

2014

# Turán Problems on Non-uniform Hypergraphs

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TURÁN PROBLEMS ON NON-UNIFORM HYPERGRAPHS

by

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Bachelor of Science  
University of Nebraska-Lincoln 2009

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Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2014

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## ACKNOWLEDGMENTS

First, I must thank my advisor, Dr. Linyuan Lu. All the work in this dissertation is joint work with him. I want to thank him for being patient with me, encouraging me, helping me find fun problems to study, and for giving me numerous opportunities to share our research with others. Dr. Lu's perpetual optimism and obvious love of solving problems are infectious. Thank you.

Second, I would like to thank my parents. They have always encouraged me to expect more of myself and to see a goal through to completion. They have always expected me to be the best I can be and they've done everything in their power to give me more opportunities to learn and grow. One doesn't get very far without the support of his parents, and I have never lacked it. Thank you.

Finally, I want thank my wife for all the sacrifices that she's made while I've been in school. For the entirety of our marriage thus far, almost 8 years, I have been a full time student. She's worked several different jobs, and for the last 3 years has additionally been a full-time mom. I'm grateful that she was willing to spend so much time with our son while I needed to be at school studying or teaching, and especially for the days that I was away at conferences. Thank you.

## ABSTRACT

A non-uniform hypergraph  $H = (V, E)$  consists of a vertex set  $V$  and an edge set  $E \subseteq 2^V$ ; unlike a uniform hypergraph, the edges in  $E$  are not required to all have the same cardinality. The set of all cardinalities of edges in  $H$  is denoted by  $R(H)$ , and is called the set of edge types. For a fixed hypergraph  $H$ , the Turán density  $\pi(H)$  is defined to be  $\lim_{n \rightarrow \infty} \max_{G_n} h_n(G_n)$ , where the maximum is taken over all  $H$ -free hypergraphs  $G_n$  on  $n$  vertices satisfying  $R(G_n) \subseteq R(H)$ , and  $h_n(G_n)$ , the Lubell function, is the expected number of edges in  $G_n$  hit by a random full chain—a set of  $n + 1$  distinct subsets of  $V$  satisfying  $\emptyset = A_0 \subsetneq A_1 \subsetneq A_2 \subsetneq \dots \subsetneq A_n = V$ . This concept, which generalizes the Turán density of  $k$ -uniform hypergraphs, is motivated by recent work on extremal poset problems.

Several properties of Turán density, such as supersaturation, blow-up, and suspension, are generalized from uniform hypergraphs to non-uniform hypergraphs. We characterize all the Turán densities of  $\{1, 2\}$ -graphs, i.e. hypergraphs whose set of edge sizes is  $\{1, 2\}$ . In the final chapters, we discuss the notion of jumps in non-uniform hypergraphs. We refine the notion of jumps to strong jumps and weak jumps. We show that every value in  $[0, 2)$  is a jump for  $\{1, 2\}$ -graphs and we determine exactly which values are the strong jumps. Using this refinement, we are able to determine which values in the interval  $[0, 2)$  are a density of a hereditary graph property of  $\{1, 2\}$ -graphs. Examples of densities of hereditary properties are Turán densities of families of graphs, Lagrangians of graphs, and others. A similarly strong classification of jumps and Turán densities of graphs with larger edge sizes remains incomplete. However, in the final chapter, we provide a sufficient condition for a value

to be a non-jump and give several examples of new non-jump values for 3-uniform hypergraphs.

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# CHAPTER 1

## INTRODUCTION

A hypergraph,  $H = (V, E)$ , is a pair of sets;  $V$  is the set of vertices, and  $E$  is the set of edges. Each element  $e \in E$  is a set of vertices. A graph, or simple graph, is a hypergraph in which every edge consists of exactly two distinct vertices. An  $r$ -uniform hypergraph, or  $r$ -graph, is a hypergraph in which every edge consists of exactly  $r$  distinct vertices. A hypergraph which is not  $r$ -uniform is called a non-uniform hypergraph. One class of graphs that we will deal with frequently is the complete graphs.  $K_n^r$  denotes the complete  $r$ -uniform hypergraph on  $n$  vertices; it is complete in the sense that every  $r$ -subset of vertices forms an edge.

We say that  $H$  is a subgraph (or sub-hypergraph) of  $G$  if there exists an injective map  $\phi : V(H) \mapsto V(G)$  that preserves edges, i.e. if  $e \in E(H)$  then  $\phi(e) \in E(G)$ . Nearly all of the questions that we will consider will have a set of forbidden subgraphs  $\mathcal{F}$ . We say that  $G$  is  $\mathcal{F}$ -free if no graph  $F \in \mathcal{F}$  is a subgraph of  $G$ . If  $\mathcal{F} = \{F\}$  then we will simply say that  $G$  is  $F$ -free.

### 1.1 THE TURÁN DENSITY OF SIMPLE GRAPHS

Turán problems on graphs (and later hypergraphs) began with the following result due to Mantel.

**Theorem 1** (Mantel, 1907, [53]). *If  $G$  is a  $K_3$ -free simple graph on  $n$  vertices then  $G$  has at most  $\frac{n^2}{4}$  edges.*

Suppose that  $\mathcal{F}$  is a family of finite forbidden simple graphs. The **extremal**

**number** of  $\mathcal{F}$  is defined as

$$\text{ex}(\mathcal{F}, n) := \max\{|E(G)| : G \text{ is an } \mathcal{F}\text{-free simple graph on } n \text{ vertices}\}.$$

The **Turán density** of  $\mathcal{F}$  is defined as

$$\pi(\mathcal{F}) := \lim_{n \rightarrow \infty} \frac{\text{ex}(\mathcal{F}, n)}{\binom{n}{2}}.$$

Let  $T^{r-1}(n)$  denote the complete  $(r-1)$ -partite graph on  $n$  vertices whose partitions differ in size by at most one vertex and let  $t_{r-1}(n)$  denote the number of edges in  $T^{r-1}(n)$ . Turán, the man whose name now describes this area of study, proved the following generalization of Mantel's theorem.

**Theorem 2** (Turán, 1941, [69]). *For any  $r \geq 2$  it follows that  $\text{ex}(K_r, n) = t_{r-1}(n)$  and this implies that  $\pi(K_r) = 1 - \frac{1}{r-1}$ .*

The problem of determining Turán densities of simple graphs was finally answered by Erdős, Stone, and later Simonovits.

**Theorem 3** (Erdős, Stone, Simonovits, 1946, [24]). *For any simple graph  $F$  with chromatic number  $\chi(F)$  the Turán density of  $F$  is  $\pi(F) = 1 - \frac{1}{\chi(F)-1}$ .*

## 1.2 THE TURÁN DENSITY OF UNIFORM HYPERGRAPHS

After the Turán densities of simple graphs were all determined, it was natural to ask the same question about uniform hypergraphs. Suppose that  $\mathcal{F}$  is a family of finite forbidden  $r$ -uniform hypergraphs. The **extremal number** of  $\mathcal{F}$  is defined as

$$\text{ex}(\mathcal{F}, n) := \max\{|E(G)| : G \text{ is an } \mathcal{F}\text{-free } r\text{-uniform hypergraph on } n \text{ vertices}\}.$$

The **Turán density** of  $\mathcal{F}$  is defined as

$$\pi(\mathcal{F}) := \lim_{n \rightarrow \infty} \frac{\text{ex}(\mathcal{F}, n)}{\binom{n}{r}}.$$

It is not obvious that this limit should always exist. The following theorem of Katona, Nemetz, and Simonovits answers this first natural question.

**Theorem 4** (Katona, Nemetz, Simonovits, 1964 [40]). *For any family of finite  $r$ -graphs,  $\mathcal{F}$ , the limit  $\lim_{n \rightarrow \infty} \frac{\text{ex}(\mathcal{F}, n)}{\binom{n}{r}}$  exists.*

Since the early work of Katona, Nemetz, and Simonovits, a lot of study has been devoted to determining Turán densities of uniform hypergraphs. The following two results (particularly the Theorem on blow-ups) are the standard tools used in finding these densities.

**Lemma 1** (Supersaturation, [23]). *Let  $H$  be an  $r$ -uniform hypergraph on  $m$  vertices. For any constant  $a > 0$  there exist positive constants  $b, n_0$  so that if  $G$  is an  $r$ -graph on  $n \geq n_0$  vertices and edge density at least  $\pi(H) + a$ , then  $G$  contains at least  $b \binom{n}{m}$  copies of  $H$ .*

Note that a copy of  $H$  is an  $m$ -set of vertices,  $M$ , so that  $G[M]$  contains  $H$  as a subgraph. The supersaturation lemma is the key ingredient in the proof of the following theorem.

**Theorem 5** (Blow-ups, [7]). *Let  $H$  be an  $r$ -uniform hypergraph. Then  $\pi(H) = \pi(H(s))$  where  $H(s)$  is the  $s$ -blow-up of  $H$ .*

The precise definition of a blow-up will be given later. The important result is that blowing-up a graph does not change its Turán density.

There are a few uniform hypergraphs whose Turán density has been determined: the Fano plane [30, 45], expanded triangles [46], 3-books, 4-books [29],  $F_5$  [25], extended complete graphs [59], etc. In particular, Baber [3] recently found the Turán density of many 3-uniform hypergraphs using flag algebra methods. For a more complete survey of methods and results on uniform hypergraphs see Keevash's survey paper [42].

Let  $K_k^r$  denote the complete  $r$ -graph on  $k$  vertices. Turán determined the value of  $\text{ex}(K_k^2, n)$  which implies that  $\pi(K_k^2) = 1 - \frac{1}{k-1}$  for all  $k \geq 3$ . However, no Turán

density  $\pi(K_k^r)$  is known for any  $k > r \geq 3$ . The most extensively studied case is when  $k = 4$  and  $r = 3$ . Turán conjectured [69] that  $\pi(K_4^3) = 5/9$ . Erdős [21] offered \$500 for determining any  $\pi(K_k^r)$  with  $k > r \geq 3$  and \$1000 for answering it for all  $k$  and  $r$ . The upper bounds for  $\pi(K_4^3)$  have been sequentially improved: 0.6213 (de Caen [16]), 0.5936 (Chung-Lu [11]), 0.56167 (Razborov [60], using the flag algebra method.)

### 1.3 HYPERGRAPH JUMPS

The problem of determining jump values, or non-jump values, is intimately related to determining Turán densities of hypergraphs.

**Definition 1.** *A real number  $\alpha$  is a **jump** for a positive integer  $r$  if there exists a  $c > 0$  such that for every  $\epsilon > 0$  and every  $t \geq r$  there exists an integer  $n_0(\alpha, r, t, \epsilon)$  such that if  $n \geq n_0$  and  $G$  is an  $r$ -uniform hypergraph on  $n$  vertices with edge density at least  $\alpha + \epsilon$  then  $G$  contains a subgraph  $H$  on  $t$  vertices with edge density at least  $\alpha + c$ .*

Erdős observed that all values in  $[0, 1)$  are jumps for 2 and all values in  $[0, \frac{r!}{r^r})$  are jumps for  $r$ . He asked whether all values in  $[0, 1)$  are jumps for any  $r \geq 2$ —this was known as the jumping constant conjecture. The question was answered negatively by Frankl and Rödl in 1984 [28]. They showed that  $1 - \frac{1}{l^{r-2}}$  is a non-jump for every  $r \geq 3$  and  $l > 2r$ . Since then, several pairs  $(\alpha, r)$  of jumps/non-jumps have been identified [2, 27].

One of the first results that suggests a connection between jump (and non-jump) values and Turán densities is the following Theorem of Frankl and Rödl.  $\lambda(H)$  is the Lagrangian of a hypergraph  $H$ . The precise definition will be given later, but it should be understood as the maximum edge density of a (very large) blow-up of  $H$ .

**Theorem 6** (Frankl, Rödl, [28]). *A value  $\alpha \in [0, 1)$  is a jump for  $r$  if and only if there exists a finite family of  $r$ -graphs  $\mathcal{F}$  such that  $\pi(\mathcal{F}) \leq \alpha$  and  $\min_{F \in \mathcal{F}} \lambda(F) > \alpha$ .*

## CHAPTER 2

### TURÁN DENSITIES OF NON-UNIFORM HYPERGRAPHS

#### 2.1 NOTATION

Let  $G = (V, E)$  be a hypergraph. In this chapter we consider non-uniform hypergraphs. Let  $R(G) = \{|e| : e \in E\}$  denote the set of edge sizes of  $G$ . Note that if  $R(G) = \{r\}$  then  $G$  is  $r$ -uniform. Throughout, we will assume that any set  $R$  is a set of non-negative integers. If  $R(G) \subseteq R$  then we will say that  $G$  is an  $R$ -graph. We will occasionally wish to speak about uniform subgraphs of an  $R$ -graph  $G$ . For any  $r \in R$  we will understand  $G^r$  to be the subgraph of  $G$  on the same vertex set as  $G$  containing all of the edges of size  $r$ . We will denote the complete  $R$ -graph on  $n$  vertices by  $K_n^R$ .

#### 2.2 THE LUBELL FUNCTION

**Definition 2.** For a hypergraph  $G$  on  $n$  vertices the **Lubell function** is defined as:

$$h_n(G) := \sum_{e \in E(G)} \frac{1}{\binom{n}{|e|}}.$$

The Lubell function takes its name from Lubell [51], who, in 1966, gave a beautiful short proof of Sperner's theorem using the function. Here are a few facts which we will use throughout our study. Let  $G$  be a hypergraph on  $n$  vertices. Let  $\pi$  be a permutation of  $[n]$ , i.e. a bijective function  $\pi : [n] \mapsto [n]$ . The full chain induced by  $\pi$  is the set

$$C_\pi = \{\emptyset, \{\pi(1)\}, \{\pi(1), \pi(2)\}, \dots, [n]\}.$$

**Fact 1.** *The Lubell value of  $G$  is precisely the expected number of times a random full chain intersects the edge set of  $G$ .*

*Proof.* Let  $G$  be a hypergraph on  $n$  vertices with edge set  $E$ . For each  $e \in E$  let  $\chi_e$  be the indicator function that is 1 precisely when  $e$  is contained in the random chain  $C_\pi$ . Note that there are  $|e|!(n - |e|)!$  chains that contain the set  $e$ . Hence  $\mathbb{P}(\chi_e = 1) = \frac{|e|!(n - |e|)!}{n!} = \frac{1}{\binom{n}{|e|}}$ . Let  $X = \sum_{e \in E} \chi_e$  denote the number of times a random chain intersects  $E$ . Then

$$\mathbb{E}(X) = \sum_{e \in E} \mathbb{E}(\chi_e) = \sum_{e \in E} \mathbb{P}(\chi_e = 1) = \sum_{e \in E} \frac{1}{\binom{n}{|e|}} = h_n(G).$$

□

**Fact 2.** *Suppose that  $k \geq \max\{r : r \in R(G)\}$ . Then the Lubell value of  $G$  is the average of the Lubell values of the  $k$ -vertex induced subgraphs of  $G$ . I.e.*

$$h_n(G) = \frac{1}{\binom{n}{k}} \sum_{K \in \binom{V}{k}} h_k(G[K]).$$

*Proof.* Note that every edge  $e \in E$  is contained in  $\binom{n - |e|}{k - |e|}$   $k$ -sets. Thus we have that

$$\begin{aligned} \frac{1}{\binom{n}{k}} \sum_{K \in \binom{V}{k}} h_k(G[K]) &= \frac{1}{\binom{n}{k}} \sum_{K \in \binom{V}{k}} \sum_{e \in E(G[K])} \frac{1}{\binom{k}{|e|}} \\ &= \frac{1}{\binom{n}{k}} \sum_{e \in E} \frac{\binom{n - |e|}{k - |e|}}{\binom{k}{|e|}} \\ &= \sum_{e \in E} \frac{1}{\binom{n}{|e|}} \\ &= h_n(G). \end{aligned}$$

□

### 2.3 GENERALIZATIONS OF CLASSIC RESULTS

We begin by extending the notion of extremal number and Turán density to non-uniform hypergraphs.

**Definition 3.** Let  $\mathcal{F}$  be a family of forbidden  $R$ -graphs. Let

$$\pi_n^R(\mathcal{F}) := \max\{h_n(G) : G \text{ is an } \mathcal{F}\text{-free } R\text{-graph on } n \text{ vertices}\}.$$

Note that if  $R = \{r\}$  then  $\pi_n(\mathcal{F}) = \frac{\text{ex}(\mathcal{F}, n)}{\binom{n}{r}}$ .

**Definition 4.** Let  $\mathcal{F}$  be a family of forbidden  $R$ -graphs. Then

$$\pi^R(\mathcal{F}) := \lim_{n \rightarrow \infty} \pi_n^R(\mathcal{F}).$$

In nearly every case, the set  $R$  in the definition of  $\pi(\mathcal{F})$  above is precisely  $R(\mathcal{F})$ . In these cases we will drop the superscript  $R$  and just write  $\pi(\mathcal{F})$ . In the event that  $R(\mathcal{F}) = \bigcup_{F \in \mathcal{F}} R(F) \subsetneq R$  we will write  $\pi^R(\mathcal{F})$  to assert that we allow the host graph  $G$  (which is  $\mathcal{F}$ -free) to contain edges of sizes not in any graph  $F \in \mathcal{F}$ . At the moment, this distinction seems quite unnecessary; however it is useful when we consider the problem of hypergraph jumps in chapter 4.

**Theorem 7.** For any family  $\mathcal{H}$  of  $R$ -graphs,  $\pi(\mathcal{H})$  is well-defined, i.e. the limit  $\lim_{n \rightarrow \infty} \pi_n(\mathcal{H})$  exists.

*Proof.* It suffices to show that  $\pi_n(\mathcal{H})$ , viewed as a sequence in  $n$ , is decreasing.

Write  $R = \{k_1, k_2, \dots, k_r\}$ . Let  $G_n \subseteq K_n^R$  be a hypergraph with  $v(G_n) = n$  not containing  $\mathcal{H}$  and with Lubell value  $h_n(G_n) = \pi_n(\mathcal{H})$ . For any  $\ell < n$ , consider a random subset  $S$  of the vertices of  $G$  with size  $|S| = \ell$ .

Let  $G_n[S]$  be the induced subgraph of  $G_n$  (whose vertex set is restricted to  $S$ ). Clearly

$$\pi_\ell(\mathcal{H}) \geq \mathbb{E}(h_\ell(G_n[S])).$$

Write  $E(G_n) = E_{k_1} \cup E_{k_2} \cup \dots \cup E_{k_r}$  where  $E_{k_i}$  contains all the edges of size  $k_i$ . Note that the expected number of edges of size  $k_i$  in  $G_n[S]$  is precisely  $\frac{\binom{\ell}{k_i}}{\binom{n}{k_i}} |E_{k_i}|$ . It follows



that

$$\begin{aligned}
\pi_\ell(\mathcal{H}) &\geq \mathbb{E}(h_\ell(G_n[S])) \\
&= \sum_{i=1}^r \frac{\mathbb{E}(|E_{k_i} \cap \binom{S}{k_i}|)}{\binom{\ell}{k_i}} \\
&= \sum_{i=1}^r \frac{\binom{\ell}{k_i} |E_{k_i}|}{\binom{\ell}{k_i}} \\
&= \sum_{i=1}^r \frac{|E_{k_i}|}{\binom{n}{k_i}} \\
&= h_n(G_n) \\
&= \pi_n(\mathcal{H}).
\end{aligned}$$

The sequence  $\pi_n(\mathcal{H})$  is non-negative and decreasing; therefore it converges.  $\square$

The following is an easy proposition which bounds the Turán density and, using the probabilistic interpretation of the Lubell function, determines the Turán density for our first family of  $R$ -graphs, flags. A  $R$ -flag is any  $R$ -graph with exactly one edge of each size in  $R$ .

**Proposition 1.** *For any hypergraph  $H$ , the following statements hold.*

1.  $|R(H)| - 1 \leq \pi_n(H) \leq |R(H)|$ .
2. For subgraph  $H'$  of  $H$ , we have  $\pi(H') \leq \pi(H) - |R(H)| + |R(H')|$ .
3. For any  $R$ -flag  $L$  on  $m$  vertices and any  $n \geq m$ , we have  $\pi_n(L) = |R| - 1$ .

*Proof.* Pick any maximal proper subset  $R'$  of  $R(H)$ . Consider the complete graph  $K_n^{R'}$ . Since  $K_n^{R'}$  misses one type of edge in  $R(H) \setminus R'$ , it does not contain  $H$  as a subgraph. Thus

$$\pi_n(H) \geq h_n(K_n^{R'}) = |R'| = |R(H)| - 1.$$

The upper bound is due to the fact  $h_n(K_n^{R(H)}) = |R(H)|$ .

Proof of item 2 is similar. Let  $S = R(H')$  and  $G_n^S$  be an extremal hypergraph for  $\pi_n(H')$ . Extend  $G_n^S$  to  $G_n^{R(H)}$  by adding all the edges with cardinalities in  $R(H) \setminus S$ . The resulting graph  $G_n^{R(H)}$  is  $H$ -free. We have

$$\pi_n(H) \geq \pi_n(G_n^{R(H)}) = \pi_n(G_n^S) + |R(H)| - |S| = |R(H)| - |R(H')| + \pi_n(H').$$

Taking the limit as  $n$  goes to infinity, we have

$$\pi(H) \geq |R(H)| - |R(H')| + \pi(H').$$

Finally, for item 3, consider an  $L$ -free hypergraph  $G_n^R$ . Pick a random permutation  $\sigma$  of  $n$  elements uniformly. Let  $X$  be the number of edges of  $G_n^R$  hit by a random flag  $\sigma(L)$ . Note that each edge  $F$  has probability  $\frac{1}{\binom{n}{|F|}}$  of being hit by  $\sigma(L)$ . We have

$$\mathbb{E}(X) = \sum_{F \in E(G)} \frac{1}{\binom{n}{|F|}} = h_n(G). \quad (2.1)$$

Since  $G_n^R$  is  $L$ -free, we have  $X \leq r - 1$ . Taking the expectation, we have

$$h_n(G_n^R) = \mathbb{E}(X) \leq r - 1.$$

Hence,  $\pi_n(H) \leq r - 1$ . The result follows after combining with item 1.  $\square$

**Lemma 2** (Supersaturation of Families). *Let  $\mathcal{H}$  be a finite family of  $R$ -graphs. For any constant  $a > 0$  there exist positive constants  $b$  and  $n_0$  so that if  $G$  is an  $R$ -graph on  $n > n_0$  vertices with  $h_n(G) > \pi(\mathcal{H}) + a$ , then  $G$  contains at least  $b \binom{n}{v(H)}$  copies of some  $H \in \mathcal{H}$ .*

*Proof.* Let  $r := |R(\mathcal{H})|$  and let  $R = R(\mathcal{H})$ . Add independent vertices to members of  $\mathcal{H}$  so that every graph in  $\mathcal{H}$  has the same number of vertices. Note that adding these vertices does not change the Turán density of  $H$ . Since  $\lim_{n \rightarrow \infty} \pi_n(\mathcal{H}) = \pi(\mathcal{H})$ , there exists an  $n_0 > 0$  so that if  $m > n_0$  then  $\pi_m(\mathcal{H}) < \pi(\mathcal{H}) + \frac{a}{2}$ . Suppose that  $n_0 < m < n$  and  $G$  is an  $R$ -graph on  $n$  vertices with  $h_n(G) > \pi(\mathcal{H}) + a$ . Then,  $G$  must contain  $\frac{a}{2r} \binom{n}{m}$   $m$ -sets  $M \subset V(G)$  satisfying  $h_m(G[M]) > \pi(\mathcal{H}) + \frac{a}{2}$ . Otherwise we have

$$\sum_M h_m(G[M]) \leq \left( \pi(\mathcal{H}) + \frac{a}{2} \right) \binom{n}{m} + \frac{a}{2r} \binom{n}{m} r = \left( \pi(\mathcal{H}) + a \right) \binom{n}{m}.$$

And we have that

$$\begin{aligned}
\sum_M h_m(G[M]) &= \sum_M \sum_{\substack{F \in E(G) \\ F \subseteq M}} \frac{1}{\binom{m}{|F|}} \\
&= \sum_{F \in E(G)} \sum_{M \supseteq F} \frac{1}{\binom{m}{|F|}} \\
&= \sum_{F \in E(G)} \frac{\binom{n-|F|}{m-|F|}}{\binom{m}{|F|}} \\
&= \sum_{F \in E(G)} \frac{\binom{n}{m}}{\binom{n}{|F|}} \\
&= \binom{n}{m} h_n(G).
\end{aligned}$$

But this implies  $h_n(G) \leq \pi(\mathcal{H}) + a$ , which is contrary to the assumption that  $h_n(G) > \pi(\mathcal{H}) + a$ .

