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Generalizations of Sperner's Theorem: Packing Posets, Families Forbidding Posets, and Supersaturation

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GENERALIZATIONS OF SPERNER'S THEOREM: PACKING POSETS,
FAMILIES FORBIDDING POSETS, AND SUPERSATURATION

by

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ABSTRACT

Sperner's Theorem is a well known theorem in extremal set theory that gives the size of the largest antichain in the poset that is the Boolean lattice. This is equivalent to finding the largest family of subsets of an n -set, $[n] := \{1, 2, \dots, n\}$, such that the family is constructed from unrelated copies of the single element poset.

For a poset P , we are interested in maximizing the size of a family \mathcal{F} of subsets of $[n]$, where each maximally connected component of \mathcal{F} is a copy of P , and finding the extreme configurations that achieve this value. For instance, Sperner showed that when P is one element, $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is the maximum number of copies of P and that this is only achieved by taking subsets of a middle size. Griggs, Stahl, and Trotter have shown that when P is a chain on k elements, $\frac{1}{2^{k-1}} \binom{n}{\lfloor \frac{n}{2} \rfloor}$ is asymptotically the maximum number of copies of P . We find the extreme families for a packing of chains, answering a conjecture of Griggs, Stahl, and Trotter, as well as finding the extreme packings of certain other posets. For the general poset P , we prove that the maximum number of unrelated copies of P is asymptotic to a constant times $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. Moreover, the constant has the form $\frac{1}{c(P)}$, where $c(P)$ is the size of the smallest convex closure over all embeddings of P into the Boolean lattice.

Sperner's Theorem has also been generalized by looking for $\text{La}(n, P)$, the size of a largest family of subsets of an n -set that does not contain a general poset P in the family. We look at this generalization, exploring different techniques for finding an upper bound on $\text{La}(n, P)$, where P is the diamond. We also find all the families that achieve $\text{La}(n, \{\mathcal{V}, \Lambda\})$, the size of the largest family of subsets that do not contain either of the posets \mathcal{V} or Λ .

We also consider another generalization of Sperner's theorem, supersaturation, where we find how many copies of P are in a family of a fixed size larger than $\text{La}(n, P)$. We seek families of subsets of an n -set of given size that contain the fewest k -chains. Erdős showed that a largest k -chain-free family in the Boolean lattice is formed by taking all subsets of the $(k - 1)$ middle sizes. Our result implies that by taking this family together with x subsets of the k -th middle size, we obtain a family with the minimum number of k -chains, over all families of this size. We prove our result using the symmetric chain decomposition method of de Bruijn, van Ebbenhorst Tengbergen, and Kruyswijk (1951).

TABLE OF CONTENTS

ACKNOWLEDGMENTS	ii
ABSTRACT	iii
LIST OF FIGURES	vii
CHAPTER 1 INTRODUCTION	1
1.1 Basic Definitions	1
1.2 Sperner's Theorem	3
1.3 Packing Posets	4
1.4 Families Forbidding Posets	5
1.5 Relationship Between Packing Posets and Families Forbidding Posets	6
1.6 Supersaturation	7
CHAPTER 2 PACKING POSETS	8
2.1 Previous Results	8
2.2 Exact Values and Extreme Families	9
2.3 Asymptotics	15
2.4 Upper Bound	17
2.5 Lower Bound Construction	19
2.6 Modifying proof for strong embeddings	24

2.7	Extending theorems for a finite collection of posets	24
CHAPTER 3 FAMILIES FORBIDDING POSETS		26
3.1	Previous Results	26
3.2	Forbidding both \mathcal{V} and Λ	30
3.3	Lubell function	30
3.4	Diamond-free families in the middle three levels	31
CHAPTER 4 SUPERSATURATION		33
4.1	Previous Results	33
4.2	Symmetric Chain Decomposition	35
4.3	Supersaturation of k -Chains	36
CHAPTER 5 CONCLUSION		40
BIBLIOGRAPHY		42

LIST OF FIGURES

Figure 1.1	The Hasse diagram of \mathcal{B}_3	1
Figure 1.2	$\text{Pa}(3, \{\Lambda, \mathcal{B}_0\}) = 4$	5
Figure 3.1	Diamond-free family	32

CHAPTER 1

INTRODUCTION

1.1 BASIC DEFINITIONS

In order to state Sperner's Theorem and introduce generalizations of it, we begin with some basic definitions. A *partially ordered set*, or *poset*, usually denoted $P = (P, \leq)$, is a set of elements P along with a partial order relation \geq amongst the elements of the set P , and \geq is reflexive, antisymmetric, and transitive, i.e., for all elements a, b , or c in P , the following hold: $a \leq a$; if a and b are not the same element then either $a < b$, $a > b$, or a and b are unrelated; and if $a < b$ and $b < c$ then $a < c$. A *Hasse diagram* of a poset P is a drawing where every element of P is a point in the diagram, and there is an upward path from element x to element y if and only if $x < y$ in P .

The n -set is a set of n elements, the integers from 1 to n , denoted $[n] = \{1, 2, \dots, n\}$. The subsets of an n -set form a poset with the partial ordering of inclusion. Let $2^{[n]}$ denote the set of all subsets of $[n]$. Let the *Boolean lattice* \mathcal{B}_n be the poset $(2^{[n]}, \subseteq)$ of all subsets of the set $[n] = \{1, \dots, n\}$, ordered by inclusion.

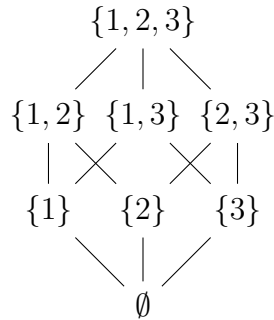


Figure 1.1 The Hasse diagram of \mathcal{B}_3 .

For a set S , the collection of all k -subsets of S is denoted by $\binom{S}{k}$. In the Boolean lattice \mathcal{B}_n , $\binom{[n]}{k}$ is called a *level*. The interval in \mathcal{B}_n from A to B is denoted and defined as $[A, B] := \{C \subseteq [n] \mid A \subseteq C \subseteq B\}$; this is all sets both including A and included in B . The level in \mathcal{B}_n of *rank* k is all the sets in $[n]$ of size k , denoted $\binom{[n]}{k}$.

By $\mathcal{B}(n, k)$ and $\Sigma(n, k)$ we mean the families of subsets of $[n]$ of the k middle sizes and the size of the families. More precisely,

$$\mathcal{B}(n, k) = \binom{[n]}{\lfloor \frac{n-k+1}{2} \rfloor} \cup \dots \cup \binom{[n]}{\lfloor \frac{n+k-1}{2} \rfloor} \quad \text{or} \quad \mathcal{B}(n, k) = \binom{[n]}{\lceil \frac{n-k+1}{2} \rceil} \cup \dots \cup \binom{[n]}{\lceil \frac{n+k-1}{2} \rceil},$$

and $\Sigma(n, k) = |\mathcal{B}(n, k)|$

(so, depending on the parity of n and k , $\mathcal{B}(n, k)$ can be either one or two different families).

Given two posets $P = (P, \leq)$ and $P' = (P', \leq')$, we say that there exists a *weak embedding* of P in P' , or P' *contains a copy of* P , or P *is a (weak) subposet of* P' , if there is an embedding $f : P \rightarrow P'$ that preserves the partial ordering, i.e. if $a \leq b$ in P , then $f(a) \leq' f(b)$ in P' . We say that P' is *P -free* if it does not contain P . Similarly we may define strong embeddings: Given two posets $P = (P, \leq)$ and $P' = (P', \leq')$, we say that there exists a *strong embedding* of P in P' , or P is an *induced subposet* of P' , if there is an embedding $f : P \rightarrow P'$ such that $a \leq b$ in P if and only if $f(a) \leq' f(b)$ in P' .

A poset P that is total ordered, i.e., no two elements in P being unrelated, is called a *chain* (or *path*). A chain is called a k -chain when it has k elements. Let \mathcal{C}_k denote that poset that is a k -element chain. It is often easier to denote a chain as \mathcal{P}_k to be the $(k+1)$ -element chain. In this way, \mathcal{P}_k embedded into \mathcal{B}_k is a full chain in \mathcal{B}_k . A *full chain* in \mathcal{B}_n is a collection of subsets of $[n]$ with exactly one set of each possible size. In contrast to a chain, an *antichain* \mathcal{A} is an induced subposet of a poset such that \mathcal{A} has no two related elements.

In extremal set theory, the question is to find the extreme (either largest or smallest) induced subposets of a poset P such that the subposet has some fixed properties.

For instance, Sperner's Theory asks: What are the largest induced subposets of \mathcal{B}_n such that the subposet is an antichain? We call an induced subposet of \mathcal{B}_n a *family*, often denoted $\mathcal{F} \subseteq \mathcal{B}_n$. A family $\mathcal{F} \subseteq \mathcal{B}_n$ is nothing more than a collection of subsets of $[n]$. So specifically, we look for the extreme families of \mathcal{B}_n (or collections of subsets of $[n]$) that have some property. Varying these properties leads to some different extensions of Sperner's Theorem.

1.2 SPERNER'S THEOREM

Sperner's Theorem finds all the antichains of maximum size in \mathcal{B}_n .

Theorem 1.2.1 (Sperner, 1928). *The size of the largest antichain in \mathcal{B}_n is $\binom{n}{\lfloor \frac{n}{2} \rfloor}$; specifically, the largest antichain is the middle level when n is even or one of the two middle levels when n is odd.*

Many different proofs for this theorem have arisen over the years since Sperner's original proof. Here we include a method devised independently by Lubell, Yamamoto, and Mešalkin now called the LYM-inequality.

Proof of most of Sperner's Theorem. Let $\mathcal{F} \subseteq \mathcal{B}_n$ be a largest antichain in \mathcal{B}_n . Each $A \in \mathcal{F}$ meets exactly $|A|!(n - |A|)!$ full chains. Each chain meets the family \mathcal{F} at most once, else the family would not be an antichain. Now the sum of all the chains that meet the family is at most the total number of chains, $n!$. This gives the inequality

$$\sum_{A \in \mathcal{F}} |A|!(n - |A|)! \leq n!.$$

Since $|A|!(n - |A|)!$ is minimized by $\lfloor n/2 \rfloor!(n - \lfloor n/2 \rfloor)!$, we see the following:

$$\begin{aligned} \sum_{A \in \mathcal{F}} \lfloor n/2 \rfloor!(n - \lfloor n/2 \rfloor)! &\leq n!, \\ |\mathcal{F}| \lfloor n/2 \rfloor!(n - \lfloor n/2 \rfloor)! &\leq n!, \text{ or} \\ |\mathcal{F}| &\leq \frac{n!}{\lfloor n/2 \rfloor!(n - \lfloor n/2 \rfloor)!} = \binom{n}{\lfloor \frac{n}{2} \rfloor}. \end{aligned}$$

□

Notice that the proof using the LYM-inequality does not give us all the extreme antichains like in the statement of the theorem, just that the size of an antichain is at most $\binom{n}{\lfloor \frac{n}{2} \rfloor}$. The following is a specific case of one of the main results discussed in Lemma 2.2.1. This gives the form of all the extreme antichains and is a nice continuation of the LYM-inequality proof above.

Proof of the rest of Sperner's Theorem. The antichains in the statement of the theorem are of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$; we just need to prove these are the only extreme antichains.

Let \mathcal{F} be an antichain such that $|\mathcal{F}| = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Notice that this implies that the number of chains that meets the family is $n!$ by making the inequalities in the proof above tight. This implies that every chain meets the family exactly once. If A is in \mathcal{F} , then $A \setminus \{a\} \cup \{b\}$ for $a \in A$ and $b \notin A$ is also in the family, else any chain that meets $A \setminus \{a\}$, $A \setminus \{a\} \cup \{b\}$, and $A \cup \{b\}$ does not meet the family at all.

Now the entire level containing A is also in the family \mathcal{F} . Since \mathcal{F} is an antichain, \mathcal{F} consists of only elements from this level. The only levels of size $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ are of rank $n/2$ if n is even and $\lfloor n/2 \rfloor$ or $\lceil n/2 \rceil$ if n is odd. \square

There are many generalizations to Sperner's Theorem. We discuss three of them: packing posets into the Boolean lattice, families of the Boolean lattice that forbid posets, and supersaturation in the Boolean lattice.

