Independence Polynomials

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Independence Polynomials

by

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Abstract

In this thesis, we investigate the independence polynomial of a simple graph $G$. In addition to giving several tools for computing these polynomials and giving closed-form representations of these polynomials for common classes of graphs, we prove two results concerning the roots of independence polynomials. The first result gives us the unique root of smallest modulus of the independence polynomial of any graph. The second result tells us that all the roots of the independence polynomial of a claw-free graph fall on the real line.
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Chapter 1

Definitions

Definition 1.1. A simple graph is a pair $G = (V, E)$ where $V$ is a set of elements called vertices, and $E$ is a set of elements called edges which are unordered pairs of vertices from $V$.

![Figure 1.1 A simple graph with 5 vertices and 6 edges.](image)

The figure above gives an example of a simple graph with vertex set $V = \{1, 2, 3, 4, 5\}$ and edge set $E = \{(1, 2), (1, 3), (1, 5), (2, 5), (3, 5), (4, 5)\}$. In this thesis we only consider simple graphs, and so we will refer to them only as graphs. If there are multiple graphs in consideration, we use the notation $V(G)$ and $E(G)$ to designate the vertex set and edge set of a graph $G$ respectively.

Definition 1.2. Let $G = (V, E)$, and $u, v \in V$. The vertices $u$ and $v$ are said to be adjacent if there is an edge between $u$ and $v$, i.e. $(u, v) \in E$.

Definition 1.3. An edge $e = (x, y)$ is incident to a vertex $v$ if $v = x$ or $v = y$. 
The above two definitions establish some terminology for describing the structure of graph with respect to how vertices and edges relate to one another. In the figure above, we can see that the vertex 1 is adjacent to the vertex 3. Conversely, we can see that the edge (1, 3) is incident to vertices 1 and 3.

**Definition 1.4.** A subset $X$ of vertices is called *independent* if the vertices in $X$ are pairwise non-adjacent.

In the above example, the subset of vertices $X = \{2, 3, 4\}$ is an independent set of cardinality of 3. Furthermore, one can check that there is no independent set of cardinality 4. So $|X|$ is maximum. The following definition provides some notation for this value.

**Definition 1.5.** The *independence number* of a graph $G$ is the maximum cardinality of an independent set in $G$. We denote this value as $\alpha(G)$.

Since the maximum cardinality of an independent set in our example is 3, we say that its independence number is 3. When working with independent sets it is necessary to consider the neighborhoods of the vertices in the graph, defined as follows.

**Definition 1.6.** Let $v$ be a vertex. The *open neighborhood*, or just *neighborhood*, of $v$ is defined to be the set of all vertices adjacent to $v$. We denote it as follows

$$N(v) := \{u \in V | (u, v) \in E\}.$$ 

The *closed neighborhood* of $v$ is defined to be $N[v] := N(v) \cup \{v\}$.

It is necessary to consider these sets since if one vertex is in the neighborhood of another, those two vertices cannot appear in an independent set together. Another type of set related to independent sets is called a *clique*. A clique is a set of vertices in which every vertex is adjacent to every other vertex.
One important family of graphs is the family of *empty graphs*. An empty graph is simply a graph with no edges. In this family, the neighborhood of every vertex is empty. We denote the empty graph on \( n \) vertices by \( E_n \). In the special case where \( n = 0 \), we call this graph the *null graph* and denote \( \emptyset := E_0 \). This family is important since, as we will see later, the graphs in this family serve as a basis for computing independence polynomials.

The independence polynomial of a graph \( G \) is the polynomial whose coefficient on \( x^k \) is given by the number of independent sets of order \( k \) in \( G \). We denote this polynomial \( I(G; x) \). So,

\[
I(G; x) = \sum_{k=0}^{\alpha(G)} c_k x^k
\]

where \( c_k \) is the number of independent sets of order \( k \) in \( G \). This definition of the independence polynomial is the one usually found in the literature. However, it is sometimes more convenient to work with a version of the polynomial given as an alternating series as we see in chapter 4. The name assigned to this polynomial also differs between sources and can be referred to as the independent set polynomial or stable set polynomial. In order to get a feel for the definition, we consider the independence polynomial of the null graph, \( I(\emptyset, x) \). Since the null graph does not have any vertices, \( c_i = 0 \) for all \( i \geq 1 \). We get that \( c_0 = 1 \) since every graph has a unique subset of cardinality 0, the emptyset. So we have \( I(\emptyset, x) = 1 \). We will make heavy use of this fact in the computation of other graphs. The next lemma gives the independence polynomial of the single vertex graph known as the *singleton*.

**Lemma 1.7.** The independence polynomial of a singleton is given by \( 1 + x \).

**Proof.** Let \( P(x) = \sum_{k=0}^{\alpha(G)} c_k x^k \) be the independence polynomial of the singleton. Since the singleton only has a single vertex, \( \alpha(G) \leq 1 \). So we must only find \( c_0 \) and \( c_1 \). All graphs have exactly one independent set of cardinality 0, the empty set, making
\( c_0 = 1 \). Since the singleton only has one vertex, there is only one independent set of cardinality 1, graph itself, making \( c_1 = 1 \). Thus, \( P(x) = c_0 x^0 + c_1 x^1 = 1 + x \). 

With this, we handle the independence polynomials of the most trivial graphs. In the next chapter, we build up tools for calculating the polynomial for graphs in general.
Chapter 2

Computing the Independence Polynomial

In order to effectively compute the independence polynomial of a graph, we need to build some tools to reduce the calculations to recursively smaller graphs. The following results establish 3 tools which, when used in concert, provide an algorithm for computing the independence polynomial of finite simple graphs. The goal of creating such tools is to be able to find recurrence relations between the independence polynomials within a given family of graphs. The first tool we build relates the independence polynomials of disjoint graphs.

Theorem 2.1. Let $G_1$ and $G_2$ be two vertex disjoint graphs. Then $I(G_1 \cup G_2; x) = I(G_1; x) \cdot I(G_2; x)$.

Proof. Let $G_1$ and $G_2$ be vertex disjoint simple graphs. An independent set of cardinality $k$ in $G_1 \cup G_2$ is obtained by taking an independent set of cardinality $i$ from $G_1$ and an independent set of cardinality $j$ where $i + j = k$. Denote the number of independent sets of cardinality $k$ in $G_1$ by $a_k$ and similarly the independent sets of cardinality $k$ in $G_2$ by $b_k$. Then the coefficient on $x^k$ in $I(G_1 \cup G_2; x)$, call it $c_k$, is given by $\sum_{m=0}^{k} a_m b_{k-m}$. So we have that
\[ I(G_1 \cup G_2; x) = \sum_{k=0}^{\alpha(G_1)+\alpha(G_2)} \sum_{m=0}^{k} a_m b_{k-m} x^k \]
\[ = \left( \sum_{k=0}^{\alpha(G_1)} a_k x^k \right) \left( \sum_{k=0}^{\alpha(G_2)} b_k x^k \right) \]
\[ = I(G_1; x) I(G_2; x). \]

With this, we have reached the desired conclusion. \qed

The next result is the most useful of the three. It provides a recurrence which allows us to decompose the independence polynomial of a graph vertex by vertex. That is, it relates the independence polynomial of \( G = (V, E) \) to the independence polynomial of \( G - v := (V - v, E - \{(u, v) | u \in N(v)\}) \) for some vertex \( v \in V(G) \). In general, if \( U \subseteq V, G - U := (V - U, E - \cup_{v \in U}\{(u, v) | u \in N(v)\}) \).

**Theorem 2.2.** Let \( G \) be a simple graph and \( v \in V(G) \). Then \( I(G; x) = I(G - v; x) + xI(G - N[v]; x) \).

**Proof.** Let \( G \) be a simple graph and \( v \in V(G) \). We separate the independent sets of \( G \) into two sets. In the first set, we have independent sets which do not include \( v \). In the second set, we have independent sets which include \( v \). Clearly these two sets form a partition over all the independent sets in \( G \). The polynomial \( I(G - v; x) \) counts the independent sets without \( v \). The polynomial \( I(G - N[v]; x) \) counts the independent sets which do not include \( v \) or any of its neighbors. In order to recover the independent sets which include \( v \), we take independent sets in \( G - N[v] \) and add in \( v \). When we add in \( v \), we get back independent sets because \( G - N[v] \) does not include any neighbors of \( v \). Then we have that \( xI(G - N[v]; x) \) counts the independent sets which include \( v \). We multiply by \( x \) since the addition of \( v \) increases the cardinality of each set by 1. Now that we have expressions which count the independent sets with and without \( v \), we can conclude that \( I(G; x) = I(G - v; x) + xI(G - N[v]; x) \). \qed
**Corollary 2.3.** Let $K$ be a clique of a graph $G$. Then

$$I(G; x) = I(G - K; x) + \sum_{v \in K} xI(G - N[v]; x).$$

*Proof.* We prove this by induction on $|K|$. In the base case, $|K| = 1$, and the problem reduces to theorem 2.2. Assume the result holds for $|K| = n$. Let $K$ be a clique with $|K| = n + 1$ and let $v \in K$. Since $K$ is a clique, $K - v$ is also a clique. Then by the inductive hypothesis

$$I(G) = I(G - (K - v); x) + \sum_{u \in K - v} xI(G - N[u]; x).$$

By theorem 2.2,

$$I(G - (K - v); x) = I(G - K; x) + xI(G - (K - v) - N[v]; x)$$
$$= I(G - K; x) + xI(G - N[v]; x).$$

Combining these two we get

$$I(G) = I(G - K; x) + \sum_{u \in K} xI(G - N[u]; x).$$

This completes the induction. $\square$

When used together, theorem 2.1 and theorem 2.2 are enough to compute the independence polynomial of a graph $G$. Removing a vertex from $G$, as is necessary in the above result, can cause $G$ to be separated into two connected components. At this point, we must use theorem 2.1 and take the product of the independence polynomials of the connected components. It is, however, not always more advantageous to decompose $G$ by removing vertices. There are some classes of graphs for which decomposing $G$ be removing edges is better and yields a more obvious recurrence.
Our third result gives us a tool to do just that. First we introduce notation for removing edges from a graph. We define \( G \setminus e := (V, E - e) \) to be the graph obtained by removing some edge \( e \) from the edgeset of \( G \).

**Theorem 2.4.** Let \( G \) be a graph and \( e = (u, v) \in E(G) \). Then \( I(G; x) = I(G \setminus e; x) - x^2 I(G - (N[u] \cup N[v]); x) \).

**Proof.** Let \( G \) be a graph and \( e = (u, v) \in E(G) \). The polynomial \( I(G \setminus e; x) \) counts every independent set in \( G \) as well as new independent sets which include both vertices \( u \) and \( v \). So, we must adjust \( I(G \setminus e; x) \) by removing these sets from the count. The polynomial \( I(G - (N[u] \cup N[v]); x) \) counts all the independent sets which do not include \( u, v \), or any of their neighbors. Then, if we add back in \( u \) and \( v \), we get all the independent sets which involve both \( u \) and \( v \). These sets are counted by the polynomial \( x^2 I(G - (N[u] \cup N[v]); x) \). Then we have that \( I(G; x) = I(G \setminus e; x) - x^2 I(G - (N[u] \cup N[v])) \) as desired. \( \square \)

As mentioned above, employing these tools together gives us a method for computing the independence polynomials of finite graphs recursively. In order for this recursion to stop, we must eventually reach graphs which have known independence polynomials. To this end, we prove a short lemma concerning the independence polynomial of empty graphs.

**Lemma 2.5.** The independence polynomial of an empty graph \( G \) of order \( n \) is given by \( I(G; x) = (1 + x)^n \).

**Proof.** Let \( G \) be an empty graph of order \( n \), \( E_n \). Then \( G \) is the union of \( n \) disjoint singleton graphs. Now, if we apply induction on \( n \) we can use theorem 2.1 and lemma 1.7 to reach the desired result. \( \square \)

Using the above lemma in combination with theorem 2.4 gives us a recursive algorithm for computing the independence polynomial of a graph by removing only edges.
The previous lemma ensures that this algorithm does indeed terminate since recursively removing edges will eventually lead to empty graphs. The lemma also leads to an early termination of the algorithm described using theorem 2.2 and theorem 2.1.

