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The Use of Applets For Developing Understanding In Mathematics: A Case Study Using Maplets For Calculus With Continuity Concepts.

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THE USE OF APPLETS FOR DEVELOPING UNDERSTANDING IN MATHEMATICS: A CASE STUDY USING MAPLETS FOR CALCULUS WITH CONTINUITY CONCEPTS

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Dedication

Andrew and Jonathan Patenaude, my sons, are the inspiration behind, and constant reminders of why I sought the Doctor of Philosophy degree.

This work is dedicated to my wife, Laurie Ann Patenaude, for your commitment to and support of this goal.

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Abstract

The Common Core State Standards for Mathematics (CCSSM) are founded on a long history of mathematics education research emphasizing the importance of teaching mathematics for understanding. The CCSSM along with the National Council of Teachers of Mathematics (NCTM) recommend the use of technology in the teaching of mathematics. New mobile technologies in society facilitate use in mathematics classrooms, and these technologies rely on software applications called *applets.* Certain applets have been developed for use in teaching mathematics.

This study investigated the questions: Is it possible to determine the characteristics of applets that lead students toward greater understanding of mathematical concepts? And, can we determine specific actions and strategies learners develop while using applets that increase their understanding?

Using a case study methodology, continuous motion, screen capture and audio recordings of seven high-school AP Calculus students were made while each used five Maplets for Calculus applets developed for continuity concepts. Audio and screen capture recordings were transcribed and analyzed to determine increases in understanding of continuity concepts using a rubric based on Tall's *Three Worlds* model of mathematics understanding. Using Drijvers and Trouché's *Instrumental Approach* theory this evidence was also analyzed to determine the features of the Maplets and strategies used by the students that contributed to the increases in understanding.

The findings relevant to teachers of mathematics included: evidence about the features of and strategies used by the students with the Maplets that developed students' embodied and symbolic understanding of left and right continuity; evidence for how the proceptual-symbolic understanding of the definition of continuity is developed; evidence of students using the concepts of left and right continuity to develop a formal 'rule' for determining the overall continuity of a function; evidence of formal thinking in the embodied world for epsilon-delta continuity; and evidence that supports the contributions of Maplets in developing procedural understanding.

A finding of relevance to applet developers included recommendations based on evidence for the sequencing of Maplets along with features and learner strategies that contribute to understanding of continuity in the symbolic world.

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I. Introduction to the Problem

Introduction

During the early 20th Century, mathematics education in the United States focused on repetition and the increase in proficiency and accuracy of arithmetic procedures on numbers (National Research Council [NRC], 2001). However, research by Brownell (1935) questioned the reliance on rote memorization and repetition. In reviewing the performance of students on arithmetic tasks, he found that those instructed with methods that focused on the conceptual understanding of addition and subtraction, performed as well as students who had been exposed to drill and repetition of addition facts (p. 9). "In the 1950s and 1960s, the new math movement defined successful mathematics learning primarily in terms of understanding the structure of mathematics together with its unifying ideas, and not just as computational skill. This emphasis was followed by a 'back to basics' movement that proposed returning to the view that success in mathematics meant being able to compute accurately and quickly" (NRC, 2001, p. 115).

In its *Agenda for Action: Recommendations for School Mathematics of the 1980's,* the National Council of Teachers of Mathematics (NCTM) recommended that school mathematics curriculum should be problem solving based; part of their reasoning for this emphasis being, "true problem-solving power requires a wide repertoire of knowledge, not only of particular skills and concepts but also of the relationships among them and the fundamental principle that unify them" (NCTM, 1980). The second recommendation in the *Agenda for Action, "*The Concept of Basic Skills in Mathematics

Must Encompass More Than Computational Facility," includes this statement regarding basic skills:

> The higher-order mental processes of logical reasoning, information processing, and decision making should be considered basic to the application of mathematics. Mathematics curricula and teachers should set as objectives the development of logical processes, concepts, and language, including: the identification of likenesses and differences leading to classification; understanding, making, and applying definitions; the development of a feeling for informal proof including counterexamples and generalizations (NCTM, 1980).

This emphasis on mathematical understanding included in the *Agenda for Action* continued in subsequent reports released by the NCTM, other organizations, and researchers. An outcome of these reports was the release of the *Curriculum and Evaluation Standards for School Mathematics* by the NCTM in 1989. Often referred to as the NCTM Standards, this report included thirteen curriculum standards that addressed both content and emphasis in teaching mathematics. A common theme throughout the standards is summarized by the statement, "the study of mathematics should emphasize reasoning so that students can believe that mathematics makes sense" (NCTM, 1989, p. 29).