Since  $m > n_0$  it follows that each of the  $\frac{a}{2r} \binom{n}{m}$   $m$ -sets  $M \subset V(G)$  satisfying  $h_m(G[M]) > \pi(\mathcal{H}) + \frac{a}{2}$  contains some member of  $\mathcal{H}$ . Let  $v = v(H) = |V(H)|$  for each  $H \in \mathcal{H}$ . Recall that we added vertices to members of  $\mathcal{H}$  as necessary so that every graph had the same number of vertices. Then, counted by multiplicity, there are at least  $\frac{a}{2r} \binom{n}{m} / \binom{n-v}{m-v} = \frac{a}{2r} \binom{m}{v}^{-1} \binom{n}{v}$  members of  $\mathcal{H}$  in  $G$ . By the pigeon hole principle, at least one member of  $\mathcal{H}$  appears in  $G$  at least  $b \binom{n}{v}$  times where  $b = \frac{a}{2r|\mathcal{H}|} \binom{m}{v}^{-1}$ .  $\square$

Supersaturation can be used to show that “blowing up” does not change the Turán density  $\pi(H)$  just like in the uniform cases.

**Definition 5.** For any hypergraph  $H_n$  and positive integers  $s_1, s_2, \dots, s_n$ , the **blowup** of  $H$  is a new hypergraph  $(V, E)$ , denoted by  $H_n(s_1, s_2, \dots, s_n)$ , satisfying

1.  $V := \sqcup_{i=1}^n V_i$ , where  $|V_i| = s_i$ .
2.  $E = \cup_{F \in E(H)} \prod_{i \in F} V_i$ .

When  $s_1 = s_2 = \dots = s_n = s$ , we simply write it as  $H(s)$ . The belowup of a family  $\mathcal{H}$  of finite many non-uniform hypergraphs is defined as  $\mathcal{H}(s) = \{H(s) : H \in \mathcal{H}\}$ .

**Theorem 8** (Blow-ups of Families). *Let  $\mathcal{H}$  be a finite family of hypergraphs and let  $s \geq 2$ . Then  $\pi(\mathcal{H}(s)) = \pi(\mathcal{H})$ .*

*Proof.* First, it is clear that  $\pi(\mathcal{H}) \leq \pi(\mathcal{H}(s))$  since any  $\mathcal{H}$ -free graph  $G$  is also  $\mathcal{H}(s)$ -free. We will now show that for any  $\epsilon > 0$  that  $\pi(\mathcal{H}(s)) < \pi(\mathcal{H}) + \epsilon$ .

Let  $\epsilon > 0$ . Let  $R = R(\mathcal{H})$ . Again, add independent vertices as necessary to the graphs in  $\mathcal{H}$  so that they all have the same number of vertices. Let  $v$  denote the common size of the vertex set of every element in  $\mathcal{H}$ . By the supersaturation lemma, there exists an  $n_0$  and a  $b > 0$  so that if  $G$  is an  $R$ -graph on  $n > n_0$  vertices, then  $G$  contains at least  $b \binom{n}{v}$  copies of some  $H \in \mathcal{H}$ . We will show that  $H(s)$  is contained in  $G$ .

Consider an auxiliary  $v$ -uniform hypergraph  $U$  with  $V(U) = V(G)$ . A  $v$ -set of the vertices forms an edge in  $U$  if and only if the corresponding  $v$ -set in  $G$  is a copy of  $H$ . Note that  $U$  contains at least  $b \binom{n}{v}$  edges. For  $n$  large enough, and any  $S > 1$  it follows that  $U$  contains a copy of  $K = K_v^v(S)$ . This follows since  $K$  is  $v$ -uniform and  $v$ -partite implying that  $\pi(K) = 0$  and  $h_n(U) \geq b > 0$ . Fix one such  $K$  in  $U$ . Color each of the edges of  $K$  with one of the  $v!$  colors corresponding to the possible orderings with which the vertices of  $H$  are mapped into the parts of  $K$ . By the pigeon hole principle, one of the color classes contains at least  $S^v/v!$  edges. For large enough  $S$  (such that  $S^v/v! \geq s$ ) it follows that  $U$  contains a monochromatic copy of  $K_v^v(s)$ . This monochromatic copy of  $K_v^v(s)$  in  $U$  corresponds to a copy of  $H(s)$  in  $G$ . Hence  $\pi(\mathcal{H}(s)) < \pi(\mathcal{H}) + \epsilon$  as desired.  $\square$

**Corollary 1** (Squeeze Theorem). *Let  $H$  be any hypergraph. If there exists a hypergraph  $H'$  and integer  $s \geq 2$  such that  $H' \subseteq H \subseteq H'(s)$  then  $\pi(H) = \pi(H')$ .*

*Proof.* One needs only observe that for any hypergraphs  $H_1 \subseteq H_2 \subseteq H_3$  it follows that  $\pi(H_1) \leq \pi(H_2) \leq \pi(H_3)$ . If  $H_3 = H_1(s)$  for some  $s \geq 2$  then  $\pi(H_1) = \pi(H_3)$  by the previous theorem.  $\square$

## 2.4 $\{1, 2\}$ -HYPERGRAPHS

In this section we will determine the Turán density for any hypergraph  $H$  with  $R(H) = \{1, 2\}$ . We begin with the following more general result.

**Theorem 9.** *Let  $H = H^1 \cup H^k$  be a hypergraph with  $R(H) = \{1, k\}$  and  $E(H^1) = V(H^k)$ . Then*

$$\pi(H) = \begin{cases} 1 + \pi(H^k) & \text{if } \pi(H^k) \geq 1 - \frac{1}{k}; \\ 1 + \left(\frac{1}{k(1-\pi(H^k))}\right)^{1/(k-1)} \left(1 - \frac{1}{k}\right) & \text{otherwise.} \end{cases}$$

*Proof.* For each  $n \in \mathbb{N}$ , let  $G_n$  be any  $H$ -free graph  $n$  vertices with  $h_n(G_n) = \pi(H)$ . Partition the vertices of  $G_n$  into  $X_n = \{v \in V(G_n) : \{v\} \in E(G)\}$  and  $\bar{X}_n$  containing everything else. Say that  $|X_n| = x_n n$  and  $|\bar{X}_n| = (1 - x_n)n$  for some  $x_n \in [0, 1]$ . Since  $(x_n)$  is a sequence in  $[0, 1]$  it has a convergent subsequence. Consider  $(x_n)$  to be the convergent subsequence, and say that  $x_n \rightarrow x \in [0, 1]$ . With the benefit of hindsight, we know that  $x > 0$ , however, for the upper bound portion of this proof we will not assume this knowledge.

Since there is no copy of  $H$  in  $G_n$ , it follows that  $G_n[X_n]$  contains no copy of  $H^k$ .

We have that

$$\begin{aligned} \pi(H) &= \lim_{n \rightarrow \infty} h_n(G_n) \\ &= \lim_{n \rightarrow \infty} \sum_{F \in G^1} \frac{1}{\binom{n}{1}} + \sum_{F \in G^k} \frac{1}{\binom{n}{k}} \\ &\leq \lim_{n \rightarrow \infty} \frac{x_n n}{\binom{n}{1}} + \frac{\binom{n}{k} - (1 - \pi_{x_n n}(H^k)) \binom{x_n n}{k}}{\binom{n}{k}} \\ &= \lim_{n \rightarrow \infty} 1 + x_n - (1 - \pi_{x_n n}(H^k)) \frac{\binom{x_n n}{k}}{\binom{n}{k}} \\ &\leq \lim_{n \rightarrow \infty} \begin{cases} 1 + \frac{1}{\sqrt{n}} & \text{if } x_n n \leq \sqrt{n}, \\ 1 + x_n - (1 - \pi_{x_n n}(H^k)) x_n^k & \text{if } x_n n > \sqrt{n}, \end{cases} \\ &\leq \max\{1, 1 + x - (1 - \pi(H^k)) x^k\}. \end{aligned}$$

Let  $f(x) = 1 + x - (1 - \pi(H^k))x^k$  and then note that

$$\pi(H) = \lim_{n \rightarrow \infty} h_n(G) \leq \max_{x \in [0,1]} f(x).$$

An easy calculus exercise shows that  $f''(x) < 0$  for all  $x > 0$ , and  $f'(x) = 0$  when  $x = \left(\frac{1}{k(1-\pi(H^k))}\right)^{\frac{1}{k-1}}$ . If  $\frac{1}{k(1-\pi(H^k))} \geq 1$  then  $f'(x) > 0$  when  $x \in [0, 1)$  and hence  $f(x)$  is maximized when  $x = 1$ . Note that  $f(1) = 1 + \pi(H^k)$ . If, on the other hand,  $\frac{1}{k(1-\pi(H^k))} < 1$  it follows that  $f(x)$  is maximized at  $x = \left(\frac{1}{k(1-\pi(H^k))}\right)^{1/(k-1)}$ . Together, this gives us

$$\pi(H) \leq \begin{cases} 1 + \pi(H^k) & \text{if } \pi(H^k) \geq 1 - \frac{1}{k}; \\ 1 + \left(\frac{1}{k(1-\pi(H^k))}\right)^{1/(k-1)} \left(1 - \frac{1}{k}\right) & \text{otherwise.} \end{cases}$$

To get equality, take  $x$  that maximizes  $f(x)$  as above. For any  $n \in \mathbb{N}$  (thinking of  $n \rightarrow \infty$ ) partition  $[n]$  into two sets  $X$  and  $\bar{X}$  with  $|X| = xn$  and  $|\bar{X}| = (1-x)n$ . Let  $E(G^1) = \{\{v\} : v \in X\}$  and let  $g^k$  be a  $k$ -uniform graph on  $xn$  vertices attaining  $|E(g^k)| = \text{ex}(H^k, xn)$  and  $g^k$  is  $H^k$ -free. Then

$$E(G^k) = \left\{ F \in \binom{[n]}{k} : \text{either } F \in E(g^k) \text{ or } F \cap \bar{X} \neq \emptyset \right\}.$$

Then  $G = G^1 \cup G^k$  is  $H$ -free and (by choice of  $x$ ) we have that  $\lim_{n \rightarrow \infty} h_n(G)$  attains the upper bound of  $\pi(H)$ .  $\square$

Let us now return to the task of determining  $\pi(H)$  when  $H = H^1 \cup H^2$ .

**Proposition 2.** *Let  $H = H^1 \cup H^2$ . If  $H^2$  is not bipartite, then*

$$\pi(H) = 1 + \pi(H^2) = 1 + \left(1 - \frac{1}{\chi(H^2) - 1}\right) = 2 - \frac{1}{\chi(H^2) - 1}.$$

*Proof.* First,  $\pi(H) \geq 1 + \pi(H^2)$  since one can construct an  $H$ -free graph  $G_n$  by letting

$$E(G_n) = \{\{v\} : v \in V(G_n)\} \cup E(G'_n)$$

where  $G'_n$  attains  $h_n(G'_n) = \pi_n(H^2)$  and  $G'_n$  is  $H^2$ -free. Then

$$\pi(H) \geq \lim_{n \rightarrow \infty} h_n(G_n) = \lim_{n \rightarrow \infty} 1 + \pi_n(H^2) = 1 + \pi(H^2).$$

To get the upper-bound, first add every missing 1-edge into  $H$ , call the new graph  $H'$ . Note that  $\pi(H) \leq \pi(H')$ . Note that we didn't change the edge set  $H^2$ . The Erdős-Stone-Simonovits theorem states that if  $H^2$  is not bipartite, then  $\pi(H^2) = 1 - \frac{1}{\chi(H^2)-1}$ . Also, if  $H^2$  is not bipartite, then  $\chi(H^2) \geq 3$ . With the added vertices, taking  $k = 2$ , we apply the previous theorem. Since

$$\pi(H^2) = 1 - \frac{1}{\chi(H^2) - 1} \geq 1 - \frac{1}{2}$$

we may conclude that  $\pi(H) \leq \pi(H') = 1 + \pi(H^2)$ . □

It remains to investigate the cases when  $H^2$  is bipartite.

**Proposition 3.** *Let  $H = H^1 \cup H^2$ . If  $H^2$  is bipartite and  $K_2^{\{1,2\}} \subseteq H$  then  $\pi(H) = \frac{5}{4}$ .*

*Proof.* First, in example 1, we computed  $\pi(K_2^{\{1,2\}}) = \frac{5}{4}$ . Second,  $H$  must be contained in some blow-up of  $K_2^{\{1,2\}}$  since  $H^2$  is bipartite, i.e. there exists some  $s > 2$  such that  $H \subseteq K_2^{\{1,2\}}(s)$ . So, by the squeeze theorem we have

$$\frac{5}{4} = \pi(K_2^{\{1,2\}}) \leq \pi(H) \leq \pi(K_2^{\{1,2\}}(s)) = \frac{5}{4}.$$

Hence  $\pi(H) = \frac{5}{4}$  as claimed. □

**Definition 6.** *We will say that  $H = H^1 \cup H^2$  is a **closed path** (from  $x_1$  to  $x_k$ ) of length  $k$  if  $V(H) = \{x_1, x_2, \dots, x_k\}$  and  $E(H^1) = \{\{x_1\}, \{x_k\}\}$  and  $E(H^2) = \{\{x_i, x_{i+1}\} : 1 \leq i \leq k-1\}$ . We will denote a closed path of length  $k$ , or a closed  $k$ -path, by  $\bar{P}_k$ .*

Pictorially, we view a closed path as follows:

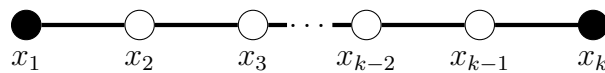


Figure 2.1 A closed path of length  $k$ .

**Proposition 4.** *Let  $H = H^1 \cup H^2$ . If  $H^2$  is bipartite,  $H$  does not contain a copy of  $K_2^{\{1,2\}}$ , and  $H$  contains a closed path of length  $2k$ , then  $\pi(H) = \frac{9}{8}$ .*

*Proof.* First, we will give a construction giving us the lower bound. For any  $n \in \mathbb{N}$  let  $G_n$  have vertex set  $[n]$ . Partition the vertices of  $G_n$  into two sets  $X$  and  $\bar{X}$  where  $|X| = \frac{3n}{4}$  and  $|\bar{X}| = \frac{n}{4}$ . Let

$$E(G) = \{\{x\} : x \in X\} \cup \{\{x, \bar{x}\} : x \in X \text{ and } \bar{x} \in \bar{X}\}.$$

It is clear that  $G_n$  contains no closed paths of length  $2k$  when  $k \geq 1$ . Also,

$$\begin{aligned} \lim_{n \rightarrow \infty} h_n(G_n) &= \lim_{n \rightarrow \infty} \frac{|X|}{\binom{n}{1}} + \frac{|X| \cdot |\bar{X}|}{\binom{n}{2}} \\ &= \lim_{n \rightarrow \infty} \frac{3}{4} + \frac{\frac{3}{16}n^2}{\binom{n}{2}} \\ &= \frac{3}{4} + \frac{3}{8} = \frac{9}{8}. \end{aligned}$$

Thus  $\pi(H) \geq \frac{9}{8}$  for any  $H$  containing a closed path of length  $2k$  for any  $k \geq 1$ .

Since  $H^2$  is bipartite, and  $H^2$  does not contain a copy of  $K_2^{\{1,2\}}$ , then  $H$  is contained in a blow-up of a closed 4-path. To see this, note that there is a bipartition of the vertices of  $H$ ,  $V(H) = A \cup B$ , (with respect to the 2-edges in  $H$ ). Furthermore, we can partition  $A$  into  $A_1 \cup A_2$  where  $v \in A_1$  if  $\{v\} \in E(H)$  and  $v \in A_2$  if  $v \in A \setminus A_1$ . And similarly partition  $B$  into  $B_1 \cup B_2$  with  $v \in B_1$  if  $\{v\} \in E(H)$  and  $v \in B_2$ . Then note that there are no edges from  $A_1$  to  $B_1$  since  $H$  contains no copy of  $K_2^{\{1,2\}}$ . So  $H \subset \bar{P}_4(\max\{|A_1|, |A_2|, |B_1|, |B_2|\})$ —a blow-up of  $\bar{P}_4$ . Below is a graphical representation of  $H$ , illustrating that  $H$  is contained in a blow-up of  $\bar{P}_4$ .

Since  $\pi(H) \leq \pi(\bar{P}_4(s)) = \pi(\bar{P}_4)$  we need only show that  $\pi(\bar{P}_4) \leq \frac{9}{8}$ . Let  $G_n$  be a family of  $\bar{P}_4$ -free graphs such that  $h_n(G_n) = \pi_n(\bar{P}_4)$ . Partition the vertices of  $G_n$  as



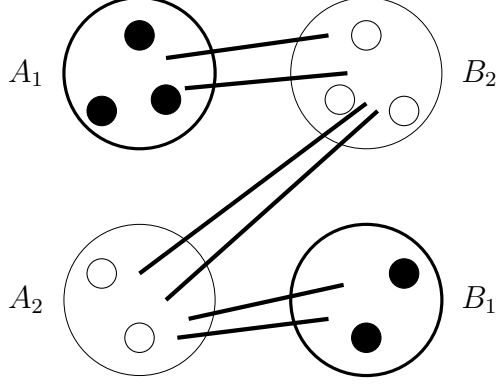


Figure 2.2 A blow-up of  $\bar{P}_4$ .

follows:

$$X_n = \{v : \{v\} \in E(G_n)\},$$

$$Y_n = \{v : \{v\} \notin E(G_n) \text{ and } \exists x_1 \neq x_2 \in X_n \text{ with } \{x_1, v\}, \{x_2, v\} \in E(G_n)\},$$

$$Z_n = V(G) \setminus (X_n \cup Y_n).$$

Let us say that  $|X_n| = xn$ ,  $|Y_n| = yn$  and hence  $|Z_n| = (1 - x - y)n$ .

First, note that  $E(G) \cap \binom{Y_n}{2} = \emptyset$ . Otherwise, since each vertex in  $Y_n$  has at least 2 neighbors in  $X_n$ ,  $G_n$  would contain a closed path of length 4. Also, each vertex in  $Z_n$  has at most 1 neighbor in  $X_n$ . It follows that

$$\begin{aligned} \pi(\bar{P}_4) &= \lim_{n \rightarrow \infty} \pi_n(\bar{P}_4) \\ &= \lim_{n \rightarrow \infty} h_n(G_n) \\ &\leq \lim_{n \rightarrow \infty} \frac{|X_n|}{\binom{n}{1}} + \frac{|X_n| \cdot |Y_n|}{\binom{n}{2}} + \frac{|Y_n| \cdot |Z_n|}{\binom{n}{2}} + \frac{\binom{|Z_n|}{2}}{\binom{n}{2}} + \frac{|Z_n|}{\binom{n}{2}} \\ &\leq \lim_{n \rightarrow \infty} \frac{xn}{\binom{n}{1}} + \frac{xy n^2}{\binom{n}{2}} + \frac{y(1-x-y)n^2}{\binom{n}{2}} + \frac{(1-x-y)^2 n^2}{\binom{n}{2}} + \frac{(1-x-y)n}{\binom{n}{2}} \\ &\leq \max_{\substack{0 \leq x \leq 1 \\ 0 \leq y \leq 1-x}} x + 2xy + 2y(1-x-y) + (1-x-y)^2 \\ &= \frac{9}{8}. \end{aligned}$$

The last inequality is an easy multivariate calculus exercise. One can also verify it with software, such as *Mathematica*, the syntax being:

Maximize [ $\{x^2 - x - y^2 + 2xy + 1, 0 \leq x \leq 1, 0 \leq y \leq 1 - x\}, \{x, y\}$ ].

It may be of interest to note that the maximum value of the function is obtained when  $x = \frac{3}{4}$  and  $y = \frac{1}{4}$ . In this case  $Z_n$  is empty. Since our upper bound matches the lower bound, we have the desired result.  $\square$

**Proposition 5.** *Let  $H = H^1 \cup H^2$ . If  $H^2$  is bipartite and  $H^2$  does not contain a closed  $2k$ -path for any  $k \geq 1$ , then  $\pi(H) = 1$ .*

*Proof.* First, since  $|R(H)| = 2$  we have, trivially, that  $\pi(H) \geq 1$ . Since  $H$  contains no path of length  $2k$  for any  $k \geq 1$  it must be the case that  $H$  is contained in a blow-up of a chain  $C^{\{1,2\}} = \{\{x\}, \{x, y\}\}$ . This is most clearly seen by again, considering the previous illustration. The difference is, in this case,  $B_1$  (or  $A_1$ ) is empty. It is clear

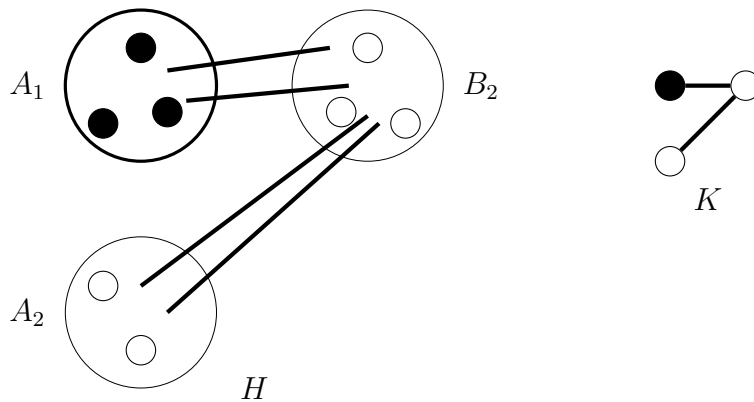


Figure 2.3  $H$  does not contain a closed path of even length.

that  $H$  is contained in a blow-up of  $K$  where

$$K = \{\{x\}, \{x, y\}, \{y, z\}\} \subseteq C^{\{1,2\}}(2, 1) = \{\{x\}, \{z\}, \{x, y\}, \{z, y\}\}.$$

It follows that  $\pi(H) \leq \pi(C^{\{1,2\}}) = 1$ .  $\square$

These propositions completely determine  $\pi(H)$  when  $R(H) = \{1, 2\}$ . The results are summarized by the following theorem.

**Theorem 10.** *For any hypergraph  $H$  with  $R(H) = \{1, 2\}$ , we have*

$$\pi(H) = \begin{cases} 2 - \frac{1}{\chi(H^2)-1} & \text{if } H^2 \text{ is not bipartite;} \\ \frac{5}{4} & \text{if } H^2 \text{ is bipartite and } \min\{k : \bar{P}_{2k} \subseteq H\} = 1; \\ \frac{9}{8} & \text{if } H^2 \text{ is bipartite and } \min\{k : \bar{P}_{2k} \subseteq H\} \geq 2; \\ 1 & \text{if } H^2 \text{ is bipartite and } \bar{P}_{2k} \not\subseteq H \text{ for any } k \geq 1. \end{cases}$$

## 2.5 DEGENERATE HYPERGRAPHS

We say that a hypergraph  $H$  is **degenerate** if  $\pi(H) = |R(H)| - 1$ . For an  $r$ -uniform hypergraph  $H$ ,  $H$  is degenerate if and only if  $H$  is  $r$ -partite. From Proposition 1 and Theorem 8, we have the following proposition.

**Proposition 6.** *Suppose  $H$  is a degenerate hypergraph. Then the following properties hold.*

- *Every subgraph of  $H$  is degenerate.*
- *Every blowup of  $H$  is degenerate.*
- *Any subgraph of the blowup of a flag is degenerate.*

Note that every flag is a subgraph of some blowup of a chain with the same edge type. Is every degenerate hypergraph a subgraph of some blowup of a chain? The answer is yes for uniform hypergraphs and  $\{1, 2\}$ -hypergraphs. This follows from Theorem 10, which completely determined  $\pi(H)$  when  $R(H) = \{1, 2\}$ , and from the fact that a  $k$ -uniform hypergraph is degenerate if and only if it is  $k$ -partite (a subgraph of a blowup of a single edge). However, the answer in general is false. We will show

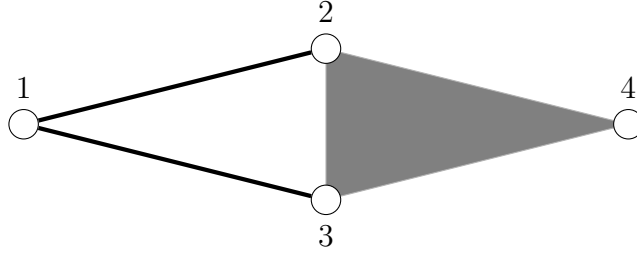


Figure 2.4 A degenerate hypergraph not contained in the blow-up of a chain.

that the following hypergraph  $H_1$  with edge set  $E(H_1) = \{\{1, 2\}, \{1, 3\}, \{2, 3, 4\}\}$  is degenerate.

This result is a special case of the following theorem.

**Definition 7.** Let  $H$  be a hypergraph containing some 2-edges. The 2-subdivision of  $H$  is a new hypergraph  $H'$  obtained from  $H$  by subdividing each 2-edge simultaneously. Namely, if  $H$  contains  $t$  2-edges, add  $t$  new vertices  $x_1, x_2, \dots, x_t$  to  $H$  and for  $i = 1, 2, \dots, t$  replace the 2-edge  $\{u_i, v_i\}$  with  $\{u_i, x_i\}$  and  $\{x_i, v_i\}$ .