1.3 PACKING POSETS

Let P be any poset. Let $f : P \rightarrow \mathcal{B}_n$ be a weak embedding of the poset P into \mathcal{B}_n , i.e., if $a < b \in P$, then $f(a) \subset f(b)$. We call $f(P)$ a copy of P in \mathcal{B}_n . Let $\{\mathcal{F}_i\}_{i \geq 1}$ be pairwise unrelated copies of P , i.e., if $A_i \in \mathcal{F}_i$, $A_j \in \mathcal{F}_j$, and $i \neq j$, then A_i and A_j are unrelated. We say the family $\mathcal{F} = \cup_i \mathcal{F}_i$ is a family constructed from pairwise unrelated copies of P . Let $\text{Pa}(n, P)$ denote the maximum size of a family constructed from pairwise unrelated copies of P in \mathcal{B}_n . This quantity can be generalized to apply

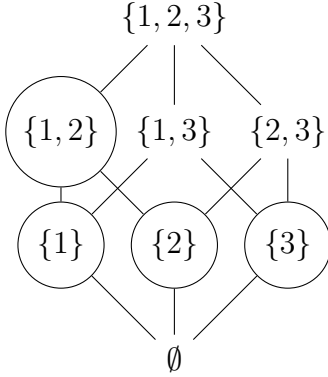


Figure 1.2 $\text{Pa}(3, \{\Lambda, \mathcal{B}_0\}) = 4$.

to a collection of posets; let $\text{Pa}(n, \{P_i\}_{i \geq 1})$ denote the maximum size of a family in \mathcal{B}_n constructed from pairwise unrelated copies of posets chosen from the collection of posets $\{P_i\}$, for $i \geq 1$, possibly finite. For example, Figure 1.2 demonstrates an example where $\{P_i\}_{i \geq 1} = \{\Lambda, \mathcal{B}_0\}$, and Λ is the poset on $\{a, b, c\}$ with $a > b$ and $a > c$. The circled sets represent a packing of posets from $\{\Lambda, \mathcal{B}_0\}$ in \mathcal{B}_3 .

We may also ask the similar question, what is the maximum number of pairwise unrelated *induced* copies of P in \mathcal{B}_n , where each copy is a strong embedding of P ? A strong embedding f of P is such that for $a, b \in P$, $a < b$ if and only if $f(a) \subset f(b)$. We will denote the maximum size of a family in \mathcal{B}_n constructed from induced copies of P as $\text{Pa}^*(n, P)$. We can also define the more general quantity $\text{Pa}^*(n, \{P_i\}_{i \geq 1})$.

An antichain in a Boolean lattice is a packing of the poset \mathcal{B}_0 in the Boolean lattice. So Sperner's Theorem is a special case of packing posets; $\text{Pa}(n, \mathcal{B}_0) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. Poset packing is the topic of Chapter 2.

1.4 FAMILIES FORBIDDING POSETS

The classic interpretation of Sperner's Theorem is the problem of finding the largest collection \mathcal{F} of subsets of an n set that does not contain any two elements $A, B \in \mathcal{F}$ that are related, $A \subset B$. This amounts to finding the maximum sized family that forbids the poset \mathcal{B}_1 , the two element chain, since any $A, B \in \mathcal{F}$ where $A \subset B$ defines

a two element chain. We may generalize this to apply to other posets other than the two element chain. For a general poset P , the quantity $\text{La}(n, P)$ is defined as the size of the maximum family $\mathcal{F} \subseteq \mathcal{B}_n$ restricted to the condition that P may not be weakly embedded into \mathcal{F} . Sperner's Theorem is then a special case giving $\text{La}(n, \mathcal{B}_1) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. We may generalize this quantity even more; for a collection of posets $\{P_i\}_{i \geq 1}$, we may define $\text{La}(n, \{P_i\}_{i \geq 1})$ to be the size of the largest family $\mathcal{F} \in \mathcal{B}_n$ such that no poset P_i from the collection may be weakly embedded into \mathcal{F} . Finding families that forbid certain posets is the topic of Chapter 3.

1.5 RELATIONSHIP BETWEEN PACKING POSETS AND FAMILIES FORBIDDING POSETS

The motivation for finding $\text{Pa}(n, \{P_i\}_{i \geq 1})$ comes from a question under intensive study in recent years, that of finding the maximum size $\text{La}(n, Q)$, which is the maximum size of a family $\mathcal{F} \subseteq \mathcal{B}_n$ that contains no copy of poset Q . This seems to be a challenging problem in extremal set theory, even determining the asymptotic growth of $\text{La}(n, Q)$, as $n \rightarrow \infty$, for posets as simple as the four element diamond (which is \mathcal{B}_2).

It is natural to extend this notion to collections of posets $\{Q_j\}_{j \geq 1}$, seeking to find the maximum size $\text{La}(n, \{Q_j\}_{j \geq 1})$ of a family $\mathcal{F} \in \mathcal{B}_n$ that contains no copy of any poset Q_j in the collection. We noticed that for the collection $\{\mathcal{V}, \Lambda\}$, where \mathcal{V} is the poset on $\{a, b, c\}$ with $a < b$ and $a < c$, and Λ is the poset on $\{a, b, c\}$ with $a > b$ and $a > c$, $\text{La}(n, \{\mathcal{V}, \Lambda\})$ is the same as $\text{Pa}(n, \{\mathcal{B}_0, \mathcal{B}_1\})$, since any collection of subsets that contains no copy of \mathcal{V} or Λ has components consisting only of single sets and/or two-element chains, all unrelated to each other. We recently learned that Katona and Tarján solved this very problem years ago, showing that $\text{La}(n, \{\mathcal{V}, \Lambda\}) = 2 \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$. We were able to derive the same result, applying a 1984 result of Griggs, Stahl, and Trotter that gives $\text{Pa}(n, \mathcal{B}_1)$.

More generally, for any collection $\{Q_j\}_{j \geq 1}$, the quantity $\text{La}(n, \{Q_j\}_{j \geq 1})$ is equivalent to $\text{Pa}^*(n, \{P_i\}_{i \geq 1})$, where $\{P_i\}_{i \geq 1}$ is the collection of all possible connected posets that do not contain any of the posets in $\{Q_j\}_{j \geq 1}$ as a subposet. Note that the collection $\{P_i\}_{i \geq 1}$ may be infinite. For instance, $\text{La}(n, \mathcal{V})$ is the same as $\text{Pa}^*(n, \{P_i\}_{i \geq 0})$ where P_i is the i -fork consisting of one set that contains i (unrelated) sets, $i \geq 0$. So the problem of determining $\text{Pa}^*(n, \{P_i\}_{i \geq 1})$ can be viewed as more general than the $\text{La}(n, \{Q_j\}_{j \geq 1})$ problem.

1.6 SUPERSATURATION

Unlike the other two generalizations of Sperner's Theorem, questions in the topic of supersaturation are usually not looking for the size of an extreme family, but are instead interested in the following type of question: Given a poset P and a family size m , over all choices of $\mathcal{F} \subseteq \mathcal{B}_n$ such that $|\mathcal{F}| = m$, what is the minimum number of weak embeddings of P in \mathcal{F} ? Sperner's Theorem is a special case of this question, if $P = \mathcal{B}_1$ and $m \leq \binom{n}{\lfloor \frac{n}{2} \rfloor}$, over all choices of $\mathcal{F} \subseteq \mathcal{B}_n$ such that $|\mathcal{F}| = m$, what is the minimum number of weak embeddings of P in \mathcal{F} ? The answer is 0. This is a difficult question in general; it is not even known for any poset other than $P = \mathcal{B}_1$ for general m . In Chapter 4 we answer this question for k -chains and specific values of m .

CHAPTER 2

PACKING POSETS

2.1 PREVIOUS RESULTS

To start us off, here are some examples in the literature of finding $\text{Pa}(n, P)$ for specific P .

Sperner's Theorem was the first result of this type, finding the value of $\text{Pa}(n, P)$ for $P = \mathcal{B}_0$, the single element poset; $\text{Pa}(n, \mathcal{B}_0) = \binom{n}{\lfloor \frac{n}{2} \rfloor}$. The proof of Sperner's Theorem in Section 1.2 was generalized by Griggs, Stahl, and Trotter in their paper *A Sperner Theorem on Unrelated Chains of Subsets* (1984) for $P = \mathcal{P}_k$, the chain (or path) on $k + 1$ elements.

Theorem 2.1.1. *The value of $\text{Pa}(n, \mathcal{P}_k)$ is $(k + 1) \binom{n - k}{\lfloor \frac{n - k}{2} \rfloor}$. For fixed k , as n goes to infinity, $\text{Pa}(n, \mathcal{P}_k)$ is asymptotically $\frac{k + 1}{2^k} \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

Proof. For a chain \mathcal{P} , with a minimum set $A \subseteq [n]$ and maximum $B \subseteq [n]$, define $\mathcal{I}_{\mathcal{P}}$ as $[A, B] = \{C \subseteq [n] \mid A \subseteq C \subseteq B\}$, the interval from A to B . For two chains \mathcal{P} and \mathcal{P}' to be pairwise unrelated copies of \mathcal{P}_k , a full chain in \mathcal{B}_n meets at most one of $\mathcal{I}_{\mathcal{P}}$ or $\mathcal{I}_{\mathcal{P}'}$. A full chain that meets $\mathcal{I}_{\mathcal{P}}$ is a chain constructed from all the elements of A before any of the elements of $[n] \setminus B$. There are $n - |B \setminus A|$ elements in A or not in B . The order elements are added to a full chain are each equally likely. Therefore, the number of full chains that meet $\mathcal{I}_{\mathcal{P}}$ is

$$\frac{n!}{\binom{n - |B \setminus A|}{|A|}} \geq \frac{n!}{\binom{n - k}{\lfloor \frac{n - k}{2} \rfloor}}$$

so each of the $\frac{\text{Pa}(n, \mathcal{P}_k)}{|\mathcal{P}_k|}$ intervals from the copies of \mathcal{P}_k meets at least $n! / \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}$ full chains. This gives that

$$\frac{\text{Pa}(n, \mathcal{P}_k)}{|\mathcal{P}_k|} \frac{n!}{\binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}} \leq n!, \text{ or}$$

$$\text{Pa}(n, \mathcal{P}_k) \leq |\mathcal{P}_k| \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} = (k+1) \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor} \sim \frac{k+1}{2^k} \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

The bound is tight, as the following construction demonstrates. Fix a set $S \subseteq [n]$ such that $|S| = k$. The A 's corresponding to the chains are all the $\lfloor \frac{n-k}{2} \rfloor$ -sets of $[n] \setminus S$, and the \mathcal{P}_k 's are chosen as any full chain in the interval $[A, A \cup S]$ for each A . \square

Notice the steps in the proof above. For the upper bound, first, each copy of \mathcal{P}_k is contained in a larger set system \mathcal{I}_P , where a full chain meets at most one \mathcal{I}_P . Second, a lower bound on the number of full chains that meet an \mathcal{I}_P is found. Now $n!$ divided by this lower bound is an upper bound on $\frac{\text{Pa}(n, P)}{|P|}$, the number of copies of P . As for the lower bound, a construction is found. In this particular example, the construction is multiple copies of \mathcal{P}_k , where each copy's minimum is on a base rank $\lfloor \frac{n-k}{2} \rfloor$, and the minimum is constructed from just the elements of $[n] \setminus S$. The set S restricts the choices of which sets in the base level to include in the packing. Each copy of P can then easily be built on top of its minimum using the elements of S .

In their paper, Griggs, Stahl, and Trotter conjecture that the *only* maximum-sized collections of pairwise unrelated chains are those obtained in the same manner as the construction for the lower bound in the proof above. This conjecture will be answered in the following section.

2.2 EXACT VALUES AND EXTREME FAMILIES

In Sperner's proof of Sperner's Theorem, he not only finds the exact maximum size of a packing of \mathcal{B}_0 , he also identifies all the extreme families, the families that witness

the maximum size of $\text{Pa}(n, \mathcal{B}_0)$. These families are of course the families that are a middle level of \mathcal{B}_n . In their paper named in the previous section, Griggs, Stahl, and Trotter find the exact value of $\text{Pa}(n, \mathcal{P}_k)$, but only conjecture on what the extreme families are. In this section, we find $\text{Pa}(n, P)$ exactly for a class of posets including the chain poset, as well as identifying all the extreme families that witness $\text{Pa}(n, P)$. This will resolve the conjecture by Griggs, Stahl, and Trotter.

