1-1-2013

Shimura Images of A Family of Half-Integral Weight Modular Forms

Kenneth Allan Brown
University of South Carolina - Columbia

Shimura Images of a Family of Half-Integral Weight Modular Forms.

by

Kenneth A. Brown

Bachelor of Science
Mississippi State University, 2008

Submitted in Partial Fulfillment of the Requirements

for the Degree of Doctor of Philosophy in

Mathematics

College of Arts and Sciences

University of South Carolina

2013

Accepted by:

Matthew Boylan, Major Professor

Michael Filaseta, Committee Member

George McNulty, Committee Member

Max Alekseyev, Committee Member

Lacy Ford, Vice Provost and Dean of Graduate Studies

Dedication

For Grace and Allan.

Abstract

In 1973, Shimura introduced a family of maps between modular forms of half-integral weight and modular forms of even integral weight. We will give explicit formulas for the images of two different classes of modular forms under these maps. In contrast to Shimura's difficult analytic construction, our formulas will fall out of relatively simple combinatorial derivations. Using the Shimura correspondence, we will prove congruences for the eigenvalues of a family of eigenforms introduced by Garvan. Using deep results of Eichler and Shimura, we state these congruences in terms of the number of points on associated elliptic curves, and we provide a table of these congruences for reference. Finally, we find a family of integral weight modular forms analogous to Garvan's half-integral weight family.

Table of Contents

List of Tables

Chapter 1

Background

In this chapter, we will review some background material on modular forms that will be relevant to the later chapters. The curious reader is encouraged to look at (Iwaniec, 1997), (Koblitz, 1993), (Diamond and Shurman, 2005), and (Diamond and Im, 1995) for more detailed information.

1.1 Modular Forms

Define the *modular group* $\mathrm{SL}_2(\mathbb{Z})$ to be

$$\mathrm{SL}_2(\mathbb{Z}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

This group acts on the complex numbers $\mathbb{C}$ with the action

$$\gamma z = \frac{az + b}{cz + d} \tag{1.1.1}$$

for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$. We extend this to the Riemann sphere $\hat{\mathbb{C}} := \mathbb{C} \cup \{\infty\}$ by defining $\gamma(\infty) = a/c$ and $\gamma(-d/c) = \infty$. Let $\mathcal{H}$ denote the upper half-plane $\mathcal{H} := \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}$. Note that any $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ preserves $\mathcal{H}$, so $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathcal{H}$ by the transformation (1.1.1).

For a positive integer N, we define the *congruence subgroups* $\Gamma_0(N), \Gamma_1(N)$, and

$\Gamma(N)$ by

$$\Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : c \equiv 0 \pmod N \right\},$$
$$\Gamma_1(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N,\, c \equiv 0 \pmod N \right\},$$
$$\Gamma(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : a \equiv d \equiv 1 \pmod N,\, b \equiv c \equiv 0 \pmod N \right\}.$$

We have $\Gamma(N) \subseteq \Gamma_1(N) \subseteq \Gamma_0(N) \subseteq \mathrm{SL}_2(\mathbb{Z})$, and these subgroups all have finite index in $\mathrm{SL}_2(\mathbb{Z})$. For a congruence subgroup Γ, we say that Γ has level N if N is the smallest positive integer with $\Gamma(N) \subseteq \Gamma$. Note that $\Gamma(1) = \Gamma_1(1) = \Gamma_0(1) = \mathrm{SL}_2(\mathbb{Z})$.

Let k be an integer. A holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is a *modular form* of level 1 and weight k if

- $f(\gamma z) = (cz+d)^k f(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ and $z \in \mathcal{H}$,
- f is holomorphic at ∞ in a sense explained below.

Since $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we see that $f(z+1) = f(z)$. Thus we can write the Fourier expansion of f as

$$f(z) = \sum_{n \gg -\infty} a(n) q^n,$$

where $q := e^{2\pi i z}$. This is called the Fourier expansion of f at ∞, or the q-expansion at ∞. Written this way, holomorphy at ∞ is equivalent to $a(n) = 0$ for $n < 0$. If in addition we have $a(0) = 0$, then we call f a *cusp form*. We also note that the map $z \mapsto q = e^{2\pi i z}$ maps $\mathcal{H}$ conformally to the open punctured unit disk, $\{q \in \mathbb{C} : 0 < |q| < 1\}$. In this sense, holomorphy of f at $z = i\infty$ is equivalent to the removability of the singularity of f at $q = 0$. Hence, when $\lim f$ as $q \to \infty$ exists and is finite, we assign the value of this limit to f at $z = i\infty$. In particular, we see that a cusp form on $\mathrm{SL}_2(\mathbb{Z})$ is a holomorphic modular form which vanishes at $z = i\infty$. The $\mathbb{C}$-vector space of modular forms of level 1 and weight k is written as $M_k(\mathrm{SL}_2(\mathbb{Z}))$, and the corresponding subspace of cusp forms is written as $S_k(\mathrm{SL}_2(\mathbb{Z}))$.

Since $\begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, there are no nonzero modular forms of level 1 with odd weight.

The holomorphy condition at ∞ ensures, via the Riemann-Roch Theorem, that $M_k(\mathrm{SL}_2(\mathbb{Z}))$ has finite $\mathbb{C}$-dimension. More generally, in order that spaces of modular forms for congruence subgroups of $\mathrm{SL}_2(\mathbb{Z})$ be finite-dimensional, we require holomorphy on $\mathcal{H}$ and at all points in $\mathbb{Q} \cup \{\infty\}$, the $\mathrm{SL}_2(\mathbb{Z})$ orbit of infinity. Hence, for a congruence subgroup Γ, we adjoin $\mathbb{Q} \cup \{\infty\}$ to $\mathcal{H}$. We identify these adjoined points under Γ-equivalence, and call these equivalence classes the *cusps* of Γ. Since $\mathrm{SL}_2(\mathbb{Z}) \cdot \infty = \mathbb{Q} \cup \{\infty\}$, it follows that $\mathrm{SL}_2(\mathbb{Z})$ has only one cusp, represented by ∞. When Γ is a proper congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$, its action on $\mathbb{Q} \cup \{\infty\}$ is not transitive, so Γ will have cusps inequivalent to $\{\infty\}$. The number of cusps is finite, bounded by the finite index of the subgroup.

Let N be a positive integer, and let χ be a Dirichlet character modulo N. A modular form of level N, weight k, and character χ is a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ satisfying

- $f(\gamma z) = \chi(d)(cz + d)^k f(z)$ for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$ and $z \in \mathcal{H}$,
- $f(\gamma z)$ is holomorphic at ∞ for all $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.

The vector space of modular forms of level N, weight k, and character χ is denoted by $M_k(\Gamma_0(N), \chi)$. The subspace of cusp forms is denoted by $S_k(\Gamma_0(N), \chi)$. If χ is the trivial character modulo N, we write $M_k(\Gamma_0(N))$ or $S_k(\Gamma_0(N))$ instead.

A few modifications allow us to define modular forms of half-integral weight. For details, see (Shimura, 1973) and Chapter 4 of (Koblitz, 1993). For odd primes d, we let $\left(\frac{c}{d}\right)$ be the usual Legendre symbol, and we extend to all odd $d > 0$ by multiplicativity.

For odd $d < 0$, we define

$$\left(\frac{c}{d}\right) := \begin{cases} \left(\frac{c}{|d|}\right) & \text{if } c > 0, \\ -\left(\frac{c}{|d|}\right) & \text{if } c < 0, \end{cases}$$

and we let $\left(\frac{0}{\pm 1}\right) = 1$.

Next, we define for odd integers d,

$$\epsilon_d := \begin{cases} 1 & \text{if } d \equiv 1 \pmod 4, \\ i & \text{if } d \equiv 3 \pmod 4. \end{cases}$$

We let $\sqrt{z}$ denote the branch of the square root having argument in $(-\pi/2, \pi/2]$.

For integers $\lambda \geq 0$ and $N > 0$, and for χ a Dirichlet character modulo $4N$, a holomorphic function $f : \mathcal{H} \to \mathbb{C}$ is a modular form of level $4N$, weight $\lambda + \frac{1}{2}$, and character χ if

- $f(\gamma z) = \chi(d) \left(\frac{c}{d}\right)^{2\lambda+1} \epsilon_d^{-1-2\lambda} (cz+d)^{\lambda+\frac{1}{2}} f(z)$ for $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$,
- f is holomorphic at the cusps of $\Gamma_0(4N)$ (in a sense analogous to that for integer weights).

A few remarks:

1. For $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(4N)$, the expression $j(\gamma, z) = \chi(d) \left(\frac{c}{d}\right) \epsilon_d^{-1}$ is an example of a *multiplier system* for half-integral weights. It is called the *theta multiplier system* since the theta series $\theta(z) = \sum_{n \in \mathbb{Z}} q^{n^2}$ satisfies $\theta(\gamma z)/\theta(z) = j(\gamma, z)$ for all $\gamma \in \Gamma_0(4)$. Hence, half-integral weight modular forms with theta-multiplier and weight $k/2$ transform like kth powers of θ. Shimura's 1973 paper laid the foundations for the theory of half-integral weights with theta-multiplier.

2. Other non-trivial multiplier systems exist for half-integral weights. For example, one can study half-integral weight modular forms on $\mathrm{SL}_2(\mathbb{Z})$ with eta-multiplier $\epsilon_{a,b,c,d}$, a 24th root of unity, where $\eta(z) = q^{1/24} \prod_{m=1}^{\infty} (1 - q^m)$ is Dedekind's

eta-function (see below). For all $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$, we have $\eta(\gamma z)/\eta(z) = \epsilon_{a,b,c,d}$.

3. When the multiplier system for a family of weights consists only of $\chi(\cdot)$, we say that it is trivial. The classical integer weight modular forms, which we study here, have trivial multiplier system. The comparative complexity of the half-integral weight multiplier systems arises from the need to explicitly and compatibly choose a branch of the complex square root. It is primarily for this reason that the study of half-integral weights is more computationally and theoretically technical than the study of integral weights.

The $\mathbb{C}$-vector space of these half-integral weight forms is denoted by $M_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$. As in the integral weight setting, this is a finite-dimensional space. Moreover, half-integral weight modular forms have Fourier expansions at ∞, and the definition of a cusp form is the same. The subspace of half-integral weight cusp forms is denoted by $S_{\lambda+\frac{1}{2}}(\Gamma_0(4N), \chi)$.

1.2 Examples of Modular Forms

Let $k \geq 2$ be even. We define the weight k Eisenstein series $E_k(z)$ by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n) q^n, \tag{1.2.1}$$

where B_k is the kth Bernoulli number, defined by

$$\frac{t}{e^t - 1} = \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} = 1 - \frac{1}{2}t + \frac{1}{12}t^2 + \cdots ,$$

and $\sigma_{k-1}(n)$ is the divisor function

$$\sigma_{k-1}(n) := \sum_{d|n} d^{k-1}.$$

For $k \geq 4$, we have $E_k(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, and we have $M_k(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_k$ for $k \in \{4, 6, 8, 10, 14\}$.

Let $\mathcal{O}$ be the ring of integers of a number field, and let $\mathfrak{p}$ be a prime ideal in $\mathcal{O}$. Suppose that $f(z) = \sum a(n)q^n$ and $g(z) = \sum b(n)q^n$ lie in $M_k(\Gamma) \cap \mathcal{O}[[q]]$ for some $k \in \mathbb{Z}$ and for some congruence subgroup Γ. Then we have $f(z) \equiv g(z) \pmod{\mathfrak{p}}$ if and only if we have $a(n) \equiv b(n) \pmod{\mathfrak{p}}$ for all n. We now give standard explicit examples of modular form congruences which are foundational to the theory of modular forms modulo prime ideals.

Proposition 1.1. *Suppose that $k \geq 2$ is even.*

(1) If ℓ is prime and $(\ell - 1) \mid k$, then $E_k(z) \equiv 1 \pmod{\ell}$.

(2) If $\ell \geq 3$ is prime, then $E_{\ell+1}(z) \equiv E_2(z) \pmod{\ell}$.

The proofs of these follow from the definition of the Eisenstein series (1.2.1) and well-known congruences of Von Staudt and Clausen (for (1)) and Kummer (for (2)) for Bernoulli numbers (see Chapter 15 of (Ireland and Rosen, 1990)).

The series $E_2(z)$ is holomorphic, but does not satisfy the transformation properties necessary to be a modular form; in fact, we have the transformation

$$z^{-2}E_2(-1/z) = E_2(z) + \frac{12}{2\pi i z}.$$

On the other hand, it can be shown that the non-holomorphic function defined by

$$E_2^*(z) := E_2(z) - \frac{3}{\pi \operatorname{Im}(z)}$$

transforms like a weight 2 modular form on $\mathrm{SL}_2(\mathbb{Z})$. Using these two functions $E_2(z)$ and $E_2^*(z)$, we can construct weight 2 modular forms on $\Gamma_0(N)$ (see, for example (Diamond and Im, 1995)). We have

Proposition 1.2. *Let $N \geq 2$ be an integer, and suppose that for each $d \mid N$ there are $c_d \in \mathbb{C}$ with $\sum_{d|N} \frac{c_d}{d} = 0$. Then*

$$\sum_{d|N} c_d E_2^*(dz) = \sum_{d|N} c_d E_2(dz) \in M_2(\Gamma_0(N)).$$

Define the Dedekind eta function $\eta(z)$ by

$$\eta(z) := q^{1/24} \prod_{n=1}^{\infty} (1 - q^n). \tag{1.2.2}$$

This is a holomorphic function from $\mathcal{H}$ to $\mathbb{C}$ that is nonzero on $\mathcal{H}$ and vanishes at ∞, and satisfies the transformation properties

$$\eta(z+1) = e^{\frac{2\pi i}{24}} \eta(z) \quad \text{and} \quad \eta\left(-\frac{1}{z}\right) = (-iz)^{1/2} \eta(z).$$

Many of the functions we will investigate will be constructed using $\eta(z)$. For $N \geq 1$ and integers r_δ, we call functions of the form

$$f(z) = \prod_{\delta | N} \eta(\delta z)^{r_\delta}$$

eta-quotients. The following theorems on eta-quotients will be useful. See (Ono, 2004) for more information.

Theorem 1.3. *Suppose that $f(z) = \prod_{\delta|N} \eta(\delta z)^{r_\delta}$ is an eta-quotient with $k = \frac{1}{2} \sum_{\delta|N} r_\delta \in \mathbb{Z}$ and such that*

$$\sum_{\delta|N} \delta r_\delta \equiv 0 \pmod{24} \quad \textit{and} \quad \sum_{\delta|N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}.$$

Then for every $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(N)$, $f(z)$ satisfies

$$f(\gamma z) = \chi(d)(cz+d)^k f(z).$$

Here, the character χ is defined by $\chi(d) := \left(\frac{(-1)^k s}{d}\right)$, where $s = \prod_{\delta|N} \delta^{r_\delta}$.

To verify that an eta-quotient satisfying the above conditions is a modular form, we need to check the behavior at the cusps of $\Gamma_0(N)$. We may use the following theorem for this.

Theorem 1.4. *Let c, d, and N be positive integers with $d \mid N$ and $\gcd(c,d) = 1$. If $f(z)$ is an eta-quotient satisfying the conditions of Theorem 1.3 for N, then the order of vanishing of $f(z)$ at the cusp $\frac{c}{d}$ is*

$$\frac{N}{24} \sum_{\delta|N} \frac{\gcd(d,\delta)^2 r_\delta}{\gcd(d, \frac{N}{d}) d\delta}.$$

Using these two theorems, it is straightforward to verify that certain eta-quotients are indeed modular forms.