A Visual Representation

When computing the independence polynomial of a graph using the above method, it can be very easy to get lost in the notation. After only one or two levels of recursion, it becomes difficult to readily distinguish one subgraph from another when using a hand-written, text-based notation. To address this problem, we use a visual aid in the form of a rooted tree of subgraphs. At the root of the tree, we have a node which is the original graph whose independence polynomial we are trying to calculate. On the next level of the tree, we introduce two nodes. The first node represents the first term in the sum from theorem 2.2, and the second node represents the second term in the sum. So in the first node, we place the subgraph $G - v$, and in the second node we place the subgraph $G - N[v]$. Since the second term in the sum comes with an additional factor of $x$, we must provide some notation for this in our tree. To accommodate the extra factor, we place an $x$ along the edge connecting the node for $G$ and the node for $G - N[v]$. Continue level by level, applying the process to each non-empty graph within the same level until each leaf of the tree is an empty graph. As an example, we use this method to compute the independence polynomial of the random graph shown in Figure 1.1.

We can read off an unsimplified expression of the independence polynomial from the tree above. To do so, we look at the leaves (the empty graphs). Each of these leaves is an empty graph, so we can use lemma 2.5 to express their independence polynomials individually. For every leaf, we trace the path from the root node down to the leaf and count how many factors of $x$ we accumulate along the way. We then multiply the independence polynomial of the empty graph in the leaf node by the
appropriate power of $x$ and add the resulting expression to a running sum. Once we have visited every leaf, the final sum gives us the independence polynomial of the original graph. In the above example, the independence polynomial is given by $(1 + x)^3 + x(1 + x) + x = x^3 + 4x^2 + 5x + 1$.

This visual representation can also be easily augmented to work with theorem 2.4 as well by appropriately changing which subgraphs get placed in the child nodes and using $-x^2$ instead of $x$.

In addition to the independence polynomial, one might also be interested in calculating its derivative. The following theorem gives us a formula to do so.

**Theorem 2.6.** The derivative of $I(G; x)$ is given by

$$I'(G; x) = \sum_{v \in V(G)} I(G - N[v]; x).$$

**Proof.** We prove the statement by induction on the order of $G$. If $|G| = 0$, $I(G; x) = 1$ and $I'(G; x) = 0$. If $|G| = 1$, $I(G; x) = 1 + x$ and $I'(G; x) = 1 = I(\emptyset; x)$ =
$I(G - N[v]; x)$. Assume the statement holds for graphs of order less than $n$, and let $G$ be a graph of order $n$. We use theorem 2.2 to get the following:

\[
I(G; x) = I(G - v; x) + xI(G - N[v]; x)
\]

\[
\Downarrow
\]

\[
I'(G; x) = I'(G - v; x) + I(G - N[v]; x) + xI'(G - N[v]; x)
\]

\[
\overset{(IH)}{=} I(G - N[v]; x) + \sum_{u \in V(G - v)} I(G - v - N[u]; x)
\]

\[
+ \sum_{u \in V(G - N[v])} xI(G - N[v] - N[u]; x)
\]

\[
= I(G - N[v]; x) + \sum_{u \in N(v)} I(G - v - N[u]; x)
\]

\[
+ \sum_{u \in V(G - N[v])} \left( I(G - v - N[u]; x) + xI(G - N[u] - N[v]; x) \right)
\]

\[
= I(G - N[v]; x) + \sum_{u \in N(v)} I(G - N[u]; x) + \sum_{u \in V(G - N[v])} I(G - N[u]; x)
\]

\[
= \sum_{w \in V(G)} I(G - N[w]; x).
\]

This completes the proof. \qed
Chapter 3

Independence Polynomials of Common Graphs

In this chapter, we define several common classes of graphs and compute their independence polynomials. Depending on the complexity of the graph, its independence polynomial can have anywhere from a trivial recurrence relation to a much more complicated one. The families of graphs to follow are ordered based on the complexity of the recurrence relation for their independence polynomials.

The Star graph of order $n$ is a graph on $n + 1$ vertices. This graph is formed by starting with a single vertex and adjoining $n$ leaves. We denote this graph $S_n$. Below we give a representative example of the graph, $S_5$, along with a table of the first several star graphs.

![Figure 3.1 The star graph of order 5, $S_5$](image)

In order to calculate the independence polynomial for the graph, we apply theorem 2.2, choosing the central vertex as the vertex to remove. We can see this visually below where we apply it to $S_3$.

In the left leaf, we have an empty graph on 3 vertices, and on the right we have
Table 3.1 The first 5 Star graphs

<table>
<thead>
<tr>
<th>$S_0$</th>
<th>$S_1$</th>
<th>$S_2$</th>
<th>$S_3$</th>
<th>$S_4$</th>
</tr>
</thead>
</table>

Figure 3.2 Tree for $S_3$.

the null graph. So, the independence polynomial for $S_3$ is given by $(1 + x)^3 + x = x^3 + 3x^2 + 4x + 1$. Likewise, if we start with $S_n$, we get an empty graph on $n$ vertices in the left leaf, and the null graph in the right leaf. Hence, the independence polynomial for $S_n$ is given by $I(S_n; x) = (1 + x)^n + x$.

While it is not necessary to come up with a recurrence relation to calculate $I(S_n; x)$, we give it for the sake of completion as we will give a table of common graphs along with their polynomials and recurrence relations at the end of this chapter. To see the recurrence we apply theorem 2.4 to $S_n$ and any of its edges. As demonstration, we apply the theorem to $S_3$.

From the above figure, we can see that $I(S_3; x) = (1 + x)I(S_2; x) - x^2$. Similarly, if we apply this to $S_n$ we see that $I(S_n; x) = (1 + x)I(S_{n-1}; x) - x^2$. With this, we have a single term linear recurrence relation for the star graph with $I(S_0; x) = 1 + x$ as the base case.
The Complete graph on $n$ vertices, denoted $K_n$, is the graph where every vertex is adjacent to every other vertex. Below we give a table of $K_n$ for $n$ up to 5.

In order to see a recurrence relation for the independence polynomial of the complete graph, we use theorem 2.2. Since $K_n$ is symmetric, any vertex we choose to re-
move will yield the same result. So, we pick any vertex $v \in V(K_n)$. Then we have that
\[ I(K_n; x) = I(K_n - v; x) + xI(K_n - N[v]; x) = I(K_{n-1}; x) + xI(\emptyset; x) = I(K_{n-1}; x) + x. \]
This recurrence can be seen a bit more easily in the figure below.

![Figure 3.5 Tree for $K_4$.](image)

In the figure above, we calculate the independence polynomial of $K_4$. The recurrence relation can be seen on the first level of the tree, where the left node is a $K_3$ and the right node is the null graph. The recurrence $I(K_n; x) = I(K_{n-1}; x) + x$ can readily be solved, as it only adds a single $x$ term on each step in the recursion. Solving the recursion gives us that $I(K_n; x) = nx + 1$. We can quickly check this against the independence polynomial given for $K_4$ above to see that the two agree with each other.

The Barbell graph of order $n$ is a graph on $2n$ vertices which is formed by joining two copies of $K_n$ by a single edge, known as a bridge. We denote this graph $Bar_n$. In the table below we give $Bar_n$ for $n \in \{3, 4, 5\}$. However, the construction extends to $n = 1, 2$ and is consistent with the equation we find for $I(Bar_n, x)$.
We start by calculating the independence polynomial directly using theorem 2.4. We apply the theorem choosing the bridge, call it \( e = (u,v) \), as our edge to remove. Then we have that 
\[
I(Bar_n; x) = I(Bar_n - e; x) - x^2I(Bar_n - (N[u] \cup N[v]); x) = I(K_n \cup K_n; x) - x^2I(\emptyset; x) = (1 + nx)^2 - x^2.
\]
We demonstrate this visually in the figure below.

In the tree above, we calculate the independence polynomial of \( Bar_3 \). The left leaf is \( Bar_3 \) with the bridge removed, and in the right leaf is \( Bar_3 \) with the neighborhoods of the vertices of the bridge removed. Since the neighborhoods of each vertex contains
$K_3$, the right leaf in the tree above contains the null graph. Now, in order to see the recurrence relation, we apply theorem 2.2, removing one vertex from each side of the barbell. We illustrate this below using $\text{Bar}_4$.

![Tree for Bar_4](image)

Figure 3.8 Tree for $\text{Bar}_4$.

In the figure above, the leaves of the tree contain $\text{Bar}_3, K_3,$ and $K_4$. Reading off the independence polynomial from the tree, we get 

$$I(\text{Bar}_4; x) = I(\text{Bar}_3; x) + x(I(K_4; x) + I(K_3; x)).$$

If we were to do the same to $\text{Bar}_n$, we would get

$$I(\text{Bar}_n; x) = I(\text{Bar}_{n-1}; x) + x(I(K_n; x) + I(K_{n-1}; x)) = I(\text{Bar}_{n-1}; x) + x(1 + nx + 1 + (n - 1)x) = I(\text{Bar}_{n-1}; x) + 2x + (2n - 1)x^2.$$ 

With this, we have a recurrence relation for the Barbell graph.

The Book graph is a graph on $2(n + 1)$ vertices formed by taking $n + 1$ copies of $K_2$, one acting as a central hub for the others, and the other $n$ copies joined to the central hub by two edges, one per vertex. We denote the Book graph as $B_n$. A more concise way to define this graph is by the Cartesian product $B_n = S_n \times K_2$, where $S_n$ is the Star graph and $K_2$ is the Complete graph on 2 vertices. Below we show the
Book graph for \( n \in \{3, 4, 5\} \), but the graph is defined for \( n \) as low as 0.

![Figure 3.9 The book graph of order 3, \( B_3 \)](image)

Table 3.4 Examples of the Book graph

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<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>( B_3 )</td>
<td>( B_4 )</td>
<td>( B_5 )</td>
</tr>
</tbody>
</table>

We would like to be able to find a recurrence between \( I(B_n; x) \) and the polynomials of previous instances of the graph with coefficients that do not depend on \( n \). However, such a recurrence is not obvious using theorems 2.2 and 2.4 alone. So, to the end of finding such a recurrence, we start by calculating the polynomial directly, without a recurrence relation. To calculate the independence polynomial of the Book graph directly, we make use an auxiliary graph which call the 2-Star graph. We define the 2-Star graph of order \( n \) to be a graph on \( 2n + 1 \) vertices which is formed by extending the Star graph by extruding an additional leaf from each leaf of the Star graph. We denote this graph by \( S^2_n \). To see the connection between the independence polynomial of the Book graph and that of the 2-Star, we apply theorem 2.2, removing one of the vertices in the central hub of the book. This can be seen in the illustration below.

In the figure, the left node contains \( S^2_3 \) and the right node contains the empty graph of order 3. So the independence polynomial for \( B_3 \) is given by \( I(B_3; x) = I(S^2_3; x) + x(1 + x)^3 \). Removing the same vertex in \( B_n \) gives us that \( I(B_n; x) = I(S^2_n; x) + x(1 + \)
This reduces the problem to computing \( I(S_n^2; x) \). We approach this the same way as we approached computing \( I(S_n; x) \). That is, we apply theorem 2.2 choosing to remove the central vertex of \( S_n^2 \). We can see this below using \( S_3^2 \) as an example.

In the figure above, the left node contains 3 disjoint copies of \( K_2 \) and the right node contains the empty graph of order 3. This gives us that \( I(S_3^2; x) = (1+2x)^3 + x(1+x)^3 \).

Doing the same to \( S_n^2 \) yields a tree with \( n \) disjoint copies of \( K_2 \) in the left node and the empty graph of order \( n \) in the right node. Hence, \( I(S_n^2; x) = (1+2x)^n + x(1+x)^n \).