In 2000, the NCTM published the *Principles and Standards for School Mathematics* (PSSM). The Learning Principle includes the statement, "unfortunately, learning mathematics without understanding has long been a common outcome of school mathematics instruction" (p. 20). Cangelosi (2003) further highlighted this absence of

teaching for understanding in describing many math classroom lessons as ones where students are told about facts or steps in a procedure, guided through some practice problems, then given exercises or problems to complete on their own (p. v).

According to the PSSM Learning Principle, the practice of teaching mathematics by focusing on definitions and procedures opposes research findings:

> In recent decades, psychological and educational research on learning complex subjects, such as mathematics has solidly established the important role of conceptual understanding in the knowledge and activity of persons who are proficient….One of the most robust findings of research is that conceptual understanding is an important component of proficiency (NCTM, 2000, p. 20).

In the late 2000's, the Common Core State Standards (CCSS) were developed by the National Governors Association Center for Best Practices (NGA) and the Council of Chief State School Officers (CCSSO) in order to provide a consistent framework for primary and secondary school curricula throughout the United States. The standards for English language arts and mathematics were presented in 2010.

The CCSS Mathematics (CCSSM) focuses on students' understanding of mathematics structure and concepts. The educational reform group, Achieve, commenting on the mathematics standards, stated: "the high school standards set a rigorous definition of college and career readiness, not by piling topic upon topic, but by demanding that students develop a depth of understanding" (CCSSO & NGA, 2010, p. 2). The CCSSM section on connecting mathematical practice to the standards for mathematical content states:

> Students who lack understanding of a topic may rely on procedures too heavily. Without a flexible base from which to work, they may be less likely to consider analogous problems coherently, justify conclusions, apply the mathematics to practical situations, use technology mindfully to work with the mathematics, explain the mathematics accurately to other students, step back for an overview, or deviate from known procedure to find a shortcut. In short, a lack of understanding prevents a student from engaging in the mathematical practices (p. 8).

The CCSSM being grounded in "evidence and research" (CCSSO & NGA, 2010, p. 1) in formulation, emphasize the need for building the conceptual knowledge base of students. Given this renewed emphasis on understanding mathematical concepts, through the CCSSM, how can teachers move toward developing greater mathematical understanding with their students?

One of the tools that can be used to help foster greater understanding of mathematical concepts, endorsed by both the CCSSM and the PSSM, is technology. "Students can learn more mathematics, more deeply with the appropriate use of technology," states the PSSM Technology Principle. "Students who have trouble with basic procedures can develop and demonstrate other mathematical understandings, which in turn can help them learn the procedures" (NCTM, 2000, p. 27). The CCSSM Standard for Mathematical Practice #5: Use Appropriate Tools Strategically, states that, "mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include…a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software" (CCSSO

& NGA, 2010, p. 7). Throughout the CCSSM standards for mathematical content, uses of technology are encouraged in the understanding of mathematical concepts. One example, from the high school algebra, "reasoning with equations and inequalities" domain, under the "represent and solve equations and inequalities graphically" cluster suggests using technology to graph functions or make a table of values to find and explain why the x-coordinates of the points where the graph of $y = f(x)$ and $y = g(x)$ intersect are solutions to the equation $f(x) = g(x)$ (p. 66).

One result of the proliferation of mobile technology during the early 2000's is the increased use of *applets*. Applets are software applications that are executed in the context of another program, usually a web browser or other application (Trigo, Oguin, & Matai, 2010). Of the many applets developed, Maplets for Calculus (M4C) are a series of applets (over 140 as of this writing) providing users with typical examples and exercises on a variety of topics covered in precalculus and the traditional three-semester calculus courses. The M4C applets allow for computer generated or user entered problems. Maplets provide users with an interactive graphic interface that provide immediate feedback, hints, and step by step checking of solutions/entries. These applets are designed using Maple software (the fusion of Maple and applets accounts for the name *Maplets*) and have been developed to increase student technical skill and understanding (Meade & Yasskin, March 2008 and December 2008).

The CCSSM includes an end note on transitions, one of which is the transition from high school to post-secondary/college education (CCSSO & NGA, 2010, p. 84). Precalculus and calculus are typical courses offered to high school students planning on continuing mathematics studies in college. While a study considering the development

for understanding of all concepts within these courses would be worthwhile, it is beyond the scope of any single study. Continuity of functions is an important topic in calculus, yet is often difficult to learn and challenging to teach (Tall & Vinner, 1981; Robert, 1982; Núñez, 1993). In preparation of this manuscript, a search for studies considering continuity yielded a limited number of results when compared to the number of studies regarding topics such as functions and limits. It appears that investigations into the understanding of the concepts of continuity are underrepresented. M4C offer five different Maplets that address topics included in the study of continuity, including addressing continuity from a graphic, algebraic, and numeric perspective. For these reasons, it is the intent of this study to investigate the use of Maplets for Calculus for developing understanding of the concept of continuity.