Suppose that $f(z) = \sum a(n)q^n$ and $g(z) = \sum b(n)q^n$ lie in $M_k(\Gamma)$ for some congruence subgroup Γ. Then we have $f(z) = g(z)$ if and only if $a(n) = b(n)$ for all $n \leq \dim_{\mathbb{C}}(M_k(\Gamma)) + 1$. Hence, finite-dimensionality allows us to computationally verify whether two modular forms are equal. A theorem of Sturm extends this criterion for equality to congruence in the setting of modular forms in the ring of integers $\mathcal{O}$ of a number field, with coefficients reduced modulo a prime ideal $\mathfrak{p}$.

Theorem 1.5 (Sturm)**.** *Let $\mathfrak{p}$ be a prime ideal in the ring of integers $\mathcal{O}$ of a number field. Suppose that $f(z) = \sum a(n)q^n \in M_k(\Gamma_0(N), \chi) \cap \mathcal{O}[[q]]$ is a modular form, and assume that*

$$a(n) \equiv 0 \pmod{\mathfrak{p}} \quad \textit{for all} \quad n \leq \frac{km}{12},$$

where

$$m = [\mathrm{SL}_2(\mathbb{Z}) : \Gamma_0(N)] = N \cdot \prod_{p \mid N} \left(1 + \frac{1}{p}\right).$$

Then we have $f \equiv 0 \pmod{\mathfrak{p}}$.

To see an example of a half-integral weight modular form, consider the function $\eta(24z)$ given by

$$\eta(24z) = q \prod_{n=1}^{\infty} (1 - q^{24n}) \in S_{1/2}(\Gamma_0(576), \chi_3), \tag{1.2.3}$$

where $\chi_3(n)$ is the Dirichlet character modulo 12 given by

$$\chi_3(n) := \begin{cases} 1 & \text{if } n \equiv \pm 1 \pmod{12}, \\ -1 & \text{if } n \equiv \pm 5 \pmod{12}, \\ 0 & \text{otherwise.} \end{cases} \tag{1.2.4}$$

This form has the convenient Fourier expansion

$$\eta(24z) = \sum_{n=1}^{\infty} \chi_3(n) q^{n^2} = \frac{1}{2} \sum_{n \in \mathbb{Z}} \chi_3(n) q^{n^2}. \tag{1.2.5}$$

Another half-integral weight form we will see is $\eta(8z)^3$. We have

$$\eta(8z)^3 = q\prod_{n=1}^{\infty}(1-q^{8n})^3 \in S_{3/2}(\Gamma_0(64)). \tag{1.2.6}$$

We can write the Fourier expansion of this form as

$$\eta(8z)^3 = \sum_{m=0}^{\infty}(-1)^m(2m+1)q^{(2m+1)^2} = \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^m(2m+1)q^{(2m+1)^2}. \tag{1.2.7}$$

These modular forms are examples of Shimura theta-series. Their series expansions follow from classical identities of Euler, Gauss, and Jacobi.

1.3 Operators on Modular Forms

There are several important operators acting on spaces of modular forms. For positive integers d, we define the operators U_d and V_d on formal power series in q by

$$\sum_{n=0}^{\infty}a(n)q^n \mid U_d := \sum_{n=0}^{\infty}a(dn)q^n \tag{1.3.1}$$

$$\sum_{n=0}^{\infty}a(n)q^n \mid V_d := \sum_{n=0}^{\infty}a(n)q^{dn} \tag{1.3.2}$$

The following propositions describe the behavior of these operators when acting on spaces of modular forms.

Proposition 1.6. *Suppose that $f(z) \in M_k(\Gamma_0(N), \chi)$.*

(1) If d is a positive integer, then $f(z) \mid V_d \in M_k(\Gamma_0(Nd), \chi)$. Moreover, if $f(z)$ is a cusp form, then so is $f(z) \mid V_d$.

(2) If $d \mid N$, then $f(z) \mid U_d \in M_k(\Gamma_0(N), \chi)$. Moreover, if $f(z)$ is a cusp form, then so is $f(z) \mid U_d$.

Proposition 1.7. *Suppose that $f(z) \in M_{\lambda+1/2}(\Gamma_0(4N), \chi)$.*

(1) If d is a positive integer, then $f(z) \mid V_d \in M_{\lambda+1/2}\left(\Gamma_0(4Nd), \left(\frac{4d}{\bullet}\right)\chi\right)$. Moreover, if $f(z)$ is a cusp form, then so is $f(z) \mid V_d$.

(2) If $d \mid N$, then $f(z) \mid U_d \in M_{\lambda+1/2}(\Gamma_0(4N), \left(\frac{4d}{\bullet}\right)\chi)$. Moreover, if $f(z)$ is a cusp form, then so is $f(z) \mid U_d$.

We can also define a family of operators called Hecke operators that are indexed by positive integers. For an integer weight modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ and a prime p, the action of the Hecke operator $T_{p,k,\chi}$ on $f(z)$ is defined by

$$f(z) \mid T_{p,k,\chi} := \sum_{n=0}^{\infty} \left(a(pn) + \chi(p)p^{k-1}a(n/p)\right) q^n, \tag{1.3.3}$$

where we agree that $a(n/p) = 0$ if $p \nmid n$. More generally, for positive integers m, we define the action of $T_{m,k,\chi}$ by

$$f(z) \mid T_{m,k,\chi} := \sum_{n=0}^{\infty} \left(\sum_{d \mid (m,n)} \chi(d)d^{k-1}a(mn/d^2) \right) q^n. \tag{1.3.4}$$

The Hecke operators are endomorphisms on spaces of modular forms, and they preserve cusp forms. When the weight and character are clear from context, we will denote the operator $T_{m,k,\chi}$ by T_m. Since $\chi(n) = 0$ if $\gcd(n, N) \neq 1$, we have $T_p = U_p$ for $p \mid N$.

We can also define Hecke operators in the half-integral weight setting. For a half-integer weight modular form $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_{\lambda+1/2}(\Gamma_0(4N), \chi)$ and a prime p, the Hecke operator $T_{p^2,\lambda,\chi}$ is defined by

$$f(z) \mid T_{p^2,\lambda,\chi} := \sum_{n=0}^{\infty} \left(a(p^2n) + \chi^*(p) \left(\frac{n}{p}\right) p^{\lambda-1}a(n) + \chi^*(p^2)p^{2\lambda-1}a(n/p^2) \right) q^n, \tag{1.3.5}$$

where χ^* is the Dirichlet character given by $\chi^*(n) := \left(\frac{(-1)^\lambda}{n}\right)\chi(n)$. Just as in the integer weight setting, the Hecke operators are endomorphisms on spaces of modular forms that preserve cusp forms.

When k is an integer, a modular form $f(z) \in M_k(\Gamma_0(N), \chi)$ is called an *eigenform* for T_n if there is a complex number $\lambda(n)$ with

$$f(z) \mid T_n = \lambda(n)f(z). \tag{1.3.6}$$

If $f(z)$ is an eigenform for T_n for all $n \in \mathbb{N}$ with $\gcd(n, N) = 1$, then we call $f(z)$ a *Hecke eigenform*. A cusp form which is a Hecke eigenform is said to be *normalized* if $a(1) = 1$, in which case $\lambda(n) = a(n)$ for all n with $\gcd(n, N) = 1$. The following multiplicative relation involving the coefficients of normalized Hecke eigenforms will be useful.

Proposition 1.8. *Let $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ be a normalized Hecke eigenform. Then for all $m, n \geq 0$, we have*

$$a(m)a(n) = \sum_{d|(m,n)} \chi(d)d^{k-1}a\left(\frac{mn}{d^2}\right).$$

The Hecke operators on $S_k(\Gamma_0(N), \chi)$ form a commuting family, and they are normal with respect to the Petersson inner product on this space. Therefore, the Hecke operators are simultaneously diagonalizable on $S_k(\Gamma_0(N), \chi)$, which implies that $S_k(\Gamma_0(N), \chi)$ has a basis all of whose elements are Hecke eigenforms. It turns out, however, that the corresponding eigenspaces do not have to be one-dimensional. For the purpose of analogy, we recall that a Dirichlet character χ is imprimitive with modulus n if and only if there exists $m \mid n$ with $m < n$ such that χ is a Dirichlet character modulo m, in which case, we say that m is the conductor of χ. Similarly, modular forms of level N can arise from forms of level dividing N. For example, if M is a divisor of N and $f \in S_k(\Gamma_0(M))$, then it turns out that $f(dz) \in S_k(\Gamma_0(N))$ for any divisor d of N/M. We let $S_k^{\text{old}}(\Gamma_0(N), \chi)$ denote the subspace of $S_k(\Gamma_0(N), \chi)$ spanned by all forms $f(dz)$ arising from forms of lower level, and we let $S_k^{\text{new}}(\Gamma_0(N), \chi)$ denote the orthogonal complement of $S_k^{\text{old}}(\Gamma_0(N), \chi)$ with respect to the Petersson inner product. A normalized Hecke eigenform in this space is called a *newform*. A key fact is that the eigenspaces of these newforms are one-dimensional. We say that "Multiplicity One" holds in $S_k^{\text{new}}(\Gamma_0(N), \chi)$, but note that this property does not hold in half-integral weight settings: one has a similar notion of eigenform, but there is no distinguished subspace of multiplicity one.

We define the *Fricke involution* W_N on $M_k(\Gamma_0(N), \chi)$ by

$$f \mid W_N = N^{s/2}(Nz)^{-s} f\left(-\frac{1}{Nz}\right) = N^{-s/2} z^{-s} f\left(-\frac{1}{Nz}\right).$$

We have the following commutation relation.

Proposition 1.9. *Let $f(z) \in M_k(\Gamma_0(N), \chi)$. For primes $p \nmid N$, we have*

$$f \mid T_p \mid W_N = \chi(p) f \mid W_N \mid T_p.$$

Let $f(z) = \sum_{n=0}^{\infty} a(n)q^n$ and let ε be a Dirichlet character. We define the *twist* of f by ε as

$$f(z) \otimes \varepsilon := \sum_{n=0}^{\infty} \varepsilon(n) a(n) q^n.$$

The following is proved in (Atkin and Li, 1978):

Proposition 1.10. *Suppose that $f(z) = \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$, where χ is a character of conductor N'. Let ε be a character of modulus M. Then we have the twisted series*

$$f(z) \otimes \varepsilon := \sum_{n=0}^{\infty} \varepsilon(n) a(n) q^n \in M_k(\Gamma_0(T), \varepsilon^2 \chi),$$

where $T := \mathrm{lcm}(N, N'M, M^2)$.

Moreover, it can be shown that if $f(z)$ is a cusp form, then so is $f(z) \otimes \varepsilon$.

We now recall the action of Ramanujan's differential operator. The Ramanujan Θ-operator is defined by

$$\Theta\left(\sum_{n=h}^{\infty} a(n)q^n\right) := \sum_{n=h}^{\infty} n a(n) q^n. \tag{1.3.7}$$

We will denote by $\Theta^n(f)$ the application of the Θ-operator n times. It is well known that if f is a non-constant modular form of weight k for a congruence subgroup Γ, then $\Theta(f)$ is not a modular form. In fact, we have the following:

Proposition 1.11. *If $f(z)$ is a weight k modular form on a congruence subgroup Γ of* $\mathrm{SL}_2(\mathbb{Z})$*, then*

$$\Theta(f(z)) = \frac{\tilde{f}(z) + kf(z)E_2(z)}{12},$$

where $\tilde{f}$ is a weight $k+2$ modular form on Γ.

However, in the modulo ℓ setting, the theta-operator behaves nicely. In (Swinnerton-Dyer, 1973), it was proved that if $\ell \geq 5$ is prime and $\sum_{n=0}^{\infty} a(n)q^n$ is a weight k modular form with integer coefficients, then there is a weight $k+\ell+1$ modular form $\sum_{n=0}^{\infty} \alpha(n)q^n$ on $\mathrm{SL}_2(\mathbb{Z})$ with integer coefficients whose Fourier expansion satisfies

$$\sum_{n=0}^{\infty} \alpha(n)q^n \equiv \sum_{n=0}^{\infty} na(n)q^n \pmod{\ell}. \tag{1.3.8}$$

1.4 Elliptic Curves

This section gives a brief overview of elliptic curves and the definitions necessary to understand the later results. The reader curious for more information is encouraged to look at (Knapp, 1992), (Koblitz, 1993), and (Silverman, 2009).

An *elliptic curve* over a field K is a nonsingular cubic projective curve E defined over K, together with a point O with coordinates in K. The set of projective points on E with coordinates in K is called the set of K-rational points of E and is denoted by $E(K)$. With the appropriate change of variables, such a curve can always be given by the (affine) equation

$$y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6, \tag{1.4.1}$$

the point O being the point at infinity. We call this the *generalized Weierstrass form* of the curve. When the characteristic of K is not equal to 2 or 3, we can use the simpler Weierstrass form

$$y^2 = x^3 + Ax + B. \tag{1.4.2}$$

One aspect that makes the theory of elliptic curves so rich is that the set $E(K)$ of K-rational points on E can be equipped with a group structure, which is geometric in nature. There is a "chord-tangent" addition law for points which can be made explicit in terms of rational functions in the coordinates. A fundamental property of elliptic curves is that this set $E(K)$ forms an abelian group. The following theorem, originally due to Mordell, was generalized to number fields by Weil.

Theorem 1.12 (Mordell-Weil). *Let K be a number field, and let E be an elliptic curve defined over K. Then the group $E(K)$ is finitely generated. In particular,*

$$E(K) \equiv \mathbb{Z}^{r_K(E)} \oplus E(K)_{\text{tors}},$$

where $r_K(E)$ is the rank and $E(K)_{\text{tors}}$ is the torsion subgroup of points of finite order.

The Mordell-Weil Theorem implies that the group $E(K)_{\text{tors}}$ is always finite. One might wonder which finite abelian groups can arise in this context. Alternatively, given a prime p, one might wish to know if there exists an elliptic curve $E/\mathbb{Q}$ such that $E(\mathbb{Q})$ contains a point of order p. The following deep theorem, first conjectured by Ogg, addresses these questions.

Theorem 1.13 (Mazur). *Let $E/\mathbb{Q}$ be an elliptic curve. Then the torsion subgroup $E(\mathbb{Q})_{\text{tors}}$ of $E(\mathbb{Q})$ is isomorphic to one of the following fifteen groups:*

$$\begin{aligned} \mathbb{Z}/N\mathbb{Z} \quad &\textit{with} \quad 1 \leq N \leq 10 \textit{ or } N = 12, \\ \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2N\mathbb{Z} \quad &\textit{with} \quad 1 \leq N \leq 4. \end{aligned}$$

Further, each of these groups occurs as $E(\mathbb{Q})_{\text{tors}}$ for some elliptic curve $E/\mathbb{Q}$.

For an elliptic curve E given by $y^2 = x^3+Ax+B$, we define $\Delta(E)$, the *discriminant* of E by

$$\Delta(E) := -16(4A^3 + 27B^2).$$

Note that this agrees with the discriminant of the cubic polynomial up to the factor -16. If the discriminant $\Delta(E)$ is not divisible by a prime p, we say that E has *good reduction* at p. For primes p where there is bad reduction, the cubic $f(x) = x^3+Ax+B$ has multiple roots modulo p. If $f(x)$ has a triple root, we say that E has *additive reduction* at p. If $f(x)$ has a double root, we say that E has *multplicative reduction* at p. More precisely, if E has a node at which the two tangent lines are defined over $\mathbb{F}_p$, the finite field with p elements, then E is said to have *split multiplicative reduction* at p. If these two tangent lines are not defined over $\mathbb{F}_p$, then E is said to have *nonsplit multiplicative reduction* at p.

We can associate a positive integer $N = N(E)$ to E, called the *conductor* of E. The conductor is similar to the discriminant in the sense that E has bad reduction at p if $p \mid N$. For primes $p \geq 5$, the power of p dividing N is 0, 1, or 2, according to whether the reduction at p is good, multiplicative, or additive. For $p = 2$ and $p = 3$, the exponent is more subtle, and can be computed by an algorithm due to Tate (see Algorithm 7.5.1 in (Cohen, 1993)).