Let $\{P_i\}_{i \geq 1}$ be a collection of posets. Let $\mathcal{F} \subseteq \mathcal{B}_n$ be a packing of weak embeddings of posets from $\{P_i\}_{i \geq 1}$ such that $|\mathcal{F}| = \text{Pa}(n, \{P_i\}_{i \geq 1})$. For $a, b \in [n]$ and $\mathcal{G} \subseteq \mathcal{B}_n$, define $\mathcal{G} - a := \{A \setminus \{a\} \mid A \in \mathcal{G}\}$ and $\mathcal{G} + b := \{A \cup \{b\} \mid A \in \mathcal{G}\}$ and $\cup \mathcal{G} := \cup_{A \in \mathcal{G}} A$ and $\cap \mathcal{G} := \cap_{A \in \mathcal{G}} A$.

The following lemma reveals that if the family \mathcal{F} has the property that every full chain meets the family, much may be said about its structure.

Lemma 2.2.1. *Let $\mathcal{F} \subseteq \mathcal{B}_n$ be such that it intersects every full chain of \mathcal{B}_n . For every maximally connected $\mathcal{G} \subseteq \mathcal{F}$, if there exists $a, b \in [n]$ such that $a \in \cap \mathcal{G}$ and $b \notin \cup \mathcal{G}$, then $(\mathcal{G} + b) - a \subseteq \mathcal{F}$.*

Proof. Assume otherwise, that there is a set $A \in \mathcal{G}$ such that $(A + b) - a \notin \mathcal{F}$. The interval $[\emptyset, A - a]$ is all below A and not in \mathcal{G} so $[\emptyset, A - a] \cap \mathcal{F} = \emptyset$, and the interval $[A + b, [n]]$ is all above A and not in \mathcal{G} so $[A + b, [n]] \cap \mathcal{F} = \emptyset$ so any full chain in $[\emptyset, A - a] \cup \{(A + b) - a\} \cup [A + b, [n]]$ does not intersect \mathcal{F} , a contradiction. \square

What does this Lemma tell us about the structure of such an $\mathcal{F} \subseteq \mathcal{B}_n$? Let us assume every connected component is an embedding of the same poset, \mathcal{B}_k . For some embedding \mathcal{G} with minimum A and maximum B , let $S = B \setminus A$. Now every embedding of \mathcal{B}_k in \mathcal{F} is a translation of \mathcal{G} ; each has the same set S and has its minimum element on the same rank as A . Specifically, \mathcal{F} is $\{E \in [C, C \cup S] \mid C \subseteq ([n] \setminus S), |C| = |A|\}$.

We first use this lemma to find all the extreme families that are packings of the poset \mathcal{B}_k . Let $\mathcal{F} \subseteq \mathcal{B}_n$ be constructed from unrelated embeddings of \mathcal{B}_k . From the

proof of Theorem 2.1.1, we have that each copy of \mathcal{B}_k meets at least $n!/\binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}$ full chains. We also have that there is a construction that allows each copy of \mathcal{B}_k to meet exactly $n!/\binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}$ full chains. This means that any extreme family in \mathcal{B}_n constructed from embeddings of \mathcal{B}_k meets every full chain of \mathcal{B}_n . Now we can use this fact and the lemma to describe all the extreme families that are packings of the poset \mathcal{B}_k .

If $n = k$, the only extremal family is the whole Boolean lattice, $\mathcal{F} = \mathcal{B}_n = \mathcal{B}_k$. If $n = k + 1$, there can only be one embedding of \mathcal{B}_k into \mathcal{B}_n so any embedding will be an extreme family. A class of extreme families for $n \geq k + 2$ can be constructed as follows: Select a set $S \subseteq [n]$, $|S| = k$, and let

$$\mathcal{F} = \{B \in [A, A \cup S] \mid |A| = \lfloor \frac{n-k}{2} \rfloor, A \subseteq [n] \setminus S\}.$$

Also, it works if all $|A| = \lceil \frac{n-k}{2} \rceil$ for odd $n - k$. We want to show using Lemma 2.2.1 that these constructions are the only extreme families.

Theorem 2.2.2. *The constructions above are the only extreme families. Specifically, for $n \geq k + 2$, if \mathcal{F} is a family in \mathcal{B}_n consisting of the maximum number of unrelated embeddings of \mathcal{B}_k , then there exists a set $S \subset [n]$, $|S| = k$, such that for each embedding of $\mathcal{B}_k \subseteq \mathcal{F}$, the embedding has a minimum member A and a maximum member B with $B = A \cup S$, and the minimum elements of each embedding have the same size.*

Proof. Clear for $n = k$ or $n = k + 1$. Let $n \geq k + 2$. Important to this proof is the fact that every full chain in \mathcal{B}_n must intersect the family if the family is to be an extremal family. Now we can apply Lemma 2.2.1. Since $n \geq k + 2$, there is a construction that allows for at least two embeddings of \mathcal{B}_k ; each embedding of \mathcal{B}_k will include neither \emptyset nor $[n]$. Then each embedding of \mathcal{B}_k has an $a \in [n]$ in its minimum and a $b \in [n]$ that is not in its maximum. Now the lemma implies the form of the extreme family. \square

The theorem above gives all the extreme families constructed from unrelated embeddings of \mathcal{B}_k . We can use this result to classify all extreme families constructed from

unrelated embeddings of other posets, including \mathcal{P}_k , the $k + 1$ element chains. The idea is that every embedding of \mathcal{P}_k corresponds to a minimal interval that contains it. These intervals must also be unrelated. The question is now finding all extreme families constructed from unrelated embeddings of \mathcal{B}_k , the problem just solved by the theorem.

If a poset P is such that embeddings of P into \mathcal{B}_n being unrelated implies the minimum intervals containing an embedding are also unrelated, then we may classify all extreme families constructed from unrelated embeddings of P . For now, we define the following: Classify a poset P as *weakly bound* if P is such that *weak* embeddings of P into \mathcal{B}_n being unrelated implies the minimum intervals containing a weak embedding are also unrelated.

Classify a poset P as *weakly bound* if there exist an integer k such that each *weak* embedding of P into any \mathcal{B}_n is contained in an embedding of \mathcal{B}_k , and *weak* embeddings of P into \mathcal{B}_n being unrelated implies these containing embeddings of \mathcal{B}_k are also unrelated. For a weakly bound poset P , let $k(P)$ be this integer k . Similarly, classify a poset P as *strongly bound* if there exist an integer k such that each *strong* embedding of P into any \mathcal{B}_n is contained in an embedding of \mathcal{B}_k , and *strong* embeddings of P into \mathcal{B}_n being unrelated implies these containing embeddings of \mathcal{B}_k are also unrelated. For a strongly bound poset P , let $k^*(P)$ be this integer k .

Theorem 2.2.3. *Let the poset P be weakly bound with $k(P) = k$. The extreme families constructed from unrelated weak embeddings of P are the extreme families of unrelated embeddings of \mathcal{B}_k with any P chosen from each embedding of \mathcal{B}_k . Now the size of this family is $\text{Pa}(n, P) = |P| \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}$.*

Theorem 2.2.4. *Similarly, let the poset P be strongly bound with $k^*(P) = k$. The extreme families constructed from unrelated strong embeddings of P are the extreme*

families of unrelated embeddings of \mathcal{B}_k with any P chosen from each embedding of \mathcal{B}_k . Now the size of this family is $\text{Pa}^*(n, P) = |P| \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}$.

The two theorems above are so similar, they will be proved simultaneously below.

Proof. Each embedding of P may be viewed inside an embedding of \mathcal{B}_k , where each of the \mathcal{B}_k 's are unrelated, so each family of pairwise unrelated embeddings of P in \mathcal{B}_n is contained in a family of pairwise unrelated embeddings of \mathcal{B}_k . Similarly, each embedding of \mathcal{B}_k contains an embedding of P , so each family of pairwise unrelated embeddings of \mathcal{B}_k in \mathcal{B}_n contains a family of pairwise unrelated embeddings of P . Now finding the extreme families for packings of P is equivalent to finding the extreme families for packings of \mathcal{B}_k and embedding a copy of P into each embedding of \mathcal{B}_k . The size of this family is the size of the poset, $|P|$, times the number of embeddings of P , $\binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}$. \square

This confirms the conjecture by Griggs, Stahl, and Trotter:

Corollary 2.2.5. *Where \mathcal{P}_k is the $k + 1$ element chain, the extreme families constructed from unrelated embeddings of \mathcal{P}_k are the extreme families of unrelated embeddings of \mathcal{B}_k with any \mathcal{P}_k chosen from each embedding of \mathcal{B}_k .*

Now we consider families where each connected component is an embedding from a collection $\{P_i\}$ and not just the single poset P . Let us start with considering the collection of posets to be $\{\mathcal{B}_0, \dots, \mathcal{B}_k\}$. The question is: What is the size of the largest $\mathcal{F} \subseteq \mathcal{B}_n$, where each connected component in \mathcal{F} is an embedding of a boolean lattice no bigger than \mathcal{B}_k , and what are the families that achieve this size?

Theorem 2.2.6. *The value of $\text{Pa}(n, \{\mathcal{B}_0, \dots, \mathcal{B}_k\})$ is exactly $2^k \binom{n-k}{\lfloor \frac{n-k}{2} \rfloor}$, and the families that achieve this bound are the extreme families constructed from embeddings of \mathcal{B}_k and when n and k have different parity, the extreme families constructed from embeddings of \mathcal{B}_{k-1} .*

The idea behind this proof is to count full chains: associate each element in a family to some number of full chains such that no chain is associated with more than one element, and the total sum of these chains over all elements in the family is at most the total number of full chains in \mathcal{B}_n , $n!$.

Proof. Let $\mathcal{F} \subseteq \mathcal{B}_n$ be a family constructed from unrelated embeddings of posets in the collection $\{\mathcal{B}_0, \dots, \mathcal{B}_k\}$ with $|\mathcal{F}| = \text{Pa}(n, \{\mathcal{B}_0, \dots, \mathcal{B}_k\})$, the maximum size possible. An embedding of \mathcal{B}_i meets at least $n!/\binom{n-i}{\lfloor \frac{n-i}{2} \rfloor}$ full chains. Split these full chains evenly among the elements in an embedding of \mathcal{B}_i . In this way, each element in an embedding of \mathcal{B}_i is associated with at least $n!/(2^i \binom{n-i}{\lfloor \frac{n-i}{2} \rfloor})$ full chains. Let $s(i)$ denote the number of elements in the family \mathcal{F} that are in an embedding of \mathcal{B}_i . Now

$$\sum_{i=0}^k s(i) = |\mathcal{F}| \quad \text{and} \quad \sum_{i=0}^k s(i) \frac{n!}{2^i \binom{n-i}{\lfloor \frac{n-i}{2} \rfloor}} \leq n!.$$

We need to minimize the number of full chains associated with an element so we may pack more elements into the family. The minimum of $n!/(2^i \binom{n-i}{\lfloor \frac{n-i}{2} \rfloor})$ over $0 \leq i \leq k$ is achieved when $i = k$ or also if n and k have a different parity, when $i = k - 1$. So using the notation above, if $n - k$ is even, then $s(k) = |\mathcal{F}|$ (i.e., every embedding is an embedding of \mathcal{B}_k), and if $n - k$ is odd, then $s(k - 1) + s(k) = |\mathcal{F}|$ (i.e., every embedding is either an embedding of \mathcal{B}_{k-1} or \mathcal{B}_k). If $n - k$ is even, we are done. For $n - k$ being odd, in order to say that \mathcal{F} may only use embeddings of at most one of \mathcal{B}_{k-1} or \mathcal{B}_k , we need to use Lemma 2.2.1.

We know that to allow for the equality $|\mathcal{F}| = 2^i \binom{n-i}{\lfloor \frac{n-i}{2} \rfloor}$, all $n!$ full chains must be accounted for by intersecting the family. Now we may use the lemma. Let's consider \mathcal{G} , a connected component of largest size in \mathcal{F} . Using the Lemma 2.2.1, we notice that \mathcal{F} is just translations of \mathcal{G} , i.e., each connected component of \mathcal{F} is an embedding of the poset induced from \mathcal{G} . In other words, when the largest connected component of \mathcal{F} is an embedding of \mathcal{B}_k , and \mathcal{F} is such that each full chain meets \mathcal{F} , then each connected component in \mathcal{F} is an embedding of \mathcal{B}_k . The extreme families that achieve

$\text{Pa}(n, \{\mathcal{B}_0, \dots, \mathcal{B}_k\})$ contain embeddings of at most one poset, \mathcal{B}_k or also if $n - k$ is odd, \mathcal{B}_{k-1} . \square

This theorem is interesting in and of itself, but it will also be used to prove Theorem 3.2.1 in Section 3.2, a parallel $\text{La}(n, P)$ question.

