b)$ (or $N_2(b)$ as the case may be).

We now turn our attention to the case that $b = 6$ or $b = 14$. In these two cases, we have that the constant term of $f(x)$ is not necessarily the largest coefficient of $f(x)$. We modify the above method to compensate for this.

Now, the polynomial

$$x^{20} + 2x^3 + 13519269991320x^2 + 610418402115746x + 610418402115527$$

given in Table 2.11 has the property that it is reducible and has a prime value when $x = 6$. We consider a polynomial $f(x)$ of degree 20 having non-negative coefficients bounded above by 610418402115745 and satisfying $f(6)$ is prime. Our goal is to show that such an $f(x)$ must be irreducible. The example above shows that the bound 610418402115745 on the coefficients of $f(x)$ for this conclusion is sharp.

Assume $f(x)$ is reducible. We have already seen from Table 2.2 for Theorem 2.1 that necessarily $f(x)$ is divisible by $g(x) = x^2 - 12x + 37$.

For the purposes of considering the approach in more generality, we consider $g(x) = x^2 - Ax + B$, and recall the definition of J. In the case that $A = 12$ and $B = 37$, we note that $J = 18$ from Table 2.17. We focus on the case that $\beta_J < \beta_{J-1}$.

We set, as before, $b_0, \dots, b_s \in \mathbb{Z}$, with $b_0 > 0$, satisfying

$$(b_0x^s + b_1x^{s-1} + \cdots + b_{s-1}x + b_s)(x^2 - Ax + B)$$

is a polynomial with non-negative coefficients. Further, we take $b_j = 0$ for $j < 0$ and $j > s$. Recall that we have shown that under these conditions

$$b_j \geq \beta_j b_0 \quad \text{for all integers } j \leq J + 1.$$

Since $b_J \geq \beta_J > 0$, we deduce $s \geq J$.

As before, we define

$$\kappa_j = b_j - Ab_{j-1} + Bb_{j-2} \quad \text{for all } j \in \mathbb{Z},$$

and

$$\kappa'(u,t) = \sum_{j=0}^{u} \beta_j \kappa_{t-j} \quad \text{for all } u \text{ and } t \text{ in } \mathbb{Z}.$$

Observe that $\kappa_j \geq 0$ for every j. For an arbitrary integer $t \in [1, J+1]$, we have

$$\begin{aligned}
b_t &= Ab_{t-1} - Bb_{t-2} + \kappa_t \\
&= A(Ab_{t-2} - Bb_{t-3} + \kappa_{t-1}) - Bb_{t-2} + \kappa_t \\
&= \beta_2 b_{t-2} - B\beta_1 b_{t-3} + \kappa_t + \beta_1 \kappa_{t-1} \\
&= \cdots = \beta_{t-2} b_2 - B\beta_{t-3} b_1 + \kappa'(t-3, t) \\
&= \beta_{t-1} b_1 - B\beta_{t-2} b_0 + \kappa'(t-2, t) \\
&= \beta_t b_0 + \kappa'(t-1, t) = \beta_t + \kappa'(t-1, t).
\end{aligned}$$

We restrict now to the example given above with $b = 6$, $A = 12$ and $B = 37$. Thus, we are assuming $f(x)$ has degree 20, has non-negative coefficients bounded above by

$$M = N_1(6) = 610418402115745,$$

where $N_1(b)$ is given in Table 2.12, satisfies $f(6)$ is prime, and satisfies $f(x)$ is divisible by $g(x) = x^2 - 12x + 37$. In this case, $J = 18$ and $0 < \beta_{18} < \beta_{17}$. The coefficient of

x and the constant term of $f(x)$ are

$$37b_{17} - 12b_{18} \quad \text{and} \quad 37b_{18},$$

respectively. Each of these is necessarily $\leq M$, so a weighted average of them must also be $\leq M$. In particular, we observe that

$$\frac{37^2}{49} b_{17} = \frac{37}{49}(37b_{17} - 12b_{18}) + \frac{12}{49}(37b_{18}) \leq M.$$

We apply our formula for b_t above with $t = 17$ to deduce

$$\frac{37^2}{49}\Big(\beta_{17} + \sum_{j=0}^{16} \beta_j \kappa_{17-j}\Big) = \frac{37^2}{49}(\beta_{17} + \kappa'(16,17)) \leq M.$$

Thus, making use of the specific values of M and β_{17}, we obtain

$$\beta_0\kappa_{17} + \beta_1\kappa_{16} + \cdots + \beta_{16}\kappa_1 \leq (49/37^2)M - \beta_{17} = \frac{5317}{1369}.$$

Since

$$\frac{5317}{1369\beta_j} \leq \frac{5317}{1369 \cdot 12} < 1 \quad \text{for } 1 \leq j \leq 16$$

and $5317/1369 < 4$, we deduce

$$\kappa_1 = \kappa_2 = \cdots = \kappa_{16} = 0 \quad \text{and} \quad \kappa_{17} \in \{0, 1, 2, 3\}.$$

We still need to determine a small list of possibilities for κ_{18}. We set $\kappa_j = 0$ for $1 \leq j \leq 16$. For each choice of $\kappa_{17} \in \{0, 1, 2, 3\}$, we consider $\kappa^* = \kappa^*(\kappa_{17}) \in \mathbb{Z}^+$ for which $\kappa_{18} = \kappa^*$ makes $37b_{17} - 12b_{18}$ and $37b_{18}$ as close as possible. In other words, κ^* is a positive integer for which the choice $\kappa_{18} = \kappa^*$ minimizes $|37b_{17} - 49b_{18}|$. Observe that with $\kappa_j = 0$ for $1 \leq j \leq 16$, we have

$$b_{17} = \beta_{17} + \kappa_{17} \quad \text{and} \quad b_{18} = \beta_{18} + \kappa_{18} + 12\kappa_{17}.$$

Thus, we want to minimize

$$|37\beta_{17} - 49\beta_{18} - 551\kappa_{17} - 49\kappa_{18}|.$$

A direct computation gives

$$\kappa^*(0) = 13519269991347, \quad \kappa^*(1) = 13519269991336,$$

$$\kappa^*(2) = 13519269991324 \quad \text{and} \quad \kappa^*(3) = 13519269991313.$$

The roll of κ^* is that, as we will see momentarily, κ_{18} must be very close to κ^*. Set $\kappa_{18} = \kappa^* + t$. Since the coefficients of $f(x)$ must be $\le M$, we deduce that

$$37\beta_{17} - 12\beta_{18} - 107\kappa_{17} - 12\kappa_{18} = 37b_{17} - 12b_{18} \le M$$

and

$$37\beta_{18} + 37\kappa_{18} + 444\kappa_{17} = 37b_{18} \le M.$$

Making the substitutions $\kappa_{17} \in \{0, 1, 2, 3\}$ and $\kappa_{18} = \kappa^* + t$ as well as the values of the β_j and M, the first of these inequalities implies $t \ge -9$ for each value of κ_{17} and the second implies $t \le 2$ for each value of κ_{17}. Thus, $-9 \le t \le 2$, and there are no more than 12 values of $\kappa_{18} = \kappa^* + t$ to consider for each value of κ_{17}.

We are left with testing the various values of $h(x)$ and, hence, $f(x)$ that arise when considering $\kappa_j = 0$ for $1 \le j \le 16$, $\kappa_{17} \in \{0, 1, 2, 3\}$, and $\kappa_{18} = \kappa^* + t$ with $-9 \le t \le 2$. In each case, the $f(x)$ found for which $f(6)$ is prime has a coefficient which exceeds M. The polynomial example given with maximal coefficient $M + 1$ arises with $\kappa_{17} = 2$ and $t = -4$.

This same procedure can be used to verify that the examples in Table 2.8, Table 2.10, and Table 2.11 for bases $b = 6$ and $b = 14$ provide sharp bounds for $N_1(b)$ and $N_2(b)$ as stated in Theorem 2.8.

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Appendix A

Supplementary Material for Chapter 2

A.1 Definition of $P_b(x, y)$

Let b be an integer with $2 \le b \le 20$, $z = x + iy$, and ζ_n be the n-th root of unity $e^{\frac{2\pi i}{n}}$. We define

$$P_b(x, y) = D_b(x, y) - N_b(x, y)$$

where

$$D_b(x, y) = |b - x - iy|^{4(e_3+e_4+e_6)+2(e_1+d+1)},$$

$$N_b(x, y) = |b - 1 - x - iy|^{2e_1} \left(|b + \zeta_3 - x - iy| \left|b + \overline{\zeta_3} - x - iy\right|\right)^{2e_3} \cdot (|b + i - x - iy| \, |b - i - x - iy|)^{2e_4} \left(|b + \zeta_6 - x - iy| \left|b + \overline{\zeta_6} - x - iy\right|\right)^{2e_6},$$

and $e_1 = e_1(b), e_3 = e_3(b), e_4 = e_4(b), e_6 = e_6(b)$ and $d = d(b)$ are all non-negative integers. The numbers e_1, e_3, e_4, e_6 and d for a given base $2 \le b \le 20$ are given in Table 2.4. For bases $6 \le b \le 20$, the same numbers as for base $b = 6$ are used. It is believed that the numbers for base $b = 6$ will work for any base $b > 20$, but these computations have not been done.

We can write

$$D_b(x, y) = |b - x - iy|^{4(e_3+e_4+e_6)+2(e_1+d+1)} = \left(y^2 + (b - x)^2\right)^{2(e_3+e_4+e_6)+e_1+d+1}.$$

Similarly, we consider $N_b(x, y)$.

The first factor of $N_b(x, y)$ can be written as

$$|b - 1 - x - iy|^{2e_1} = \left(y^2 + (b - 1 - x)^2\right)^{e_1}.$$

The second factor of $N_b(x,y)$ can be written as

$$\left(|b+\zeta_3-x-iy|\left|b+\overline{\zeta_3}-x-iy\right|\right)^{2e_3} = \\ \left(y^4+\left(2b^2-4bx+2x^2-2b+2x-1\right)y^2+\left(b^2-2bx+x^2-b+x+1\right)^2\right)^{e_3}.$$

The third factor of $N_b(x,y)$ can be written as

$$\left(|b+i-x-iy|\,|b-i-x-iy|\right)^{2e_4} = \\ \left(y^4+2(b-x+1)(b-x-1)y^2+\left(b^2-2bx+x^2+1\right)^2\right)^{e_4}.$$

The fourth factor of $N_b(x,y)$ can be written as

$$\left(|b+\zeta_6-x-iy|\left|b+\overline{\zeta_6}-x-iy\right|\right)^{2e_6} = \\ \left(y^4+\left(2b^2-4bx+2x^2+2b-2x-1\right)y^2+\left(b^2-2bx+x^2+b-x+1\right)^2\right)^{e_6}.$$

Combining all of these calculations, we have that $P_b(x,y) = D_b(x,y) - N_b(x,y)$ is a polynomial in $\mathbb{Z}[b,x,y]$. Moreover, $P_b(x,y)$ is even in y. Thus, we can write

$$P_b(x,y) = \sum_{j=0}^{r} a_j(b,x)y^{2j},$$

where $r = 2\left(e_3+e_4+e_6\right)+e_1+d+1$ and each $a_j(b,x) \in \mathbb{Z}[b,x]$ for all j.

In fact, each $a_j(b,x)$ can be computed directly, but this process is tedious and the majority of the $a_j(b,x)$'s are very large.

A.2 Definition of $\overleftarrow{P_b}(x,y)$

We consider five sets of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ for each of the bases $b = 2$, $b = 3$, $b = 4$, $b = 5$, and and one set for a base $6 \le b \le 20$. The numbers e_1, e_3, e_4, e_6 and d for a given base $2 \le b \le 20$ are given in Table 2.4. Again, we note that it is believed that the same set of numbers for base $b = 6$ will work for bases $b > 20$.

In the proof of Lemma 2.4, we define $\overleftarrow{P_b}(x,y) = \sum_{j=0}^{r} p_j(b,x) y^j$ for a fixed base $2 \leq b \leq 20$ such that

$$\overleftarrow{P_b}\left(x, y^2\right) = P_b\left(x+b, y\right),$$

where $r = 2\left(e_3 + e_4 + e_6\right) + e_1 + d + 1$.

Definition of $\overleftarrow{P_2}(x,y)$

For the case of base $b=2$, we have the numbers $e_1 = 20$, $e_3 = 4$, $e_4 = 0$, $e_6 = 0$, and $d = 0$. In this case, $\overleftarrow{P_2}(x,y)$ is of degree 29 in y. We compute each $p_j(2,x)$ directly:

$$\begin{aligned}
p_0(2,x) = {} & x^{58} - x^{56} - 48\,x^{55} - 1136\,x^{54} - 17672\,x^{53} - 203256\,x^{52} - 1843312\,x^{51} \\
& - 13727468\,x^{50} - 86328144\,x^{49} - 467874268\,x^{48} - 2219417464\,x^{47} \\
& - 9327056784\,x^{46} - 35064432560\,x^{45} - 118864863822\,x^{44} \\
& - 365731347024\,x^{43} - 1027041689472\,x^{42} - 2644578598968\,x^{41} \\
& - 6268897720278\,x^{40} - 13726693343952\,x^{39} - 27844786467884\,x^{38} \\
& - 52457586670752\,x^{37} - 91978546411120\,x^{36} - 150372738595624\,x^{35} \\
& - 229575102375384\,x^{34} - 327727318669808\,x^{33} - 437916772320589\,x^{32} \\
& - 548185808672784\,x^{31} - 643280702149520\,x^{30} - 707956119521472\,x^{29} \\
& - 730906849197204\,x^{28} - 707956119521472\,x^{27} - 643280702149520\,x^{26} \\
& - 548185808672784\,x^{25} - 437916772320589\,x^{24} - 327727318669808\,x^{23} \\
& - 229575102375384\,x^{22} - 150372738595624\,x^{21} - 91978546411120\,x^{20} \\
& - 52457586670752\,x^{19} - 27844786467884\,x^{18} - 13726693343952\,x^{17} \\
& - 6268897720278\,x^{16} - 2644578598968\,x^{15} - 1027041689472\,x^{14} \\
& - 365731347024\,x^{13} - 118864863822\,x^{12} - 35064432560\,x^{11} \\
& - 9327056784\,x^{10} - 2219417464\,x^{9} - 467874268\,x^{8} - 86328144\,x^{7} \\
& - 13727468\,x^{6} - 1843312\,x^{5} - 203256\,x^{4} - 17672\,x^{3} - 1136\,x^{2} - 48\,x - 1
\end{aligned}$$

$$\begin{aligned}
p_1(2,x) = {} & 29\,x^{56} - 28\,x^{54} - 1296\,x^{53} - 29552\,x^{52} - 442544\,x^{51} - 4895216\,x^{50} \\
& - 42653872\,x^{49} - 304882324\,x^{48} - 1838251648\,x^{47} - 9541097376\,x^{46} \\
& - 43292005064\,x^{45} - 173809129296\,x^{44} - 623433636192\,x^{43} \\
& - 2013652802724\,x^{42} - 5895066281808\,x^{41} - 15727966768512\,x^{40} \\
& - 38418079165536\,x^{39} - 86253113637432\,x^{38} - 178581194116464\,x^{37} \\
& - 341941894427468\,x^{36} - 606983560940352\,x^{35} - 1000931328276832\,x^{34} \\
& - 1535996305316104\,x^{33} - 2196677397430888\,x^{32} - 2931243231236096\,x^{31} \\
& - 3653110574321744\,x^{30} - 4255188039424496\,x^{29} - 4634982792069072\,x^{28} \\
& - 4722715997715584\,x^{27} - 4501998664289208\,x^{26} - 4014759878194112\,x^{25} \\
& - 3348421387770032\,x^{24} - 2610630613406144\,x^{23} - 1901443485039388\,x^{22} \\
& - 1292606637887056\,x^{21} - 819226783178584\,x^{20} - 483387135146736\,x^{19} \\
& - 265102956987872\,x^{18} - 134865280903584\,x^{17} - 63494029748628\,x^{16} \\
& - 27587567332992\,x^{15} - 11026340994096\,x^{14} - 4038557378952\,x^{13} \\
& - 1349383115904\,x^{12} - 409096076448\,x^{11} - 111814981572\,x^{10} \\
& - 27338282672\,x^{9} - 5922107184\,x^{8} - 1123073248\,x^{7} - 183612016\,x^{6} \\
& - 25360624\,x^{5} - 2878028\,x^{4} - 257696\,x^{3} - 17072\,x^{2} - 744\,x - 16
\end{aligned}$$

$$
\begin{aligned}
p_2(2,x) = {} & 406\,x^{54} - 378\,x^{52} - 16848\,x^{51} - 369616\,x^{50} - 5320200\,x^{49} - 56508504\,x^{48} \\
& - 472290368\,x^{47} - 3234482832\,x^{46} - 18663263808\,x^{45} - 92587732656\,x^{44} \\
& - 401025338136\,x^{43} - 1534814173584\,x^{42} - 5240536649232\,x^{41} \\
& - 16088966185698\,x^{40} - 44701048725408\,x^{39} - 113001811663104\,x^{38} \\
& - 261096128092944\,x^{37} - 553513557177684\,x^{36} - 1080144207726576\,x^{35} \\
& - 1945629247540788\,x^{34} - 3242492839479456\,x^{33} - 5009498690757488\,x^{32} \\
& - 7186540248686400\,x^{31} - 9586113494260800\,x^{30} - 11902357736845696\,x^{29} \\
& - 13767470871920664\,x^{28} - 14844574782711696\,x^{27} \\
& - 14925831498157392\,x^{26} - 13997062582641216\,x^{25} \\
& - 12241649522840220\,x^{24} - 9982302704447104\,x^{23} - 7585708020467808\,x^{22} \\
& - 5368125516474912\,x^{21} - 3534196297839578\,x^{20} - 2162051360712336\,x^{19} \\
& - 1227126259558536\,x^{18} - 645002792151144\,x^{17} - 313275735343152\,x^{16} \\
& - 140232191200128\,x^{15} - 57673558999728\,x^{14} - 21712833146304\,x^{13} \\
& - 7450215710184\,x^{12} - 2317751447064\,x^{11} - 649654298496\,x^{10} \\
& - 162816141168\,x^{9} - 36142821906\,x^{8} - 7023015648\,x^{7} - 1176556320\,x^{6} \\
& - 166560080\,x^{5} - 19381416\,x^{4} - 1780464\,x^{3} - 121108\,x^{2} - 5424\,x - 120
\end{aligned}
$$

$$
\begin{aligned}
p_3(2,x) = {} & 3654\,x^{52} - 3276\,x^{50} - 140400\,x^{49} - 2958800\,x^{48} - 40868800\,x^{47} \\
& - 416104896\,x^{46} - 3329827392\,x^{45} - 21807790128\,x^{44} - 120179577216\,x^{43} \\
& - 568651120992\,x^{42} - 2345833520040\,x^{41} - 8538222227664\,x^{40} \\
& - 27681914566464\,x^{39} - 80565268674648\,x^{38} - 211833511178016\,x^{37} \\
& - 505875335718144\,x^{36} - 1102118377834080\,x^{35} - 2198757663408504\,x^{34} \\
& - 4029626476269648\,x^{33} - 6802186653062676\,x^{32} - 10599868889882112\,x^{31} \\
& - 15276693222635776\,x^{30} - 20393818009226048\,x^{29} \\
& - 25248748889520576\,x^{28} - 29017767697193472\,x^{27} \\
& - 30979052598346992\,x^{26} - 30735567777694320\,x^{25} \\
& - 28344399282585360\,x^{24} - 24295522226699520\,x^{23} \\
& - 19350562475294640\,x^{22} - 14313219522252416\,x^{21} \\
& - 9824561853879712\,x^{20} - 6251113579378368\,x^{19} - 3681939407469996\,x^{18} \\
& - 2004171328063152\,x^{17} - 1006087536982920\,x^{16} - 464622775239744\,x^{15} \\
& - 196805945207424\,x^{14} - 76191776208768\,x^{13} - 26845348231056\,x^{12} \\
& - 8564706593664\,x^{11} - 2459070814800\,x^{10} - 630654159144\,x^{9} \\
& - 143137024128\,x^{8} - 28418137536\,x^{7} - 4861995192\,x^{6} - 702716256\,x^{5} \\
& - 83479008\,x^{4} - 7830496\,x^{3} - 544112\,x^{2} - 24912\,x - 564
\end{aligned}
$$

$$
\begin{aligned}
p_4(2,x) = {} & 23751\,x^{50} - 20475\,x^{48} - 842400\,x^{47} - 17024800\,x^{46} - 225262000\,x^{45} \\
& - 2194385104\,x^{44} - 16780279648\,x^{43} - 104876579896\,x^{42} \\
& - 550777992352\,x^{41} - 2479861879864\,x^{40} - 9719350702792\,x^{39} \\
& - 33554750721904\,x^{38} - 103010482975888\,x^{37} - 283365634300426\,x^{36} \\
& - 702886138898512\,x^{35} - 1580381389278592\,x^{34} - 3234968825592664\,x^{33} \\
& - 6050540265106078\,x^{32} - 10371977340026176\,x^{31} - 16337323227028912\,x^{30} \\
& - 23695783660015744\,x^{29} - 31701780744977088\,x^{28} \\
& - 39175990030243296\,x^{27} - 44765556317741472\,x^{26} \\
& - 47336004493341504\,x^{25} - 46342251206172780\,x^{24} \\
& - 42014292764865360\,x^{23} - 35272105723882320\,x^{22} \\
& - 27412118951951040\,x^{21} - 19709347480425780\,x^{20} \\
& - 13098899429698624\,x^{19} - 8037273995655088\,x^{18} - 4545944863899376\,x^{17} \\
& - 2365625076777187\,x^{16} - 1129931273682784\,x^{15} - 493975003263472\,x^{14} \\
& - 196976387408464\,x^{13} - 71348753610208\,x^{12} - 23359274516416\,x^{11} \\
& - 6870861639592\,x^{10} - 1802331001312\,x^{9} - 417788217604\,x^{8} \\
& - 84600010408\,x^{7} - 14744427136\,x^{6} - 2168541232\,x^{5} \\
& - 261912274\,x^{4} - 24961648\,x^{3} - 1761616\,x^{2} - 81912\,x - 1884
\end{aligned}
$$

$$
\begin{aligned}
p_5(2,x) = {} & 118755\,x^{48} - 98280\,x^{46} - 3875040\,x^{45} - 74965280\,x^{44} - 948324960\,x^{43} \\
& - 8820842976\,x^{42} - 64317109472\,x^{41} - 382738553736\,x^{40} \\
& - 1910847985536\,x^{39} - 8165721101792\,x^{38} - 30323155469304\,x^{37} \\
& - 99008744467632\,x^{36} - 286915466049248\,x^{35} - 743532606379476\,x^{34} \\
& - 1733805218749392\,x^{33} - 3656576664927104\,x^{32} - 7004314209941376\,x^{31} \\
& - 12229400770082400\,x^{30} - 19519331043108800\,x^{29} \\
& - 28549175000159280\,x^{28} - 38339401961515776\,x^{27} \\
& - 47347991317923456\,x^{26} - 53837715188981088\,x^{25} \\
& - 56413044169607520\,x^{24} - 54503172883799040\,x^{23} \\
& - 48564828236258640\,x^{22} - 39907706149134000\,x^{21} \\
& - 30232261429706640\,x^{20} - 21099407718683520\,x^{19} \\
& - 13552438575351240\,x^{18} - 8000508288984896\,x^{17} \\
& - 4333151584516752\,x^{16} - 2148405302825856\,x^{15} - 972462807932216\,x^{14} \\
& - 400529119525152\,x^{13} - 149503542181104\,x^{12} - 50327231153312\,x^{11} \\
& - 15188055843648\,x^{10} - 4079172175296\,x^{9} - 966196279832\,x^{8} \\
& - 199525509504\,x^{7} - 35394916176\,x^{6} - 5288710040\,x^{5} \\
& - 647756160\,x^{4} - 62493600\,x^{3} - 4457156\,x^{2} - 209136\,x - 4848
\end{aligned}
$$

$$
\begin{aligned}
p_6(2,x) = {} & 475020\,x^{46} - 376740\,x^{44} - 14208480\,x^{43} - 262593760\,x^{42} - 3169239920\,x^{41} \\
& - 28084347984\,x^{40} - 194796058688\,x^{39} - 1100924332816\,x^{38} \\
& - 5211262953792\,x^{37} - 21076050929904\,x^{36} - 73929518988648\,x^{35} \\
& - 227556747314928\,x^{34} - 620322733172656\,x^{33} - 1508811822967142\,x^{32} \\
& - 3294403829376384\,x^{31} - 6489427841911808\,x^{30} - 11579947280800960\,x^{29} \\
& - 18782176010857200\,x^{28} - 27766933676218560\,x^{27} \\
& - 37499711219899920\,x^{26} - 46346865231003264\,x^{25} \\
& - 52492721706548160\,x^{24} - 54537068893149120\,x^{23} \\
& - 52007944666069440\,x^{22} - 45535809212158080\,x^{21} \\
& - 36602684836665000\,x^{20} - 27000337621674960\,x^{19} \\
& - 18263530119643920\,x^{18} - 11315027911947840\,x^{17} \\
& - 6410569187457180\,x^{16} - 3314519622962944\,x^{15} - 1559952954955328\,x^{14} \\
& - 666173542120896\,x^{13} - 257129216553452\,x^{12} - 89274155624224\,x^{11} \\
& - 27717929547408\,x^{10} - 7640272566288\,x^{9} - 1852862244576\,x^{8} \\
& - 390831887232\,x^{7} - 70650740464\,x^{6} - 10731558848\,x^{5} \\
& - 1332843240\,x^{4} - 130052360\,x^{3} - 9354880\,x^{2} - 441360\,x - 10254
\end{aligned}
$$