Suppose E is given by the equation (1.4.2) with $A, B \in \mathbb{Z}$. For primes p, let $|E(\mathbb{F}_p)|$ denote the number of points on $E/\mathbb{F}_p$; i.e. the number of solutions to the congruence $y^2 \equiv x^3 + Ax + B \pmod{p}$, not counting the point at infinity. We define the quantity

$$\lambda(p) := \begin{cases} p + 1 - |E(\mathbb{F}_p)| & \text{if } E \text{ has good reduction at } p, \\ 1 & \text{if } E \text{ has split multiplicative reduction at } p, \\ -1 & \text{if } E \text{ has nonsplit multiplicative reduction at } p, \\ 0 & \text{if } E \text{ has additive reduction at } p, \end{cases}$$

and define an L-function for the elliptic curve E by the following Euler product:

$$L(E, s) := \sum_{n=1}^{\infty} \frac{\lambda(n)}{n^s}. = \prod_{p \mid N} (1 - \lambda(p)p^{-s})^{-1} \cdot \prod_{p \nmid N} (1 - \lambda(p)p^{-s} + p^{1-2s})^{-1}. \tag{1.4.3}$$

The product that defines $L(E,s)$ converges absolutely and gives an analytic function for all $\mathrm{Re}(s) > 3/2$. This follows from the Hasse Bound, which is $|\lambda(p)| \leq 2\sqrt{p}$ for all primes p. The L-function attached to a cusp form $f(z) = \sum a(n)q^n \in S_k(\Gamma_0(N),\chi)$ is defined by

$$L(f,s) := \sum_{n=1}^{\infty} \frac{a(n)}{n^s}. \tag{1.4.4}$$

In his work on the Weil Conjectures, Deligne proved that for such cusp forms, we have $|a(p)| \leq 2p^{\frac{k-1}{2}}$. As a result of this, the sum above converges for $\mathrm{Re}(s) > \frac{k+1}{2}$. If f is a Hecke eigenform, then we can expand the sum (1.4.4) as an Euler product

$$L(f,s) = \prod_p \left(1 - a(p)p^{-s} + \chi(p)p^{k-1-2s}\right)^{-1},$$

where the product is taken over all primes p. Additionally, in the case that f is a newform, the completed L-function defined by

$$\Lambda(s,f) = N^{s/2}(2\pi)^{-s}\Gamma(s)L(f,s)$$

satisfies a functional equation

$$\Lambda(s,f) = \omega\Lambda(k-s,f),$$

where ω is a complex number of absolute value 1 that can be specified precisely, and $L(f,s)$ has analytic continuation to $\mathbb{C}$.

Work of Eichler and Shimura establishes a link between $\lambda(n)$, the coefficients of $L(E,s)$ when E is an elliptic curve over $\mathbb{Q}$, and Fourier coefficients of weight 2 newforms.

Theorem 1.14 (Eichler-Shimura). *Let $f(z)$ be a newform in $S_2^{\mathrm{new}}(\Gamma_0(N))$ such that $a(n) \in \mathbb{Z}$ for all n. Then there exists an elliptic curve E over $\mathbb{Q}$ with conductor N such that $L(E,s) = L(f,s)$.*

In the other direction we have the famous Shimura-Taniyama conjecture, which was proved by Breuil, Conrad, Diamond, Taylor, and Wiles.

Theorem 1.15 (Modularity Theorem)**.** *If E is an elliptic curve over $\mathbb{Q}$ of conductor N, then there is a newform $f(z) \in S_2^{\mathrm{new}}(\Gamma_0(N))$ for which $L(E,s) = L(f,s)$.*

For elliptic curves $E/\mathbb{Q}$, Theorem 1.15 implies that $L(E,s)$ has analytic continuation to $\mathbb{C}$, and so the behavior of $L(E,s)$ at $s = 1$ is well-defined. The Birch and Swinnerton-Dyer Conjecture, one of the remaining Clay Millenium Problems, predicts that the order of vanish of the Taylor series for $L(E,s)$ at $s = 1$ (called the analytic rank of E) is equal to the geometric (or algebraic) rank of K, which is $r_E(\mathbb{Q})$, the rank of $E(\mathbb{Q})$, the Mordell-Weil group of rational points.

Chapter 2

Formulas for Shimura Images

2.1 Background and Statement of Results

In a 1973 paper in Annals of Mathematics, Shimura began a systematic study of modular forms of half-integral weight with θ-multiplier system, as described in the introduction. A crucial aspect of this theory entailed a linear map, called the Shimura Correspondence, or Shimura Lift, from spaces of modular forms with half-integral weight to spaces of modular forms with even integral weight. Shortly after Shimura's work, Shintani studied the dual map from integer weight forms to half-integral weight forms. While the Shimura Lift is interesting in its own right on theoretical grounds, it has also proved to be very useful in applications to number theory and arithmetic geometry, for example. In particular, it allows for interplay between spaces that have different features and whose functions encode objects of different types as generating functions.

On one hand, integral weight modular forms enjoy a nice spectral theory characterized by the "Multiplicity One" property from the introduction. Furthermore, modular forms whose coefficients contain information on solution counts to Diophantine equations modulo p for all primes p lie in spaces of integer weight, the prototype being newforms of weight 2 with integer coefficients, which correspond to elliptic curves over $\mathbb{Q}$ via the work of Wiles et al and Eichler-Shimura. Integer weight newforms have similarly deep connections to the arithmetic of number fields through the theory of "modularity of Galois representations" pioneered by Deligne and Serre.

On the other hand, the spectral theory of Hecke operators in half-integral weight is more complicated. Multiplicity one does not hold, nor is there a direct connection to number fields via Galois representations. However, deep work of Waldspurger shows that the Fourier coefficients of half-integral weight modular forms often interpolate square roots of central critical values of L-functions in arithmetic geometry.

For example, one may recast important results of Bump, Friedberg, and Hoffstein, of Iwaniec, of Murty and Murty, and of Waldspurger, on non-vanishing of central critical L-values of quadratic twists of elliptic curves and their derivatives as statements about the non-vanishing of coefficients of half-integral weight modular forms. It turns out that through the Shimura Lift, these non-vanishing theorems together with work of Kolyvagin and Gross and Zagier imply the known cases of the Birch and Swinnerton-Dyer Conjecture.

In a different direction, spaces of half-integral weight modular forms provide a setting for generating functions of interest in additive combinatorics such as the generating function for the ordinary partition function $p(n)$, studied by Euler, Gauss, and Ramanujan. It turns out the generating function for $p(n)$ is a modular form of weight $-1/2$ on the group $\Gamma_0(576)$ which is holomorphic on the upper half-plane, but has a simple pole at every cusp. Work of Ono et al shows how to use the Shimura Lift to transfer the study of $p(n)$ to the integer weight setting. Facts about Galois representations coming from the Chebotarev Density Theorem then allowed Ahlgren and Ono to prove that for every $M \geq 1$ coprime to 6, there exist infinitely many arithmetic progressions $An + B$, none contained in any other, such that $p(An + B) \equiv 0 \pmod{M}$ for all n.

We now define the Shimura Lift. Let $f(z) := \sum_{n=1}^{\infty} a(n)q^n$. Let χ be a Dirichlet character modulo $4N$, let t be a squarefree positive integer, and let $\lambda \geq 1$ be an integer. Define

$$S_t(f(z)) := \sum_{n=1}^{\infty} A_t(n)q^n$$

by

$$\sum_{n=1}^{\infty} \frac{A_t(n)}{n^s} := L(s-\lambda+1, \psi_t) \cdot \sum_{n=1}^{\infty} \frac{a(tn^2)}{n^s},$$

where $\psi_t(m) := \chi(m) \left(\frac{-1}{m}\right)^t \left(\frac{t}{m}\right)$.

Multiplying out the series on the right side and equating coefficients, we find that the values $A_t(n)$ are given by

$$A_t(n) := \sum_{d|n} \psi\chi_{-1}^{\lambda}\chi_t(d) d^{\lambda-1} a\left(\frac{tn^2}{d^2}\right). \tag{2.1.1}$$

In (Shimura, 1973), the following was proved:

Theorem 2.1 (Shimura)**.** *Suppose that $f(z) \in S_{\lambda+1/2}(\Gamma_0(4N), \chi)$. Then*

$$S_t(f) \in \begin{cases} M_{2\lambda}(\Gamma_0(2N), \chi^2) & \text{if } \lambda = 1, \\ S_{2\lambda}(\Gamma_0(2N), \chi^2) & \text{if } \lambda > 1. \end{cases}$$

Moreover, S_t commutes with the Hecke operators:

$$S_t(f \mid T_{p^2,\lambda+1/2,\chi}) = S_t(f) \mid T_{p,2\lambda,\chi^2}).$$

A version of the Shimura lift was discovered earlier by Selberg, but never published. Selberg's version deals with the special case that the half-integral weight form is a theta function times an eigenform. In this case, Selberg explicitly identifies the image. Note that not every cusp form can be written in this way, so this version of the lift is not as general as Shimura's. Selberg's version is also weaker in that it only treats the lift for $t = 1$.

Theorem 2.2 (Selberg)**.** *Suppose $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_\lambda(\mathrm{SL}_2(\mathbb{Z}))$ is a normalized Hecke eigenform. Define*

$$g(z) := \theta(z) f(4z) \in S_{\lambda+1/2}(\Gamma_0(4))$$

where

$$\theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2} \in M_{1/2}(\Gamma_0(4)).$$

Then

$$S_1(g(z)) = f(z)^2 - 2^{k-1} f(2z)^2 \in S_{2\lambda}(\Gamma_0(2)).$$

Cipra generalized Selberg's work in (Cipra, 1989). Here we present a special case.

Theorem 2.3 (Cipra)**.** *Suppose* $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda}^{\text{new}}(\Gamma_0(N), \chi)$ *be a newform. Define*

$$g(z) := \theta(z) f(4z) \in S_{\lambda+1/2}(\Gamma_0(4N))$$

where

$$\theta(z) := \sum_{n=-\infty}^{\infty} q^{n^2} \in M_{1/2}(\Gamma_0(4)).$$

Then

$$S_1(g(z)) = f(z)^2 - 2^{k-1}\chi(2) f(2z)^2 \in S_{2\lambda}(\Gamma_0(2N), \chi^2).$$

We prove the following theorem:

Theorem 2.4. *Suppose that* $g(z) := \sum_{n=0}^{\infty} a(n)q^n \in M_k(\Gamma_0(N), \chi)$ *is a normalized Hecke eigenform of integer weight.*

(1) We have

$$S_1(\eta(24z)g(24z)) = (g(z)g(6z) - g(2z)g(3z)) \otimes \chi_3,$$

and, $S_1(\eta(24z)g(24z)) \in S_{2k}(\Gamma_0(144N), \chi^2)$.

(2) We have

$$S_1(\eta(8z)^3 g(8z)) = (g(2z)\,\Theta(g(z)) - g(z)\,\Theta(g(2z))) \otimes \chi_{-1},$$

and $S_1(\eta(8z)^2 g(8z)) \in S_{2k+2}(\Gamma_0(16N), \chi^2)$.

Before we give an example of the utility of Theorem 2.4, we prove the following result which seems to not appear in the literature. This useful fact establishes a relationship between the different Shimura maps.

Proposition 2.5. *Let $f(z) = \sum_{n=1}^{\infty} a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(4N), \chi)$, and let $r > 0$ be a square-free odd integer. Then we have*

$$S_r(f) = S_1(f \mid U_r).$$

Proof. From the definition, we have $S_r(f) = \sum_{n=1}^{\infty} A_r(n)q^n$, where

$$A_r(n) := \sum_{d|n} \chi(d) \left(\frac{-1}{d}\right)^r \left(\frac{r}{d}\right) d^{\lambda-1} a\left(\frac{rn^2}{d^2}\right).$$

On the other hand, we have

$$f \mid U_r = \sum_{n=1}^{\infty} a(rn)q^n \in S_{\lambda+1/2}\left(\Gamma_0(4rN), \chi\left(\frac{r}{\cdot}\right)\right),$$

and $S_1(f \mid U_r) = \sum_{n=1}^{\infty} A_1(n)q^n$, where

$$A_1(n) := \sum_{d|n} \chi(d) \left(\frac{-1}{d}\right)^r \left(\frac{r}{d}\right) d^{\lambda-1} a\left(\frac{rn^2}{d^2}\right),$$

which is what we wanted. □

2.2 Examples

We now give an example of Theorem 2.4.

Example 2.6. Let $f_5(z) := \eta(24z)^5 \in S_{2+1/2}(\Gamma_0(576), \chi_3)$. Using the definition (2.1.1), we can compute the image of f_5 under the Shimura map S_5,

$$\begin{aligned} F_5(z) &:= S_5(f_5(z)) \in S_4(\Gamma_0(144)) \\ &= q - 6q^5 + 16q^7 + 12q^{11} + \cdots . \end{aligned}$$

Using Proposition 1.7, we have

$$\begin{aligned} h(z) &:= f_5(z) \mid U_5 \in S_{2+1/2}\left(\Gamma_0(5 \cdot 576), \chi_3\left(\frac{5}{\cdot}\right)\right) \\ &= q - 6q^{25} + 9q^{49} + 10q^{73} + \cdots . \end{aligned}$$

If we can write $h(z) = \eta(24z)g(24z)$ for some normalized Hecke eigenform $g(z) \in M_2\left(\Gamma_0(5), \left(\frac{5}{\cdot}\right)\right)$, then we can use Theorem 2.4. It is easy to verify that the form

$g(z) := \frac{\eta(z)^5}{\eta(5z)} = 1 - 5q + 5q^2 + 10q^3 + \cdots$ is a Hecke eigenform (not normalized) in this space, and that we have the relation

$$\eta(24z)^5 \mid U_5 = \eta(24z) \cdot \frac{\eta(24z)^5}{\eta(5 \cdot 24z)}.$$

Using Theorem 2.4, we have

$$S_1(h) = \frac{-1}{5}\left(g(z)g(6z) - g(2z)g(3z)\right) \otimes \chi_3.$$

Strictly speaking, we have

$$\frac{-1}{5}\left(g(z)g(6z) - g(2z)g(3z)\right) \in S_4(\Gamma_0(30)),$$

however, we can identify this form as the newform $\left(\eta(z)\eta(2z)\eta(3z)\eta(6z)\right)^2 \in S_4(\Gamma_0(6))$ using Sturm's Theorem. Twisting by χ_3 gives

$$S_5(\eta(24z)^5) = \left(\eta(z)\eta(2z)\eta(3z)\eta(6z)\right)^2 \otimes \chi_3 \in S_4(\Gamma_0(144)).$$

2.3 Proofs of Results

Proof of Theorem 2.4. By Proposition 1.6, we have $g(24z) \in M_k(\Gamma_0(24N), \chi)$. Thinking of $g(24z)$ as a half-integral weight form, we have $g(24z) \in M_k(\Gamma_0(24N), \chi\chi_{-1}^k)$. See Prop. 3 in Chapter 4 of (Koblitz, 1993) for details. Multiplying out the series gives

$$\eta(24z)g(24z) = \frac{1}{2} \cdot \sum_{t=1}^{\infty} \sum_{m \in \mathbb{Z}} \chi_3(m) a\left(\frac{t - m^2}{24}\right) q^t \in S_{k+1/2}(\Gamma_0(576N), \chi\chi_3\chi_{-1}^k).$$