Now that we know the independence polynomial of \( S_n^2 \), we can compute the polynomial for the Book graph. We have \( I(B_n; x) = I(S_n^2; x) + x(1 + x)^n = (1 + 2x)^n + x(1 + \)
We can see a recurrence relation for $I(B_n; x)$ by using theorem 2.2 twice, each time removing a vertex of the same page of the book. This will break our expression for the $I(B_n; x)$ into three parts. The first part involves $B_{n-1}$ and the other two involve $S_{n-1}^2$. We demonstrate this below using $B_3$.

![Diagram of tree showing recurrence for $I(B_3; x)$](image)

In the figure above, we see that the right-hand nodes are all the same. This behaviour is the same no matter the choice of $n$. So we see that we have the recurrence

$$I(B_n; x) = I(B_{n-1}; x) + 2xI(S_{n-1}^2; x) = I(B_{n-1}; x) + 2x[(1 + 2x)^{n-1} + x(1 + x)^{n-1}]$$

Using the fact that $I(B_n; x) = (1+2x)^n + 2x(1+x)^n$, we can manipulate the expression as follows to find a recurrence relation for $I(B_n; x)$ whose coefficients do not depend on $n$. 
\[ I(B_n; x) = I(B_{n-1}; x) + 2x^2(1 + x)^{n-1} + 2x(1 + 2x)^{n-1} \]
\[ = I(B_{n-1}; x) + [6x^2 + 2x - 2x(1 + 2x)](1 + x)^{n-1} \]
\[ + [3x + 1 - (1 + x)](1 + 2x)^{n-1} \]
\[ = I(B_{n-1}; x) + (3x + 1)[2x(1 + x)^{n-1} + (1 + 2x)^{n-1}] \]
\[ - (1 + x)(1 + 2x)[2x(1 + x)^{n-2} + (1 + 2x)^{n-2}] \]
\[ = I(B_{n-1}; x) + (3x + 1)I(B_{n-1}; x) - (1 + x)(1 + 2x)I(B_{n-2}; x) \]
\[ = (3x + 2)I(B_{n-1}; x) - (1 + x)(1 + 2x)I(B_{n-2}; x) \]

So we have found a 2-term recurrence relation for \( I(B_n; x) \) with base cases \( I(B_0; x) = 1 + 2x \) and \( I(B_1; x) = (1 + 2x) + 2x(1 + x) \).

The Cocktail Party graph of order \( n \) is a graph on \( 2n \) vertices. The graph is formed by taking \( n \) pairs of vertices such that the vertices in any one pair are adjacent to both vertices in any other pair. Furthermore, there is no edge between the two vertices within any given pair.

Table 3.5 The first 4 Cocktail graphs

<table>
<thead>
<tr>
<th>( CP_1 )</th>
<th>( CP_2 )</th>
<th>( CP_3 )</th>
<th>( CP_4 )</th>
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We find a recurrence relation for \( I(CP_n; x) \) by fixing some pair of vertices \( (u, v) \) as described in the construction on \( CP_n \) and applying theorem 2.2 removing \( u \) and

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Figure 3.13 The cocktail party graph of order 3, $CP_3$.

$v$, one for each application of the theorem. We demonstrate this on $CP_3$ in the figure below.

Figure 3.14 Tree for $CP_3$.

After the removal of the first vertex in the pair, say $u$, we get $CP_n - u$ in the left node and the singleton in the right node. On the next level, we apply theorem 2.2 to $CP_n - u$ removing $v$ this time. After the removal of $v$, we get $CP_n - \{u, v\} = CP_{n-1}$ in the left node and the null graph in the right node. This gives the recurrence relation
\[ I(CP_n; x) = I(CP_{n-1}; x) + x(1 + x) + x = I(CP_{n-1}; x) + x(2 + x). \]

We note that \( CP_0 = \emptyset \) and solve this recurrence as follows

\[
I(CP_n; x) = I(CP_{n-1}; x) + x(2 + x) \\
= I(CP_{n-2}; x) + 2x(2 + x) \\
\vdots \\
= I(CP_{n-i}; x) + ix(2 + x) \\
\equiv I(\emptyset; x) + nx(2 + x) \\
= 1 + nx(2 + x).
\]

With this we have a closed-form representation of \( I(CP_n; x) \).

The Complete Bipartite graph, denoted \( K_{m,n} \), is a graph on \( m + n \) vertices. The vertices in \( K_{m,n} \) are partitioned into two independent sets \( A \) and \( B \), where \( |A| = m \), and \( |B| = n \). Additionally, every vertex in \( A \) is adjacent to every vertex in \( B \). For our purposes, we will assume that \( m \geq n \).

Table 3.6  The first 4 Complete Bipartite graphs

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\[ K_{1,1} \quad K_{2,2} \quad K_{3,3} \quad K_{4,4} \]
We start by finding a recurrence relation for $I(K_{n,n}, x)$. Let $u \in A$ and $v \in B$. To find the recurrence, we apply theorem 2.2 twice, removing first $u$ and then $v$. This is demonstrated below on $K_{3,3}$.

We first apply the theorem to $K_{n,n}$ removing $u$ giving $K_{n,n} - u$ in the left node and the empty graph on $n - 1$ vertices in the right node. On the next level, we apply theorem 2.2 to $K_{n,n} - u$ removing $v$ this time, giving $K_{n-1,n-1}$ in the left node and the empty graph on $n - 1$ vertices in the right node. This gives us the following recurrence.

Figure 3.15 The complete bipartite graph $K_{3,3}$.

Figure 3.16 Tree for $K_{3,3}$. 
\[ I(K_{n,n}; x) = I(K_{n-1,n-1}; x) + 2x(1 + x)^{n-1}. \]

We note that \( I(K_{0,0}; x) = 1 \) and solve the recurrence as follows

\[
\begin{align*}
I(K_{n,n}; x) &= I(K_{n-1,n-1}; x) + 2x(1 + x)^{n-1} \\
&= I(K_{n-2,n-2}; x) + 2x(1 + x)^{n-2} + 2x(1 + x)^{n-1} \\
&= I(K_{n-i,n-i}; x) + 2x \sum_{k=1}^{i} (1 + x)^{n-k} \\
&\overset{i=n}{=} 1 - 2x(1 + x)^{n} + 2x \sum_{k=0}^{n} (1 + x)^{n-k} \\
&= 1 - 2x(1 + x)^{n} + 2x \left( \frac{1 - (1 + x)^{n+1}}{1 - (1 + x)} \right) \\
&= 2(1 + x)^{n} - 1.
\]

With this we have reached the desired closed-form of \( I(K_{n,n}; x) \).

The Sun graph, also known as the Trampoline graph, of order \( n \) is a graph on \( 2n \) vertices. This graph is formed by starting with a copy of \( K_n \). One then enumerates the vertices of \( K_n \) as \( v_1, v_2, \ldots, v_n \). For each pair of consecutive vertices \( v_i \) and \( v_{i+1} \), \( 1 \leq i \leq n \), one adds a new vertex \( u_i \) which is adjacent to \( v_i \) and \( v_{i+1} \). Since \( v_n \) is adjacent to \( v_1 \), we say \( v_{n+1} := v_1 \). We denote the sun graph of order \( n \) by \( T_n \).

Table 3.7  The first 4 Trampoline graphs

<table>
<thead>
<tr>
<th>T_2</th>
<th>T_3</th>
<th>T_4</th>
<th>T_5</th>
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<tbody>
<tr>
<td><img src="image" alt="Graph T_2" /></td>
<td><img src="image" alt="Graph T_3" /></td>
<td><img src="image" alt="Graph T_4" /></td>
<td><img src="image" alt="Graph T_5" /></td>
</tr>
</tbody>
</table>

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We calculate $I(T_n; x)$ by applying theorem 2.2 $n$ times, removing $v_i$ (as described in the definition) at the $i$th step. This is demonstrated on $T_3$ in the figure below.

When we remove $v_i$ during the application of theorem 2.2, we will get the empty graph on $n - 2$ vertices in the right node since the neighborhood of $v_i$ consists of all $v_1, \ldots, v_n$ (as these vertices induce a complete subgraph) and only two of the additional vertices. After the $n$th application of theorem 2.2, we get the empty graph on $n$ vertices in the left node. After the $n$th application of the theorem, then, all
the leaves in the tree are empty graphs, and so we can read off the independence polynomial. Since we only ever need to apply the theorem to the left node, we have only one left node leaf, the empty graph on \( n \) vertices, and \( n \) right node leaves that are empty graphs on \( n - 2 \) vertices. This gives us that

\[
I(T_n; x) = (1 + x)^n + nx(1 + x)^{n-2} = (1 + x)^{n-2}((1 + x)^2 + nx).
\]

To see a recurrence for \( I(T_n; x) \), we observe the following

\[
\begin{align*}
I(T_n; x) &= (1 + x)^{n-2}((1 + x)^2 + nx) \\
I(T_{n-1}; x) &= (1 + x)^{n-3}((1 + x)^2 + nx - x) \\
(1 + x)I(T_{n-1}; x) &= (1 + x)^{n-2}((1 + x)^2 + nx) - x(1 + x)^{n-2} \\
&= I(T_n; x) - x(1 + x)^{n-2}.
\end{align*}
\]

We conclude that

\[
I(T_n; x) = (1 + x)I(T_{n-1}; x) + x(1 + x)^{n-2}.
\]

With this we have both a recurrence and closed-form representation of \( I(T_n; x) \) as desired.

The Crown graph of order \( n \) is a graph on \( 2n \) vertices. The graph is formed by first taking a complete bipartite graph \( K_{n,n} \) with bipartition \( A, B \). Enumerate the vertices of \( A \) as \( \{a_1, a_2, \ldots, a_n\} \) and the vertices of \( B \) as \( \{b_1, b_2, \ldots, b_n\} \). We then remove the edges \((a_i, b_i)\) for \( 1 \leq i \leq n \), the resulting graph is the crown graph of order \( n \). We denote this graph \( Cr_n \).
In order to find a recurrence for $I(Cr_n; x)$, we apply theorem 2.2 twice, removing the vertices $a_1$ and $b_1$ respectively. We demonstrate this in the figure below with $Cr_3$.

On the first application of theorem 2.2 we remove $a_1$ and get $Cr_n - a_1$ in the left node and in the right node we get a graph which is one vertex adjacent to $n - 1$ other vertices. This is the $(n - 1)$-claw. We define the $n$-claw to be the graph which is a
single vertex adjacent to \( n \) other vertices. On the second application of the theorem we remove \( b_1 \) and get \( Cr_{n-1} \) in the left node and the empty graph on \( n - 1 \) vertices in the right node. If we denote the \( n \)-claw as \( H_n \) we have that

\[
I(Cr_n; x) = I(Cr_{n-1}; x) + x(1 + x)^{n-1} + xI(H_n; x).
\]

If we can determine a closed-form for \( I(H_n; x) \), we have found a recurrence relation for \( I(Cr_n; x) \). Indeed, we can easily determine \( I(H_n; x) \) by application of theorem 2.2 removing the central vertex of the \( n \)-claw. This gives that \( I(H_n; x) = (1 + x)^n + x \), and so

\[
I(Cr_n; x) = I(Cr_{n-1}; x) + x(1 + x)^{n-1} + x((1 + x)^n + x) = I(Cr_{n-1}; x) + 2x(1 + x)^{n-1} + x^2.
\]

We note that \( I(Cr_0; x) = 1 \) and solve this recurrence as follows:

\[
I(Cr_n; x) = I(Cr_{n-1}; x) + 2x(1 + x)^{n-1} + x^2
\]

\[
= I(Cr_{n-i}; x) + ix^2 + 2x \sum_{k=1}^{i} (1 + x)^{n-k}
\]

\[
\equiv 1 + nx^2 - 2x(1 + x)^n + 2x \sum_{k=0}^{n} (1 + x)^k
\]

\[
= 1 + nx^2 - 2x(1 + x)^n + 2x \left( \frac{1 - (1 + x)^{n+1}}{1 - (1 + x)} \right)
\]

\[
= 2(1 + x)^n + nx^2 - 1.
\]

This completes the computation of \( I(Cr_n; x) \).