**Theorem 11.** Let  $H'$  be the 2-subdivision of  $H$ . If  $H$  is degenerate, then so is  $H'$ .

For example,  $H_1$  can be viewed as the 2-division of the chain  $C^{\{2,3\}}$ . Since any chain is degenerate, so is  $H_1$ . To prove this theorem, we need a Lemma on graphs, which has independent interest.

**Definition 8.** Let  $G$  be any simple graph. Then  $G^{(2)}$ , a variation of the square of  $G$ , will be defined as follows:

- $V(G^{(2)}) := V(G)$ ,
- $E(G^{(2)}) := \{\{u, v\} \mid \exists w \in V(G) \text{ with } \{u, w\}, \{v, w\} \in E(G)\}$ .

Note that an edge of  $G$  may or may not be an edge of  $G^{(2)}$ . For example, if  $G$  is the complete graph, then  $G^{(2)}$  is also the complete graph. However, if  $G$  is a complete

bipartite graph with partite set  $V_1 \cup V_2$ , then  $G^{(2)}$  is the disjoint union of two complete graphs on  $V_1$  and  $V_2$ . In this case,  $G^{(2)}$  is the complement graph of  $G$ ! We also note that  $G^{(2)}$  is the empty graph if and only if  $G$  is a matching. Surprisingly, we have the following Lemma on the difference of the number of edges in  $G$  and  $G^{(2)}$ .

**Lemma 3.** *For any simple graph  $G$  on  $n$  vertices,*

$$|E(G)| - |E(G^{(2)})| \leq \left\lfloor \frac{n}{2} \right\rfloor. \quad (2.2)$$

*Furthermore, equality holds if and only if  $G$  is the vertex-disjoint union of complete bipartite graphs of balanced part-size with at most one component having odd number of vertices, i.e.*

$$G = K_{t_1, t_1} \cup K_{t_2, t_2} \cup \cdots \cup K_{t_k, t_k} \cup K_{\lfloor \frac{n}{2} \rfloor - \sum_{i=1}^k t_i, \lceil \frac{n}{2} \rceil - \sum_{i=1}^k t_i},$$

*for some positive integers  $t_1, t_2, \dots, t_k$  satisfying  $\sum_{i=1}^k t_i \leq \lfloor \frac{n}{2} \rfloor$ .*

*Proof.* First, we will show that inequality (2.2) holds for any forest. Let  $G$  be a forest. Since  $G$  is a forest, if  $\{a, b\} \in E(G^{(2)})$  then  $a$  and  $b$  have a unique common neighbor in  $G$ . Furthermore, given any vertex  $c \in V(G)$ , it follows that any pair of neighbors of  $c$  is in  $E(G^{(2)})$ . Thus we have

$$\begin{aligned} |E(G)| - |E(G^{(2)})| &= \frac{1}{2} \sum_{v \in V(G)} \deg(v) - \sum_{v \in V(G)} \binom{\deg(v)}{2} \\ &= \sum_{v \in V(G)} \frac{1}{2} \deg(v) - \binom{\deg(v)}{2} \\ &= \sum_{v \in V(G)} -\frac{1}{2} \deg(v)^2 + \deg(v) \\ &\leq \sum_{v \in V(G)} \frac{1}{2} \\ &= \frac{n}{2}. \end{aligned}$$

The inequality above comes from the fact that  $-\frac{1}{2}x^2 + x \leq \frac{1}{2}$ , attaining its maximum when  $x = 1$ . Since  $|E(G)| - |E(G^{(2)})|$  is an integer, we have that

$$|E(G)| - |E(G^{(2)})| \leq \left\lfloor \frac{n}{2} \right\rfloor \quad (2.3)$$

as claimed.

Now we will prove the statement  $|E(G)| - |E(G^{(2)})| \leq \lfloor \frac{|V(G)|}{2} \rfloor$  for general graphs using induction on the number of vertices. It holds trivially for  $n = 1, 2$ .

Assume that the statement holds for all graphs with at fewer than  $n$  vertices. Consider a graph  $G$  with  $n$  vertices. If  $G$  is a forest, then the statement holds. Otherwise,  $G$  contains a cycle. Choose  $C_g$  to be a minimal cycle in  $G$ , i.e. one with no chords. If  $G = C_g$ , then  $|E(C_g) - E(C_g^{(2)})| = 0$  if  $g \neq 4$  or  $2$  if  $g = 4$ . The statement holds.

Now assume  $V(C) \subsetneq V(G)$ . Let  $V_1 := V(C) = \{x_1, x_2, \dots, x_g\}$ , where  $x_i$  is adjacent to  $x_{i+1}$ , and let  $V_2 := V(G) \setminus V_1 = \{v_1, v_2, \dots, v_{n-g}\}$ .

The edges of  $G$  can be partitioned into three parts: the induced graph  $G[V_1] = C_g$ , the induced graph  $G[V_2]$ , and the bipartite graph  $G[V_1, V_2]$ . Similarly, the edges of  $G^{(2)}$  can be partitioned into three parts:  $G^{(2)}[V_1]$ , the induced graph  $G^{(2)}[V_2]$ , and the bipartite graph  $G^{(2)}[V_1, V_2]$ . Now we compare term by term.

1. Note  $|E(G^{(2)}[V_1])| \geq |E(C_g^{(2)})|$ , and  $|E(C_g^{(2)})| = g$  if  $g \neq 4$  or  $2$  if  $g = 4$ . We have

$$|E(G[V_1])| - |E(G^{(2)}[V_1])| \leq |E(C_g)| - |E(C_g^{(2)})| \leq \left\lfloor \frac{g}{2} \right\rfloor. \quad (2.4)$$

2. By inductive hypothesis, we have  $|E(G[V_2])| - |E((G[V_2])^{(2)})| \leq \lfloor \frac{n-g}{2} \rfloor$ . Combining with the fact  $|E(G^{(2)}[V_2])| \geq |E((G[V_2])^{(2)})|$ , we have

$$|E(G[V_2])| - |E(G^{(2)}[V_2])| \leq \left\lfloor \frac{n-g}{2} \right\rfloor. \quad (2.5)$$

3. We claim  $|E(G[V_1, V_2])| \leq |E(G^{(2)}[V_1, V_2])|$ . We define a map

$$f: E(G[V_1, V_2]) \rightarrow E(G^{(2)}[V_1, V_2])$$

as follows. For any edge  $x_i v \in E(G)$  with  $v \in V_2$  and  $x_i \in V_1$ , define  $f(x_i v) = v x_{i+1}$  (with the convention  $x_{g+1} = x_1$ ). Since  $x_i v \in E(G)$  and  $x_i x_{i+1} \in E(G)$ ,

we have  $vx_{i+1} \in E(G^{(2)})$ . The map  $f$  is well-defined. We also observe that  $f$  is an injective map. Thus

$$|E(G[V_1, V_2])| \leq |E(G^{(2)}[V_1, V_2])|. \quad (2.6)$$

Combining equations (2.4), (2.5), and (2.6), we get

$$|E(G)| - |E(G^{(2)})| \leq \left\lfloor \frac{n-g}{2} \right\rfloor + \left\lfloor \frac{g}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

The inductive step is finished.

Now we check when equality holds. It is straightforward to verify the sufficient condition; we omit the computation here.

Now we prove the necessary condition. Assume that  $G$  has  $k+1$  connected components  $G_1, G_2, \dots, G_{k+1}$ . Then we have

$$|E(G)| - |E(G^{(2)})| \leq \sum_{i=1}^{k+1} (|E(G_i)| - |E(G_i^{(2)})|) \leq \sum_{i=1}^{k+1} \left\lfloor \frac{|V(G_i)|}{2} \right\rfloor \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

If equality holds, then all but possibly one component has an even number of vertices. It remains to show each component is a balanced complete bipartite graph.

Without loss of generality, we assume  $G$  is connected. If  $G$  is a tree, then equality in Equation (2.3) either forces the degree of every vertex to be 1, or all the degrees are 1 with a single exceptional vertex of degree 2. Since  $G$  is assumed to be connected,  $G$  is either  $P_2 = K_{1,1}$  or  $P_3 = K_{1,2}$ .

Suppose that  $G$  contains cycles, and the equalities hold in Equations (2.4), (2.5), and (2.6). First we show that  $C_4$  is the only possible chordless cycle in  $G$ . Suppose not; let  $C_g$  ( $g \neq 4$ ) be a chordless cycle. We have  $|E(C_g)| - |E(C_g^{(2)})| = 0$ ; which contradicts the assumption that equality holds in Equation (2.4). Thus  $G$  is a bipartite graph. Furthermore, the equality in (2.5) forces each vertex  $v$  to be connected to at least 2 vertices of  $C_4$ . Hence  $G$  is 2-connected. Now  $G$  must be a complete bipartite graph. Otherwise, say  $uv$  is a non-edge crossing the partite sets. Since  $G$  is 2-connected, there exists a cycle containing both  $u$  and  $v$ . Let  $C$  be such a cycle with minimum

length;  $C$  is cordless but not a  $C_4$ . This is a contradiction. Finally we show  $G = K_{st}$  is balanced. Note that

$$|E(G)| - |E(G^{(2)})| = st - \binom{s}{2} - \binom{t}{2} = \frac{n}{2} - \frac{(s-t)^2}{2} \leq \left\lfloor \frac{n}{2} \right\rfloor.$$

The equality holds only if  $|s - t| \leq 1$ . So  $G$  is balanced.  $\square$

We now prove Theorem 11 (which we previously stated).

*Proof.* The proof is by contradiction. Let  $R := R(H) = R(H')$  be the common set of edge types of  $H$  and  $H'$ . Suppose that  $H'$  is not degenerate, then  $\pi(H') > |R| - 1 + \epsilon$  for some  $\epsilon > 0$ . Thus, there exists an  $n_0$  satisfying  $\pi_n(H') > |R| - 1 + \epsilon/2$  for any  $n \geq n_0$ . Let  $G_n^R$  be a  $H'$ -free hypergraph with  $\pi_n(G) > |R| - 1 + \epsilon/2$ . Define a new hypergraph  $G'_n$  over the same vertex set of  $G$  with a new edge set  $E(G'_n) = E(G_n) \setminus E(G_n^2) \cup E((G_n^2)^{(2)})$ . The hypergraph  $G'_n$  is obtained from  $G_n$  by replacing all 2-edges by the edges in its square graph while keeping all the edges of other types. By Lemma 3, we have

$$h_n(G'_n) \geq h_n(G) - \frac{\lfloor \frac{n}{2} \rfloor}{\binom{n}{2}} \geq |R| - 1 + \epsilon/2 - \frac{1}{n}. \quad (2.7)$$

Suppose that  $H$  has  $t$  2-edges. Since  $H$  is degenerate, so is the blowup hypergraph  $H(t+1)$ . For sufficiently large  $n$ ,  $G'_n$  contains a subhypergraph  $H(t+1)$ . By the definition of  $G'$ , for every copy of  $H \subseteq H(t+1)$  and every 2-edge  $u_i v_i$  (for  $1 \leq i \leq t$ ) of  $H$ , there exists a vertex  $x_i := x_i(u_i, v_i)$  satisfying  $u_i x_i$  and  $v_i x_i$  are 2-edges of  $G$ . Our goal is to force  $x_1, x_2, \dots, x_t$  to be distinct from the vertices of  $H$  and from each other. This can be done by a greedy algorithm. Suppose that the vertices of  $H$  are listed by  $y_1, y_2, y_3, \dots$ , and so on. Each vertex  $y_i$  has  $t+1$  copies in  $H(t+1)$ . For  $i = 1, 2, 3, \dots$ , select a vertex  $y'_i$  from the  $t+1$  copies of  $y_i$  so that  $y'_i$  is not the same vertex as  $x_j(u_j, v_j)$  for some 2-edge  $u_j v_j$  where  $u_j, v_j$  have been selected. This is always possible since  $H$  has only  $t$  2-edges. Thus, we found a copy of  $H'$  as a subgraph of  $G$  which is a contradiction.  $\square$



It remains an open question to classify all non-degenerate hypergraphs. In the remainder of this section, we generalize the following theorem due to Erdős on the Turán density of complete  $k$ -partite  $k$ -uniform hypergraphs.

**Theorem 12** (Erdős). *Let  $K_k^{(k)}(s_1, \dots, s_k)$  be the complete  $k$ -partite  $k$ -uniform hypergraph with partite sets of size  $s_1, \dots, s_k$ . Then any  $K_k^{(k)}(s_1, \dots, s_k)$ -free  $r$ -uniform hypergraph can have at most  $O(n^{k-\delta})$  edges, where  $\delta = \left(\prod_{i=1}^{k-1} s_i\right)^{-1}$ .*

We have the following theorem.

**Theorem 13.** *Let  $L(s_1, s_2, \dots, s_{v(L)})$  be a blowup of a flag  $L^R$ , we have*

$$\pi_n(L(s_1, s_2, \dots, s_{v(L)})) = r - 1 + O(n^{-\delta}),$$

where  $\delta = \frac{\max\{s_i : 1 \leq i \leq v(L)\}}{\prod_{i=1}^{v(L)} s_i}$ .

Using the concept of  $H$ -density, we can say a lot more about avoiding a blowup of any hypergraph  $H$ .

Given two hypergraphs  $H$  and  $G$  with the same edge-type  $R(H) = R(G)$ , the **density** of  $H$  in  $G$ , denoted by  $\mu_H(G)$ , is defined as the probability that a random injective map  $f: V(H) \rightarrow V(G)$  satisfies  $H \xrightarrow{f} G$  (i.e.  $f$  maps  $H$  to a labelled copy of  $H$  in  $G$ ). We have the following theorem.

**Theorem 14.** *Let  $H$  be a fixed hypergraph on  $m$  vertices and let  $s_1, s_2, \dots, s_m$  be  $m$  positive integers. If  $G$  is a sufficiently large with edge type  $R(G) = R(H)$  and if  $G$  contains no subgraph  $H(s_1, s_2, \dots, s_m)$ , then  $\mu_H(G) = O(n^{-\delta})$ , where  $\delta = \frac{\max\{s_i : 1 \leq i \leq m\}}{\prod_{i=1}^m s_i}$ .*

*Proof.* The proof is by contradiction. We assume  $\mu_H(G) \geq Cn^{-\delta}$  for some constant  $C$  to be chosen later. By reordering the vertices of  $H$ , we can assume  $s_1 \leq s_2 \leq \dots \leq s_m$ . Without loss of generality, we assume  $n$  is divisible by  $m$ . Consider a random  $m$ -partition of  $V(G) = V_1 \cup V_2 \cup \dots \cup V_m$  where each part has size  $\frac{n}{m}$ . For any  $m$ -set  $S$  of  $V(G)$ , we say  $S$  is a **transversal** (with respect to this partition of  $V(G)$ ) if

$S$  intersects each  $V_i$  exactly once. The probability that  $S$  is transversal is given by  $\frac{\binom{n}{m}^m}{\binom{n}{m}}$ .

We say a labelled copy of  $H$  in  $G$ ,  $f(H)$ , is **transversal ordered** if  $f(v_i) \in V_i$  for each vertex  $v_i$  of  $H$ . Clearly, the probability that  $f(H)$  is transversal ordered is exactly  $\frac{1}{m!}$  of the probability that  $f(H)$  is a transversal. By the definition of  $\mu_H(G)$ ,  $G$  contains  $\mu_H(G) \binom{n}{m} m!$  labelled copies of  $H$ . Thus, the expected number of transversal ordered copies of  $H$  is

$$\mu_H(G) \binom{n}{m}^m \geq \frac{C}{m^m} n^{m-\delta}.$$

There exists a partition so that the number of transversal ordered copies of  $H$  is at least  $Cm^{-m}n^{m-\delta}$ . Now we fix this partition  $[n] = V_1 \cup \dots \cup V_m$ .

For  $t_i \in \{1, s_i\}$  with  $i = 1, 2, \dots, m$ , let  $f(t_1, t_2, \dots, t_m)$  count the number of labelled copies of  $H' = H(t_1, t_2, \dots, t_m)$  such that the first  $t_1$  vertices of  $H'$  are in  $V_1$ , the second  $t_2$  vertices of  $H'$  are in  $V_2$ , and so on.

**Claim a:** For  $0 \leq l \leq m - 1$ , we have

$$f(s_1, \dots, s_l, 1, \dots, 1) \geq (1 + o(1)) \frac{(Cn^{-\delta})^{s_1 s_2 \dots s_l}}{\prod_{j=1}^l (s_j!)^{s_{j+1} \dots s_l}} \left(\frac{n}{m}\right)^{m-l+s_1+\dots+s_l}.$$

We prove claim (a) by induction on  $l$ . For the initial case  $l = 0$ , the claim is trivial since  $f(1, 1, \dots, 1)$  counts the number of transversal ordered copies of  $H$ . We have

$$f(1, 1, \dots, 1) \geq Cn^{-\delta} \left(\frac{n}{m}\right)^m.$$

Thus the statement holds for  $l = 0$ .

Now we assume claim (a) holds for  $l > 0$ . Consider the case  $l + 1$ , for some  $l \geq 0$ . For any  $S \in \binom{V_1}{s_1} \times \dots \times \binom{V_l}{s_l} \times \binom{V_{l+2}}{1} \times \dots \times \binom{V_m}{1}$ , let  $d_S$  be the number of vertices  $v$  in  $V_{l+1}$  such that the induced subgraph of  $H'$  on  $S \times \{v\}$  is  $H(s_1, \dots, s_l, 1, \dots, 1)$ . We have

$$f(s_1, \dots, s_l, 1, 1, \dots, 1) = \sum_S d_S; \tag{2.8}$$

$$f(s_1, \dots, s_l, s_{l+1}, 1, \dots, 1) = \sum_S \binom{d_S}{s_{l+1}}. \tag{2.9}$$

Let  $\bar{d}_l$  be the average of  $d_S$ . By equation (2.8) and the inductive hypothesis, we have

$$\bar{d}_l = \frac{\sum_S d_S}{\binom{n/m}{s_1} \cdots \binom{n/m}{s_l} \binom{n/m}{1} \cdots \binom{n/m}{1}} \geq \frac{f(s_1, \dots, s_l, 1, \dots, 1)}{\left(\frac{n}{m}\right)^{m-l-1+s_1+\dots+s_l}} \quad (2.10)$$

$$\geq (1 + o(1)) \frac{n (Cn^{-\delta})^{s_1 s_2 \cdots s_l}}{m \prod_{j=1}^l (s_j!)^{s_{j+1} \cdots s_l}} \quad (2.11)$$

Applying the convex inequality, we have

$$f(s_1, \dots, s_l, s_{l+1}, 1, \dots, 1) = \sum_S \binom{d_S}{s_{l+1}} \quad (2.12)$$

$$\geq \left(\frac{n}{m}\right)^{m-l-1+s_1+\dots+s_l} \binom{\bar{d}_l}{s_{l+1}} \quad (2.13)$$

$$= \left(1 + O\left(\frac{1}{\bar{d}_l}\right)\right) \frac{\bar{d}_l^{s_{l+1}}}{s_{l+1}!} \left(\frac{n}{m}\right)^{m-l-1+s_1+\dots+s_l}. \quad (2.14)$$

Combining inequalities (2.11) and (2.14), we have

$$f(s_1, \dots, s_l, s_{l+1}, 1, \dots, 1) \geq (1 + o(1)) \frac{(Cn^{-\delta})^{s_1 s_2 \cdots s_{l+1}}}{\prod_{j=1}^{l+1} (s_j!)^{s_{j+1} \cdots s_{l+1}}} \left(\frac{n}{m}\right)^{m-l-1+s_1+s_2+\dots+s_{l+1}}.$$

This concludes the proof of the claim.

Applying claim (a) with  $l = m - 1$ , we get

$$f(s_1, s_2, \dots, s_{m-1}, 1) \geq (1 + o(1)) \frac{(Cn^{-\delta})^{s_1 s_2 \cdots s_{m-1}}}{\prod_{j=1}^{m-1} (s_j!)^{s_{j+1} \cdots s_{m-1}}} \left(\frac{n}{m}\right)^{1+s_1+\dots+s_{m-1}} \quad (2.15)$$

$$= (1 + o(1)) \frac{m^{-1} C^{s_1 s_2 \cdots s_{m-1}} \left(\frac{n}{m}\right)^{s_1+s_2+\dots+s_{m-1}}}{\prod_{j=1}^{m-1} (s_j!)^{s_{j+1} \cdots s_{m-1}}}. \quad (2.16)$$

For any  $S \in \binom{V_1}{s_1} \times \cdots \times \binom{V_{m-1}}{s_{m-1}}$ , let  $d_S$  be the number of vertices  $v$  in  $V_m$  such that the induced subgraph  $G[S \times \{v\}]$  contains a transversal ordered copy of  $H(s_1, \dots, s_{m-1}, 1)$ . Since  $G$  is  $H(s_1, \dots, s_m)$ -free, we have  $d_S \leq s_m$ . It implies

$$f(s_1, s_2, \dots, s_{m-1}, 1) = \sum_S d_S \leq s_m \left(\frac{n}{m}\right)^{s_1+s_2+\dots+s_{m-1}}. \quad (2.17)$$

Choosing  $C$  sufficiently large, say  $C > \left((ms_m)^{s_1^{-1} \cdots s_{m-1}^{-1}}\right) \cdot \prod_{j=1}^{m-1} (s_j!)^{s_1^{-1} \cdots s_j^{-1}}$ , equations (2.16) and (2.17) contradict each other.  $\square$

We now give the proof of Theorem 13.

*Proof.* Let  $r = |R|$  and consider a hypergraph  $G := G_n^R$  with

$$h_n(G) = r - 1 + Cn^{-\delta}.$$

It suffices to show that  $\mu_H(G) \geq C'n^{-\delta}$ .

Given a random permutation  $\sigma$ , let  $X$  be the number of edges on a random full chain  $\sigma(L)$ . By the definition of the Lubell function, we have  $h_n(G) = \mathbb{E}(X)$ . Note  $X$  only takes integer values  $0, 1, \dots, r$ . Since  $\mathbb{E}(X) > r - 1$ , there is non-zero probability that  $X = r$ . In fact, we have

$$\begin{aligned} \mathbb{E}(X) &= \sum_{i=0}^r i\mathbb{P}(X = i) \\ &\leq r\mathbb{P}(X = r) + (r - 1)(1 - \mathbb{P}(X = r)) \\ &= r - 1 + \mathbb{P}(X = r). \end{aligned}$$

Thus, we get

$$\mathbb{P}(X = r) \geq \frac{C}{n^\delta}. \quad (2.18)$$

Every flag  $\sigma(L)$  contributes an equal share of the probability of the event that  $X = r$ , namely,

$$\frac{|Aut(L)|}{\binom{n}{v(L)}v(L)!}. \quad (2.19)$$

Here  $Aut(L)$  is the automorphism of  $L$ . Thus, the number of such flags is at least

$$\frac{C}{|Aut(L)|n^\delta} \binom{n}{v(L)} v(L)!. \quad (2.20)$$

It follows that  $\mu_H(G) \geq C'n^{-\delta}$ , where  $C' := C/|Aut(L)|$ . □

## CHAPTER 3

### APPLICATIONS TO FORBIDDEN SUBPOSET PROBLEMS

#### 3.1 SUSPENSIONS

**Definition 9.** The *suspension* of a hypergraph  $H$ , denoted  $S(H)$ , is the hypergraph with  $V = V(H) \cup \{*\}$  where  $\{*\}$  is an element not in  $V(H)$ , and edge set  $E = \{F \cup \{*\} : F \in E(H)\}$ . We write  $S^t(H)$  to denote the hypergraph obtained by iterating the suspension operation  $t$ -times, i.e.  $S^2(H) = S(S(H))$  and  $S^3(H) = S(S(S(H)))$ , etc.

In this section we will investigate the relationship between  $\pi(H)$  and  $\pi(S(H))$  and look at limits such as  $\lim_{t \rightarrow \infty} \pi(S^t(H))$ .

**Definition 10.** Given a graph  $G$  with vertex set  $v_1, \dots, v_n$  the *link* hypergraph  $G^{v_i}$  is the hypergraph with vertex set  $V(G) \setminus \{v_i\}$  and edge set  $E = \{F \setminus \{v_i\} : v_i \in F \text{ and } F \in E(G)\}$ .