For the sake of simplicity, write

$$b(t) := \frac{1}{2} \cdot \sum_{m \in \mathbb{Z}} \chi_3(m) a\left(\frac{t - m^2}{24}\right),$$

so that $\eta(24z)g(24z) = \sum_{t=1}^{\infty} b(t)q^t$. Now, by the definition of the Shimura correspondence, we have

$$S_1(\eta(24z)g(24z)) = \sum_{n=1}^{\infty} A(n)q^n \in S_{2k}(\Gamma_0(288N), \chi^2),$$

where (2.1.1) gives

$$\begin{aligned}
A(n) &= \sum_{d|n} \chi\chi_3(d) d^{k-1} b\left(\frac{n^2}{d^2}\right) \\
&= \frac{1}{2} \cdot \sum_{d|n} \chi\chi_3(d) d^{k-1} \sum_{m\in\mathbb{Z}} \chi_3(m) a\left(\frac{\frac{n^2}{d^2} - m^2}{24}\right) \\
&= \frac{1}{2} \cdot \sum_{d|n} \chi\chi_3(d) d^{k-1} \sum_{m\in\mathbb{Z}} \chi_3(m) a(n,d,m), \qquad (2.3.1)
\end{aligned}$$

where we have written $a(n,d,m) := a\left(\frac{\frac{n^2}{d^2}-m^2}{24}\right)$ for brevity. At this point, we need the following lemma:

Lemma 2.7. *With notations and definitions as above, we have*

$$\sum_{m\in\mathbb{Z}} \chi_3(m) a(n,d,m) = 2\chi_3(n)\chi_3(d)\left(\sum_{\substack{s\in\mathbb{Z}\\ d|s}} a\left(\frac{s(n-6s)}{d^2}\right) - \sum_{\substack{t\in\mathbb{Z}\\ d|t\\ t\ odd}} a\left(\frac{t\left(\frac{n-3t}{2}\right)}{d^2}\right)\right)$$

Proof. Because of the $\chi_3(m)$ factor in each term, the sum on the left is supported on integers m coprime to 12. For such m, we have $m^2 \equiv 1 \pmod{24}$. Furthermore, we have

$$\frac{n^2}{d^2} - m^2 \equiv \frac{n^2}{d^2} - 1 \equiv 0 \pmod{24}.$$

It follows that $n/d \equiv 1, 5, 7, 11 \pmod{12}$. The sum of interest is indexed by integers m, so we will split the sum into residue classes. By the previous observation, we have that

$$m \equiv \pm\frac{n}{d}, \quad \pm\frac{n}{d} + 6 \pmod{12}.$$

When $m \equiv \pm n/d \pmod{12}$, we see that $\chi_3(m) = \chi_3(\pm n/d) = \chi_3(\pm 1)\chi_3(n/d) = \chi_3(n/d)$ since $\chi_3(1) = \chi_3(-1) = 1$. When $m \equiv \pm n/d + 6 \pmod{12}$, we see that $\chi_3(m) = \chi_3(\pm n/d + 6) = -\chi_3(\pm n/d) = -\chi_3(n/d)$. Thus, our sum becomes

$$\sum_{m\in\mathbb{Z}} \chi_3(m) a(n,d,m) = \chi_3(n/d) \sum_{m\equiv \pm n/d \ (\mathrm{mod}\ 12)} a(n,d,m) - \chi_3(n/d) \sum_{m\equiv \pm n/d+6 \ (\mathrm{mod}\ 12)} a(n,d,m) \qquad (2.3.2)$$

We analyze each of these two sums individually.

First, consider the case that $m \equiv n/d \pmod{12}$. For each such m, we have $m = n/d - 12j$ for some $j \in \mathbb{Z}$. So we obtain

$$\frac{n}{d} - m = 12j, \quad \frac{n}{d} + m = 2\left(\frac{n}{d} - 6j\right).$$

Thus, we conclude that

$$\frac{1}{24}\left(\frac{n^2}{d^2} - m^2\right) = \frac{1}{24}\left(\frac{n}{d} - m\right)\left(\frac{n}{d} + m\right) = j\left(\frac{n}{d} - 6j\right).$$

Summing over all such m is equivalent to summing over all j, so we get

$$\sum_{m \equiv n/d \ (\mathrm{mod}\ 12)} a(n,d,m) = \sum_{j \in \mathbb{Z}} a\left(j\left(\frac{n}{d} - 6j\right)\right). \tag{2.3.3}$$

Next, consider the case $m \equiv -n/d \pmod{12}$. For each m of this form, we have $m = -n/d + 12i$ for some $i \in \mathbb{Z}$, so

$$\frac{n}{d} + m = 12i, \quad \frac{n}{d} - m = 2\left(\frac{n}{d} - 6i\right).$$

By a similar manipulation as before, we get

$$\sum_{m \equiv -n/d \ (\mathrm{mod}\ 12)} a(n,d,m) = \sum_{i \in \mathbb{Z}} a\left(i\left(\frac{n}{d} - 6i\right)\right). \tag{2.3.4}$$

Now, we look at the case where $m \equiv n/d + 6 \pmod{12}$. For these m, we have $m = n/d + 6 - 12\ell$ for some $\ell \in \mathbb{Z}$. This gives us

$$\frac{n}{d} - m = 6(2\ell - 1), \quad \frac{n}{d} + m = 2\left(\frac{n}{d} - 3(2\ell - 1)\right)$$

so that

$$\frac{1}{24}\left(\frac{n^2}{d^2} - m^2\right) = \frac{(2\ell - 1)\left(\frac{n}{d} - 3(2\ell - 1)\right)}{2}.$$

This gives

$$\sum_{m \equiv n/d+6 \ (\mathrm{mod}\ 12)} a(n,d,m) = \sum_{\ell \in \mathbb{Z}} a\left(\frac{(2\ell - 1)\left(\frac{n}{d} - 3(2\ell - 1)\right)}{2}\right). \tag{2.3.5}$$

Finally, in the case that $m \equiv -n/d + 6 \pmod{12}$, we get $m = -n/d + 6 + 12u$ for some $u \in \mathbb{Z}$. Rewriting, we get

$$\frac{n}{d} + m = 6(2u + 1), \quad \frac{n}{d} - m = 2\left(\frac{n}{d} - 3(2u + 1)\right).$$

Similar manipulation as the last case and reindexing by replacing u with $\ell-1$ gives

$$\sum_{m\equiv -n/d+6 \pmod{12}} a(n,d,m) = \sum_{\ell\in\mathbb{Z}} a\left(\frac{(2\ell-1)\left(\frac{n}{d}-3(2\ell-1)\right)}{2}\right). \tag{2.3.6}$$

Using these sums (2.3.3), (2.3.4), (2.3.5), and (2.3.6) in (2.3.2), we arrive at

$$\begin{aligned}&\sum_{m\in\mathbb{Z}} \chi_3(m)a(n,d,m)\\ &\qquad = 2\chi_3(n/d)\left(\sum_{j\in\mathbb{Z}} a\left(\frac{jd(n-6jd)}{d^2}\right) - \sum_{\ell\in\mathbb{Z}} a\left(\frac{(2\ell-1)d\left(\frac{n-3(2\ell-1)d}{2}\right)}{d^2}\right)\right).\end{aligned} \tag{2.3.7}$$

Because of the constant factor of $\chi_3(d)$ in the sum (2.3.1) defining $A(n)$, we can split up this character as $\chi_3(n/d)=\chi_3(n)\chi_3(d)$, as all terms in the sum are zero when d is odd. Moreover, since n is odd, we must have t odd, otherwise $\frac{n-3t}{2}\notin\mathbb{Z}$. Thus, (2.3.7) becomes

$$\sum_{m\in\mathbb{Z}} \chi_3(m)a(n,d,m) = 2\chi_3(n)\chi_3(d)\left(\sum_{\substack{s\in\mathbb{Z}\\ d\mid s}} a\left(\frac{s(n-6s)}{d^2}\right) - \sum_{\substack{t\in\mathbb{Z}\\ d\mid t\\ t \text{ odd}}} a\left(\frac{t\left(\frac{n-3t}{2}\right)}{d^2}\right)\right),$$

which is what we wanted. □

Using Lemma 2.7 in our expression (2.3.1), we get

$$\begin{aligned} A(n) &= \chi_3(n)\sum_{d\mid n}\chi(d)d^{k-1}\left(\sum_{\substack{s\in\mathbb{Z}\\ d\mid s}} a\left(\frac{s(n-6s)}{d^2}\right) - \sum_{\substack{t\in\mathbb{Z}\\ d\mid t\\ t \text{ odd}}} a\left(\frac{t\left(\frac{n-3t}{2}\right)}{d^2}\right)\right)\\ &= \chi_3(n)\left(\sum_{s\in\mathbb{Z}}\sum_{d\mid(s,n)}\chi(d)d^{k-1}a\left(\frac{s(n-6s)}{d^2}\right) - \sum_{\substack{t\in\mathbb{Z}\\ t \text{ odd}}}\sum_{d\mid(t,n)}\chi(d)d^{k-1}a\left(\frac{t\left(\frac{n-3t}{2}\right)}{d^2}\right)\right)\\ &= \chi_3(n)\left(\sum_{s\in\mathbb{Z}}\sum_{d\mid(s,n-6s)}\chi(d)d^{k-1}a\left(\frac{s(n-6s)}{d^2}\right) - \sum_{\substack{t\in\mathbb{Z}\\ t \text{ odd}}}\sum_{d\mid(t,\frac{n-3t}{2})}\chi(d)d^{k-1}a\left(\frac{t\left(\frac{n-3t}{2}\right)}{d^2}\right)\right),\end{aligned}$$

were we used the fact that $\gcd(s,n)=\gcd(s,n-6s)$ and $\gcd(t,n)=\gcd(t,n-3t)=\gcd(t,\frac{n-3t}{2})$, since t,n both odd give us that $n-3t$ is even.

Now, using Proposition 1.8 we get

$$\begin{aligned}
A(n) &= \chi_3(n)\left(\sum_{s\in\mathbb{Z}} a(s)a(n-6s) - \sum_{\substack{t\in\mathbb{Z}\\ t \text{ odd}}} a(t)a\left(\frac{n-3t}{2}\right)\right)\\
&= \chi_3(n)\left(\sum_{s\in\mathbb{Z}} a(s)a(n-6s) - \sum_{t\in\mathbb{Z}} a(t)a\left(\frac{n-3t}{2}\right)\right),
\end{aligned}$$

where we removed the stipulation that t be odd since $\frac{n-3t}{2} \notin \mathbb{Z}$ unless t is odd. We conclude with

$$\begin{aligned}
S_1(\eta(24z)g(24z)) &= \sum_{n=1}^{\infty} A(n)q^n\\
&= \sum_{n=1}^{\infty} \chi_3(n)\left(\sum_{s\in\mathbb{Z}} a(s)a(n-6s) - \sum_{t\in\mathbb{Z}} a(t)a\left(\frac{n-3t}{2}\right)\right)\\
&= \left(\sum_{n=1}^{\infty}\left(\sum_{s\in\mathbb{Z}} a(s)a(n-6s)\right)q^n - \sum_{n=1}^{\infty}\left(\sum_{t\in\mathbb{Z}} a(t)a\left(\frac{n-3t}{2}\right)\right)q^n\right)\otimes\chi_3\\
&= (g(z)g(6z) - g(2z)g(3z))\otimes\chi_3,
\end{aligned}$$

which gives us what we wanted. After this, a straightforward application of Proposition 1.10 shows that the level of the image divides $144N$.

Now, we prove the second part of the theorem. Recall that

$$\eta(8z)^3 = \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^m(2m+1)q^{(2m+1)^2} \in S_{3/2}(\Gamma_0(64)). \tag{2.3.8}$$

As $g(z) \in M_k(\Gamma_0(N), \chi)$, we have $g(8z) \in M_k(\Gamma_0(8N), \chi\chi_{-1}^k)$ for the same reasons as in the first part of the proof. Multiplying out the series gives

$$\eta(8z)^3 g(8z) = \sum_{t=1}^{\infty} b(t)q^t \in S_{k+1+1/2}(\Gamma_0(64N), \chi\chi_{-1}^k),$$

where

$$b(t) = \frac{1}{2}\sum_{m\in\mathbb{Z}}(-1)^m(2m+1)a\left(\frac{t-(2m+1)^2}{8}\right).$$

Now, by definition of the Shimura map S_1, we have

$$S_1(\eta(8z)^3 g(8z)) = \sum_{n=1}^{\infty} A(n)q^n \in S_{2k+2}(\Gamma_0(32N), \chi^2),$$

where by (2.1.1) we get

$$
\begin{aligned}
A(n) &= \sum_{d|n} \chi\chi_{-1}(d) d^k b\left(\frac{n^2}{d^2}\right) \\
&= \frac{1}{2}\sum_{d|n} \chi\chi_{-1}(d) d^k \sum_{m\in\mathbb{Z}} (-1)^m (2m+1) a\left(\frac{\frac{n^2}{d^2}-(2m+1)^2}{8}\right). \qquad (2.3.9)
\end{aligned}
$$

We now prove a lemma.

Lemma 2.8. *With notations and definitions as above, we have*

$$A(n) = \chi_{-1}(n) \sum_{d|n} \chi(d) d^k \sum_{j\in\mathbb{Z}} \left(\frac{n}{d}-4j\right) a\left(j\left(\frac{n}{d}-2j\right)\right).$$

Proof. Observe that $(2m+1)^2 \equiv 1 \pmod 8$ for all integers m. So $\frac{n^2}{d^2} - (2m+1)^2 \equiv \frac{n^2}{d^2} - 1 \equiv 0 \pmod 8$, and this is true for $n/d \equiv 1, 3 \pmod 4$. We split the sum (2.3.9) defining $A(n)$ into residue classes accordingly. We have

$$
\begin{aligned}
2A(n) = &\sum_{\substack{d|n \\ \frac{n}{d}\equiv 1 \pmod 4}} \chi\chi_{-1}(d) d^k \sum_{m\in\mathbb{Z}} (-1)^m (2m+1) a\left(\frac{\frac{n^2}{d^2}-(2m+1)^2}{8}\right) \qquad (2.3.10) \\
+ &\sum_{\substack{d|n \\ \frac{n}{d}\equiv 3 \pmod 4}} \chi\chi_{-1}(d) d^k \sum_{m\in\mathbb{Z}} (-1)^m (2m+1) a\left(\frac{\frac{n^2}{d^2}-(2m+1)^2}{8}\right). \qquad (2.3.11)
\end{aligned}
$$

First, consider (2.3.10). We will look at the even- and odd-indexed terms separately. When m is even, the inner sum in (2.3.10) becomes

$$\sum_{i\in\mathbb{Z}} (4i+1) a\left(\frac{\frac{n^2}{d^2}-(4i+1)^2}{8}\right). \qquad (2.3.12)$$

Since $n/d \equiv 1 \pmod 4$, we have $n/d = 4j + 4i + 1$ for some $j \in \mathbb{Z}$. So

$$\frac{n}{d} + (4i+1) = \frac{n}{d}\left(\frac{n}{d} - 4j\right) = 2\left(\frac{n}{d} - 2j\right), \quad \frac{n}{d} - (4i+1) = 4j,$$

so that

$$\frac{n^2}{d^2} - (4i+1)^2 = \left(\frac{n}{d} - (4i+1)\right)\left(\frac{n}{d} + (4i+1)\right) = 8j\left(\frac{n}{d} - 2j\right).$$

Thus, (2.3.12) becomes

$$\sum_{j\in\mathbb{Z}} \left(\frac{n}{d} - 4j\right) a\left(j\left(\frac{n}{d} - 2j\right)\right). \qquad (2.3.13)$$

Similarly, for m odd, the inner sum in (2.3.10) becomes

$$-\sum_{i\in\mathbb{Z}}(4i+3)a\left(\frac{\frac{n^2}{d^2}-(4i+3)^2}{8}\right). \tag{2.3.14}$$

Since $n/d \equiv 1 \pmod 4$, we have $n/d = 4j-(4i+3)$ for some $j \in \mathbb{Z}$, so

$$\frac{n}{d}+(4i+3)=4j, \quad \frac{n}{d}-(4i+3)=2\left(\frac{n}{d}-2j\right)$$

Thus, (2.3.12) becomes

$$\sum_{j\in\mathbb{Z}}\left(\frac{n}{d}-4j\right)a\left(j\left(\frac{n}{d}-2j\right)\right) \tag{2.3.15}$$