$$
\begin{aligned}
p_7(2,x) = {} & 1560780\,x^{44} - 1184040\,x^{42} - 42625440\,x^{41} - 750944480\,x^{40} - 8626692800\,x^{39} \\
& - 72650638016\,x^{38} - 478096990272\,x^{37} - 2559068989488\,x^{36} \\
& - 11450785245312\,x^{35} - 43689542839584\,x^{34} - 144270421526232\,x^{33} \\
& - 417099871392432\,x^{32} - 1065419558657792\,x^{31} - 2422096297783328\,x^{30} \\
& - 4929727965094784\,x^{29} - 9026318690343936\,x^{28} - 14926734988118400\,x^{27} \\
& - 22365382452263904\,x^{26} - 30441423800256576\,x^{25} \\
& - 37715071722998160\,x^{24} - 42599142101045760\,x^{23} \\
& - 43914308280426240\,x^{22} - 41346121452649920\,x^{21} \\
& - 35564692549001280\,x^{20} - 27946064928468480\,x^{19} \\
& - 20050648739387760\,x^{18} - 13123532717686320\,x^{17} \\
& - 7825333344569040\,x^{16} - 4243144064570880\,x^{15} - 2087229195219360\,x^{14} \\
& - 928624441607936\,x^{13} - 372275850141632\,x^{12} - 133851318946688\,x^{11} \\
& - 42915311001912\,x^{10} - 12182217239904\,x^{9} - 3034378457232\,x^{8} \\
& - 655677982272\,x^{7} - 121105990272\,x^{6} - 18746802048\,x^{5} \\
& - 2366438864\,x^{4} - 234022016\,x^{3} - 17007920\,x^{2} - 807864\,x - 18816
\end{aligned}
$$

$$
\begin{aligned}
p_8(2,x) = {} & 4292145\,x^{42} - 3108105\,x^{40} - 106563600\,x^{39} - 1785269200\,x^{38} \\
& - 19471234200\,x^{37} - 155413732584\,x^{36} - 967525194768\,x^{35} \\
& - 4889513933172\,x^{34} - 20612969414064\,x^{33} - 73931393744868\,x^{32} \\
& - 228947904173232\,x^{31} - 619160723943840\,x^{30} - 1475416077725152\,x^{29} \\
& - 3120102276971244\,x^{28} - 5889237463321248\,x^{27} - 9967757034317568\,x^{26} \\
& - 15184662238233072\,x^{25} - 20882230061445900\,x^{24} \\
& - 25985308732307040\,x^{23} - 29310801578182440\,x^{22} \\
& - 30007094349862080\,x^{21} - 27903452304329760\,x^{20} \\
& - 23576175895889520\,x^{19} - 18097504619850000\,x^{18} \\
& - 12613732641812640\,x^{17} - 7974238489240590\,x^{16} - 4565320958787120\,x^{15} \\
& - 2361868827745200\,x^{14} - 1101092558437440\,x^{13} - 460918775750940\,x^{12} \\
& - 172468116829504\,x^{11} - 57364172506608\,x^{10} - 16841072758512\,x^{9} \\
& - 4325637406179\,x^{8} - 961107642288\,x^{7} - 182031865272\,x^{6} - 28816220040\,x^{5} \\
& - 3710062512\,x^{4} - 373228704\,x^{3} - 27519628\,x^{2} - 1322448\,x - 31062
\end{aligned}
$$

$$
\begin{aligned}
p_9(2,x) = {} & 10015005\,x^{40} - 6906900\,x^{38} - 224967600\,x^{37} - 3574485200\,x^{36} \\
& - 36908146000\,x^{35} - 278355526416\,x^{34} - 1634017821392\,x^{33} \\
& - 7769320470604\,x^{32} - 30743448922368\,x^{31} - 103237334067776\,x^{30} \\
& - 298515507520976\,x^{29} - 751628607988512\,x^{28} - 1662445392417984\,x^{27} \\
& - 3252433745636488\,x^{26} - 5659570924484256\,x^{25} - 8797959149208320\,x^{24} \\
& - 12260659964288320\,x^{23} - 15358651666137360\,x^{22} \\
& - 17329350062124320\,x^{21} - 17636972617645800\,x^{20} \\
& - 16205465607123840\,x^{19} - 13447750390900800\,x^{18} \\
& - 10076700471036080\,x^{17} - 6813462358658160\,x^{16} \\
& - 4151985663784960\,x^{15} - 2276017277579120\,x^{14} - 1119497706375120\,x^{13} \\
& - 492438300430960\,x^{12} - 192888803253120\,x^{11} - 66917191341000\,x^{10} \\
& - 20420686423616\,x^{9} - 5434105276112\,x^{8} - 1246989138624\,x^{7} \\
& - 243187796908\,x^{6} - 39525623696\,x^{5} - 5210379576\,x^{4} \\
& - 535274992\,x^{3} - 40208864\,x^{2} - 1964448\,x - 46836
\end{aligned}
$$

$$
\begin{aligned}
p_{10}(2,x) = {} & 20030010\,x^{38} - 13123110\,x^{36} - 404941680\,x^{35} - 6084123760\,x^{34} \\
& - 59285500120\,x^{33} - 421047014344\,x^{32} - 2322116643968\,x^{31} \\
& - 10347223852064\,x^{30} - 38268769238144\,x^{29} - 119765782025568\,x^{28} \\
& - 321758336428528\,x^{27} - 750239333618592\,x^{26} - 1531232891821984\,x^{25} \\
& - 2753929442606836\,x^{24} - 4387451500798912\,x^{23} - 6217275547997696\,x^{22} \\
& - 7861140060886112\,x^{21} - 8889715717975928\,x^{20} - 9005718283931424\,x^{19} \\
& - 8181021300513368\,x^{18} - 6666846701763264\,x^{17} - 4872606021394208\,x^{16} \\
& - 3191296500731072\,x^{15} - 1870214562485696\,x^{14} - 978531471517312\,x^{13} \\
& - 455723653716136\,x^{12} - 188164331188912\,x^{11} - 68523685037808\,x^{10} \\
& - 21864465498304\,x^{9} - 6061037530548\,x^{8} - 1443779074688\,x^{7} \\
& - 291301978976\,x^{6} - 48826550816\,x^{5} - 6617530330\,x^{4} \\
& - 696930640\,x^{3} - 53523304\,x^{2} - 2667144\,x - 64752
\end{aligned}
$$

$$
\begin{aligned}
p_{11}(2,x) = {} & 34597290\,x^{36} - 21474180\,x^{34} - 625818960\,x^{33} - 8861905520\,x^{32} \\
& - 81203130240\,x^{31} - 540995707776\,x^{30} - 2791555272832\,x^{29} \\
& - 11605282391904\,x^{28} - 39922816620288\,x^{27} - 115831574149056\,x^{26} \\
& - 287477730268560\,x^{25} - 616877015627808\,x^{24} - 1153932923751296\,x^{23} \\
& - 1893695872438032\,x^{22} - 2739772976551104\,x^{21} - 3507590896016896\,x^{20} \\
& - 3984565727867712\,x^{19} - 4023909996309264\,x^{18} - 3616557786488928\,x^{17} \\
& - 2893979148880536\,x^{16} - 2061180859175424\,x^{15} - 1305308132593408\,x^{14} \\
& - 733681285899456\,x^{13} - 365045910545472\,x^{12} - 160195062470144\,x^{11} \\
& - 61703522911056\,x^{10} - 20729349290832\,x^{9} - 6024358442544\,x^{8} \\
& - 1498414963968\,x^{7} - 314482190256\,x^{6} - 54635889280\,x^{5} \\
& - 7649208480\,x^{4} - 829464000\,x^{3} - 65381660\,x^{2} - 3333360\,x - 82536
\end{aligned}
$$

$$
\begin{aligned}
p_{12}(2,x) = {} & 51895935\,x^{34} - 30421755\,x^{32} - 834425280\,x^{31} - 11094765760\,x^{30} \\
& - 95216007200\,x^{29} - 592485453024\,x^{28} - 2846932108608\,x^{27} \\
& - 10985659869072\,x^{26} - 34954849745088\,x^{25} - 93450055020816\,x^{24} \\
& - 212832173108880\,x^{23} - 417232361056992\,x^{22} - 709594820475424\,x^{21} \\
& - 1053216717427316\,x^{20} - 1370352475693344\,x^{19} \\
& - 1568029267234560\,x^{18} - 1581361484387568\,x^{17} - 1407377779905036\,x^{16} \\
& - 1105782162713664\,x^{15} - 766698151505136\,x^{14} - 468514612583040\,x^{13} \\
& - 251777236548032\,x^{12} - 118605735081184\,x^{11} - 48759689206176\,x^{10} \\
& - 17389997661504\,x^{9} - 5338355400300\,x^{8} - 1395961043568\,x^{7} \\
& - 306674065776\,x^{6} - 55542087744\,x^{5} - 8075533596\,x^{4} \\
& - 906198464\,x^{3} - 73669840\,x^{2} - 3860496\,x - 97869
\end{aligned}
$$

$$
\begin{aligned}
p_{13}(2,x) = {} & 67863915\,x^{32} - 37442160\,x^{30} - 962798400\,x^{29} - 11969604800\,x^{28} \\
& - 95768724800\,x^{27} - 553821133376\,x^{26} - 2464629349952\,x^{25} \\
& - 8775180593648\,x^{24} - 25657947236096\,x^{23} - 62755365935552\,x^{22} \\
& - 130126525846256\,x^{21} - 231033684957536\,x^{20} - 353817811832512\,x^{19} \\
& - 469928051102344\,x^{18} - 543382741270048\,x^{17} - 548422490331904\,x^{16} \\
& - 483817660319104\,x^{15} - 373232846438368\,x^{14} - 251624663379136\,x^{13} \\
& - 148014783018352\,x^{12} - 75763777743616\,x^{11} - 33612503275136\,x^{10} \\
& - 12853987037792\,x^{9} - 4205797046624\,x^{8} - 1165738400768\,x^{7} \\
& - 270050599664\,x^{6} - 51326524304\,x^{5} - 7796656688\,x^{4} \\
& - 910353536\,x^{3} - 76726712\,x^{2} - 4155072\,x - 108528
\end{aligned}
$$

$$
\begin{aligned}
p_{14}(2,x) = {} & 77558760\,x^{30} - 40116600\,x^{28} - 962798400\,x^{27} - 11137556800\,x^{26} \\
& - 82640196000\,x^{25} - 441575718624\,x^{24} - 1808491036288\,x^{23} \\
& - 5899932997536\,x^{22} - 15730988606592\,x^{21} - 34901624213344\,x^{20} \\
& - 65270650362576\,x^{19} - 103856956089312\,x^{18} - 141554702340704\,x^{17} \\
& - 166045819530348\,x^{16} - 168140910139776\,x^{15} - 147222959934464\,x^{14} \\
& - 111506144941248\,x^{13} - 72993225786096\,x^{12} - 41212796076992\,x^{11} \\
& - 20001871347024\,x^{10} - 8303142765696\,x^{9} - 2927848146112\,x^{8} \\
& - 868725647424\,x^{7} - 214098643008\,x^{6} - 43043610496\,x^{5} \\
& - 6879644184\,x^{4} - 841045488\,x^{3} - 73886896\,x^{2} - 4155072\,x - 112404
\end{aligned}
$$

$$
\begin{aligned}
p_{15}(2,x) = {} & 77558760\,x^{28} - 37442160\,x^{26} - 834425280\,x^{25} \\
& - 8931440960\,x^{24} - 61081832320\,x^{23} - 299540132736\,x^{22} \\
& - 1120593688192\,x^{21} - 3322050565856\,x^{20} - 8003072768256\,x^{19} \\
& - 15942029984832\,x^{18} - 26581420014000\,x^{17} - 37419238979424\,x^{16} \\
& - 44734973112064\,x^{15} - 45588290150432\,x^{14} - 39678055964544\,x^{13} \\
& - 29504172858368\,x^{12} - 18721654557568\,x^{11} - 10110262927968\,x^{10} \\
& - 4626397382976\,x^{9} - 1782505624272\,x^{8} - 573154352640\,x^{7} \\
& - 151915964672\,x^{6} - 32619155008\,x^{5} - 5532757440\,x^{4} \\
& - 713597440\,x^{3} - 65773136\,x^{2} - 3860496\,x - 108528
\end{aligned}
$$

$$
\begin{aligned}
p_{16}(2,x) = \;& 67863915\,x^{26} - 30421755\,x^{24} - 625818960\,x^{23} - 6157749520\,x^{22} \\
& - 38535411640\,x^{21} - 172045952584\,x^{20} - 582685235088\,x^{19} \\
& - 1554085227156\,x^{18} - 3344961999792\,x^{17} - 5907212098404\,x^{16} \\
& - 8656992441432\,x^{15} - 10607846552016\,x^{14} - 10919268071792\,x^{13} \\
& - 9463991815334\,x^{12} - 6908558063888\,x^{11} - 4240500652416\,x^{10} \\
& - 2180746275288\,x^{9} - 934251983646\,x^{8} - 330662071632\,x^{7} \\
& - 95559132684\,x^{6} - 22176580512\,x^{5} - 4033546480\,x^{4} \\
& - 553945000\,x^{3} - 54018520\,x^{2} - 3333360\,x - 97869
\end{aligned}
$$

$$
\begin{aligned}
p_{17}(2,x) = \;& 51895935\,x^{24} - 21474180\,x^{22} - 404941680\,x^{21} - 3634476560\,x^{20} \\
& - 20633566800\,x^{19} - 83060576016\,x^{18} - 251912257872\,x^{17} \\
& - 597072038508\,x^{16} - 1132249145472\,x^{15} - 1744702185504\,x^{14} \\
& - 2206640677608\,x^{13} - 2304668667024\,x^{12} - 1993495276000\,x^{11} \\
& - 1428367241940\,x^{10} - 845822077968\,x^{9} - 411928825728\,x^{8} \\
& - 163728374112\,x^{7} - 52518391416\,x^{6} - 13378535664\,x^{5} \\
& - 2643742764\,x^{4} - 390908736\,x^{3} - 40726880\,x^{2} - 2667144\,x - 82536
\end{aligned}
$$

$$
\begin{aligned}
p_{18}(2,x) = \;& 34597290\,x^{22} - 13123110\,x^{20} - 224967600\,x^{19} - 1824737200\,x^{18} \\
& - 9299622200\,x^{17} - 33354388584\,x^{16} - 89368975168\,x^{15} \\
& - 185338835792\,x^{14} - 304182252864\,x^{13} - 400625705968\,x^{12} \\
& - 426914649976\,x^{11} - 369489056208\,x^{10} - 259770623824\,x^{9} \\
& - 147878887498\,x^{8} - 67708902816\,x^{7} - 24671156480\,x^{6} - 7042404112\,x^{5} \\
& - 1538988372\,x^{4} - 248568944\,x^{3} - 27987988\,x^{2} - 1964448\,x - 64752
\end{aligned}
$$

$$
\begin{aligned}
p_{19}(2,x) = \;& 20030010\,x^{20} - 6906900\,x^{18} - 106563600\,x^{17} - 772257200\,x^{16} \\
& - 3487352000\,x^{15} - 10978548416\,x^{14} - 25541276992\,x^{13} \\
& - 45424415344\,x^{12} - 63021551488\,x^{11} - 69004014816\,x^{10} - 59944824392\,x^{9} \\
& - 41324722704\,x^{8} - 22506188608\,x^{7} - 9593192536\,x^{6} - 3152288032\,x^{5} \\
& - 780436736\,x^{4} - 140554592\,x^{3} - 17380664\,x^{2} - 1322448\,x - 46836
\end{aligned}
$$

$$
\begin{aligned}
p_{20}(2,x) = \;& 10015005\,x^{18} - 3108105\,x^{16} - 42625440\,x^{15} - 272066080\,x^{14} \\
& - 1070675760\,x^{13} - 2901578064\,x^{12} - 5728915808\,x^{11} \\
& - 8503965624\,x^{10} - 9655737504\,x^{9} - 8451999288\,x^{8} \\
& - 5706102312\,x^{7} - 2952661680\,x^{6} - 1155506768\,x^{5} \\
& - 334457250\,x^{4} - 69114000\,x^{3} - 9618304\,x^{2} - 807864\,x - 31062
\end{aligned}
$$

$$
\begin{aligned}
p_{21}(2,x) = \;& 4292145\,x^{16} - 1184040\,x^{14} - 14208480\,x^{13} - 78409760\,x^{12} \\
& - 263079520\,x^{11} - 597897696\,x^{10} - 970767072\,x^{9} \\
& - 1157460744\,x^{8} - 1025931648\,x^{7} - 676769376\,x^{6} - 329346456\,x^{5} \\
& - 115929840\,x^{4} - 28516960\,x^{3} - 4621220\,x^{2} - 441360\,x - 18816
\end{aligned}
$$

$$
\begin{aligned}
p_{22}(2,x) = \;& 1560780\,x^{14} - 376740\,x^{12} - 3875040\,x^{11} - 18035680\,x^{10} - 50057200\,x^{9} \\
& - 91927504\,x^{8} - 117199168\,x^{7} - 105888592\,x^{6} - 67981888\,x^{5} \\
& - 30621616\,x^{4} - 9386824\,x^{3} - 1848880\,x^{2} - 209136\,x - 10254
\end{aligned}
$$

$$
\begin{aligned}
p_{23}(2,x) = {} & 475020\,x^{12} - 98280\,x^{10} - 842400\,x^{9} - 3192800\,x^{8} - 7012800\,x^{7} - 9832896\,x^{6} \\
& - 9142592\,x^{5} - 5672688\,x^{4} - 2303616\,x^{3} - 581792\,x^{2} - 81912\,x - 4848
\end{aligned}
$$

$$
\begin{aligned}
p_{24}(2,x) = {} & 118755\,x^{10} - 20475\,x^{8} - 140400\,x^{7} - 410800\,x^{6} - 665000\,x^{5} \\
& - 646104\,x^{4} - 382448\,x^{3} - 133516\,x^{2} - 24912\,x - 1884
\end{aligned}
$$

$$p_{25}(2,x) = 23751\,x^{8} - 3276\,x^{6} - 16848\,x^{5} - 34736\,x^{4} - 36272\,x^{3} - 19952\,x^{2} - 5424\,x - 564$$

$$p_{26}(2,x) = 3654\,x^{6} - 378\,x^{4} - 1296\,x^{3} - 1552\,x^{2} - 744\,x - 120$$

$$p_{27}(2,x) = 406\,x^{4} - 28\,x^{2} - 48\,x - 16$$

$$p_{28}(2,x) = 29\,x^{2} - 1$$

$$p_{29}(2,x) = 1$$

Definition of $\overleftarrow{P_3}(x,y)$

For the case of base $b = 3$, we have the numbers $e_1 = 0$, $e_3 = 15$, $e_4 = 2$, $e_6 = 0$, and $d = 3$. In this case, $\overleftarrow{P_3}(x,y)$ is of degree 38 in y. We compute each $p_j(3,x)$ directly:

$$
\begin{aligned}
p_0(3,x) = {} & x^{76} - x^{68} - 30\,x^{67} - 469\,x^{66} - 5050\,x^{65} - 41886\,x^{64} - 284206\,x^{63} - 1637669\,x^{62} \\
& - 8220054\,x^{61} - 36603776\,x^{60} - 146608486\,x^{59} - 533853171\,x^{58} \\
& - 1782456754\,x^{57} - 5494958599\,x^{56} - 15730773456\,x^{55} - 42020795956\,x^{54} \\
& - 105166278600\,x^{53} - 247457064699\,x^{52} - 549082890330\,x^{51} \\
& - 1151915516271\,x^{50} - 2289982771350\,x^{49} - 4322480753340\,x^{48} \\
& - 7760277543150\,x^{47} - 13271700338685\,x^{46} - 21650201225550\,x^{45} \\
& - 33728167129215\,x^{44} - 50230147187376\,x^{43} - 71575588941690\,x^{42} \\
& - 97662903707344\,x^{41} - 127686764332515\,x^{40} - 160050261273286\,x^{39} \\
& - 192425270319841\,x^{38} - 221985461550294\,x^{37} - 245791863305269\,x^{36} \\
& - 261263375603276\,x^{35} - 266630465571096\,x^{34} - 261263375603276\,x^{33} \\
& - 245791863305269\,x^{32} - 221985461550294\,x^{31} - 192425270319841\,x^{30} \\
& - 160050261273286\,x^{29} - 127686764332515\,x^{28} - 97662903707344\,x^{27} \\
& - 71575588941690\,x^{26} - 50230147187376\,x^{25} - 33728167129215\,x^{24} \\
& - 21650201225550\,x^{23} - 13271700338685\,x^{22} - 7760277543150\,x^{21} \\
& - 4322480753340\,x^{20} - 2289982771350\,x^{19} - 1151915516271\,x^{18} \\
& - 549082890330\,x^{17} - 247457064699\,x^{16} - 105166278600\,x^{15} \\
& - 42020795956\,x^{14} - 15730773456\,x^{13} - 5494958599\,x^{12} - 1782456754\,x^{11} \\
& - 533853171\,x^{10} - 146608486\,x^{9} - 36603776\,x^{8} - 8220054\,x^{7} \\
& - 1637669\,x^{6} - 284206\,x^{5} - 41886\,x^{4} - 5050\,x^{3} - 469\,x^{2} - 30\,x - 1
\end{aligned}
$$

$$
\begin{aligned}
p_1(3,x) = {} & 38\,x^{74} - 34\,x^{66} - 990\,x^{65} - 14989\,x^{64} - 156040\,x^{63} - 1249436\,x^{62} \\
& - 8173114\,x^{61} - 45344751\,x^{60} - 218863724\,x^{59} - 936007780\,x^{58} \\
& - 3596011526\,x^{57} - 12544073759\,x^{56} - 40070544216\,x^{55} - 118026481124\,x^{54} \\
& - 322384910640\,x^{53} - 820498491876\,x^{52} - 1953591625440\,x^{51} \\
& - 4366449861126\,x^{50} - 9188299623450\,x^{49} - 18249471693975\,x^{48} \\
& - 34286350813800\,x^{47} - 61047497953860\,x^{46} - 103180939642650\,x^{45} \\
& - 165778643312535\,x^{44} - 253502115096300\,x^{43} - 369323898934590\,x^{42} \\
& - 513078988958184\,x^{41} - 680181306831330\,x^{40} - 860953681203064\,x^{39} \\
& - 1040977850661840\,x^{38} - 1202663781626986\,x^{37} - 1327899542097679\,x^{36} \\
& - 1401275358127076\,x^{35} - 1413115051623214\,x^{34} - 1361511300066180\,x^{33} \\
& - 1252788657573536\,x^{32} - 1100247924462904\,x^{31} - 921531325012676\,x^{30} \\
& - 735318973476194\,x^{29} - 558198460818315\,x^{28} - 402412475260556\,x^{27} \\
& - 274857264666750\,x^{26} - 177313355251656\,x^{25} - 107576595297810\,x^{24} \\
& - 61007664271800\,x^{23} - 32042227642440\,x^{22} - 15349901615250\,x^{21} \\
& - 6518239248315\,x^{20} - 2297331581700\,x^{19} - 532767407100\,x^{18} \\
& + 63390756450\,x^{17} + 181176566361\,x^{16} + 146109512160\,x^{15} \\
& + 87777303456\,x^{14} + 44567667960\,x^{13} + 19917427244\,x^{12} + 7961331936\,x^{11} \\
& + 2862608054\,x^{10} + 925961126\,x^{9} + 268402345\,x^{8} + 69200624\,x^{7} \\
& + 15690396\,x^{6} + 3077734\,x^{5} + 509981\,x^{4} + 68860\,x^{3} + 7144\,x^{2} + 510\,x + 19
\end{aligned}
$$

$$
\begin{aligned}
p_2(3,x) = {} & 703\,x^{72} - 561\,x^{64} - 15840\,x^{63} - 232016\,x^{62} - 2332440\,x^{61} - 18006276\,x^{60} \\
& - 113395166\,x^{59} - 604815645\,x^{58} - 2802627066\,x^{57} - 11491513460\,x^{56} \\
& - 42270025956\,x^{55} - 140981497026\,x^{54} - 429977371380\,x^{53} \\
& - 1207435127526\,x^{52} - 3139561671120\,x^{51} - 7594576114956\,x^{50} \\
& - 17158712909400\,x^{49} - 36330159329775\,x^{48} - 72292001083800\,x^{47} \\
& - 135522728597700\,x^{46} - 239849390721600\,x^{45} - 401461294998540\,x^{44} \\
& - 636482280385350\,x^{43} - 957025695965265\,x^{42} - 1366220741302650\,x^{41} \\
& - 1853359402900365\,x^{40} - 2390801319265536\,x^{39} - 2934284890489140\,x^{38} \\
& - 3427637821244616\,x^{37} - 3811629581253990\,x^{36} - 4035237670531826\,x^{35} \\
& - 4066480093890051\,x^{34} - 3899722853233470\,x^{33} - 3557218267950401\,x^{32} \\
& - 3084329325840960\,x^{31} - 2539827457030416\,x^{30} - 1984081401378056\,x^{29} \\
& - 1468363754619780\,x^{28} - 1027809548154006\,x^{27} - 679134628697745\,x^{26} \\
& - 422672681496186\,x^{25} - 247184528265765\,x^{24} - 135523356628800\,x^{23} \\
& - 69555637739340\,x^{22} - 33447042073800\,x^{21} - 15169605661290\,x^{20} \\
& - 6610756129650\,x^{19} - 2874602632275\,x^{18} - 1315321457550\,x^{17} \\
& - 656802745200\,x^{16} - 352061076600\,x^{15} - 191691706956\,x^{14} \\
& - 100589705640\,x^{13} - 49219893036\,x^{12} - 22064960040\,x^{11} - 8973428196\,x^{10} \\
& - 3287216016\,x^{9} - 1077068345\,x^{8} - 312918516\,x^{7} - 79678110\,x^{6} \\
& - 17495756\,x^{5} - 3236556\,x^{4} - 486930\,x^{3} - 56231\,x^{2} - 4470\,x - 186
\end{aligned}
$$