Problem Statement

The NCTM *Principles and Standards for School Mathematics* (PSSM) state that, "students must learn mathematics with understanding" (NCTM, 2000, p. 20). Building on this and released in 2010, the *Common Core State Standards for Mathematics* (CCSSM) endeavor to stress conceptual understanding and the organization/structure of mathematical ideas.

Research supports the use of technology as a way for students to focus on understanding the underlying structure and concepts of mathematics (Borwein & Bailey, 2003; Kaput, 1992; NCTM, 2000). Further, the NCTM PSSM asserts technology enhances learning, supports effective teaching, and influences what mathematics is taught (2000, p. 24). The CCSSM also include Standards for Mathematical Practice that suggests various technologies be available to all students in stating:

> Mathematically proficient students consider the available tools when solving a mathematical problem. These tools might include pencil and paper, concrete models, a ruler, a protractor, a calculator, a spreadsheet, a computer algebra system, a statistical package, or dynamic geometry software (CCSSO & NGA, 2010, p. 7).

More and more, these technologies include applets that are available on computers, tablets, and smart phones. Maplets for Calculus are specific applets that have been developed to assist students in the learning of calculus, and five Maplets have been developed for addressing the conceptual understanding of continuous functions.

Is it possible to determine the characteristics of applets that lead students toward greater understanding of mathematical concepts? Can we determine specific actions and strategies learners develop while using applets that increase their understanding? In particular, which features of Maplets for Calculus lead students toward greater understanding of continuity of functions? Can we determine specific actions and strategies students develop while using Maplets that increase their understanding?

Conceptual Framework

In this study, the following approaches and theories provided the framework for investigating the problem stated and its objectives: *instrumental approach*, *think aloud methodology* and David Tall's *Three Worlds of mathematics.*

The *instrumental approach* as developed by Drijvers and Trouché (2008) is grounded, in part, by the work of Vygotsky (1930/1985) and Rabardel (2002). It concerns the use of *artifacts,* a material or abstract 'tool' that can be used to sustain a certain activity or problem situation, and the development of *schemes* by a user for

employing the artifact. During this process, the artifact develops into an *instrument* whose use becomes second nature to the user (Drijvers & Trouché, 2008, p. 368). Though the ideas for the instrumental approach were developed before the proliferation of technology, researchers of mathematics education have found this framework useful for investigating the teaching and learning of mathematics using technology. Though discussed in a study investigating the use of computer algebra systems (CAS) to learn algebraic concepts, Drijvers and Trouché justified their use of the instrumental approach theory in writing:

> It allows for an analysis of the learning process in technological environments of increasing complexity, and takes into account the non-trivial character of using computerized environments. Furthermore, it stresses the subtle relationship between machine technique and mathematical insight, and provides a conceptual framework for investigating the development of schemes, in which both aspects are included. This is helpful for designing student activities, for observing interaction between students and the computer algebra environment, for interpreting it and for understanding what works well and what does not (2008, p. 375).

As the primary question to be answered by this research is exactly the "what works well and what does not" for students using applets to learn about concepts, the instrumental approach provides the researcher with a means for examining the impact of the applets.

Drijvers, Doorman, Boon, Van Gisbergen, and Gravemeijer (2007) state that instrumentation theory may be used by researchers in priori hypothetical investigations to formulate hypotheses and focus research direction, or a posteriori as a guide for data

analysis and forming conclusions (p. 118). Qualitative research approaches involve systematic investigation of human behavior, social phenomena, and interaction; they rely on verbal and visual communication to answer questions. Case study research can be indicated when a researcher sets out to investigate particular people, programs, curriculum, or techniques; and can provide rich detail and insight into the cases being studied (Lichtman, 2013).

One method used to collect and analyze data from subjects while solving problems is the *think aloud method*. According to van Someren, Barnard, and Sandberg (1994):

> The think aloud method consists of asking people to think aloud while solving a problem and analyzing the resulting verbal protocols. This method has applications in psychological and educational research on cognitive processes but also for the knowledge acquisition in the context of building knowledge-based computer systems. In many cases the think aloud method is a unique source of information on cognitive processes. Think aloud protocols are collected by instructing people to solve one or more problems while saying 'what goes through their head', stating directly what they think (p. 1).