In this section, we found the exact values of $\text{Pa}(n, \{P_i\}_{i \geq 1})$ and $\text{Pa}^*(n, \{P_i\}_{i \geq 1})$ for specific collections of posets. This seems to be a difficult problem in general. A problem that is resolvable is the the behavior of $\text{Pa}(n, \{P_i\}_{i \geq 1})$ and $\text{Pa}^*(n, \{P_i\}_{i \geq 1})$ asymptotically for a finite collection of posets. That is the subject of the next section.

2.3 ASYMPTOTICS

In this section we now concentrate on finding the asymptotic behavior of $\text{Pa}(n, P)$ for single posets P . We hope that these ideas might help with solving the more difficult problem of finding $\text{La}(n, \{P_j\})$.

Looking back at the proof of Theorem 2.1.1, the proof that

$$\text{Pa}(n, \mathcal{P}_k) = (k + 1) \binom{n - k}{\lfloor \frac{n-k}{2} \rfloor} \sim \frac{k + 1}{2^k} \binom{n}{\lfloor \frac{n}{2} \rfloor},$$

a similar method may be used to find $\text{Pa}(n, P)$ for general P . One important concept is how each copy of \mathcal{P}_k is contained in a larger set system $\mathcal{I}_{\mathcal{P}}$, where a full chain meets at most one $\mathcal{I}_{\mathcal{P}}$. Similarly, we contain a copy of P inside the convex closure of that copy. The convex closure of a set system is defined as follows: Let $\mathcal{F} \subseteq \mathcal{B}_n$. In \mathcal{B}_n , \mathcal{F} generates an ideal (or down-set) and a filter (or up-set) denoted as follows:

$$D(\mathcal{F}) = \{S \in \mathcal{B}_n \mid S \subseteq A \text{ for some } A \in \mathcal{F}\}, \text{ and}$$

$$U(\mathcal{F}) = \{S \in \mathcal{B}_n \mid A \subseteq S \text{ for some } A \in \mathcal{F}\}.$$

We define a closure operator on \mathcal{F} as $\overline{\mathcal{F}} := D(\mathcal{F}) \cap U(\mathcal{F})$. Another definition would be

$$\overline{\mathcal{F}} := \{S \in \mathcal{B}_n \mid A \subseteq S \subseteq B \text{ for some } A, B \in \mathcal{F}\}.$$

Here S , A , and B could be equal, so clearly $\mathcal{F} \subseteq \overline{\mathcal{F}}$. A family \mathcal{F} such that $\mathcal{F} = \overline{\mathcal{F}}$ is called convex. Note that convex families appear in the literature, including the conjecture by P. Frankl and J. Akiyama:

Conjecture 2.3.1 (Frankl-Akiyama, 1987). *For every convex family $\mathcal{F} \subseteq \mathcal{B}_n$, there exists an antichain $\mathcal{A} \subseteq \mathcal{F}$ such that $|\mathcal{A}| / |\mathcal{F}| \geq \binom{n}{\lfloor \frac{n}{2} \rfloor} / 2^n$.*

If two copies of P are unrelated, then their closures must be unrelated as well. Therefore, we are more interested in the size and structure of the closure of a copy of P than of the copy of P itself. For a weak embedding f of P into \mathcal{B}_k , there exists a minimum value of $|\overline{f(P)}|$ over all choices of f and k . Denote this minimum as $c(P)$.

Here are some examples. If $P = \mathcal{V}$, the poset on $\{a, b, c\}$, where $a < b$ and $a < c$, $\overline{\mathcal{V}} = \mathcal{V}$ so $c(\mathcal{V}) = |\mathcal{V}| = 3$. In the proof of Theorem 2.1.1, the closure of an embedding of a chain \mathcal{P}_k is the smallest interval in which it is enclosed, $\mathcal{I}_{\mathcal{P}}$ in the proof. The smallest size of this interval is 2^k so $c(\mathcal{P}_k) = 2^k$.

Here is one of the two main theorems, finding $\text{Pa}(n, P)$ asymptotically for any P in terms of $c(P)$ and $|P|$.

Theorem 2.3.1. *For any poset P , as $n \rightarrow \infty$, $\text{Pa}(n, P) \sim \frac{|P|}{c(P)} \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

We can similarly define $c^*(P)$ as the minimum size of the closure of a strong embedding of P in \mathcal{B}_n over all possible n . In general, $c^*(P) \neq c(P)$. Take for instance the poset $J = \{a, b, c, d\}$, $a < b < c$, and $a < d$; J may be weakly embedded into \mathcal{B}_2 so $c(J) = 4$. As for f , a strong embedding of J into \mathcal{B}_k , there exists a set $B' \in \mathcal{B}_k$, $f(b) \neq B'$, such that $f(a) \subset B' \subset f(c)$ so $B' \in \overline{f(J)}$, but $f(d) \neq B'$ because $f(d) \not\subseteq f(c)$. Therefore, $c^*(J) \geq 5$. Also, a strong embedding of J into \mathcal{B}_3 is easy to find such that $c^*(J) = 5$.

The second main theorem finds $\text{Pa}^*(n, P)$ asymptotically for any P in terms of $c^*(P)$ and $|P|$.

Theorem 2.3.2. *For any poset P , as $n \rightarrow \infty$, $\text{Pa}^*(n, P) \sim \frac{|P|}{c^*(P)} \binom{n}{\lfloor \frac{n}{2} \rfloor}$.*

We recently learned that this problem of determining asymptotically the maximum number of unrelated copies of a poset P in \mathcal{B}_n was already proposed by Katona at a conference lecture in 2010. We also learned that Katona and Nagy have recently (and independently) obtained results essentially equivalent to our two main, Theorem 2.3.1 and Theorem 2.3.2.

The following two sections are a proof of Theorem 2.3.1. The proof of Theorem 2.3.2 will require only a few alterations. This will be demonstrated after the main proof.

2.4 UPPER BOUND

We obtain the upper bound on the number of unrelated copies of poset P from an asymptotic lower bound on the number of full chains that meet the *closure* of a copy of P . For a family \mathcal{F} of subsets of $[n]$, let $a(\mathcal{F})$ be the number of full chains in \mathcal{B}_n that intersect \mathcal{F} . While $a(\mathcal{F})$ will be as large as $n!$, if, say, \mathcal{F} contains \emptyset , we are interested in how small it can get. If \mathcal{F} consists of m subsets of size k , then $a(\mathcal{F})$ will be $mk!(n-k)! = m(n!/\binom{n}{k})$, which is at least $m\left(n!/\binom{n}{\lfloor \frac{n}{2} \rfloor}\right)$. For fixed m , as n grows we expect this last formula to be the minimum asymptotically. Let us denote by $\bar{a}(m, n)$ the minimum of $a(\mathcal{F})$, over all families $\mathcal{F} \subseteq \mathcal{B}_n$ with $|\mathcal{F}| = m$.

Proposition 2.4.1. *Let integer $m \geq 1$. Then as $n \rightarrow \infty$ the minimum number of full chains in \mathcal{B}_n that meet a family of m subsets in \mathcal{B}_n , $\bar{a}(m, n) \sim m\left(n!/\binom{n}{\lfloor \frac{n}{2} \rfloor}\right)$.*

Proof. Let $\mathcal{F} = \{A_1, \dots, A_m\}$ be a family of m subsets of $[n]$. For convenience let us assume that the subsets are labeled so that for all $i < j$, $|A_i| \leq |A_j|$. For any $1 \leq i_1 < \dots, i_k \leq m$ let $b(i_1, \dots, i_k)$ denote the number of full chains that pass through all of A_{i_1}, \dots, A_{i_k} . Of course, $b(i_1, \dots, i_k)$ is nonzero if and only if the sets A_{i_1}, \dots, A_{i_k} form a chain. Inclusion-exclusion gives us that $a(\mathcal{F})$ is the sum of the

$b(i_1)$ minus the sum of the $b(i_1, i_2)$ plus the sum of the $b(i_1, i_2, i_3)$ minus and so on. Our difficulty now is that some terms $b(i_1, \dots, i_k)$ with $k \geq 2$ can actually be large compared to some singleton terms $b(i_1)$, so we cannot immediately dismiss them. For instance, if $n = 100$ and \mathcal{F} happens to be a chain with $|A_i| = i$ for all i , then $b(1, 2) = 1!1!98!$ is much larger than $b(50) = 50!50!$. However, we can exploit the fact that terms $b(i_1, \dots, i_k)$ with $k \geq 2$ are considerably smaller than some $b(i_1)$ terms. In the example, we could instead compare $b(1, 2)$ to $b(1) = 1!99!$.

By making all signs for terms with $k \geq 2$ negative, our alternating sum lower bound above is at least the sum of the $b(i_1)$ minus the sum over all $k \geq 2$ of the terms $b(i_1, \dots, i_k)$. For the $2^m - m - 1$ terms being subtracted, we assign each one to a particular positive singleton term $b(j)$ as follows: For a term $b(i_1, \dots, i_k)$ with $k \geq 2$, by our labeling we have $|A_{i_1}| \leq \dots \leq |A_{i_k}|$. Let $u := |A_{i_1}|$ and $v := |A_{i_k}|$. We assign this term to one of $b(i_1)$ or $b(i_k)$, resp., according to whether $|u - (n/2)|$ is at least (less than, resp.) $|v - (n/2)|$. For instance in the example above, the terms $b(20, 28)$ and $b(20, 30, 80)$ are assigned to $b(20)$, while $b(20, 30, 81)$ is assigned to $b(81)$.

We have then each singleton term $b(j) = |A_j|!(n - |A_j|)!$. There are less than 2^{m-1} terms $b(i_1, \dots, i_k)$ with $k \geq 2$ assigned to $b(j)$. For those terms that are nonzero, it means that $A_{i_1} \subset \dots \subset A_{i_k}$ and either i_1 or i_k is j , according to which is farther from $n/2$. Suppose $j = i_1$ (so $i_1 < n/2$). Then this term $b(i_1, \dots, i_k)$ is a product of factorials that refines $b(i_1)$: While $i_1!$ is still a factor, $(n - i_1)!$ is replaced by a product of factorials no more than $1!(n - i_1 - 1)!$, so in total, we get at most $b(i_1)$ divided by $(n - i_1)$, which is at least $n/2$. In this case, and similarly when $j = i_k$, we see that the term $b(i_1, \dots, i_k)$ is at most $b(j)$ divided by $n/2$. Therefore, the sum of all the terms assigned to $b(j)$ is at most $b(j)$ times $2^m/n$. Hence,

$$a(\mathcal{F}) \geq \sum_{j=1}^m b(j)(1 - (2^m/n)).$$

Since each term $b(j) = j!(n-j)! \geq n! / \binom{n}{\lfloor \frac{n}{2} \rfloor}$, and this bound holds independent of \mathcal{F} , we see that as $n \rightarrow \infty$ for fixed m , $\bar{a}(m, n) \sim m \left(n! / \binom{n}{\lfloor \frac{n}{2} \rfloor} \right)$. \square

Now we consider our poset packing problem. Assume that we have $\text{Pa}(n, P)/|P|$ unrelated copies \mathcal{F}_i of our poset P contained in the Boolean lattice \mathcal{B}_n . In fact, if a full chain passes through the closure $\overline{\mathcal{F}_i}$ of one of these families \mathcal{F}_i , it does not pass through the closure of any other \mathcal{F}_j , since \mathcal{F}_i and \mathcal{F}_j are unrelated. That is, the closures $\overline{\mathcal{F}_i}$ are also unrelated. Each closure $\overline{\mathcal{F}_i}$ has at least $m = c(P)$ subsets in it so it meets at least $\bar{a}(m, n)$ full chains.

Altogether, the number of full chains that meet some closure $\overline{\mathcal{F}_i}$ is then at least $\bar{a}(m, n) \text{Pa}(n, P)/|P|$. This is in turn at most the total number of full chains, $n!$. Hence, $\text{Pa}(n, P)/|P|$ is at most $n!/\bar{a}(m, n)$, which is asymptotic to $(1/m) \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for large n . This gives the desired asymptotic upper bound.