**Proposition 7.** For any hypergraph  $H$  we have that  $\pi(S(H)) \leq \pi(H)$ .

*Proof.* Let  $G_n$  be a graph on  $n$  vertices containing no copy of  $S(H)$  such that  $h_n(G_n) = \pi_n(S(H))$ . Say  $V(G_n) = \{v_1, v_2, \dots, v_n\}$ . Note that for any  $v_i \in V(G_n)$ , we have that Lubell value of the corresponding link graph is

$$h_{n-1}(G_n^{v_i}) = \sum_{F \in G_n, v_i \in F} \frac{1}{\binom{n-1}{|F|-1}}.$$

Also, note that  $G_n^{v_i}$  contains no copy of  $H$ . If it did, then  $S(H) \subset S(G_n^{v_i}) \subseteq G_n$ ; but  $S(H)$  is not contained in  $G_n$ . Thus  $h_{n-1}(G_n^{v_i}) \leq \pi_{n-1}(H)$ . We then have the

following:

$$\begin{aligned}
\pi_n(S(H)) &= h_n(G_n) \\
&= \sum_{F \in E(G_n)} \frac{1}{\binom{n}{|F|}} \\
&= \sum_{F \in E(G_n)} \frac{1}{|F|} \sum_{v_i \in F} \frac{1}{\binom{n}{|F|}} \\
&= \sum_{i=1}^n \sum_{F \in E(G_n), v_i \in F} \frac{1}{|F|} \cdot \frac{1}{\binom{n}{|F|}} \\
&= \sum_{i=1}^n \sum_{F \in E(G_n), v_i \in F} \frac{1}{n} \cdot \frac{1}{\binom{n-1}{|F|-1}} \\
&= \frac{1}{n} \sum_{i=1}^n h_{n-1}(G_n^{v_i}) \\
&\leq \frac{1}{n} \sum_{i=1}^n \pi_{n-1}(H) \\
&= \pi_{n-1}(H).
\end{aligned}$$

Thus, for any  $n$ ,  $\pi_n(S(H)) \leq \pi_{n-1}(H)$ ; taking the limit as  $n \rightarrow \infty$  we get the result as claimed.  $\square$

**Corollary 2.** *If  $H$  is degenerate, so is  $S(H)$ .*

**Conjecture 1.** *For all  $H$ ,  $\lim_{t \rightarrow \infty} \pi(S^t(H)) = |R(H)| - 1$ .*

To conclude this section, we prove a special case of this conjecture.

**Theorem 15.** *Suppose that  $H$  is a subgraph of the blowup of a chain. Let  $k_1$  be the minimum number in  $R(H)$ . Suppose  $k_1 \geq 2$ , and  $H'$  is a new hypergraph obtained by adding finitely many edges of type  $k_1 - 1$  arbitrarily to  $H$ . Then*

$$\lim_{t \rightarrow \infty} \pi(S^t(H')) = |R(H')| - 1.$$

*Proof.* Without loss of generality, we can assume that  $H$  is a blowup of a chain and  $V(H') = V(H)$ . (This can be done by taking blowup of  $H$  and adding more edges.)

Suppose that  $H$  has  $v$  vertices and its edge type is  $R(H) := \{k_1, k_2, \dots, k_r\}$ . Set  $k_0 := k_1 - 1$  so that  $R(H') := \{k_0, k_1, \dots, k_r\}$ . For convenience, we write  $R$  for  $R(H)$  and  $R'$  for  $R(H')$ , and

$$\begin{aligned} R + t &:= \{k_1 + t, k_2 + t, \dots, k_r + t\}, \\ R' + t &:= \{k_0 + t, k_1 + t, \dots, k_r + t\}. \end{aligned}$$

For any small  $\epsilon > 0$ , let  $n_0 = \lfloor \epsilon^{-t} \rfloor$ . For any  $n \geq n_0$  and any hypergraph  $G_n^{R+t}$  with

$$\pi_n(G) > |R(H)| - 1 + \epsilon = r + \epsilon,$$

we will show  $G$  contains a subhypergraph  $S^t(H')$ .

Take a random permutation  $\sigma \in S_n$  and let  $X$  be the number of edges in  $G$  hit by the random full chain  $C_\sigma$ :

$$\emptyset \subset \{\sigma(1)\} \subset \{\sigma(1), \sigma(2)\} \cdots \subset \{\sigma(1), \sigma(2), \dots, \sigma(i)\} \subset \cdots \subset [n].$$

We have

$$\mathbb{E}(X) = \pi_n(G) > r + \epsilon.$$

Since  $X \leq r + 1$ , we have

$$\mathbb{E}(X) = \sum_{i=0}^{r+1} i \Pr(X = i) \leq (r + 1) \Pr(X = r + 1) + r.$$

Thus, we get

$$\mathbb{P}(X = r + 1) \geq \frac{\mathbb{E}(X) - r}{r + 1} > \frac{\epsilon}{r + 1}. \quad (3.1)$$

Recall that the density  $\mu_H(G)$  is the probability that a random injective map  $f: V(H) \rightarrow V(G)$  such that  $H \xrightarrow{f} G$ . Applying to  $H = C^{R'+t}$ , we have

$$\mu_{C^{R'+t}}(G) = \mathbb{P}(X = r + 1) > \frac{\epsilon}{r + 1}.$$

Every copy of the chain  $C^{R'+t}$  will pass through a set  $A_1 \in E^{k_1+t}(G)$ . Let  $\mu_{C^{R'+t}, A_1}(G)$  be the conditional probability that a random injective map  $f: V(C^{R'+t}) \rightarrow V(G)$

satisfies  $C^{R+t} \xrightarrow{f} G$  given that the chain  $C^{R+t}$  passes through  $A_1$ . Let  $d_-(A_1)$  be the number of sets  $A_0$  satisfying  $A_0 \in E^{k_0+t}(G)$  and  $A_0 \subset A_1$ . Then, we have

$$\mu_{C^{R+t}}(G) = \frac{1}{\binom{n}{k_1+t}} \sum_{A_1 \in E^{k_1+t}(G)} \mu_{C^{R+t}, A_1}(G) \cdot \frac{d_-(A_1)}{k_1+t}.$$

Setting  $\eta = \frac{\epsilon}{2(r+1)}$ , define a family

$$\mathcal{A} = \{A_1 \in E^{k_1+t}(G) : \mu_{C^{R+t}, A_1}(G) > \eta \text{ and } d_-(A_1) > \eta(k_1+t)\}.$$

We claim  $|\mathcal{A}| > \eta \binom{n}{k_1+t}$ . Otherwise, we have

$$\begin{aligned} \mu_{C^{R+t}}(G) &= \frac{1}{\binom{n}{k_1+t}} \sum_{A_1 \in E^{k_1+t}(G)} \mu_{C^{R+t}, A_1}(G) \cdot \frac{d_-(A_1)}{k_1+t} \\ &= \frac{1}{\binom{n}{k_1+t}} \sum_{A_1 \in \mathcal{A}} \mu_{C^{R+t}, A_1}(G) \cdot \frac{d_-(A_1)}{k_1+t} + \frac{1}{\binom{n}{k_1+t}} \sum_{A_1 \notin \mathcal{A}} \mu_{C^{R+t}, A_1}(G) \cdot \frac{d_-(A_1)}{k_1+t} \\ &\leq \eta + \eta < \frac{\epsilon}{t+1}. \end{aligned}$$

But this is a contradiction!

A  $k_1$ -configuration is a pair  $(S, A_1)$  satisfying  $A_1 \in \mathcal{A}$ ,  $S = A_1 \setminus \{i_1, i_2, \dots, i_{k_1}\}$ , and  $A_1 \setminus \{i_j\} \in E^{k_0+t}(G)$  for any  $1 \leq j \leq k_1$ .

For any  $A_1 \in \mathcal{A}$ , the number of  $S$  such that  $(S, A_1)$  forms a  $k_1$ -configuration is at least

$$\binom{d_-(A_1)}{k_1} \geq \binom{\eta(k_1+t)}{k_1} > \left(\frac{\eta}{2}\right)^{k_1} \binom{k_1+t}{k_1}.$$

In the above inequality, we use the assumption  $t > \frac{2}{\eta}k_1$ .

By an averaging argument, there exists an  $S$  so that the number of  $k_1$ -configurations  $(S, A_1)$  is at least

$$\frac{|\mathcal{A}| \left(\frac{\eta}{2}\right)^{k_1} \binom{k_1+t}{k_1}}{\binom{n}{t}} \geq \frac{\eta^{k_1+1}}{2^{k_1}} \binom{n-t}{k_1}.$$

Now consider the link graph  $G^S$ . The inequality above implies

$$\mu_{C^R}(G^S) \geq \frac{\eta^{k_1+2}}{2^{k_1}}.$$



This implies  $G^S$  contains a blow up of  $C^R$ . Thus  $G^S$  has a subhypergraph  $H$ . By the definition of  $k_1$ -configuration, this  $H$  can be extended to  $H'$  in  $G^S$ . In another words,  $G$  contains  $S^t(H')$ .  $\square$

### 3.2 FORBIDDEN SUBPOSET PROBLEMS

Let  $\mathcal{B}_n = (2^{[n]}, \subseteq)$  be the  $n$ -dimensional Boolean lattice. Under the partial relation  $\subseteq$ , any family  $\mathcal{F} \subseteq 2^{[n]}$  can be viewed as a subposet of  $\mathcal{B}_n$ .

For posets  $P = (P, \leq)$  and  $P' = (P', \leq')$ , we say  $P'$  is a **weak subposet** of  $P$  if there exists an injection  $f: P' \rightarrow P$  that preserves the partial ordering, meaning that whenever  $u \leq' v$  in  $P'$ , we have  $f(u) \leq f(v)$  in  $P$ . If  $P'$  is not a weak poset of  $P$ , we say  $P$  is  $P'$ -free. The following problems originate from Sperner's theorem, which states that the largest antichain of  $\mathcal{B}_n$  is  $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ .

**Question.** *Given a fixed poset  $P$ , what is the largest size of a  $P$ -free family  $\mathcal{F} \subset \mathcal{B}_n$ ?*

Let  $\text{La}(n, P)$  be the largest size of a  $P$ -free family  $\mathcal{F} \subseteq \mathcal{B}_n$ . The value of  $\text{La}(n, P)$  is known for only a few posets  $P$ . Let  $\mathcal{P}_k$  be the (poset) chain of size  $k$ . Then  $\text{La}(n, \mathcal{P}_2) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$  by Sperner's theorem. Erdős [19] proved that  $\text{La}(n, \mathcal{P}_k) = \Sigma(n, k - 1)$ , where  $\Sigma(n, k)$  is the sum of  $k$  largest binomial coefficients. De Boinis-Katona-Swanepoel [14] proved  $\text{La}(n, \mathcal{O}_4) = \Sigma(n, 2)$ . Here  $\mathcal{O}_4$  is the butterfly poset  $(A, B \subset C, D)$ , or the crown poset of size 4.








The asymptotic value of  $\text{La}(n, P)$  has been discovered for various posets (see Table 3.1). Let  $e(P)$  be the largest integer  $k$  so that the family of  $k$  middle layers of  $\mathcal{B}_n$  is  $P$ -free. Griggs and Lu [37] first conjectured that  $\lim_{n \rightarrow \infty} \frac{\text{La}(n, P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$  exists and is an integer, and it slowly involves into the following conjecture.

**Conjecture 2.** *For any fixed poset  $P$ ,  $\lim_{n \rightarrow \infty} \frac{\text{La}(n, P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} = e(P)$ .*

We overload the notation  $\pi(P)$  for the limit  $\lim_{n \rightarrow \infty} \frac{\text{La}(n, P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ , where  $P$  is a poset. The conjecture is based on the observation of several previous known results, which

are obtained by Katona and others [10, 13, 14, 33, 40, 41, 68]. We summarize the known posets  $P$ , for which the conjecture has been verified in Table 3.1.

Table 3.1 Conjecture 2 has been verified for the following posets  $P$ .

$P$	Meaning	Hasse Diagram	$e(P)$	References
fork $\mathcal{V}_r$	$A < B_1, \dots, B_r$		1	[41] for $r = 2$ ; [13] for all $r$ .
$\mathcal{N}$	$A < B, B > C$ , and $C < D$ .		1	[33]
butterfly $\mathcal{O}_4$	$A, C < B$ and $A, C < D$		2	[14]
Diamond $\mathcal{D}_r$	$A < B_1, \dots, B_r < C$ $r \in \{3, 4, 7, 8, 9, 15, \dots\}$		varies	[36]
baton $P_k(s, t)$	$A_1, \dots, A_s < B_1 < B_2 < \dots < B_{k-2} < C_1, \dots, C_t$		$k - 1$	[68] if $s = 1, k, t \geq 2$ ; [37] for $k, s, t \geq 2$ .
tree $T$	A poset whose Hasse diagram is a tree of height $h(T)$ .		$h(T) - 1$	[37] for $h(T) = 2$ [8] for general cases.
Crowns $\mathcal{O}_{2t}$	A height-2 poset, whose Hasse diagram is a cycle $C_{2t}$ .		1	[37] for even $t \geq 2$ ; [50] for odd $t \geq 7$ .

The posets in Table 3.1 are far from complete. Let  $\lambda_n(P) = \max\{h_n(\mathcal{F}) : \mathcal{F} \subseteq 2^{[n]}, P\text{-free}\}$ . A poset  $P$  is called **uniform-L-bounded** if  $\lambda_n(P) \leq e(P)$  for all  $n$ . Griggs-Li [35] proved  $\text{La}(n, P) = \sum_{i=\lfloor \frac{n-e(P)+1}{2} \rfloor}^{\lfloor \frac{n+e(P)-1}{2} \rfloor} \binom{n}{i}$  if  $P$  is uniform-L-bounded. The uniform-L-bounded posets include  $\mathcal{P}_k$  (for any  $k \geq 1$ ), diamonds  $\mathcal{D}_k$  (for  $k \in [2^{m-1} - 1, 2^m - \binom{m}{\lfloor \frac{m}{2} \rfloor}] - 1$  where  $m := \lceil \log_2(k + 2) \rceil$ ), harps  $\mathcal{H}(l_1, l_2, \dots, l_k)$  (for  $l_1 > l_2 > \dots > l_k$ ), and other posets. Notably, Griggs-Li [35] provide a method to construct

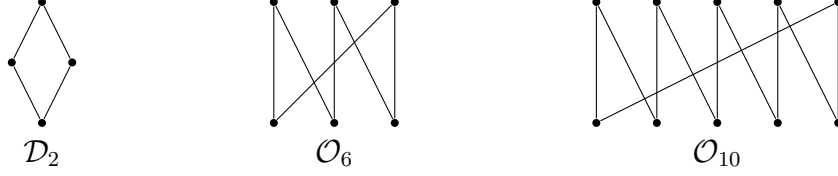


Figure 3.1 The three most wanted posets for Conjecture 2.

large uniform-L-bounded posets from smaller uniform-L-bounded posets. Thus there are infinitely many posets  $P$  so that  $\pi(P) = e(P)$  holds.

Although there is no counterexample found yet for Conjecture 2, some posets have resisted efforts to determine their  $\pi$  value. The most studied, yet unsolved, poset is the diamond poset  $\mathcal{D}_2$  (or  $\mathcal{B}_2$ ,  $Q_2$  in some papers) as shown in Figure 3.1. Griggs and Lu first observed  $\pi(\mathcal{D}_2) \in [2, 2.296]$ . Axenovich, Manske, and Martin [1] came up with a new approach which improves the upper bound to 2.283. Griggs, Li, and Lu [36] further improved the upper bound to  $2.27\dot{3} = 2\frac{3}{11}$ . Very recently, Kramer-Martin-Young [48] recently proved  $\pi(\mathcal{D}_2) \leq 2.25$ .

While it seems to be hard to prove the conjecture  $\pi(\mathcal{D}_2) = 2$ , several groups of researchers have considered restricting the problem to three consecutive layers. Let  $\text{La}^c(n, P)$  be the largest size a  $P$ -free family  $\mathcal{F} \subseteq \mathcal{B}_n$  such that  $\mathcal{F}$  is in  $e(P) + 1$  consecutive layers. Let  $\pi^c(P) = \lim_{n \rightarrow \infty} \frac{\text{La}^c(n, P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}$ , if the limit exists. Here is a weaker conjecture (of consecutive layers).

**Conjecture 3.** *For any fixed poset  $P$ ,  $\pi^c(P) = e(P)$ .*

Axenovich-Manske-Martin [1] first proved  $\pi^c(\mathcal{D}_2) \leq 2.207$ ; it was recently improved to 2.1547 (Manske-Shen [52]) and 2.15121 (Balogh-Hu-Lidický-Liu [4]).

We say a hypergraph  $H$  **represents** a poset  $P$  if the set of edges of  $H$  (as a poset) is isomorphic to  $P$ . For any fixed finite poset  $P$ , by the definition of  $e(P)$ , there exists a hypergraph  $H \subseteq \mathcal{B}_{n_0}$  with  $|R(H)| = e(P) + 1$  representing a superposet of  $P$ .

**Theorem 16.** *Let  $H$  be a hypergraph with  $|R(H)| = e(P) + 1$  and  $R(H)$  is a set of consecutive integers. If  $H$  represents a superposet of  $P$  then, for any integer  $t \geq 0$ , we have*

$$\pi^c(P) \leq \pi(S^t(H)).$$

*Proof.* Let  $x := \pi(S^t(H))$ . For any  $\epsilon > 0$ , there exists an  $n_1$  so that  $\pi_n(S^t(H)) \leq x + \epsilon$  for all  $n \geq n_1$ . We claim

$$\pi^c(P) \leq x + 2\epsilon.$$

Otherwise, for any sufficiently large  $n$ , there exists a family  $\mathcal{F} \subset \mathcal{B}_n$  which is in  $e(P) + 1$  consecutive layers with  $|\mathcal{F}| > (x + \epsilon) \binom{n}{\lfloor \frac{n}{2} \rfloor}$ . Let  $k_0$  be the smallest size of edges in  $S^t(H)$ . Let  $k_1$  be the integer that  $\mathcal{F}$  is in  $k_1$ -th to  $(k_1 + e(P))$ -th layer. Since  $e(P) \leq x < e(P) + 1$ , we have

$$|\mathcal{F}| > (x + \epsilon) \binom{n}{\lfloor \frac{n}{2} \rfloor} \geq (e(P) + \epsilon) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Note any layer below  $\frac{n}{2} - 2\sqrt{n \ln n}$  can only contribute  $\frac{2^n}{n^2}$ , which is less than  $\epsilon \binom{n}{\lfloor \frac{n}{2} \rfloor}$  for sufficiently large  $n$ . We get

$$k_1 \geq \frac{n}{2} - 2\sqrt{n \ln n}.$$

Choose  $n$  large enough so that  $k_1 \geq k_0$  and  $n - k_1 + k_0 \geq n_1$ . We observe that

$$h_n(\mathcal{F}) \geq \frac{|\mathcal{F}|}{\binom{n}{\lfloor \frac{n}{2} \rfloor}} > x + 2\epsilon.$$

By the property of Lubell function,  $h_n(\mathcal{F})$  is the average of  $h_{n+k_0-k_1}(\mathcal{F}_S)$  over all  $S \in \binom{[n]}{k_1-k_0}$ , where  $\mathcal{F}_S$  is the link hypergraph over  $S$ . Therefore, there exists a set  $S \in \binom{[n]}{k_1-k_0}$  so that  $h_{n+k_0-k_1}(\mathcal{F}_S) > x + 2\epsilon$ . Thus,  $\mathcal{F}_S$  contains a subhypergraph  $S^{t_0}(H)$ . In particular,  $\mathcal{F}$  contains a subposet  $P$ .

Thus, we have

$$\pi^c(P) \leq x + 2\epsilon.$$

Since this holds for any  $\epsilon > 0$ , we have  $\pi^c(P) \leq x$ . □

**Corollary 3.** *Conjecture 2 implies Conjecture 3.*

Note that the complete hypergraph  $K_2^{\{0,1,2\}}$  has 4 edges  $\emptyset, \{1\}, \{2\}, \{1, 2\}$ ; which form the diamond poset  $\mathcal{D}_2$ . In particular, for any  $t \geq 0$ , we have

$$\pi^c(\mathcal{D}_2) \leq \pi(S^t(K_2^{\{0,1,2\}})).$$

This provides a possible way to improve the bounds of  $\pi^c(\mathcal{D}_2)$ . Using the results on  $\{1, 2\}$ -graphs from the previous chapter, we can show that  $\pi(K_2^{\{0,1,2\}}) = 2.25$  implying that  $\pi^c(\mathcal{D}_2) \leq 2.25$ . This is not a new result just an indication of what might be possible. In particular, we conjecture that  $\pi(S(K_2^{\{0,1,2\}})) = 2 + \frac{1}{2^3} = 2.125$  and in general  $\pi(S^t(K_2^{\{0,1,2\}})) = 2 + \frac{1}{2^{2+t}}$ .

## CHAPTER 4

### JUMPS IN NON-UNIFORM HYPERGRAPHS

#### 4.1 THE UNDERPINNINGS

Recall that

**Definition.** A real number  $\alpha$  is a **jump** for a positive integer  $r$  if there exists a  $c > 0$  such that for every  $\epsilon > 0$  and every  $t \geq r$  there exists an integer  $n_0(\alpha, r, t, \epsilon)$  such that if  $n \geq n_0$  and  $G$  is an  $r$ -uniform hypergraph on  $n$  vertices with edge density at least  $\alpha + \epsilon$  then  $G$  contains a subgraph  $H$  on  $t$  vertices with edge density at least  $\alpha + c$ .

Erdős observed that all values in  $[0, 1)$  are jumps for 2 and all values in  $[0, \frac{r!}{r})$  are jumps for  $r$ . He asked whether all values in  $[0, 1)$  are jumps for any  $r \geq 2$ —this was known as the jumping constant conjecture. The question was answered negatively by Frankl and Rödl in 1984. They showed that  $1 - \frac{1}{r-2}$  is a non-jump for every  $r \geq 3$  and  $l > 2r$ .

The same question, whether  $\alpha$  is a jump or not, can be asked of non-uniform hypergraphs as well. Using the Lubell function we can extend the definition of jump to non-uniform hypergraphs as follows.

**Definition 11.** The value  $\alpha \in [0, |R|]$  is a **jump** for  $R$  if there exists a  $c > 0$  such that for every  $\epsilon > 0$  and every  $t \geq \max\{r : r \in R\}$  there exists an integer  $n_0$  such that if  $n \geq n_0$  and  $G_n$  is an  $R$ -graph on  $n$  vertices with  $h_n(G_n) \geq \alpha + \epsilon$  then there exists a subgraph  $H_t$  of  $G_n$  on  $t$  vertices with  $h_t(H_t) \geq \alpha + c$ .

Knowing whether a value  $\alpha$  is a jump or not for  $r$  rarely gives much information regarding Turán densities. One notable exception is the following example. Suppose that one is trying to determine the Turán density of a family of  $R$ -graphs  $\mathcal{H}$ . It is known that  $\alpha$  is a jump for  $R$  with jumping constant  $c$ . If it can be shown that  $\pi(\mathcal{H}) < \alpha + c$  then the fact that  $\alpha$  is a jump implies that  $\pi(\mathcal{H}) \leq \alpha$ . This observation may be particularly helpful if when using a method such as Razborov's flag algebras.

**Definition 12.** *The value  $\alpha \in [0, |R|]$  is a **strong jump** for  $R$  if there exists  $c > 0$  such that for every  $t \geq \max\{r : r \in R\}$  there exists an integer  $n_0$  such that if  $n \geq n_0$  and  $G_n$  is an  $R$ -graph on  $n$  vertices with  $h_n(G_n) \geq \alpha - c$  then there exists a subgraph  $H_t$  of  $G_n$  on  $t$  vertices with  $h_t(H_t) \geq \alpha + c$ .*

If a value  $\alpha$  is a strong jump for  $R$  it is also a jump for  $R$ ; the converse statement is not true. A value  $\alpha$  is a **weak jump** for  $R$  if it is a jump but not strong jump. Refining the notion of jumps in this way turns out to have several nice consequences. For example, the set of all strong jumps forms an open set (see Proposition 8). Its complement, the set of not strong-jump values, has an algebraic structure (see Theorem 17) and is closely related to Turán density. We will show that 0 is always a jump for  $R$ . Furthermore, 0 cannot be a strong jump; hence it is a weak jump. Notice that  $|R|$  is a weak jump for  $R$ ; this is a degenerate case of the definition of jump.

**Proposition 8.** *For any fixed finite set  $R$  of non-negative integers, the set of all strong jumps for  $R$  is an open subset of  $(0, |R|)$ .*

*Proof.* Suppose  $\alpha$  is a strong jump. Let  $c > 0$  be the positive constant (from the definition) whose existence is guaranteed by the fact that  $\alpha$  is a strong jump. For every  $\beta \in (\alpha - \frac{c}{2}, \alpha + \frac{c}{2})$ , we can choose a new constant  $c' := \frac{c}{2}$ . For every  $t \geq \max\{r : r \in R\}$  and every  $R$ -graph  $G_n$  with  $n \geq n_0$  and  $h_n(G_n) \geq \beta - c' > \alpha - c$ , there exists a

subgraph  $H_t$  of  $G_n$  on  $t$  vertices with  $h_t(H_t) \geq \alpha + c > \beta + c'$ . Thus  $\beta$  is a strong jump. Hence the set of strong jumps is open.  $\square$

In the 1984 paper of Frankl and Rödl (where they first proved the existence of non-jumps) they introduce an equivalent definition of jump using admissible sequences and upper densities (for  $r$ -uniform hypergraphs). Now we generalize it to  $R$ -graphs.