While developed before the proliferation of technology and its use in studying and learning mathematics, the think aloud method has been used to analyze the thought processes of students while solving mathematics problems (e.g. Kintsch & Greeno, 1985; Riley, Greeno, & Heller, 1983; Sandberg & de Ruiter, 1985). More recently, researchers have employed think aloud procedures in their investigations of student use of technology to learn mathematics (i.e. Drijvers, 2003; Doorman, Drijvers, Gravemeijer, Boon, &

Reed, 2012). The think aloud method is indicated for this research by Drivers and Trouché's (2008) observation regarding the use of instrumental approach: "A difficulty is that we cannot observe mental schemes directly. Our observations are limited to techniques students carry out with the artifact, and to the way they report on this in written or oral form. From these data we try to construct schemes..." (p. 371). As the think aloud method calls for the subject to verbalize their thought processes while solving problems, application to this study is indicated for analyzing both the subjects' development of schema while using Maplets to solve continuity problems, and in order to determine students' understanding of continuity. The determination of this level of understanding leads to the third framework that guides this study: Tall's *Three Worlds* of mathematics.

Núñez, Edwards, and Matos (1999) discussed two ways in which continuity is addressed in textbooks and classroom teaching. A *natural continuity* refers to the informal/intuitive definition that characterizes a continuous function as one that can be graphed without gaps or jumps. *Formal definitions* of continuity are presented using limits, limit and function symbolism, and are 'rigorous' or precise in definition. Núñez describes most teaching patterns for continuity as introducing students to the idea of natural continuity using ideas and examples from their lived experiences, then moving towards more formal definitions. Núñez et al. argues that students' difficulty with continuity concepts is inherent, as the two definitions of continuity are "radically different cognitive concepts" (p. 55) and that the natural continuity definition involves an *embodied cognition* that would serve teachers and students better than formal treatments. This discrepancy and the challenges presented students because of it, calls for a

framework that accounts for provisions of both the natural and formal continuity definitions.

Tall's (2008) *Three Worlds of Mathematics* framework guided this research in documenting students' understanding of continuity. Tall's *Three Worlds* include: (1) *conceptual-embodied world* based on the perception, reflection, and investigation of properties of objects seen and experienced in the real world; (2) *proceptual-symbolic world* developed from the embodied world through actions (processes) that can be symbolically represented and can themselves be thought of as concepts, thus the term *procept;* and (3) *axiomatic-formal world* is characterized by the formal world of mathematical knowledge construction by using set-theoretic definitions and deducing other properties and schema using formal proof (pp. 7-8). The *Three Worlds* model accounts for overlap between the embodied-conceptual and the proceptual-symbolic as understanding in either or both of these worlds develops toward the axiomatic-formal level of understanding, which can inform this study with regard to the Núñez et al. (1999) concerns listed. Further reasoning for the use of the *Three Worlds* of mathematics comes from Tall's (2008) discussion of calculus. "Calculus builds in three very different worlds," Tall states before providing that calculus is a blend of the world of embodiment (drawing graphs) and symbolism (manipulating formulae) and formalism (proof) (p. 15). While Tall provided examples of limit and derivative concepts, a parallel blending of worlds involving continuity could include: embodiment (investigating graphs); symbolism (limit and function notation); and formalism (proving continuity at a point).

Objectives

Global Objective

This study sought to determine the properties of applets and the actions of students while using applets that foster the development of conceptual understanding of mathematics.

Local Objectives

The following 'local' objectives informed the global objective of this study:

1. Determine the particular characteristics of the Maplets for Calculus applets that promote student understanding of the mathematical concept of continuity of a function.
2. Determine the particular actions and strategies a student develops while using the Maplets, which promote the understanding of continuity.

Research Questions

Questions investigated during the course of this research were:

1. The Maplets for Calculus that present continuity exercises include interactive graphics, hints, "check" answer, and other features. To what degree do each of these features help promote conceptual understanding of continuity with respect to Tall's *Three Worlds*?
2. These Maplets on continuity also allowed students to use multiple features simultaneously. Are there particular combinations of features, e.g. utilization schemes, students developed that lead to a more 'formal' understanding of continuity? Are there utilization schemes that inhibited understanding of

continuity? With respect to Drijvers and Trouché (2008), “what works well and what does not” (p. 375)?

3. In addition to the computer and Maplet software, students were allowed the use of paper, pencil, and a calculator. Are there any other patterns of behavior or thought that students exhibited while engaged with the Maplets that promote/inhibit the development of conceptual understanding?