$$
\begin{aligned}
p_3(3,x) = {} & 8436\,x^{70} - 5984\,x^{62} - 163680\,x^{61} - 2316816\,x^{60} - 22462600\,x^{59} \\
& - 166955100\,x^{58} - 1010661194\,x^{57} - 5173804055\,x^{56} - 22976493816\,x^{55} \\
& - 90154288920\,x^{54} - 316874387220\,x^{53} - 1008338096034\,x^{52} \\
& - 2929592273400\,x^{51} - 7824393977556\,x^{50} - 19318116691440\,x^{49} \\
& - 44295919063260\,x^{48} - 94696343606400\,x^{47} - 189361962676440\,x^{46} \\
& - 355174826418360\,x^{45} - 626321126716500\,x^{44} - 1040436957862440\,x^{43} \\
& - 1630874118191220\,x^{42} - 2415534378576450\,x^{41} - 3384455905040355\,x^{40} \\
& - 4489979122776360\,x^{39} - 5643987280241700\,x^{38} - 6725661949407024\,x^{37} \\
& - 7600295509191540\,x^{36} - 8145756427077416\,x^{35} - 8279750879202840\,x^{34} \\
& - 7979705548619310\,x^{33} - 7288804151407909\,x^{32} - 6306021638970720\,x^{31} \\
& - 5163257694713936\,x^{30} - 3996731559456320\,x^{29} - 2921065549595760\,x^{28} \\
& - 2012650158078264\,x^{27} - 1304998860461700\,x^{26} - 794650612529826\,x^{25} \\
& - 453373790427675\,x^{24} - 241714775799960\,x^{23} - 120049452956700\,x^{22} \\
& - 55310840634000\,x^{21} - 23472758203500\,x^{20} - 9030293218440\,x^{19} \\
& - 3010924213200\,x^{18} - 733014295710\,x^{17} + 17191446435\,x^{16} \\
& + 196041085200\,x^{15} + 185948503440\,x^{14} + 130089823560\,x^{13} \\
& + 77441743236\,x^{12} + 40670348160\,x^{11} + 19032763824\,x^{10} + 7943376600\,x^{9} \\
& + 2945714140\,x^{8} + 963834816\,x^{7} + 275308020\,x^{6} + 67598924\,x^{5} \\
& + 13949490\,x^{4} + 2337280\,x^{3} + 300436\,x^{2} + 26610\,x + 1239
\end{aligned}
$$

$$
\begin{aligned}
p_4(3,x) = {} & 73815\,x^{68} - 46376\,x^{60} - 1227600\,x^{59} - 16771000\,x^{58} - 156605800\,x^{57} \\
& - 1118962100\,x^{56} - 6500377086\,x^{55} - 31881795645\,x^{54} - 135429769710\,x^{53} \\
& - 507475497240\,x^{52} - 1700626327530\,x^{51} - 5151135032901\,x^{50} \\
& - 14221399477350\,x^{49} - 36029997048045\,x^{48} - 84230378907360\,x^{47} \\
& - 182531640599640\,x^{46} - 368062834742640\,x^{45} - 692786421335970\,x^{44} \\
& - 1220465967171300\,x^{43} - 2016821531985510\,x^{42} - 3132055418626260\,x^{41} \\
& - 4577991476486820\,x^{40} - 6305827180692930\,x^{39} - 8193233835182955\,x^{38} \\
& - 10049288549401530\,x^{37} - 11641392914793525\,x^{36} \\
& - 12740731432552656\,x^{35} - 13174789600157850\,x^{34} \\
& - 12870767154654920\,x^{33} - 11875161171269835\,x^{32} \\
& - 10342419643518040\,x^{31} - 8496378945668836\,x^{30} - 6577513443197240\,x^{29} \\
& - 4792866190425060\,x^{28} - 3282641497657200\,x^{27} - 2109753860361840\,x^{26} \\
& - 1269991339578936\,x^{25} - 714504714881880\,x^{24} - 374813987669550\,x^{23} \\
& - 182856898178565\,x^{22} - 82742890584390\,x^{21} - 34649771220435\,x^{20} \\
& - 13431717447120\,x^{19} - 4869115232670\,x^{18} - 1722934489320\,x^{17} \\
& - 669003675525\,x^{16} - 333634070220\,x^{15} - 212056531650\,x^{14} \\
& - 145365393180\,x^{13} - 95135444040\,x^{12} - 56664228660\,x^{11} - 30232591506\,x^{10} \\
& - 14352406380\,x^{9} - 6029091450\,x^{8} - 2225130840\,x^{7} - 714176540\,x^{6} \\
& - 196415856\,x^{5} - 45288265\,x^{4} - 8465530\,x^{3} - 1213535\,x^{2} - 120030\,x - 6276
\end{aligned}
$$

$$
\begin{aligned}
p_5(3,x) = {} & 501942\,x^{66} - 278256\,x^{58} - 7120080\,x^{57} - 93762104\,x^{56} - 842003400\,x^{55} \\
& - 5773980156\,x^{54} - 32131723050\,x^{53} - 150691992759\,x^{52} \\
& - 611005657860\,x^{51} - 2181542532156\,x^{50} - 6953463023130\,x^{49} \\
& - 19996299396801\,x^{48} - 52315771981200\,x^{47} - 125360343350424\,x^{46} \\
& - 276630629698080\,x^{45} - 564674674678776\,x^{44} - 1070193661159104\,x^{43} \\
& - 1888972064647884\,x^{42} - 3113094078234756\,x^{41} - 4800354059690406\,x^{40} \\
& - 6937648095521304\,x^{39} - 9410356563289500\,x^{38} - 11992790312923446\,x^{37} \\
& - 14371453886583105\,x^{36} - 16202520978266844\,x^{35} \\
& - 17190815571599670\,x^{34} - 17166142521250632\,x^{33} \\
& - 16130358923622930\,x^{32} - 14257613449053408\,x^{31} \\
& - 11847353612561280\,x^{30} - 9247214386939112\,x^{29} - 6772683151554204\,x^{28} \\
& - 4648536968975088\,x^{27} - 2985530557040136\,x^{26} - 1791089955384912\,x^{25} \\
& - 1001705936750064\,x^{24} - 521098615805976\,x^{23} - 251522266814676\,x^{22} \\
& - 112333620222954\,x^{21} - 46275840664599\,x^{20} - 17512723479636\,x^{19} \\
& - 6043069576530\,x^{18} - 1857879939384\,x^{17} - 458407079190\,x^{16} \\
& - 29776927536\,x^{15} + 84655578960\,x^{14} + 100043398476\,x^{13} \\
& + 83831586930\,x^{12} + 59448351144\,x^{11} + 36861060600\,x^{10} + 20106097596\,x^{9} \\
& + 9632775438\,x^{8} + 4031580384\,x^{7} + 1460616192\,x^{6} + 451754856\,x^{5} \\
& + 116813508\,x^{4} + 24444000\,x^{3} + 3921142\,x^{2} + 434670\,x + 25653
\end{aligned}
$$

$$
\begin{aligned}
p_6(3,x) = {} & 2760681\,x^{64} - 1344904\,x^{56} - 33227040\,x^{55} - 421177904\,x^{54} - 3631564440\,x^{53} \\
& - 23857894116\,x^{52} - 126932231790\,x^{51} - 568001757141\,x^{50} \\
& - 2193220505970\,x^{49} - 7442780517900\,x^{48} - 22503715295880\,x^{47} \\
& - 61265088281124\,x^{46} - 151429303563720\,x^{45} - 342076581457884\,x^{44} \\
& - 710042370115680\,x^{43} - 1360177772508936\,x^{42} - 2413337253271056\,x^{41} \\
& - 3977695557192546\,x^{40} - 6104956088986776\,x^{39} - 8742147947840580\,x^{38} \\
& - 11698027853161104\,x^{37} - 14644937885721780\,x^{36} \\
& - 17168277025570554\,x^{35} - 18857667715618095\,x^{34} \\
& - 19413842466703230\,x^{33} - 18733802746149855\,x^{32} \\
& - 16941531923866368\,x^{31} - 14351869537936560\,x^{30} \\
& - 11381696141394208\,x^{29} - 8442270674815080\,x^{28} - 5850239675310632\,x^{27} \\
& - 3782291658282396\,x^{26} - 2277736706182392\,x^{25} - 1275301193660676\,x^{24} \\
& - 662470806291840\,x^{23} - 318518675761104\,x^{22} - 141372190485864\,x^{21} \\
& - 57751945149444\,x^{20} - 21643269975054\,x^{19} - 7417116144165\,x^{18} \\
& - 2323511327466\,x^{17} - 678353595885\,x^{16} - 207224994816\,x^{15} \\
& - 92126593800\,x^{14} - 68296268880\,x^{13} - 59579055900\,x^{12} \\
& - 48676198116\,x^{11} - 35181201510\,x^{10} - 22251993036\,x^{9} \\
& - 12271175280\,x^{8} - 5871789144\,x^{7} - 2418385788\,x^{6} - 846433224\,x^{5} \\
& - 246821148\,x^{4} - 58113720\,x^{3} - 10482892\,x^{2} - 1308720\,x - 87767
\end{aligned}
$$

$$
\begin{aligned}
p_7(3,x) = {} & 12620256\,x^{62} - 5379616\,x^{54} - 128161440\,x^{53} - 1561368816\,x^{52} \\
& - 12903850440\,x^{51} - 81056096316\,x^{50} - 411403998330\,x^{49} \\
& - 1752436374975\,x^{48} - 6427457365680\,x^{47} - 20673989434800\,x^{46} \\
& - 59119527703080\,x^{45} - 151883696089476\,x^{44} - 353454441809040\,x^{43} \\
& - 749962924280856\,x^{42} - 1458539755956000\,x^{41} - 2611096844363496\,x^{40} \\
& - 4317771309162624\,x^{39} - 6613752235292280\,x^{38} - 9405192013269624\,x^{37} \\
& - 12439229543419380\,x^{36} - 15322193049607704\,x^{35} \\
& - 17594882569693260\,x^{34} - 18848726460990270\,x^{33} \\
& - 18843744437980605\,x^{32} - 17582348204609760\,x^{31} \\
& - 15308323278646320\,x^{30} - 12431456683693632\,x^{29} - 9409172811608880\,x^{28} \\
& - 6631285711116832\,x^{27} - 4346441472811680\,x^{26} - 2645594965622232\,x^{25} \\
& - 1492859768780484\,x^{24} - 779398104569760\,x^{23} - 375632254346544\,x^{22} \\
& - 166693373383680\,x^{21} - 67915451199024\,x^{20} - 25321182897816\,x^{19} \\
& - 8606097232980\,x^{18} - 2653446306186\,x^{17} - 734877074295\,x^{16} \\
& - 174707848656\,x^{15} - 23654567880\,x^{14} + 16755086880\,x^{13} \\
& + 28219908600\,x^{12} + 29793619440\,x^{11} + 26023757760\,x^{10} + 19468399236\,x^{9} \\
& + 12541121190\,x^{8} + 6944170896\,x^{7} + 3284807760\,x^{6} + 1312497576\,x^{5} \\
& + 434959668\,x^{4} + 116026560\,x^{3} + 23689008\,x^{2} + 3351240\,x + 257588
\end{aligned}
$$

$$
\begin{aligned}
p_8(3,x) = {} & 48903492\,x^{60} - 18156204\,x^{52} - 416524680\,x^{51} - 4869131436\,x^{50} \\
& - 38496978000\,x^{49} - 230722303500\,x^{48} - 1114501390470\,x^{47} \\
& - 4507272002025\,x^{46} - 15658004184030\,x^{45} - 47589505095600\,x^{44} \\
& - 128279170369350\,x^{43} - 309882689186499\,x^{42} - 676345390822530\,x^{41} \\
& - 1342360887641631\,x^{40} - 2435219141474160\,x^{39} - 4054829436007740\,x^{38} \\
& - 6217499659130136\,x^{37} - 8802741200709105\,x^{36} - 11531427756421014\,x^{35} \\
& - 13999206685193145\,x^{34} - 15768321009041250\,x^{33} \\
& - 16491861671776920\,x^{32} - 16023066041087820\,x^{31} \\
& - 14463154530537570\,x^{30} - 12126634453559340\,x^{29} - 9439968066435270\,x^{28} \\
& - 6817510540590048\,x^{27} - 4563097986798900\,x^{26} - 2826888646113792\,x^{25} \\
& - 1618418350161990\,x^{24} - 854680240301940\,x^{23} - 415449974520126\,x^{22} \\
& - 185426444148660\,x^{21} - 75778598355174\,x^{20} - 28265105480040\,x^{19} \\
& - 9586793193120\,x^{18} - 2943922582224\,x^{17} - 814458739770\,x^{16} \\
& - 202675081986\,x^{15} - 47508520995\,x^{14} - 15124265730\,x^{13} \\
& - 11912973345\,x^{12} - 13793335920\,x^{11} - 14671298070\,x^{10} - 13291724160\,x^{9} \\
& - 10215607545\,x^{8} - 6660865926\,x^{7} - 3671689905\,x^{6} - 1695421494\,x^{5} \\
& - 645313500\,x^{4} - 196816230\,x^{3} - 45870903\,x^{2} - 7409610\,x - 659907
\end{aligned}
$$

$$
\begin{aligned}
p_9(3,x) = {} & 163011640\,x^{58} - 52451256\,x^{50} - 1157013000\,x^{49} - 12955039500\,x^{48} \\
& - 97785129000\,x^{47} - 557834647500\,x^{46} - 2557744092930\,x^{45} \\
& - 9792254492475\,x^{44} - 32117685900300\,x^{43} - 91917864881700\,x^{42} \\
& - 232674461598150\,x^{41} - 526363689945951\,x^{40} - 1072765394929800\,x^{39} \\
& - 1982236188598380\,x^{38} - 3337483061661840\,x^{37} - 5140777385715660\,x^{36} \\
& - 7267005076060704\,x^{35} - 9450838452899610\,x^{34} - 11328992640080310\,x^{33} \\
& - 12534797886538785\,x^{32} - 12813112359075600\,x^{31} \\
& - 12106990642666920\,x^{30} - 10576373177893380\,x^{29} - 8540510430922470\,x^{28} \\
& - 6371853839122200\,x^{27} - 4388717152420380\,x^{26} - 2787557736789072\,x^{25} \\
& - 1630497688655940\,x^{24} - 876781836687120\,x^{23} - 432581564731680\,x^{22} \\
& - 195360328594860\,x^{21} - 80542744110114\,x^{20} - 30219631093080\,x^{19} \\
& - 10281650713620\,x^{18} - 3158777364600\,x^{17} - 871911920880\,x^{16} \\
& - 214854782376\,x^{15} - 46636001100\,x^{14} - 7996953510\,x^{13} \\
& + 857660895\,x^{12} + 3992477580\,x^{11} + 5927471550\,x^{10} + 6839993160\,x^{9} \\
& + 6489023970\,x^{8} + 5114177640\,x^{7} + 3355243320\,x^{6} + 1821402726\,x^{5} \\
& + 807908985\,x^{4} + 285171900\,x^{3} + 76729380\,x^{2} + 14284530\,x + 1494753
\end{aligned}
$$

$$
\begin{aligned}
p_{10}(3,x) = {} & 472733756\,x^{56} - 131128140\,x^{48} - 2776831200\,x^{47} - 29723313360\,x^{46} \\
& - 213696795000\,x^{45} - 1157343862740\,x^{44} - 5022201665910\,x^{43} \\
& - 18142227488385\,x^{42} - 55979308246290\,x^{41} - 150264339937140\,x^{40} \\
& - 355673508993660\,x^{39} - 750020806600494\,x^{38} - 1420257919654860\,x^{37} \\
& - 2430112989684666\,x^{36} - 3775430800792560\,x^{35} - 5346214989485604\,x^{34} \\
& - 6920782649236872\,x^{33} - 8208704684195301\,x^{32} - 8935635206282928\,x^{31} \\
& - 8937208256011464\,x^{30} - 8218596650227392\,x^{29} - 6950648287038456\,x^{28} \\
& - 5405433360623868\,x^{27} - 3863691548739594\,x^{26} - 2536182696036132\,x^{25} \\
& - 1527066919869426\,x^{24} - 842136738124032\,x^{23} - 424573615907064\,x^{22} \\
& - 195259615959408\,x^{21} - 81702781241796\,x^{20} - 31011113711292\,x^{19} \\
& - 10639833225450\,x^{18} - 3286417808388\,x^{17} - 909521179710\,x^{16} \\
& - 224256681792\,x^{15} - 48925596720\,x^{14} - 9401899416\,x^{13} \\
& - 1790437740\,x^{12} - 955395714\,x^{11} - 1588301715\,x^{10} - 2474838366\,x^{9} \\
& - 3067495431\,x^{8} - 3047588544\,x^{7} - 2458751724\,x^{6} - 1609034856\,x^{5} \\
& - 849297306\,x^{4} - 352693770\,x^{3} - 111249879\,x^{2} - 24142950\,x - 3022656
\end{aligned}
$$

$$
\begin{aligned}
p_{11}(3,x) = {} & 1203322288\,x^{54} - 286097760\,x^{46} - 5806101600\,x^{45} \\
& - 59286748560\,x^{44} - 404975586600\,x^{43} - 2076138630540\,x^{42} \\
& - 8498257985490\,x^{41} - 28859593077315\,x^{40} - 83430639269640\,x^{39} \\
& - 209110950133800\,x^{38} - 460565580709260\,x^{37} - 900505238831406\,x^{36} \\
& - 1575260610030120\,x^{35} - 2480379062291676\,x^{34} - 3532026429706320\,x^{33} \\
& - 4565050643016084\,x^{32} - 5370071863673088\,x^{31} - 5761147643385264\,x^{30} \\
& - 5644749023133168\,x^{29} - 5055501895764840\,x^{28} - 4140334483124112\,x^{27} \\
& - 3100516742146824\,x^{26} - 2122058284205652\,x^{25} - 1326266830205694\,x^{24} \\
& - 755983799316720\,x^{23} - 392364816583896\,x^{22} - 185044424786208\,x^{21} \\
& - 79103605174776\,x^{20} - 30561494968368\,x^{19} - 10634373893520\,x^{18} \\
& - 3319420404612\,x^{17} - 925140728070\,x^{16} - 229002782112\,x^{15} \\
& - 50044902960\,x^{14} - 9593962560\,x^{13} - 1600457040\,x^{12} \\
& - 192063144\,x^{11} + 171973620\,x^{10} + 529347546\,x^{9} + 977671695\,x^{8} \\
& + 1322239464\,x^{7} + 1388142756\,x^{6} + 1138925424\,x^{5} + 738792756\,x^{4} \\
& + 369359640\,x^{3} + 139633104\,x^{2} + 35817210\,x + 5497479
\end{aligned}
$$

$$
\begin{aligned}
p_{12}(3,x) = {} & 2707475148\,x^{52} - 548354040\,x^{44} - 10644519600\,x^{43} - 103445376840\,x^{42} \\
& - 669475770600\,x^{41} - 3238186163940\,x^{40} - 12456300857910\,x^{39} \\
& - 39598709247825\,x^{38} - 106752259040310\,x^{37} - 248544582834600\,x^{36} \\
& - 506497492550010\,x^{35} - 912565150948269\,x^{34} - 1464851371420710\,x^{33} \\
& - 2107275544989069\,x^{32} - 2729008134216000\,x^{31} - 3192472898354064\,x^{30} \\
& - 3382083190061472\,x^{29} - 3250478519896140\,x^{28} - 2837321978275512\,x^{27} \\
& - 2250684323645940\,x^{26} - 1622514227056632\,x^{25} - 1062540765880776\,x^{24} \\
& - 631539745295940\,x^{23} - 340227478412406\,x^{22} - 165829866951060\,x^{21} \\
& - 72958437978426\,x^{20} - 28891377339552\,x^{19} - 10262292489540\,x^{18} \\
& - 3256313157552\,x^{17} - 918625667310\,x^{16} - 229153304952\,x^{15} \\
& - 50252385300\,x^{14} - 9635010840\,x^{13} - 1610967540\,x^{12} \\
& - 233753520\,x^{11} - 33480720\,x^{10} - 41690376\,x^{9} - 164477040\,x^{8} \\
& - 359912826\,x^{7} - 555384375\,x^{6} - 611717106\,x^{5} - 515920041\,x^{4} \\
& - 321200880\,x^{3} - 150649746\,x^{2} - 46449000\,x - 9044451
\end{aligned}
$$

$$
\begin{aligned}
p_{13}(3,x) = {} & 5414950296\,x^{50} - 927983760\,x^{42} - 17194993200\,x^{41} - 158628154440\,x^{40} \\
& - 969606561000\,x^{39} - 4408514155500\,x^{38} - 15868427628690\,x^{37} \\
& - 46994602770675\,x^{36} - 117501464843460\,x^{35} - 252598017980700\,x^{34} \\
& - 473134898469450\,x^{33} - 779869928104281\,x^{32} - 1139733933767520\,x^{31} \\
& - 1485268298811024\,x^{30} - 1733385098907840\,x^{29} - 1817433174465360\,x^{28} \\
& - 1715862293486208\,x^{27} - 1460893976073000\,x^{26} - 1122603502987512\,x^{25} \\
& - 778755081473460\,x^{24} - 487518206157360\,x^{23} - 275172313510584\,x^{22} \\
& - 139832376858540\,x^{21} - 63842328016866\,x^{20} - 26117853111000\,x^{19} \\
& - 9541412129340\,x^{18} - 3099579367248\,x^{17} - 890818980660\,x^{16} \\
& - 225144266592\,x^{15} - 49706058240\,x^{14} - 9524243400\,x^{13} \\
& - 1579201260\,x^{12} - 228249840\,x^{11} - 28755720\,x^{10} - 6385680\,x^{9} \\
& + 4077360\,x^{8} + 35304696\,x^{7} + 128224740\,x^{6} + 218841714\,x^{5} \\
& + 271852035\,x^{4} + 222208980\,x^{3} + 137563986\,x^{2} + 51985080\,x + 13519494
\end{aligned}
$$

$$
\begin{aligned}
p_{14}(3,x) = {} & 9669554100\,x^{48} - 1391975640\,x^{40} - 24564276000\,x^{39} - 214503198000\,x^{38} \\
& - 1233982683000\,x^{37} - 5251697536500\,x^{36} - 17601530587110\,x^{35} \\
& - 48286961202825\,x^{34} - 111263981819850\,x^{33} - 219288561561900\,x^{32} \\
& - 374585997826800\,x^{31} - 560040446543544\,x^{30} - 738266333602800\,x^{29} \\
& - 862838349046920\,x^{28} - 897734846007360\,x^{27} - 833976659899440\,x^{26} \\
& - 693140852338272\,x^{25} - 516026696084460\,x^{24} - 344278484768880\,x^{23} \\
& - 205797490364520\,x^{22} - 110121763062240\,x^{21} - 52663924210536\,x^{20} \\
& - 22455944303220\,x^{19} - 8509850805150\,x^{18} - 2853914196060\,x^{17} \\
& - 842414200110\,x^{16} - 217375840512\,x^{15} - 48628591920\,x^{14} \\
& - 9342940320\,x^{13} - 1529559720\,x^{12} - 214105800\,x^{11} - 25675020\,x^{10} \\
& - 3639960\,x^{9} - 1676340\,x^{8} + 3974400\,x^{7} - 7518960\,x^{6} - 31330296\,x^{5} \\
& - 93648780\,x^{4} - 109774170\,x^{3} - 103161975\,x^{2} - 48660030\,x - 18422631
\end{aligned}
$$