**Definition 13.** Let  $\mathbf{G} := \{G_{n_i}^R\}_{i=1}^\infty$  be a sequence of  $R$ -graphs. We say that  $\mathbf{G}$  is an **admissible** sequence if  $n_i \rightarrow \infty$  as  $i \rightarrow \infty$  and  $\lim_{i \rightarrow \infty} h_{n_i}(G_{n_i}^R)$  exists.

The limit  $\lim_{i \rightarrow \infty} h_{n_i}(G_{n_i}^R)$ , denoted by  $h(\mathbf{G})$ , is called the **density** of the sequence  $\mathbf{G}$ . Note  $0 \leq h(\mathbf{G}) \leq |R|$  holds for any  $R$ -admissible sequence  $\mathbf{G}$ . The converse also holds: for any  $\alpha \in [0, |R|]$  there exists an  $R$ -admissible sequence  $\mathbf{G}$  with density  $\alpha$ .

**Definition 14.** The **upper density** of an admissible sequence of  $R$ -graphs  $\mathbf{G}$ , denoted by  $\bar{h}(\mathbf{G})$ , is defined as  $\lim_{t \rightarrow \infty} \sigma_t(\mathbf{G})$ , where  $\sigma_t(\mathbf{G}) := \sup_i \sup_{T \in \binom{[n_i]}{t}} \{h_t(G_{n_i}^R[T])\}$  is the supremum of the density of all induced subgraphs on  $t$  vertices among all the graphs in the sequence.

Note for any  $t \geq \max\{r : r \in R\}$ ,  $\sigma_t(\mathbf{G})$  is a decreasing function on  $t$ . Thus, the limit,  $\lim_{t \rightarrow \infty} \sigma_t(\mathbf{G})$  exists. We also note that  $\sup$  can be replaced by  $\max$  in the definition.

**Lemma 4.** A value  $\alpha \in [0, |R|]$  is a jump for  $R$  if and only if there exists a constant  $c := c(\alpha) > 0$  such that if  $\mathbf{G}$  is an admissible sequence of  $R$ -graphs with  $h(\mathbf{G}) > \alpha$ , we have that  $\bar{h}(\mathbf{G}) \geq \alpha + c$ .

*Proof.* We first prove that it is necessary. Suppose that  $\alpha$  is a jump for  $R$ . There is a constant  $c > 0$  such that for any  $\epsilon > 0$  and any integer  $t \geq \max\{r : r \in R\}$ , there is an integer  $n_0$  such that if  $n \geq n_0$  and  $G_n$  is an  $R$ -graph on  $n$  vertices with  $h_n(G_n) \geq \alpha + \epsilon$  then there exists a subgraph  $H_t$  of  $G_n$  on  $t$  vertices with  $h_t(H_t) \geq \alpha + c$ .



Consider any admissible sequence of  $R$ -graphs  $\mathbf{G} := \{G_{n_i}^R\}_{i=1}^\infty$  with  $h(\mathbf{G}) > \alpha$ . Choose  $\epsilon = \frac{h(\mathbf{G}) - \alpha}{2}$ . There exists an  $i_0$  such that  $h_{n_i}(G_{n_i}) > h(\mathbf{G}) - \epsilon = \alpha + \epsilon$  for all  $i \geq i_0$ . Since  $\alpha$  is a jump, for any  $t$ , for sufficiently large  $i$ ,  $G_{n_i}$  contains a subgraph  $H_t$  with  $h_t(H_t) \geq \alpha + c$ . Thus, we have  $\bar{h}(\mathbf{G}) \geq \alpha + c$ .

We now prove the contrapositive of the reverse implication. Suppose that  $\alpha$  is not a jump. For any  $c > 0$ , there exist  $\epsilon > 0$  and  $t > \max\{r : r \in R\}$ , so that for any  $i$  there exists a graph  $G_{n_i}$  with  $n_i \geq i$  satisfying  $h_{n_i}(G_{n_i}) \geq \alpha + \epsilon$  and  $G_{n_i}$  contains no subgraph  $H_t$  with density  $h_t(H_t) \geq \alpha + c$ . The sequence  $\mathbf{G}$  formed by the  $G_{n_i}$ 's may not be admissible because the limit  $\lim_{i \rightarrow \infty} h_{n_i}(G_{n_i})$  may not exist. However, by deleting some edges, each  $G_{n_i}$  contains a spanning subgraph  $G'_{n_i}$  with  $h_{n_i}(G'_{n_i}) = \alpha + \epsilon \pm O(\frac{1}{n})$ . Now  $\mathbf{G}' := \{G'_{n_i}\}_{i=1}^\infty$  is an admissible sequence with  $h(\mathbf{G}') = \alpha + \epsilon > \alpha$ . Note  $G'_{n_i}$  contains no subgraph  $H_t$  with density  $h_t(H_t) \geq \alpha + c$ . Thus we have  $\bar{h}(\mathbf{G}') < \alpha + c$ .  $\square$

**Lemma 5.** *A value  $\alpha \in [0, |R|)$  is a strong jump for  $R$  if and only if there exists a constant  $c := c(\alpha) > 0$  such that every admissible sequence of  $R$ -graphs  $\mathbf{G}$  with  $h(\mathbf{G}) = \alpha$  has upper density  $\bar{h}(\mathbf{G}) \geq \alpha + c$ .*

*Proof.* This is similar to the proof of Lemma 4. Suppose that  $\alpha$  is a strong jump for  $R$ . There is a constant  $c > 0$  such that for any integer  $t \geq \max\{r : r \in R\}$ , there is an integer  $n_0$  such that if  $n \geq n_0$  and  $G_n$  is an  $R$ -graph on  $n$  vertices with  $h_n(G_n) \geq \alpha - c$  then there exists a subgraph  $H_t$  of  $G_n$  on  $t$  vertices with  $h_t(H_t) \geq \alpha + c$ .

Consider any admissible sequence of  $R$ -graphs  $\mathbf{G} := \{G_{n_i}^R\}_{i=1}^\infty$  with  $h(\mathbf{G}) = \alpha$ . There exists an  $i_0$  such that  $h_{n_i}(G_{n_i}) > h(\mathbf{G}) - c$  for all  $i \geq i_0$ . Since  $\alpha$  is a strong jump, for any  $t$ , for sufficiently large  $i$ ,  $G_{n_i}$  contains a subgraph  $H_t$  with  $h_t(H_t) \geq \alpha + c$ . Thus, we have  $\bar{h}(\mathbf{G}) \geq \alpha + c$ .

Now we prove the contrapositive of the reverse implication. Assume that  $\alpha$  is not a strong jump. For any  $c > 0$ , there exists a  $t > \max\{r : r \in R\}$ , and for any  $i$  there exists a graph  $G_{n_i}$  with  $n_i \geq i$  satisfying  $h_{n_i}(G_{n_i}) \geq \alpha - c$  and  $G_{n_i}$  contains no

subgraph  $H_t$  with density  $h_t(H_t) \geq \alpha + c$ . In particular, for  $k = 1, 2, \dots$ , we choose  $c = \frac{1}{k}$  and  $i = k$ . We obtain a sequence of  $R$ -graphs  $\mathbf{G} = \{G_{n_k}\}_{k=1}^\infty$ . By deleting some edges,  $\mathbf{G}$  contains a spanning subgraph  $\{G'_{n_k}\}$  with  $h_{n_k}(G'_{n_k}) = \alpha - \frac{1}{k} \pm O(\frac{1}{n})$ . Now  $\mathbf{G}' := \{G'_{n_k}\}_{k=1}^\infty$  is an admissible sequence with  $h(\mathbf{G}') = \alpha$ . Note  $G'_{n_i}$  contains no subgraph  $H_t$  with density  $h_t(H_t) \geq \alpha + \frac{1}{k}$  for  $i$  sufficiently large. We have  $\bar{h}(\mathbf{G}') \leq \alpha$ .  $\square$

**Corollary 4.** *The following statements are equivalent.*

1. *An value  $\alpha \in [0, |R|]$  is NOT a strong jump for  $R$ .*
2. *There exists an admissible sequence of  $R$ -graphs  $\mathbf{G}$  satisfying  $h(\mathbf{G}) = \bar{h}(\mathbf{G}) = \alpha$ .*
3. *For a given increasing sequence of positive integers  $n_1 < n_2 < \dots$ , there exists an admissible sequence of  $R$ -graphs  $\mathbf{G} := \{G_{n_i}\}_{i=1}^\infty$  satisfying  $h(\mathbf{G}) = \bar{h}(\mathbf{G}) = \alpha$ .*

*Proof.* (1)  $\Rightarrow$  (2): See the proof of Lemma 5.

(2)  $\Rightarrow$  (3): Suppose that there an admissible sequence of  $R$ -graphs  $\mathbf{G} := \{G_{n_i}\}_{i=1}^\infty$  satisfying  $h(\mathbf{G}) = \bar{h}(\mathbf{G}) = \alpha$ . For each  $i = 1, 2, \dots$ , find an index  $n'_j > n_i$  so that  $h_{n_j}(G_{n'_j}) > \alpha - \frac{1}{i}$ . There is subgraph of  $G_{n'_j}$  on  $n_i$  vertices whose density is at least  $h_{n_j}(G_{n'_j}) > \alpha - \frac{1}{i}$ . By deleting some edges if necessary, there exists an subgraph  $G'_{n_i} \subset G_{n'_j}$  satisfying  $h_{n_i}(G'_{n_i}) = \alpha - \frac{1}{i} + O(\frac{1}{n_i})$ . Let  $\mathbf{G}' := \{G'_{n_i}\}_{i=1}^\infty$ . We have

$$h(\mathbf{G}') = \lim_{i \rightarrow \infty} h_{n_i}(G'_{n_i}) = \alpha,$$

and

$$\bar{h}(\mathbf{G}') \leq \bar{h}(\mathbf{G}) = \alpha.$$

Since  $h(\mathbf{G}') \leq \bar{h}(\mathbf{G}')$ , we have  $\bar{h}(\mathbf{G}') = h(\mathbf{G}') = \alpha$ .

(3)  $\Rightarrow$  (1): This is the contrapositive of Lemma 5.  $\square$

**Theorem 17.** *Consider any two finite sets  $R_1$  and  $R_2$  of non-negative integers. Suppose that  $R_1 \cap R_2 = \emptyset$  and  $R = R_1 \cup R_2$ . If  $\alpha_1$  is not a strong jump for  $R_1$  and  $\alpha_2$  is not a strong jump for  $R_2$ , then  $\alpha_1 + \alpha_2$  is not a strong jump for  $R_1 \cup R_2$ .*

*Proof.* For  $j \in \{1, 2\}$ , since  $\alpha_j$  is not a strong jump for  $R_j$ , by corollary 4, there exists an admissible  $R_j$  sequence of graphs  $\mathbf{G}^{R_j} := \{G_n^{R_j}\}_{n=1}^\infty$  satisfying  $h(\mathbf{G}^{R_j}) = \bar{h}(\mathbf{G}^{R_j}) = \alpha_j$ .

For  $n = 1, 2, 3, \dots$ , construct a new sequence of graphs  $\mathbf{H}^R := \{H_n^R\}_{n=1}^\infty$  as follows. The vertex set of  $H_n^R$  is the common vertex set  $[n]$  while the edge set of  $H_n^R$  is the union of  $E(G_n^{R_1})$  and  $E(G_n^{R_2})$ . Since  $R_1 \cap R_2 = \emptyset$ , we have

$$\begin{aligned} h(\mathbf{H}) &= h(\mathbf{G}^{R_1}) + h(\mathbf{G}^{R_2}) = \alpha_1 + \alpha_2 \\ \bar{h}(\mathbf{H}) &\leq \bar{h}(\mathbf{G}^{R_1}) + \bar{h}(\mathbf{G}^{R_2}) = \alpha_1 + \alpha_2. \end{aligned}$$

Since  $h(\mathbf{H}) \leq \bar{h}(\mathbf{H})$ , we have  $\bar{h}(\mathbf{H}) = \alpha_1 + \alpha_2$ . Hence  $\alpha_1 + \alpha_2$  is not a strong jump for  $R_1 \cup R_2$ .  $\square$

**Lemma 6.** *If  $\alpha = \bar{h}(\mathbf{G})$  for some  $R$ -admissible sequence  $\mathbf{G}$ , then  $\alpha$  is not a strong jump for  $R$ .*

*Proof.* It suffices to find an  $R$ -admissible sequence  $\mathbf{F}$  such that  $h(\mathbf{F}) = \bar{h}(\mathbf{F}) = \alpha$ . For each  $t \geq \max\{r : r \in R\}$  there is some  $t$ -subset  $T$  of vertices of  $G_{i_t}$  such that  $h_t(G_{i_t}[T]) = \sigma_t(\mathbf{G})$ . Let  $F_t = G_{i_t}[T]$  and create the  $R$ -admissible sequence  $\mathbf{F} = \{F_t\}$ . By construction  $\lim_{t \rightarrow \infty} h_t(F_t) = \lim_{t \rightarrow \infty} \sigma_t(\mathbf{G}) = \alpha$ . Furthermore,  $\bar{h}(\mathbf{F}) = h(\mathbf{F}) = \alpha$ . Hence  $\alpha$  is not a strong jump.  $\square$

**Lemma 7.** *If for some  $c > 0$  every value in the interval  $(\alpha, \alpha + c)$  is a strong jump for  $R$ , then  $\alpha$  is a jump for  $R$ .*

*Proof.* Consider any admissible sequence  $\mathbf{G}$  with  $h(\mathbf{G}) > \alpha$ . We need to show that there exists a constant  $c' > 0$  such that  $\bar{h}(\mathbf{G}) \geq \alpha + c'$ . Take  $c' = c > 0$ . If  $h(\mathbf{G}) \geq \alpha + c$ , then we are done since  $\bar{h}(\mathbf{G}) \geq h(\mathbf{G})$ . Otherwise,  $h(\mathbf{G})$  is in the interval  $(\alpha, \alpha + c)$ . By hypothesis,  $h(\mathbf{G})$  is a strong jump. Since  $\bar{h}(\mathbf{G}) \geq h(\mathbf{G})$  and  $\bar{h}(\mathbf{G})$  is not a strong jump, we have that  $\bar{h}(\mathbf{G}) \geq \alpha + c$ . Therefore,  $\alpha$  is jump.  $\square$

**Theorem 18.** *Let  $R$  be a finite set of positive integers. If  $\mathcal{F}$  is a family (finite or infinite) of  $R$ -graphs then  $\pi^R(\mathcal{F})$  is not a strong jump for  $R$ .*

*Proof.* Let  $\mathcal{H}$  be a family of finite  $R$ -graphs and  $\alpha := \pi(\mathcal{H})$ . By the definition of Turán density, there exists an admissible sequence of  $\mathcal{H}$ -free  $R$ -graphs  $\mathbf{G} := \{G_n^R\}_{n=1}^\infty$  with  $h(\mathbf{G}) = \alpha$ . Observe any subgraph of  $G_n^R$  is also  $\mathcal{H}$ -free. We have  $\bar{h}(\mathbf{G}) = \alpha$ . By Corollary 4,  $\alpha$  is not a strong jump.  $\square$

## 4.2 THE LAGRANGIAN

Frequently we need to describe a sequence of hypergraphs whose size grows to infinity. We begin this section by giving a formal way of describing a family of hypergraphs, specifically, a hypergraph pattern.

**Definition 15.** *A hypergraph **pattern**,  $P$ , is a pair  $P = (V, E)$ .  $V = \{v_1, \dots, v_n\}$  is a vertex set and the edge set  $E$  is a finite set of multisets of vertices. A typical element  $e \in E$  will have the form  $e = \{k_1 \cdot v_1, k_2 \cdot v_2, \dots, k_n \cdot v_n\}$  where  $k_i$  is a non-negative integer for each  $1 \leq i \leq n$ . We say that  $|e| = \sum_{i=1}^n k_i$ .*

**Definition 16.** *Suppose that  $P$  is a hypergraph pattern on  $n$ -vertices and  $m$ -edges. Let  $\vec{s} = (s_1, \dots, s_n)$  be a non-negative vector of integers. A hypergraph  $H = P(\vec{s})$  is a **realization** of a pattern  $P$  if:*

- $V(H) = \bigcup_{i=1}^n V_i$  with  $|V_i| = s_i$  for  $1 \leq i \leq n$
- $E(H) = \bigcup_{e \in E(P)} \binom{V_1}{k_1} \times \binom{V_2}{k_2} \times \dots \times \binom{V_n}{k_n}$

We view a realization of  $P$  essentially as a blow-up of the pattern  $P$ . Note that any hypergraph can also be viewed as a pattern—but not every pattern can be viewed as a hypergraph.

Let  $P$  be a hypergraph pattern on  $n$  vertices. Suppose that we want a realization of  $P$  with  $N$  vertices. We can choose a vector  $\vec{x} \in S_n$  such that  $x_i N \in \mathbb{Z}$  for each

$i$ . Then  $H = P(N\vec{x})$  is a realization of  $P$  on  $N$  vertices and  $|V_i| = x_i N$  for each  $1 \leq i \leq n$ . Let  $e = \{k_1 \cdot v_1, \dots, k_n \cdot v_n\}$  be an edge in  $E(P)$ . The edges of  $H$  that correspond to  $e$  contribute

$$\frac{\prod_{i=1}^n \binom{|V_i|}{k_i}}{\binom{N}{|e|}} = \frac{\prod_{i=1}^n \frac{(x_i N)^{k_i}}{k_i!}}{\frac{N^{|e|}}{|e|!}} + o(1) = \binom{|e|}{k_1, k_2, \dots, k_n} \prod_{i=1}^n x_i^{k_i} + o(1)$$

to the Lubell function of  $H$ .

**Definition 17.** Let  $P$  be a hypergraph pattern on  $n$  vertices. The **polynomial form** of  $P$ , denoted by  $\lambda(P, \vec{x})$ , is defined as

$$\lambda(P, \vec{x}) := \sum_{e \in E(P)} \binom{|e|}{k_1, k_2, \dots, k_n} \prod_{i=1}^n x_i^{k_i}.$$

**Definition 18.** The **standard simplex** of  $\mathbb{R}^n$  is

$$S_n := \{\vec{x} \in \mathbb{R}^n : \sum_{i=1}^n x_i = 1 \text{ and each } x_i \geq 0\}.$$

Note that  $\lambda(P, \vec{x})$  can be viewed as a polynomial in  $\mathbb{Z}[x_1, \dots, x_n]$  when the vector  $\vec{x}$  is unknown, or as a real number when  $\vec{x}$  is specified. Since  $S_n$  is compact, it follows that (the polynomial)  $\lambda(P, \vec{x})$  attains a maximum value on  $S_n$ .

**Definition 19.** The **Lagrangian** or **blow-up density** of a hypergraph pattern  $P$  is

$$\lambda(P) := \max_{\vec{x} \in S_n} \lambda(P, \vec{x}).$$

Typically, the polynomial form (for  $r$ -uniform hypergraphs) is defined so that every term has coefficient 1 and the blow-up density is defined to be the largest edge density (in the limit) one can obtain by blowing up a given hypergraph. For  $r$ -uniform hypergraphs the Lagrangian and blow-up density differ by a constant ( $r!$ ). When the graph is uniform, differing by a constant is easy to work around. However, if we generalize the Lagrangian to non-uniform graphs and leave every term monic, then the blow-up density and the Lagrangian no longer differ by a constant. This is

unacceptable for the applications we have in mind. For this reason, we adjust the coefficients of the polynomial form so that the value of the Lagrangian is meaningful.

We note that our treatment of the Lagrangian or blow-up density is not new—merely a generalization to non-uniform hypergraphs and graph patterns. For  $r$ -uniform graphs, the Lagrangian has been used extensively for graphs, and recently hypergraphs [9, 67]. Additionally, Kim and Vu introduce a random version of the Lagrangian [47].

The following proposition is the first reason for our interest in Lagrangians.

**Proposition 9.** *Let  $P$  be a hypergraph pattern on  $n$  vertices. Let  $\mathcal{F}$  be a family of  $R$ -graphs. If every realization  $H$  of  $P$  that is a hypergraph has the properties that  $R(H) \subseteq R$  and  $H$  is  $\mathcal{F}$ -free, then  $\lambda(P) \leq \pi^R(\mathcal{F})$ .*

For a vector  $\vec{x} = (x_1, \dots, x_n)$  we denote by  $\text{supp}(\vec{x})$ , the support of  $\vec{x}$ , the set of indices  $i$  such that  $x_i \neq 0$ . Let  $J \subseteq [n]$  be a set of indices. Then  $S_J = \{\vec{x} \in S_n : \text{supp}(\vec{x}) = J\}$ . When we refer to  $S_J$  we will always assume that  $J \neq \emptyset$  (otherwise  $S_J = \emptyset$  since  $\vec{0} \notin S_n$ ). The following lemmas are generalizations of results due to Frankl and Rödl. The proofs are similar, in some case with no essential difference. The one thing to keep in mind is that the way we have defined a Lagrangian differs slightly from the *standard* definition, primarily because one typically considers only uniform hypergraphs.

**Lemma 8.** *Let  $H$  be a hypergraph and suppose that  $\vec{y} \in S_J$  satisfies  $\lambda(H, \vec{y}) = \lambda$  and  $|J|$  is minimal. Then for any  $a, b \in J$  there exists an edge  $e \in E(H)$  with  $\{a, b\} \subseteq e \subseteq J$ .*

*Proof.* Towards a contradiction, assume that there is no edge satisfying  $\{a, b\} \subseteq e \subseteq J$ . We will use  $\vec{x}$  to denote variables, and  $\vec{y}$  as an assignment of those variables. Since there is no edge  $e \subseteq J$  with  $\{a, b\} \subseteq e$ , it follows that

$$\frac{\partial^2}{\partial x_a \partial x_b} \lambda(H, \vec{y}) = 0.$$

Without loss of generality, assume that

$$\frac{\partial}{\partial x_a} \lambda(H, \vec{y}) \leq \frac{\partial}{\partial x_b} \lambda(H, \vec{y}).$$

Set  $\delta = \min\{y_a, 1 - y_b\} \geq 0$ . Create a new vector  $\vec{z}$  as follows:  $z_a = y_a - \delta \geq 0$ ,  $z_b = y_b + \delta \leq 1$ , and  $z_i = y_i$  for every value of  $i$ . Note that  $\vec{z} \in S_n$  and  $z_i = 0$  if  $i \notin J$  and  $z_a = 0$ . The last follows from the fact that if  $\delta \neq y_a$  then  $z_b = 1$ . We will now show that  $\lambda(H, \vec{z}) \geq \lambda(H, \vec{y}) = \lambda(H)$  contradicting the minimality of  $|J|$ .

$$\begin{aligned} \lambda(H, \vec{z}) &= \sum_{e \in H} |e|! \prod_{i \in e} z_i \\ &= \sum_{\substack{e \in H \\ a, b \notin e}} |e|! \prod_{i \in e} z_i + \sum_{\substack{e \in H \\ a \in e}} |e|! \prod_{i \in e} z_i + \sum_{\substack{e \in H \\ b \in e}} |e|! \prod_{i \in e} z_i \\ &= \sum_{\substack{e \in H \\ a, b \notin e}} |e|! \prod_{i \in e} y_i + z_a \sum_{\substack{e \in H \\ a \in e}} |e|! \prod_{\substack{i \in e \\ i \neq a}} y_i + z_b \sum_{\substack{e \in H \\ b \in e}} |e|! \prod_{\substack{i \in e \\ i \neq b}} y_i \\ &= \sum_{\substack{e \in H \\ a, b \notin e}} |e|! \prod_{i \in e} y_i + (y_a - \delta) \sum_{\substack{e \in H \\ a \in e}} |e|! \prod_{\substack{i \in e \\ i \neq a}} y_i + (y_b + \delta) \sum_{\substack{e \in H \\ b \in e}} |e|! \prod_{\substack{i \in e \\ i \neq b}} y_i \\ &= \lambda(H, \vec{y}) + \delta \left( \sum_{\substack{e \in H \\ b \in e}} |e|! \prod_{\substack{i \in e \\ i \neq b}} y_i - \sum_{\substack{e \in H \\ a \in e}} |e|! \prod_{\substack{i \in e \\ i \neq a}} y_i \right) \\ &= \lambda(H, \vec{y}) + \delta \left( \frac{\partial}{\partial x_b} \lambda(H, \vec{y}) - \frac{\partial}{\partial x_a} \lambda(H, \vec{y}) \right) \\ &\geq \lambda(H, \vec{y}). \end{aligned}$$

We have created a new optimal vector  $\vec{z}$  with  $|\text{supp}(\vec{z})| < |\text{supp}(\vec{y})| = |J|$ . Contradiction.  $\square$

**Definition 20.** Let  $H^k$  be a  $k$ -uniform hypergraph. We say that two vertices  $i, j$  are equivalent if for every  $e \in \binom{V(H) - \{i, j\}}{k-1}$  it follows that  $e \cup \{i\} \in E(H^k)$  if and only if  $e \cup \{j\} \in E(H^k)$ .