Logical Structure

The structure for this investigation derived logically from the instrumental approach that accounts for actions and schema development of the student-user as he or she engaged with Maplets to complete exercises about continuity. The instrumental approach has framed investigations using a variety of technologies (spreadsheets, dynamic geometry software, CAS, etc.) and is beneficial to this study, as the use of computer applets is the primary artifact being employed (Drijvers & Trouché, 2008). In this study, actions of students, and characteristics of the Maplets that lead toward greater understanding for continuity were documented.

The development of more sophisticated schemes might indicate, as it did in Drijvers’ (2003) study, a higher level of understanding. For example, if a student used a particular series of keystrokes and inputs to solve an exercise, and later repeated this sequence to answer subsequent exercises; this could be considered a utilization scheme that contributes to the understanding of the concept. A schema developed using particular features of the Maplet repeatedly provided evidence of that characteristic of the applet as contributing to the increase in understanding. The think aloud method provided for the data upon which student understanding was determined, as well as the action

schema developed that helped increase understanding. For example, during a Maplet session, one student stated a function was not continuous because the value of the left and right limits did not equal the value of the function for the given value of *x*; demonstrating understanding of the definition of continuity. Tall's *Three Worlds* informed this study by providing the framework for which student understanding of the concepts of continuity. The example regarding left/right limits and the function value above indicates understanding in the *proceptual-symbolic world* for the concept of continuity – the use of properties and mathematical objects to form a new/greater understanding of the concept.

Definitions

Terms and definitions used throughout this study include:

Applets - Applets are software applications that are executed in the context of another program, usually a web browser or other application (Trigo, Oguin, & Matai, 2010).

Artifact – An artifact is a particular thing or an abstract object that can be used to perform particular tasks (Rabardel, 2002).

Axiomatic-formal world – Part of Tall's *Three Worlds* of mathematics theory. The *formal world* (abbreviated title) represents mathematics based on set-theoretical definitions of concepts. Knowledge and properties about these concepts are represented by theorems (axioms) and are developed through the use of formal proof (Tall, 2008).

Concept-embodied world – Part of Tall's *Three Worlds* of mathematics theory. The *embodied world* (abbreviated title) is based on the perception of and reflection on properties of objects. Initially sensed physically, these properties become part of a

mental image (embodied). It refers to the perceptual representations of concepts (Tall, 2008).

Conceptual knowledge – Conceptual knowledge refers to:

> …knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking of relationships are as prominent as the discrete pieces of information. Relationships pervade the individual facts and propositions so that all pieces of information are linked to some network (Hiebert & Lefevre, 1986, pp. 3-4).

Conceptual understanding – "Conceptual understanding refers to an integrated and functional grasp of mathematical ideas. Students with conceptual understanding know more than isolated facts and methods. They understand why a mathematical idea is important and the kinds of contexts in which is it useful. Such students have organized their knowledge into a coherent whole, which enables them to learn new ideas by connecting those ideas to what they already know" (NRC, 2001, p. 118).

Formal definition of continuity – The definition of continuity that involves the limit of a function and the value of a function being equal at a given point included the use of limit and function notation and implies a more rigorous definition than natural continuity (Núñez et al., 1999).

Instrument – An instrument represents the combination of an artifact and the mental schemes a user develops while using the artifact to perform a specific task (Drijvers & Trouché, 2008).

Instrumental approach theory - Concerns the use of *artifacts*, a material or abstract 'tool' that can be used to sustain a certain activity or problem situation, and the

development of *schemes* by a user for employing the artifact. During this process, the artifact develops into an *instrument* whose use becomes second nature to the user (Drijvers & Trouché, 2008, p. 368).

Maplet - A Maplet is a computer applet designed using the computer software *Maple*. Maplets use an interactive graphic user interface to provide typical examples and exercises on topics in single variable calculus. Individual Maplets are designed to focus on a particular concept or topic to encourage understanding and procedural skill (Meade & Yasskin, 2008, March).

Maplets for Calculus – The collective set of over 140 Maplets designed to provide exercises for topics of single variable calculus (Meade & Yasskin, 2012, March).

Natural continuity – Continuity defined by intuitively thinking of a function without gaps, jumps, or holes (Núñez et al., 1999).

Procedural knowledge – Procedural knowledge: "consists of rules or procedures for solving mathematical problems. Many of the procedures that students possess are probably chains of prescriptions for manipulating symbols" (Hiebert & Lefevre, 1986, pp. 7-8).

Proceptual-symbolic world – Part of Tall's *Three Worlds* of mathematics theory. The *symbolic world* (abbreviated title) "grows out of the embodied world through action (such as counting) and is symbolized as thinkable concepts (such as number) that function both as processes to do and concepts to think about (procepts)" (Tall, 2008, p. 7). Key to this world is the use of mathematical symbols that can represent a process to be carried out, or the concept that process represents (Tall, 2008).