$$
\begin{aligned}
p_{15}(3,x) = {} & 15471286560\,x^{46} - 1855967520\,x^{38} - 31114749600\,x^{37} \\
& - 256366679280\,x^{36} - 1382484897000\,x^{35} - 5480628184620\,x^{34} \\
& - 17005320431010\,x^{33} - 42926878484955\,x^{32} - 90466559540640\,x^{31} \\
& - 162082585487520\,x^{30} - 250136737744560\,x^{29} - 335754732086136\,x^{28} \\
& - 394826510989920\,x^{27} - 408940405776144\,x^{26} - 374528530157760\,x^{25} \\
& - 304138837416816\,x^{24} - 219370053364992\,x^{23} - 140659256582064\,x^{22} \\
& - 80175683973168\,x^{21} - 40590733129416\,x^{20} - 18220126919472\,x^{19} \\
& - 7231036165272\,x^{18} - 2527146948348\,x^{17} - 773470693818\,x^{16} \\
& - 205784248032\,x^{15} - 47122992432\,x^{14} - 9165635136\,x^{13} \\
& - 1490248368\,x^{12} - 200697504\,x^{11} - 20836512\,x^{10} - 2022456\,x^{9} \\
& - 2459988\,x^{8} + 2210976\,x^{7} + 850608\,x^{6} - 6800256\,x^{5} + 13262832\,x^{4} \\
& + 24530040\,x^{3} + 59923428\,x^{2} + 34921170\,x + 22941507
\end{aligned}
$$

$$
\begin{aligned}
p_{16}(3,x) = {} & 22239974430\,x^{44} - 2203961430\,x^{36} - 35004093300\,x^{35} - 271157971230\,x^{34} \\
& - 1364452485300\,x^{33} - 5010063219720\,x^{32} - 14293323270090\,x^{31} \\
& - 32935380588495\,x^{30} - 62902540629090\,x^{29} - 101396632096800\,x^{28} \\
& - 139772457599010\,x^{27} - 166363175740689\,x^{26} - 172204647538950\,x^{25} \\
& - 155841277903389\,x^{24} - 123770853647280\,x^{23} - 86487615813996\,x^{22} \\
& - 53247710001528\,x^{21} - 28893723944301\,x^{20} - 13807893829158\,x^{19} \\
& - 5799825840465\,x^{18} - 2134018661562\,x^{17} - 684302610348\,x^{16} \\
& - 189807239142\,x^{15} - 45053598033\,x^{14} - 9009771750\,x^{13} \\
& - 1478846187\,x^{12} - 193690224\,x^{11} - 20150802\,x^{10} + 1754256\,x^{9} \\
& - 2499255\,x^{8} - 856566\,x^{7} + 4817943\,x^{6} - 5717286\,x^{5} + 46323\,x^{4} \\
& + 11803860\,x^{3} - 24195528\,x^{2} - 12726180\,x - 26151753
\end{aligned}
$$

$$
\begin{aligned}
p_{17}(3,x) = {} & 28781143380\,x^{42} - 2333606220\,x^{34} - 35004093300\,x^{33} - 253903428270\,x^{32} \\
& - 1185896251800\,x^{31} - 4006213345620\,x^{30} - 10422714572910\,x^{29} \\
& - 21708787481325\,x^{28} - 37150517116260\,x^{27} - 53197005617100\,x^{26} \\
& - 64588355948610\,x^{25} - 67147291789461\,x^{24} - 60212392030440\,x^{23} \\
& - 46822374924156\,x^{22} - 31691757157200\,x^{21} - 18713006466876\,x^{20} \\
& - 9647562000672\,x^{19} - 4340188987290\,x^{18} - 1699992830502\,x^{17} \\
& - 577359262665\,x^{16} - 168889567752\,x^{15} - 42079486932\,x^{14} \\
& - 8798843970\,x^{13} - 1515897747\,x^{12} - 181375740\,x^{11} - 18294822\,x^{10} \\
& - 15744456\,x^{9} + 15186150\,x^{8} - 851256\,x^{7} - 12370320\,x^{6} + 12727494\,x^{5} \\
& - 3146823\,x^{4} - 5010660\,x^{3} + 7619922\,x^{2} - 12726180\,x + 27315792
\end{aligned}
$$

$$
\begin{aligned}
p_{18}(3,x) = {} & 33578000610\,x^{40} - 2203961430\,x^{32} - 31114749600\,x^{31} \\
& - 210354564720\,x^{30} - 906334941000\,x^{29} - 2794339471500\,x^{28} \\
& - 6562555017690\,x^{27} - 12204141509175\,x^{26} - 18447271108110\,x^{25} \\
& - 23090860232700\,x^{24} - 24267440780700\,x^{23} - 21637320674814\,x^{22} \\
& - 16495010626380\,x^{21} - 10810999345146\,x^{20} - 6113181558960\,x^{19} \\
& - 2987356731540\,x^{18} - 1261256005608\,x^{17} - 458819271945\,x^{16} \\
& - 143109337272\,x^{15} - 37973071860\,x^{14} - 8383294080\,x^{13} \\
& - 1511152188\,x^{12} - 246843870\,x^{11} + 6009003\,x^{10} + 1231230\,x^{9} \\
& - 24915345\,x^{8} + 28094976\,x^{7} - 9510900\,x^{6} - 17794056\,x^{5} \\
& + 33124410\,x^{4} - 21091170\,x^{3} - 11395947\,x^{2} + 34921170\,x - 26151753
\end{aligned}
$$

$$
\begin{aligned}
p_{19}(3,x) = {} & 35345263800\,x^{38} - 1855967520\,x^{30} - 24564276000\,x^{29} - 153960942000\,x^{28} \\
& - 607469583000\,x^{27} - 1692486820500\,x^{26} - 3542254462110\,x^{25} \\
& - 5788501316325\,x^{24} - 7585277598600\,x^{23} - 8130684455400\,x^{22} \\
& - 7239939864300\,x^{21} - 5418870142686\,x^{20} - 3437384550600\,x^{19} \\
& - 1857654178380\,x^{18} - 858077258640\,x^{17} - 338914811460\,x^{16} \\
& - 113837668992\,x^{15} - 32422713720\,x^{14} - 7886446680\,x^{13} \\
& - 1410236100\,x^{12} - 181938120\,x^{11} - 157249092\,x^{10} + 76306230\,x^{9} \\
& + 49114065\,x^{8} - 113753640\,x^{7} + 60425820\,x^{6} + 43037904\,x^{5} \\
& - 87859380\,x^{4} + 42694680\,x^{3} + 27028200\,x^{2} - 48660030\,x + 22941507
\end{aligned}
$$

$$
\begin{aligned}
p_{20}(3,x) = {} & 33578000610\,x^{36} - 1391975640\,x^{28} - 17194993200\,x^{27} - 99296743560\,x^{26} \\
& - 355623723000\,x^{25} - 884295806940\,x^{24} - 1621707908730\,x^{23} \\
& - 2278978527735\,x^{22} - 2524520396970\,x^{21} - 2257144529640\,x^{20} \\
& - 1661362933230\,x^{19} - 1019909529639\,x^{18} - 525090856290\,x^{17} \\
& - 228143906991\,x^{16} - 84527462880\,x^{15} - 25966869144\,x^{14} - 6319628784\,x^{13} \\
& - 1774129266\,x^{12} - 307674276\,x^{11} + 306972666\,x^{10} - 211495284\,x^{9} \\
& - 103579476\,x^{8} + 242215974\,x^{7} - 109828719\,x^{6} - 88282194\,x^{5} \\
& + 140033439\,x^{4} - 48681360\,x^{3} - 42493794\,x^{2} + 51985080\,x - 18422631
\end{aligned}
$$

$$
\begin{aligned}
p_{21}(3,x) = {} & 28781143380\,x^{34} - 927983760\,x^{26} - 10644519600\,x^{25} - 56222417160\,x^{24} \\
& - 180795552600\,x^{23} - 394788375540\,x^{22} - 619754262270\,x^{21} \\
& - 726113467365\,x^{20} - 656402767020\,x^{19} - 474649074900\,x^{18} \\
& - 282757533630\,x^{17} - 138710611611\,x^{16} - 55349133840\,x^{15} \\
& - 19454851416\,x^{14} - 6112185120\,x^{13} - 607906104\,x^{12} \\
& - 168282816\,x^{11} - 588380364\,x^{10} + 332730684\,x^{9} + 166507770\,x^{8} \\
& - 350568504\,x^{7} + 132545556\,x^{6} + 128663106\,x^{5} - 164255949\,x^{4} \\
& + 41295540\,x^{3} + 49188594\,x^{2} - 46449000\,x + 13519494
\end{aligned}
$$

$$
\begin{aligned}
p_{22}(3,x) = {} & 22239974430\,x^{32} - 548354040\,x^{24} - 5806101600\,x^{23} - 27804775440\,x^{22} \\
& - 79188774600\,x^{21} - 148533709740\,x^{20} - 192793008330\,x^{19} \\
& - 179351718975\,x^{18} - 125897170230\,x^{17} - 72442527300\,x^{16} \\
& - 35127273480\,x^{15} - 12305953764\,x^{14} - 3296546760\,x^{13} \\
& - 1918882524\,x^{12} - 231784800\,x^{11} + 618743736\,x^{10} - 406817424\,x^{9} \\
& - 190720530\,x^{8} + 383292936\,x^{7} - 126477780\,x^{6} - 139870224\,x^{5} \\
& + 153836124\,x^{4} - 28686450\,x^{3} - 45280131\,x^{2} + 35817210\,x - 9044451
\end{aligned}
$$

$$
\begin{aligned}
p_{23}(3,x) = {} & 15471286560\,x^{30} - 286097760\,x^{22} - 2776831200\,x^{21} - 11929154640\,x^{20} \\
& - 29556423000\,x^{19} - 46119352500\,x^{18} - 46714111470\,x^{17} \\
& - 31548678525\,x^{16} - 16302152880\,x^{15} - 8458023600\,x^{14} - 3220050600\,x^{13} \\
& + 98056764\,x^{12} - 272098320\,x^{11} - 608773464\,x^{10} + 383816160\,x^{9} \\
& + 154266840\,x^{8} - 331883136\,x^{7} + 103769640\,x^{6} + 118311336\,x^{5} \\
& - 119604420\,x^{4} + 17451720\,x^{3} + 34412196\,x^{2} - 24142950\,x + 5497479
\end{aligned}
$$

$$
\begin{aligned}
p_{24}(3,x) = {} & 9669554100\,x^{28} - 131128140\,x^{20} - 1157013000\,x^{19} - 4400155500\,x^{18} \\
& - 9256104000\,x^{17} - 11415748500\,x^{16} - 8017692930\,x^{15} - 3127781475\,x^{14} \\
& - 1462459050\,x^{13} - 1169953200\,x^{12} + 44134350\,x^{11} + 345744711\,x^{10} \\
& - 312087750\,x^{9} - 85338045\,x^{8} + 231744240\,x^{7} - 76005540\,x^{6} - 79343784\,x^{5} \\
& + 78615225\,x^{4} - 9936810\,x^{3} - 21954855\,x^{2} + 14284530\,x - 3022656
\end{aligned}
$$

$$
\begin{aligned}
p_{25}(3,x) = {} & 5414950296\,x^{26} - 52451256\,x^{18} - 416524680\,x^{17} - 1378738764\,x^{16} \\
& - 2377135800\,x^{15} - 2105736516\,x^{14} - 696046806\,x^{13} \\
& + 17255511\,x^{12} - 280263204\,x^{11} - 188682156\,x^{10} + 192044814\,x^{9} \\
& + 22580883\,x^{8} - 130121784\,x^{7} + 49588812\,x^{6} + 42076944\,x^{5} \\
& - 43902612\,x^{4} + 5559840\,x^{3} + 11810682\,x^{2} - 7409610\,x + 1494753
\end{aligned}
$$

$$
\begin{aligned}
p_{26}(3,x) = {} & 2707475148\,x^{24} - 18156204\,x^{16} - 128161440\,x^{15} - 361052784\,x^{14} \\
& - 482547240\,x^{13} - 240265116\,x^{12} + 66170286\,x^{11} + 22098069\,x^{10} \\
& - 98921574\,x^{9} + 6773220\,x^{8} + 56921724\,x^{7} - 27904338\,x^{6} - 17136756\,x^{5} \\
& + 20702682\,x^{4} - 3039120\,x^{3} - 5331132\,x^{2} + 3351240\,x - 659907
\end{aligned}
$$

$$
\begin{aligned}
p_{27}(3,x) = {} & 1203322288\,x^{22} - 5379616\,x^{14} - 33227040\,x^{13} - 77227696\,x^{12} - 72243640\,x^{11} \\
& - 781956\,x^{10} + 30059834\,x^{9} - 11926145\,x^{8} - 18199416\,x^{7} + 12864040\,x^{6} \\
& + 4927244\,x^{5} - 8095122\,x^{4} + 1531880\,x^{3} + 1990268\,x^{2} - 1308720\,x + 257588
\end{aligned}
$$

$$\begin{aligned}p_{28}(3,x) = {}& 472733756\,x^{20} - 1344904\,x^{12} - 7120080\,x^{11} - 13039096\,x^{10}\\ & - 6652600\,x^{9} + 5686900\,x^{8} + 3381806\,x^{7} - 4554555\,x^{6} - 729266\,x^{5}\\ & + 2535080\,x^{4} - 663470\,x^{3} - 597863\,x^{2} + 434670\,x - 87767\end{aligned}$$

$$\begin{aligned}p_{29}(3,x) = {}& 163011640\,x^{18} - 278256\,x^{10} - 1227600\,x^{9} - 1643000\,x^{8}\\ & - 55800\,x^{7} + 1095900\,x^{6} - 102086\,x^{5} - 596145\,x^{4}\\ & + 231540\,x^{3} + 137260\,x^{2} - 120030\,x + 25653\end{aligned}$$

$$\begin{aligned}p_{30}(3,x) = {}& 48903492\,x^{16} - 46376\,x^{8} - 163680\,x^{7} - 138384\,x^{6} + 80600\,x^{5}\\ & + 91524\,x^{4} - 60770\,x^{3} - 21539\,x^{2} + 26610\,x - 6276\end{aligned}$$

$$\begin{aligned}p_{31}(3,x) = {}& 12620256\,x^{14} - 5984\,x^{6} - 15840\,x^{5} - 5584\,x^{4}\\ & + 10760\,x^{3} + 1564\,x^{2} - 4470\,x + 1239\end{aligned}$$

$$p_{32}(3,x) = 2760681\,x^{12} - 561\,x^{4} - 990\,x^{3} + 139\,x^{2} + 510\,x - 186$$

$$p_{33}(3,x) = 501942\,x^{10} - 34\,x^{2} - 30\,x + 19$$

$$p_{34}(3,x) = 73815\,x^{8} - 1$$

$$p_{35}(3,x) = 8436\,x^{6}$$

$$p_{36}(3,x) = 703\,x^{4}$$

$$p_{37}(3,x) = 38\,x^{2}$$

$$p_{38}(3,x) = 1$$

Definition of $\overleftarrow{P_4}(x,y)$

For the case of base $b = 4$, we have the numbers $e_1 = 0$, $e_3 = 9$, $e_4 = 2$, $e_6 = 3$, and $d = 3$. In this case, $\overleftarrow{P_4}(x,y)$ is of degree 32 in y. We compute each $p_j(4,x)$ directly:

$$\begin{aligned}p_0(4,x) = {}& x^{64} - x^{56} - 12\,x^{55} - 88\,x^{54} - 472\,x^{53} - 2052\,x^{52} - 7564\,x^{51} - 24436\,x^{50}\\ & - 70596\,x^{49} - 185338\,x^{48} - 447184\,x^{47} - 1000920\,x^{46} - 2093044\,x^{45}\\ & - 4113654\,x^{44} - 7634988\,x^{43} - 13436736\,x^{42} - 22496928\,x^{41}\\ & - 35937531\,x^{40} - 54903336\,x^{39} - 80385068\,x^{38} - 112985484\,x^{37}\\ & - 152683340\,x^{36} - 198617348\,x^{35} - 248979348\,x^{34} - 301025144\,x^{33}\\ & - 351278747\,x^{32} - 395868600\,x^{31} - 431015840\,x^{30} - 453528340\,x^{29}\\ & - 461283546\,x^{28} - 453528340\,x^{27} - 431015840\,x^{26} - 395868600\,x^{25}\\ & - 351278747\,x^{24} - 301025144\,x^{23} - 248979348\,x^{22} - 198617348\,x^{21}\\ & - 152683340\,x^{20} - 112985484\,x^{19} - 80385068\,x^{18} - 54903336\,x^{17}\\ & - 35937531\,x^{16} - 22496928\,x^{15} - 13436736\,x^{14} - 7634988\,x^{13}\\ & - 4113654\,x^{12} - 2093044\,x^{11} - 1000920\,x^{10} - 447184\,x^{9} - 185338\,x^{8}\\ & - 70596\,x^{7} - 24436\,x^{6} - 7564\,x^{5} - 2052\,x^{4} - 472\,x^{3} - 88\,x^{2} - 12\,x - 1\end{aligned}$$

$$
\begin{aligned}
p_1(4,x) = {} & 32\,x^{62} - 28\,x^{54} - 324\,x^{53} - 2272\,x^{52} - 11632\,x^{51} - 48104\,x^{50} - 168220\,x^{49} \\
& - 513788\,x^{48} - 1398464\,x^{47} - 3445056\,x^{46} - 7766096\,x^{45} - 16161744\,x^{44} \\
& - 31256904\,x^{43} - 56482356\,x^{42} - 95761356\,x^{41} - 152826696\,x^{40} \\
& - 230135904\,x^{39} - 327542220\,x^{38} - 440986584\,x^{37} - 561647396\,x^{36} \\
& - 675973608\,x^{35} - 766902968\,x^{34} - 816337076\,x^{33} - 808462420\,x^{32} \\
& - 733370528\,x^{31} - 589897024\,x^{30} - 387115432\,x^{29} - 143588616\,x^{28} \\
& + 115199256\,x^{27} + 361252260\,x^{26} + 568727596\,x^{25} + 718443064\,x^{24} \\
& + 800490368\,x^{23} + 815217964\,x^{22} + 771755192\,x^{21} + 685413668\,x^{20} \\
& + 573984360\,x^{19} + 454563752\,x^{18} + 340895748\,x^{17} + 242203284\,x^{16} \\
& + 162950112\,x^{15} + 103708152\,x^{14} + 62324160\,x^{13} + 35287608\,x^{12} \\
& + 18763464\,x^{11} + 9336108\,x^{10} + 4324844\,x^{9} + 1854816\,x^{8} + 730400\,x^{7} \\
& + 261704\,x^{6} + 84052\,x^{5} + 23804\,x^{4} + 5752\,x^{3} + 1144\,x^{2} + 168\,x + 16
\end{aligned}
$$

$$
\begin{aligned}
p_2(4,x) = {} & 496\,x^{60} - 378\,x^{52} - 4212\,x^{51} - 28184\,x^{50} - 137400\,x^{49} - 539028\,x^{48} \\
& - 1782608\,x^{47} - 5128752\,x^{46} - 13096368\,x^{45} - 30122424\,x^{44} - 63066576\,x^{43} \\
& - 121160616\,x^{42} - 214852860\,x^{41} - 353210586\,x^{40} - 539923224\,x^{39} \\
& - 768740064\,x^{38} - 1019874144\,x^{37} - 1259297178\,x^{36} - 1442557368\,x^{35} \\
& - 1523415588\,x^{34} - 1465965948\,x^{33} - 1256872300\,x^{32} - 913684704\,x^{31} \\
& - 485560608\,x^{30} - 44819680\,x^{29} + 328914024\,x^{28} + 568206024\,x^{27} \\
& + 634250736\,x^{26} + 526423572\,x^{25} + 280358946\,x^{24} - 42304024\,x^{23} \\
& - 371619552\,x^{22} - 645659280\,x^{21} - 824285350\,x^{20} - 893344200\,x^{19} \\
& - 863352396\,x^{18} - 760610916\,x^{17} - 618248796\,x^{16} - 466550256\,x^{15} \\
& - 328120368\,x^{14} - 215325120\,x^{13} - 131917956\,x^{12} - 75327840\,x^{11} \\
& - 40023600\,x^{10} - 19701444\,x^{9} - 8949834\,x^{8} - 3721224\,x^{7} - 1406544\,x^{6} \\
& - 475712\,x^{5} - 142284\,x^{4} - 36300\,x^{3} - 7724\,x^{2} - 1212\,x - 132
\end{aligned}
$$

$$
\begin{aligned}
p_3(4,x) = {} & 4960\,x^{58} - 3276\,x^{50} - 35100\,x^{49} - 223600\,x^{48} - 1035200\,x^{47} \\
& - 3840672\,x^{46} - 11968464\,x^{45} - 32301904\,x^{44} - 76998144\,x^{43} \\
& - 164371744\,x^{42} - 317306576\,x^{41} - 557709792\,x^{40} - 896561264\,x^{39} \\
& - 1321538872\,x^{38} - 1786990584\,x^{37} - 2212589872\,x^{36} - 2495982368\,x^{35} \\
& - 2539675212\,x^{34} - 2285655944\,x^{33} - 1744865260\,x^{32} - 1007414976\,x^{31} \\
& - 225122112\,x^{30} + 431443104\,x^{29} + 826694880\,x^{28} + 904178656\,x^{27} \\
& + 702953184\,x^{26} + 339222744\,x^{25} - 37570776\,x^{24} - 297938640\,x^{23} \\
& - 370157400\,x^{22} - 255634616\,x^{21} - 12774736\,x^{20} + 270598080\,x^{19} \\
& + 513901788\,x^{18} + 663599176\,x^{17} + 705191196\,x^{16} + 653655136\,x^{15} \\
& + 543350240\,x^{14} + 410071056\,x^{13} + 283398800\,x^{12} + 179714656\,x^{11} \\
& + 104859048\,x^{10} + 56116480\,x^{9} + 27540104\,x^{8} + 12286896\,x^{7} + 4971752\,x^{6} \\
& + 1790680\,x^{5} + 571872\,x^{4} + 154848\,x^{3} + 35496\,x^{2} + 5916\,x + 740
\end{aligned}
$$

$$
\begin{aligned}
p_4(4,x) = {} & 35960\,x^{56} - 20475\,x^{48} - 210600\,x^{47} - 1274000\,x^{46} - 5584400\,x^{45} \\
& - 19525528\,x^{44} - 57099064\,x^{43} - 143859144\,x^{42} - 318226664\,x^{41} \\
& - 625947236\,x^{40} - 1103953968\,x^{39} - 1754316424\,x^{38} - 2516691004\,x^{37} \\
& - 3254777922\,x^{36} - 3773704748\,x^{35} - 3874907936\,x^{34} - 3435836928\,x^{33} \\
& - 2481541327\,x^{32} - 1208060320\,x^{31} + 66462672\,x^{30} + 1013227024\,x^{29} \\
& + 1420189584\,x^{28} + 1274171408\,x^{27} + 755406480\,x^{26} + 145967808\,x^{25} \\
& - 299338596\,x^{24} - 454363032\,x^{23} - 349567104\,x^{22} - 121371156\,x^{21} \\
& + 72770070\,x^{20} + 134263460\,x^{19} + 47601920\,x^{18} - 129669800\,x^{17} \\
& - 318826757\,x^{16} - 451012272\,x^{15} - 499037896\,x^{14} - 466430728\,x^{13} \\
& - 383382360\,x^{12} - 280675112\,x^{11} - 185638568\,x^{10} - 110707536\,x^{9} \\
& - 59980402\,x^{8} - 29185888\,x^{7} - 12838368\,x^{6} - 4974068\,x^{5} \\
& - 1716074\,x^{4} - 495300\,x^{3} - 123544\,x^{2} - 21744\,x - 3146
\end{aligned}
$$

$$
\begin{aligned}
p_5(4,x) = {} & 201376\,x^{54} - 98280\,x^{46} - 968760\,x^{45} - 5549440\,x^{44} - 22952160\,x^{43} \\
& - 75329232\,x^{42} - 205733528\,x^{41} - 481095384\,x^{40} - 980543424\,x^{39} \\
& - 1761046912\,x^{38} - 2803724208\,x^{37} - 3962602128\,x^{36} - 4953883880\,x^{35} \\
& - 5419133604\,x^{34} - 5063132460\,x^{33} - 3815570920\,x^{32} - 1927136640\,x^{31} \\
& + 77480976\,x^{30} + 1598408608\,x^{29} + 2224706544\,x^{28} + 1920923040\,x^{27} \\
& + 1025050080\,x^{26} + 54370512\,x^{25} - 564661872\,x^{24} - 682561056\,x^{23} \\
& - 433335168\,x^{22} - 85393224\,x^{21} + 143705784\,x^{20} + 188567208\,x^{19} \\
& + 121748508\,x^{18} + 54303748\,x^{17} + 61276680\,x^{16} + 137481216\,x^{15} \\
& + 240484568\,x^{14} + 312199728\,x^{13} + 331416552\,x^{12} + 295521520\,x^{11} \\
& + 231039216\,x^{10} + 157969272\,x^{9} + 96937880\,x^{8} + 52334496\,x^{7} + 25488264\,x^{6} \\
& + 10728640\,x^{5} + 4057032\,x^{4} + 1252104\,x^{3} + 344492\,x^{2} + 63612\,x + 10752
\end{aligned}
$$