The path graph of order \( n \) is a graph on \( n \) vertices, denoted \( P_n \). This graph is defined by \( P_n = (V, E) \) where \( V = \{v_i | 1 \leq i \leq n\} \) and \( E = \{(v_i, v_{i+1}) | 1 \leq i < n\} \).
The order of the path graph is often defined to be the number of edges in the path. We define the order to be the number of vertices since it makes the indexing behave a little nicer in the sense that we can define \( P_0 = (\emptyset, \emptyset) \). This makes the calculation of the independence polynomial of \( P_n \) a bit simpler.

![Figure 3.21 The path graph of order 3, \( P_3 \)](image)

Table 3.9 The first 4 Path graphs

<table>
<thead>
<tr>
<th></th>
<th>( P_1 )</th>
<th>( P_2 )</th>
<th>( P_3 )</th>
<th>( P_4 )</th>
</tr>
</thead>
</table>

We can very quickly find a recurrence relation for \( I(P_n; x) \) by applying theorem 2.2 removing \( v_1 \). We see this below with \( P_5 \).

![Figure 3.22 Tree for \( P_5 \).](image)

On the application of theorem 2.2, we get \( P_{n-1} \) in the left node and \( P_{n-2} \) in the right node. This gives us the recurrence relation

\[
I(P_n; x) = I(P_{n-1}; x) + xI(P_{n-2}; x).
\]

In order to solve this recurrence we use a characteristic equation. The characteristic equation corresponding to the above recurrence is
\[ r^2 - r - x = 0. \]

This gives

\[ r = \frac{1 \pm \sqrt{1 + 4x}}{2} = \frac{1 \pm s}{2} \]

where \( s = \sqrt{1 + 4x} \). Then we have the following:

\[
\begin{align*}
I(P_n; x) &= c_1 \left( \frac{1 + s}{2} \right)^n + c_2 \left( \frac{1 - s}{2} \right)^n \\
I(P_0; x) &= c_1 + c_2 = 1 \\
I(P_1; x) &= c_1 \left( \frac{1 + s}{2} \right) + c_2 \left( \frac{1 - s}{2} \right) = 1 + x.
\end{align*}
\]

Solving the system of equations given by \( I(P_0; x) \) and \( I(P_1; x) \) gives that

\[
\begin{align*}
c_1 &= \frac{1 + 2x + s}{2s} \\
c_2 &= \frac{s - 1 - 2x}{2s}
\end{align*}
\]

And so

\[
I(P_n; x) = \left( \frac{1 + 2x + s}{2} \right) \left( \frac{1 + s}{2} \right)^n + \left( \frac{s - 1 - 2x}{2} \right) \left( \frac{1 - s}{2} \right)^n
\]

\[
= \frac{1}{2^{n+1}} \left[(1 + 2x + s)(1 + s)^n + (s - 1 - 2x)(1 - s)^n\right]
\]

This is a closed-form for \( I(P_n; x) \) as desired.
The Cycle graph of order $n$ is a graph on $n$ vertices formed by taking a Path graph on $n$ vertices, $P_n$, and identifying the first vertex of $P_n$ with the last vertex so that they become a single vertex.

![Figure 3.23 The cycle graph of order 5, $C_5$.](image)

Table 3.10 The first 3 Cycle graphs

<table>
<thead>
<tr>
<th></th>
<th>$C_3$</th>
<th>$C_4$</th>
<th>$C_5$</th>
</tr>
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</table>

We find an expression for $I(C_n; x)$ by applying theorem 2.2 removing any vertex. On application of the theorem we get $P_{n-1}$ in the left node and $P_{n-3}$ in the right node. We see this with $C_5$ in the figure below.

So we have

$$I(C_n; x) = I(P_{n-1}; x) + xI(P_{n-3}; x).$$

Since we have a closed-form representation of $I(P_n; x)$, we have a closed-form for $I(C_n; x)$ as well. Substituting the closed-forms in the above expression gives us
Figure 3.24  Tree for $C_5$.

\[
I(C_n; x) = I(P_{n-1}; x) + xI(P_{n-3}; x)
\]
\[
= \frac{1}{s^{2n}}[(1 + 2x + s)(1 + s)^{n-1} + (s - 2x - 1)(1 - s)^{n-1}]
\]
\[
+ \frac{4x}{s^{2n}}[(1 + 2x + s)(1 + s)^{n-3} + (s - 2x - 1)(1 - s)^{n-3}]
\]
\[
= \frac{1}{2^{n-1}}[(1 + 2x + s)(1 + s)^{n-2} + (1 + 2x - s)(1 - s)^{n-2}]
\]

where $s = \sqrt{1 + 4x}$.

The wheel graph of order $n$ is a graph on $n + 1$ vertices. This graph is formed by taking a copy of $C_n$ and adding a central vertex which is adjacent to every vertex in $C_n$. We denote the wheel graph of order $n$ by $W_n$.

Table 3.11  The first 3 Wheel graphs

<table>
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<tr>
<th>$W_3$</th>
<th>$W_4$</th>
<th>$W_5$</th>
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</table>

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To find an expression for $I(W_n; x)$, we apply theorem 2.2 removing the central vertex. We demonstrate this below with $W_3$.

On application of the theorem we get $C_n$ in the left node and the null graph in the right node. This gives us that

$$I(W_n; x) = I(C_n; x) + x.$$ 

Since we have a closed-form representation of $I(C_n; x)$, we have one for $I(W_n; x)$ as well. Making the substitution for $I(C_n; x)$ gives us
\[ I(W_n; x) = I(C_n; x) + x \]
\[ = \frac{1}{2^{n-1}} [(1 + 2x + s)(1 + s)^{n-2} + (1 + 2x - s)(1 - s)^{n-2}] + x \]

where \( s = \sqrt{1 + 4x} \).

The pan graph of order \( n \) is a graph on \( n + 1 \) vertices. This graph is formed by taking a copy of \( C_n \) and adjoining a leaf to one of its vertices. We denote this graph by \( Pan_n \).

![Figure 3.27 The pan graph of order 5, \( Pan_5 \)]

Table 3.12 The first 3 Pan graphs

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<th>Pan_3</th>
<th>Pan_4</th>
<th>Pan_5</th>
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We find an expression for \( I(Pan_n; x) \) by applying theorem 2.2 removing the leaf vertex. This is demonstrated on \( Pan_3 \) in the figure below.
On the application of the theorem we get $C_n$ in the left node, and we get $P_{n-1}$ in the right node. Then

$$I(Pan_n; x) = I(C_n; x) + xI(P_{n-1}; x).$$

Since we have closed-form representations for both $I(C_n; x)$ and $I(P_n; x)$, we have a closed form for $I(Pan_n; x)$ as well. Substituting $I(C_n; x)$ and $I(P_{n-1}; x)$ we get

$$I(Pan_n; x) = I(C_n; x) + xI(P_{n-1}; x)$$

$$= \frac{1}{2^{n-1}} [(1 + 2x + s)(1 + s)^{n-2} + (1 + 2x - s)(1 - s)^{n-2}]$$

$$+ \frac{x}{2^n} [(1 + 2x + s)(1 + s)^{n-1} + (s - 2x - 1)(1 - s)^{n-1}]$$

$$= \frac{1}{2^n} [(1 + 2x + s)(1 + s)^{n-2}(2 + x(1 + s))$$

$$+(1 + 2x - s)(1 - s)^{n-2}(2 - x(1 - s))]$$

This gives us a closed form for $I(Pan_n; x)$ as desired.

The $d$-regular Caterpillar graph of order $n$ is a graph on $d \cdot n$ vertices formed by taking a copy of the path graph of order $n$, $P_n$, and adjoining $d$ vertices (pairwise
disjoint) to each vertex along the path. We call the path portion of the caterpillar the spine and the leaves adjoined to the spine the legs. Denote this graph as $Cat_n^d$.

![Figure 3.29 The 3-regular caterpillar graph of order 3](image)

**Table 3.13** The first 3 non-trivial 3-regular caterpillar graphs

<table>
<thead>
<tr>
<th>$Cat_1^3$</th>
<th>$Cat_2^3$</th>
<th>$Cat_3^3$</th>
</tr>
</thead>
</table>

We can find a recurrence relation for $I(Cat_n^d; x)$ by applying theorem 2.2 removing the first vertex in the spine. This can be seen in the figure below with $Cat_3^3$.

![Figure 3.30 Tree for $Cat_3^3$.](image)

On application of the theorem we get $Cat_{n-1}^d$ together with $d$ pairwise disjoint vertices in the left node, and we get $Cat_{n-2}^d$ together with $d$ pairwise disjoint vertices in the right node. This gives us that
\[ I(Cat^d_n; x) = (1 + x)^d I(Cat^d_{n-1}; x) + x(1 + x)^d I(Cat^d_{n-2}; x). \]

We solve this recurrence using its characteristic equation:

\[ r^2 - (1 + x)^d r - x(1 + x)^d = 0. \]

Solving this yields

\[ r = \frac{(1 + x)^d \pm \sqrt{(1 + x)^d ((1 + x)^d + 4x)}}{2} = \frac{(1 + x)^d \pm t}{2} \]

where \( t = \sqrt{(1 + x)^d ((1 + x)^d + 4x)} \). Then we have the following

\[
\begin{align*}
I(Cat^d_n; x) &= c_1 \left( \frac{(1 + x)^d + t}{2} \right)^n + c_2 \left( \frac{(1 + x)^d - t}{2} \right)^n, \\
I(Cat^d_0; x) &= c_1 + c_2 = 1, \\
I(Cat^d_1; x) &= c_1 \left( \frac{(1 + x)^d + t}{2} \right) + c_2 \left( \frac{(1 + x)^d - t}{2} \right) = (1 + x)^d + x.
\end{align*}
\]

Solving the system of equations given by \( I(Cat^d_0; x) \) and \( I(Cat^d_1; x) \) gives

\[
\begin{align*}
c_1 &= \frac{(1 + x)^d + 2x + t}{2t}, \\
c_2 &= \frac{t - 2x - (1 + x)^d}{2t}.
\end{align*}
\]

And so,

\[
I(Cat^d_n; x) = \left( \frac{(1 + x)^d + 2x + t}{2t} \right) \left( \frac{(1 + x)^d + t}{2} \right)^n + \left( \frac{t - 2x - (1 + x)^d}{2t} \right) \left( \frac{(1 + x)^d - t}{2} \right)^n.
\]
With this we have a closed form for $I(Cat_d^n; x)$ as desired.

The Sunlet graph of order $n$ is a graph on $2n$ vertices. This graph is formed by taking a copy of $C_n$, and adding a leaf to each of its vertices.

Table 3.14 The first 3 Sunlet graphs

![Sunlet graphs](image)

We calculate the independence polynomial of $SuL_n$ directly. To do this, we apply theorem 2.2 removing some arbitrary vertex from the cycle in $SuL_n$. This is demonstrated with $Sun_4$ in the figure below.

When we apply theorem 2.2 to $Sun_n$ as described, we get the disjoint union of a singleton and $Cat_{n-1}^1$ in the left node, and we get the disjoint union of the empty graph on 2 vertices and $Cat_{n-3}^1$ in the right node. We found a closed-form representation of $I(Cat_d^1; x)$ above, so we have a closed form for $I(Sun_n; x)$ as well. Substituting what we know about $I(Cat_1^1; x)$, we get

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\[ I(Sun_n; x) = (1 + x)I(Cat^1_{n-1}; x) + x(1 + x)^2I(Cat^3_{n-3}; x) \]

\[ = (1 + x) \left( \frac{1 + 3x + t}{2t} \right) \left( \frac{1 + x + t}{2} \right)^{n-1} \]

\[ + (1 + x) \left( \frac{t - 3x - 1}{2t} \right) \left( \frac{1 + x - t}{2} \right)^{n-1} \]

\[ + x(1 + x)^2 \left( \frac{1 + 3x + t}{2t} \right) \left( \frac{1 + x + t}{2} \right)^{n-3} \]

\[ + x(1 + x)^2 \left( \frac{t - 3x - 1}{2t} \right) \left( \frac{1 + x - t}{2} \right)^{n-3} \]

\[ = \left( \frac{1 + x}{t^{2n}} \right) \left[ (1 + x + t)^n (1 + 3x + t) \left( \frac{(1 + x + t)^2 + 4x(1 + x)}{(1 + x + t)^3} \right) \right] \]

\[ + (1 + x - t)^n (t - 3x - 1) \left( \frac{(1 + x - t)^2 + 4x(1 + x)}{(1 + x - t)^3} \right) \]

where \( t = \sqrt{(1 + x)(1 + 5x)} \). We omit the intermediate steps in the calculation above as they are lengthy and nothing useful is gained by having them.