Now, $\chi_{-1}(d) = \chi_{-1}(d)\chi_{-1}(n/d) = \chi_{-1}(n)$ since $n/d \equiv 1 \pmod 4$, so that using (2.3.13) and (2.3.15), (2.3.10) becomes

$$2\chi_{-1}(n)\sum_{\substack{d|n\\ \frac{n}{d}\equiv 1 \pmod 4}}\chi(d)d^k\sum_{j\in\mathbb{Z}}\left(\frac{n}{d}-4j\right)a\left(j\left(\frac{n}{d}-2j\right)\right) \tag{2.3.16}$$

Next, we look at (2.3.11). As before, we will split the inner sum over m into even- and odd-indexed terms. For m even, this sum becomes

$$\sum_{i\in\mathbb{Z}}(4i+1)\,a\left(\frac{\left(\frac{n}{d}\right)^2-(4i+1)^2}{8}\right) \tag{2.3.17}$$

Here, $n/d \equiv 3 \pmod 4$, so we have $n/d = 4j-(4i+1)$ for some $j \in \mathbb{Z}$. We get

$$\begin{aligned}
&\frac{n}{d}+(4i+1)=4j\\
&\frac{n}{d}-(4i+1)=\frac{n}{d}-\left(4j-\frac{n}{d}\right)=2\left(\frac{n}{d}-2j\right)
\end{aligned}$$

so that

$$\left(\frac{n}{d}\right)^2-(4i+1)^2=8j\left(\frac{n}{d}-2j\right).$$

Thus, (2.3.17) becomes

$$\sum_{j\in\mathbb{Z}}\left(4j-\frac{n}{d}\right)a\left(j\left(\frac{n}{d}-2j\right)\right) \tag{2.3.18}$$

Similarly, for odd m, the inner sum in (2.3.11) becomes

$$\sum_{i\in\mathbb{Z}}(4i+1)\,a\left(\frac{\left(\frac{n}{d}\right)^2-(4i+3)^2}{8}\right). \tag{2.3.19}$$

Since $n/d \equiv 3 \pmod 4$, we have $n/d = 4j+4i+3$ for some $j\in\mathbb{Z}$. We get

$$\begin{aligned}
&\frac{n}{d}-(4i+3)=4j\\
&\frac{n}{d}+(4i+3)=\frac{n}{d}+\left(\frac{n}{d}-4j\right)=2\left(\frac{n}{d}-2j\right)
\end{aligned}$$

so that

$$\left(\frac{n}{d}\right)^2-(4i+3)^2=8j\left(\frac{n}{d}-2j\right).$$

Thus, (2.3.19) becomes

$$\sum_{j\in\mathbb{Z}}\left(\frac{n}{d}-4j\right)a\left(j\left(\frac{n}{d}-2j\right)\right). \tag{2.3.20}$$

For $n/d\equiv 3 \pmod 4$, we have that $-\chi_{-1}(d)=\chi_{-1}(d)\chi_{-1}(n/d)=\chi_{-1}(n)$, so that by combining (2.3.18) and (2.3.20), (2.3.11) becomes

$$2\chi_{-1}(n)\sum_{\substack{d\mid n\\ \frac{n}{d}\equiv 3 \pmod 4}}\chi(d)\,d^k\sum_{j\in\mathbb{Z}}\left(\frac{n}{d}-4j\right)a\left(j\left(\frac{n}{d}-2j\right)\right). \tag{2.3.21}$$

Now from (2.3.16) and (2.3.21), it follows that

$$A(n)=\chi_{-1}(n)\sum_{d\mid n}\chi(d)d^k\sum_{j\in\mathbb{Z}}\left(\frac{n}{d}-4j\right)\,a\left(j\left(\frac{n}{d}-2j\right)\right),$$

which is what we wanted to show. $\square$

Using our Lemma 2.8 in (2.3.9), we obtain

$$\begin{aligned}
A(n) &= \chi_{-1}(n)\sum_{d|n}\chi(d)d^k\sum_{j\in\mathbb{Z}}\left(\frac{n}{d}-4j\right)a\left(j\left(\frac{n}{d}-2j\right)\right)\\
&= \chi_{-1}(n)\sum_{d|n}\chi(d)d^k\sum_{\substack{s\in\mathbb{Z}\\ d|s}}\frac{1}{d}(n-4s)a\left(\frac{s(n-2s)}{d^2}\right) \quad \text{replacing } s=jd\\
&= \chi_{-1}(n)\sum_{s\in\mathbb{Z}}(n-4s)\sum_{d|(s,n)}\chi(d)d^{k-1}a\left(\frac{s(n-2s)}{d^2}\right)\\
&= \chi_{-1}(n)\sum_{s\in\mathbb{Z}}(n-4s)\sum_{d|(s,n-2s)}\chi(d)d^{k-1}a\left(\frac{s(n-2s)}{d^2}\right)\\
&= \chi_{-1}(n)\sum_{s\in\mathbb{Z}}(n-4s)a(s)a(n-2s),
\end{aligned} \tag{2.3.22}$$

where the last equality comes from the fact that g is a Hecke eigenform and Proposition 1.8.

We need another lemma.

Lemma 2.9. *We have*

$$g(2z)\Theta(g(z))-g(z)\Theta(g(2z))=\sum_{t=0}^{\infty}\left(\sum_{s\in\mathbb{Z}}(t-4s)a(s)a(t-2s)\right)q^t.$$

Proof. We have $g(z)=\sum_{n=0}^{\infty}a(n)q^n$. It follows that

$$g(2z)=\sum_{n=0}^{\infty}a(n)q^{2n},\quad \Theta(g(z))=\sum_{n=0}^{\infty}na(n)q^n,\quad \Theta(g(2z))=\sum_{n=0}^{\infty}2na(n)q^{2n}.$$

From this, we see that

$$\begin{aligned}
g(2z)\Theta(g(z)) &= \sum_{t=0}^{\infty}\left(\sum_{2s+r=t}a(s)ra(r)\right)q^t=\sum_{t=0}^{\infty}\left(\sum_{s\in\mathbb{Z}}(t-2s)a(s)a(t-2s)\right)q^t,\\
g(z)\Theta(g(2z)) &= \sum_{t=0}^{\infty}\left(\sum_{2s+r=t}a(r)2sa(s)\right)q^t=\sum_{t=0}^{\infty}\left(\sum_{s\in\mathbb{Z}}2sa(s)a(t-2s)\right)q^t.
\end{aligned}$$

The desired result follows. □

Lemma 2.9 establishes that we have the q-series identity

$$\sum_{n=1}^{\infty}A(n)q^n=(g(2z)\Theta(g(z))-g(z)\Theta(g(2z)))\otimes\chi_{-1}.$$

It is not clear that the expression on the right is a modular form, since it contains derivatives of modular forms. Letting $h(z) = g(2z)$ and applying Proposition 1.11, we see that

$$
\begin{aligned}
&g(2z)\Theta(g(z)) - g(z)\Theta(g(2z)) \\
&\quad = \frac{1}{12}h(z)\left(\tilde{g}(z) + kg(z)E_2(z)\right) - g(z)\left(\tilde{h}(z) + kh(z)E_2(z)\right) \\
&\quad = \frac{1}{12}\left(\tilde{g}(z)h(z) = kg(z)h(z)E_2(z) - g(z)\tilde{h}(z) - kg(z)h(z)E_2(z)\right) \\
&\quad = \frac{1}{12}\left(\tilde{g}(z)h(z) - g(z)\tilde{h}(z)\right),
\end{aligned}
$$

which is modular of weight $2k+2$

Theorem 2.1 tells us that the image will have level dividing $32N$. However an application of Proposition 1.10 shows us that the level will in fact divide $16N$. This completes the proof. □

Chapter 3

Elliptic Curve Congruences

3.1 Framework

Let r be an odd integer with $1 \leq r \leq 23$, let $s \geq 0$ be even, and let χ_3 be the Dirichlet character modulo 12 as defined in (1.2.4). In (Garvan, 2010) it was proved that the subspace

$$\mathcal{S}_{r,s} := \{\eta(24z)^r f(24z) \mid f(z) \in M_s(\mathrm{SL}_2(\mathbb{Z}))\} \subseteq S_{r/2+s}(\Gamma_0(576), \chi_3) \tag{3.1.1}$$

is stable under the Hecke operators $T_{\ell^2} := T_{\ell^2, r/2+s, \chi_3}$ for primes $\ell \geq 5$. That is, for primes $\ell \geq 5$ and $f \in \mathcal{S}_{r,s}$, we have $f \mid T_{\ell^2} \in \mathcal{S}_{r,s}$. We have $M_s(\mathrm{SL}_2(\mathbb{Z})) = \mathbb{C}E_s(z)$ when $s \in \{0, 4, 6, 8, 10, 14\}$, so for these values of s, the form

$$f_{r,s}(z) := \eta(24z)^r E_s(24z) = \sum_{n=1}^{\infty} a_{r,s}(n) q^n = q^r + \cdots \tag{3.1.2}$$

is an eigenform. Considering all such possibilities for r and s, there are 72 eigenforms in total.

For a fixed squarefree $t \geq 1$, the Shimura lift S_t maps

$$S_t : \mathcal{S}_{r,s} \subseteq S_{r/2+s}(\Gamma_0(576), \chi_3) \to S_{r-1+2s}(\Gamma_0(N)),$$

with $N \mid 288$. In this section, we will stick to the convention of using the Shimura map S_r on the form $f_{r,s}$. We define

$$\mathcal{F}_{r,s}(z) := \sum_{n=1}^{\infty} A_{r,s}(n) q^n = S_r(f_{r,s}(z)) \in S_{r-1+2s}(\Gamma_0(N)) \tag{3.1.3}$$

and recall that the map S_r is Hecke-equivariant. In our case, this means that for all primes $\ell \geq 5$, we have

$$S_r(f_{r,s}(z) \mid T_{\ell^2}) = S_r(f_{r,s}(z)) \mid T_\ell = \mathcal{F}_{r,s}(z) \mid T_\ell. \tag{3.1.4}$$

We now outline the general idea for connecting elliptic curves and weight 2 newforms on one hand with half-integral weight eigenforms $f_{r,s}(z)$ as in (3.1.2) on the other. To make the connection, start with $G(z) \in S_2^{\mathrm{new}}(\Gamma_0(6\ell^j)) \cap \mathbb{Z}[[q]]$. By Theorem 1.14, there exists an elliptic curve

$$E_G/\mathbb{Q} : y^2 + a_1xy + a_3y = x^3 + a_2x^2 + a_4x + a_6$$

with conductor $N(E_G) = 6\ell^j$ such that for primes $p \notin \{2, 3, \ell\}$, we have

$$c_G(p) = p + 1 - |E_G(\mathbb{F}_p)|.$$

For all $0 \leq a \leq \ell - 1$, we find that $\Theta^a G(z)$ is congruent modulo ℓ to a form in $S_{2+a(\ell+1)}(\Gamma_0(6\ell^j))$. Furthermore, a theorem of Serre asserts that for some $b \leq a + \left\lfloor \frac{2a}{\ell-1} \right\rfloor$, we have $\Theta^a G(z)$ congruent modulo ℓ to a form in $S_{2+a(\ell+1)-b(\ell-1)}(\Gamma_0(6))$. To verify this congruence when $b \geq 0$, it suffices to compute

$$\frac{(2 + a(\ell + 1))\, 6\ell^j}{12} \left(1 + \frac{1}{2}\right) \left(1 + \frac{1}{3}\right) \left(1 + \frac{1}{\ell}\right) = (2 + a(\ell + 1))(\ell + 1)\ell^j$$

coefficients of the forms modulo ℓ. Next, we twist $\Theta^a G(z)$ by χ_3 via Proposition 1.10 to obtain

$$\Theta^a G(z) \otimes \chi_3 = \sum_{n=1}^{\infty} n^a c_G(n)\chi_3(n)q^n \in S_{2+a(\ell+1)-b(\ell-1)}(\Gamma_0(144))$$

modulo ℓ. To connect with the half-integral weight eigenform $f_{r,s}(z)$, we try to find $G(z)$, and hence $E_G/\mathbb{Q}$, for which

$$\mathcal{F}_{r,s}(z) = S_r(f_{r,s}(z)) \equiv \Theta^a G(z) \otimes \chi_3 \pmod{\ell}.$$

The form on the left has weight $r - 1 + 2s$ and level $N \mid 288$. Experiments suggest that $N = 144$ as in the speculation above. The form on the right has weight $2 + a(\ell + 1) - b(\ell - 1)$ and level 144. Therefore, for such a congruence to hold, we must have

$$r - 1 + 2s = 2 + a(\ell + 1) - b(\ell - 1) = (a - b)\ell + a + b + 2.$$

To verify the congruence, we must compute

$$\frac{(r-1+2s)288}{12}\left(1+\frac{1}{2}\right)\left(1+\frac{1}{3}\right)=48(r-1+2s)$$

coefficients of the forms modulo ℓ at level $288 = 24 \cdot 12$.

3.2 Examples

We will demonstrate a few examples to illustrate the method of proof outlined.

Example 3.1. Let $f_{17}(z) = \eta(24z)^{17} = \sum_{n=0}^{\infty} a_7(n)q^n$. We see that $f_{17}(z) \in S_{17/2}(\Gamma_0(576), \chi_3)$, and

$$f_{17}(z) = q^{17} - 17q^{41} + 119q^{65} - 408q^{89} + \cdots .$$

Letting $\mathcal{F}_7(z) := \sum_{n=1}^{\infty} A_7(n)q^n$ denote the image of $f_{17}(z)$ under the Shimura map S_{17}, we find that $\mathcal{F}_{17}(z) \in S_{16}(\Gamma_0(288))$, and that

$$\mathcal{F}_{17}(z) = q + 114810q^5 + 3034528q^7 - 103451700q^{11} + \cdots .$$

Let E denote the elliptic curve

$$y^2 + xy + y = x^3 + x^2 - 4x + 5$$

defined over $\mathbb{Q}$, and let $G(z) = \sum_{n=1}^{\infty} c_G(n)q^n \in S_2(\Gamma_0(42))$ be the cuspform associated to the curve E (which has conductor 42). We will show that

$$\mathcal{F}_7(z) \equiv \sum_{n=1}^{\infty} \chi_3(n) n c_G(n) q^n \pmod 7.$$

From (1.3.8), we find that

$$E_6 \cdot \Theta G \equiv \Theta G = \sum_{n=1}^{\infty} n c_G(n) q^n \pmod 7$$

is congruent to a modular form in $S_{16}(\Gamma_0(42))$, and one can show that that is congruent to a modular form in $S_{16}(\Gamma_0(6))$. Using Proposition 1.10, we find that the χ_3 quadratic twist

$$\Theta G \otimes \chi_3 = \sum_{n=1}^{\infty} n \left(\frac{12}{n}\right) c_G(n) q^n$$

is congruent to a modular form in $S_{16}(\Gamma_0(864))$. Since $\mathcal{F}_7(z) \in S_{16}(\Gamma_0(288)) \subseteq S_{16}(\Gamma_0(864))$, we may check that

$$\sum_{n=1}^{\infty} n \left(\frac{12}{n}\right) c_G(n) q^n \equiv \mathcal{F}_7(z) \pmod 7$$

by checking that the first 2304 coefficients agree modulo 7.