$$
\begin{aligned}
p_6(4,x) = {} & 906192\,x^{52} - 376740\,x^{44} - 3552120\,x^{43} - 19207760\,x^{42} - 74665360\,x^{41} \\
& - 228969048\,x^{40} - 580746320\,x^{39} - 1251750192\,x^{38} - 2329834416\,x^{37} \\
& - 3775740344\,x^{36} - 5337435312\,x^{35} - 6545101176\,x^{34} - 6846507988\,x^{33} \\
& - 5866985870\,x^{32} - 3672014496\,x^{31} - 856041856\,x^{30} + 1656770688\,x^{29} \\
& + 3039538632\,x^{28} + 2981564576\,x^{27} + 1832649648\,x^{26} + 367892496\,x^{25} \\
& - 673994544\,x^{24} - 970394400\,x^{23} - 681307488\,x^{22} - 212363232\,x^{21} \\
& + 104699832\,x^{20} + 178318440\,x^{19} + 107110992\,x^{18} + 24634356\,x^{17} \\
& - 15418782\,x^{16} - 29669392\,x^{15} - 63044288\,x^{14} - 115713888\,x^{13} \\
& - 175599700\,x^{12} - 204265104\,x^{11} - 201279000\,x^{10} - 163897576\,x^{9} \\
& - 118236504\,x^{8} - 72246960\,x^{7} - 39935600\,x^{6} - 18439552\,x^{5} \\
& - 7794084\,x^{4} - 2575520\,x^{3} - 796080\,x^{2} - 152988\,x - 30614
\end{aligned}
$$

$$
\begin{aligned}
p_7(4,x) = {} & 3365856\,x^{50} - 1184040\,x^{42} - 10656360\,x^{41} - 54202720\,x^{40} - 197137600\,x^{39} \\
& - 561838112\,x^{38} - 1314502992\,x^{37} - 2589316048\,x^{36} - 4351166592\,x^{35} \\
& - 6261334176\,x^{34} - 7667952816\,x^{33} - 7820981376\,x^{32} - 6274518848\,x^{31} \\
& - 3261087392\,x^{30} + 254141408\,x^{29} + 2979158976\,x^{28} + 3976379776\,x^{27} \\
& + 3184581840\,x^{26} + 1403676768\,x^{25} - 257491440\,x^{24} - 1063168320\,x^{23} \\
& - 977416896\,x^{22} - 465514656\,x^{21} - 25968480\,x^{20} + 134822496\,x^{19} \\
& + 96140064\,x^{18} + 18673080\,x^{17} - 16387800\,x^{16} - 16370784\,x^{15} - 2899920\,x^{14} \\
& + 13298608\,x^{13} + 48424096\,x^{12} + 83040896\,x^{11} + 117036696\,x^{10} \\
& + 119621776\,x^{9} + 108047928\,x^{8} + 76518240\,x^{7} + 49886688\,x^{6} + 25483728\,x^{5} \\
& + 12391376\,x^{4} + 4376096\,x^{3} + 1558360\,x^{2} + 308352\,x + 74360
\end{aligned}
$$

$$
\begin{aligned}
p_8(4,x) = {} & 10518300\,x^{48} - 3108105\,x^{40} - 26640900\,x^{39} - 126955400\,x^{38} \\
& - 429745800\,x^{37} - 1131010188\,x^{36} - 2420939796\,x^{35} - 4311093996\,x^{34} \\
& - 6441411996\,x^{33} - 8039574006\,x^{32} - 8186391072\,x^{31} - 6351147504\,x^{30} \\
& - 2881031752\,x^{29} + 974677764\,x^{28} + 3667605288\,x^{27} + 4255120896\,x^{26} \\
& + 2957647680\,x^{25} + 925206426\,x^{24} - 602407344\,x^{23} - 1069077672\,x^{22} \\
& - 739239336\,x^{21} - 226087080\,x^{20} + 60307560\,x^{19} + 90127944\,x^{18} \\
& + 25873296\,x^{17} - 13624722\,x^{16} - 15339816\,x^{15} - 5226528\,x^{14} \\
& + 2145060\,x^{13} - 1895742\,x^{12} - 11153548\,x^{11} - 39030432\,x^{10} \\
& - 54802056\,x^{9} - 71284821\,x^{8} - 60119640\,x^{7} - 49532004\,x^{6} - 28114548\,x^{5} \\
& - 16463196\,x^{4} - 6166188\,x^{3} - 2622092\,x^{2} - 526152\,x - 156651
\end{aligned}
$$

$$
\begin{aligned}
p_9(4,x) = {} & 28048800\,x^{46} - 6906900\,x^{38} - 56241900\,x^{37} - 249964000\,x^{36} - 782579600\,x^{35} \\
& - 1887305112\,x^{34} - 3658069316\,x^{33} - 5805218276\,x^{32} - 7546478976\,x^{31} \\
& - 7865323904\,x^{30} - 6126550304\,x^{29} - 2672940192\,x^{28} + 1132629488\,x^{27} \\
& + 3637217624\,x^{26} + 3949284456\,x^{25} + 2485948400\,x^{24} + 555389120\,x^{23} \\
& - 661496616\,x^{22} - 846159952\,x^{21} - 449954744\,x^{20} - 61764720\,x^{19} \\
& + 73419184\,x^{18} + 43991624\,x^{17} - 1507896\,x^{16} - 13077472\,x^{15} \\
& - 6470464\,x^{14} + 563976\,x^{13} + 1161512\,x^{12} - 1526552\,x^{11} + 4476828\,x^{10} \\
& + 9626996\,x^{9} + 30798280\,x^{8} + 31561344\,x^{7} + 38362876\,x^{6} + 23967640\,x^{5} \\
& + 18333108\,x^{4} + 7142632\,x^{3} + 3828648\,x^{2} + 761508\,x + 289652
\end{aligned}
$$

$$
\begin{aligned}
p_{10}(4,x) = {} & 64512240\,x^{44} - 13123110\,x^{36} - 101235420\,x^{35} - 417439880\,x^{34} \\
& - 1199634920\,x^{33} - 2625794908\,x^{32} - 4547498912\,x^{31} - 6304556896\,x^{30} \\
& - 6893708896\,x^{29} - 5580308208\,x^{28} - 2579813920\,x^{27} + 833251056\,x^{26} \\
& + 3057425384\,x^{25} + 3262116572\,x^{24} + 1926681328\,x^{23} + 321134528\,x^{22} \\
& - 551929664\,x^{21} - 567863516\,x^{20} - 224768016\,x^{19} + 13725192\,x^{18} \\
& + 54307320\,x^{17} + 18327896\,x^{16} - 5018656\,x^{15} - 6624992\,x^{14} \\
& - 862624\,x^{13} + 1246360\,x^{12} - 1022120\,x^{11} + 401040\,x^{10} + 2621724\,x^{9} \\
& - 6467466\,x^{8} - 7005272\,x^{7} - 22165088\,x^{6} - 14340880\,x^{5} \\
& - 17043302\,x^{4} - 6596392\,x^{3} - 4878556\,x^{2} - 926772\,x - 474188
\end{aligned}
$$

$$
\begin{aligned}
p_{11}(4,x) = {} & 129024480\,x^{42} - 21474180\,x^{34} - 156454740\,x^{33} - 594914320\,x^{32} \\
& - 1554741120\,x^{31} - 3051001152\,x^{30} - 4636884640\,x^{29} - 5450617248\,x^{28} \\
& - 4719665664\,x^{27} - 2448453696\,x^{26} + 336591840\,x^{25} + 2215502016\,x^{24} \\
& + 2429196640\,x^{23} + 1388542896\,x^{22} + 184602288\,x^{21} - 389720864\,x^{20} \\
& - 334406592\,x^{19} - 92545896\,x^{18} + 30858384\,x^{17} + 31073304\,x^{16} \\
& + 6491328\,x^{15} - 2316992\,x^{14} - 1992096\,x^{13} + 238368\,x^{12} + 263584\,x^{11} \\
& - 663264\,x^{10} + 1299336\,x^{9} - 483144\,x^{8} - 3105744\,x^{7} + 8557224\,x^{6} \\
& + 3899528\,x^{5} + 13048752\,x^{4} + 4440000\,x^{3} + 5435276\,x^{2} + 921768\,x + 691692
\end{aligned}
$$

$$
\begin{aligned}
p_{12}(4,x) = {} & 225792840\,x^{40} - 30421755\,x^{32} - 208606320\,x^{31} - 726259040\,x^{30} \\
& - 1705698400\,x^{29} - 2952225168\,x^{28} - 3836573328\,x^{27} - 3635941232\,x^{26} \\
& - 2168497968\,x^{25} - 113440184\,x^{24} + 1388155808\,x^{23} + 1639583088\,x^{22} \\
& + 929962696\,x^{21} + 109800860\,x^{20} - 245856312\,x^{19} - 178033856\,x^{18} \\
& - 26289536\,x^{17} + 22317114\,x^{16} + 11373280\,x^{15} + 4947344\,x^{14} \\
& + 120912\,x^{13} - 3332912\,x^{12} + 1642256\,x^{11} + 1662480\,x^{10} \\
& - 2655104\,x^{9} + 744892\,x^{8} + 1170552\,x^{7} - 1350080\,x^{6} + 2467844\,x^{5} \\
& - 7964670\,x^{4} - 1431684\,x^{3} - 5282496\,x^{2} - 688152\,x - 903083
\end{aligned}
$$

$$
\begin{aligned}
p_{13}(4,x) = {} & 347373600\,x^{38} - 37442160\,x^{30} - 240699600\,x^{29} - 760729600\,x^{28} \\
& - 1580891200\,x^{27} - 2358024032\,x^{26} - 2515281104\,x^{25} - 1732432464\,x^{24} \\
& - 377098624\,x^{23} + 731127296\,x^{22} + 1011015840\,x^{21} + 582817120\,x^{20} \\
& + 57631088\,x^{19} - 139513256\,x^{18} - 81618872\,x^{17} - 5877392\,x^{16} \\
& + 10848384\,x^{15} + 5334608\,x^{14} + 1435424\,x^{13} - 805584\,x^{12} - 198752\,x^{11} \\
& + 1486304\,x^{10} - 1120304\,x^{9} - 1343216\,x^{8} + 2972320\,x^{7} - 811008\,x^{6} \\
& - 3085336\,x^{5} + 3542312\,x^{4} - 1175976\,x^{3} + 4440868\,x^{2} + 255708\,x + 1058552
\end{aligned}
$$

$$
\begin{aligned}
p_{14}(4,x) = {} & 471435600\,x^{36} - 40116600\,x^{28} - 240699600\,x^{27} - 683468000\,x^{26} \\
& - 1230242400\,x^{25} - 1525857168\,x^{24} - 1239276064\,x^{23} - 445988064\,x^{22} \\
& + 311847456\,x^{21} + 560743504\,x^{20} + 342235296\,x^{19} + 34143888\,x^{18} \\
& - 81806312\,x^{17} - 32837436\,x^{16} + 8271456\,x^{15} - 1412992\,x^{14} \\
& - 1600896\,x^{13} + 6506184\,x^{12} - 1167584\,x^{11} - 4043664\,x^{10} \\
& + 3151056\,x^{9} + 1098640\,x^{8} - 2725344\,x^{7} + 1122144\,x^{6} + 203360\,x^{5} \\
& - 720696\,x^{4} + 2375544\,x^{3} - 3165392\,x^{2} + 255708\,x - 1115898
\end{aligned}
$$

$$
\begin{aligned}
p_{15}(4,x) = {} & 565722720\,x^{34} - 37442160\,x^{26} - 208606320\,x^{25} - 525378880\,x^{24} \\
& - 794011520\,x^{23} - 768812352\,x^{22} - 394866208\,x^{21} + 80837856\,x^{20} \\
& + 294948096\,x^{19} + 183339968\,x^{18} + 11814048\,x^{17} - 37229184\,x^{16} \\
& - 12251968\,x^{15} + 3852640\,x^{14} + 463584\,x^{13} - 1419328\,x^{12} + 1430400\,x^{11} \\
& - 58224\,x^{10} - 777760\,x^{9} + 1369680\,x^{8} - 909504\,x^{7} - 1171264\,x^{6} \\
& + 2421664\,x^{5} - 562848\,x^{4} - 1987552\,x^{3} + 1822944\,x^{2} - 688152\,x + 1058552
\end{aligned}
$$

$$
\begin{aligned}
p_{16}(4,x) = {} & 601080390\,x^{32} - 30421755\,x^{24} - 156454740\,x^{23} - 343814120\,x^{22} \\
& - 415132520\,x^{21} - 272287708\,x^{20} - 24515700\,x^{19} + 134573428\,x^{18} \\
& + 107322564\,x^{17} - 83334\,x^{16} - 28170768\,x^{15} + 4864584\,x^{14} \\
& + 6328828\,x^{13} - 8205422\,x^{12} - 136060\,x^{11} + 5998656\,x^{10} \\
& - 2593184\,x^{9} - 3233823\,x^{8} + 3369624\,x^{7} + 742260\,x^{6} - 2597676\,x^{5} \\
& + 888724\,x^{4} + 655228\,x^{3} - 728404\,x^{2} + 921768\,x - 903083
\end{aligned}
$$

$$
\begin{aligned}
p_{17}(4,x) = {} & 565722720\,x^{30} - 21474180\,x^{22} - 101235420\,x^{21} - 189972640\,x^{20} \\
& - 167283600\,x^{19} - 41417112\,x^{18} + 52319388\,x^{17} + 51938172\,x^{16} \\
& + 5986368\,x^{15} - 17353536\,x^{14} - 1956240\,x^{13} + 9033840\,x^{12} - 2177704\,x^{11} \\
& - 4998948\,x^{10} + 3203556\,x^{9} + 1654104\,x^{8} - 2415456\,x^{7} + 347796\,x^{6} \\
& + 759144\,x^{5} - 795204\,x^{4} + 687384\,x^{3} + 35144\,x^{2} - 926772\,x + 691692
\end{aligned}
$$

$$\begin{aligned}p_{18}(4,x) = {} & 471435600\,x^{28} - 13123110\,x^{20} - 56241900\,x^{19} - 87487400\,x^{18} \\ & - 45185800\,x^{17} + 22669812\,x^{16} + 28995824\,x^{15} - 2003056\,x^{14} \\ & - 8484336\,x^{13} + 2153384\,x^{12} + 2788592\,x^{11} - 2089416\,x^{10} \\ & - 810924\,x^{9} + 1867358\,x^{8} - 462744\,x^{7} - 1416160\,x^{6} + 1298336\,x^{5} \\ & + 567270\,x^{4} - 1411576\,x^{3} + 274428\,x^{2} + 761508\,x - 474188\end{aligned}$$

$$\begin{aligned}p_{19}(4,x) = {} & 347373600\,x^{26} - 6906900\,x^{18} - 26640900\,x^{17} - 32890000\,x^{16} \\ & - 2833600\,x^{15} + 21073888\,x^{14} + 4222064\,x^{13} - 11695376\,x^{12} + 93184\,x^{11} \\ & + 7015776\,x^{10} - 2148368\,x^{9} - 3877344\,x^{8} + 2615696\,x^{7} + 1709768\,x^{6} \\ & - 2256248\,x^{5} - 321328\,x^{4} + 1433056\,x^{3} - 321452\,x^{2} - 526152\,x + 289652\end{aligned}$$

$$\begin{aligned}p_{20}(4,x) = {} & 225792840\,x^{24} - 3108105\,x^{16} - 10656360\,x^{15} - 9735440\,x^{14} \\ & + 4675440\,x^{13} + 9053352\,x^{12} - 3244472\,x^{11} - 6000456\,x^{10} \\ & + 3296664\,x^{9} + 3237852\,x^{8} - 2899056\,x^{7} - 1241448\,x^{6} + 2021812\,x^{5} \\ & + 122118\,x^{4} - 1042716\,x^{3} + 244960\,x^{2} + 308352\,x - 156651\end{aligned}$$

$$\begin{aligned}p_{21}(4,x) = {} & 129024480\,x^{22} - 1184040\,x^{14} - 3552120\,x^{13} - 2104960\,x^{12} + 2955040\,x^{11} \\ & + 2392368\,x^{10} - 2459160\,x^{9} - 1520728\,x^{8} + 1939776\,x^{7} + 573824\,x^{6} \\ & - 1251824\,x^{5} - 6864\,x^{4} + 586104\,x^{3} - 145332\,x^{2} - 152988\,x + 74360\end{aligned}$$

$$\begin{aligned}p_{22}(4,x) = {} & 64512240\,x^{20} - 376740\,x^{12} - 968760\,x^{11} - 263120\,x^{10} \\ & + 1039600\,x^{9} + 346472\,x^{8} - 865168\,x^{7} - 145392\,x^{6} + 564112\,x^{5} \\ & - 30104\,x^{4} - 258288\,x^{3} + 69992\,x^{2} + 63612\,x - 30614\end{aligned}$$

$$\begin{aligned}p_{23}(4,x) = {} & 28048800\,x^{18} - 98280\,x^{10} - 210600\,x^{9} + 10400\,x^{8} + 244800\,x^{7} - 672\,x^{6} \\ & - 181904\,x^{5} + 24816\,x^{4} + 87936\,x^{3} - 27424\,x^{2} - 21744\,x + 10752\end{aligned}$$

$$\begin{aligned}p_{24}(4,x) = {} & 10518300\,x^{16} - 20475\,x^{8} - 35100\,x^{7} + 13000\,x^{6} + 38600\,x^{5} \\ & - 11028\,x^{4} - 22124\,x^{3} + 8556\,x^{2} + 5916\,x - 3146\end{aligned}$$

$$p_{25}(4,x) = 3365856\,x^{14} - 3276\,x^{6} - 4212\,x^{5} + 2912\,x^{4} + 3728\,x^{3} - 2024\,x^{2} - 1212\,x + 740$$

$$p_{26}(4,x) = 906192\,x^{12} - 378\,x^{4} - 324\,x^{3} + 328\,x^{2} + 168\,x - 132$$

$$p_{27}(4,x) = 201376\,x^{10} - 28\,x^{2} - 12\,x + 16$$

$$p_{28}(4,x) = 35960\,x^{8} - 1$$

$$p_{29}(4,x) = 4960\,x^{6}$$

$$p_{30}(4,x) = 496\,x^{4}$$

$$p_{31}(4,x) = 32\,x^{2}$$

$$p_{32}(4,x) = 1$$

Definition of $\overleftarrow{P_5}(x,y)$

For the case of base $b=5$, we have the numbers $e_1=0$, $e_3=6$, $e_4=2$, $e_6=3$, and $d=3$. In this case, $\overleftarrow{P_5}(x,y)$ is of degree 26 in y. We compute each $p_j(5,x)$ directly:

$$\begin{aligned} p_0(5,x) = \; & x^{52} - x^{44} - 6\,x^{43} - 31\,x^{42} - 110\,x^{41} - 351\,x^{40} - 932\,x^{39} - 2286\,x^{38} \\ & - 4980\,x^{37} - 10185\,x^{36} - 19082\,x^{35} - 33909\,x^{34} - 56162\,x^{33} \\ & - 88827\,x^{32} - 132336\,x^{31} - 189144\,x^{30} - 256384\,x^{29} - 334461\,x^{28} \\ & - 415630\,x^{27} - 498131\,x^{26} - 570294\,x^{25} - 630515\,x^{24} - 666932\,x^{23} \\ & - 681678\,x^{22} - 666932\,x^{21} - 630515\,x^{20} - 570294\,x^{19} - 498131\,x^{18} \\ & - 415630\,x^{17} - 334461\,x^{16} - 256384\,x^{15} - 189144\,x^{14} - 132336\,x^{13} \\ & - 88827\,x^{12} - 56162\,x^{11} - 33909\,x^{10} - 19082\,x^{9} - 10185\,x^{8} \\ & - 4980\,x^{7} - 2286\,x^{6} - 932\,x^{5} - 351\,x^{4} - 110\,x^{3} - 31\,x^{2} - 6\,x - 1 \end{aligned}$$

$$\begin{aligned} p_1(5,x) = \; & 26\,x^{50} - 22\,x^{42} - 126\,x^{41} - 607\,x^{40} - 2024\,x^{39} - 5972\,x^{38} - 14684\,x^{37} \\ & - 32866\,x^{36} - 65176\,x^{35} - 119498\,x^{34} - 199546\,x^{33} - 310465\,x^{32} \\ & - 445632\,x^{31} - 596576\,x^{30} - 738672\,x^{29} - 845512\,x^{28} - 883744\,x^{27} \\ & - 823046\,x^{26} - 651094\,x^{25} - 358203\,x^{24} + 19336\,x^{23} + 462788\,x^{22} \\ & + 892404\,x^{21} + 1285870\,x^{20} + 1559336\,x^{19} + 1723818\,x^{18} + 1730218\,x^{17} \\ & + 1634321\,x^{16} + 1427056\,x^{15} + 1183720\,x^{14} + 910944\,x^{13} + 667640\,x^{12} \\ & + 452352\,x^{11} + 291694\,x^{10} + 172150\,x^{9} + 96443\,x^{8} + 48520\,x^{7} \\ & + 23092\,x^{6} + 9524\,x^{5} + 3710\,x^{4} + 1160\,x^{3} + 346\,x^{2} + 66\,x + 13 \end{aligned}$$

$$\begin{aligned} p_2(5,x) = \; & 325\,x^{48} - 231\,x^{40} - 1260\,x^{39} - 5630\,x^{38} - 17556\,x^{37} - 47562\,x^{36} \\ & - 107404\,x^{35} - 216954\,x^{34} - 386148\,x^{33} - 622873\,x^{32} - 903888\,x^{31} \\ & - 1189416\,x^{30} - 1408464\,x^{29} - 1483896\,x^{28} - 1358256\,x^{27} \\ & - 1004888\,x^{26} - 477984\,x^{25} + 127905\,x^{24} + 658108\,x^{23} + 963654\,x^{22} \\ & + 956484\,x^{21} + 577090\,x^{20} - 50028\,x^{19} - 871362\,x^{18} - 1609364\,x^{17} \\ & - 2240031\,x^{16} - 2523288\,x^{15} - 2584844\,x^{14} - 2320872\,x^{13} - 1957500\,x^{12} \\ & - 1470528\,x^{11} - 1049976\,x^{10} - 665328\,x^{9} - 403621\,x^{8} - 213204\,x^{7} \\ & - 108738\,x^{6} - 46252\,x^{5} - 19326\,x^{4} - 6108\,x^{3} - 1994\,x^{2} - 372\,x - 87 \end{aligned}$$

$$\begin{aligned} p_3(5,x) = \; & 2600\,x^{46} - 1540\,x^{38} - 7980\,x^{37} - 32870\,x^{36} - 95304\,x^{35} - 235044\,x^{34} \\ & - 482324\,x^{33} - 866470\,x^{32} - 1356096\,x^{31} - 1868784\,x^{30} - 2253200\,x^{29} \\ & - 2334120\,x^{28} - 2002688\,x^{27} - 1251136\,x^{26} - 260016\,x^{25} \\ & + 688312\,x^{24} + 1289792\,x^{23} + 1339812\,x^{22} + 885148\,x^{21} + 139134\,x^{20} \\ & - 536568\,x^{19} - 801628\,x^{18} - 589108\,x^{17} + 169554\,x^{16} + 1020256\,x^{15} \\ & + 1954616\,x^{14} + 2412936\,x^{13} + 2665108\,x^{12} + 2375248\,x^{11} \\ & + 2018232\,x^{10} + 1427104\,x^{9} + 986264\,x^{8} + 559488\,x^{7} + 318084\,x^{6} \\ & + 141036\,x^{5} + 65478\,x^{4} + 20968\,x^{3} + 7748\,x^{2} + 1404\,x + 394 \end{aligned}$$

$$p_4(5,x) = 14950\,x^{44} - 7315\,x^{36} - 35910\,x^{35} - 135375\,x^{34} - 362406\,x^{33} - 805035\,x^{32} - 1479952\,x^{31} - 2315896\,x^{30} - 3084560\,x^{29} - 3441236\,x^{28} - 3105912\,x^{27} - 1980988\,x^{26} - 355480\,x^{25} + 1206588\,x^{24} + 2108368\,x^{23} + 1997000\,x^{22} + 1061056\,x^{21} - 135269\,x^{20} - 947354\,x^{19} - 984961\,x^{18} - 430618\,x^{17} + 229979\,x^{16} + 557304\,x^{15} + 161524\,x^{14} - 462952\,x^{13} - 1439222\,x^{12} - 1829780\,x^{11} - 2175482\,x^{10} - 1807284\,x^{9} - 1539646\,x^{8} - 957184\,x^{7} - 641544\,x^{6} - 298448\,x^{5} - 160917\,x^{4} - 51946\,x^{3} - 22521\,x^{2} - 3930\,x - 1345$$

$$p_5(5,x) = 65780\,x^{42} - 26334\,x^{34} - 122094\,x^{33} - 417639\,x^{32} - 1023264\,x^{31} - 2019600\,x^{30} - 3259376\,x^{29} - 4302408\,x^{28} - 4586400\,x^{27} - 3623704\,x^{26} - 1510392\,x^{25} + 1015956\,x^{24} + 2878144\,x^{23} + 3193056\,x^{22} + 1987152\,x^{21} + 129304\,x^{20} - 1246560\,x^{19} - 1431450\,x^{18} - 690754\,x^{17} + 143295\,x^{16} + 472464\,x^{15} + 231496\,x^{14} - 118008\,x^{13} + 60684\,x^{12} + 344944\,x^{11} + 1155324\,x^{10} + 1246284\,x^{9} + 1559630\,x^{8} + 1082256\,x^{7} + 933912\,x^{6} + 454048\,x^{5} + 302184\,x^{4} + 97152\,x^{3} + 51630\,x^{2} + 8574\,x + 3663$$