The Helm graph of order \( n \) is a graph on \( 2n + 1 \) vertices. This graph is formed by taking a Sunlet graph of order \( n \) and adding a central vertex which is connected to each vertex along the cycle in the Sunlet graph. We denote this graph \( H_n \).
We find an expression for $I(H_n; x)$ by applying theorem 2.2 removing the central vertex. As an example, we apply the theorem in the way described to $H_3$ in the figure below.

Upon applying the theorem to $H_n$, we get $SuL_n$ in the left node, and we get the empty graph on $n$ vertices in the right node. This gives that

$$I(H_n; x) = I(SuL; x) + x(1 + x)^n.$$
We have (a rather lengthy) closed-form representation of $I(SuL_n; x)$, and so this gives a closed form for $I(H_n; x)$ as well. Due to the length of the expression for $I(SuL_n; x)$ and the simplicity of the connection to $I(H_n; x)$, we omit the expression we get for $I(H_n; x)$.   


CHAPTER 4

SMALLEST ROOT OF THE INDEPENDENCE POLYNOMIAL

When dealing with polynomials, one will often want to determine what its roots are, or to give some bound on where its roots lie if one cannot determine them. In this chapter, we reproduce the results by Peter Csikvari (Csikvari 2013) to determine the smallest root of an independence polynomial. For convenience in the proofs to come, we consider an alternative definition of the independence polynomial. We define \( \bar{I}(G; x) := I(G; -x) \). Since the roots of \( \bar{I}(G; x) \) are exactly the negatives of the roots of \( I(G; x) \), we may speak of the roots of these polynomials synonymously. The goal of this chapter is to establish the value of the smallest root of \( \bar{I}(G; x) \) and its uniqueness.

We start by introducing a couple lemmas from complex analysis.

Lemma 4.1. (Pringsheim’s theorem). If \( f(z) \) is representable at the origin by a power series expansion that has non-negative coefficients and radius of convergence \( R \), then the point \( z = R \) is a singularity of \( f(z) \).

Proof. Suppose that \( f(z) \) is analytic at \( z = R \). Then there exists some \( \epsilon > 0 \) such that \( f(z) \) is a sum of a convergent power series on \( |z - R| < \epsilon \) with coefficients \( b_i \).

Then, since \(|(R - \frac{\epsilon}{4}) - R| < \epsilon\), we have that

\[
f(z) = \sum_{n \geq 0} b_n \left(z + \frac{\epsilon}{4}\right)^n
\]

is convergent on a small disk around \( R - \frac{\epsilon}{4} \). Let \( f(z) = \sum_{n \geq 0} a_n z^n \) be the representation of \( f(z) \) at the origin. Then we have that on the small disk around \( R - \frac{\epsilon}{4} \)

\[
\sum_{n \geq 0} a_n z^n = \sum_{n \geq 0} b_n \left(z + \frac{\epsilon}{4}\right)^n.
\]
Now, for any natural number $k$, if we take the $k$th derivative we have that

$$
\left( \sum_{n \geq 0} a_n z^n \right)^{(k)} = \left( \sum_{n \geq 0} b_n \left( z - R + \frac{\epsilon}{4} \right)^n \right)^{(k)}
$$

\[ \downarrow \]

$$
\sum_{n \geq 0} \frac{n!}{(n-k)!} a_n z^{n-k} = \sum_{n \geq 0} \frac{n!}{(n-k)!} b_n \left( z - R + \frac{\epsilon}{4} \right)^{n-k}
$$

Evaluating the above expression at $z = R - \frac{\epsilon}{4}$ gives us that

$$
b_k = \sum_{n \geq 0} \binom{n}{k} a_n \left( R - \frac{\epsilon}{4} \right)^{n-k}.
$$

Next, we substitute $b_n$ in our representation of $f(z)$ about $R - \frac{\epsilon}{4}$ and evaluate at $z = R + \frac{\epsilon}{4}$ to get that

$$
f \left( R + \frac{\epsilon}{4} \right) = \sum_{k \geq 0} b_k \left( \frac{\epsilon}{2} \right)^k = \sum_{k \geq 0} \sum_{n \geq 0} \binom{n}{k} a_n \left( R - \frac{\epsilon}{4} \right)^{n-k} \left( \frac{\epsilon}{2} \right)^k.
$$

The above sum is absolutely convergent as all terms are non-negative, and so the order of summation can be changed. By the binomial theorem we have that

$$
f \left( R + \frac{\epsilon}{4} \right) = \sum_{n \geq 0} a_n \left( R + \frac{\epsilon}{4} \right)^n.
$$

As for $|z| \leq R + \frac{\epsilon}{4}$,

$$
\left| \sum_{n \geq 0} a_n z^n \right| \leq \sum_{n \geq 0} a_n |z|^n \leq \sum_{n \geq 0} a_n \left( R + \frac{\epsilon}{4} \right)^n = f \left( R + \frac{\epsilon}{4} \right) < \infty,
$$

meaning the radius of convergence of $\sum a_n z^n$ is at least $R + \frac{\epsilon}{4}$. This is a contradiction, therefore $z = R$ is a singularity of $f$. \qed

An alternative proof of the above theorem can be found in a book by Philippe Flajolet (Flajolet and Sedgewick 2009). Flajolet also provides for us the following definition and theorem. For a sequence $(f_n)$, we define $\text{Supp}(f) := \{ k | f_k \neq 0 \}$. We say that $(f_n)$ admits a span $d$ if for some $r$, we have
Supp(f) ⊆ \{r, r + d, r + 2d, \ldots \}.

We call the largest such span, p, the period of the sequence, and all other spans are divisors of p. When p = 1, we say that the sequence is aperiodic. This connection is key in determining the value of the smallest root of the independence polynomial, as we will see later in this chapter.

**Lemma 4.2.** Let f(z) be analytic in |z| < R and have non-negative coefficients at 0. Assume that f does not reduce to a monomial and that for some non-zero s with |s| < R, we have |f(s)| = f(|s|). Then the following holds,

i. the argument of s is commensurable to 2\pi, that is s = |s|e^{i\theta} with \(\theta/2\pi = \frac{r}{p} \in \mathbb{Q}\) and \(r < p\) with gcd\((r, p) = 1\);

ii. f admits p as a span.

**Proof.** For part (i) of the statement we investigate when the equality |f(z)| = f(|z|) holds. Let the power series expansion of f(z) be

\[
\sum_{n=0}^{\infty} a_n z^n.
\]

We know that f does not reduce to a monomial, so we may assume that there are at least two monomials in the expansion of f. Let \(s = |s|e^{i\theta}\) be a complex number satisfying |f(s)| = f(|s|). We claim that for all \(n \in \text{Supp}(f)\) we have that the numbers \(a_n|s|^n e^{i\theta n}\) all fall on a common ray through the origin. We first show the claim for two terms and show it extends to the general case. For the equality

\[
|a_jz^j + a_lz^l| = a_j|z|^j + a_l|z|^l
\]
to hold, it must be that \(a_j z^j\) and \(a_l z^l\) are parallel. This is so that the modulus of the sum achieves its upper bound given by the triangle inequality. In general we have that

\[
\left| \sum_{n=0}^{\infty} a_n s^n \right| \leq \left| \sum_{n=0, n \neq j, l} a_n s^n \right| + |a_j s^j + a_l s^l| \leq \left| \sum_{n=0, n \neq j, l} a_n s^n \right| + a_j |s|^j + a_l |s|^l.
\]

Since we are assuming we have equality, we may repeatedly apply the argument for the two term case. Let \(n_1, n_2 \in \text{Supp}(f)\). Since \(a_{n_1} |s| e^{in_1 \theta}\) and \(a_{n_2} |s| e^{in_2 \theta}\) are on the same ray, we must have that \(n_1 \theta \equiv n_2 \theta \mod 2\pi\). This means that \((n_1 - n_2) \theta = (2\pi) m\) for some integer \(m\). From this we deduce that \(\frac{\theta}{2\pi}\) must be some rational number \(\frac{r}{p}\), with \(0 \leq r < p\), otherwise the equality we obtain from the congruence is invalid.

For part (ii) we use that \(\frac{\theta}{2\pi} = \frac{r}{p}\). Let arbitrary \(n_1, n_2 \in \text{Supp}(f)\). Then, as above, we have that

\[
(n_1 - n_2) \frac{\theta}{2\pi} = (n_1 - n_2) \frac{r}{p} = m,
\]

which only holds true if \(p \mid (n_1 - n_2)\). Therefore, \(f\) admits \(p\) as a span. \(\square\)

**Lemma 4.3.** Let \(G\) and \(H\) be graphs and set

\[
\frac{\bar{I}(H; z)}{\bar{I}(G; z)} = \sum_{k=0}^{\infty} r_k(H, G) z^k.
\]

Then \(r_0(H, G) = 1\) and

1. If \(H\) is a proper induced subgraph of \(G\), then \(r_k(H, G) > 0\) for \(k \geq 0\),

2. If \(H\) is a proper subgraph of \(G\), then \(r_k(H, G) > 0\) for \(k \geq 2\) and \(r_1(H, G) \geq 0\).

**Proof.** We start by proving part (i). To prove the statement, we induct on the order of \(G\). In the base case we have \(|V(G)| = 1\) and \(|V(H)| = 0\). In this case,

\[
\frac{\bar{I}(H; z)}{\bar{I}(G; z)} = \frac{1}{1 - z} = \sum_{k=0}^{\infty} z^k.
\]
It suffices to prove the statement for $H = G - v$ for some vertex $v \in V(G)$ since if $S = \{v_1, v_2, \ldots, v_k\} \subseteq V(G)$, we have that

$$\frac{\bar{I}(G - S; z)}{\bar{I}(G; z)} = \frac{\bar{I}(G - v_1; z)}{\bar{I}(G; z)} \frac{\bar{I}(G - \{v_1, v_2\}; z)}{\bar{I}(G - v_1; z)} \cdots \frac{\bar{I}(G - \{v_1, \ldots, v_k\}; z)}{\bar{I}(G - \{v_1, \ldots, v_{k-1}\}; z)}.$$ 

By inductive hypothesis, each term except the first has a power series expansion in which all coefficients are positive. So, it remains only to show the statement holds for $H = G - v$. To this end, we apply theorem 2.2 to get

$$\frac{\bar{I}(G - v; z)}{\bar{I}(G; z)} = \frac{\bar{I}(G - v; z)}{\bar{I}(G - v; z) - z\bar{I}(G - N[v]; z)} = \frac{1}{1 - z \frac{\bar{I}(G - N[v]; z)}{\bar{I}(G - v; z)}} = \sum_{k=0}^{\infty} \left( z \frac{\bar{I}(G - N[v]; z)}{\bar{I}(G - v; z)} \right)^k.$$ 

We now have two cases. In the first case, $G - N[v]$ is a proper subgraph of $G - v$, and in the second, $G - N[v] = G - v$. We first consider the case when $G - N[v] \neq G - v$. Since $G - N[v]$ is a proper subgraph of $G - v$, we may apply the inductive hypothesis to see that all the coefficients are positive. On the other hand, if $G - N[v] = G - v$, we have that $\frac{\bar{I}(G - v; z)}{\bar{I}(G; z)} = \frac{1}{1 - z}$, and we again have all positive coefficients.