**Definition 21.** Let  $H$  be a non-uniform hypergraph. We say that two vertices  $i$  and  $j$  are equivalent if for every  $k \in R(H)$ ,  $i$  and  $j$  are equivalent in  $H^k$ .

**Lemma 9.** *Let  $H$  be a hypergraph whose vertex set is  $[n]$  and suppose that  $a$  and  $b$  are equivalent vertices. Then there exists a  $\vec{y} \in S_n$  satisfying  $\lambda(H, \vec{y}) = \lambda(H)$  and  $y_a = y_b$ . Moreover, for any vector  $\vec{y} \in S_n$  satisfying  $\lambda(H, \vec{y}) = \lambda(H)$  if there exists an edge  $e \in H$  such that  $\{a, b\} \subseteq e \subseteq \text{supp}(\vec{y}) \cup \{a\}$  then  $y_a = y_b$ .*

*Proof.* Suppose that  $y_a \neq y_b$ . Define a new vector  $\vec{z}$  so that  $z_a = z_b = \frac{y_a + y_b}{2}$  and  $z_v = y_v$  otherwise. Clearly,  $\vec{z} \in S_n$ . We just need to check that  $\lambda(H, \vec{z}) \geq \lambda(H, \vec{y})$ .

$$\begin{aligned}
\lambda(H, \vec{z}) &= \sum_{e \in H} |e|! \prod_{i \in e} z_i \\
&= \sum_{\substack{e \in H \\ a, b \notin e}} |e|! \prod_{i \in e} y_i + \sum_{\substack{e \in H \\ a \in e, b \notin e}} |e|! z_a \prod_{i \in e, i \neq a} y_i + \sum_{\substack{e \in H \\ b \in e, a \notin e}} |e|! z_b \prod_{i \in e, i \neq b} y_i + \sum_{\substack{e \in H \\ \{a, b\} \subseteq e}} |e|! z_a z_b \prod_{\substack{i \in e \\ a, b \neq i}} y_i \\
&= \sum_{\substack{e \in H \\ a, b \notin e}} |e|! \prod_{i \in e} y_i + 2 \left( \frac{y_a + y_b}{2} \right) \sum_{\substack{e \in H \\ a \oplus b \in e}} |e|! \prod_{\substack{i \in e \\ i \neq a, b}} y_i + \left( \frac{y_a + y_b}{2} \right)^2 \sum_{\substack{e \in H \\ \{a, b\} \subseteq e}} |e|! \prod_{\substack{i \in e \\ a, b \neq i}} y_i \\
&\geq \lambda(H, \vec{y})
\end{aligned}$$

since  $\left( \frac{y_a + y_b}{2} \right)^2 \geq y_a y_b$  (with equality if and only if  $y_a = y_b$ ).  $\square$

The following theorem is a generalization of a theorem due to Rödl and Frankl [28].

**Theorem 19.** *Let  $R$  be a finite set of positive integers, and let  $\alpha \in [0, |R|)$ . Then  $\alpha$  is a jump for  $R$  if and only if there exists a finite family of  $R$ -graphs  $\mathcal{F}$  such that*

(i)  $\pi(\mathcal{F}) \leq \alpha$  and

(ii)  $\min_{F \in \mathcal{F}} \lambda(F) > \alpha$

Moreover,  $\alpha$  is a strong jump if the condition (i) is replaced by

(i')  $\pi(\mathcal{F}) < \alpha$ .

*Proof.* First, let us suppose that  $\alpha \in [0, |R|)$  is a jump for  $R$ . By definition, there exists some  $\Delta > 0$  so that for any  $k$  and any  $\epsilon > 0$  there exists an  $n_0(R, k, \epsilon)$  so that if



$G$  is an  $R$ -graph on  $n \geq n_0$  vertices, with  $h_n(G) \geq \alpha + \epsilon$  then  $G$  contains a subgraph  $H$  on  $k$  vertices with  $h_k(H) > \alpha + \Delta$ . We will find a finite family of graph  $\mathcal{F}$  with properties (i) and (ii) above.

Suppose that  $R = \{r_1, \dots, r_t\}$  with  $r_1 < r_2 < \dots < r_t$ . Fix  $k$  large enough that the constant

$$c = c(R) := \left(1 - \frac{1}{k}\right) \left(1 - \frac{2}{k}\right) \dots \left(1 - \frac{r_t - 1}{k}\right) > \frac{\alpha + \frac{\Delta}{2}}{\alpha + \Delta}.$$

Let  $\mathcal{F}$  be the set of all hypergraphs  $F$  on exactly  $k$  vertices satisfying the following two conditions:

(i)  $R(F) \subseteq R$

(ii)  $\lambda\left(F, k^{-1}(1, 1, \dots, 1)\right) = \sum_{r \in R} \frac{r!}{k^r} |E^r(F)| \geq \alpha + \frac{\Delta}{2}$ .

Note that  $\min_{F \in \mathcal{F}} \lambda(F) \geq \alpha + \frac{\Delta}{2} > \alpha$ . It remains to show that  $\pi(\mathcal{F}) \leq \alpha$ .

Let  $\epsilon > 0$  be given. Let  $G_n$  be a graph on  $n \geq n_0(R, k, \epsilon)$  vertices (enough vertices) with  $h_n(G) \geq \alpha + \epsilon$ . We need to show that  $G_n$  contains a member of  $\mathcal{F}$ . First, by hypothesis that  $\alpha$  is a jump (and  $G_n$  has enough vertices) we know that  $G_n$  contains a graph  $H_k$  on  $k$ -vertices with  $h_k(H_k) \geq \alpha + \Delta$ . We will now show that  $H_k \in \mathcal{F}$ . First, it is clear that  $R(H) \subseteq R$  since  $R(G) \subseteq R$ .

$$\begin{aligned} \alpha + \Delta &\leq h_k(H_k) \\ &= \sum_{r \in R} \frac{|E^r(H_k)|}{\binom{k}{r}} \\ &= \sum_{r \in R} \frac{r!}{k(k-1)\dots(k-r+1)} |E^r(H_k)| \\ &= \sum_{r \in R} \frac{r!}{k^r \left(1 - \frac{1}{k}\right) \left(1 - \frac{2}{k}\right) \dots \left(1 - \frac{r-1}{k}\right)} |E^r(H_k)| \\ &\leq \frac{1}{c} \sum_{r \in R} \frac{r!}{k^r} |E^r(H_k)|. \end{aligned}$$

Rearranging terms, we have that

$$\sum_{r \in R} \frac{r!}{k^r} |E^r(H_k)| \geq c(\alpha + \Delta) > \alpha + \frac{\Delta}{2}.$$

Thus,  $H_k$  is a member of  $\mathcal{F}$ . Hence  $\pi(\mathcal{F}) \leq \alpha$  as desired.

Now, suppose that we have a finite family  $\mathcal{F}$  with the properties that  $R(\mathcal{F}) \subseteq R$ ,  $\min_{F \in \mathcal{F}} \lambda(F) > \alpha$ , and  $\pi(\mathcal{F}) \leq \alpha$ . We need to show that  $\alpha$  is a jump for  $R$ . Write  $R = \{r_1, \dots, r_t\}$  with  $r_1 < r_2 < \dots < r_t$  and fix  $\epsilon > 0$  and  $k \geq r_t$ . Let  $\Delta = \min_{F \in \mathcal{F}} \lambda(F) - \alpha > 0$ . Choose  $n_0$  large enough that if  $n \geq n_0$  it follows that each  $\lambda(F)$  for  $F \in \mathcal{F}$  can be approximated by some vector

$$\frac{\vec{x}_F}{n} = \left( \frac{x_1}{n}, \dots, \frac{x_{|F|}}{n} \right) \in S_{|F|}$$

where each  $\vec{x} \in \mathbb{N}^{|F|}$  satisfying  $\lambda(F, \frac{\vec{x}_F}{n}) \geq \alpha + \frac{\Delta}{2}$ . Now, consider the family  $\mathcal{F}'$  where

$$\mathcal{F}' := \{F(\vec{x}_F) : F \in \mathcal{F}\}$$

obtained by blowing-up each graph in  $\mathcal{F}$  so as to maximize the Lubell value of each graph. Since  $\mathcal{F}'$  is obtained by blowing up graphs in  $\mathcal{F}$ , it follows that  $\pi(\mathcal{F}') = \pi(\mathcal{F}) \leq \alpha$ . Suppose that  $G$  has  $N$  vertices and  $h_N(G) \geq \alpha + \epsilon$  and  $N$  is large enough that  $G$  must contain some member of  $\mathcal{F}'$ . We will now show that  $G$  has some subgraph  $H_k$  on exactly  $k$ -vertices with  $h_k(H_k) \geq \alpha + \frac{\Delta}{2}$ .

Suppose that  $G$  contains  $F' \in \mathcal{F}'$ . Note that  $F'$  (by construction) has  $n \geq n_0$  vertices. Consider  $G[F']$ ; we have that  $h_n(G[F']) \geq h_n(F') \geq \alpha + \frac{\Delta}{2}$ . Let  $K$  be a random  $k$ -subset of the vertices of  $G[F']$ . Since  $\mathbb{E}(h_k(G[K])) = h_n(G[F']) \geq \alpha + \frac{\Delta}{2}$  it follows that there is some  $k$ -subset of  $G[F']$  satisfying  $h_k(G[K]) \geq \alpha + \frac{\Delta}{2}$ . Then  $G[K]$  is a  $k$ -vertex subgraph of  $G$  with  $h_k(G[K]) \geq \alpha + \frac{\Delta}{2}$ ; we have found a subgraph with the desired properties. Hence  $\alpha$  is a jump for  $R$ .  $\square$

**Proposition 10.** *If  $\alpha$  is a jump for  $R$  and there exists an  $R$ -graph  $F$  with  $\lambda(F) = \alpha$  then  $\alpha$  is a weak jump.*

*Proof.* We need to show that  $\alpha$  is not a strong jump. For  $n$  larger than  $|F|$ , let  $F_n$  be a blow-up of  $F$  on  $n$  vertices such that  $h_n(F_n)$  is as large as possible. The Lagrangian is constructed in such a way that  $\lim_{n \rightarrow \infty} h_n(F_n) = \lambda(F)$ . Hence we have a sequence  $\mathbf{F} = \{F_n\}$  with  $h(\mathbf{F}) = \lambda(F) = \alpha$ . By construction, we also have that  $\bar{h}(\mathbf{F}) = \lambda(F)$ . Hence  $\alpha$  is not a strong jump.  $\square$

### 4.3 JUMPS FOR $\{1, 2\}$ -GRAPHS

This section is devoted to the proof of the following theorem.

**Theorem 20.** *Every  $\alpha \in [0, 2]$  is a jump for  $\{1, 2\}$ . Furthermore,  $\alpha$  is a weak jump if and only if it is in one of the following sets.*

- $\left\{ \frac{k}{k+1} : k \in \mathbb{Z}, k \geq 0 \right\}$
- $\left\{ 1 + \frac{k}{4(k+1)} : k \in \mathbb{Z}, k \geq 0 \right\}$
- $\left\{ \frac{5}{4} \right\}$
- $\left\{ \frac{2k+1}{k+1} : k \in \mathbb{Z}, k \geq 1 \right\}$
- $\{2\}$ .

**Case:**  $\alpha \in [0, 1)$

There is a unique integer  $t \geq 2$  such that  $\alpha \in [1 - \frac{1}{t-1}, 1 - \frac{1}{t})$ . Let  $\mathcal{F} = \{K_1^{\{1\}}, K_t^{\{2\}}\}$ .

If  $G$  is an  $R$ -graph that is  $\mathcal{F}$ -free, then  $G$  is also a  $\{2\}$ -graph. Hence

$$\pi^R(\mathcal{F}) = \pi^{\{2\}}(K_t^{\{2\}}) = 1 - \frac{1}{t-1} \leq \alpha.$$

We note that  $\lambda(K_1^{\{1\}}) = 1$ . We now compute  $\lambda(K_t^{\{2\}})$ .

$$\begin{aligned}
\lambda(K_t^{\{2\}}) &= \max_{\vec{x} \in S_t} \lambda(K_t^{\{t\}}, \vec{x}) \\
&= \max_{\vec{x} \in S_t} \sum_{1 \leq i < j \leq t} 2x_i x_j \\
&= \sum_{1 \leq i < j \leq t} \frac{2}{t^2} \\
&= \binom{t}{2} \frac{2}{t^2} \\
&= \frac{t-1}{t} \\
&> \alpha.
\end{aligned}$$

Note that the third line follows since every vertex is equivalent (see Lemma 9). Hence, by Theorem 19 we have that  $\alpha$  is a jump. Furthermore, we have that for all  $t \geq 2$  the value  $1 - \frac{1}{t-1}$  is a weak jump and every other value in  $[0, 1)$  is a strong jump.

**Case:**  $\alpha \in [1, \frac{9}{8})$

Let  $F$  be a chain on two vertices, i.e. edges  $\{1\}$  and  $\{1, 2\}$ . Since  $F$  is a chain (of length 2),  $\pi^R(F) = 1 \leq \alpha$ . We now compute  $\lambda(F)$ .

$$\begin{aligned}
\lambda(F) &= \max_{\vec{x} \in S_2} \lambda(F, \vec{x}) \\
&= \max_{\vec{x} \in S_2} x_1 + 2x_1 x_2 \\
&= \max_{x_1 \in [0, 1]} x_1 + 2x_1(1 - x_1) \\
&= \frac{9}{8}.
\end{aligned}$$

Note that 1 is not a strong jump by theorem 18 since  $\pi^R(F) = 1$ . Also, note that  $\frac{9}{8}$  is not a strong jump since  $\lambda(F) = \frac{9}{8}$ .

**Case:**  $\alpha \in [\frac{9}{8}, \frac{5}{4})$

Note that there is a unique  $t \geq 3$  such that  $\alpha \in [\frac{5}{4} - \frac{1}{4(t-1)}, \frac{5}{4} - \frac{1}{4t})$ . Let  $K_t^*$  denote the  $\{1, 2\}$ -graph with vertex set  $V = [t]$  and edge set  $E = \{1\} \cup \binom{[t]}{2}$ . First, we will show that  $\pi(K_t^*, K_2^{\{1,2\}}) \leq \frac{5}{4} - \frac{1}{4(t-1)}$  (for  $t \geq 3$ ).

Let  $G_n$  be any graph on  $n$  vertices which forbids both  $K_t^*$  and  $K_2^{\{1,2\}}$ . Partition the vertex set of  $G_n$  into two sets  $X$  and  $\bar{X}$  where a vertex  $v \in X$  if and only if  $\{v\} \in E$ . We will say that  $|X| = xn$  and  $|\bar{X}| = (1-x)n$ . For each  $v \in X$  define the set  $N_v$  to be the set of vertices  $u \in \bar{X}$  such that  $\{v, u\} \in E$ . Since  $G_n$  is  $K_t^*$ -free, it follows that for each  $v \in X$  the graph  $G[N_v]$  is  $K_{t-1}^{\{2\}}$ -free.

Note that  $\pi(K_{t-1}^{\{2\}}) = 1 - \frac{1}{t-2}$  so the number of edges in  $G[N_v]$  is at most  $(1 - \frac{1}{t-2}) \binom{|N_v|}{2} + o(1)$ . In other words, in  $G[N_v]$  there are at least  $\frac{1}{t-2} \binom{|N_v|}{2} - o(1)$  non-edges. Fix  $v \in X$  such that  $|N_v|$  is as large as possible, and say that  $|N_v| = \alpha(1-x)n$ . We then have that

$$\begin{aligned} h_n(G_n) &\leq \frac{xn}{\binom{n}{1}} + \frac{\binom{(1-x)n}{2} + (xn)\alpha(1-x)n - \frac{1}{t-2} \binom{\alpha(1-x)n}{2}}{\binom{n}{2}} \\ &= x + (1-x)^2 + 2\alpha x(1-x) - \frac{\alpha^2(1-x)^2}{t-2} + o(1) \\ &= x + 2\alpha x(1-x) + (1-x)^2 \left(1 - \frac{\alpha^2}{t-2}\right) + o(1) \\ &\leq \max_{x \in [0,1], \alpha \in [0,1]} x + 2\alpha x(1-x) + (1-x)^2 \left(1 - \frac{\alpha^2}{t-2}\right) + o(1) \\ &= \frac{5}{4} - \frac{1}{4(t-1)} + o(1). \end{aligned}$$

The last line of the inequality above is achieved when  $x = \frac{t}{2(t-1)}$  and  $\alpha = 1$ . With that observation, we have actually proven that  $\pi(K_t^*, K_2^{\{1,2\}}) = \frac{5}{4} - \frac{1}{4(t-1)}$ .

We now show that  $\lambda(K_2^{\{1,2\}}) = \frac{3}{2} > \alpha$ .

$$\begin{aligned}
\lambda(K_2^{\{1,2\}}) &= \max_{\vec{x} \in S_2} \lambda(K_2^{\{1,2\}}, \vec{x}) \\
&= \max_{\vec{x} \in S_2} x_1 + x_2 + 2x_1x_2 \\
&= 1 + 2 \binom{1}{2} \binom{1}{2} \\
&= \frac{3}{2}.
\end{aligned}$$

Note again that the third line follows since the two vertices are equivalent. Now, we bound  $\lambda(K_t^*)$ .

Note that

$$\lambda(K_t^*, \vec{x}) = x_1 + \sum_{1 \leq i < j \leq t} 2x_i x_j.$$

To get a lower bound on  $\lambda(K_t^*)$ , choose  $x_1 = \frac{t+1}{2t}$  and  $x_i = \frac{1}{2t}$  for  $2 \leq i \leq t$ . It follows that

$$\begin{aligned}
\lambda(K_t^*) &\geq \frac{t+1}{2t} + 2 \binom{t+1}{2t} \sum_{i=2}^t \frac{1}{2t} + \sum_{2 \leq i < j \leq t} \left(\frac{1}{2t}\right)^2 \\
&= \frac{t+1}{2t} + 2 \binom{t+1}{2t} \binom{t-1}{2t} + 2 \binom{t-1}{2} \left(\frac{1}{2t}\right)^2 \\
&= \frac{5}{4} - \frac{1}{4t} \\
&> \alpha.
\end{aligned}$$

Hence by Theorem 19  $\alpha$  is a jump for  $R$ . By the same arguments we've previously seen, the interior of the interval:  $[\frac{5}{4} - \frac{1}{4(t-1)}, \frac{5}{4} - \frac{1}{4t})$  are strong jumps, and the endpoints are weak jumps.

**Case:**  $\alpha \in [\frac{5}{4}, \frac{3}{2})$

Note that  $\pi^R(K_2^{\{1,2\}}) = \frac{5}{4} \leq \alpha$ . (This is a result from [38].) Additionally, we already saw that  $\lambda(K_2^{\{1,2\}}) = \frac{3}{2} > \alpha$ . Hence  $\alpha$  is a jump. Again,  $\frac{5}{4}$  and  $\frac{3}{2}$  are weak jumps, and the interval  $(\frac{5}{4}, \frac{3}{2})$  is comprised of strong jumps.

**Case:**  $\alpha \in [\frac{3}{2}, 2)$

There is a unique  $t \geq 3$  such that  $\alpha \in [2 - \frac{1}{t-1}, 2 - \frac{1}{t})$ . Note that  $\pi^R(K_t^{\{1,2\}}) = 2 - \frac{1}{t-1}$ ; again, this is a result from [38]. Furthermore, every vertex is equivalent. Thus

$$\begin{aligned}
\lambda(K_t^{\{1,2\}}) &= \max_{\vec{x} \in S_t} \lambda(K_t^{\{1,2\}}) \\
&= \max_{\vec{x} \in S_t} \sum_{i=1}^t x_i + \sum_{1 \leq i < j \leq t} 2x_i x_j \\
&= 1 + 2 \binom{t}{2} \left(\frac{1}{t}\right)^2 \\
&= 1 + \frac{t-1}{t} \\
&= 2 - \frac{1}{t} \\
&> \alpha.
\end{aligned}$$

Hence  $\alpha$  is a jump. As before,  $2 - \frac{1}{t-1}$  and  $2 - \frac{1}{t}$  are weak jumps, and the interval  $(2 - \frac{1}{t-1}, 2 - \frac{1}{t})$  is comprised of strong jumps.

#### 4.4 A FINAL NOTE ON NON-STRONG JUMPS

In this last section, we will reveal the relationship between the non-strong-jump values and hereditary properties. Hereditary properties have been well-studied for graphs and  $r$ -uniform hypergraphs [5, 6, 18, 57]. This concept can be naturally extended to  $R$ -graphs. A **property** of  $R$ -graphs is a family of  $R$ -graphs closed under isomorphism. A property is called **hereditary** if it is closed under taking induced subgraphs. A typical hereditary property can be obtained by forbidding a set of  $R$ -graphs as induced sub-hypergraphs. Given a hereditary property  $\mathcal{P}$  of  $R$ -graphs, let  $\mathcal{P}_n$  be the set of  $R$ -graphs in  $\mathcal{P}$  with  $n$  vertices, and set

$$\pi_n(\mathcal{P}) = \max_{G \in \mathcal{P}_n} h_n(G).$$

We have the following proposition.

**Proposition 11.** *For any hereditary property  $\mathcal{P}$  of  $R$ -graphs, the limit  $\lim_{n \rightarrow \infty} \pi_n(\mathcal{P})$  always exists.*

The limit,  $\pi(\mathcal{P})$ , is called the density of  $\mathcal{P}$ :

$$\pi(\mathcal{P}) = \lim_{n \rightarrow \infty} \pi_n(\mathcal{P}).$$

This proposition can be proved using the average argument, first shown in Katona-Nemetz-Simonovits's theorem [40] for the existence of the Turán density of any  $r$ -uniform hypergraph. For non-uniform hypergraphs, the proof of Proposition 11 is actually identical to the proof of existence of the Turán density  $\pi(\mathcal{H})$  (see Theorem 1 in [38]), and is omitted here.

**Theorem 21.** *For any fixed set  $R$  of finite positive integers, a value  $\alpha \in [0, |R|]$  is not a strong jump for  $R$  if and only if there exists a hereditary property  $\mathcal{P}$  of  $R$ -graphs such that  $\pi(\mathcal{P}) = \alpha$ .*

*Proof.* By Corollary 4,  $\alpha$  is not a strong jump for  $R$  if and only if there exists an admissible sequence of  $R$ -graphs  $\mathbf{G} := \{G_{n_i}\}_{i=1}^{\infty}$  satisfying  $h(\mathbf{G}) = \bar{h}(\mathbf{G}) = \alpha$ .

Now we show that it is a sufficient condition. Consider a hereditary property  $\mathcal{P}$  with  $\pi(\mathcal{P}) = \alpha$ . Let  $G_n \in \mathcal{P}_n$  be an  $R$ -graph achieving the maximum Lubell value, and  $\mathbf{G} := \{G_n\}$ . By definition of  $\pi(\mathcal{P})$ , we have

$$h(\mathbf{G}) = \pi(\mathcal{P}) = \alpha.$$

Since  $\mathcal{P}$  is hereditary, any induced subgraphs of  $\mathbf{G}$  are still in  $\mathcal{P}$ . Thus

$$\bar{h}(\mathbf{G}) \leq \pi(\mathcal{P}) = \alpha.$$

Since  $\bar{h}(\mathbf{G}) \geq h(\mathbf{G})$ , it forces  $\bar{h}(\mathbf{G}) = h(\mathbf{G}) = \alpha$ . By Corollary 4,  $\alpha$  is not a strong jump for  $R$ .

Now we show that it is also a necessary condition. We define a property

$$\mathcal{P} := \{H : H \text{ is an induced subgraph of } G_n \in \mathbf{G}\}.$$



It is clear that  $\mathcal{P}$  is hereditary. Since  $\bar{h}(\mathbf{G}) = \alpha$ , we have  $\pi(\mathcal{P}) = \alpha$ . □

## CHAPTER 5

### JUMPS IN UNIFORM HYPERGRAPHS

#### 5.1 A SUFFICIENT CONDITION FOR NON-JUMPS

In this section, we will prove a general theorem giving sufficient conditions for non-jump values. Throughout this section the graphs (and patterns) we will consider will all be  $r$ -uniform. It is motivated by previous results of non-jump values.