Three Worlds of mathematics theory - A framework to represent the transition in thinking from "school mathematics" (elementary and secondary school) to "pure" or "formal" mathematics represented axiomatic systems and mathematical proof (university and research level) (Tall, 2008).

Think aloud method – Think aloud method is characterized by asking a subject to think (talk) out loud while engaged in a problem solving exercise. What they say is recorded and used as data for analysis. It is used to gain insight to the cognitive process of a subject during the problems solving task (van Someren et al., 1994, pp. 1-2).

Think aloud protocols – The transcribed verbalizations of subjects obtained after using the think aloud method. (van Someren et al., 1994)

Tool – A tool may either be a physical object, such as a hammer, calculator, or computer, or it may be a non-physical cognitive tool such, such as a letters, equations, or language (Vygotsky, 1930/1985).

Utilization scheme - "A mental scheme that involves the global solution strategy, the technical means that the artifact offers, and the mathematical concepts that underpin the strategy." This definition is used in the context of solving mathematics problems with a tool, which in this study is applet technology (Drijvers & Trouché, 2008, p. 369).

Verbal cues – Verbal statements, fragments, or utterances of subjects that may be used to determine underlying cognitive processes (van Someren et al., 1994).

Significance of the Study

This study is significant for high school and college instructors, software developers, researchers investigating the use of technology for building understanding, and researchers investigating learners' understanding of mathematics.

This study has implication for mathematics teachers both at the high school and college levels. Technology is a tool that if used strategically can help educators teach math for understanding, as called for by the CCSSM. Tablets, notebook computers, and smart phones are all technologies that can support applet use. In order to effectively and strategically harness the potential these devices offer for mathematics classes, instructors can use the findings regarding the features found to promote understanding when evaluating and selecting applets for use in their mathematics classes. For example, based on the findings of this study, a teacher may consider selecting applets that allow students to check their answers and change incorrect responses. The successful strategies that students employed while using Maplets can help teachers specifically with regard to the CCSSM Standard for Mathematical Practice number 5 (Use appropriate tools strategically). This knowledge can inform the guidance given to students by their teachers when introducing applet technologies, and for giving advice to students using applets during class sessions. For example: the findings of this study suggest students who read and followed the directions and prompts given by the applets saw an increase in their understanding of continuity concepts.

This study also has important implications for the developers and innovators of mathematics education software, as the documented knowledge of features of applet technology leading to increased understanding for mathematics concepts provides a guide for improving current applets and in the development of new ones, comparable to the work of Jacobse and Harskamp's (2009) whose investigation showing that cognitive hints included in one particular computerized learning environment increased elementary students' performance in solving word problems. Findings from this study also suggest

features that appear to be worthwhile to developers, may provide little evidence of effectiveness when used by students or teachers. The findings regarding the presentation and variety of exercises included in applets can inform software developers of the importance of these ‘subtle’ features.

Researchers investigating the use of technology benefit from this systematic investigation of applet technology; it fills a documented need by the mathematics education community. Researchers have raised concern for the proliferation of technology, programs, and money spent on software prior to its proper evaluation and documented effectiveness in classroom use (Zbiek, 2003; Epper & Baker, 2009). Others have raised concerns for studies that use technology as a before and after treatment without delving into how students use technology and the impact it has on their understanding of mathematics (Zbiek, Heid, Blume, & Dick, 2007; Drijvers et al., 2007). This work helps fill both needs, as well as providing guidance to the methods used in future research considering other software and different populations of students. In documenting schemes developed by students as they used applet technology, this research moves the knowledge base established by Drijvers and Trouché (2008) by providing results tied to the *instrumental approach*.

This work benefits researchers investigating learners’ understanding of mathematics by providing model for documenting understanding of the concepts of continuity using Tall’s (2008) *Three Worlds* framework. The methods presented for building this model and its use for determining student understanding can be used by researchers considering the development of understanding of mathematical concepts other than continuity.

Delimitations

This study sought to determine the properties of computer applets, and the actions of student users, as they used these technologies, and the contributions the properties had to the increase of students' understanding of mathematical concepts. In this particular study, the properties of Maplets that foster greater understanding of the concept of continuity of a function were the focus. The results of this investigation can only be applied to the Maplets used in this study and the individual subjects using them. Any generalization to other technologies or software packages cannot be assumed. Likewise, given the limited number of subjects in this case study, any documented changes in understanding are limited to the particular subjects of this study, and hence, may not be generalized to other populations.