Next, we prove part (ii). As in part (i), we use proof by induction. This time we induct on the number of edges in $G$. In the base case, $G$ is the empty graph on $n$ vertices and $H$ is the empty graph on $k < n$ vertices. Then

$$\frac{\bar{I}(H; z)}{\bar{I}(G; z)} = \frac{(1 - z)^k}{(1 - z)^n} = (1 - z)^{k-n} = \sum_{i=0}^{\infty} \binom{k-n}{i} (-z)^i = \sum_{i=0}^{\infty} \binom{n-k-1+i}{n-k-1} z^i.$$ 

So the base case holds. As in (i), it suffices to prove the statement only for $\frac{\bar{I}(G \setminus E; z)}{\bar{I}(G; z)}$ since if $E(H) = E(G) - \{e_1, \ldots, e_k\}$ and $|V(G)| - |V(H)| = s$ we have
\[
\frac{\bar{I}(H; z)}{\bar{I}(G; z)} = \frac{\bar{I}(G \setminus e_1; z) \bar{I}(G \setminus \{e_1, e_2\}) \cdots \bar{I}(G \setminus \{e_1, \ldots, e_k\}; z)}{\bar{I}(G \setminus e_1; z) \bar{I}(G \setminus \{e_1, \ldots, e_{k-1}\}; z)} \left(1 - z\right)^s
\]

We know by the inductive hypothesis that each term above except for the first has a power series expansion with non-negative coefficients. So it remains to show that the first term satisfies (ii). To do this, we use theorem 2.4 to give us

\[
\frac{\bar{I}(G \setminus e; z)}{\bar{I}(G; z)} = \frac{\bar{I}(G \setminus e; z) - z^2 \bar{I}(G - (N[u] \cup N[v]); z)}{1 - z^2 \bar{I}(G - (N[u] \cup N[v]); z) / \bar{I}(G \setminus e; z)} = \sum_{k=0}^{\infty} \left(z^2 \frac{\bar{I}(G - (N[u] \cup N[v]); z)}{\bar{I}(G \setminus e; z)}\right)^k,
\]

where \(e = (u, v)\). Since \(G - (N[u] \cup N[v])\) is a proper induced subgraph of \(G\), we may apply part (i) to get the desired result. In fact, the \(k = 1\) term suffices. \(\square\)

**Lemma 4.4.** Let \(\beta(G)\) be the convergence radius of \(\frac{1}{\bar{I}(G; z)}\). Then \(\beta(G)\) is a root of the independence polynomial \(\bar{I}(G; z)\), and it has the smallest modulus among the roots of \(\bar{I}(G; z)\). Let \(H\) be a subgraph of \(G\). Then \(\beta(G) \leq \beta(H)\).

**Proof.** Let

\[
\frac{1}{\bar{I}(G; z)} = \sum_{k=0}^{\infty} r_k(G) z^k.
\]

Since \(\bar{I}(\emptyset; z) = 1\), we have that \(r_k(G) = r_k(\emptyset, G) > 0\). Then by Pringsheim’s theorem \(\beta(G)\), the radius of convergence of \(\frac{1}{\bar{I}(G; z)}\), is a root of \(\bar{I}(G; z)\) with smallest possible modulus. To see the second part of claim, we consider the following identity.

\[
\frac{1}{\bar{I}(G; z)} = \frac{\bar{I}(H; z)}{\bar{I}(G; z)} \frac{1}{\bar{I}(H; z)}
\]

Suppose that \(H\) is a subgraph of \(G\). Then the power series expansion of \(\frac{\bar{I}(H; z)}{\bar{I}(G; z)}\) has positive coefficients. We rewrite the above identity as follows:
\[
\frac{1}{I(G; z)} = \frac{\bar{I}(H; z)}{I(G; z)} \frac{1}{I(H; z)}
\]

\[
\sum_{k=0}^{\infty} r_k(G) z^k = \left( \sum_{k=0}^{\infty} r_k(H, G) z^k \right) \left( \sum_{k=0}^{\infty} r_k(H) z^k \right)
\]

\[
= \sum_{k=0}^{\infty} z^k \left( \sum_{i=0}^{k} r_{k-i}(H, G) r_i(H) \right)
\]

Observe that \( r_0(H, G) = 1 \). The above equation gives us that

\[
r_k(G) = \sum_{i=0}^{k} r_{k-i}(H, G) r_i(H) \geq r_0(H, G) r_k(H) = r_k(H).
\]

And

\[
\beta(G) = \lim_{k \to \infty} r_k(G)^{-1/k} \leq \lim_{k \to \infty} r_k(H)^{-1/k} = \beta(H).
\]

\textbf{Lemma 4.5.} Let \( G \) be a connected graph and let \( H \) be a proper subgraph of \( G \). Then \( \beta(G) < \beta(H) \) and the multiplicity of the root \( \beta(G) \) in \( \bar{I}(G; z) \) is 1.

\textbf{Proof.} Let \( G \) be a connected graph and \( H \) a proper subgraph of \( G \). To prove this claim, we induct on the number of edges. We consider two base cases. When \( |E(G)| = 0 \), we must have \( |V(G)| = 1 \) forcing \( H \) to be the null graph. In this case, \( \beta(G) = 1 \) and \( \beta(H) = \infty \), and since \( \bar{I}(G; z) = 1 - z \), the multiplicity of the root \( \beta(G) \) is 1. In the second case, \( |E(G)| = 1 \) and \( H \) is either the empty graph on 1 or 2 vertices or the null graph. This gives that \( \beta(G) = 1/2 < 1 = \beta(H) \) or \( \beta(G) = 1/2 < \infty = \beta(H) \), and since \( \bar{I}(G; z) = 1 - 2z \) the multiplicity of the root \( \beta(G) \) is 1. So the claim holds in the base cases. Assume it holds for graphs with up to \( k - 1 \) edges. Let \( G \) be a graph with \( k \) edges. It suffices to prove that \( \beta(G) < \beta(G\setminus e) \) for some edge \( e = (u, v) \). The above claim gives us that \( \beta(G) \leq \beta(G\setminus e) \) so we need only prove that \( \beta(G) \neq \beta(G\setminus e) \).

Suppose for the sake of contradiction that \( \beta(G) = \beta(G\setminus e) \). By theorem 2.4 we have
\[ \bar{I}(G; z) = \bar{I}(G \setminus e; z) - z^2 \bar{I}(G - (N[u] \cup N[v]); z). \]

Since \( \beta(G) = \beta(G \setminus e) \) is a root of \( \bar{I}(G; z) \) and \( \bar{I}(G \setminus e; z) \), we find that \( \beta(G) \) is also a root of \( \bar{I}(G - (N[u] \cup N[v]); z) \). If \( G \setminus e \) is connected, \( G - (N[u] \cup N[v]) \) has less edges than \( G \setminus e \) and so \( \beta(G \setminus e) < \beta(G - (N[u] \cup N[v])) \), a contradiction. On the other hand, if \( G \setminus e \) is not connected, then without loss of generality \( G \setminus e = H_1 \cup H_2 \) for some disjoint connected graphs \( H_1 \) and \( H_2 \). Suppose that \( u \in V(H_1) \) and \( v \in V(H_2) \).

Then we have

\[ G - (N[u] \cup N[v]) = (H_1 - N[u]) \cup (H_2 - N[v]). \]

This gives

\[ \beta(G \setminus e) = \min\{\beta(H_1), \beta(H_2)\} < \min\{\beta(H_1 - N[u]), \beta(H_2 - N[v])\} = \beta(G - (N[u] \cup N[v])). \]

Again we have a contradiction. Therefore, \( \beta(G) < \beta(H) \). To see that the root \( \beta(G) \) has multiplicity 1, we show that it is not a root of the derivative. A simple modification of theorem 2.6 through the chain rule gives the following expression for the derivative of \( \bar{I}(G; z) \).

\[ -\bar{I}'(G; z) = \sum_{v \in V(G)} \bar{I}(G - N[v]; z) \]

Since \( \beta(G) < \beta(G - N[v]) \), we have that \( \bar{I}(G - N[v]; z) \) is positive for \( z \in [0, \beta(G)] \). In particular, we have have

\[ -\bar{I}'(G; \beta(G)) = \sum_{v \in V(G)} \bar{I}(G - N[v], \beta(G)) > 0. \]

Hence, \( \beta(G) \) is not a root of \( \bar{I}'(G; z) \) and so is a root of \( \bar{I}(G; z) \) of multiplicity 1. This concludes the proof of the claim.
Theorem 4.6. Let $\alpha$ be a root of the independence polynomial $\overline{I}(G; z)$ different from $\beta(G)$, then $|\alpha| > \beta(G)$.

Proof. Let $G$ be a graph. We may assume that $G$ is connected, since otherwise the roots of $\overline{I}(G; z)$ are the union of the roots of $\overline{I}(H_i; z)$ for each connected component $H_i$. If $|V(G)| \leq 1$, the claim holds trivially. Assume that $|V(G)| \geq 2$ and let $v \in V(G)$. Since $|V(G)| \geq 2$, $G - N[v]$ is a proper subgraph of $G - v$, giving us that $\beta(G - v) < \beta(G - N[v])$. Consider the identity

$$g(z) := \frac{\overline{I}(G - v; z)}{\overline{I}(G; z)} = \frac{\overline{I}(G - v; z)}{\overline{I}(G - v; z) - z\overline{I}(G - N[v]; z)} = \frac{1}{1 - z\frac{I(G - N[v]; z)}{I(G - v; z)}},$$

and define

$$f(z) := z\frac{\overline{I}(G - N[v]; z)}{\overline{I}(G - v; z)}.$$

Since $\beta(G - v) < \beta(G - N[v])$, the radius of convergence of $f(z)$ is $\beta(G - v)$. Observe that $f(z)$ has all positive coefficients by lemma 4.3(i). Let $\alpha$ be a root of $\overline{I}(G; z)$ with $|\alpha| = \beta(G)$. Since $\beta(G) < \beta(G - v)$, $\alpha$ is not a root of $\overline{I}(G - v; z)$. This gives us that $\alpha$ is a singularity of $g(z)$, and so $f(\alpha) = 1$. Immediately, we get that $|f(\alpha)| = 1$. By the same argument, $f(|\alpha|) = f(\beta(G)) = 1$. Hence, $|f(\alpha)| = f(|\alpha|)$. Since $f(z)$ has all positive coefficients, we may use lemma 4.2 to see that $\alpha = |\alpha|e^{2\pi ir/p} = \beta(G)e^{2\pi ir/p}$. We also know that $f(z)$ is aperiodic since all of its coefficients are positive, giving us that $p = 1$ (recall that $p$ is the period, and an aperiodic function has period 1). Therefore $\alpha = \beta(G)e^{2\pi ir} = \beta(G)$, giving us that $\beta(G)$ is the unique root of $\overline{I}(G; z)$ of modulus $\beta(G)$.
Chapter 5

Roots of Independence Polynomials of Claw-Free Graphs

In this chapter, we explore a very surprising property of graphs which do not have a claw as an induced subgraph. A claw is a graph with vertex set \( V = \{v_1, v_2, v_3, v_4\} \) with edge set \( E = \{(v_1, v_2), (v_1, v_3), (v_1, v_4)\} \). We say that a graph \( G \) is claw-free if no induced subgraph of \( G \) is a claw. When a graph is claw-free, the following result holds. For the duration of this chapter, we shall follow the work of Maria Chudnovsky and Paul Seymour (Chudnovsky and Seymour 2007) to build up lemmas which allow us to prove this result.

Theorem 5.1. If \( G \) is claw-free then all roots of \( I(G; x) \) are real.