$$p_6(5,x) = 230230\,x^{40} - 74613\,x^{32} - 325584\,x^{31} - 1000008\,x^{30} - 2217072\,x^{29} - 3817656\,x^{28} - 5241712\,x^{27} - 5509448\,x^{26} - 4022928\,x^{25} - 1002820\,x^{24} + 2320656\,x^{23} + 4188360\,x^{22} + 3737360\,x^{21} + 1529688\,x^{20} - 838896\,x^{19} - 1817656\,x^{18} - 1246304\,x^{17} - 165663\,x^{16} + 421928\,x^{15} + 272580\,x^{14} - 81672\,x^{13} - 84868\,x^{12} + 63912\,x^{11} - 183876\,x^{10} - 281032\,x^{9} - 956294\,x^{8} - 744120\,x^{7} - 988636\,x^{6} - 491688\,x^{5} - 444444\,x^{4} - 139072\,x^{3} - 96168\,x^{2} - 14928\,x - 8235$$

$$p_7(5,x) = 657800\,x^{38} - 170544\,x^{30} - 697680\,x^{29} - 1899240\,x^{28} - 3751968\,x^{27} - 5492496\,x^{26} - 6071312\,x^{25} - 4434872\,x^{24} - 804544\,x^{23} + 2965872\,x^{22} + 4857424\,x^{21} + 3835656\,x^{20} + 941952\,x^{19} - 1387392\,x^{18} - 1808880\,x^{17} - 798440\,x^{16} + 268672\,x^{15} + 409208\,x^{14} - 3672\,x^{13} - 119372\,x^{12} + 24528\,x^{11} - 25176\,x^{10} - 83304\,x^{9} + 266148\,x^{8} + 197728\,x^{7} + 732088\,x^{6} + 347816\,x^{5} + 514212\,x^{4} + 149680\,x^{3} + 147912\,x^{2} + 20832\,x + 15624$$

$$p_8(5,x) = 1562275\,x^{36} - 319770\,x^{28} - 1220940\,x^{27} - 2897310\,x^{26} - 4988412\,x^{25} - 5963022\,x^{24} - 4701112\,x^{23} - 1225380\,x^{22} + 2886312\,x^{21} + 4795362\,x^{20} + 3509220\,x^{19} + 640530\,x^{18} - 1526316\,x^{17} - 1564722\,x^{16} - 279504\,x^{15} + 429944\,x^{14} + 201984\,x^{13} - 69135\,x^{12} - 29922\,x^{11} - 933\,x^{10} - 54090\,x^{9} + 48915\,x^{8} + 93564\,x^{7} - 323910\,x^{6} - 104164\,x^{5} - 458967\,x^{4} - 111462\,x^{3} - 189003\,x^{2} - 22830\,x - 25389$$

$$p_9(5,x) = 3124550\,x^{34} - 497420\,x^{26} - 1763580\,x^{25} - 3569150\,x^{24} - 5173168\,x^{23} - 4694040\,x^{22} - 1691976\,x^{21} + 1939652\,x^{20} + 4084080\,x^{19} + 3202628\,x^{18} + 356356\,x^{17} - 1362966\,x^{16} - 1052480\,x^{15} - 95904\,x^{14} + 366768\,x^{13} + 161704\,x^{12} - 120032\,x^{11} - 33498\,x^{10} + 69766\,x^{9} - 29965\,x^{8} - 16920\,x^{7} + 13940\,x^{6} - 66364\,x^{5} + 294822\,x^{4} + 37800\,x^{3} + 199786\,x^{2} + 18138\,x + 35689$$

$$\begin{aligned}p_{10}(5,x) = {}& 5311735\,x^{32} - 646646\,x^{24} - 2116296\,x^{23} - 3543956\,x^{22} - 4064632\,x^{21}\\ & - 2343484\,x^{20} + 1030744\,x^{19} + 2980692\,x^{18} + 2442440\,x^{17} + 551122\,x^{16}\\ & - 1054768\,x^{15} - 895576\,x^{14} + 175824\,x^{13} + 324088\,x^{12} - 103440\,x^{11}\\ & - 57608\,x^{10} + 123424\,x^{9} - 25429\,x^{8} - 107684\,x^{7} + 76950\,x^{6}\\ & + 67844\,x^{5} - 103070\,x^{4} + 30852\,x^{3} - 171226\,x^{2} - 6948\,x - 43675\end{aligned}$$

$$\begin{aligned}p_{11}(5,x) = {}& 7726160\,x^{30} - 705432\,x^{22} - 2116296\,x^{21} - 2804932\,x^{20}\\ & - 2217072\,x^{19} - 259896\,x^{18} + 1981928\,x^{17} + 1983852\,x^{16}\\ & + 267072\,x^{15} - 631696\,x^{14} - 439920\,x^{13} - 4440\,x^{12} + 201216\,x^{11}\\ & + 26304\,x^{10} - 94224\,x^{9} + 23336\,x^{8} + 29760\,x^{7} - 21852\,x^{6}\\ & + 28412\,x^{5} - 27666\,x^{4} - 57048\,x^{3} + 112436\,x^{2} - 6948\,x + 46698\end{aligned}$$

$$\begin{aligned}p_{12}(5,x) = {}& 9657700\,x^{28} - 646646\,x^{20} - 1763580\,x^{19} - 1721590\,x^{18} - 554268\,x^{17}\\ & + 688194\,x^{16} + 1354288\,x^{15} + 584168\,x^{14} - 580944\,x^{13} - 394212\,x^{12}\\ & + 242632\,x^{11} + 105060\,x^{10} - 157784\,x^{9} + 2204\,x^{8} + 116976\,x^{7} - 41832\,x^{6}\\ & - 81984\,x^{5} + 62193\,x^{4} + 35530\,x^{3} - 46247\,x^{2} + 18138\,x - 43675\end{aligned}$$

$$\begin{aligned}p_{13}(5,x) = {}& 10400600\,x^{26} - 497420\,x^{18} - 1220940\,x^{17} - 765510\,x^{16}\\ & + 341088\,x^{15} + 647088\,x^{14} + 356048\,x^{13} - 82600\,x^{12} - 310688\,x^{11}\\ & - 12824\,x^{10} + 190920\,x^{9} - 28300\,x^{8} - 104128\,x^{7} + 44064\,x^{6}\\ & + 38512\,x^{5} - 29688\,x^{4} + 6752\,x^{3} - 4422\,x^{2} - 22830\,x + 35689\end{aligned}$$

$$\begin{aligned}p_{14}(5,x) = {}& 9657700\,x^{24} - 319770\,x^{16} - 697680\,x^{15} - 193800\,x^{14} + 511632\,x^{13}\\ & + 275400\,x^{12} - 174896\,x^{11} - 104808\,x^{10} + 62640\,x^{9} + 28364\,x^{8} - 41232\,x^{7}\\ & - 3528\,x^{6} + 41200\,x^{5} - 15192\,x^{4} - 37200\,x^{3} + 28824\,x^{2} + 20832\,x - 25389\end{aligned}$$

$$\begin{aligned}p_{15}(5,x) = {}& 7726160\,x^{22} - 170544\,x^{14} - 325584\,x^{13} + 23256\,x^{12} + 341088\,x^{11}\\ & + 26928\,x^{10} - 223312\,x^{9} + 9064\,x^{8} + 135360\,x^{7} - 31984\,x^{6}\\ & - 80208\,x^{5} + 36984\,x^{4} + 42368\,x^{3} - 30720\,x^{2} - 14928\,x + 15624\end{aligned}$$

$$\begin{aligned}p_{16}(5,x) = {}& 5311735\,x^{20} - 74613\,x^{12} - 122094\,x^{11} + 51357\,x^{10}\\ & + 149226\,x^{9} - 37587\,x^{8} - 111044\,x^{7} + 38434\,x^{6} + 65324\,x^{5}\\ & - 33249\,x^{4} - 30122\,x^{3} + 21435\,x^{2} + 8574\,x - 8235\end{aligned}$$

$$\begin{aligned}p_{17}(5,x) = {}& 3124550\,x^{18} - 26334\,x^{10} - 35910\,x^{9} + 27645\,x^{8} + 45144\,x^{7} - 24372\,x^{6}\\ & - 31804\,x^{5} + 19086\,x^{4} + 14952\,x^{3} - 11082\,x^{2} - 3930\,x + 3663\end{aligned}$$

$$\begin{aligned}p_{18}(5,x) = {}& 1562275\,x^{16} - 7315\,x^{8} - 7980\,x^{7} + 8930\,x^{6} + 9196\,x^{5}\\ & - 7434\,x^{4} - 5164\,x^{3} + 4310\,x^{2} + 1404\,x - 1345\end{aligned}$$

$$p_{19}(5,x) = 657800\,x^{14} - 1540\,x^{6} - 1260\,x^{5} + 1850\,x^{4} + 1144\,x^{3} - 1220\,x^{2} - 372\,x + 394$$

$$p_{20}(5,x) = 230230\,x^{12} - 231\,x^4 - 126\,x^3 + 229\,x^2 + 66\,x - 87$$

$$p_{21}(5,x) = 65780\,x^{10} - 22\,x^2 - 6\,x + 13$$

$$p_{22}(5,x) = 14950\,x^8 - 1$$

$$p_{23}(5,x) = 2600\,x^6$$

$$p_{24}(5,x) = 325\,x^4$$

$$p_{25}(5,x) = 26\,x^2$$

$$p_{26}(5,x) = 1$$

Definition of $\overleftarrow{P_b}(x,y)$ for bases $6 \le b \le 20$

For the case of base $6 \le b \le 20$, we have the numbers $e_1 = 0$, $e_3 = 4$, $e_4 = 2$, $e_6 = 3$, and $d = 3$. In this case, $\overleftarrow{P_b}(x,y)$ is of degree 22 in y. We compute each $p_j(b,x)$ directly:

$$\begin{aligned} p_0(b,x) = {} & x^{44} - x^{36} - 2\,x^{35} - 13\,x^{34} - 22\,x^{33} - 82\,x^{32} - 122\,x^{31} - 337\,x^{30} - 450\,x^{29} \\ & - 1014\,x^{28} - 1230\,x^{27} - 2373\,x^{26} - 2634\,x^{25} - 4475\,x^{24} - 4564\,x^{23} \\ & - 6950\,x^{22} - 6524\,x^{21} - 9008\,x^{20} - 7780\,x^{19} - 9814\,x^{18} - 7780\,x^{17} - 9008\,x^{16} \\ & - 6524\,x^{15} - 6950\,x^{14} - 4564\,x^{13} - 4475\,x^{12} - 2634\,x^{11} - 2373\,x^{10} - 1230\,x^9 \\ & - 1014\,x^8 - 450\,x^7 - 337\,x^6 - 122\,x^5 - 82\,x^4 - 22\,x^3 - 13\,x^2 - 2\,x - 1 \end{aligned}$$

$$\begin{aligned} p_1(b,x) = {} & 22\,x^{42} - 18\,x^{34} - 34\,x^{33} - 197\,x^{32} - 312\,x^{31} - 1004\,x^{30} - 1374\,x^{29} - 3155\,x^{28} \\ & - 3780\,x^{27} - 6712\,x^{26} - 7038\,x^{25} - 9737\,x^{24} - 8712\,x^{23} - 8380\,x^{22} - 5516\,x^{21} \\ & + 222\,x^{20} + 3528\,x^{19} + 14272\,x^{18} + 14940\,x^{17} + 27186\,x^{16} + 22720\,x^{15} \\ & + 32288\,x^{14} + 23100\,x^{13} + 28022\,x^{12} + 17304\,x^{11} + 18470\,x^{10} + 9726\,x^9 \\ & + 9247\,x^8 + 4032\,x^7 + 3428\,x^6 + 1170\,x^5 + 889\,x^4 + 212\,x^3 + 144\,x^2 + 18\,x + 11 \end{aligned}$$

$$\begin{aligned} p_2(b,x) = {} & 231\,x^{40} - 153\,x^{32} - 272\,x^{31} - 1384\,x^{30} - 2040\,x^{29} - 5508\,x^{28} \\ & - 6842\,x^{27} - 12353\,x^{26} - 12830\,x^{25} - 15814\,x^{24} - 12900\,x^{23} \\ & - 7878\,x^{22} - 2308\,x^{21} + 8306\,x^{20} + 10804\,x^{19} + 14630\,x^{18} + 10164\,x^{17} \\ & - 3504\,x^{16} - 9072\,x^{15} - 36744\,x^{14} - 31792\,x^{13} - 59072\,x^{12} \\ & - 39564\,x^{11} - 55758\,x^{10} - 30076\,x^9 - 35389\,x^8 - 15076\,x^7 \\ & - 15410\,x^6 - 4908\,x^5 - 4464\,x^4 - 950\,x^3 - 779\,x^2 - 82\,x - 62 \end{aligned}$$

$$\begin{aligned} p_3(b,x) = {} & 1540\,x^{38} - 816\,x^{30} - 1360\,x^{29} - 5960\,x^{28} - 8120\,x^{27} - 17612\,x^{26} - 19422\,x^{25} \\ & - 24579\,x^{24} - 20104\,x^{23} - 9344\,x^{22} + 268\,x^{21} + 16570\,x^{20} + 18424\,x^{19} \\ & + 15396\,x^{18} + 6604\,x^{17} - 11182\,x^{16} - 13600\,x^{15} - 10048\,x^{14} - 1904\,x^{13} \\ & + 36184\,x^{12} + 29888\,x^{11} + 75808\,x^{10} + 42316\,x^{9} + 71518\,x^{8} + 29584\,x^{7} \\ & + 40180\,x^{6} + 11900\,x^{5} + 13822\,x^{4} + 2608\,x^{3} + 2716\,x^{2} + 246\,x + 235 \end{aligned}$$

$$\begin{aligned} p_4(b,x) = {} & 7315\,x^{36} - 3060\,x^{28} - 4760\,x^{27} - 17500\,x^{26} - 21840\,x^{25} - 35308\,x^{24} \\ & - 32994\,x^{23} - 19829\,x^{22} - 4994\,x^{21} + 24034\,x^{20} + 29958\,x^{19} \\ & + 24009\,x^{18} + 10074\,x^{17} - 16669\,x^{16} - 20296\,x^{15} - 9148\,x^{14} + 456\,x^{13} \\ & + 15072\,x^{12} + 9960\,x^{11} - 31108\,x^{10} - 19928\,x^{9} - 77472\,x^{8} - 30900\,x^{7} \\ & - 65810\,x^{6} - 17676\,x^{5} - 29189\,x^{4} - 4850\,x^{3} - 6809\,x^{2} - 534\,x - 666 \end{aligned}$$

$$\begin{aligned} p_5(b,x) = {} & 26334\,x^{34} - 8568\,x^{26} - 12376\,x^{25} - 36764\,x^{24} - 41496\,x^{23} - 42588\,x^{22} \\ & - 28886\,x^{21} + 17281\,x^{20} + 34980\,x^{19} + 44232\,x^{18} + 29862\,x^{17} \\ & - 14523\,x^{16} - 25744\,x^{15} - 17080\,x^{14} - 5112\,x^{13} + 14508\,x^{12} \\ & + 10032\,x^{11} - 12480\,x^{10} - 8344\,x^{9} + 31916\,x^{8} + 11584\,x^{7} + 66656\,x^{6} \\ & + 15348\,x^{5} + 43650\,x^{4} + 6232\,x^{3} + 12934\,x^{2} + 870\,x + 1491 \end{aligned}$$

$$\begin{aligned} p_6(b,x) = {} & 74613\,x^{32} - 18564\,x^{24} - 24752\,x^{23} - 56056\,x^{22} - 56056\,x^{21} - 20020\,x^{20} \\ & + 4862\,x^{19} + 58795\,x^{18} + 59730\,x^{17} + 9426\,x^{16} - 13848\,x^{15} - 32004\,x^{14} \\ & - 21112\,x^{13} + 16156\,x^{12} + 16104\,x^{11} - 9012\,x^{10} - 9528\,x^{9} + 10080\,x^{8} + 6416\,x^{7} \\ & - 33928\,x^{6} - 5168\,x^{5} - 45760\,x^{4} - 5188\,x^{3} - 19066\,x^{2} - 1060\,x - 2723 \end{aligned}$$

$$\begin{aligned} p_7(b,x) = {} & 170544\,x^{30} - 31824\,x^{22} - 38896\,x^{21} - 60632\,x^{20} - 51480\,x^{19} + 26884\,x^{18} \\ & + 48906\,x^{17} + 53889\,x^{16} + 34320\,x^{15} - 36288\,x^{14} - 40248\,x^{13} \\ & + 4956\,x^{12} + 13872\,x^{11} + 2728\,x^{10} - 1896\,x^{9} - 1948\,x^{8} + 480\,x^{7} \\ & - 3008\,x^{6} - 3312\,x^{5} + 30552\,x^{4} + 1856\,x^{3} + 21856\,x^{2} + 900\,x + 4138 \end{aligned}$$

$$\begin{aligned} p_8(b,x) = {} & 319770\,x^{28} - 43758\,x^{20} - 48620\,x^{19} - 41470\,x^{18} - 25740\,x^{17} \\ & + 61776\,x^{16} + 66066\,x^{15} + 3597\,x^{14} - 13398\,x^{13} - 27138\,x^{12} \\ & - 16866\,x^{11} + 21021\,x^{10} + 14794\,x^{9} - 14309\,x^{8} - 7076\,x^{7} + 10946\,x^{6} \\ & + 2484\,x^{5} - 7344\,x^{4} + 1500\,x^{3} - 18966\,x^{2} - 356\,x - 5296 \end{aligned}$$

$$\begin{aligned} p_9(b,x) = {} & 497420\,x^{26} - 48620\,x^{18} - 48620\,x^{17} - 7150\,x^{16} + 5720\,x^{15} + 55484\,x^{14} \\ & + 44902\,x^{13} - 33913\,x^{12} - 31548\,x^{11} + 10968\,x^{10} + 10158\,x^{9} - 3951\,x^{8} \\ & - 3176\,x^{7} + 4596\,x^{6} + 2884\,x^{5} - 7786\,x^{4} - 2488\,x^{3} + 11264\,x^{2} - 356\,x + 5746 \end{aligned}$$

$$\begin{aligned} p_{10}(b,x) = {} & 646646\,x^{24} - 43758\,x^{16} - 38896\,x^{15} + 21736\,x^{14} + 24024\,x^{13} \\ & + 20020\,x^{12} + 11154\,x^{11} - 26147\,x^{10} - 16170\,x^{9} + 18446\,x^{8} + 8892\,x^{7} \\ & - 11990\,x^{6} - 3780\,x^{5} + 7570\,x^{4} + 932\,x^{3} - 2514\,x^{2} + 900\,x - 5296 \end{aligned}$$

$$
\begin{aligned}
p_{11}(b,x) = {} & 705432\,x^{22} - 31824\,x^{14} - 24752\,x^{13} + 31304\,x^{12} + 24024\,x^{11} \\
& - 9828\,x^{10} - 7514\,x^{9} - 425\,x^{8} + 760\,x^{7} + 1088\,x^{6} \\
& - 660\,x^{5} + 1098\,x^{4} + 1240\,x^{3} - 3404\,x^{2} - 1060\,x + 4138
\end{aligned}
$$

$$
\begin{aligned}
p_{12}(b,x) = {} & 646646\,x^{20} - 18564\,x^{12} - 12376\,x^{11} + 24388\,x^{10} + 14560\,x^{9} - 17108\,x^{8} \\
& - 8502\,x^{7} + 10617\,x^{6} + 4250\,x^{5} - 7274\,x^{4} - 2198\,x^{3} + 5119\,x^{2} + 870\,x - 2723
\end{aligned}
$$

$$
\begin{aligned}
p_{13}(b,x) = {} & 497420\,x^{18} - 8568\,x^{10} - 4760\,x^{9} + 12740\,x^{8} + 5880\,x^{7} - 10612\,x^{6} \\
& - 3762\,x^{5} + 7051\,x^{4} + 1756\,x^{3} - 3944\,x^{2} - 534\,x + 1491
\end{aligned}
$$

$$
\begin{aligned}
p_{14}(b,x) = {} & 319770\,x^{16} - 3060\,x^{8} - 1360\,x^{7} + 4600\,x^{6} + 1560\,x^{5} \\
& - 3708\,x^{4} - 854\,x^{3} + 2041\,x^{2} + 246\,x - 666
\end{aligned}
$$

$$
p_{15}(b,x) = 170544\,x^{14} - 816\,x^{6} - 272\,x^{5} + 1112\,x^{4} + 248\,x^{3} - 724\,x^{2} - 82\,x + 235
$$

$$
p_{16}(b,x) = 74613\,x^{12} - 153\,x^{4} - 34\,x^{3} + 163\,x^{2} + 18\,x - 62
$$

$$
p_{17}(b,x) = 26334\,x^{10} - 18\,x^{2} - 2\,x + 11
$$

$$
p_{18}(b,x) = 7315\,x^{8} - 1
$$

$$
p_{19}(b,x) = 1540\,x^{6}
$$

$$
p_{20}(b,x) = 231\,x^{4}
$$

$$
p_{21}(b,x) = 22\,x^{2}
$$

$$
p_{22}(b,x) = 1
$$

A.3 Properties of $\overleftarrow{P_b}(x,y)$

In the proof of Lemma 2.4, we define $\overleftarrow{P_b}(x,y) = \sum_{j=0}^{r} p_j(b,x)y^j$ such that

$$
\overleftarrow{P_b}\left(x, y^2\right) = P_b\left(x+b, y\right),
$$

where $r = 2\left(e_3 + e_4 + e_6\right) + e_1 + d + 1$, and each of e_1, e_3, e_4, e_6 and d is given in Table 2.4. We note that these numbers are the same for bases $6 \le b \le 20$, so we will only show results for bases $b \leq 6$, with everything being identical to base 6 for bases $7 \leq b \leq 20$.

Tables A.1, A.2, A.3, A.4, and A.5 show the distinct roots, accurate to the digits shown, for each $p_j(b, x)$ and bases $b = 2$, $b = 3$, $b = 4$, $b = 5$, and $6 \leq b \leq 20$, respectively.

For the purposes of Table 2.6 and these tables, computations were done in Maple 17 using

```
[> sturm(sturmseq(p[j](x), x), x, -infinity, infinity);
```

and

```
[> sturm(sturmseq(p[j](x), x), x, -s, t);
```

where $s = \hat{a}_0$ and $t = \hat{a}_1$, to determine the exact number of real roots and verify that they are all in the interval $[-\hat{a}_0, \hat{a}_1]$. Then using

```
[> fsolve(p[j](x)=0,x);
```

we obtained the decimal approximations of these roots.