II. Literature Review

Introduction

This study sought to determine the properties of applets and the actions of students while using applets that foster the development of conceptual understanding for mathematics, by considering the following objectives:

1. Determine the particular characteristics of the Maplets for Calculus applets that promote student understanding of the mathematical concept of continuity of a function.
2. Determine the particular actions and strategies a student develops while using the Maplets, which promote the understanding of continuity.

Questions guiding this investigation included:

1. The Maplets for Calculus that present continuity exercises include interactive graphics, hints, "check" answer, and other features. To what degree does each of these features help promote conceptual understanding of continuity with respect to Tall's *Three Worlds* (embodied, symbolic, and formal)?
2. Maplets on continuity also allow students to use multiple features simultaneously. Are there particular combinations of features, e.g. utilization schemes, students develop that lead to a more 'formal' understanding of continuity? Are there utilization schemes that inhibit understanding of continuity?

3. In addition to the computer and Maplet software, students were allowed the use of paper, pencil, and a calculator. Are there any other patterns of behavior or thought that students exhibit while engaged with the Maplets that promote/inhibit the development of conceptual understanding?

The review of the research literature and scholarship relevant to this study will begin with a brief historical outline of United States mathematics educators and researchers work emphasizing the importance of mathematics teaching to focus on the understanding of concepts, not just proficiency with computation and procedures. Next, a section reviews studies and reports on technology, applets, and mobile technology that call for the inclusion of technology in the teaching of mathematics as well as examples of research considering applets and mobile technologies that informed this study. Specific to the conceptual framework on which this study is ground, a review of the instrumental approach and the use of this theory for investigating the relationship between technology and the users of technology in learning mathematics will then be presented. Think aloud methodology, its development as a research tool and investigations using this method will also be presented. Finally, since developing methods for evaluating a level of understanding of continuity was a goal of this study, research about mathematical understanding and continuity will be reviewed.

A History of Mathematics Educators Focusing on Understanding

> During the twentieth century, the meaning of successful mathematics learning underwent several shifts in response to changes in both society and schooling. For roughly the first half of the century, success in learning the mathematics from pre-kindergarten to eighth grade usually meant facility in using the computational

> procedures of arithmetic, with many educators emphasizing the need for skilled performance and others emphasizing the need for students to learn procedures with understanding (NRC, 2001, p. 115).

The work of W. A. Brownell is often credited for leading this call for teaching understanding of mathematical concepts behind the procedures (i.e. NRC, 2001). His 1929 work included a case study which documented work with four students (ages 7 – 10) identified by their teachers as having ‘special difficulty’ (p. 100) in arithmetic. Intelligence test classified the children in the ‘normal’ to ‘above normal’ (86 – 141), yet evaluation using Pittsburg Arithmetic Scale, Form A determined their ‘arithmetic age’ to be one to three years below their chronological age. As intervention, Brownell and his team met with each student 30 – 45 minutes daily and devised learning activities for arithmetic that emphasized “systemization, recognition of relationships, and generalization” (p. 105). This approach opposed popular classroom approaches that emphasized ‘drill and practice’ for learning arithmetic. While Brownell was careful to point out that this experiment ended before “the most desirable degree of improvement had been attained” (p 103), final evaluation of the students using the Pittsburg Arithmetic Scale saw students improve their ‘arithmetic age’ by one to two years.

Thiele’s (1938) work provided evidence that addition facts were learned better by children who developed relationships with numbers. In this experiment, one group of students were given a specific set of addition facts and prompted to discover for themselves generalizations about numbers and operations. This experimental group was compared with students who engage in repetitive drill of addition facts. The students

who developed generalizations performed better on tests of math facts than those who learned by drill.

Brownell's 1938 review of studies considered the readiness of children to learn arithmetic. Citing work by Ballard (1912), Taylor (1916), Wilson (1930), and Benezet (1935), Brownell stated that young children (age 7 or less) are "incapable of learning (they are *unready* to learn) abstract arithmetic when presented through the usual mechanical drill techniques and devices. On the other hand...primary-grade children can learn (are *ready* for) much arithmetic when that arithmetic is met incidentally and informally..." (p. 348-349) and suggested arithmetic curriculum be rearranged so that 'abstract' approaches (i.e. drill) be reserved for later grades and replaced with approaches emphasizing "concrete numbers experiences for children in the first grades" (p. 351). Brownell offered Wilson's "social uses of number" and Thiele's (1935) "understanding of arithmetic" as theory upon which this curriculum could be constructed.