This property is particularly nice since it is not very computationally intensive to determine if a graph a claw-free; we only need to check \( \binom{n}{4} \) induced subgraphs of \( G \), which is just a polynomial number. The above theorem is a generalization of a result by Ole Heilmann and Elliot Lieb (Heilmann and Lieb 1972) which proves that the independence polynomial of a line graph has all real roots. One can see that all line graphs are claw-free by examining the definition of a line graph. The line graph, \( L(G) \), of a graph \( G \) is formed by interchanging the edge set and vertex set, and two vertices in \( L(G) \) are adjacent if and only if the corresponding edges in \( G \) share a vertex. If a line graph contains a claw, then there is an edge in \( G \) that is incident to 3 other edges. However, the vertices in \( L(G) \) corresponding to these 4 edges will induce a \( K_4 \), and hence \( L(G) \) cannot have a claw.
In the proof of theorem 5.1 we will induct on the order of $G$. Then applying theorem 2.2, we have $I(G; x) = I(G - v; x) + xI(G - N[v]; x)$. By the inductive hypothesis, each term in this sum would have all real roots. The problem we encounter is that it is not in general true that the sum of two polynomials with real roots yields a polynomial which has all real roots. An example of this is $f(x) = x^2$ and $g(x) = x + 1$. We have $f(0) = 0$ and $g(-1) = 0$, but the equation $f(x) + g(x) = x^2 + x + 1$ has only non-real roots. To get around this problem, we prove a stronger statement involving the notion of compatibility.

Let $f_1(x), \ldots, f_k(x)$ be a set of polynomials with real coefficients. We say that the polynomials are compatible if for all $c_i \geq 0$ we have that all the roots of $\sum_{i=1}^{k} c_i f_i(x)$ are real. Additionally, we say that the functions are pairwise compatible if for all $i, j \in \{1, \ldots, k\}$ we have that the $f_i(x)$ and $f_j(x)$ are compatible. Now we prove several lemmas regarding pairwise compatible polynomials.

**Lemma 5.2.** If the polynomials $f$ and $g$ are compatible then so are their derivatives.

**Proof.** We first observe that between each pair of roots in a polynomial $p(x)$, there is a root of its derivative $p'(x)$. Since for all $c_1$ and $c_2 \geq 0$ we have that $c_1 f(x) + c_2 g(x)$ is a polynomial with all real roots, its derivative $c_1 f'(x) + c_2 g'(x)$ is a polynomial with all real roots. Hence, $f'(x)$ and $g'(x)$ are compatible. \hfill $\square$

**Lemma 5.3.** If $f$ and $g$ are compatible polynomials with positive leading coefficients then $|\deg(f) - \deg(g)| \leq 1$.

**Proof.** We induct on $\min\{\deg(f), \deg(g)\}$. In the base case, we have $f(x) = c$ is a constant function. Since the leading coefficient of $f(x)$ is positive, we have $c > 0$. Then, if we choose $c_1$ large enough, the polynomial $c_1 c + 1 \cdot g(x)$ has only one real zero. This is a consequence of $g(x)$ also having a positive leading coefficient. This implies that $g(x)$ cannot have a degree greater than 1. So, $\deg(g) \leq 1$ and $|\deg(f) - \deg(g)| \leq 1$.  

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Now assume that both $f$ and $g$ have degree at least 1. By the above lemma, $f'$ and $g'$ are compatible. Then we have
\[|\deg(f) - \deg(g)| = |\deg(f') - \deg(g')| \leq 1,\]
thus completing the induction and proving the lemma.

For the next lemma, we introduce a new notation. Let $f(x)$ be a polynomial, we denote $n_f(x)$ to be the number of real roots of $f(x)$ in the interval $[x, \infty)$, counting multiplicities. We say that $f$ and $g$ agree at a point $a$ if $f$ and $g$ are both non-zero and both have the same sign.

**Lemma 5.4.** If $f$ and $g$ are compatible polynomials that agree at $a$ and $b$ for some $a < b$, then
\[n_f(b) - n_f(a) = n_g(b) - n_g(a).\]

**Proof.** Let $t \in [0, 1]$ and define $p_t(x) = tf(x) + (1 - t)g(x)$. Since $f$ and $g$ agree at $a$ and $b$, $p_t(x)$ cannot have a root at $a$ or $b$ since the following equality would hold for any root $x$ of $p_t(x)$.
\[f(x) = \frac{(t - 1)g(x)}{t}.\]
Evaluating the above expression at $x = a, b$ gives a contradiction with the sign of $f(a)$ and $f(b)$ since $(t - 1) \leq 0$, $t \geq 0$, and $g$ agrees with $f$ at $x = a, b$. For $t \in [0, 1]$, the roots of $p_t(x)$ move continuously with $t$ in the complex plane. Since $f$ and $g$ are compatible, the roots of $p_t(x)$ move continuously in the real line. This implies that the number of roots of $p_t(x)$ in the interval $(a, b)$ does not depend on our choice of $t$, otherwise we would have that for some $t$, either $a$ or $b$ is a root of $p_t(x)$. Next we consider the polynomials $p_0(x)$ and $p_1(x)$. The polynomial $p_0(x) = g(x)$ has $n_g(b) - n_g(a)$ roots in $(a, b)$, and $p_1(x) = f(x)$ has $n_f(b) - n_f(a)$ roots in $(a, b)$. Since the number of roots of $p_t(x)$ is independent of $t$, we have that $n_f(b) - n_f(a) = n_g(b) - n_g(a)$. \qed
Lemma 5.5. Let $f, g$ be compatible polynomials with positive leading coefficients. Then $|n_f(x) - n_g(x)| \leq 1$ for all $x$.

Proof. We shall proceed by induction on $\max\{\deg(f), \deg(g)\}$. Clearly the base cases hold since if $\max\{\deg(f), \deg(g)\} \leq 1$ then $|n_f(x) - n_g(x)| \leq 1$. Assume the result holds for $\max\{\deg(f), \deg(g)\} < n$ and let $f$ and $g$ be polynomials such that $\max\{\deg(f), \deg(g)\} = n$. Without loss of generality, we may assume that $f(x)$ and $g(x)$ have no roots in common, since otherwise we could factor out the greatest common divisor which preserves compatibility. Suppose to the contrary that $n_f(x_0) - n_g(x_0) \geq 2$ for some $x_0$. We can safely assume that $x_0$ is a root of $f$. Furthermore, we can choose $x_0$ such that it is the largest such root. Since $x_0$ is a root of $f$ we know that $x_0$ is not a root of $g$ as we have assumed that $f$ and $g$ have no common factors.

Next we show that $n_f(x_0) - n_g(x_0) = 2$ by contradiction. Suppose that $n_f(x_0) - n_g(x_0) \geq 3$. Recall that between every two real roots of a polynomial lies a root of its derivative. Then $n_f'(x_0) = n_f'(x_0) - 1$ and $n_g'(x_0) \leq n_g(x_0)$. This gives that

$$n_f'(x_0) - n_g'(x_0) \geq n_f(x) - n_g(x) - 1 \geq 2.$$

But, since $\max\{\deg(f'), \deg(g')\} < n$ and so by the inductive hypothesis $|n_f'(x_0) - n_g'(x_0)| \leq 1$, a contradiction. Hence, $|n_f(x_0) - n_g(x_0)| = 2$.

Now choose $y_1$ strictly greater than all the roots of both $f$ and $g$. Since both $f$ and $g$ have positive leading coefficients we have that $f$ and $g$ must agree at $y_1$. Since both $f$ and $g$ have positive leading coefficients and $|n_f(x_0) - n_g(x_0)|$ is even, we can find some $y_2 < x_0$ such that $f$ and $g$ agree at $y_2$ and neither $f$ nor $g$ have a root in the interval $[y_2, x_0)$. This implies - as $n_f(y_1) = n_g(y_1) = 0$ - that

$$n_f(y_2) - n_f(y_1) \neq n_g(y_2) - n_g(y_1)$$
since \( f \) has an extra root in the interval \([y_2, y_1]\). This contradicts the previous lemma, thus \(|n_f(x_0) - n_g(x_0)| \neq 2\). Therefore, \(|n_f(x_0) - n_g(x_0)| \leq 1\) completing the proof. \(\square\)

For the next lemma, we introduce the idea of an \textit{interleaver}. We say that for two decreasing sequences \(\{a_i\}_{i=1}^m\) and \(\{b_i\}_{i=1}^n\) that the first interleaves the second if \(n \leq m \leq n + 1\) and \(\{a_1, b_1, a_2, b_2, \ldots\}\) is another decreasing sequence.

Let \(\{r_1, \ldots, r_{\deg(f)}\}\) be the decreasing sequence of roots of \(f\), call this this the \textit{root sequence} of \(f\). A \textit{common interleaver} for a set of \(k \in \mathbb{N}\) functions \(f_1, \ldots, f_k\) is a sequence which interleaves the root sequence of each of these functions.

\textbf{Lemma 5.6.} Let \(f(x), g(x)\) be polynomials with all real roots. They have a common interleaver if and only if \(|n_f(x) - n_g(x)| \leq 1\) for all \(x\).

\textbf{Proof.} (\(\Rightarrow\)) Let \(x_0\) be a number bigger than all the roots of \(f\) and \(g\). Then \(n_f(x_0) = n_g(x_0) = 0\). Then as we move \(x\) to the left of \(x_0\) we clearly have that \(|n_f(x) - n_g(x)| \leq 1\).

(\(\Leftarrow\)) We can form a decreasing sequence by merging the root sequences of \(f\) and \(g\). Since \(|n_f(x) - n_g(x)| \leq 1\) for all \(x\), we can find a common interleaver by taking every second term in this sequence. \(\square\)

\textbf{Lemma 5.7.} Let \(f_1(x), \ldots, f_k(x)\) be polynomials with positive leading coefficients and all real roots. Then the following statements are equivalent:

1. \(f_1, \ldots, f_k\) are pairwise compatible,

2. for all \(s, t\) such that \(1 \leq s < t \leq k\), the polynomials \(f_s, f_t\) have a common interleaver,

3. \(f_1, \ldots, f_k\) have a common interleaver,

4. \(f_1, \ldots, f_k\) are compatible.
Proof. Let \( f_1(x), \ldots, f_k(x) \) be polynomials with positive leading coefficients. For convenience we denote \( d_i = \deg(f_i) \). For each \( 1 \leq i \leq k \) let \( \{r^i_j\}_{j=1}^{d_i} \) be the decreasingly ordered root sequence for \( f_i \). When \( d_i \geq 1 \) we define the intervals \( I_i^1, \ldots, I_i^{d_i+1} \) by \( I_i^1 = [r^i_1, \infty), I_i^{d_i+1} = (-\infty, r^i_{d_i+1}] \), and \( I_i^j = [r^i_j, r^i_{j-1}] \) for \( 2 \leq j \leq d_i \). On the other hand, when \( d_i = 0 \), let \( r^i_1 = R \).

\( (1 \Rightarrow 2) \)

Let \( 1 \leq s < t \leq k \) be distinct integers. By lemma 5.3 we have that \( \min\{d_s, d_t\} \geq \max\{d_s, d_t\} - 1 \). If we can show that for any \( 1 \leq j \leq \min\{d_s, d_t\} + 1 \) we have \( I_j^s \cap I_j^t \neq \emptyset \), we may choose a point from each of these intersections to construct a suitable interleaver. Suppose to the contrary that at least one of these intersection is empty. Let \( j \) be the smallest such that \( I_j^s \cap I_j^t = \emptyset \). By definition of \( I_j^s \) and \( I_j^t \), their intersection can never be empty, so \( j \geq 2 \). Without loss of generality, assume that \( r^s_{j-1} \leq r^t_{j-1} \). Then, since \( I_j^s \cap I_j^t = \emptyset \) we have that \( r^s_j \leq r^s_{j-1} < r^t_j \leq r^t_{j-1} \). However, then we have \( |n_{f_s}(r^t_j) - n_{f_t}(r^s_j)| \geq 2 \), contradicting lemma 5.6. Therefore, we can find an interleaver of \( f_s \) and \( f_t \).