Table A.1 The Real Roots of $p_j(2,x)$

$p_j(2,x)$	**# of Real Roots**	**Real Roots**
$p_1(2,x)$	2	-0.55204, 9.74905
$p_2(2,x)$	2	-0.55149, 9.43156
$p_3(2,x)$	2	-0.55066, 9.11258
$p_4(2,x)$	2	-0.54946, 8.79206
$p_5(2,x)$	2	-0.54776, 8.46991
$p_6(2,x)$	2	-0.54535, 8.14605
$p_7(2,x)$	2	-0.54188, 7.82040
$p_8(2,x)$	4	-0.53662, -0.39777, -0.22711, 7.49286
$p_9(2,x)$	4	-0.52749, -0.46505, -0.22760, 7.16332
$p_{10}(2,x)$	2	-0.28224, 6.83166
$p_{11}(2,x)$	2	-0.37688, 6.49775
$p_{12}(2,x)$	2	-0.44499, 6.16145
$p_{13}(2,x)$	4	-0.48448, -0.30553, -0.20969, 5.82257
$p_{14}(2,x)$	4	-0.48981, -0.38762, -0.21494, 5.48094
$p_{15}(2,x)$	4	-0.47138, -0.45710, -0.26256, 5.13635
$p_{16}(2,x)$	2	-0.34006, 4.78853
$p_{17}(2,x)$	2	-0.40891, 4.43721
$p_{18}(2,x)$	4	-0.44546, -0.30008, -0.20140, 4.08205
$p_{19}(2,x)$	4	-0.44587, -0.37409, -0.21977, 3.72265
$p_{20}(2,x)$	2	-0.27060, 3.35854
$p_{21}(2,x)$	2	-0.33992, 2.98914
$p_{22}(2,x)$	2	-0.38964, 2.61374
$p_{23}(2,x)$	4	-0.39600, -0.30874, -0.20594, 2.23145
$p_{24}(2,x)$	2	-0.23030, 1.84115
$p_{25}(2,x)$	2	-0.27398, 1.44145
$p_{26}(2,x)$	2	-0.31084, 1.03072
$p_{27}(2,x)$	2	-0.31077, 0.60819
$p_{28}(2,x)$	2	-0.18570, 0.18570
$p_{29}(2,x)$	0	

Table A.2 The Real Roots of $p_j(3,x)$

$p_j(3,x)$	**# of Real Roots**	**Real Roots**
$p_1(3,x)$	2	0.47410, 3.34602
$p_2(3,x)$	2	-1.02267, 3.25209
$p_3(3,x)$	2	0.47466, 3.15798
$p_4(3,x)$	2	-0.99018, 3.06367
$p_5(3,x)$	2	0.47414, 2.96916
$p_6(3,x)$	2	-0.96437, 2.87444
$p_7(3,x)$	2	0.47267, 2.77951
$p_8(3,x)$	2	-0.94212, 2.68436
$p_9(3,x)$	2	0.47047, 2.58899
$p_{10}(3,x)$	2	-0.92190, 2.49338
$p_{11}(3,x)$	2	0.46771, 2.39753
$p_{12}(3,x)$	2	-0.90273, 2.30144
$p_{13}(3,x)$	2	0.46451, 2.20509
$p_{14}(3,x)$	2	-0.88388, 2.10848
$p_{15}(3,x)$	2	0.46094, 2.01160
$p_{16}(3,x)$	2	-0.86470, 1.91445
$p_{17}(3,x)$	2	0.45708, 1.81701
$p_{18}(3,x)$	2	-0.84452, 1.71929
$p_{19}(3,x)$	2	0.45304, 1.62128
$p_{20}(3,x)$	2	-0.82250, 1.52298
$p_{21}(3,x)$	2	0.44891, 1.42438
$p_{22}(3,x)$	2	-0.79749, 1.32549
$p_{23}(3,x)$	2	0.44481, 1.22633
$p_{24}(3,x)$	2	-0.76769, 1.12690
$p_{25}(3,x)$	2	0.44081, 1.02722
$p_{26}(3,x)$	2	-0.72997, 0.92734
$p_{27}(3,x)$	2	0.43702, 0.82730
$p_{28}(3,x)$	2	-0.67835, 0.72721
$p_{29}(3,x)$	2	0.43469, 0.62665
$p_{30}(3,x)$	2	-0.60034, 0.53186
$p_{31}(3,x)$	0	
$p_{32}(3,x)$	2	-0.46986, 0.39309
$p_{33}(3,x)$	0	
$p_{34}(3,x)$	2	-0.24631, 0.24631
$p_{35}(3,x)$	1	0
$p_{36}(3,x)$	1	0
$p_{37}(3,x)$	1	0
$p_{38}(3,x)$	0	

Table A.3 The Real Roots of $p_j(4,x)$

$p_j(4,x)$	**# of Real Roots**	**Real Roots**
$p_1(4,x)$	2	1.00000, 2.35497
$p_2(4,x)$	2	-1.20129, 2.26585
$p_3(4,x)$	2	0.94204, 2.17720
$p_4(4,x)$	2	-1.12256, 2.08899
$p_5(4,x)$	2	0.90246, 2.00117
$p_6(4,x)$	2	-1.06697, 1.91372
$p_7(4,x)$	2	0.86419, 1.82660
$p_8(4,x)$	2	-1.01910, 1.73981
$p_9(4,x)$	2	0.83314, 1.65331
$p_{10}(4,x)$	2	-0.97379, 1.56709
$p_{11}(4,x)$	2	0.80283, 1.48114
$p_{12}(4,x)$	2	-0.92834, 1.39544
$p_{13}(4,x)$	2	0.76702, 1.30999
$p_{14}(4,x)$	2	-0.88134, 1.22477
$p_{15}(4,x)$	2	0.72914, 1.13977
$p_{16}(4,x)$	2	-0.83230, 1.05500
$p_{17}(4,x)$	2	0.69477, 0.97038
$p_{18}(4,x)$	2	-0.78088, 0.88632
$p_{19}(4,x)$	2	0.67035, 0.79997
$p_{20}(4,x)$	2	-0.72571, 0.72708
$p_{21}(4,x)$	0	
$p_{22}(4,x)$	2	-0.66364, 0.61641
$p_{23}(4,x)$	0	
$p_{24}(4,x)$	2	-0.58730, 0.53006
$p_{25}(4,x)$	0	
$p_{26}(4,x)$	2	-0.47585, 0.43135
$p_{27}(4,x)$	0	
$p_{28}(4,x)$	2	-0.26948, 0.26948
$p_{29}(4,x)$	1	0
$p_{30}(4,x)$	1	0
$p_{31}(4,x)$	1	0
$p_{32}(4,x)$	0	

Table A.4 The Real Roots of $p_j(5,x)$

$p_j(5,x)$	**# of Real Roots**	**Real Roots**
$p_1(5,x)$	2	1.07763, 1.93578
$p_2(5,x)$	2	-1.24204, 1.83236
$p_3(5,x)$	2	1.02294, 1.73055
$p_4(5,x)$	2	-1.12553, 1.63008
$p_5(5,x)$	2	0.97012, 1.53061
$p_6(5,x)$	2	-1.04770, 1.43197
$p_7(5,x)$	2	0.92640, 1.33378
$p_8(5,x)$	2	-0.97807, 1.23601
$p_9(5,x)$	2	0.88969, 1.13744
$p_{10}(5,x)$	2	-0.90968, 1.04040
$p_{11}(5,x)$	2	0.87744, 0.92144
$p_{12}(5,x)$	2	-0.84127, 0.86593
$p_{13}(5,x)$	0	
$p_{14}(5,x)$	2	-0.77196, 0.75628
$p_{15}(5,x)$	0	
$p_{16}(5,x)$	2	-0.69857, 0.66857
$p_{17}(5,x)$	0	
$p_{18}(5,x)$	2	-0.61481, 0.57997
$p_{19}(5,x)$	0	
$p_{20}(5,x)$	2	-0.50339, 0.47368
$p_{21}(5,x)$	0	
$p_{22}(5,x)$	2	-0.30073, 0.30073
$p_{23}(5,x)$	1	0
$p_{24}(5,x)$	1	0
$p_{25}(5,x)$	1	0
$p_{26}(5,x)$	0	

Table A.5 The Real Roots of $p_j(6,x)$

$p_j(6,x)$	**# of Real Roots**	**Real Roots**
$p_1(6,x)$	4	-1.39271, -1.22149, 1.13848, 1.62883
$p_2(6,x)$	2	-1.29580, 1.50885
$p_3(6,x)$	2	1.09413, 1.38634
$p_4(6,x)$	2	-1.12133, 1.27396
$p_5(6,x)$	2	1.07551, 1.12285
$p_6(6,x)$	2	-1.00850, 1.06265
$p_7(6,x)$	0	
$p_8(6,x)$	2	-0.91270, 0.92053
$p_9(6,x)$	0	
$p_{10}(6,x)$	2	-0.82346, 0.81779
$p_{11}(6,x)$	0	
$p_{12}(6,x)$	2	-0.73550, 0.72442
$p_{13}(6,x)$	0	
$p_{14}(6,x)$	2	-0.64139, 0.62791
$p_{15}(6,x)$	0	
$p_{16}(6,x)$	2	-0.52561, 0.51341
$p_{17}(6,x)$	0	
$p_{18}(6,x)$	2	-0.32883, 0.32883
$p_{19}(6,x)$	1	0
$p_{20}(6,x)$	1	0
$p_{21}(6,x)$	1	0
$p_{22}(6,x)$	0	

A.4 The Proof of Lemma 2.9

In Section 2.5, we make use of the following lemma, which utilizes an idea introduced in [8] and formalized in [1] and [2]. The proof is reproduced here from [9] for the readers' benefit.

Lemma A.1. *Let* $f(x) = \sum_{j=0}^{n} a_j x^j \in \mathbb{Z}[x]$*, where* $a_j \geq 0$ *for* $j \in \{0, 1, \ldots, n\}$*. Suppose* $\alpha = re^{i\theta}$ *is a root of* $f(x)$ *with* $0 < \theta < \pi/2$ *and* $r > 1$*. Let*

$$B = \max_{\pi/(2\theta) < k < \pi/\theta} \left\{ \frac{r^k(r-1)}{1 + \cot(\pi - k\theta)} \right\},$$

where the maximum is over $k \in \mathbb{Z}$*. Then there is some* $j \in \{0, 1, \ldots, n-1\}$ *such that* $a_j > Ba_n$*.*

Proof. Observe that $\pi/\theta > 2$ implies there is an integer in $(\pi/(2\theta), \pi/\theta)$. Fix an arbitrary integer $k \in (\pi/(2\theta), \pi/\theta)$. It suffices to show that there is a $j \in \{0, 1, \ldots, n-1\}$ such that

$$a_j > B_k a_n, \qquad \text{where } B_k = \frac{r^k(r-1)}{1 + \cot(\pi - k\theta)}.$$

Assume $a_j \leq B_k a_n$ for all $j \in \{0, 1, \ldots, n-1\}$. Set

$$\gamma = \frac{a_n}{1 + \cot(\pi - k\theta)} \qquad \text{and} \qquad k' = \left\lfloor \frac{\pi}{2\theta} \right\rfloor.$$

Then $k' < k$. Also,

$$j\theta \in (0, \pi/2] \qquad \text{for } j \in \{1, 2, \ldots, k'\}$$

and

$$j\theta \in (\pi/2, \pi) \qquad \text{for } j \in \{k'+1, k'+2, \ldots, k\}.$$

Since

$$\begin{aligned} \alpha^{-j} = r^{-j} e^{-ij\theta} &= r^{-j}\left(\cos(-j\theta) + i\sin(-j\theta)\right) \\ &= r^{-j}\left(\cos(j\theta) - i\sin(j\theta)\right), \end{aligned}$$

we can conclude that

$$\Re(\alpha^{-j}) \geq 0 \qquad \text{for } j \in \{1, 2, \ldots, k'\}, \tag{A.4.1}$$

$$\Re(\alpha^{-j}) < 0 \qquad \text{for } j \in \{k'+1, k'+2, \ldots, k\} \tag{A.4.2}$$

and

$$\Im(\alpha^{-j}) < 0 \qquad \text{for } j \in \{1, 2, \ldots, k\}. \tag{A.4.3}$$

Since

$$0 < \pi - k\theta \leq \pi - j\theta < \frac{\pi}{2} \quad \text{for } j \in \{k'+1, k'+2, \ldots, k\},$$

we obtain

$$0 < \tan(\pi - k\theta) \leq \tan(\pi - j\theta) \quad \text{for } j \in \{k'+1, k'+2, \ldots, k\}.$$

Now, we derive an inequality relating the imaginary and real parts of α^{-j}. For $j \in \{k'+1, k'+2, \ldots, k\}$, we have

$$\begin{aligned} \left|\Im(\alpha^{-j})\right| &= r^{-j}\sin(j\theta) = r^{-j}\sin(\pi - j\theta) \\ &= r^{-j}\tan(\pi - j\theta)\cos(\pi - j\theta) \\ &= \tan(\pi - j\theta)\left|\Re(\alpha^{-j})\right| \\ &\geq \tan(\pi - k\theta)\left|\Re(\alpha^{-j})\right|. \end{aligned} \tag{A.4.4}$$

Motivated by the approach in [**Polya1964**], we consider

$$\left|\frac{f(\alpha)}{\alpha^n}\right| = \left|a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k}}{\alpha^k} + \sum_{j=k+1}^{n} \frac{a_{n-j}}{\alpha^j}\right|,$$

where, in the case $k > n$, we interpret a_{n-j} to be zero for all $j > n$. We consider two possible cases.

Case 1: $\left|\Re\left(\frac{a_{n-k'-1}}{\alpha^{k'+1}} + \cdots + \frac{a_{n-k}}{\alpha^k}\right)\right| \leq a_n - \gamma.$

In this case, we use that

$$\begin{aligned} \left|\frac{f(\alpha)}{\alpha^n}\right| &\geq \left|a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k}}{\alpha^k}\right| - \sum_{j=k+1}^{n}\left|\frac{a_{n-j}}{\alpha^j}\right| \\ &\geq \left|a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k}}{\alpha^k}\right| - \sum_{j=k+1}^{n}\frac{a_n B_k}{r^j} \\ &> \left|\Re\left(a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k}}{\alpha^k}\right)\right| - \sum_{j=k+1}^{\infty}\frac{a_n B_k}{r^j} \\ &\geq \left|\Re\left(a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k'}}{\alpha^{k'}}\right)\right| \\ &\qquad - \left|\Re\left(\frac{a_{n-k'-1}}{\alpha^{k'+1}} + \cdots + \frac{a_{n-k}}{\alpha^k}\right)\right| - \sum_{j=k+1}^{\infty}\frac{a_n B_k}{r^j}. \end{aligned}$$

From (A.4.1), we have

$$\left|\Re\left(a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k'}}{\alpha^{k'}}\right)\right| = \Re\left(a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k'}}{\alpha^{k'}}\right) \geq a_n.$$

Combining this with the condition of the case we are considering and summing the geometric series $\sum_{j=k+1}^{\infty} 1/r^j$, we deduce

$$\left|\frac{f(\alpha)}{\alpha^n}\right| > a_n - (a_n - \gamma) - \frac{a_n B_k}{r^k(r-1)} = \gamma - \frac{a_n B_k}{r^k(r-1)} = 0.$$

Case 2: $\left|\Re\left(\frac{a_{n-k'-1}}{\alpha^{k'+1}} + \cdots + \frac{a_{n-k}}{\alpha^k}\right)\right| > a_n - \gamma.$

In this case, we use that

$$\begin{aligned}
\left|\frac{f(\alpha)}{\alpha^n}\right| &\geq \left|a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k}}{\alpha^k}\right| - \sum_{j=k+1}^{n} \left|\frac{a_{n-j}}{\alpha^j}\right| \\
&\geq \left|a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k}}{\alpha^k}\right| - \sum_{j=k+1}^{n} \frac{a_n B_k}{r^j} \\
&> \left|a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k}}{\alpha^k}\right| - \sum_{j=k+1}^{\infty} \frac{a_n B_k}{r^j} \\
&\geq \left|\Im\left(a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k}}{\alpha^k}\right)\right| - \sum_{j=k+1}^{\infty} \frac{a_n B_k}{r^j}.
\end{aligned}$$

As a consequence of (A.4.2), (A.4.3) and (A.4.4), we have

$$\begin{aligned}
&\left|\Im\left(a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k}}{\alpha^k}\right)\right| \\
&\quad = \left|\Im\left(a_n + \frac{a_{n-1}}{\alpha} + \cdots + \frac{a_{n-k'}}{\alpha^{k'}}\right)\right| + \left|\Im\left(\frac{a_{n-k'-1}}{\alpha^{k'+1}} + \cdots + \frac{a_{n-k}}{\alpha^k}\right)\right| \\
&\quad \geq \left|\Im\left(\frac{a_{n-k'-1}}{\alpha^{k'+1}} + \cdots + \frac{a_{n-k}}{\alpha^k}\right)\right| \\
&\quad \geq \tan(\pi - k\theta)\left|\Re\left(\frac{a_{n-k'-1}}{\alpha^{k'+1}} + \cdots + \frac{a_{n-k}}{\alpha^k}\right)\right|.
\end{aligned}$$

Note that

$$a_n - \gamma = a_n - \frac{a_n}{1 + \cot(\pi - k\theta)} = \frac{a_n \cot(\pi - k\theta)}{1 + \cot(\pi - k\theta)}.$$

The condition of the current case under consideration now implies

$$\begin{aligned}
\left|\frac{f(\alpha)}{\alpha^n}\right| &> \tan(\pi - k\theta)(a_n - \gamma) - \sum_{j=k+1}^{\infty} \frac{a_n B_k}{r^j} \\
&= \tan(\pi - k\theta)\left(\frac{a_n \cot(\pi - k\theta)}{1 + \cot(\pi - k\theta)}\right) - \frac{a_n B_k}{r^k(r-1)} \\
&= \gamma - \frac{a_n B_k}{r^k(r-1)} = 0.
\end{aligned}$$

Therefore, in both Case 1 and Case 2, we conclude that $\left|\frac{f(\alpha)}{\alpha^n}\right| > 0$, contradicting that α is a root of $f(x)$. As a consequence, our assumption was incorrect so that there exists a $j \in \{0, 1, \ldots, n-1\}$ for which $a_j > B_k a_n$, completing the proof. □

A.5 Choice of $r_l = \tan(\theta_l)$ for Various Bases $b \geq 3$

In Section 2.5 in Table 2.16, we showed our selection of $r_l = \tan(\theta_l)$ for base $b = 2$ to be used with Lemma 2.9 to prove Corollary 2.10. Here, we show the choices of r_l for bases $3 \leq b \leq 20$. The calculations were done with Maple 17, with the Maple code given in Appendix A.6 using the GetBound function.

We note that in the case of base $b = 3$, the bounds given are sharp. In the cases for bases $4 \leq b \leq 20$, the bounds are accurate to the digits shown. With much tedious work, however, sharp bounds could be calculated by carefully refining the sectors of R_b.

Also, for the cases of bases $b = 3$ and $b = 4$, each individual sector calculation is shown. For bases $5 \leq b \leq 20$, we list the starting r_l and ending r_l along with the number of subdivisions and the overall bound for that sector. For instance, with base $b = 5$, we list $r_{start} = .1485$, $r_{end} = .149$, 100 subdivisions, and $B'_5(\arctan(r_{start}), \arctan(r_{end})) = 4.149 \times 10^{11}$. We divided the sector from .1485 to 149 into 100 equal-sized subdivisions and calculated the bound for each subdivision. $B'_5(\arctan(.1485), \arctan(.149)) = 4.149 \times 10^{11}$ was the best bound over all 100 divisions.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 3$

Consider base $b = 3$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_3$, which we divide into sectors as shown in Table A.6 and Table A.7.

Table A.6 Values of $B_3'(\theta_l, \theta_{l+1})$ for $0 \le l \le 49$

l	$r_l = \tan(\theta_l)$	$B_3'(\theta_l, \theta_{l+1})$
0	$0 = 0$	1.99×10^{483}
1	$\frac{1}{1000} = 0.001$	8.44936×10^{593}
2	$\frac{3}{2000} = 0.0015$	6.16345×10^{444}
3	$\frac{1}{500} = 0.002$	5.05173×10^{295}
4	$\frac{3}{1000} = 0.003$	1.62979×10^{221}
5	$\frac{1}{250} = 0.004$	3.40789×10^{176}
6	$\frac{1}{200} = 0.005$	5.83854×10^{146}
7	$\frac{3}{500} = 0.006$	3.21683×10^{125}
8	$\frac{7}{1000} = 0.007$	3.81023×10^{109}
9	$\frac{1}{125} = 0.008$	1.51095×10^{97}
10	$\frac{9}{1000} = 0.009$	1.93848×10^{87}
11	$\frac{1}{100} = 0.01$	4.13801×10^{57}
12	$\frac{3}{200} = 0.015$	6.53920×10^{42}
13	$\frac{1}{50} = 0.02$	1.20423×10^{28}
14	$\frac{3}{100} = 0.03$	6.06373×10^{20}
15	$\frac{1}{25} = 0.04$	2.63744×10^{16}
16	$\frac{1}{20} = 0.05$	3.73730×10^{13}
17	$\frac{3}{50} = 0.06$	3.32212×10^{11}
18	$\frac{7}{100} = 0.07$	1.11539×10^{10}
19	$\frac{2}{25} = 0.08$	7.62345×10^{8}
20	$\frac{9}{100} = 0.09$	9.76572×10^{7}
21	$\frac{1}{10} = 0.1$	132053
22	$\frac{3}{20} = 0.15$	10512.8
23	$\frac{1}{5} = 0.2$	6477.69
24	$\frac{1}{4} = 0.25$	11266.7

l	$r_l = \tan(\theta_l)$	$B_3'(\theta_l, \theta_{l+1})$
25	$\frac{3}{10} = 0.3$	7064.82
26	$\frac{7}{20} = 0.35$	8477.23
27	$\frac{3}{8} = 0.375$	4908.52
28	$\frac{2}{5} = 0.4$	5030.40
29	$\frac{21}{50} = 0.42$	5501.07
30	$\frac{43}{100} = 0.43$	5252.67
31	$\frac{11}{25} = 0.44$	4819.68
32	$\frac{9}{20} = 0.45$	4932.77
33	$\frac{91}{200} = 0.455$	4997.0
34	$\frac{183}{400} = 0.4575$	4784.55
35	$\frac{23}{50} = 0.46$	4864.11
36	$\frac{461}{1000} = 0.461$	4766.86
37	$\frac{231}{500} = 0.462$	4778.71
38	$\frac{37}{80} = 0.4625$	4726.97
39	$\frac{463}{1000} = 0.463$	4744.90
40	$\frac{579}{1250} = 0.4632$	4723.68
41	$\frac{2317}{5000} = 0.4634$	4726.24
42	$\frac{927}{2000} = 0.4635$	4715.52
43	$\frac{1159}{2500} = 0.4636$	4723.11
44	$\frac{57953}{125000} = 0.463624$	4720.51
45	$\frac{14489}{31250} = 0.463648$	4717.93
46	$\frac{57959}{125000} = 0.463672$	4715.34
47	$\frac{28981}{62500} = 0.463696$	4712.75
48	$\frac{11593}{25000} = 0.46372$	4715.49
49	$\frac{231861}{500000} = 0.463722$	4715.28

Table A.7 Values of $B_3'(\theta_l, \theta_{l+1})$ for $50 \le l \le 77$

l	$r_l = \tan(\theta_l)$	$B_3'(\theta_l, \theta_{l+1})$	l	$r_l = \tan(\theta_l)$	$B_3'(\theta_l, \theta_{l+1})$
50	$\frac{115931}{250000} = 0.463724$	4715.06	64	$\frac{57969}{125000} = 0.463752$	4712.38
51	$\frac{231863}{500000} = 0.463726$	4714.85	65	$\frac{231877}{500000} = 0.463754$	4712.56
52	$\frac{28983}{62500} = 0.463728$	4714.63	66	$\frac{115939}{250000} = 0.463756$	4712.73
53	$\frac{46373}{100000} = 0.46373$	4714.41	67	$\frac{231879}{500000} = 0.463758$	4712.90
54	$\frac{115933}{250000} = 0.463732$	4714.20	68	$\frac{5797}{12500} = 0.46376$	4712.47
55	$\frac{231867}{500000} = 0.463734$	4713.98	69	$\frac{23189}{50000} = 0.46378$	4714.18
56	$\frac{57967}{125000} = 0.463736$	4713.76	70	$\frac{2319}{5000} = 0.4638$	4713.23
57	$\frac{231869}{500000} = 0.463738$	4713.55	71	$\frac{4639}{10000} = 0.4639$	4721.83
58	$\frac{23187}{50000} = 0.46374$	4713.33	72	$\frac{58}{125} = 0.464$	4717.04
59	$\frac{231871}{500000} = 0.463742$	4713.11	73	$\frac{929}{2000} = 0.4645$	4761.29
60	$\frac{7246}{15625} = 0.463744$	4712.90	74	$\frac{93}{200} = 0.465$	4790.06
61	$\frac{231873}{500000} = 0.463746$	4712.68	75	$\frac{233}{500} = 0.466$	4780.85
62	$\frac{115937}{250000} = 0.463748$	4712.46	76	$\frac{47}{100} = 0.47$	4170.96
63	$\frac{371}{800} = 0.46375$	4712.25	77	$\frac{1}{2} = .5$	

This gives us the bound $B_3 = 4172$ as shown in Table 2.14.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 4$

Consider base $b = 4$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_4$, which we divide into sectors as shown in Table A.8, Table A.9, and Table A.10.