2.5 LOWER BOUND CONSTRUCTION

Let m, k , and f be such that f embeds P into \mathcal{B}_k , and $|\overline{f(P)}| = m = c(P)$. We will construct an $\mathcal{F} \subseteq \mathcal{B}_n$ from pairwise unrelated copies of $f(P)$ so that the number of copies of P in \mathcal{F} is $\frac{|\mathcal{F}|}{|P|} \sim \frac{1}{m} \binom{n}{\lfloor \frac{n}{2} \rfloor}$.

We will construct \mathcal{F} through a finite number of iterations. Fix an $i \in \mathbb{N}$. This i is the number of iterations for which we construct asymptotically $\frac{(2^k - m)^j}{(2^k)^{j+1}} \binom{n}{\lfloor \frac{n}{2} \rfloor}$ unrelated copies of P for each $0 \leq j \leq i - 1$. For fixed i , as n goes to infinity, we have

$$\frac{|\mathcal{F}|}{|P|} \sim \sum_{j=0}^{i-1} \left(\frac{(2^k - m)^j}{(2^k)^{j+1}} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Now as we increase i to infinity, we will have

$$\begin{aligned} \frac{|\mathcal{F}|}{|P|} &\sim \sum_{j=0}^{\infty} \left(\frac{(2^k - m)^j}{(2^k)^{j+1}} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor} \\ &= \frac{1}{2^k} \left[\frac{1}{1 - \frac{2^k - m}{2^k}} \right] \binom{n}{\lfloor \frac{n}{2} \rfloor} = \frac{1}{m} \binom{n}{\lfloor \frac{n}{2} \rfloor}. \end{aligned}$$

Let's now create such an $\mathcal{F} \subseteq \mathcal{B}_n$ for each i and n . For the rest of the argument, let $(A+x)$ be the translation $\{a+x \mid a \in A\}$ for a set $A \subseteq [n]$ and an integer x . For the ease of notation, define $S_j := [k(j+1)] \setminus [kj] = \{kj+1, kj+2, \dots, kj+k\} = ([k]+kj)$, the set $[k]$ translated by a multiple of k .

A level (or row) of \mathcal{B}_n is all subsets of $[n]$ of the same size, the rank of the level. The level of rank r is often denoted as $\binom{[n]}{r}$. Define a *layer* of \mathcal{B}_n (denoted as ℓ) to be $k+1$ consecutive levels of \mathcal{B}_n . We call the smallest rank in layer ℓ its base rank, b_ℓ . Specifically, $\ell = \binom{[n]}{b_\ell} \cup \binom{[n]}{b_\ell+1} \cup \dots \cup \binom{[n]}{b_\ell+k}$. We define our layers by taking the base ranks to be $\lfloor n/2 \rfloor + z(k+1)$ for all integers z ; in this way, we partition the levels of \mathcal{B}_n and any two layers are disjoint. We construct \mathcal{F} by populating certain layers with many copies of $f(P)$. A layer ℓ that is populated corresponds to a triple (j_ℓ, R_ℓ, b_ℓ) ; ℓ has base rank b_ℓ , the iteration in which it is populated j_ℓ (ranges from 0 to $i-1$), and a restriction set $R_\ell \subseteq [kj_\ell]$, which defines which elements of ℓ are in \mathcal{F} . The following is exactly how \mathcal{F} is constructed in a layer ℓ :

$$\begin{aligned} \ell \cap \mathcal{F} = \{ & R_\ell \cup A \cup B \mid A \subseteq S_{j_\ell}, (A - kj_\ell) \in f(P), \\ & B \subseteq [n] \setminus [k(j_\ell + 1)], \text{ and} \\ & |R_\ell| + |B| = b_\ell \}. \end{aligned}$$

Our choice for the R_ℓ and the order of the b_ℓ 's, as we will show later, prevents any two copies of P in different layers from having any related sets. For a fixed B , the family of all the A 's forms a copy of P that is $f(P)$ translated, from using the elements in $[k]$ to using the elements from S_{j_ℓ} . There is then one copy of P in ℓ for each choice of B . The purpose of B is to combine with R_ℓ to be in the base level of the layer, i.e., $B \cup R_\ell \in \binom{[n]}{b_\ell}$. There are $\binom{n - k(j_\ell + 1)}{b_\ell - |R_\ell|}$ choices for B . Notice that copies of P within a layer are unrelated; every set in a copy of P has the same base set $R_\ell \cup B$, and the copies of P in a layer have unrelated base sets.

For each iteration j , we will be populating $(2^k - m)^j$ layers. This gives a total of only $L := 1$ populated layers if $2^k - m = 1$, or $L := \sum_{j=0}^{i-1} (2^k - m)^j = \frac{(2^k - m)^i - 1}{(2^k - m) - 1}$ populated layers otherwise. The order of the b_ℓ 's of the populated layers is important in preventing any two copies of P from being related, but as long as the order of the populated layers is maintained, the b_ℓ 's for the populated layers may be chosen close to the middle level, i.e., $|b_\ell - \lfloor n/2 \rfloor| \leq (k+1)L$, where L is a constant defined above that does not depend on n and $k+1$ is the number of levels in each layer. Each layer has $\binom{n - k(j_\ell + 1)}{b_\ell - |R_\ell|}$ copies of P . This is now asymptotic to $\frac{1}{(2^k)^{j_\ell + 1}} \binom{n}{\lfloor \frac{n}{2} \rfloor}$ copies since $b_\ell - |R_\ell|$ is at most a fixed, finite distance from $n/2$. This results in our desired number of copies of P ,

$$\frac{|\mathcal{F}|}{|P|} \sim \sum_{j=0}^{i-1} \left(\frac{(2^k - m)^j}{(2^k)^{j+1}} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

We will now demonstrate how each R_ℓ is chosen and in what order are the populated layers to ensure that the copies of P are pairwise unrelated.

Let's start with $j = 0$. We start by populating one layer of F ; let $\mathcal{F} \supseteq \{A \cup B \mid A \in f(P), B \in [n] \setminus [k], |B| = \lfloor n/2 \rfloor\}$. In other words, the layer ℓ with $b_\ell = \lfloor n/2 \rfloor$ is populated with $R_\ell = \emptyset$ and $j_\ell = 0$. Now $|\mathcal{F}| \geq |P| \binom{n - k}{\lfloor n/2 \rfloor}$, which is asymptotically $\frac{1}{2^k} \binom{n}{\lfloor \frac{n}{2} \rfloor}$ copies of P . So if $m = 2^k$, (i.e., $\overline{f(P)} = \mathcal{B}_k$), we are done. If not, we would like to add more copies of P than just those in our middle layer so we will need to know which of the elements of \mathcal{B}_n are available to include in the family; we consider which elements of \mathcal{B}_n are unrelated to any element of this middle, populated layer. Consider a set $B \in \mathcal{B}_n$ and $B_{[k]} := B \cap [k]$ and $b := |B \setminus B_{[k]}|$. This set B is unrelated to all sets in \mathcal{F} if and only if one of the following is true:

1. $B_{[k]}$ is unrelated to all sets in $f(P)$;
2. $B_{[k]} \not\subseteq C$ for all $C \in f(P)$ and $b < \lfloor n/2 \rfloor$; or
3. $B_{[k]} \not\supseteq C$ for all $C \in f(P)$ and $b > \lfloor n/2 \rfloor$.

The choices for $B_{[k]}$ that can provide more sets to add to the family are exactly the sets $B_{[k]} \in \mathcal{B}_k \setminus \overline{f(P)}$. In fact, each one of the $B_{[k]} \in \mathcal{B}_k \setminus \overline{f(P)}$ can lead to a distinct *layer* of copies of P by choosing the base levels correctly; the new layers are the layers from the second iteration (so would have $j_\ell = 1$), and the layer's restriction set would be $B_{[k]}$. The next step is identifying appropriate base levels for each new layer and then demonstrating how this process iterates.

Let's order the elements of $U := \mathcal{B}_k \setminus \overline{f(P)}$. First, split U into two sets, U^+ and U^- :

$$\begin{aligned}
U^+ &:= \mathcal{B}_k \setminus U(f(P)) \\
&= \{V \in U \mid V \not\supseteq C \text{ for all } C \in \overline{f(P)}\}, \text{ and} \\
U^- &:= U \setminus U^+ \\
&= U(f(P)) \setminus \overline{f(P)} \\
&\subseteq \{V \in U \mid V \not\subseteq C \text{ for all } C \in \overline{f(P)}\}.
\end{aligned}$$

The set U^+ contains both the elements of U contained in some element of $f(P)$ and the subsets of $[k]$ that are unrelated to any element of $f(P)$. On the other hand, U^- contains the elements of U containing some element of $f(P)$. Let \leq_U be any ordering of the elements in U such that if $V_1 \in U^-$ and $V_2 \in U^+$, then $V_1 \leq_U V_2$, else if $V_1 \supseteq V_2$, then $V_1 \leq_U V_2$. We will use this ordering \leq_U to order the base ranks to guarantee all copies of P remain unrelated.

For $j = 0$, we have the populated layer corresponding to $(0, \emptyset, \lfloor n/2 \rfloor)$. For $j = 1$, we populate the layers corresponding to $(1, V, b_V)$ for each $V \in U$. We can choose the b_V 's such that if $V \in U^-$, then $b_V < \lfloor n/2 \rfloor$, and if $V \in U^+$, then $b_V > \lfloor n/2 \rfloor$, and if $V_1 <_U V_2$, then $b_{V_1} < b_{V_2}$. For an iteration $j > 1$, for each layer corresponding to $(j-1, R, b)$ populated in iteration $j-1$, we can populate $2^k - m$ new layers, one for each set in U . These new layers correspond to $\left(j, R \cup (V + k(j-1)), b_\ell\right)$ for each $V \in U$. Inductively, there are then $(2^k - m)^j$ layers populated in iteration j , each

with asymptotically $\frac{1}{(2^k)^{j+1}} \binom{n}{\lfloor \frac{n}{2} \rfloor}$ copies of P , for a total of $\frac{(2^k - m)^j}{(2^k)^{j+1}} \binom{n}{\lfloor \frac{n}{2} \rfloor}$ copies of P associated with iteration j . All that is left to prove is that we can put the layers in an appropriate order, i.e., the base ranks (b_ℓ) may be chosen in such a way as to prevent any two copies of P from being related.

Let $(\ell_s)_{1 \leq s \leq L}$ be the sequence of populated layers, ℓ_s corresponding to (j_s, R_s, b_s) , in the order of the rank of the base levels, i.e., for all $s_1 < s_2$, $b_{s_1} < b_{s_2}$. Let's consider our ordered set U again. Let's add to U the character E to indicate the 'end' of a word. Let E be between U^- and U^+ in \leq_U . Consider words $V_0 V_1 \dots V_{j-1} E$, where the letters come from U , the words always end in E , and E is only at the end of a word. We only consider words of length $j + 1$, where $0 \leq j \leq i - 1$. There is a bijection between the layers (ℓ_s) and the possible words of length at most i . Specifically, given a word $V_0 V_1 \dots V_{j-1} E$, its corresponding j_s is j and $R_s = \cup_{p=0}^{j-1} (V_p + kp)$. Let W be the set of all words of length at most i . Order these words lexicographically using \leq_U . Specifically, for any two words in W , $w_p = U_0 \dots U_s$ and $w_q = V_0 \dots V_t$, we say $w_1 < w_2$ if and only if $U_i = V_i$ for $0 \leq i \leq j - 1$ and $U_j <_U V_j$ for some $j \geq 0$. Use this ordering of W and the bijection between the words and the layers to directly define the corresponding ordering of the (b_s) . Specifically, for two layers ℓ_1 and ℓ_2 with base level ranks b_1 and b_2 respectively, $b_1 < b_2$ if and only if the word corresponding to ℓ_1 is less than the word corresponding to ℓ_2 .

Now we show that no two copies of P are related. We have already seen that no two copies of P in the same layer can be related. For two copies of P , P_p in layer ℓ_p (with base rank b_p) and P_q in layer ℓ_q (with base rank b_q), consider their corresponding words, $w_p = U_0 \dots U_s$ and $w_q = V_0 \dots V_t$. Without loss of generality, let $w_p <_U w_q$ so $b_p < b_q$. Consider the subscript c for the first character where w_p and w_q differ, i.e., $U_0 \dots U_{c-1} = V_0 \dots V_{c-1}$ and $U_c \neq V_c$, $U_c <_U V_c$. Choose any representatives of the copies of P , $A_p \in P_p$ and $A_q \in P_q$, and define $B_p := A_p \cap S_c$ and $B_q = A_q \cap S_c$. Since $b_p < b_q$, we have that $A_q \not\subseteq A_p$; next we show that $A_p \not\subseteq A_q$.