\( (2 \Rightarrow 3) \)

From (1), we have that \( I_j^s \cap I_j^t \neq \emptyset \) for all \( 1 \leq s < t \leq k \) and for all \( 1 \leq j \leq \max_i\{d_i\} \). Since intervals in \( \mathbb{R} \) have the Helly property, we can apply Helly’s theorem (Helly 1923) to show that for all \( j \) \((1 \leq j \leq \max_i\{d_i\}) \cap \cup_{i=1}^k I_j^i \neq \emptyset \). Then we may choose points \( p_j \) from these intersections to construct an interleaver \( \{p_j\}_{j=1}^{\max_i\{d_i\}} \) for \( f_1, \ldots, f_k \).

\( (3 \Rightarrow 4) \)

To prove this we induct on \( \max_i\{d_i\} \). In the base case, the max degree is 1. Let \( c_1, \ldots, c_k \geq 0 \) and \( f(x) = \sum_{i=1}^k c_i f_i(x) \). If \( c_i = 0 \) for all \( i \) then \( f \) trivially has all real

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roots. Assume that \( c_i \neq 0 \) for all least one \( i \). Since the max degree is 1, every \( f_i \) has the form \( f_i = m_i x + b_i \) and at least one \( m_i \neq 0 \). Solving \( f(x) = 0 \) gives

\[
x = -\frac{\sum_{i=1}^{k} c_i b_i}{\sum_{i=1}^{k} c_i m_i}.
\]

Since the leading coefficient of each \( f_i \) is positive, and at least one \( c_i \neq 0 \), \( x \) exists and is real. So \( f_1, \ldots, f_k \) are compatible. Assume that the result holds for \( \max_i\{d_i\} = d \).

Let \( \max_i\{d_i\} = d + 1 \). If each \( f_i \) has a common root \( x_0 \), then by the inductive hypothesis \( \frac{f(x)}{x-x_0} \) has all real roots, as by lemma 5.6 they do have a common interleaver. Assume that there is no common factor. By lemma 5.6, we have that \( d - 1 \leq d_i \leq d \).

Let \( \{p_i\}_{i=1}^{d} \) be a common interleaver for the \( f_i \).

Fix \( 1 \leq i \leq k \). Recall \( f(x) = \sum_{j=1}^{k} c_j f_j(x) \). We can assume without loss of generality that all \( c_j \)'s are positive. Since the leading coefficient of \( f_i(x) \) is positive, we have for \( 1 \leq j \leq d \) that \( f(p_j) \geq 0 \) if \( j \) is odd and \( f(p_j) \leq 0 \) if \( j \) is even. Since we assume that there is no common root between the \( f_i \), \( f_i(p_j) \neq 0 \) and so \( f(p_j) > 0 \) if \( j \) is odd and \( f(p_j) < 0 \) if \( j \) is even. This implies that between each \( p_j \) and \( p_{j+1} \) there is a root \( r_j \). So \( f(x) \) has at least \( d \) real roots. This implies that that \( f(x) \) has exactly \( d + 1 \) real roots since \( \deg(f(x)) = d + 1 \) and complex roots come in pairs.

This completes the induction, and we have that \( f_1, \ldots, f_k \) are compatible.

( 4 \( \Rightarrow 1 \) )

Let \( 1 \leq s < t \leq k \). Take \( c_s, c_t \geq 0 \) and \( c_i = 0 \) for \( i \notin \{s,t\} \). Since \( f_1, \ldots, f_k \) are compatible the polynomial \( f(x) = \sum_{i=1}^{k} c_i f_i(x) = c_s f_s(x) + c_t f_t(x) \) has all real roots. Then \( f_s \) and \( f_t \) are compatible. Therefore, \( f_1, \ldots, f_k \) are pairwise compatible.

For the next lemma, we expand on the definition of a clique and introduce the idea of a simplicial clique. A simplicial clique \( K \) is a clique in which for every \( k \in K \), we have that \( N[k] - K \) is itself a clique.
Lemma 5.8. Let $G$ be a claw-free graph and let $K$ be a simplicial clique in $G$. Then $N[v] - K$ is a simplicial clique in $G - K$ for all $v \in K$.

**Proof.** Let $K$ be a simplicial clique in $G$. Then $N[k] - K$ is a clique for every $k \in K$. Suppose that $N[k] - K$ is not a simplicial clique in $G - K$. Then there exists some $v \in N[k] - K$ such that $N[v] - (K \cup N[k]) = N[v] - N[k]$ is not a clique. Since $N[v] - N[k]$ is not a clique, there exist two non-adjacent vertices $x$ and $y$ in $N[v] - N[k]$. Observe that neither $x$ nor $y$ is adjacent to $k$ since otherwise, $x, y \not\in N[v] - N[k]$. However, in this case, the vertices $\{v, k, x, y\}$ induce a claw in $G$. This contradicts our assumption that $G$ is claw-free. Therefore $N[k] - K$ is a simplicial clique in $G - K$ as desired. 

We say that $G$ is **real-rooted** if for every induced subgraph $H$ of $G$, all roots of $I(H; x)$ are real. We wish to prove that every claw-free graph is real-rooted.

Lemma 5.9. Let $G$ be a real-rooted claw-free graph, then

i. for every two simplicial cliques $K, L$ in $G$, the polynomials $I(G - K; x)$ and $I(G - L; x)$ are compatible,

ii. for every simplicial clique $K$, the polynomials $I(G; x)$ and $xI(G - K; x)$ are compatible.

**Proof.** We prove the statement by induction on $|V(G)|$. In the base case there is nothing to prove. Let $|V(G)| = n$ and assume that both results hold for graphs of smaller order. Let $K$ and $L$ be simplicial cliques in $G$. If $K \cup L = \emptyset$, $I(G - K; x) = I(G; x) = I(G - L; x)$. Since $G$ is real-rooted $I(G; x)$ has real roots, and so it is compatible with itself. Assume that $K \cup L \neq \emptyset$. For convenience, let $H = G - (K \cup L)$. Then by corollary 2.3, applied to $G - L$ and $G - K$ respectively, we have

$$I(G - L; x) = I(G - H; x) + \sum_{v \in K - L} xI(H - N_H[v]; x)$$
and

\[ I(G - K; x) = I(G - H; x) + \sum_{v \in L - K} xI(H - N_H[v]; x). \]

By lemma 5.7, in order to show that \( I(G - K; x) \) and \( I(G - L; x) \) are compatible, it is enough to show that for every \( u, v \in K \cup L \) that \( I(G - H; x), xI(H - N_H[u]; x) \), and \( xI(H - N_H[v]; x) \) are pairwise compatible since each of these polynomials have positive leading coefficients. Since \( |V(H)| < |V(G)| \), we may apply the inductive hypotheses. By lemma 5.8, \( N_H[v] \) is a simplicial clique and by assumption of (ii), \( I(H; x) \) and \( xI(H - N_H[v]; x) \) are compatible. By assumption of (i), \( xI(H - N_H[u]; x) \) and \( xI(H - N_H[v]; x) \) are compatible. Hence \( I(H; x), xI(H - N_H[u]; x), \) and \( xI(H - N_H[v]; x) \) are compatible, proving (i).

Next we prove (ii). Since \( K \cup L \neq \emptyset \), either \( K \neq \emptyset \) or \( L \neq \emptyset \). Without loss of generality, assume that \( K \neq \emptyset \). By corollary 2.3 we have

\[ I(G; x) = I(G - K; x) + \sum_{v \in K} xI(G - N[v]; x). \]

Then by lemma 5.7 it is enough to show that \( I(G - K; x) \), \( xI(G - K; x) \), \( xI(G - N[u]; x) \), and \( xI(G - N[v]; x) \) are pairwise compatible for \( u, v \in K \). Since \( G \) is real-rooted, the roots of \( I(G - K; x) \) are real and so \( I(G - K; x) \) and \( xI(G - K; x) \) are compatible. By lemma 5.8, \( N[u] - K \) is a simplicial clique in \( G - K \), and applying the assumption of (ii) to \( G - K \), we get that \( xI(G - K; x) \) and \( xI(G - N[u]; x) \) are compatible. Applying (i) with \( L = \emptyset \) gives that \( xI(G - K; x) \) and \( xI(G - N[u]; x) \) are compatible. Also by lemma 5.8 and (i) we have \( xI(G - N[u]; x) \) and \( xI(G - N[v]; x) \) are compatible. Therefore, the functions as a whole are compatible, completing the proof of (ii).

\[ \square \]

**Lemma 5.10.** Let \( G \) be a claw-free graph, and let \( v \in V(G) \) such that \( G - v \) is real-rooted. Then the polynomials \( I(G - v; x) \) and \( xI(G - N[v]; x) \) are compatible.
Proof. We prove the statement by induction on \(|G|\). In the base case \(|V(G)| = 2\). Then \(I(G - v; x) = 1 + x\) and either \(xI(G - N[v]; x) = x(1 + x)\) or \(xI(G - N[v]; x) = x\). Either way, the two functions are compatible. Let \(|V(G)| = n\) and assume the result holds for smaller orders. Let \(v \in V(G)\). If \(v\) has no neighbors then \(G - v = G - N[v]\). Since \(G - v\) is real-rooted, \(I(G - v; x)\) has all real roots and so \(I(G - v; x)\) and \(xI(G - N[v]; x)\) are compatible. Assume that there is some vertex \(u\) adjacent to \(v\). Let \(H = G - (N[u] \cap N[v])\).

We claim that \(N_H[u]\) and \(N_H[v]\) are simplicial cliques in \(H\). We first show that \(N_H[u]\) (and similarly \(N_H[v]\)) is a clique. Suppose that there are two non-adjacent vertices \(x, y \in N_H[u]\). Since \(x, y \in H\), \(x, y \notin N[v]\). So \(\{u, v, x, y\}\) induces a claw, a contradiction. Hence \(N_H[u]\) and \(N_H[v]\) are cliques. Suppose that \(N_H[u]\) is not a simplicial clique. Then there exists some \(w \in N_H[u]\) such that \(N_H[w] - N_H[u]\) is not a clique. Since \(N_H[w] - N_H[u]\) is not a clique, we can find two non-adjacent vertices \(s\) and \(t\). Then \(\{w, u, s, t\}\) induces a claw, another contradiction. Argue similarly for \(N_H[v]\). Therefore \(N_H[u]\) and \(N_H[v]\) are both simplicial cliques.

By theorem 2.2 applied to \(G - v\), we know that

\[
I(G - v; x) = I(G - \{u, v\}; x) + xI(G - N[u]; x).
\]

Since these functions all have positive leading coefficients, by lemma 5.7 it is enough to show that \(I(G - N[v]; x)\), \(I(G - \{u, v\}; x)\), and \(xI(G - N[u]; x)\) are pairwise compatible. We already have that \(I(G - \{u, v\}; x)\) is compatible with both \(I(G - N[u]; x)\) and \(I(G - N[v]; x)\) by the inductive hypothesis applied to \(G - u\) and \(G - v\) respectively. Since \(N_H[u]\) and \(N_H[v]\) are both simplicial cliques in \(H\) by our claim, we can apply lemma 5.9 to get that \(I(G - N[u]; x)\) and \(I(G - N[v]; x)\) are compatible. Therefore, the functions are compatible, completing the proof.

Proof of theorem 5.1
Proof. We prove the statement by induction on $|V(G)|$. In the base case $|V(G)| = 1$ and $I(G; x) = 1 + x$ has real roots. Let $|V(G)| = n$ and assume that the result holds for claw-free graphs of smaller order. Let $v \in V(G)$. If $v$ has no neighbors, then $I(G - v; x) = I(G - N[v]; x)$ and $I(G; x) = (1 + x)I(G - v; x)$. By the inductive hypothesis, $I(G - v; x)$ has all real roots and so does $I(G; x)$.

Assume that $u$ is some vertex adjacent to $v$. Again by the inductive hypothesis, $I(G - v; x)$ has all real roots. Now, by the previous lemma, $I(G - v; x)$ and $xI(G - N[v]; x)$ are compatible. Therefore, $I(G; x) = I(G - v; x) + xI(G - N[v]; x)$ has all real roots.
BIBLIOGRAPHY