Chapter 4

Pseudo Eigenforms

4.1 Background and Statement of Results

For an integer $1 \leq r \leq 12$, we let δ_r denote the least positive integer such that $24 \mid 2r\delta_r$. Note that we have $\delta_r = 12/\gcd(12, r)$. Theorem 1.3 tells us that

$$\eta(\delta_r z)^{2r} \in S_r\left(\Gamma_0(\delta_r^2), \left(\frac{-1}{\cdot}\right)^r\right).$$

In (Dummit et al., 1985), a complete classification of eta-products that are also Hecke eigenforms was given. In particular, they proved that $\eta(az)^b$ is a newform if and only there exists $r \in \mathbb{Z}$ with $b = 2r$ and $a = \delta_r$ such that $r \mid 12$. We note that this holds if and only if

$$(r, \delta_r) \in A := \{(1,12), (2,6), (3,4), (4,3), (6,2), (12,1)\}.$$

It turns out that these forms all have order of vanish at the cusp infinity equal to one, and they lie in one-dimensional spaces. Later, in (Martin, 1996), all eta-quotient newforms were classified. We will prove that the forms $\eta(\delta_r z)^{2r}$ are eigenforms for primes $p \equiv 1 \pmod{\delta_r}$ when (r, δ_r) come from the set

$$B := \{(5,12), (7,12), (8,3), (9,2), (10,6), (11,12)\},$$

and we will describe Hecke invariant spaces of forms analogous to the space $\mathcal{S}_{r,s}$ as defined in (3.1.1).

Let $f_r(z) := \eta(\delta_r z)^{2r}$, let $b_r := 2r\delta_r/24$, and observe that

$$f_r(z) = q^{b_r} \prod_{n=1}^{\infty} (1 - q^{\delta_r n})^{2r} = q^{b_r} + \cdots$$

is supported on exponents $b_r + \delta_r n \equiv b_r \pmod{\delta_r}$.

Lemma 4.1. *Let $1 \le r \le 12$, and let f_r, δ_r, and b_r be as above. Suppose that p is prime with $p \nmid \delta_r$, and that $f_r \mid T_p \neq 0$. Then the q-expansion of $f_r \mid T_p$ has support on exponents $pb_r \pmod{\delta_r}$.*

Proof. Writing $f_r(z) = \sum_{n=0}^{\infty} a_r(n)q^n$, we have

$$f_r \mid T_p = \sum_{n=0}^{\infty} \left(a_r(pn) + p^{r-1} \left(\frac{-1}{p} \right)^r a_r(n/p) \right) q^n,$$

where $a_r(n/p) = 0$ if $p \nmid n$. It is clear that the image has support on exponents n with $a_r(pn) \neq 0$ or $a_r(n/p) \neq 0$. Observe that if an integer m is coprime to 24, then $m^2 \equiv 1 \pmod{24}$. Since $\delta_r \mid 24$, it follows that $m^2 \equiv 1 \pmod{\delta_r}$ for integers m coprime to δ_r. We also recall that $a_r(m) \neq 0$ implies that $m \equiv b_r \pmod{\delta_r}$. Hence, if we suppose that $a_r(pn) \neq 0$, then we must have $pn \equiv b_r \pmod{\delta_r}$. Since $p \nmid \delta_r$, we multiply both sides by p and use our observation above to obtain $n \equiv pb_r \pmod{\delta_r}$. Similarly, we suppose that $p \mid n$. We see that $a(n/p) \neq 0$ implies that $n/p \equiv b_r \pmod{\delta_r}$. It follows that $n \equiv pb_r \pmod{\delta_r}$, as desired. $\square$

The vital input for the main results in this chapter is the following technical proposition. We state the proposition and its corollaries now; we defer the proofs and examples to Section 4.2.

Proposition 4.2. *Let $1 \le r \le 12$, let δ_r and b_r be as above, and let p be prime with $p \nmid \delta_r$. Define $j \ge 1$ to be $j = r$ if $r \mid 12$ and to be the least positive residue of $pb_r \pmod{12}$ if $r \nmid 12$. Suppose that $k \ge 0$ is even and that $F(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$. Then there exists $K(z) \in M_{k+r-j}(\mathrm{SL}_2(\mathbb{Z}))$ such that*

$$H(z) := \frac{(\eta(\delta_r z)^{2r} F(\delta_r z)) \mid T_p}{\eta(\delta_r z)^{2j}} = K(\delta_r z).$$

Let $s \ge 0$ be an integer, and define

$$\mathcal{A}_{r,s} := \left\{ \eta(\delta_r z)^{2r} F_s(\delta_r z) \mid (r, \delta_r) \in A \quad \text{and} \quad F(z) \in M_s(\mathrm{SL}_2(\mathbb{Z})) \right\}$$

$$\mathcal{B}_{r,s} := \left\{ \eta(\delta_r z)^{2r} F_s(\delta_r z) \mid (r, \delta_r) \in B \quad \text{and} \quad F(z) \in M_s(\mathrm{SL}_2(\mathbb{Z})) \right\}$$

Then $\mathcal{A}_{r,s}$, $\mathcal{B}_{r,s} \subseteq M_{r+s}\left(\Gamma_0(\delta_r^2), \left(\frac{-1}{\cdot}\right)^r\right)$ For $(r, \delta_r) \in A$, we see that $j = r$ in the statement of Proposition 4.2, and so

$$f_r \mid T_p = \eta(\delta_r z)^{2r} F(\delta_r z) \mid T_p = \eta(\delta_r z)^{2r} K(\delta_r z),$$

where $F(z)$ and $K(z) \in M_s(\mathrm{SL}_2(\mathbb{Z}))$. We have proved the following.

Corollary 4.3. *The subspace $\mathcal{A}_{r,s}$ is stable under the Hecke operators T_p for primes $p \nmid \delta_r$. If $s \in \{0, 4, 6, 8, 10, 14\}$ and $f(z) \in A_{r,s}$, then $f(z)$ is a Hecke eigenform.*

Corollary 4.4. *The subspace $\mathcal{B}_{r,s}$ is stable under the Hecke operators T_p for primes $p \equiv 1 \pmod{\delta_r}$. Given $f \in \mathcal{B}_{r,s}$, we have, $f \mid T_p \mid T_p \in \mathcal{B}_{r,s}$ for primes $p \nmid \delta_r$.*

For example, let

$$\begin{aligned} f(z) &= \eta(12z)^{10} E_4(12z) \in M_9\left(\Gamma_0(144), \left(\frac{-1}{\cdot}\right)\right) \\ &= \sum_{n=0}^{\infty} a(n) q^n = q^5 + 230 q^{17} - 205 q^{29} + \cdots \end{aligned}$$

Here, we have $(r, \delta_r) = (5, 12) \in B$, so that f is an eigenform for primes $p \equiv 1 \pmod{12}$. Indeed, we find that

$$f \mid T_{13} = 478 f = a(13 \cdot 5) f$$

$$f \mid T_{37} = 925922 f = a(37 \cdot 5) f.$$

4.2 Proofs

Proof of Proposition 4.2. Define

$$g(z) := g_{r,F}(z) = \eta(\delta_r z)^{2r} F(\delta_r z) \in S_{r+k}\left(\Gamma_0(\delta_r^2), \left(\frac{-1}{\cdot}\right)^r\right),$$

and let

$$H(z) := H_{p,r,F}(z) = \frac{g_{r,F}(z) \mid T_p}{\eta(\delta_r z)^{2j}}$$

Note that $H(z)$ is modular on $\Gamma_0(\delta_r^2)$. Since $F(z) \in M_k(\mathrm{SL}_2(\mathbb{Z}))$, we have

$$F\left(-\frac{1}{\delta_r z}\right) = (\delta_r z)^k F(\delta_r z),$$

and the transformation for the Dedekind eta-function implies that

$$\eta\left(-\frac{1}{\delta_r z}\right)^{2j} = (-i\delta_r z)^j \eta(\delta_r z)^{2j}.$$

Next, we observe that

$$\begin{aligned}
g \mid W_{\delta_r^2} &= (\delta_r z)^{-(r+k)} \eta\left(-\frac{1}{\delta_r z}\right)^{2r} F\left(-\frac{1}{\delta_r z}\right) \\
&= (\delta_r z)^{-(r+k)} (-i\delta_r z)^r \eta(\delta_r z)^{2r} (\delta_r z)^k F(\delta_r z) \\
&= (-i)^r \eta(\delta_r z)^{2r} F(\delta_r z) \\
&= i^{-r} g(z).
\end{aligned}$$

We conclude that $g \mid W_{\delta_r^2} = i^{-r} g(z)$. Also, we have

$$\begin{aligned}
(g \mid T_p)\left(-\frac{1}{\delta_r^2 z}\right) &= (\delta_r z)^{r+k} (g \mid T_p) \mid W_{\delta_r^2} \\
&= (\delta_r z)^{r+k} \left(\frac{-1}{p}\right)^r (g \mid W_{\delta_r^2}) \mid T_p \\
&= (\delta_r z)^{r+k} \left(\frac{-1}{p}\right)^r i^{-r} g(z) \mid T_p \\
&= (\delta_r z)^{r+k} i^{(p-1)r-r} g \mid T_p,
\end{aligned}$$

the last equality coming from the fact that $(-1)^{\frac{p-1}{2}} = \left(\frac{-1}{p}\right)$. Rearranging, we see that

$$(g \mid T_p) \mid W_{\delta_r^2} = (\delta_r z)^{-(r+k)} (g \mid T_p)\left(-\frac{1}{\delta_r^2 z}\right).$$

To finish the proof, we compute

$$
\begin{aligned}
H\left(-\frac{1}{\delta_r^2 z}\right) &= \frac{(g \mid T_p)\left(-\frac{1}{\delta_r^2 z}\right)}{\eta\left(-\frac{1}{\delta_r z}\right)^{2j}} \\
&= \frac{i^{(p-1)r-r}(\delta_r z)^{k+r-j} g(z) \mid T_p}{i^{-j}\eta(\delta_r z)^{2j}} \\
&= i^{2r(p-1)}(\delta_r z)^{r+k-j} H(z) \\
&= (\delta_r z)^{r+k-j} H(z).
\end{aligned}
$$

Our earlier Lemma implies that $H(z)$ has q-series supported on multiples of δ_r. It follows that $K(z) = K_{p,r,F}(z) = H(z/\delta_r)$ is a q-series supported on integral exponents, and hence, that $K(z+1) = K(z)$. Further, we find that

$$
K\left(-\frac{1}{z}\right) = H\left(-\frac{1}{\delta_r z}\right) = z^{k+r-j} H\left(\frac{z}{\delta_r}\right) = z^{k+r-j} K(z).
$$

Therefore, the series $K(z)$ is modular on $\mathrm{SL}_2(\mathbb{Z})$. It remains to see that $K(z)$ is holomorphic on $\mathcal{H}$ and at the cusp infinity. Since $\eta(z) \neq 0$ on $\mathcal{H}$, $K(z)$ is holomorphic there. By the minimality of j, the order of vanish of $g \mid T_p$ at infinity is $\delta_r n_0 + \frac{2j\delta_r}{24}$ for some $n_0 \geq 0$. We conclude that $K(z)$ is holomorphic at infinity. $\square$

Chapter 5

Appendix

5.1 Efficient Computation of q-expansions

In this section, we will describe some of the methods used to compute the q-expansions of the forms $f_{r,s}(z) := \eta(24z)^r E_s(24z)$. Throughout this section and the next, we will stick to the convention of $f_{r,s}(z) := \sum_{n=0}^{\infty} a(n)q^n \in S_{\lambda+1/2}(\Gamma_0(576), \chi_3)$, and $\mathcal{F}_{r,s}(z)$ will denote the image $S_r(f_{r,s}(z)) := \sum_{n=1}^{\infty} A(n)q^n \in S_{2\lambda}(\Gamma_0(288))$ under the shimura map S_r. By (2.1.1), we see that computing the Shimura image $\mathcal{F}_{r,s}(z)$ from the definition requires the ability to compute many coefficients of $f_{r,s}(z)$.

First, we outline methods used to compute powers $\eta(24z)^r$. We have

$$\eta(24z)^r = q^r \prod_{n=1}^{\infty} \left(1 - q^{24n}\right)^r = q^r + \cdots = \sum_{n=0}^{\infty} a(n)q^n. \tag{5.1.1}$$

This power series is supported on terms q^{24n+r}. Because of this sparseness, it is more memory efficient to compute the q-expansion

$$\prod_{n=1}^{\infty} (1 - q^n)^r = \sum_{n=0}^{\infty} c(n)q^n = 1 + \cdots . \tag{5.1.2}$$

This can be done effectively using the classical Pentagonal Number Theorem, which gives

$$\prod_{n=1}^{\infty} (1 - q^n) = 1 + \sum_{n=1}^{\infty} (-1)^n \left(q^{\frac{n(3n-1)}{2}} + q^{\frac{n(3n+1)}{2}}\right) = \sum_{n=0}^{\infty} b(n)q^n. \tag{5.1.3}$$

A useful identity commonly attributed to J.C.P. Miller (see (Zeilberger, 1995) and §4.7 of (Knuth, 1997)) allows us to compute the coefficients of (5.1.2) in terms of the coefficients of (5.1.3). We have $c(0) = b(0)^r = 1$, and for $k \geq 1$, we have

$$c(k) = \frac{1}{k} \sum_{j=1}^{k} \left((r+1)j - k\right) b(j)c(k-j). \tag{5.1.4}$$

This allows us to compute the coefficients $c(m)$, and we can use the relationship $a(m) = c\left(\frac{m-r}{24}\right)$ to get the coefficients $a(m)$ of (5.1.1).

Next, we outline the method used to compute $E_s(24z)$. Again, for reasons of sparseness, we will work with $E_s(z)$ and reindex as necessary. From the definition (1.2.1), we have

$$E_s(z) := 1 - \frac{2s}{B_s}\sum_{n=1}^{\infty} \sigma_{s-1}(n)q^n.$$

Instead of computing $\sigma_{s-1}(n)$ for each n, we use the fact that if $n = p_1^{e_1} \cdots p_r^{e_r}$, then

$$\sigma_k(n) = \prod_{i=1}^{r} \left(\frac{p_i^{e_i+1)k} - 1}{p_i^k - 1} \right). \tag{5.1.5}$$

So to compute the first k coefficients of $E_s(z)$, we can compute the primes up to k, and construct the coefficients using this multiplicative property.

5.2 Efficient Computation of Shimura Images

In this section, we will describe some methods to compute Shimura images. Our ultimate goal here is to devise a computational scheme allowing us to efficiently compute the Fourier coefficients of $\mathcal{F}_{r,s}(z)$, so that we may computationally verify the congruences from Chapter 3.

Our first task is to compute each Shimura image to the precision required to uniquely identify it as a modular form in $S_{2\lambda}(\Gamma_0(288))$. Using 8 1.5, we see that it is necessary to compute 96λ coefficients of $\mathcal{F}_{r,s}(z)$ to do this. Due to the Hecke-equivariance of the Shimura map, computing the ℓth coefficient of the Shimura image is equivalent to computing an eigenvalue of the half-integral weight form $f_{r,s}(z)$. Let λ_ℓ be the eigenvalue $f_{r,s} \mid T_{\ell^2} = \lambda_\ell f_{r,s}$. It follows that the definition of $A_{r,s}(n)$ given by (2.1.1) is equivalent to the following:

- $A_{r,s}(1) = 1$
- $A_{r,s}(\ell) = \lambda_\ell \quad$ for all primes $\ell \nmid 576$

- $A_{r,s}(\ell^k) = A_{r,s}(\ell)A_{r,s}(\ell^{k-1}) - \ell^{2\lambda-1}A_{r,s}(\ell^{k-2}) \quad$ for $\ell \nmid 576$ and $k \geq 2$

- $A_{r,s}(mn) = A_{r,s}(m)A_{r,s}(n)$ for m and n relatively prime

Using this, we can compute the first k coefficients of $\mathcal{F}_{r,s}$ by computing $A_{r,s}(\ell)$ for primes $\ell \leq k$, since the other coefficients can be reconstructed using these relationships. Since $f_{r,s}(z) = q^r + \cdots$ and r is square-free, we have from (1.3.5) that

$$\lambda_\ell = a(\ell^2 r) + \left(\frac{(-1)^\lambda 12r}{\ell}\right)\ell^{\lambda-1} \tag{5.2.1}$$

The difficulty here is in computing $a(\ell^2 r)$, and we do this via the methods of the previous section. In particular, we see that $A_{r,s}(\ell)$ depends on having $r\ell^2$ coefficients of $f_{r,s}(z)$ at our disposal. One can see that for large values of ℓ, this becomes impractical.