The order of the words, and hence the order of the b_ℓ 's, was chosen specifically to prevent any copies of P from being pairwise related. If $U_c = E$, then $(B_p - kc) \in f(P)$ and $V_c \in U^+$ so $V_c \not\subseteq C$ for all $C \in f(P)$, i.e., $(V_c + kc) = B_q \not\subseteq B_p$ for any B_p such that $(B_p - kc) \in f(P)$. But $B_q \not\subseteq B_p$ implies $A_q \not\subseteq A_p$. For similar reasoning, if $V_c = E$, then $A_p \not\subseteq A_q$. If neither $U_c = E$ nor $V_c = E$, then $U_c < V_c$ implies $U_c \not\subseteq V_c$, but $U_c = (B_p + kc)$ and $V_c = (B_q + kc)$ so $B_p \not\subseteq B_q$ so $A_p \not\subseteq A_q$. Either way, no set from P_p is related to any set from P_q . This completes the proof of Theorem 2.3.1.

2.6 MODIFYING PROOF FOR STRONG EMBEDDINGS

We now explain how we may modify the proof above to prove Theorem 2.3.2. In proving the upper bound of Theorem 2.3.1, we use the fact that the closure of a copy of P meets at least $\bar{a}(c(P), n)$ full chains in \mathcal{B}_n . For Theorem 2.3.2, using only strong embeddings, we have that a copy of P meets at least $\bar{a}(c^*(P), n)$, which similarly gives us the upper bound. In the lower bound of Theorem 2.3.1, we created a family $\mathcal{F} \subseteq \mathcal{B}_n$ constructed from multiple copies of $f(P)$, a weak embedding of P into \mathcal{B}_k such that $\overline{f(P)} = c(P)$. If we instead take f to be a strong embedding such that $\overline{f(P)} = c^*(P)$, then the same method of construction will achieve the asymptotic lower bound.

2.7 EXTENDING THEOREMS FOR A FINITE COLLECTION OF POSETS

The previous sections focused on finding $\text{Pa}(n, P)$ and $\text{Pa}^*(n, P)$ asymptotically for a single poset. For a finite collection of posets, the quantities $\text{Pa}(n, \{P_1, \dots, P_k\})$ and $\text{Pa}^*(n, \{P_1, \dots, P_k\})$ may be found asymptotically as well.

Theorem 2.7.1. *As n goes to infinity,*

$$\text{Pa}(n, \{P_1, P_2, \dots, P_k\}) \sim \max_{1 \leq i \leq k} \left(\frac{|P_i|}{c(P_i)} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}, \text{ and}$$

$$\text{Pa}^*(n, \{P_1, P_2, \dots, P_k\}) \sim \max_{1 \leq i \leq k} \left(\frac{|P_i|}{c^*(P_i)} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}.$$

Let's look at just the proof of finding $\text{Pa}(n, \{P_1, P_2, \dots, P_k\})$ asymptotically first, and then just as in Section 2.6, the proof of finding $\text{Pa}^*(n, \{P_1, P_2, \dots, P_k\})$ asymptotically will be clear.

Proof. The lower bound is clear; use the same construction that asymptotically achieves $\text{Pa}(n, P)$, where P is a poset in $\{P_1, \dots, P_k\}$ such that $\frac{|P|}{c(P)}$ (or $\frac{|P|}{c^*(P)}$) is maximized. For the upper bound, we do a similar chain counting argument. Let $\mathcal{F} \subseteq \mathcal{B}_n$ be a family, where each connected component of \mathcal{F} is an embedding of one of the posets in $\{P_1, P_2, \dots, P_k\}$. Each embedding of a poset has a closure. Since no chain may meet more than one of these closures, we may partition these full chains and assign them to the elements of \mathcal{F} . Specifically, each element $A \in \mathcal{F}$ is a part of an embedding of some P_i , the size of which is at least $c(P_i)$ (or $c^*(P_i)$). The number of full chains that meet this embedding is at least $\bar{a}(c(P_i), n)$ (as defined in the proof of the upper bound Section 2.4) so assign to A at least $\frac{\bar{a}(c(P_i), n)}{|P_i|}$ of these full chains. Define $s(i)$ to be the number of elements of \mathcal{F} that are a part of an embedding of P_i . Now by counting the full chains, we have

$$\sum_{i=1}^k s(i) = |\mathcal{F}| \quad \text{and} \quad \sum_{i=1}^k s(i) \frac{\bar{a}(c(P_i), n)}{|P_i|} \leq n!.$$

Continuing,

$$\min_{1 \leq i \leq k} \left(\frac{\bar{a}(c(P_i), n)}{|P_i|} \right) \sum_{i=1}^k s(i) \leq n!,$$

$$\sum_{i=1}^k s(i) \leq \max_{1 \leq i \leq k} \left(\frac{|P_i| n!}{\bar{a}(c(P_i), n)} \right) \sim \max_{1 \leq i \leq k} \left(\frac{|P_i|}{c(P_i)} \right) \binom{n}{\lfloor \frac{n}{2} \rfloor}$$

by Theorem 2.4.1. □

Since the process of packing posets into a Boolean lattice and finding families forbidding certain posets is directly linked, we may now use this result to calculate the value of various $\text{La}(n, \{P_i\}_{i \geq 1})$ as we do in the next chapter.

CHAPTER 3

FAMILIES FORBIDDING POSETS

3.1 PREVIOUS RESULTS

What is the largest size $\text{La}(n, \{P_i\}_{i \geq 1})$ of a family $\mathcal{F} \subseteq \mathcal{B}_n$ that does not contain any $P_i \in \{P_i\}_{i \geq 1}$ as a weak embedding? Recall that Sperner's Theorem from 1928 answers this question for a two-element chain: $\text{La}(n, \mathcal{P}_1) = \binom{n}{\lfloor \frac{n}{2} \rfloor} = \Sigma(n, 1)$. Moreover, the value $\text{La}(n, \mathcal{P}_1)$ is attained only by the family $\mathcal{B}(n, 1)$, which consists of subsets all of (a) middle size. Erdős in 1945 generalized this to $P = \mathcal{P}_k$, the $k + 1$ element chain, showing that $\text{La}(n, \mathcal{P}_k) = \Sigma(n, k)$, which is attained only by $\mathcal{B}(n, k)$. The proof of this result is included to demonstrate a common technique as well as give a new proof that the family $\mathcal{B}(n, k)$ is the only extreme family.

Theorem 3.1.1 (Erdős, 1945). *The quantity $\text{La}(n, \mathcal{P}_k) = \Sigma(n, k)$, which is attained only by $\mathcal{B}(n, k)$.*

Just as in the proof of Sperner's Theorem, we include a method devised independently by Lubell, Yamamoto, and Mešalkin now called the LYM-inequality.

Proof. Let $\mathcal{F} \subseteq \mathcal{B}_n$ be a largest family that does not contain a weak embedding of \mathcal{P}_k in \mathcal{B}_n . Each $A \in \mathcal{F}$ meets exactly $|A|!(n - |A|)!$ full chains. Each chain meets the family \mathcal{F} at most k times, else the family would contain a \mathcal{P}_k . Now the sum of all the times a full chain meets the family is at most the total number of full chains multiplied by the number of possible times the chain meets the family, giving a total

of $kn!$. This gives the inequality

$$\sum_{A \in \mathcal{F}} |A|!(n - |A|)! \leq kn!, \text{ or}$$

$$\sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq k.$$

Since $|\mathcal{F}|$ is maximized by maximizing $\binom{n}{|A|}$ as much as possible for A , we see that it is best to take A to be as close to the middle as possible. The maximum number of terms is then $\Sigma(n, k)$, the sum of the sizes of the middle k levels of \mathcal{B}_n .

Notice that the proof using the LYM-inequality does not give us all the extreme families like in the statement of the theorem, just that the size of the family is at most $\Sigma(n, k)$. The rest of the argument gives the form of all the extreme antichains and is a nice continuation of the LYM-inequality part of the proof above.

The families $\mathcal{B}(n, k)$ are of size $\Sigma(n, k)$; we just need to prove these are the only extreme families forbidding a \mathcal{P}_k . Let \mathcal{F} be an family forbidding \mathcal{P}_k such that $|\mathcal{F}| = \Sigma(n, k)$. Notice that this implies that each full chain meets the family exactly k times by making the inequalities in the proof above tight. If A is in \mathcal{F} , then $A \setminus \{a\} \cup \{b\}$ for $a \in A$ and $b \notin A$ is also in the family, else any chain that meets $A \setminus \{a\}$, $A \setminus \{a\} \cup \{b\}$, and $A \cup \{b\}$ meets the family only $k - 1$ times.

Now the entire level containing A is also in the family \mathcal{F} . Since \mathcal{F} forbids an embedding of \mathcal{P}_k , \mathcal{F} consists of all elements from at most k levels. The only k levels that sum to $\Sigma(n, k)$ are $\mathcal{B}(n, k)$. \square

In 1983, Katona and Tarján brought the topic back under focus. They showed results involving the posets \mathcal{V} and Λ . The poset \mathcal{V} is $\{a, b, c\}$ with $a < b$ and $a < c$. The poset Λ is $\{a, b, c\}$ with $a > b$ and $a > c$.

Theorem 3.1.2 (Katona-Tarján, 1983). *The quantity $\text{La}(n, \{\mathcal{V}, \Lambda\}) = 2 \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$, and $\text{La}(n, \mathcal{V}) \sim \binom{n}{\lfloor \frac{n}{2} \rfloor}$ as n goes to infinity.*

Notice the exact value of $\text{La}(n, \{\mathcal{V}, \Lambda\})$ is found, but only an asymptotic approximation of $\text{La}(n, \mathcal{V})$ in terms of $\binom{n}{\lfloor \frac{n}{2} \rfloor}$ is found. In general, the exact size of extreme families will not be easy to find, even for a poset as simple as \mathcal{V} . Therefore, it is appropriate to find asymptotic bounds on this value.

Specifically, if it exists, define

$$\pi(P) := \lim_{n \rightarrow \infty} \frac{\text{La}(n, P)}{\binom{n}{\lfloor \frac{n}{2} \rfloor}}.$$

It was conjectured by Griggs and Lu that $\pi(P)$ exists and is an integer for all posets.

Conjecture 3.1.1 (Griggs-Lu, 2009). *For any poset P , the limit $\pi(P)$ exists and is an integer.*

Specifically, it is believed that the integer value of $\pi(P)$ would be $e(P)$, defined to be the largest value such that $\mathcal{B}(n, e(P))$ does not contain a weak embedding of P for all values of n . It is clear that $\pi(P)$ would be at least the size of $e(P)$, since $\mathcal{B}(n, e(P))$ does not contain P .

Some recent results have been made in the area. Bukh in 2009 summarized many of the previous results in the area with the following theorem:

Theorem 3.1.3 (Bukh, 2009). *If the Hasse diagram of a poset P is a tree, then $\pi(P) = e(P)$.*

The value of $\pi(P)$ has been found for other posets that do not have a Hasse diagram of a tree. Consider the butterfly poset, denoted \mathcal{O}_4 , consisting of $\{a, b, c, d\}$ with $a < c$, $a < d$, $b < c$, and $b < d$.

Theorem 3.1.4 (De Bonis et al., 2005). *Not only is $\pi(\mathcal{O}_4) = e(\mathcal{O}_4) = 2$, but $\text{La}(n, \mathcal{O}_4) = \Sigma(n, 2)$ for $n \geq 3$.*

This can be expanded to include other *crowns*, \mathcal{O}_{2k} , where \mathcal{O}_{2k} is the poset of height 2 with a Hasse diagram of a cycle of length $2k$.

Theorem 3.1.5 (Griggs-Lu, 2009). *For even $k \geq 4$, $\pi(\mathcal{O}_{2k}) = 1$.*

For odd values of $k \geq 3$, the value (if it exists) is unknown. Another type of poset slightly understood is the diamond poset, \mathcal{D}_k , with elements $\{a, b_1, \dots, b_k, c\}$ with $a < b_i < c$ for all $1 \leq i \leq k$.