Consider a  $r$ -uniform hypergraph pattern  $P$  on  $n$  vertices. We say  $P$  is **irreducible** if the Lagrange polynomial  $\lambda(P, \vec{x})$  achieves the maximum value only at some interior point of the standard simplex. For any pattern  $P$ , if the Lagrangian  $\lambda(P)$  is achieved on some boundary point, we can fix an optimal vertex weighting then delete the vertices whose weights are 0 in that optimal weighting. The resulting smaller pattern  $P'$  satisfies  $\lambda(P') = \lambda(P)$ . We can iterate this process until we are left with an irreducible pattern  $P''$  which satisfies  $\lambda(P'') = \lambda(P)$ .

The following Lemma generalizes Lemma 2.2(c) of [27].

**Lemma 10.** *Let  $P$  be an irreducible  $r$ -uniform hypergraph pattern. If the Lagrange polynomial  $\lambda(P, \vec{x})$  achieves the maximum value  $\lambda(P)$  at  $\vec{x}^*$ , then for any  $i$ , we have  $\frac{\partial \lambda(P, \vec{x})}{\partial x_i}(\vec{x}^*) = r\lambda(P)$ .*

*Proof.* We use the Lagrange multiplier method to find the maximum value of  $\lambda(P, \vec{x})$ . Let  $F(\vec{x}) = \lambda(P, \vec{x}) - \mu \sum_{i=1}^n x_i$ . Since  $P$  is irreducible every  $x_i^*$  is positive. The point  $\vec{x}^*$  satisfies the equations  $\frac{\partial F}{\partial x_i}(\vec{x}^*) = 0$  for every  $i$ . Thus,  $\frac{\partial \lambda(P, \vec{x})}{\partial x_i}(\vec{x}^*) = \mu$  for all  $i$ .

Recall the definition of the Lagrange polynomial

$$\lambda(P, \vec{x}) := \sum_{e \in E(P)} \binom{|e|}{k_1, k_2, \dots, k_n} \prod_{i=1}^n x_i^{k_i}.$$

Now consider the new polynomial  $H(\vec{x}) := \sum_{i=1}^n x_i \frac{\partial \lambda(P, \vec{x})}{\partial x_i}(\vec{x}^*)$ . For each multiset  $e = k_1 \cdot v_1, \dots, k_n \cdot v_n$  of  $P$ , the coefficient of the monomial  $x_e := \prod_{i=1}^n x_i^{k_i}$  in  $H(\vec{x})$  is given by  $\sum_{i=1}^n \binom{|e|}{k_1, k_2, \dots, k_n} k_i = r \binom{|e|}{k_1, k_2, \dots, k_n}$ . Here we use the assumption  $P$  is  $r$ -uniform,  $\sum_{i=1}^n k_i = r$ . Thus, we observed

$$H(\vec{x}) = r\lambda(P, \vec{x}).$$

Evaluating at the maximum point  $\vec{x}^*$ , we get

$$\mu = \sum_{i=1}^n x_i^* \mu = \sum_{i=1}^n x_i^* \frac{\partial \lambda(P, \vec{x})}{\partial x_i}(\vec{x}^*) = H(\vec{x}^*) = r\lambda(P, \vec{x}^*) = r\lambda(P).$$

□

We say the ( $r$ -uniform) pattern  $P$  contains a  $s$ -cycle if there exist  $s$  vertices  $v_{i_1}, v_{i_2}, \dots, v_{i_s}$  so that for each  $1 \leq j \leq s$ ,  $P$  contains the edge  $\{(r-1)v_{i_j}, v_{i_{j+1}}\}$ . Here  $v_{i_{s+1}} := v_{i_1}$ .

**Theorem 22.** *Consider an irreducible  $r$ -uniform hypergraph pattern  $P$  on  $n$  vertices. Suppose that  $P$  contains a  $s$ -cycle for  $s \geq 2$ . Without loss of generality, we assume the vertices of the  $s$ -cycle are  $v_1, v_2, \dots, v_s$ . If  $P$  satisfies the following two properties:*

1.  $\sum_{i=1}^s \frac{\partial \lambda(P, \vec{x})}{\partial x_i}(\vec{z}) \leq rs\lambda(P)$  for every point  $\vec{z}$  in the simplex
2. for every  $1 \leq i \leq s$  it follows that  $\frac{r!}{r^r} \sum_{j=1}^s x_j^r + \lambda(P, \vec{x}) \leq \lambda(P)$  for all  $\vec{x}$  in the simplex satisfying  $x_i = 0$

then  $\lambda(P)$  is a non-jump value for  $r$ .

Before we give the proof of this theorem, we need a Lemma. The following Lemma is a direct generalization of Lemma 3.2 of [27] and the proof was given in [28].

**Lemma 11.** *Let  $k, r$  be any fixed integer and  $c \geq 0$  be any fixed real number. Then there exists  $t_0(k, c)$  such that for every  $t > t_0(r, k, c)$ , there exists a  $r$ -uniform graph  $A$  satisfying:*

1.  $|V(A)| = t$  ;
2.  $|E(A)| \geq ct^{r-2}$ ;
3. *for all  $V_0 \subset V(A)$ ,  $r \leq |V_0| \leq k$  we have  $|E(A) \cap \binom{V_0}{r}| \leq |V_0| - r + 1$ .*

We now prove Theorem 22.

*Proof.* Towards a contradiction, assume that  $\lambda(P)$  is a jump. Then, there exists a finite family  $\mathcal{F}$  with  $\min_{F \in \mathcal{F}} \lambda(F) > \lambda(P)$  and  $\pi(\mathcal{F}) \leq \lambda(P)$ . Fix  $k := \max_{F \in \mathcal{F}} |V(F)|$  making certain that  $k \geq r$ .

For a real number  $c > 0$  fix  $t > t_0(r, k, c)$  as given in the previous lemma. Let  $A$  be the  $r$ -graph on  $t$  vertices whose existence is guaranteed by the lemma. Let  $G^*(t, k, c)$  be the  $r$ -uniform hypergraph obtained from  $P(t)$  by adding a copy of  $E(A)$  into each vertex class  $V_i$  of  $P(t)$  for each  $1 \leq i \leq s$ . Let  $M$  be a subgraph of  $G^*(t, k, c)$  containing exactly  $k$  vertices from each vertex class. Note that if there exists an  $F \in \mathcal{F}$  that is contained in  $G^*$ , then some  $M$  (as described) will contain it. Furthermore, if there exists an  $F \in \mathcal{F}$  that is contained in  $M$ , then  $\lambda(M) \geq \lambda(F) > \lambda(P)$ .

Following the proof strategy in [27] (and their notation as much as possible) our immediate goal is to show that  $\lambda(M) \leq \lambda(P)$ . This will be the contradiction by which we may conclude that  $\lambda(P)$  is not a jump. We assume that  $M$  is an induced subgraph of  $G^*$  and for each  $1 \leq i \leq n$  we set  $U_i = V_i \cap V(M)$ . We will say that  $U_i = \{v_1^i, \dots, v_k^i\}$  recalling that  $|U_i| = k$  for each  $i$ .

**Lemma 12.** *If  $N$  is the  $r$ -uniform hypergraph formed from  $M$  by removing the edges contained in each  $U_i$  and inserting edges  $\{\{v_1^i, \dots, v_{r-1}^i, v_j^i\} : 1 \leq i \leq s, r \leq j \leq k\}$  then  $\lambda(M) \leq \lambda(N)$ .*

The proof of this lemma is identical to the proof of claim 3.4 in [27].

It now remains to show that  $\lambda(N) \leq \lambda(P)$ . For each  $1 \leq i \leq s$ ,  $U_i$  contains vertices in two equivalence classes. We obtain a  $\vec{z} \in S$  satisfying  $\lambda(N, \vec{z}) = \lambda(N)$  such that

$$z_1^i = z_2^i = \cdots = z_{r-1}^i = a_i, \quad z_r^i = z_{r+1}^i = \cdots = z_k^i = b_i.$$

Each  $U_i$  for  $s < i \leq n$  has a single equivalence class. For  $i > s$  we will arbitrarily pick  $r - 1$  vertices to assign to the  $a_i$  equivalence class, and the remaining  $(k - r + 1)$  vertices will be in the  $b_i$  equivalence class. Note that if  $i > s$  then  $a_i = b_i$  since we really only have one equivalence class. Let  $w_i = (r - 1)a_i + (k - r + 1)b_i$  be the weight of  $U_i$ . Note that  $\vec{w} = (w_1, \dots, w_n) \in S_n$ . Let  $\vec{ab}^*$  denote the vector  $(a_1, b_1, \dots, a_n, b_n)$  that comes from this optimal solution  $\vec{z}$ .

For the moment, consider the  $a_i$ 's and  $b_i$ 's as variables, and consider the variables  $x_j^i$  (for  $1 \leq j \leq k$  and  $1 \leq i \leq n$ ) corresponding to vertices of  $N$  as functions of the variables  $a_i$  and  $b_i$ , specifically,

$$x_j^i(a_1, b_1, \dots, a_n, b_n) = \begin{cases} a_i & \text{if } j \leq r - 1 \\ b_i & \text{if } j \geq r. \end{cases}$$

Towards our goal of showing that  $\lambda(N) \leq \lambda(P)$ , let us first consider the case that each  $b_i$  (for  $1 \leq i \leq n$ ) in  $\vec{ab}^*$  is non-zero. Note that

$$\begin{aligned} \frac{\partial \lambda(N, \vec{x})}{\partial b_i} &= \sum_{j=1}^k \frac{\partial \lambda(N, \vec{x})}{\partial x_j^i} \cdot \frac{\partial x_j^i}{\partial b_i} \\ &= \sum_{j=r}^k \frac{\partial \lambda(N, \vec{x})}{\partial x_j^i} \end{aligned}$$

So that

$$\begin{aligned} \frac{\partial \lambda(N, \vec{x})}{\partial b_i} (\vec{ab}^*) &= \sum_{j=r}^k \frac{\partial \lambda(N, \vec{x})}{\partial x_j^i} (\vec{ab}^*) \\ &= (k - r + 1)r\lambda(N, \vec{z}) \\ &= (k - r + 1)r\lambda(N). \end{aligned}$$

Note that the second equality above holds by Lemma 10. We then have that

$$\begin{aligned}
\lambda(N) &= \lambda(N, \vec{z}) \\
&= \frac{1}{rs} \sum_{i=1}^s \frac{1}{k-r+1} \frac{\partial \lambda(N, \vec{x})}{\partial b_i} (\vec{ab}^*) \\
&= \frac{1}{rs} \sum_{i=1}^s \frac{1}{k-r+1} \left[ \frac{\partial \lambda(P(t), \vec{x})}{\partial b_i} (\vec{ab}^*) \frac{\partial}{\partial b_i} (r!(k-r+1)a_i^{r-1}b_i) \right] \\
&= \frac{1}{rs} \sum_{i=1}^s \frac{1}{k-r+1} \left[ \frac{\partial \lambda(P, \vec{w})}{\partial b_i} (\vec{ab}^*) + r!(k-r+1)a_i^{r-1} - \frac{\partial}{\partial b_i} \{ \dots \} \right] \\
&= \frac{1}{rs} \sum_{i=1}^s \frac{1}{k-r+1} \left[ \frac{\partial \lambda(P, \vec{w})}{\partial w_i} \frac{\partial w_i}{\partial b_i} (\vec{ab}^*) + r!(k-r+1)a_i^{r-1} - \frac{\partial}{\partial b_i} \{ \dots \} \right] \\
&\leq \frac{1}{rs} \sum_{i=1}^s \left[ \frac{\partial \lambda(P, \vec{w})}{\partial w_i} (\vec{ab}^*) + r!a_i^{r-1} - r \left[ (r-1)^{r-1} - (r-1)! \right] a_i^{r-1} \right] \\
&= \frac{1}{rs} \sum_{i=1}^s \underbrace{\frac{\partial \lambda(P, \vec{w})}{\partial w_i} (\vec{ab}^*)}_{\leq \lambda(P) \text{ by assumption}} + \frac{1}{rs} \sum_{i=1}^s a_i^{r-1} \underbrace{\left[ r! - r((r-1)^{r-1} - (r-1)!) \right]}_{\leq 0 \text{ if } r \geq 3} \\
&\leq \lambda(P).
\end{aligned}$$

The  $\{ \dots \}$  above represents edges in the pattern which use non-distinct vertices. We select out the terms  $w_i^{r-1}w_{i+1}$  arising from the cycle and consider the numbers of ways of choosing non-distinct vertices from the  $a_i$  class in  $w_i$  connected to each of the  $b_{i+1}$  class of vertices in  $w_{i+1}$ .

We now consider the case that some of the  $b_i$ 's are zero. First, we will show that if  $b_i = 0$  and  $\lambda(N, \vec{w})(\vec{ab}^*) = \lambda(N)$  then  $a_i = 0$  too, and hence  $w_i = 0$ . Towards a contradiction, assume that  $a_i \neq 0$ . We will shortly make use of two facts, namely that if  $1 < r \leq u < v$  are integers then  $\lambda(K_u^r) < \lambda(K_v^r)$  and that  $\lambda(K_v^r, \vec{x})$  attains its maximum precisely at  $\vec{x} = \left( \frac{1}{v}, \frac{1}{v}, \dots, \frac{1}{v} \right)$ .

Let  $E_1, E_{>1}$ , and  $E_r$  be disjoint sets of edges of  $N$  satisfying the following:

- $e \in E_1$  if and only if  $|e \cap U_i| = 1$
- $e \in E_{>1}$  if and only if  $1 < |e \cap U_i| < r$
- $e \in E_r$  if and only if  $|e \cap U_i| = r$

We will change the weight of the vertices in  $U_i$  so that each vertex carries the same weight. Specifically, set  $a'_i = b'_i = \frac{(r-1)a_i}{k}$  so that  $w_i = w'_i$ . Note that if  $e \cap U_i = \emptyset$  then this change doesn't affect the contribution of  $e$  to the lagrangian of  $N$ . Suppose that  $e$  is an edge that intersects  $U_i$ . Let  $S = \prod_{v \in e \setminus U_i} x_v$ . If  $e \in E_1$  then the contribution of  $e$  to the lagrangian is:

$$r!S \sum_{j=1}^k x_j^i = r!S((r-1)a_i + (k-r+1)b_i) = r!S w_i = r!S w'_i.$$

So the re-weighting doesn't affect the contribution of  $e$  to the lagrangian of  $N$ . If  $e \in E_{>1}$  then the contribution before re-weighting is:

$$r!S \sum_{1 \leq i_1 < i_2 < \dots < i_{r-|S|} \leq k} x_{i_1} x_{i_2} \dots x_{i_{r-|S|}} = \frac{r!S \lambda(K_{r-1}^{r-|S|})}{(r-|S|)!}.$$

After the re-weighting, the contribution of  $e$  to the lagrangian is:

$$r!S \sum_{1 \leq i_1 < i_2 < \dots < i_{r-|S|} \leq k} x_{i_1} x_{i_2} \dots x_{i_{r-|S|}} = \frac{r!S \lambda(K_k^{r-|S|})}{(r-|S|)!}$$

which is larger than the first contribution. Finally, if  $e \in E_r$ , then when  $b_i = 0$  the contribution of  $e$  to the lagrangian is 0. So, re-weighting makes the contribution to the lagrangian positive. This is a contradiction, since we assumed that the original choice of weights was optimal (and we've shown it is strictly sub-optimal).

We now have that if  $b_i = 0$ , then necessarily,  $a_i = 0$  and hence  $w_i = 0$ . Let us now consider, for each  $j \neq i$ , the contribution to the lagrangian of edges contained in  $U_j$ . The edges in  $U_j$  contribute  $r!(k-r+1)a_j^{r-1}b_j = r!(w_j a_j^{r-1} - (r-1)a_j^r)$  to the lagrangian. An easy calculus exercise shows that  $a_j = w_j/r$  maximizes this function and the maximum value of the function is  $\frac{r!w_j^r}{r^r}$ . Thus, the total contribution of edges contained in some  $U_j$  is:

$$\frac{r!}{r^r} \sum_{1 \leq j \neq i \leq s} w_j^r.$$

By the second hypothesis, we have that if  $w_i = 0$ , then

$$\lambda(N) = \lambda(N, \vec{z}) = \frac{r!}{r^r} \sum_{j=1}^s w_j^r + \lambda(P, \vec{w}) \leq \lambda(P).$$

And the proof is complete. □

## 5.2 NEW NON-JUMP VALUES FOR $R = \{3\}$

Frankl et. al. [27] give the following construction which we re-state in terms of a hypergraph pattern. Let  $P_{\ell,s}$  be a pattern with vertices  $v_1, \dots, v_\ell$ . The edges of  $P_{\ell,s}$ , all of which contain exactly 3 vertices, are one of two forms. If  $e$  is an edge of  $P_{\ell,s}$  then either all of the vertices of  $e$  are distinct, or  $e = \{2v_i, 1v_{i+j}\}$  where  $1 \leq j \leq s$  and the addition is performed modulo  $\ell$ . Every multiset of vertices of  $P_{\ell,s}$  with exactly 3 vertices that satisfies one of these previous two conditions is an edge in the pattern. The Lagrange polynomial of  $P_{\ell,s}$  is the following:

$$\lambda(P_{\ell,s}, \vec{x}) = 6 \sum_{1 \leq i < j < k \leq \ell} x_i x_j x_k + 3 \sum_{i=1}^{\ell} \sum_{j=1}^s x_i^2 x_{i+j}.$$

The polynomial is maximized at  $\vec{x} = \frac{1}{\ell}(1, 1, \dots, 1)$  and

$$\lambda(P_{\ell,s}) = 1 - \frac{3}{\ell} + \frac{3s+2}{\ell^2}.$$

In their paper, Frankl et. al. prove that  $\lambda(P_{\ell,s})$  is a non-jump value for  $r = 3$  if  $s \geq 1$  and  $\ell \geq 9s + 6$ . They note that if  $s = 1$  then the condition on  $\ell$  can be relaxed to  $\ell \geq 2$ . They conjecture that the condition  $\ell \geq 9s + 6$  can be relaxed to  $\ell \geq s + 1$  for general  $s$ . We now demonstrate how the sufficient condition given in the previous section could answer their question.

### Verifying the First Hypothesis

The following short list of facts will prove useful for verifying the first hypothesis of the sufficient condition for a non-jump. We assume, without further mentioning it, that  $x_i \geq 0$  for each  $i$  and that  $\sum_{i=1}^{\ell} x_i = 1$ . For shorthand throughout, we will write  $i < j$  to mean  $1 \leq i < j \leq \ell$  and addition of subscripts is always modulo  $\ell$ .

$$\text{I. } \sum_{i=1}^{\ell} x_i^2 = 1 - 2 \sum_{i < j} x_i x_j$$



II. For any  $a, b \geq 1$ ,

$$\begin{aligned} 4x_i(x_{i-a} + x_{i+b}) &\leq 4x_i^2 + (x_{i-a} + x_{i+b})^2 \\ &= 4x_i^2 + x_{i-a}^2 + x_{i+b}^2 + 2x_{i-a}x_{i+b} \end{aligned}$$

III. By summing the previous inequality over  $1 \leq i \leq \ell$  we obtain:

$$\begin{aligned} 4 \sum_{i=1}^{\ell} x_i(x_{i-a} + x_{i+b}) &\leq 6 \sum_{i=1}^{\ell} x_i^2 + 2 \sum_{i=1}^{\ell} x_{i-a}x_{i+b} \\ &= 6 \left( 1 - 2 \sum_{i < j} x_i x_j \right) + 2 \sum_{i=1}^{\ell} x_i x_{i+a+b} \end{aligned}$$

Note that  $\sum_{i=1}^{\ell} x_i x_{i-a} = \sum_{i=1}^{\ell} x_i x_{i+a}$  for any  $a$ .

IV. Fact (III) implies that

$$\epsilon \left( \sum_{i=1}^{\ell} x_i x_{i+a} + \sum_{i=1}^{\ell} x_i x_{i+b} \right) \leq \frac{3\epsilon}{2} \left( 1 - 2 \sum_{i < j} x_i x_j \right) + \frac{\epsilon}{2} \sum_{i=1}^{\ell} x_i x_{i+a+b}$$

and a special case occurs when  $a = b$ , in which case we have

$$\epsilon \sum_{i=1}^{\ell} x_i x_{i+a} \leq \frac{3\epsilon}{4} \left( 1 - 2 \sum_{i < j} x_i x_j \right) + \frac{\epsilon}{4} \sum_{i=1}^{\ell} x_i x_{i+2a}.$$

V.  $2 \sum_{i < j} x_i x_j \leq 1 - \frac{1}{\ell}.$

Note also that every inequality listed above achieves equality when each  $x_i = \frac{1}{\ell}$ .

We set

$$\begin{aligned} f_{\ell,s} &:= \sum_{i=1}^{\ell} \frac{\partial \lambda(P_{\ell,s}, x)}{\partial x_i} \\ &= 6(\ell - 2) \sum_{1 \leq i < j \leq \ell} x_i x_j + 6 \sum_{i=1}^{\ell} \sum_{j=1}^s x_i x_{i+j} + 3s \sum_{i=1}^{\ell} x_i^2 \\ &= 6(\ell - s - 2) \sum_{1 \leq i < j \leq \ell} x_i x_j + 6 \sum_{i=1}^{\ell} \sum_{j=1}^s x_i x_{i+j} + 3s. \end{aligned}$$

The first condition we must verify is that  $f_{\ell,s}(\vec{x}) \leq 3\ell\lambda(P_{\ell,s}) = 3\ell - 9 + \frac{9s+6}{\ell}$  for any  $\vec{x} \in S_{\ell}$ . Our first goal is to show that for any  $x \in S_{\ell}$  it follows that  $f_{\ell,s}(x) \leq 3\ell\lambda(P_{\ell,s})$ .

We will prove the following Lemma.

**Lemma 13.** *If  $\ell$  and  $s$  are positive integers satisfying any of the following, then  $f_{\ell,s}(\vec{x}) \leq 3\ell\lambda(P_{\ell,s})$  for any  $\vec{x} \in S_\ell$ .*

(i)  $\ell = 5$  and  $1 \leq s < 5$ ,

(ii)  $\ell = 7$  and  $s \in \{1, 2, 3, 4, 6\}$ ,

(iii)  $\ell = 4$  and  $s \in \{1, 3\}$ ,

(iv)  $\ell = 6$  and  $s \in \{1, 2, 3\}$ ,

(v)  $\ell = 2s + 1$ ,

(vi)  $\ell = 2s$  and  $s \geq 2$ ,

(vii)  $\ell = s + 1$ ,

(viii)  $\ell \geq 3s + 2$ .

We will handle the proofs of each of these cases separately.

**Proof of Lemma 13 case:  $\ell = 5, s = 1$**

Our goal is to show that  $f_{5,1}(\vec{x}) \leq 15\lambda(P_{5,1})$  for any  $\vec{x} \in S_5$ .

$$\begin{aligned}
f_{5,1}(\vec{x}) &:= 12 \sum_{1 \leq i < j \leq 5} x_i x_j + 6 \sum_{i=1}^5 x_i x_{i+1} + 3 \\
&= 12 \sum_{1 \leq i < j \leq 5} x_i x_j + (6 - \epsilon) \sum_{i=1}^5 x_i x_{i+1} + \epsilon \sum_{i=1}^5 x_i x_{i+1} + 3 \\
&\leq 12 \sum_{1 \leq i < j \leq 5} x_i x_j + (6 - \epsilon) \sum_{i=1}^5 x_i x_{i+1} + \frac{3\epsilon}{4} \left( 1 - 2 \sum_{1 \leq i < j \leq 5} x_i x_j \right) \\
&\quad + \frac{\epsilon}{4} \sum_{i=1}^5 x_i x_{i+2} + 3 \\
&\star = \left( 12 - \frac{3\epsilon}{2} \right) \sum_{1 \leq i < j \leq 5} x_i x_j + (6 - \epsilon) \sum_{i=1}^5 x_i x_{i+1} + \frac{\epsilon}{4} \sum_{i=1}^5 x_i x_{i+2} + \frac{3\epsilon}{4} + 3
\end{aligned}$$

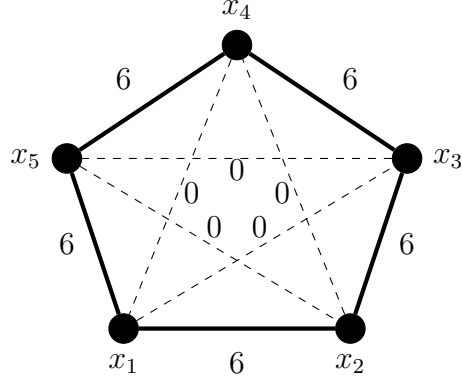


Figure 5.1 The initial edge weighting of  $K_5$ .