Next, we will outline a method to compute these images to near arbitrary precision in an efficient manner by finding the image as a linear combination of forms that are much simpler to compute. In the course of our computations, we observed that the images $\mathcal{F}_{r,s}(z)$ are actually χ_3-twists of forms in $S_{2\lambda}^{\text{new}}(\Gamma_0(6))$. A non-computational proof of this fact is given in (Yang, 2011). Using Proposition 1.2, we construct the following forms:

$$\begin{aligned}
f_1(z) &:= E_2(z) - 2E_2(2z) - E_2(3z) + 2E_2(6z)\\
f_2(z) &:= E_2(z) - 2E_2(2z) + E_2(3z) - 2E_2(6z)\\
f_3(z) &:= E_2(z) - E_2(2z) - 2E_2(3z) + E_2(6z),
\end{aligned}$$

all of which live in $M_2(\Gamma_0(6))$. Define

$$\mathcal{C} := \left\{f_1^{e_1}f_2^{e_2}f_3^{e_3} \otimes \chi_3 \mid e_1 + e_2 + e_3 = \lambda\right\}. \tag{5.2.2}$$

The forms in $\mathcal{C}$ all live in $M_{2\lambda}(\Gamma_0(6)) \otimes \chi_3 \subseteq M_{2\lambda}(\Gamma_0(144))$. We were able to find each shimura image $\mathcal{F}_{r,s}$ as a linear combination of elements in $\mathcal{C}$. Since the forms in $\mathcal{C}$ are relatively easy to compute to high precision, this allows us to compute each image $\mathcal{F}_{r,s}$ to the precision necessary to verify the congruences we're interested in.

5.3 Congruence Tables

This section lists information about congruences of the form

$$\mathcal{F}_{r,s}(z)E_{\ell-1}(z)^{m_1} \equiv \Theta^a(M(z))E_{\ell-1}(z)^{m_2} \otimes \chi_3 \pmod{\ell},$$

where $M(z)$ is a weight 2 newform associated to an elliptic curve, and $\mathcal{F}_{r,s}(z)$ is the Shimura image of the form $f_{r,s}(z)$ of weight $\lambda + 1/2$ defined earlier. Here, we have the relationship

$$2\lambda + m_1(\ell - 1) = 2 + a(\ell + 1) + m_2(\ell - 1),$$

where $m_1 m_2 = 0$ and $a \geq 0$.

The values of $(r, s), m_1, m_2,$ and a are as above. The label column gives the Cremona label of the elliptic curve associated to $M(z)$. This label uniquely identifies the curve, as well as gives the conductor. The twist column gives the Cremona label of the quadratic twist of the curve having minimal conductor. If there is more than one curve with minimal conductor, then the one with smallest label in the database is returned.

Table 5.1 Congruences for $p = 5$

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(1, 6)	0	1	1	150a1	150a1	3600	12	8640
(1, 6)	2	0	3	150b1	150a1	3600	20	14400
(1, 10)	0	3	1	150a1	150a1	3600	20	14400
(1, 10)	0	0	3	150b1	150a1	3600	20	14400
(1, 14)	0	5	1	150a1	150a1	3600	28	20160
(1, 14)	0	2	3	150b1	150a1	3600	28	20160
(5, 6)	0	2	1	150a1	150a1	3600	16	11520
(5, 6)	1	0	3	150b1	150a1	3600	20	14400
(5, 10)	0	4	1	150a1	150a1	3600	24	17280
(5, 10)	0	1	3	150b1	150a1	3600	24	17280
(5, 14)	0	6	1	150a1	150a1	3600	32	23040
(5, 14)	0	3	3	150b1	150a1	3600	32	23040
(7, 0)	0	1	0	30a1	30a1	720	6	864
(7, 4)	0	3	0	30a1	30a1	720	14	2016
(7, 6)	0	1	2	30a1	30a1	720	18	2592

Continued on next page

Table 5.1 – continued from the previous page

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(7, 6)	0	4	0	150c1	30a1	3600	18	12960
(7, 8)	0	5	0	30a1	30a1	720	22	3168
(7, 10)	0	3	2	30a1	30a1	720	26	3744
(7, 10)	0	6	0	150c1	30a1	3600	26	18720
(7, 14)	0	5	2	30a1	30a1	720	34	4896
(7, 14)	0	8	0	150c1	30a1	3600	34	24480
(11, 0)	0	2	0	30a1	30a1	720	10	1440
(11, 4)	0	4	0	30a1	30a1	720	18	2592
(11, 6)	0	2	2	30a1	30a1	720	22	3168
(11, 6)	0	5	0	150c1	30a1	3600	22	15840
(11, 8)	0	6	0	30a1	30a1	720	26	3744
(11, 10)	0	4	2	30a1	30a1	720	30	4320
(11, 10)	0	7	0	150c1	30a1	3600	30	21600
(11, 14)	0	6	2	30a1	30a1	720	38	5472
(11, 14)	0	9	0	150c1	30a1	3600	38	27360
(13, 0)	0	1	1	30a1	30a1	720	12	1728
(13, 0)	2	0	3	150c1	30a1	3600	20	14400
(13, 4)	0	3	1	30a1	30a1	720	20	2880
(13, 4)	0	0	3	150c1	30a1	3600	20	14400
(13, 6)	0	1	3	30a1	30a1	720	24	3456
(13, 6)	0	4	1	150c1	30a1	3600	24	17280
(13, 8)	0	5	1	30a1	30a1	720	28	4032
(13, 8)	0	2	3	150c1	30a1	3600	28	20160
(13, 10)	0	3	3	30a1	30a1	720	32	4608
(13, 10)	0	6	1	150c1	30a1	3600	32	23040
(13, 14)	0	5	3	30a1	30a1	720	40	5760
(13, 14)	0	8	1	150c1	30a1	3600	40	28800
(17, 0)	0	2	1	30a1	30a1	720	16	2304
(17, 0)	1	0	3	150c1	30a1	3600	20	14400
(17, 4)	0	4	1	30a1	30a1	720	24	3456
(17, 4)	0	1	3	150c1	30a1	3600	24	17280
(17, 6)	0	2	3	30a1	30a1	720	28	4032
(17, 6)	0	5	1	150c1	30a1	3600	28	20160
(17, 8)	0	6	1	30a1	30a1	720	32	4608
(17, 8)	0	3	3	150c1	30a1	3600	32	23040
(17, 10)	0	4	3	30a1	30a1	720	36	5184
(17, 10)	0	7	1	150c1	30a1	3600	36	25920
(17, 14)	0	6	3	30a1	30a1	720	44	6336

Continued on next page

Table 5.1 – continued from the previous page

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(17, 14)	0	9	1	150c1	30a1	3600	44	31680
(19, 0)	0	1	2	150a1	150a1	3600	18	12960
(19, 0)	0	4	0	150b1	150a1	3600	18	12960
(19, 4)	0	3	2	150a1	150a1	3600	26	18720
(19, 4)	0	6	0	150b1	150a1	3600	26	18720
(19, 6)	0	7	0	150a1	150a1	3600	30	21600
(19, 6)	0	4	2	150b1	150a1	3600	30	21600
(19, 8)	0	5	2	150a1	150a1	3600	34	24480
(19, 8)	0	8	0	150b1	150a1	3600	34	24480
(19, 10)	0	9	0	150a1	150a1	3600	38	27360
(19, 10)	0	6	2	150b1	150a1	3600	38	27360
(19, 14)	0	11	0	150a1	150a1	3600	46	33120
(19, 14)	0	8	2	150b1	150a1	3600	46	33120
(23, 0)	0	2	2	150a1	150a1	3600	22	15840
(23, 0)	0	5	0	150b1	150a1	3600	22	15840
(23, 4)	0	4	2	150a1	150a1	3600	30	21600
(23, 4)	0	7	0	150b1	150a1	3600	30	21600
(23, 6)	0	8	0	150a1	150a1	3600	34	24480
(23, 6)	0	5	2	150b1	150a1	3600	34	24480
(23, 8)	0	6	2	150a1	150a1	3600	38	27360
(23, 8)	0	9	0	150b1	150a1	3600	38	27360
(23, 10)	0	10	0	150a1	150a1	3600	42	30240
(23, 10)	0	7	2	150b1	150a1	3600	42	30240
(23, 14)	0	12	0	150a1	150a1	3600	50	36000
(23, 14)	0	9	2	150b1	150a1	3600	50	36000

Table 5.2 Congruences for $p = 7$

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(1, 4)	0	1	0	42a1	42a1	1008	8	1536
(1, 8)	3	0	4	294a1	294b1	7056	34	45696
(1, 8)	0	1	1	294b1	294b1	7056	16	21504
(1, 10)	0	3	0	42a1	42a1	1008	20	3840
(1, 14)	1	0	4	294a1	294b1	7056	34	45696
(1, 14)	0	3	1	294b1	294b1	7056	28	37632
(5, 4)	1	0	2	294d1	294d1	7056	18	24192
(5, 4)	5	0	5	294e1	294d1	7056	42	56448
(5, 4)	5	0	5	294f1	294g1	7056	42	56448
(5, 4)	1	0	2	294g1	294g1	7056	18	24192

Continued on next page

Table 5.2 – continued from the previous page

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(5, 8)	0	3	0	294d1	294d1	7056	20	26880
(5, 8)	1	0	3	294e1	294d1	7056	26	34944
(5, 8)	1	0	3	294f1	294g1	7056	26	34944
(5, 8)	0	3	0	294g1	294g1	7056	20	26880
(5, 10)	0	1	2	294d1	294d1	7056	24	32256
(5, 10)	3	0	5	294e1	294d1	7056	42	56448
(5, 10)	3	0	5	294f1	294g1	7056	42	56448
(5, 10)	0	1	2	294g1	294g1	7056	24	32256
(5, 14)	0	5	0	294d1	294d1	7056	32	43008
(5, 14)	0	1	3	294e1	294d1	7056	32	43008
(5, 14)	0	1	3	294f1	294g1	7056	32	43008
(5, 14)	0	5	0	294g1	294g1	7056	32	43008
(7, 4)	0	2	0	42a1	42a1	1008	14	2688
(7, 8)	2	0	4	294a1	294b1	7056	34	45696
(7, 8)	0	2	1	294b1	294b1	7056	22	29568
(7, 10)	0	4	0	42a1	42a1	1008	26	4992
(7, 14)	0	0	4	294a1	294b1	7056	34	45696
(7, 14)	0	4	1	294b1	294b1	7056	34	45696
(11, 4)	0	0	2	294d1	294d1	7056	18	24192
(11, 4)	4	0	5	294e1	294d1	7056	42	56448
(11, 4)	4	0	5	294f1	294g1	7056	42	56448
(11, 4)	0	0	2	294g1	294g1	7056	18	24192
(11, 8)	0	4	0	294d1	294d1	7056	26	34944
(11, 8)	0	0	3	294e1	294d1	7056	26	34944
(11, 8)	0	0	3	294f1	294g1	7056	26	34944
(11, 8)	0	4	0	294g1	294g1	7056	26	34944
(11, 10)	0	2	2	294d1	294d1	7056	30	40320
(11, 10)	2	0	5	294e1	294d1	7056	42	56448
(11, 10)	2	0	5	294f1	294g1	7056	42	56448
(11, 10)	0	2	2	294g1	294g1	7056	30	40320
(11, 14)	0	6	0	294d1	294d1	7056	38	51072
(11, 14)	0	2	3	294e1	294d1	7056	38	51072
(11, 14)	0	2	3	294f1	294g1	7056	38	51072
(11, 14)	0	6	0	294g1	294g1	7056	38	51072
(13, 0)	5	0	5	294d1	294d1	7056	42	56448
(13, 0)	1	0	2	294e1	294d1	7056	18	24192
(13, 0)	1	0	2	294f1	294g1	7056	18	24192
(13, 0)	5	0	5	294g1	294g1	7056	42	56448

Continued on next page

Table 5.2 – continued from the previous page

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(13, 4)	1	0	3	294d1	294d1	7056	26	34944
(13, 4)	0	3	0	294e1	294d1	7056	20	26880
(13, 4)	0	3	0	294f1	294g1	7056	20	26880
(13, 4)	1	0	3	294g1	294g1	7056	26	34944
(13, 6)	3	0	5	294d1	294d1	7056	42	56448
(13, 6)	0	1	2	294e1	294d1	7056	24	32256
(13, 6)	0	1	2	294f1	294g1	7056	24	32256
(13, 6)	3	0	5	294g1	294g1	7056	42	56448
(13, 8)	0	3	1	294d1	294d1	7056	28	37632
(13, 8)	1	0	4	294e1	294d1	7056	34	45696
(13, 8)	1	0	4	294f1	294g1	7056	34	45696
(13, 8)	0	3	1	294g1	294g1	7056	28	37632
(13, 10)	0	1	3	294d1	294d1	7056	32	43008
(13, 10)	0	5	0	294e1	294d1	7056	32	43008
(13, 10)	0	5	0	294f1	294g1	7056	32	43008
(13, 10)	0	1	3	294g1	294g1	7056	32	43008
(13, 14)	0	5	1	294d1	294d1	7056	40	53760
(13, 14)	0	1	4	294e1	294d1	7056	40	53760
(13, 14)	0	1	4	294f1	294g1	7056	40	53760
(13, 14)	0	5	1	294g1	294g1	7056	40	53760
(17, 0)	0	1	1	42a1	42a1	1008	16	3072
(17, 0)	3	0	4	294c1	42a1	7056	34	45696
(17, 4)	3	0	5	294a1	294b1	7056	42	56448
(17, 4)	0	1	2	294b1	294b1	7056	24	32256
(17, 6)	0	3	1	42a1	42a1	1008	28	5376
(17, 6)	1	0	4	294c1	42a1	7056	34	45696
(17, 8)	0	1	3	42a1	42a1	1008	32	6144
(17, 8)	0	5	0	294c1	42a1	7056	32	43008
(17, 10)	1	0	5	294a1	294b1	7056	42	56448
(17, 10)	0	3	2	294b1	294b1	7056	36	48384
(17, 14)	0	3	3	42a1	42a1	1008	44	8448
(17, 14)	0	7	0	294c1	42a1	7056	44	59136
(19, 0)	4	0	5	294d1	294d1	7056	42	56448
(19, 0)	0	0	2	294e1	294d1	7056	18	24192
(19, 0)	0	0	2	294f1	294g1	7056	18	24192
(19, 0)	4	0	5	294g1	294g1	7056	42	56448
(19, 4)	0	0	3	294d1	294d1	7056	26	34944
(19, 4)	0	4	0	294e1	294d1	7056	26	34944

Continued on next page

Table 5.2 – continued from the previous page

(r,s)	m_1	m_2	a	label	twist	level	weight	bound
(19, 4)	0	4	0	294f1	294g1	7056	26	34944
(19, 4)	0	0	3	294g1	294g1	7056	26	34944
(19, 6)	2	0	5	294d1	294d1	7056	42	56448
(19, 6)	0	2	2	294e1	294d1	7056	30	40320
(19, 6)	0	2	2	294f1	294g1	7056	30	40320
(19, 6)	2	0	5	294g1	294g1	7056	42	56448
(19, 8)	0	4	1	294d1	294d1	7056	34	45696
(19, 8)	0	0	4	294e1	294d1	7056	34	45696
(19, 8)	0	0	4	294f1	294g1	7056	34	45696
(19, 8)	0	4	1	294g1	294g1	7056	34	45696
(19, 10)	0	2	3	294d1	294d1	7056	38	51072
(19, 10)	0	6	0	294e1	294d1	7056	38	51072
(19, 10)	0	6	0	294f1	294g1	7056	38	51072
(19, 10)	0	2	3	294g1	294g1	7056	38	51072
(19, 14)	0	6	1	294d1	294d1	7056	46	61824
(19, 14)	0	2	4	294e1	294d1	7056	46	61824
(19, 14)	0	2	4	294f1	294g1	7056	46	61824
(19, 14)	0	6	1	294g1	294g1	7056	46	61824
(23, 0)	0	2	1	42a1	42a1	1008	22	4224
(23, 0)	2	0	4	294c1	42a1	7056	34	45696
(23, 4)	2	0	5	294a1	294b1	7056	42	56448
(23, 4)	0	2	2	294b1	294b1	7056	30	40320
(23, 6)	0	4	1	42a1	42a1	1008	34	6528
(23, 6)	0	0	4	294c1	42a1	7056	34	45696
(23, 8)	0	2	3	42a1	42a1	1008	38	7296
(23, 8)	0	6	0	294c1	42a1	7056	38	51072
(23, 10)	0	0	5	294a1	294b1	7056	42	56448
(23, 10)	0	4	2	294b1	294b1	7056	42	56448
(23, 14)	0	4	3	42a1	42a1	1008	50	9600
(23, 14)	0	8	0	294c1	42a1	7056	50	67200