Theorem 3.1.6 (Griggs-Li-Lu, 2011). *For $k \geq 2$, define $m := \lceil \log_2(k+2) \rceil$. If $2^{m-1} - 1 \leq k \leq 2^m - \binom{m}{\lfloor \frac{m}{2} \rfloor} - 1$, then $\text{La}(n, \mathcal{D}_k) = \Sigma(n, m)$, and hence $\pi(\mathcal{D}_k) = e(\mathcal{D}_k)$.*

Notice that many values of k (including $k = 2$) do not satisfy the conditions of the theorem above.

There are many other posets P where it has been found that $\pi(P) = e(P)$, but probably the most glaring collection of posets for which it is unknown is the collection of Boolean lattices, \mathcal{B}_k for $k \geq 2$. The most tantalizing is the poset $\mathcal{B}_2 = \mathcal{D}_2$, the diamond. The current best asymptotic upper bound to $\text{La}(n, \mathcal{B}_2)$ is due to Lucas Kramer, Ryan Martin, and Michael Young.

Theorem 3.1.7 (Kramer-Martin-Young, 2012). *If it exists, the value of $\pi(\mathcal{B}_2)$ is at most 2.25.*

They believe they can push their bound to achieve $\pi(\mathcal{B}_2) \leq \frac{53}{24} < 2.2084$. Even if we restrict the question to only consider families in $\mathcal{B}(n, 3)$, the middle three levels of \mathcal{B}_n , the question of the asymptotic size of the largest diamond-free family is still open. The current best bound is from Balogh, Hu, Lidický, and Liu.

Theorem 3.1.8 (Balogh-Hu-Lidický-Liu, 2012). *If it exists, the value of $\pi(\mathcal{B}_2)$ restricted to just the middle three levels of \mathcal{B}_n is at most 2.15121.*

The last two theorems use a proof technique called flag algebras, which has been used to improve many bounds in extremal graph theory.

3.2 FORBIDDING BOTH \mathcal{V} AND Λ

This section demonstrates the usefulness of finding the extreme families for some $\text{Pa}(n, \{P_i\}_{i \geq 1})$ as in Chapter 2, for now we may find the extreme families for some values of $\text{La}(n, \{P_i\}_{i \geq 1})$. This was our motivation for looking at $\mathcal{P}(n, \{P_i\}_{i \geq 1})$ originally. In 1983, Katona and Tarján originally found the value of $\text{La}(n, \{\mathcal{V}, \Lambda\})$, but they did not identify all the extreme families. We do that here.

Theorem 3.2.1. *Not only is $\text{La}(n, \{\mathcal{V}, \Lambda\}) = 2 \binom{n-1}{\lfloor \frac{n-1}{2} \rfloor}$ as seen by Katona and Tarján in 1983, but the extreme families of \mathcal{B}_n that contain neither weak embeddings of \mathcal{V} nor Λ are the extreme families that are constructed from unrelated embeddings of \mathcal{B}_1 or when n is even, the family that is the middle level.*

Proof. Notice that a family that contains neither a \mathcal{V} nor a Λ is constructed from unrelated embeddings of either \mathcal{B}_0 or \mathcal{B}_1 . In this way, $\text{La}(n, \{\mathcal{V}, \Lambda\}) = \text{Pa}(n, \{\mathcal{B}_0, \mathcal{B}_1\})$. Based on Theorem 2.2.6, the extreme families that achieve $\text{Pa}(n, \{\mathcal{B}_0, \mathcal{B}_1\})$ are the extreme families that are constructed from unrelated embeddings of \mathcal{B}_1 or when n is even, the family that is the middle level. \square

As shown in Theorem 2.2.2, the extreme packings of \mathcal{B}_1 are the families in the middle two levels, the level of smaller rank consisting of all sets without the element x and the level of larger rank consisting of all the sets with the element x for some fixed element $x \in [n]$.

3.3 LUBELL FUNCTION

A useful technique to improve the bound of $\text{La}(n, P)$ is the Lubell function, defined for a family $\mathcal{F} \subseteq \mathcal{B}_n$ as

$$\bar{h}(\mathcal{F}) := \frac{\sum_c |\mathcal{F} \cap \mathcal{C}|}{n!} = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}}$$

for every full chain \mathcal{C} in \mathcal{B}_n . This is the expected number of times a random full chain will meet the family \mathcal{F} . Just as in the proof of Theorem 3.1.1, for any real k , if

$$\bar{h}(\mathcal{F}) = \sum_{A \in \mathcal{F}} \frac{1}{\binom{n}{|A|}} \leq k,$$

then $|\mathcal{F}|$ is maximized by taking \mathcal{F} to be elements in the middle levels, implying $|\mathcal{F}| \leq k \binom{n}{\lfloor \frac{n}{2} \rfloor}$, and if k is an integer, $|\mathcal{F}| \leq \Sigma(n, k)$. Proving an asymptotic upper bound on $\bar{h}(\mathcal{F})$ then gives an asymptotic upper bound for $|\mathcal{F}|$. Many of the results improving $\text{La}(n, P)$ for some posets P are actually improvements on $\bar{h}(\mathcal{F})$ for P -free families $\mathcal{F} \subseteq \mathcal{B}_n$.

3.4 DIAMOND-FREE FAMILIES IN THE MIDDLE THREE LEVELS

Finding the largest \mathcal{B}_2 -free family should help us find the largest P -free family for a more general P . Remember the middle k levels of \mathcal{B}_n is denoted $\mathcal{B}(n, k)$. Finding the largest \mathcal{B}_2 -free family in $\mathcal{B}(n, 3)$ may help us understand how to find the largest \mathcal{B}_2 -free family in \mathcal{B}_n . Let us consider some \mathcal{B}_2 -free families, their sizes, and the value of their Lubell function.

The first family is $\mathcal{B}(n, 2)$. Clearly this is \mathcal{B}_2 -free since $e(\mathcal{B}_2) = 2$. The size of $\mathcal{B}(n, 2)$ is $\Sigma(n, 2)$, which is $2 \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for odd values of n and $(2 - \frac{2}{n+2}) \binom{n}{\lfloor \frac{n}{2} \rfloor}$ for even values of n . The Lubell value of this family is 2 for both even and odd values of n . This family is the best known family in terms of size for n being odd.

Now let us consider a new \mathcal{B}_2 -free family denoted $\mathcal{F}' \subseteq \mathcal{B}(n, 3)$. For $n \geq 2$, fix A to be the elements $\{1, 2\}$. Let the three levels be of rank $\lfloor n/2 \rfloor - 1$, $\lfloor n/2 \rfloor$, and $\lfloor n/2 \rfloor + 1$. Of the sets of size $\lfloor n/2 \rfloor - 1$, include in \mathcal{F}' all the sets that do not have both the elements of A . Of the sets of size $\lfloor n/2 \rfloor$, include in \mathcal{F}' all the sets that do not have exactly one of the elements of A . Of the sets of size $\lfloor n/2 \rfloor + 1$, include in \mathcal{F}' all the sets that have either of (or both of) the elements of A . This is depicted in Figure 3.1.

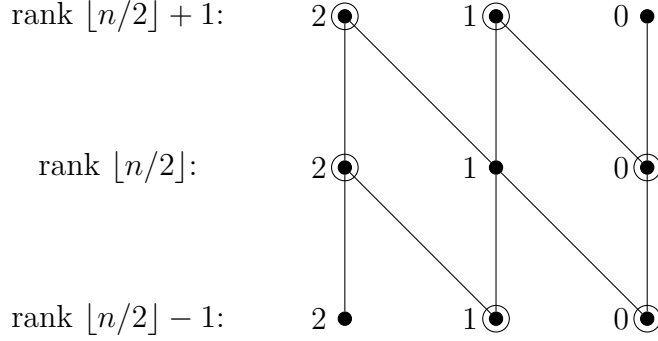


Figure 3.1 The family \mathcal{F}' , for $n \geq 2$, each node represents all the sets on its layer with exactly the labeled number of elements from a fixed, two element set. A node is circled if every set the node represents is in the family. This family is \mathcal{B}_2 -free.

For odd values of n , this family \mathcal{F}' has size $(2 - \frac{1}{n}) \binom{n}{\lfloor \frac{n}{2} \rfloor}$, which is strictly smaller than $\Sigma(n, 2)$, but for even values of n , $|\mathcal{F}'| = (2 - \frac{1}{n-1}) \binom{n}{\lfloor \frac{n}{2} \rfloor}$, which is strictly bigger than $\Sigma(n, 2)$ for $n \geq 6$. Also of interest is the Lubell value of this family:

$$\bar{h}(\mathcal{F}') = 2 + \frac{4}{n} - \frac{2}{n-1}$$

for $n \geq 2$ being either even or odd. After searching the middle levels of \mathcal{B}_n for small n , this family seems to be the uniquely largest \mathcal{B}_2 -free family for even $n \geq 8$ and have the uniquely largest Lubell value for all $n \geq 4$.

CHAPTER 4

SUPERSATURATION

4.1 PREVIOUS RESULTS

In Chapter 3 we consider the value of $\text{La}(n, P)$ for a fixed poset P . This is the largest a family in \mathcal{B}_n may be before it contains a weak embedding of P . If the size of a family $\mathcal{F} \subseteq \mathcal{B}_n$ exceeds the value of $\text{La}(n, P)$, then \mathcal{F} will contain at least one embedding of P . Supersaturation questions ask what is the minimum number of these embeddings, and which families achieve this minimum? These questions are parallel to a well studied topic in extremal graph theory.

A core topic of extremal graph theory is the study of “Turán-type questions”: fix a (finite) graph H and a positive integer n . What is the largest number $\text{ex}(n, H)$ of edges in an n -vertex graph that contains no copy of H ? More than a hundred years ago, Mantel answered this question in the case where H is K_3 , the triangle. About forty years later, Turán generalized this to all complete graphs. More precisely, the *Turán graph* $T(n, r)$ is the complete r -partite graph of order n with parts of size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$. Not only did Turán prove that $T(n, r)$ has the largest number of edges among all n -vertex graphs with no copies of K_{r+1} , that is, $\text{ex}(n, r) = |E(T(n, r))|$, but also he proved that all other n -vertex graphs containing no copies of K_{k+1} have strictly fewer edges than $T(n, r)$.

The theory of graph supersaturation deals with the situation beyond the threshold given by $\text{ex}(n, H)$. Specifically, define $\ell(n, H, q)$ as the least number of copies of H in an n -vertex graph with at least $\text{ex}(n, H) + q$ edges. By the definition, we know

that $\ell(n, H, q) \geq 1$ as soon as $q \geq 1$, but it turns out that an extra edge is likely to create many more copies of H . Arguably, the first result in this direction was proved in an unpublished work of Rademacher from 1941 (orally communicated to Erdős): while Mantel's theorem states that every n -vertex graph with more than $|E(T(n, 2))| = \lfloor n/2 \rfloor \lceil n/2 \rceil$ edges contains a triangle, Rademacher established that such graphs contain, actually, at least $\lfloor n/2 \rfloor$ triangle copies.

This result was generalized by Erdős, who proved that $\ell(n, K_3, q) \geq \lfloor n/2 \rfloor$ first if $q \in \{1, 2, 3\}$ in 1955 and a few years later in the case $q < c \cdot n/2$ for a fixed constant $c \in (0, 1)$. More than twenty years later, Lovász and Simonovits established the following theorem, thereby confirming a conjecture of Erdős.

Theorem 4.1.1 (Lovász and Simonovits (1983)). *Let n and q be positive integers. If $q < n/2$, then $\ell(n, K_3, q) \geq q \cdot \lfloor n/2 \rfloor$.*

In addition, Lovász and Simonovits determined $\ell(n, K_r, q)$ when $q = o(n^2)$. Their techniques do not apply, though, for the case where $q = \Omega(n^2)$. Solutions to this difficult problem were provided recently with the aid of flag algebras: first, by Razborov for $H = K_3$, then, by Nikiforov for $H = K_4$ and, finally, by Reiher for the general case $H = K_r$.

Supersaturation results have not to our knowledge been studied as extensively in other important areas of extremal combinatorics. In this paper, we pursue this direction for extremal set theory.

However, for those P for which we know the exact threshold, we can ask how many copies of P must be present in families larger than the threshold $\text{La}(n, P)$. Here we investigate the simplest instance of this problem, when P is a chain. Analogous to the way that Rademacher and Erdős (and subsequent researchers) have extended the theorems of Mantel and Turán, we present a supersaturation extension of Sperner's Theorem and its k -chain generalization by Erdős.