Table A.8 Values of $B'_4(\theta_l, \theta_{l+1})$ for $0 \le l \le 49$

l	$r_l = \tan(\theta_l)$	$B'_4(\theta_l, \theta_{l+1})$	l	$r_l = \tan(\theta_l)$	$B'_4(\theta_l, \theta_{l+1})$
0	$0 = 0$	1.99×10^{483}	25	0.1675	6.41056×10^{07}
1	0.001	4.57405×10^{873}	26	0.17	6.44083×10^{07}
2	0.0015	4.14170×10^{654}	27	0.171	6.32421×10^{07}
3	0.002	4.21479×10^{435}	28	0.1715	5.93692×10^{07}
4	0.003	1.40291×10^{326}	29	0.172	5.91299×10^{07}
5	0.004	3.17840×10^{260}	30	0.1722	5.94366×10^{07}
6	0.005	5.18517×10^{216}	31	0.17225	5.90000×10^{07}
7	0.006	3.25027×10^{185}	32	0.1723	5.88351×10^{07}
8	0.007	1.19021×10^{162}	33	0.17233	5.88462×10^{07}
9	0.008	7.80634×10^{143}	34	0.17234	5.88268×10^{07}
10	0.009	2.12278×10^{129}	35	0.172345	5.88240×10^{07}
11	0.01	4.31426×10^{85}	36	0.172347	5.88062×10^{07}
12	0.015	7.75847×10^{63}	37	0.172349	5.88023×10^{07}
13	0.02	1.38438×10^{42}	38	0.17235	5.88038×10^{07}
14	0.03	2.03714×10^{31}	39	0.17235025	5.88023×10^{07}
15	0.04	8.33266×10^{24}	40	0.17235045	5.88026×10^{07}
16	0.05	4.21969×10^{20}	41	0.1723505	5.88027×10^{07}
17	0.06	4.35849×10^{17}	42	0.17235051	5.88026×10^{07}
18	0.07	2.31410×10^{15}	43	0.17235052	5.88026×10^{07}
19	0.08	4.62832×10^{13}	44	0.17235053	5.88025×10^{07}
20	0.09	1.73929×10^{12}	45	0.17235054	5.88024×10^{07}
21	0.1	1.08264×10^{08}	46	0.17235055	5.88023×10^{07}
22	0.15	9.75387×10^{07}	47	0.17235056	5.88022×10^{07}
23	0.16	7.32947×10^{07}	48	0.17235057	5.88022×10^{07}
24	0.165	7.50008×10^{07}	49	0.172350571	5.88022×10^{07}

Table A.9 Values of $B_4'(\theta_l, \theta_{l+1})$ for $50 \leq l \leq 99$

l	$r_l = \tan(\theta_l)$	$B_4'(\theta_l, \theta_{l+1})$	l	$r_l = \tan(\theta_l)$	$B_4'(\theta_l, \theta_{l+1})$
50	0.172350572	5.88022×10^{07}	75	0.172350595	5.88022×10^{07}
51	0.172350573	5.88022×10^{07}	76	0.1723506	5.88022×10^{07}
52	0.172350574	5.88022×10^{07}	77	0.172350605	5.88022×10^{07}
53	0.172350575	5.88022×10^{07}	78	0.17235061	5.88022×10^{07}
54	0.172350576	5.88022×10^{07}	79	0.172350615	5.88022×10^{07}
55	0.172350577	5.88022×10^{07}	80	0.17235062	5.88022×10^{07}
56	0.1723505772	5.88022×10^{07}	81	0.172350625	5.88022×10^{07}
57	0.1723505774	5.88022×10^{07}	82	0.17235063	5.88022×10^{07}
58	0.1723505776	5.88022×10^{07}	83	0.172350635	5.88022×10^{07}
59	0.1723505778	5.88022×10^{07}	84	0.17235064	5.88022×10^{07}
60	0.172350578	5.88022×10^{07}	85	0.172350645	5.88023×10^{07}
61	0.1723505782	5.88022×10^{07}	86	0.17235065	5.88022×10^{07}
62	0.1723505784	5.88022×10^{07}	87	0.172350675	5.88022×10^{07}
63	0.1723505786	5.88022×10^{07}	88	0.1723507	5.88023×10^{07}
64	0.1723505788	5.88022×10^{07}	89	0.172350725	5.88023×10^{07}
65	0.172350579	5.88022×10^{07}	90	0.17235075	5.88022×10^{07}
66	0.1723505795	5.88022×10^{07}	91	0.1723508	5.88023×10^{07}
67	0.17235058	5.88022×10^{07}	92	0.17235085	5.88022×10^{07}
68	0.172350581	5.88022×10^{07}	93	0.17235095	5.88024×10^{07}
69	0.172350582	5.88022×10^{07}	94	0.17235105	5.88025×10^{07}
70	0.172350583	5.88022×10^{07}	95	0.17235115	5.88026×10^{07}
71	0.172350584	5.88022×10^{07}	96	0.17235125	5.88023×10^{07}
72	0.172350586	5.88022×10^{07}	97	0.1723515	5.88026×10^{07}
73	0.172350588	5.88022×10^{07}	98	0.17235175	5.88029×10^{07}
74	0.17235059	5.88022×10^{07}	99	0.172352	5.88024×10^{07}

Table A.10 Values of $B_4'(\theta_l, \theta_{l+1})$ for $100 \leq l \leq 148$

l	$r_l = \tan(\theta_l)$	$B_4'(\theta_l, \theta_{l+1})$	l	$r_l = \tan(\theta_l)$	$B_4'(\theta_l, \theta_{l+1})$
100	0.1723525	5.88031×10^{07}	125	0.1728	5.90840×10^{07}
101	0.172353	5.88038×10^{07}	126	0.1729	5.92286×10^{07}
102	0.1723535	5.88045×10^{07}	127	0.173	5.88326×10^{07}
103	0.172354	5.88035×10^{07}	128	0.17325	5.91987×10^{07}
104	0.172355	5.88048×10^{07}	129	0.1735	5.95837×10^{07}
105	0.172356	5.88062×10^{07}	130	0.17375	5.99900×10^{07}
106	0.172357	5.88042×10^{07}	131	0.174	5.94034×10^{07}
107	0.172359	5.88069×10^{07}	132	0.1745	6.02864×10^{07}
108	0.172361	5.88096×10^{07}	133	0.175	5.89813×10^{07}
109	0.172363	5.88124×10^{07}	134	0.176	6.12853×10^{07}
110	0.172365	5.88049×10^{07}	135	0.177	6.50547×10^{07}
111	0.17237	5.88118×10^{07}	136	0.178	7.33646×10^{07}
112	0.172375	5.88186×10^{07}	137	0.179	6.95886×10^{08}
113	0.17238	5.88254×10^{07}	138	0.18	8.97427×10^{07}
114	0.172385	5.88322×10^{07}	139	0.2	7.65178×10^{07}
115	0.17239	5.88391×10^{07}	140	0.21	8.14974×10^{07}
116	0.172395	5.88120×10^{07}	141	0.215	6.63901×10^{07}
117	0.17241	5.88325×10^{07}	142	0.22	6.27922×10^{07}
118	0.172425	5.88531×10^{07}	143	0.225	7.44318×10^{07}
119	0.17244	5.88226×10^{07}	144	0.23	7.80704×10^{07}
120	0.17247	5.88638×10^{07}	145	0.235	1.90871×10^{08}
121	0.1725	5.88367×10^{07}	146	0.24	1.00423×10^{08}
122	0.17255	5.89059×10^{07}	147	0.25	3.07434×10^{07}
123	0.1726	5.88023×10^{07}	148	0.27	
124	0.1727	5.89419×10^{07}			

This gives us the bound $B_4 = 5.8802 \times 10^7$ as shown in Table 2.14.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 5$

Consider base $b = 5$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_5$, which we divide into sectors as shown in Table A.11.

Table A.11 Values of $B'_5(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_5(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	3.64413×10^{856}
0.002	0.01	8	5.54760×10^{169}
0.01	0.02	2	1.35849×10^{84}
0.02	0.1	8	2.88242×10^{16}
0.1	0.14	1	4.86474×10^{11}
0.14	0.148	4	4.36883×10^{11}
0.148	0.1485	1	4.20285×10^{11}
0.1485	0.149	100	4.14981×10^{11}
0.149	0.1494	200	4.14925×10^{11}
0.1494	0.1497	300	4.14902×10^{11}
0.1497	0.15	1000	4.14928×10^{11}
0.15	0.151	1000	4.14927×10^{11}
0.151	0.152	100	4.15908×10^{11}
0.152	0.153	10	4.17629×10^{11}
0.153	0.155	20	4.51889×10^{11}
0.155	0.16	1	2.52875×10^{12}
0.16	0.17	1	5.89955×10^{11}
0.17	0.1725	1	5.86931×10^{11}
0.1725	0.175	2	4.89033×10^{11}
0.175	0.18	1	6.44711×10^{11}

This gives us the bound $B_5 = 4.149 \times 10^{11}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = .1494899$ and $r_{end} = .14949$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 4.149793×10^{11}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 6$

Consider base $b = 6$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_6$, which we divide into sectors as shown in Table A.12.

Table A.12 Values of $B_6'(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B_6'(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	3.55409×10^{1013}
0.002	0.01	8	1.41516×10^{201}
0.01	0.02	2	7.16020×10^{99}
0.02	0.1	8	5.20449×10^{19}
0.1	0.13	1	1.17317×10^{15}
0.13	0.135	1	9.75867×10^{14}
0.135	0.14	100	6.64970×10^{14}
0.14	0.1402	20	6.63812×10^{14}
0.1402	0.1404	200	6.61627×10^{14}
0.1404	0.1405	1000	6.61612×10^{14}
0.1405	0.1406	100	6.61690×10^{14}
0.1406	0.141	40	6.62186×10^{14}
0.141	0.15	1	1.23367×10^{15}

This gives us the bound $B_6 = 6.616 \times 10^{14}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = .140447$ and $r_{end} = .140448$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 6.61629×10^{14}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 7$

Consider base $b = 7$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_7$, which we divide into sectors as shown in Table A.13.

Table A.13 Values of $B_7'(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B_7'(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	1.85832×10^{1152}
0.002	0.01	8	7.97850×10^{228}
0.01	0.02	2	5.60075×10^{113}
0.02	0.1	8	4.83366×10^{22}
0.1	0.11	1	5.43868×10^{20}
0.11	0.115	5	1.57159×10^{20}
0.115	0.116	2	9.06565×10^{19}
0.116	0.1161	10	8.83465×10^{19}
0.1161	0.1163	2000	8.76263×10^{19}
0.1163	0.1165	100	8.76341×10^{19}
0.1165	0.117	50	8.76922×10^{19}
0.117	0.118	10	8.79180×10^{19}
0.118	0.125	1	9.92730×10^{19}

This gives us the bound $B_7 = 8.762 \times 10^{19}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = .116179$ and $r_{end} = .11618$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 8.76299×10^{19}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 8$

Consider base $b = 8$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_8$, which we divide into sectors as shown in Table A.14.

Table A.14 Values of $B'_8(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_8(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	3.11157×10^{1267}
0.002	0.01	8	8.93265×10^{251}
0.01	0.02	2	1.93521×10^{125}
0.02	0.1	8	1.65059×10^{25}
0.1	0.107	7	1.86443×10^{25}
0.107	0.108	5	1.41783×10^{25}
0.108	0.10808	8	1.40525×10^{25}
0.10808	0.108096	16	1.40231×10^{25}
0.108096	0.108099	30	1.40233×10^{25}
0.108099	0.1081	10	1.40197×10^{25}
0.1081	0.10811	100	1.40123×10^{25}
0.10811	0.108115	5	1.40197×10^{25}

This gives us the bound $B_8 = 1.401 \times 10^{25}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = .1081038$ and $r_{end} = 0.1081039$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 1.401444×10^{25}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 9$

Consider base $b = 9$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_9$, which we divide into sectors as shown in Table A.15.

Table A.15 Values of $B_9'(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B_9'(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	1.09697×10^{1366}
0.002	0.01	8	5.02207×10^{271}
0.01	0.02	2	1.42805×10^{135}
0.02	0.08	6	6.17573×10^{33}
0.08	0.09	10	1.07832×10^{31}
0.09	0.095	500	2.09326×10^{30}
0.095	0.096	1000	1.41212×10^{30}

This gives us the bound $B_9 = 1.412 \times 10^{30}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = .09573$ and $r_{end} = .095731$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 1.4125×10^{30}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 10$

Consider base $b = 10$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{10}$, which we divide into sectors as shown in Table A.16.

This gives us the bound $B_{10} = 2.749 \times 10^{35}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = .08616369$ and $r_{end} = .0861637$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 2.749889×10^{35}.

Table A.16 Values of $B'_{10}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{10}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	1.35945×10^{1452}
0.002	0.01	8	9.47832×10^{288}
0.01	0.02	2	5.96751×10^{143}
0.02	0.08	6	1.33634×10^{36}
0.08	0.085	5	3.38637×10^{35}
0.085	0.086	100	2.83670×10^{35}
0.086	0.0861	10	2.75920×10^{35}
0.0861	0.08622	1000	2.74964×10^{35}

Choice of $r_l = \tan(\theta_l)$ for Base $b = 11$

Consider base $b = 11$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{11}$, which we divide into sectors as shown in Table A.17.

Table A.17 Values of $B'_{11}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{11}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	3.71629×10^{1528}
0.002	0.01	8	2.06523×10^{304}
0.01	0.02	2	2.69323×10^{151}
0.02	0.07	4	1.41855×10^{43}
0.07	0.077	7	1.27212×10^{41}
0.077	0.0775	5	1.09319×10^{41}
0.0775	0.078	50	7.38582×10^{40}
0.078	0.07827	27	5.21473×10^{40}
0.07827	0.0782845	145	5.20649×10^{40}
0.0782845	0.0783	1000	5.20316×10^{40}

This gives us the bound $B_{11} = 5.203 \times 10^{40}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = .0782872435$ and $r_{end} = 0.7828724351$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 5.20349×10^{40}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 12$

Consider base $b = 12$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{12}$, which we divide into sectors as shown in Table A.18.

Table A.18 Values of $B'_{12}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{12}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	1.99448×10^{1597}
0.002	0.01	8	1.27910×10^{318}
0.01	0.02	2	2.05825×10^{158}
0.02	0.06	4	6.97121×10^{52}
0.06	0.07	10	8.65050×10^{46}
0.07	0.071	10	3.95029×10^{46}
0.071	0.07155	55	1.15903×10^{46}
0.07155	0.07159	40	1.15923×10^{46}
0.07159	0.0716	100	1.15924×10^{46}
0.0716	0.07163	3000	1.15904×10^{46}
0.07163	0.071631	10	1.15958×10^{46}
0.071631	0.07166	29	1.15903×10^{46}
0.07166	0.0717	8	1.16123×10^{46}
0.0717	0.072	30	1.18910×10^{46}

This gives us the bound $B_{12} = 1.159 \times 10^{46}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = 0.07161$ and

$r_{end} = 0.071611$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 1.159058×10^{46}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 13$

Consider base $b = 13$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{13}$, which we divide into sectors as shown in Table A.19.

Table A.19 Values of $B'_{13}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	# *of Divisions*	$B'_{13}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	5.43711×10^{1659}
0.002	0.01	8	4.32611×10^{330}
0.01	0.02	2	3.93934×10^{164}
0.02	0.06	4	1.08354×10^{55}
0.06	0.063	3	8.68587×10^{53}
0.063	0.065	20	4.31388×10^{52}
0.065	0.066	50	7.20498×10^{51}
0.066	0.0661	100	6.97169×10^{51}
0.0661	0.0665	4000	6.96914×10^{51}

This gives us the bound $B_{13} = 6.969 \times 10^{51}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = 0.066112$ and $r_{end} = 0.0661120001$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 6.96994×10^{51}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 14$

Consider base $b = 14$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{14}$, which we divide into sectors as shown in Table A.20.

Table A.20 Values of $B'_{14}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{14}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	8.57965×10^{1716}
0.002	0.01	8	1.30203×10^{342}
0.01	0.02	2	2.29851×10^{170}
0.02	0.05	3	1.00153×10^{68}
0.05	0.06	10	5.17095×10^{58}
0.06	0.0615	15	3.34535×10^{57}
0.0615	0.06159	9	2.70508×10^{57}
0.06159	0.0616	100	2.68961×10^{57}
0.0616	0.0617	100	2.69055×10^{57}

This gives us the bound $B_{14} = 2.689 \times 10^{57}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = 0.0615913$ and $r_{end} = 0.0615913001$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 2.68998×10^{57}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 15$

Consider base $b = 15$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{15}$, which we divide into sectors as shown in Table A.21.

This gives us the bound $B_{15} = 1.590 \times 10^{63}$ as shown in Table 2.14.

Table A.21 Values of $B'_{15}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{15}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	5.07272×10^{1769}
0.002	0.01	8	5.07265×10^{352}
0.01	0.02	2	4.80766×10^{175}
0.02	0.05	3	1.48891×10^{70}
0.05	0.057	7	3.14658×10^{63}
0.057	0.0571	1	1.79490×10^{63}
0.0571	0.05718	8	1.59007×10^{63}
0.05718	0.057191	11	1.59809×10^{63}
0.057191	0.0572	90	1.59827×10^{63}
0.0572	0.05722	200	1.59811×10^{63}
0.05722	0.05725	30	1.59853×10^{63}
0.05725	0.0573	5	1.61025×10^{63}
0.0573	0.0577	4	2.30983×10^{63}

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = 0.05720$ and $r_{end} = 0.0572001$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 1.59841×10^{63}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 16$

Consider base $b = 16$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{16}$, which we divide into sectors as shown in Table A.22.

This gives us the bound $B_{16} = 1.869 \times 10^{69}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = 0.0540515$

Table A.22 Values of $B'_{16}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{16}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	4.86219×10^{1818}
0.002	0.01	8	3.43229×10^{362}
0.01	0.02	2	4.17808×10^{180}
0.02	0.05	3	1.59394×10^{72}
0.05	0.053	3	1.23667×10^{70}
0.053	0.054	10	1.99749×10^{69}
0.054	0.05403	3	1.97333×10^{69}
0.05403	0.054049	19	1.87005×10^{69}
0.054049	0.05405	10	1.87176×10^{69}
0.05405	0.054053	100	1.86925×10^{69}

and $r_{end} = 0.0540516$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 1.86958×10^{69}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 17$

Consider base $b = 17$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{17}$, which we divide into sectors as shown in Table A.23.

This gives us the bound $B_{17} = 1.269 \times 10^{75}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = .050879$, and $r_{end} = 0.05088$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 1.2697×10^{75}.

Table A.23 Values of $B'_{17}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{17}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	2.43444×10^{1864}
0.002	0.01	8	5.10066×10^{371}
0.01	0.02	2	1.69732×10^{185}
0.02	0.04	2	3.93506×10^{92}
0.04	0.05	2	3.55407×10^{75}
0.05	0.0505	1	2.20506×10^{75}
0.0505	0.0506	1	2.82779×10^{75}
0.0506	0.05083	23	1.27030×10^{75}
0.05083	0.0509	700	1.26915×10^{75}

Choice of $r_l = \tan(\theta_l)$ for Base $b = 18$

Consider base $b = 18$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{18}$, which we divide into sectors as shown in Table A.24.

Table A.24 Values of $B'_{18}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{18}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	1.64548×10^{1907}
0.002	0.01	8	2.01415×10^{380}
0.01	0.02	2	3.54679×10^{189}
0.02	0.04	2	6.06638×10^{94}
0.04	0.045	1	3.12995×10^{84}
0.045	0.047	2	2.03821×10^{82}
0.047	0.04755	10	4.09132×10^{81}
0.04755	0.0476	100	2.08384×10^{81}
0.0476	0.04764	400	2.07562×10^{81}
0.04764	0.0477	60	2.07775×10^{81}
0.0477	0.048	3	2.35470×10^{81}
0.048	0.0481	10	2.09935×10^{81}

This gives us the bound $B_{18} = 2.075 \times 10^{81}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = 0.047626$ and $r_{end} = 0.0476261$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 2.07588×10^{81}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 19$

Consider base $b = 19$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{19}$, which we divide into sectors as shown in Table A.25.

Table A.25 Values of $B'_{19}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{19}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	3.27887×10^{1947}
0.002	0.01	8	2.47192×10^{388}
0.01	0.02	2	4.12437×10^{193}
0.02	0.04	2	7.02276×10^{96}
0.04	0.044	4	8.17124×10^{89}
0.044	0.0455	15	1.98471×10^{87}
0.0455	0.045545	45	1.24544×10^{87}
0.045545	0.04555	50	1.24608×10^{87}
0.04555	0.045565	150	1.24572×10^{87}
0.045565	0.0456	35	1.24513×10^{87}

This gives us the bound $B_{19} = 1.245 \times 10^{87}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = 0.0455557$ and $r_{end} = 0.0455557001$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 1.2458×10^{87}.

Choice of $r_l = \tan(\theta_l)$ for Base $b = 20$

Consider base $b = 20$. Using the numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ from Table 2.4, have a region $\mathcal{R}_{20}$, which we divide into sectors as shown in Table A.26.

Table A.26 Values of $B'_{20}(\theta_l, \theta_{l+1})$

$r_{start} = \tan(\theta_{start})$	$r_{end} = \tan(\theta_{end})$	*# of Divisions*	$B'_{20}(\theta_{start}, \theta_{end})$
0	0.001	1	1.88×10^{483}
0.001	0.002	2	3.69063×10^{1985}
0.002	0.01	8	1.07426×10^{396}
0.01	0.02	2	2.84957×10^{197}
0.02	0.04	2	6.32328×10^{98}
0.04	0.042	2	4.85080×10^{95}
0.042	0.043	10	9.89047×10^{93}
0.043	0.0433	300	3.99299×10^{93}
0.0433	0.04331	10	3.95196×10^{93}
0.04331	0.043317	70	3.94214×10^{93}
0.043317	0.0433184	140	3.94200×10^{93}
0.0433184	0.043319	600	3.94200×10^{93}

This gives us the bound $B_{20} = 3.942 \times 10^{93}$ as shown in Table 2.14.

These choices for r_l are optimum in the sense that this is the best bound, accurate to the digits shown. By carefully analyzing the sector given by $r_{start} = 0.0433188$ and $r_{end} = 0.043318801$ with 100 and 1000 subdivisions, we see that the best we can achieve with our choice of numbers $e_1 = e_1(b)$, $e_3 = e_3(b)$, $e_4 = e_4(b)$, $e_6 = e_6(b)$, and $d = d(b)$ is 3.942031×10^{93}.

A.6 Maple Code Used for Computations

The following functions were written for use with Maple 17. The default settings used for all computations is shown at the beginning of the code.

```
#####
#Default precision for (most) computations.
# kernelopts(maxdigits) will return the maximum setting of for Digits.
#####
Digits := 100;

#####
#Round towards 0
#####
Rounding := 0;

#####
#Load necessary Maple packages
#####
with(numtheory):
with(LinearAlgebra):
with(Optimization):

#####
#Representation of complex numbers.
#####
z := x + I*y;
```

```
#####
#Define zeta_n, which is not to be confused with the
#     protected Zeta function.
#####
zeta := n -> exp(2 * Pi * I/n);


#####
#Define the root bounding function P
#####
GetFuncP := proc( base, exponents::Array )
local exp1, exp3, exp4, exp6, expd, z, n, d, P;
exp1 := 2*exponents[1];
exp3 := 2*exponents[3];
exp4 := 2*exponents[4];
exp6 := 2*exponents[6];
expd := 2*(exp3 + exp4 + exp6) + exp1 + 2*exponents[0] + 2;
n := z -> (abs(base - 1 - z))^exp1 * (abs(base + zeta(6) - z)
     * abs(base + zeta(-6) - z))^exp6 * (abs(base + zeta(3) - z)
     * abs(base + zeta(-3) - z))^exp3 * (abs(base + zeta(4) - z)
     * abs(base + zeta(-4) - z))^exp4;
d := z -> abs(base - z)^expd;
P := z -> d(z) - n(z);
return(P);
end proc:
```

```
#####
#Get the best Theta and the Approximation for tan(theta)
#####
GetTheta := proc ( P::procedure, con::integer := 2)
local d, j, roots, convergent, ss;
d := 0;
j := 2;
while d = 0 do
j := j + 1;
convergent := cfrac(cfrac(tan(Pi/j),con));
ss := sturmseq(expand(evalc(P(x+I*convergent*x))),x);
roots := sturm(ss, x, -infinity, infinity);
if roots > 0 then
d := j-1;
end if;
end do;
return([Pi/d, cfrac(cfrac(tan(Pi/d),con))]);
end proc:

#####
#Get the Maximum Coefficient based on the Recursive Relations
#####
GetMax := proc( Phi::procedure )
local A, B, J, beta, j, E, E_sol, u, v, w;
A := abs(coeff(Phi(x), x, 1));
B := abs(coeff(Phi(x), x, 0));
beta[-1] := 0;
```

```
beta[0] := 1;
beta[max] :=1;
j := 0;
while beta[j] > 0 do
j := j+1;
beta[j] := A * beta[j-1] - B * beta[j-2];
if beta[j]>beta[max] then beta[max]:=beta[j]; end if;
end do;
J := j-1;
E := Matrix(j, j+1, 0);
for j from 1 to J+2 do
E(1, j) := 1;
end do;
E(2, 1) := -A;
E(2, 2) := B;
for j from 3 to J+1 do
E(j, j-2) := 1;
E(j, j-1) := -A;
E(j, j) := B;
end do;
E_sol := LinearSolve(E);

#verify sizes
for j from 1 to J do
if not(0 <= E_sol(j) and E_sol(j) <= 1) then
printf("ERROR: Solution %d is of the wrong size : %e", j, E_sol(j));
return(NULL);
```

```
end if;
end do;
u := E_sol(1)*B;
v := E_sol(J) - E_sol(J+1)*A;
w := E_sol(J+1);
return([A, B, J, beta[max], Phi(1)*beta[J]+1, u, v, w]);
end proc:


#####
#Get the Bound B using the Lemma
#####
GetB := proc( P::procedure, mStart::numeric,
                    mEnd::numeric, count::integer)
     local j, bestj, R, r__1, r__2, k__1, k__2, c__1, c__2,
             xtemp, Btemp, Bound, B, inc, roots, ss;
     Bound := 1000^(1000);
     bestj := 0;
     inc := (mEnd - mStart)/count;
     for j from 1 to count do
          r__1 := convert(mStart + (j-1)*inc, rational, exact);
          r__2 := convert(mStart + j*inc, rational, exact);
          xtemp := min(fsolve(evalc(P(x+I*r__1*x))=0,x));
          if xtemp <> infinity then
               xtemp := cfrac(cfrac(xtemp,10));
          end if;
          ss := sturmseq(expand(evalc(P(x+I*r__1*x))),x);
          roots := sturm(ss,x,0,xtemp);
```

```
        if roots <> 0 then
            return("ERROR: Sturm Sequence");
        end if;
        R := sqrt(r__1^2+1)*xtemp;
        k__1 := floor(Pi/arctan(r__2));
        k__2 := k__1-1;
        if k__2<= evalf(Pi/(2*arctan(r__1))+10^(-10))
            then k__2:=k__1; end if;
        if k__1<= evalf(Pi/(2*arctan(r__1))+10^(-10))
            then print("Error"); end if;
        c__1 := evalf(cot(Pi-k__1*arctan(r__2))+10^(-10));
        c__2 := evalf(cot(Pi-k__2*arctan(r__2))+10^(-10));
        Btemp := evalf(max(R^k__1*(R-1)/(1+c__1),
                                R^k__2*(R-1)/(1+c__2)));
        if Btemp < Bound then
            Bound := Btemp;
            bestj := j;
        end if;
    print(j, evalf(r__1),evalf(Btemp,20));
    end do;
    return([Bound, bestj,
             convert(mStart + bestj * inc, rational, exact)]);
end proc:
```