Table 5.3 Congruences for $p = 11$

(r,s)	m_1	m_2	a	label	twist	level	weight	bound
(1, 6)	0	1	0	66a1	66a1	1584	12	3456
(1, 8)	1	0	2	726d1	726d1	17424	26	82368
(1, 8)	7	0	7	726i1	726d1	17424	86	272448
(5, 4)	0	1	0	66b1	66b1	1584	12	3456
(5, 8)	9	0	9	726b1	726g1	17424	110	348480

Continued on next page

Table 5.3 – continued from the previous page

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(5, 8)	3	0	4	726g1	726g1	17424	50	158400
(5, 14)	0	3	0	66b1	66b1	1584	32	9216
(7, 6)	2	0	3	726a1	726a1	17424	38	120384
(7, 6)	8	0	8	726f1	726a1	17424	98	310464
(7, 8)	0	2	0	66b1	66b1	1584	22	6336
(11, 6)	0	2	0	66a1	66a1	1584	22	6336
(11, 8)	0	0	2	726d1	726d1	17424	26	82368
(11, 8)	6	0	7	726i1	726d1	17424	86	272448
(13, 0)	0	1	0	66c1	66c1	1584	12	3456
(13, 4)	9	0	9	726d1	726d1	17424	110	348480
(13, 4)	3	0	4	726i1	726d1	17424	50	158400
(13, 6)	0	1	1	66a1	66a1	1584	24	6912
(13, 6)	5	0	6	726h1	66a1	17424	74	234432
(13, 8)	1	0	3	726d1	726d1	17424	38	120384
(13, 8)	7	0	8	726i1	726d1	17424	98	310464
(13, 10)	0	3	0	66c1	66c1	1584	32	9216
(13, 14)	7	0	9	726d1	726d1	17424	110	348480
(13, 14)	1	0	4	726i1	726d1	17424	50	158400
(17, 0)	1	0	2	726b1	726g1	17424	26	82368
(17, 0)	7	0	7	726g1	726g1	17424	86	272448
(17, 4)	0	1	1	66b1	66b1	1584	24	6912
(17, 4)	5	0	6	726c1	66b1	17424	74	234432
(17, 6)	7	0	8	726a1	726a1	17424	98	310464
(17, 6)	1	0	3	726f1	726a1	17424	38	120384
(17, 8)	0	3	0	726b1	726g1	17424	32	101376
(17, 8)	3	0	5	726g1	726g1	17424	62	196416
(17, 10)	0	1	2	726b1	726g1	17424	36	114048
(17, 10)	5	0	7	726g1	726g1	17424	86	272448
(17, 14)	0	3	1	66b1	66b1	1584	44	12672
(17, 14)	3	0	6	726c1	66b1	17424	74	234432
(19, 0)	8	0	8	726a1	726a1	17424	98	310464
(19, 0)	2	0	3	726f1	726a1	17424	38	120384
(19, 4)	0	0	2	726b1	726g1	17424	26	82368
(19, 4)	6	0	7	726g1	726g1	17424	86	272448
(19, 6)	2	0	4	726a1	726a1	17424	50	158400
(19, 6)	8	0	9	726f1	726a1	17424	110	348480
(19, 8)	0	2	1	66b1	66b1	1584	34	9792
(19, 8)	4	0	6	726c1	66b1	17424	74	234432

Continued on next page

Table 5.3 – continued from the previous page

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(19, 10)	6	0	8	726a1	726a1	17424	98	310464
(19, 10)	0	0	3	726f1	726a1	17424	38	120384
(19, 14)	0	2	2	726b1	726g1	17424	46	145728
(19, 14)	4	0	7	726g1	726g1	17424	86	272448
(23, 0)	0	2	0	66c1	66c1	1584	22	6336
(23, 4)	8	0	9	726d1	726d1	17424	110	348480
(23, 4)	2	0	4	726i1	726d1	17424	50	158400
(23, 6)	0	2	1	66a1	66a1	1584	34	9792
(23, 6)	4	0	6	726h1	66a1	17424	74	234432
(23, 8)	0	0	3	726d1	726d1	17424	38	120384
(23, 8)	6	0	8	726i1	726d1	17424	98	310464
(23, 10)	0	4	0	66c1	66c1	1584	42	12096
(23, 14)	6	0	9	726d1	726d1	17424	110	348480
(23, 14)	0	0	4	726i1	726d1	17424	50	158400

Table 5.4 Congruences for $p = 13$

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(1, 10)	2	0	3	1014c1	1014c1	24336	44	192192
(1, 10)	9	0	9	1014g1	1014c1	24336	128	559104
(5, 8)	9	0	9	1014a1	1014a1	24336	128	559104
(5, 8)	2	0	3	1014d1	1014a1	24336	44	192192
(7, 4)	0	1	0	78a1	78a1	1872	14	4704
(7, 8)	3	0	4	1014a1	1014a1	24336	58	253344
(7, 8)	10	0	10	1014d1	1014a1	24336	142	620256
(7, 14)	9	0	10	1014a1	1014a1	24336	142	620256
(7, 14)	2	0	4	1014d1	1014a1	24336	58	253344
(11, 4)	8	0	8	1014b1	1014f1	24336	114	497952
(11, 4)	1	0	2	1014f1	1014f1	24336	30	131040
(11, 6)	10	0	10	1014c1	1014c1	24336	142	620256
(11, 6)	3	0	4	1014g1	1014c1	24336	58	253344
(11, 14)	4	0	6	1014b1	1014f1	24336	86	375648
(11, 14)	0	3	0	1014f1	1014f1	24336	38	165984
(13, 10)	1	0	3	1014c1	1014c1	24336	44	192192
(13, 10)	8	0	9	1014g1	1014c1	24336	128	559104
(17, 8)	8	0	9	1014a1	1014a1	24336	128	559104
(17, 8)	1	0	3	1014d1	1014a1	24336	44	192192
(19, 4)	0	2	0	78a1	78a1	1872	26	8736
(19, 8)	2	0	4	1014a1	1014a1	24336	58	253344

Continued on next page

Table 5.4 – continued from the previous page

(r,s)	m_1	m_2	a	label	twist	level	weight	bound
(19, 8)	9	0	10	1014d1	1014a1	24336	142	620256
(19, 14)	8	0	10	1014a1	1014a1	24336	142	620256
(19, 14)	1	0	4	1014d1	1014a1	24336	58	253344
(23, 4)	7	0	8	1014b1	1014f1	24336	114	497952
(23, 4)	0	0	2	1014f1	1014f1	24336	30	131040
(23, 6)	9	0	10	1014c1	1014c1	24336	142	620256
(23, 6)	2	0	4	1014g1	1014c1	24336	58	253344
(23, 14)	3	0	6	1014b1	1014f1	24336	86	375648
(23, 14)	0	4	0	1014f1	1014f1	24336	50	218400

Table 5.5 Congruences for $p = 17$

(r,s)	m_1	m_2	a	label	twist	level	weight	bound
(5, 10)	11	0	11	1734d1	1734e1	41616	200	1468800
(5, 10)	2	0	3	1734e1	1734e1	41616	56	411264
(7, 6)	0	1	0	102b1	102b1	2448	18	7776
(11, 4)	0	1	0	102a1	102a1	2448	18	7776
(11, 8)	3	0	4	1734c1	1734c1	41616	74	543456
(11, 8)	12	0	12	1734f1	1734c1	41616	218	1600992
(11, 10)	5	0	6	1734d1	1734e1	41616	110	807840
(11, 10)	14	0	14	1734e1	1734e1	41616	254	1865376
(13, 6)	2	0	3	1734a1	1734a1	41616	56	411264
(13, 6)	11	0	11	1734h1	1734a1	41616	200	1468800
(13, 14)	10	0	11	1734d1	1734e1	41616	200	1468800
(13, 14)	1	0	3	1734e1	1734e1	41616	56	411264
(19, 0)	0	1	0	102c1	102c1	2448	18	7776
(19, 6)	14	0	14	1734a1	1734a1	41616	254	1865376
(19, 6)	5	0	6	1734h1	1734a1	41616	110	807840
(19, 8)	0	2	0	102a1	102a1	2448	34	14688
(19, 14)	4	0	6	1734d1	1734e1	41616	110	807840
(19, 14)	13	0	14	1734e1	1734e1	41616	254	1865376
(23, 6)	0	2	0	102b1	102b1	2448	34	14688

Table 5.6 Congruences for $p = 19$

(r,s)	m_1	m_2	a	label	twist	level	weight	bound
(1, 10)	0	1	0	114b1	114b1	2736	20	9600
(5, 8)	0	1	0	114a1	114a1	2736	20	9600
(7, 10)	2	0	3	2166e1	2166e1	51984	62	565440

Continued on next page

Table 5.6 – continued from the previous page

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(7, 10)	12	0	12	2166g1	2166e1	51984	242	2207040
(11, 10)	14	0	14	2166c1	2166f1	51984	282	2571840
(11, 10)	4	0	5	2166f1	2166f1	51984	102	930240
(13, 4)	0	1	0	114c1	114c1	2736	20	9600
(19, 10)	0	2	0	114b1	114b1	2736	38	18240
(23, 8)	0	2	0	114a1	114a1	2736	38	18240

Table 5.7 Congruences for $p = 23$

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(5, 10)	0	1	0	138a1	138a1	3312	24	13824
(5, 14)	3	0	4	3174b1	3174b1	76176	98	1298304
(5, 14)	15	0	15	3174c1	3174b1	76176	362	4795776
(13, 6)	0	1	0	138b1	138b1	3312	24	13824
(17, 4)	0	1	0	138c1	138c1	3312	24	13824
(17, 8)	3	0	4	3174e1	3174e1	76176	98	1298304
(17, 8)	15	0	15	3174f1	3174e1	76176	362	4795776
(19, 14)	0	2	0	138a1	138a1	3312	46	26496

Table 5.8 Congruences for $p = 29$

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(11, 10)	0	1	0	174c1	174c1	4176	30	21600
(13, 14)	19	0	19	5046g1	5046j1	121104	572	11943360
(13, 14)	4	0	5	5046j1	5046j1	121104	152	3173760
(17, 14)	6	0	7	5046f1	5046f1	121104	212	4426560
(17, 14)	21	0	21	5046k1	5046f1	121104	632	13196160
(19, 6)	0	1	0	174b1	174b1	4176	30	21600
(23, 4)	0	1	0	174e1	174e1	4176	30	21600
(23, 14)	9	0	10	5046c1	5046m1	121104	302	6305760
(23, 14)	24	0	24	5046m1	5046m1	121104	722	15075360

Table 5.9 Congruences for $p = 31$

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(5, 14)	0	1	0	186c1	186c1	4464	32	24576
(13, 10)	0	1	0	186a1	186a1	4464	32	24576
(17, 8)	0	1	0	186b1	186b1	4464	32	24576
(23, 10)	4	0	5	5766h1	5766h1	138384	162	3856896
(23, 10)	20	0	20	5766i1	5766h1	138384	642	15284736

Table 5.10 Congruences for $p = 37$

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(11, 14)	0	1	0	222a1	222a1	5328	38	34656
(19, 10)	0	1	0	222b1	222b1	5328	38	34656
(23, 8)	0	1	0	222d1	222d1	5328	38	34656

Table 5.11 Congruences for $p = 41$

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(23, 10)	0	1	0	246a1	246a1	5904	42	42336

Table 5.12 Congruences for $p = 43$

(r, s)	m_1	m_2	a	label	twist	level	weight	bound
(17, 14)	0	1	0	258c1	258c1	6192	44	46464

Bibliography

Atkin, A. O. L. and Wen Ch'ing Winnie Li. 1978. *Twists of newforms and pseudo-eigenvalues of W-operators*, Invent. Math. **48**, no. 3, 221–243.

Cipra, Barry A. 1989. *On the Shimura lift, après Selberg*, J. Number Theory **32**, no. 1, 58–64.

Cohen, Henri. 1993. *A course in computational algebraic number theory*, Graduate Texts in Mathematics, vol. 138, Springer-Verlag, Berlin.

Diamond, Fred and John Im. 1995. *Modular forms and modular curves*, Seminar on Fermat's Last Theorem (Toronto, ON, 1993), CMS Conf. Proc., vol. 17, Amer. Math. Soc., Providence, RI, pp. 39–133.

Diamond, Fred and Jerry Shurman. 2005. *A first course in modular forms*, Graduate Texts in Mathematics, vol. 228, Springer-Verlag, New York.

Dummit, D., H. Kisilevsky, and J. McKay. 1985. *Multiplicative products of η-functions*, Finite groups—coming of age (Montreal, Que., 1982), Contemp. Math., vol. 45, Amer. Math. Soc., Providence, RI, pp. 89–98.

Garvan, F. G. 2010. *Congruences for Andrews' smallest parts partition function and new congruences for Dyson's rank*, Int. J. Number Theory **6**, no. 2, 281–309.

Ireland, Kenneth and Michael Rosen. 1990. *A classical introduction to modern number theory*, 2nd ed., Graduate Texts in Mathematics, vol. 84, Springer-Verlag, New York.

Iwaniec, Henryk. 1997. *Topics in classical automorphic forms*, Graduate Studies in Mathematics, vol. 17, American Mathematical Society, Providence, RI.

Knapp, Anthony W. 1992. *Elliptic curves*, Mathematical Notes, vol. 40, Princeton University Press, Princeton, NJ.

Knuth, Donald E. 1997. *The art of computer programming. Vol. 2: Seminumerical algorithms*, 3rd ed., Addison-Wesley Publishing Co., Reading, Mass.

Koblitz, Neal. 1993. *Introduction to elliptic curves and modular forms*, 2nd ed., Graduate Texts in Mathematics, vol. 97, Springer-Verlag, New York.

Martin, Yves. 1996. *Multiplicative η-quotients*, Trans. Amer. Math. Soc. **348**, no. 12, 4825–4856.

Ono, Ken. 2004. *The web of modularity: arithmetic of the coefficients of modular forms and q-series*, CBMS Regional Conference Series in Mathematics, vol. 102, Published for the Conference Board of the Mathematical Sciences, Washington, DC.

Shimura, Goro. 1973. *On modular forms of half integral weight*, Ann. of Math. (2) **97**, 440–481.

Silverman, Joseph H. 2009. *The arithmetic of elliptic curves*, 2nd ed., Graduate Texts in Mathematics, vol. 106, Springer.

Swinnerton-Dyer, H. P. F. 1973. *On l-adic representations and congruences for coefficients of modular forms*, Modular functions of one variable, III (Proc. Internat. Summer School, Univ. Antwerp, 1972), Springer, Berlin, pp. 1–55. Lecture Notes in Math., Vol. 350.

Yang, Yifan. 2011. *Modular forms of half-integral weights on SL(2, Z)*, available at `http://arxiv.org/abs/1110.1810`.

Zeilberger, Doron. 1995. *The J. C. P. Miller recurrence for exponentiating a polynomial, and its q-analog*, J. Differ. Equations Appl. **1**, no. 1, 57–60.