Remember \mathcal{C}_k is the k element chain. Our initial result was a lower bound on the number of \mathcal{C}_2 's in a family $\mathcal{F} \subseteq 2^{[n]}$ of a given size that is optimal for $|\mathcal{F}| \leq \Sigma(n, 2)$, extending Sperner's Theorem. By investigating more examples, we came to believe that for any size $|\mathcal{F}|$, with $\Sigma(n, \ell) \leq |\mathcal{F}| \leq \Sigma(n, \ell + 1)$, the number of \mathcal{C}_2 's in \mathcal{F} is minimized by taking \mathcal{F} to consist of $\mathcal{B}(n, \ell)$ together with subsets of $\mathcal{B}(n, \ell + 1)$. In further exploration of problems related to poset-free families of subsets, we came across the work of Kleitman from 1968, which corroborates our findings and intuition. Indeed, Kleitman, albeit with matching theory techniques, obtained the (same) supersaturation extension of Sperner's Theorem and more. This settled a conjecture of Erdős and Katona. In particular, he determined the minimum number of pairs (A, B) with $A \subset B$ in a family $\mathcal{F} \subseteq 2^{[n]}$ of *any* given size. As we had intuited, taking the subsets of some middle sizes attains the optimum.

4.2 SYMMETRIC CHAIN DECOMPOSITION

One particularly nice way to quickly derive Sperner's Theorem and its generalization by Erdős is to employ the remarkable symmetric chain decomposition (*SCD*, for short) of all 2^n subsets of $[n]$, discovered by de Bruijn, van Ebbenhorst Tengbergen, and Kruyswijk in 1951. It is a partition of the Boolean lattice into just $\binom{n}{\lfloor n/2 \rfloor}$ disjoint chains of subsets, where for each chain there is some $k \leq n/2$ such that the chain consists of a subset of each size from k to $n - k$. For all k the decomposition induces the best possible upper bound on $|\mathcal{F}|$ for a \mathcal{C}_k -free family $\mathcal{F} \subseteq \mathcal{B}_n$. (It requires some additional arguments to obtain the extremal families.) The construction, which is obtained by a clever inductive argument, was done originally in the more general setting of a product of chains. In this way, the authors obtained the extension of Sperner's Theorem to the lattice of divisors of an integer N .

There is a large literature on the existence of SCDs in posets and other ordered/ranked set systems. Greene and Kleitman discovered an explicit SCD of the

Boolean lattice for all n , based on a simple “bracketing procedure”, as opposed to the original inductive construction.

It is not surprising then that a SCD of \mathcal{B}_n yields a lower bound on the number of paths in a family \mathcal{F} of given size. In particular, if we arbitrarily consider one particular SCD, the number of chains in \mathcal{F} that are also chains in the SCD is minimized by taking the sets of \mathcal{F} to be of the middle sizes. However, this argument does not account for the many containment relations for pairs of subsets $A \subset B$ where A and B are on different chains in the SCD. To adjust for this, and to exploit symmetry by avoiding bias towards a particular SCD, our new idea here is to take all $n!$ SCDs obtained by permutation of the ground set $[n]$. In this way, we obtain lower bounds on the number of paths in a family \mathcal{F} of given size, bounds that are best possible for small \mathcal{F} .

4.3 SUPERSATURATION OF k -CHAINS

Our main aim in this chapter is then to prove the following supersaturation extension of the theorems of Sperner and Erdős, using the above-outlined SCD approach.

Theorem 4.3.1. *If a family $\mathcal{F} \subseteq \mathcal{B}_n$ satisfies $|\mathcal{F}| = x + \Sigma(n, k - 1)$, then there must be at least*

$$x \cdot \prod_{i=1}^{k-1} \left(\left\lfloor \frac{n+k}{2} \right\rfloor - i + 1 \right)$$

copies of \mathcal{C}_k in \mathcal{F} .

Note that

$$\prod_{i=1}^{k-1} \left(\left\lfloor \frac{n+k}{2} \right\rfloor - i + 1 \right)$$

is the number of copies of \mathcal{C}_k contained in $\mathcal{B}(n, k)$, with one endpoint of the chain being a particular set in the k th middle row. Thus the family that consists of $\mathcal{B}(n, k - 1)$

and x sets from the k th middle row witnesses that the above bound is tight for

$$x \leq \binom{n}{\lfloor \frac{n}{2} \rfloor} + (-1)^k \binom{k}{\lfloor \frac{k}{2} \rfloor}.$$

More generally, Kleitman has conjectured that for any k the natural construction (that selects subsets around the middle) minimizes the number of chains \mathcal{C}_k in \mathcal{F} . Our result gives new information in support of this conjecture, verifying it for $|\mathcal{F}| \leq \Sigma(n, k)$.

As mentioned earlier, we shall use the symmetric chain decomposition of \mathcal{B}_n .

Proof of Theorem 4.3.1. Given a poset (P, \preceq) on \mathcal{B}_n , let us say that a k -chain $A_1 \subset \dots \subset A_k$ of $\mathcal{F} \subseteq \mathcal{B}_n$ is *included* in P if $A_1 \prec \dots \prec A_k$, and furthermore define $c_{\mathcal{F}}(P)$ to be the number of k -chains of \mathcal{F} included in P . For any SCD \mathcal{D} of \mathcal{B}_n , let $P_{\mathcal{D}}$ be the poset on \mathcal{B}_n defined by taking the disjoint union of the chains in \mathcal{D} . Let us fix the SCD \mathcal{D} . By the pigeonhole principle, $P_{\mathcal{D}}$ includes at least x k -chains of \mathcal{F} , i.e. $c_{\mathcal{F}}(P_{\mathcal{D}}) \geq x$. Each (non-trivial) permutation π of $[n]$ applied to \mathcal{D} results in a new unique SCD $\pi(\mathcal{D})$ for \mathcal{B}_n . Note that $\pi(\mathcal{D}) \neq \pi'(\mathcal{D})$ for distinct permutations π and π' of $[n]$. By summing over the permutations π of $[n]$, we obtain

$$n! \cdot x \leq \sum_{\pi} c_{\mathcal{F}}(P_{\pi(\mathcal{D})}).$$

Let us change the summation to sum over all k -chains of \mathcal{F} . For this, we define $N(n, A_1, \dots, A_k)$ to be the number of permutations π such that $P_{\pi(\mathcal{D})}$ includes a given chain $A_1 \subset \dots \subset A_k$ of \mathcal{F} . We obtain

$$\sum_{\pi} c_{\mathcal{F}}(P_{\pi(\mathcal{D})}) = \sum_{\substack{A_1 \subset \dots \subset A_k \\ A_i \in \mathcal{F}}} N(n, A_1, \dots, A_k).$$

Setting $a_i := |A_i|$ for each $i \in \{1, \dots, k\}$, it holds that

$$N(n, A_1, \dots, A_k) = a_1! \cdot (a_2 - a_1)! \cdots (a_k - a_{k-1})! \cdot (n - a_k)! \cdot \min \left\{ \binom{n}{a_1}, \binom{n}{a_k} \right\},$$

where the last factor comes from the number of chains in a SCD that the given chain could fit. After some manipulation, we deduce that

$$N(n, A_1, \dots, A_k) = \frac{n!}{\max \left\{ \binom{a_k}{a_{k-1}} \cdots \binom{a_2}{a_1}, \binom{n-a_1}{n-a_2} \cdots \binom{n-a_{k-1}}{n-a_k} \right\}}.$$

We shall find a general upper bound for $N(n, A_1, \dots, A_k)$ by minimizing the maximum of y defined as $\binom{a_k}{a_{k-1}} \cdots \binom{a_2}{a_1}$ and z defined as $\binom{n-a_1}{n-a_2} \cdots \binom{n-a_{k-1}}{n-a_k}$. Note the following binomial identity:

$$\binom{a+i+j}{a+i} \binom{a+i}{a} = \binom{a+i+j}{a+j} \binom{a+j}{a}.$$

As a consequence of this, the values of y and z are invariant as long as the multiset of all differences between consecutive values of a_i is invariant. By this fact, if there is some difference in this multiset that is at least 2, we may assume without loss of generality that this “large” difference is between a_{k-1} and a_k . It follows that

$$y' := y \cdot \frac{\binom{a_{k-1}+1}{a_{k-1}}}{\binom{a_k}{a_{k-1}}} = y \cdot \frac{a_{k-1}+1}{\binom{a_k}{a_{k-1}}} < y,$$

provided that $a_{k-1} > 0$. Similarly,

$$z' := z \cdot \frac{\binom{n-a_2+1}{n-a_2}}{\binom{n-a_1}{n-a_2}} = z \cdot \frac{n-a_2+1}{\binom{n-a_1}{n-a_2}} < z,$$

provided that $a_2 < n$. It follows that y and z are minimized when the multiset of differences is the multiset of all ones, i.e. with

$$y = \frac{a_k!}{(a_k - k + 1)!} \quad \text{and} \quad z = \frac{(n - a_k + k - 1)!}{(n - a_k)!}.$$

The maximum of y and z is then minimized by choosing a_k to be $\lfloor \frac{n+k}{2} \rfloor$, so

$$\begin{aligned} n! \cdot x &\leq \sum_{\substack{A_1 \subset \cdots \subset A_k \\ A_i \in \mathcal{F}}} N(n, A_1, \dots, A_k) \\ &\leq \sum_{\substack{A_1 \subset \cdots \subset A_k \\ A_i \in \mathcal{F}}} \frac{n!}{\prod_{i=1}^{k-1} \left(\lfloor \frac{n+k}{2} \rfloor - i + 1 \right)} \\ &= c_{\mathcal{F}}(\mathcal{F}) \cdot \frac{n!}{\prod_{i=1}^{k-1} \left(\lfloor \frac{n+k}{2} \rfloor - i + 1 \right)}, \end{aligned}$$

as required. □

Unfortunately, this is only tight for small values of x as indicated earlier. It may be possible to improve this result on k element chains by consider some sort of matching argument as Kleitman used for two element chains. Also, supersaturation for other posets seems to be wide open; it might be resolvable (as much as it is for the k -chains) for certain posets where $\text{La}(n, P) = \Sigma(n, m)$ for some m .

We learned that recently Das, Gan, and Sudakov have independently pursued a similar line of research and obtained results similar to Theorem 4.3.1.

CHAPTER 5

CONCLUSION

Our first generalization of Sperner's Theorem was packing posets into the Boolean lattice. We found the extreme families that were packings of the k -chain poset, and similarly for a small number of other posets as well. Also, we found the extreme families that were packings of posets from the collection $\{\mathcal{B}_0, \dots, \mathcal{B}_k\}$. We found the asymptotic approximation to the size of an extreme packing of a general poset and also for posets from a finite collection of posets. The questions that remain are as follows: 1) What is the asymptotic size of an extreme family of packings from an infinite collection of posets? 2) What is the exact value of $\text{Pa}(n, \{P_i\}_{i \geq 1})$? 3) Which families achieve this value?

Of course the same questions may be asked for strong embeddings, like what is the exact value of $\text{Pa}^*(n, \{P_i\}_{i \geq 1})$? Also, it would be nice to know how to find $c(P)$ and $c^*(P)$ quickly for any P , or at least to find the complexity of such an algorithm.

Our second generalization was on extreme families that forbid any posets from the collection $\{P_i\}$, finding $\text{La}(n, \{P_i\}_{i \geq 1})$. When we found the extreme families constructed from packings of \mathcal{B}_0 or \mathcal{B}_1 , we used this to find the extreme families that forbid both \mathcal{V} and Λ . We also looked at the Lubell function in a new light to help get closer to resolving the value of $\text{La}(n, \mathcal{B}_2)$ restricted to just the middle three levels of \mathcal{B}_n . Still unknown is $\pi(\mathcal{B}_2)$ or even if it exists. Also unknown is if it is sufficient to look for families in just the middle m levels of \mathcal{B}_n when trying to find $\text{La}(n, P)$ asymptotically. For instance, is it sufficient to look for maximum diamond-free families

in just the middle three levels of \mathcal{B}_n ? This would certainly improve the usefulness of the Lubell function.

Then there is supersaturation. We used symmetric chain decompositions of the Boolean lattice to find a lower bound on the number of k -chains a family must have, which was tight for small sizes of the family. This generalization of the $\text{La}(n, P)$ question is still wide open. No result is known for any poset other than k -chains, and even k -chains are not well understood except for the one and two element chains.

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