Now, we choose  $\epsilon = \frac{24}{5}$  so that  $6 - \epsilon = \frac{\epsilon}{4}$ .

$$\begin{aligned}
 \star &= \left(12 - \frac{36}{5}\right) \sum_{1 \leq i < j \leq 5} x_i x_j + \frac{6}{5} \sum_{i=1}^5 (x_i x_{i+1} + x_i x_{i+2}) + \frac{18}{5} + 3 \\
 &= \left(12 - \frac{36}{5} + \frac{6}{5}\right) \sum_{1 \leq i < j \leq 5} x_i x_j + \frac{33}{5} \\
 &\leq 3 \left(1 - \frac{1}{5}\right) + \frac{33}{5} \\
 &= 9 \\
 &= 15\lambda(P_{5,1}).
 \end{aligned}$$

The term that causes the difficulty is  $6 \sum_{i=1}^{\ell} \sum_{j=1}^s x_i x_{i+j}$ . There is a nice way to visualize the process that occurred in the previous string of inequalities. We began with  $6 \sum_{i=1}^5 x_i x_{i+1}$  which can be represented by the illustration in Figure 5.1. The labels on the edges represent the coefficient in front of the sum.

We then borrowed some of the weight of the  $x_i x_{i+1}$  edges, and shifted it to the  $x_i x_{i+2}$  edges resulting in the weighting depicted in Figure 5.2.

We then engineered  $\epsilon$  so that our weighting was equitable,  $\epsilon = \frac{24}{5}$ , giving us the weighting in Figure 5.3:

Having an equitably weighted complete graph was the goal since it allows us to apply fact V, namely that  $\sum_{1 \leq i < j \leq \ell} x_i x_j \leq \frac{1}{2} \left(1 - \frac{1}{\ell}\right)$ .

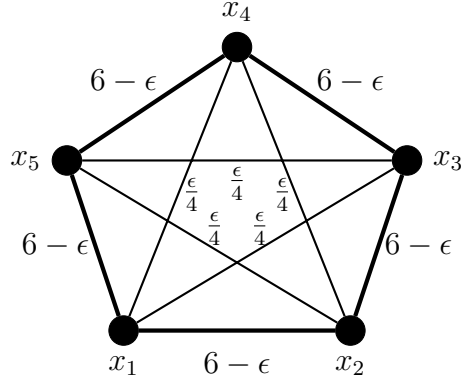


Figure 5.2 Edge weight of  $K_5$  after shifting.

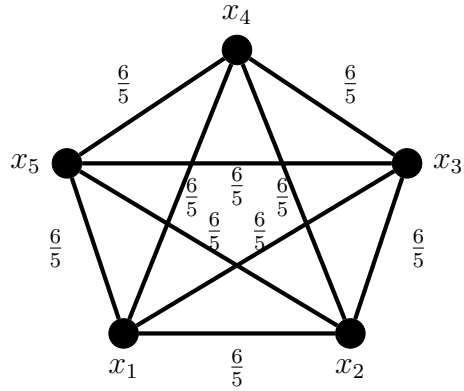


Figure 5.3 Final *equitable* weighting of  $K_5$ .

**Proof of Lemma 13 case:  $\ell = 7, s = 1$**

Our goal is to show that  $f_{7,1}(\vec{x}) \leq 21\lambda(P_{7,1})$  for any  $\vec{x} \in S_7$ . This case will be somewhat different from previous cases in that we will apply fact (IV) twice. With the benefit of hindsight, we will eventually set  $\epsilon_1 = \frac{40}{7}$  and  $\epsilon_2 = \frac{8}{7}$ . When these values of  $\epsilon$  first appear, they will be left in generic terms.

$$\begin{aligned}
f_{7,1}(\vec{x}) &:= 6(7-1-2) \sum_{i<j} x_i x_j + 6 \sum_{i=1}^7 x_i x_{i+1} + 3 \\
&= 24 \sum_{i<j} x_i x_j + (6-\epsilon_1) \sum_{i=1}^7 x_i x_{i+1} + \epsilon_1 \sum_{i=1}^7 x_i x_{i+1} + 3 \\
&\leq 24 \sum_{i<j} x_i x_j + (6-\epsilon_1) \sum_{i=1}^7 x_i x_{i+1} + \frac{3\epsilon_1}{4} \left(1 - 2 \sum_{i<j} x_i x_j\right) + \frac{\epsilon_1}{4} \sum_{i=1}^7 x_i x_{i+2} + 3 \\
&= \left(24 - \frac{3\epsilon_1}{2}\right) \sum_{i<j} x_i x_j + (6-\epsilon_1) \sum_{i=1}^7 x_i x_{i+1} + \frac{\epsilon_1}{4} \sum_{i=1}^7 x_i x_{i+2} + 3 + \frac{3\epsilon_1}{4} \\
&= \left(24 - \frac{3\epsilon_1}{2}\right) \sum_{i<j} x_i x_j + (6-\epsilon_1) \sum_{i=1}^7 x_i x_{i+1} + \left(\frac{\epsilon_1}{4} - \epsilon_2\right) \sum_{i=1}^7 x_i x_{i+2} \\
&\quad + \epsilon_2 \sum_{i=1}^7 x_i x_{i+2} + 3 + \frac{3\epsilon_1}{4} \\
&\leq \left(24 - \frac{3\epsilon_1}{2} - \frac{3\epsilon_2}{2}\right) \sum_{i<j} x_i x_j + (6-\epsilon_1) \sum_{i=1}^7 x_i x_{i+1} + \left(\frac{\epsilon_1}{4} - \epsilon_2\right) \sum_{i=1}^7 x_i x_{i+2} \\
&\quad + \frac{\epsilon_2}{4} \sum_{i=1}^7 x_i x_{i+4} + 3 + \frac{3\epsilon_1}{4} + \frac{3\epsilon_2}{4} \\
&= \frac{96}{7} \sum_{i<j} x_i x_j + \frac{2}{7} \sum_{i=1}^7 x_i x_{i+1} + \frac{2}{7} \sum_{i=1}^7 x_i x_{i+2} + \frac{2}{7} \sum_{i=1}^7 x_i x_{i+4} + \frac{57}{7} \\
&= \frac{98}{7} \sum_{i<j} x_i x_j + \frac{57}{7} \\
&\leq 7 \left(1 - \frac{1}{7}\right) + \frac{57}{7} \\
&= \frac{99}{7} \\
&= 21\lambda(P_{7,1}).
\end{aligned}$$

**Proof of Lemma 13 case:**  $\ell = 7, s = 2$

Our goal is to show that  $f_{7,2}(\vec{x}) \leq 21\lambda(P_{7,2})$  for any  $\vec{x} \in S_7$ . Again, we will apply fact (IV) twice. With the benefit of hindsight, we will chose  $\epsilon_1 = \frac{32}{7}$  and  $\epsilon_2 = \frac{40}{7}$ . We

have the following:

$$\begin{aligned}
f_{7,2}(\vec{x}) &:= 6(7-2-2) \sum_{i<j} x_i x_j + 6 \sum_{i=1}^7 \sum_{j=1}^2 x_i x_{i+j} + 6 \\
&= 18 \sum_{i<j} x_i x_j + 6 \sum_{i=1}^7 x_i x_{i+1} + 6 \sum_{i=1}^7 x_i x_{i+2} + 6 \\
&\leq 18 \sum_{i<j} x_i x_j + (6 - \epsilon_1) \sum_{i=1}^7 x_i x_{i+1} + \left(6 + \frac{\epsilon_1}{4}\right) \sum_{i=1}^7 x_i x_{i+2} \\
&\quad + \frac{3\epsilon_1}{4} \left(1 - 2 \sum_{i<j} x_i x_j\right) + 6 \\
&\leq 18 \sum_{i<j} x_i x_j + (6 - \epsilon_1) \sum_{i=1}^7 x_i x_{i+1} + \left(6 + \frac{\epsilon_1}{4} - \epsilon_2\right) \sum_{i=1}^7 x_i x_{i+2} + \frac{\epsilon_2}{4} \sum_{i=1}^7 x_i x_{i+4} \\
&\quad + \left(\frac{3\epsilon_1}{4} + \frac{3\epsilon_2}{4}\right) \left(1 - 2 \sum_{i<j} x_i x_j\right) + 6
\end{aligned}$$

Now, by our choice of  $\epsilon_1$  and  $\epsilon_2$  we have that  $6 - \epsilon_1 = 6 + \frac{\epsilon_1}{4} - \epsilon_2 = \frac{\epsilon_2}{4} = \frac{10}{7}$ . And,  $\frac{3}{4}(\epsilon_1 + \epsilon_2) = \frac{54}{7}$ . This gives us:

$$\begin{aligned}
f_{7,2}(\vec{x}) &\leq \left(18 + \frac{10}{7} - \frac{108}{7}\right) \sum_{i<j} x_i x_j + \frac{54}{7} + 6 \\
&= 4 \sum_{i<j} x_i x_j + \frac{96}{7} \\
&\leq 2 \left(1 - \frac{1}{7}\right) + \frac{96}{7} \\
&= \frac{108}{7} \\
&= 21\lambda(P_{7,2}).
\end{aligned}$$

### Proof of Lemma 13 case: $\ell = 2s + 1$

Our goal is to show that  $f_{2s+1,s}(\vec{x}) \leq 3(2s+1)\lambda(P_{2s+1,s})$  for any  $\vec{x} \in S_{2s+1}$ . We note that in the case when  $\ell = 2s + 1$  it follows that  $\sum_{i=1}^{\ell} \sum_{j=1}^s x_i x_{i+j} = \sum_{i<j} x_i x_j$ . It follows

that

$$\begin{aligned}
f_{2s+1,s}(\vec{x}) &:= 6((2s+1) - s - 2) \sum_{i<j} x_i x_j + 6 \sum_{i=1}^{2s+1} \sum_{j=1}^s x_i x_{i+j} + 3s \\
&= 6(s-1) \sum_{i<j} x_i x_j + 6 \sum_{i<j} x_i x_j + 3s \\
&= 6s \sum_{i<j} x_i x_j + 3s \\
&\leq 3s \left(1 - \frac{1}{2s+1}\right) + 3s \\
&= 3(2s+1)\lambda(P_{2s+1,s}).
\end{aligned}$$

Note that this case takes care of the  $\ell = 5, s = 2$  case and the  $\ell = 7, s = 3$  case.

### **Proof of Lemma 13 case: $\ell = 7, s = 4$**

Our goal is to show that  $f_{7,4}(\vec{x}) \leq 21\lambda(P_{7,4})$  for any  $\vec{x} \in S_7$ . This case is very similar to the case where  $\ell = 7$  and  $s = 1$ . We will choose  $\epsilon_1$  and  $\epsilon_2$  the same way as that case. We note two quick facts (that transform this case into essentially that one).

We note:

$$\sum_{i=1}^7 x_i(x_{i+1} + x_{i+2} + x_{i+4}) = \sum_{i<j} x_i x_j$$

and

$$\sum_{i=1}^7 x_i(x_{i+3} + x_{i+2(3)} + x_{i+4(3)}) = \sum_{i<j} x_i x_j.$$

Putting these together, we have the following:

$$\begin{aligned}
f_{7,4}(\vec{x}) &:= 6(7-4-2) \sum_{i<j} x_i x_j + 6 \sum_{i=1}^7 x_i (x_{i+1} + x_{i+2} + x_{i+4}) + 6 \sum_{i=1}^7 x_i x_{i+3} + 12 \\
&= 12 \sum_{i<j} x_i x_j + 6 \sum_{i=1}^7 x_i x_{i+3} + 12 \\
&\leq \left(12 - \frac{3\epsilon_1}{2}\right) \sum_{i<j} x_i x_j + (6 - \epsilon_1) \sum_{i=1}^7 x_i x_{i+3} + \frac{\epsilon_1}{4} \sum_{i=1}^7 x_i x_{i+2(3)} + \frac{3\epsilon_1}{4} + 12 \\
&\leq \left(12 - \frac{3\epsilon_1}{2} - \frac{3\epsilon_2}{2}\right) \sum_{i<j} x_i x_j + (6 - \epsilon_1) \sum_{i=1}^7 x_i x_{i+3} + \left(\frac{\epsilon_1}{4} - \epsilon_2\right) \sum_{i=1}^7 x_i x_{i+2(3)} \\
&\quad + \frac{\epsilon_2}{4} \sum_{i=1}^7 x_i x_{i+4(3)} + \frac{3(\epsilon_1 + \epsilon_2)}{4} + 12 \\
&= \left(12 - \frac{60}{7} - \frac{12}{7} + \frac{2}{7}\right) \sum_{i<j} x_i x_j + 12 + \frac{30}{7} + \frac{6}{7} \\
&= 2 \sum_{i<j} x_i x_j + \frac{120}{7} \\
&\leq \left(1 - \frac{1}{7}\right) + \frac{120}{7} \\
&= 18 \\
&= 21\lambda(P_{7,4}).
\end{aligned}$$

### Proof of Lemma 13 case: $\ell = 4, s = 1$

Our goal is to show that  $f_{4,1}(\vec{x}) \leq 12\lambda(P_{4,1})$  for any  $\vec{x} \in S_4$ . This case is slightly different in the sense that  $\sum_{i=1}^4 x_i x_{i+2}$  double counts some edges. In particular, note that  $2 \sum_{i<j} x_i x_j = 2 \sum_{i=1}^4 x_i x_{i+1} + \sum_{i=1}^4 x_i x_{i+2}$ . When we borrow  $\epsilon$  from the  $\sum_{i=1}^4 x_i x_{i+1}$  coefficient to generate  $\frac{\epsilon}{4}$  as the coefficient of  $\sum_{i=1}^4 x_i x_{i+2}$  our goal is not that  $6 - \epsilon = \frac{\epsilon}{4}$ ,



rather we need  $6 - \epsilon = 2 \left( \frac{\epsilon}{4} \right)$ . With this in mind, we will choose  $\epsilon = 4$ .

$$\begin{aligned}
f_{4,1}(\vec{x}) &:= 6(4 - 1 - 2) \sum_{i < j} x_i x_j + 6 \sum_{i=1}^4 x_i x_{i+1} + 3 \\
&\leq 6 \sum_{i < j} x_i x_j + (6 - \epsilon) \sum_{i=1}^4 x_i x_{i+1} + \frac{\epsilon}{4} \sum_{i=1}^4 x_i x_{i+2} + \frac{3\epsilon}{4} \left( 1 - 2 \sum_{i < j} x_i x_j \right) + 3 \\
&= 2 \sum_{i < j} x_i x_j + 6 \\
&\leq \left( 1 - \frac{1}{4} \right) + 6 \\
&= \frac{27}{4} \\
&= 12\lambda(P_{4,1}).
\end{aligned}$$

**Proof of Lemma 13 case:**  $\ell = 6, s = 2$

Our goal is to show that  $f_{6,2}(\vec{x}) \leq 18\lambda(P_{6,2})$  for any  $\vec{x} \in S_6$ . This case is similar to the previous in the sense that, because  $\ell$  is even there is eventual double-counting of terms. This means that  $\sum_{i < j} x_i x_j = \sum_{i=1}^6 x_i(x_{i+1} + x_{i+2}) + \frac{1}{2} \sum_{i=1}^6 x_i x_{i+3}$ . The difference is that for the first time, we will use the first inequality of fact (IV) instead of the second inequality (as we have always previously used). In this case we will want  $6 - \epsilon = 2 \left( \frac{\epsilon}{2} \right)$  so we will choose  $\epsilon = 3$ .

$$\begin{aligned}
f_{6,2}(\vec{x}) &:= 6(6-2-2) \sum_{i<j} x_i x_j + 6 \sum_{i=1}^6 x_i(x_{i+1} + x_{i+2}) + 6 \\
&\leq 12 \sum_{i<j} x_i x_j + (6-\epsilon) \sum_{i=1}^6 x_i(x_{i+1} + x_{i+2}) + \frac{\epsilon}{2} \sum_{i=1}^6 x_i x_{i+3} \\
&\quad + \frac{3\epsilon}{2} \left( 1 - 2 \sum_{i<j} x_i x_j \right) + 6 \\
&= 12 \sum_{i<j} x_i x_j + 3 \sum_{i<j} x_i x_j + \frac{9}{2} - 9 \sum_{i<j} x_i x_j + 6 \\
&= 6 \sum_{i<j} x_i x_j + \frac{21}{2} \\
&\leq 3 \left( 1 - \frac{1}{6} \right) + \frac{21}{2} \\
&= 18\lambda(P_{6,2}).
\end{aligned}$$

**Proof of Lemma 13 case:**  $\ell = 2s$ ,  $s \geq 2$

Our goal is to show that  $f_{2s,s}(\vec{x}) \leq 6s\lambda(P_{2s,s})$  for any  $\vec{x} \in S_{2s}$ . We begin by noting that  $\sum_{i=1}^{2s} \sum_{j=1}^s x_i x_{i+j} = \sum_{i<j} x_i x_j + \frac{1}{2} \sum_{i=1}^{2s} x_i x_{i+s}$ . We have that

$$\begin{aligned}
f_{2s,s}(\vec{x}) &:= 6(2s-s-2) \sum_{i<j} x_i x_j + 6 \sum_{i=1}^{2s} \sum_{j=1}^s x_i x_{i+j} + 3s \\
&= 6(s-2) \sum_{i<j} x_i x_j + 6 \sum_{i<j} x_i x_j + 3 \sum_{i=1}^{2s} x_i x_{i+s} + 3s \\
&\leq 6(s-1) \sum_{i<j} x_i x_j + \frac{9}{4} \left( 1 - 2 \sum_{i<j} x_i x_j \right) + \frac{3}{4} \sum_{i=1}^{2s} x_i x_{i+2s} + 3s \\
&= 6 \left( s - 1 - \frac{3}{4} \right) \sum_{i<j} x_i x_j + \frac{3}{4} \sum_{i=1}^{2s} x_i^2 + \frac{9}{4} + 3s \\
&= 6 \left( s - 1 - \frac{3}{4} \right) \sum_{i<j} x_i x_j + \frac{3}{4} \left( 1 - 2 \sum_{i<j} x_i x_j \right) + \frac{9}{4} + 3s \\
&= 6(s-2) \sum_{i<j} x_i x_j + 3 + 3s \\
&\leq 3(s-2) \left( 1 - \frac{1}{2s} \right) + 3 + 3s \\
&= 6s\lambda(P_{2s,s}).
\end{aligned}$$

Note that the final inequality only holds when  $6(s-2) \geq 0$ , hence the condition that  $s \geq 2$ .

**Proof of Lemma 13 case:**  $\ell = s + 1$

Our goal is to show that  $f_{s+1,s}(\vec{x}) \leq 3(s+1)\lambda(P_{s+1,s})$  for any  $\vec{x} \in S_{s+1}$ . This case is also special in the sense that  $\sum_{i=1}^{\ell} \sum_{j=1}^s x_i x_{i+j} = 2 \sum_{i < j} x_i x_j$ .

$$\begin{aligned}
 f_{\ell,\ell-1}(\vec{x}) &:= 6(\ell - (\ell - 1) - 2) \sum_{1 \leq i < j \leq \ell} x_i x_j + 6 \sum_{i=1}^{\ell} \sum_{j=1}^{\ell-1} x_i x_{i+j} + 3(\ell - 1) \\
 &= 6 \sum_{1 \leq i < j \leq \ell} x_i x_j + 3\ell - 3 \\
 &\leq 3 \left(1 - \frac{1}{\ell}\right) + 3\ell - 3 \\
 &= 3\ell\lambda(P_{\ell,\ell-1}).
 \end{aligned}$$

Note that this case takes care of the cases:  $\ell = 4$  and  $s = 3$ ,  $\ell = 5$  and  $s = 4$ , and the case  $\ell = 7$  and  $s = 6$ .

## Proof of Lemma 13 case: $\ell \geq 3s + 2$

In this final case, our goal is to show that  $f_{\ell,s}(\vec{x}) \geq 3\ell\lambda(P_{\ell,s})$  for any  $\vec{x} \in S_\ell$  provided that  $\ell \geq 3s + 2$ . For this case, we apply a variation of fact II, namely  $2x_i x_j \leq x_i^2 + x_j^2$ .

$$\begin{aligned}
f_{\ell,s}(\vec{x}) &:= 6(\ell - s - 2) \sum_{i < j} x_i x_j + 6 \sum_{i=1}^{\ell} \sum_{j=1}^s x_i x_{i+j} + 3s \\
&\leq 6(\ell - s - 2) \sum_{i < j} x_i x_j + 3 \sum_{i=1}^{\ell} \sum_{j=1}^s (x_i^2 + x_{i+j}^2) + 3s \\
&= 6(\ell - s - 2) \sum_{i < j} x_i x_j + 6s \sum_{i=1}^{\ell} x_i^2 + 3s \\
&= 6(\ell - s - 2) \sum_{i < j} x_i x_j + 6s \left( 1 - 2 \sum_{i < j} x_i x_j \right) + 3s \\
&= 6(\ell - 3s - 2) \sum_{i < j} x_i x_j + 9s \\
&\leq 3(\ell - 3s - 2) \left( 1 - \frac{1}{\ell} \right) + 9s \\
&= 3\ell\lambda(P_{\ell,s}).
\end{aligned}$$

Note that the final inequality only holds when  $\ell - 3s - 2 \geq 0$ , i.e. when  $\ell \geq 3s + 2$ .

## Verifying the Second Hypothesis

We have verified that the pattern given by Frankl et. al satisfies the first hypothesis of our theorem for many values of  $\ell$  and  $s$ . Because of the symmetry of these particular patterns, it suffices to verify the following:

$$\max \left\{ \lambda(P_{\ell,s}, \vec{x}) + \frac{2}{9} \sum_{i=1}^{\ell} x_i^3 \mid x_1 = 0, \vec{x} \in S_\ell \right\} \leq \lambda(P_{\ell,s}).$$

This hypothesis appears to be harder than the first to verify in general. We are, however, able to compute maximum values for specific values of  $\ell$  and  $s$  using software packages such as Maple. Table 5.1 contains the computed maximum values, the specified values of  $\ell$  and  $s$ .

These computations imply the Theorem 23.

Table 5.1 Verifying the sufficient condition for non-jump values.

$\ell$	$s$	Max LHS	$\lambda(P_{\ell,s})$	Hyp. Satisfied?
4	1	.52352343...	.5625	Yes
4	2	.80555555...	.75	<b>No</b>
4	3	.91358024...	.9375	Yes
5	1	.54900696...	.6	Yes
5	2	.67892380...	.72	Yes
5	3	.85007051...	.84	<b>No</b>
5	4	.95138888...	.96	Yes
7	1	.63483290...	.67346938...	Yes
7	2	.70598310...	.73469387...	Yes
7	3	.77230982...	.79591836...	Yes
7	4	.85214229...	.85714285...	Yes
7	5	.92760072...	.91836734...	<b>No</b>
7	6	.97839506...	.97959183...	Yes

**Theorem 23.** *If  $\alpha \in \left\{ \frac{9}{16}, \frac{15}{16}, \frac{3}{5}, \frac{18}{25}, \frac{24}{25}, \frac{33}{49}, \frac{36}{49}, \frac{39}{49}, \frac{42}{49}, \frac{48}{49} \right\}$  then  $\alpha$  is a non-jump for  $r = 3$ .*

## CHAPTER 6

### A FEW OPEN QUESTIONS

In this concluding chapter, we present a few of the questions whose solutions are the most sought after and which remain open.

**Question 1:** For any  $n > r \geq 3$  what is  $\pi(K_n^r)$ ?

Currently,  $\pi(K_n^r)$  is unknown for any case when  $n > r \geq 3$ . The smallest, and most studied case is for  $n = 4$  and  $r = 3$ . Turán conjectured that  $\pi(K_4^3) = 5/9$  giving a constructive lower bound. The current best known upperbound is approximately .56 and is due to Razborov [60]. One could ask a related question for non-uniform hypergraphs.

**Question 2:** For any  $n > \max\{r : r \in R\} \geq 3$  what is  $\pi(K_n^R)$ ?

In Chapter 3, we mentioned two open questions. They are the following:

**Question 3:** Let  $\mathcal{D}_2$  be the diamond poset. Is  $\pi(\mathcal{D}_2) = 2$ ?

**Question 4:** Does the limit  $\lim_{n \rightarrow \infty} \frac{\text{La}(n, \mathcal{P})}{\binom{n}{\lfloor n/2 \rfloor}}$  exist for any poset  $\mathcal{P}$ ? Is it an integer?

Finally, related to hypergraph jumps, the following questions are active areas of research.

**Question 5:** For  $r \geq 3$  and  $\alpha \in [0, 1)$  is  $\alpha$  a jump or non-jump for  $r$ ?

**Question 6:** For  $r \geq 3$  what is the smallest value of  $\alpha$  such that  $\alpha$  is not a jump for  $r$ ? In particular, is  $\frac{r!}{r^r}$  a jump (or non-jump) for  $r$ ?

Finally, we note that it is likely that all of these questions are quite hard. The questions are interesting (and challenging) enough that Erdős has offered a cash prize for several. He offered \$500 for determining the Turán density of any one complete graph described in Question 1 (twice as much money for a complete solution) and he

has offered \$250 for determining whether or not  $\frac{2}{9}$  is a jump for  $r = 3$ .

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