Brownell (1947) later defined and defended teaching for understanding in the context of arithmetic by stating:

> From the standpoint of the pupil *meaningful arithmetic* [emphasis added] -
>
> 1. Gives assurance of retention.
> 2. Equips him with the means to rehabilitate quickly skills that are temporarily weak.
> 3. Increases the likelihood that arithmetical ideas and skills will be used.
> 4. Contributes to ease of learning by providing a sound foundation and transferable understandings.
> 5. Reduces the amount of repetitive practice necessary to complete learning.

6. Safeguards him from answers that are mathematically absurd.

7. Encourages learning by problem-solving in place of unintelligent memorization and practice.

8. Provides him with a versatility of attack which enables him to substitute equally effective procedures for procedures normally used but not available at the time.

9. Makes him relatively independent so that he faces new quantitative situations with confidence.

10. Presents the subject in a way which makes it worthy of respect. (pp. 263-264)

The "New Math" movement of the 1950s and 1960s further emphasized the learning of mathematics in terms of understanding the structure of mathematics and relational connections (NRC, 2001, p 115). According to Klein (2003), "the New Math groups introduced curricula that emphasized coherent logical explanations for the mathematical procedures taught in the schools." Influential during this time was Jerome Bruner's (1960) work, *The Process of Education*, which emphasized that educational experiences should result in understanding, not just performance. Understanding, to Bruner, included the placement of facts and ideas within a structure of knowledge and the ability to point to such items exemplars of broader concepts and principles. Bruner's claim that "any subject can be taught effectively in some intellectually honest form to any child at any stage of development" (p. 33) is based on teaching based in logic and understanding. However, researchers point that the New Math curriculum relied too heavily on these logical structures and understanding, to the detriment of basic skills (Klein, 2003; Hekimoglu & Sloan, 2005).

In reaction to this, the ‘Back to Basics’ movement of the 1970s returned the focus of mathematical instruction to basic skills, particularly computational and procedural skills (Hekimoglu & Sloan, 2005). Skemp (1971) challenged this movement in laying blame for the widespread negative attitude towards mathematics being a result of teaching mathematics without understanding. In later work, Skemp (1977) named two types of mathematical understanding: *instrumental understanding* and *relational understanding.* Instrumental understanding regarded the learning of rules and the ability to use them without understanding the mathematics concepts underlying them; and relational understanding considered both the ability to use mathematical rules and the understanding for the concepts and reasons for which the rules were being applied.

In 1980, the NCTM released *An Agenda for Action: Recommendations for School Mathematics of the 1980s*. In this report, the NCTM described a set of eight recommendations for guiding the teaching of mathematics. The second recommendation called for redefining what ‘basic skills’ in mathematics education represent:

> There must be an acceptance of the full spectrum of basic skills and recognition that there is a wide variety of such skills beyond the mere computational if we are to design a basic skills component of the curriculum that enhances rather than undermines education...Some groups narrowly limit [basic skills] to routine computation at the expense of understanding, applications, and problem solving. This would leave little hope of developing the functionally competent student that all desire (NCTM, 1980).

This recommendation outlined the skills to be considered basic (problem solving, estimating, measurement, etc.) and also listed activities that should receive greater

classroom emphasis (i.e. mental estimation of computations, using technological aids to calculate) and a list of activities that should be de-emphasized (i.e. isolated drill with numbers separate from problem context). The final point in this recommendation called for the following action:

> The higher-order mental processes of logical reasoning, information processing, and decision making should be considered basic to the application of mathematics. Mathematics curricula and teachers should set as objectives the development of logical processes, concepts, and language (NCTM, 1980).

The call for a balance of computational/procedural skills along with understanding outlined by the *Agenda for Action* would be further supported by work of Hiebert and Lefevre (1986). The terms *conceptual knowledge* and *procedural knowledge* are attributed to them (Star, 2005). Procedural knowledge refers to the rules, steps, or procedures for solving mathematical problems (Hiebert & Lefevre, 1986, p. 8). Conceptual knowledge emphasizes the connection and relationship between ideas in mathematics (p. 3-4) and includes the abilities to reason and communicate knowledge (Davis & Barnard, 2000). Hiebert and Lefevre (1986) indicate that the distinction between procedural and conceptual knowledge is a distinction between skill and understanding, and that knowledge of symbols and procedures does not equate to 'knowledge of meaning' (p. 6). They go on to state that, "mathematical knowledge in its fullest sense includes relationships between procedural and conceptual knowledge" (p. 9). Hiebert's later works included statements regarding understanding based on structure, similar to Bruner's ideals, such as: "we understand something if we see how it is related or connected to other things we know" (Hiebert & Carpenter, 1992); and "understanding

should be the *most fundamental goal* of mathematics instruction, the goal upon which all other depend" [emphasis added] (Hiebert et al., 1997, p. 2).

In 1989, the NCTM presented their *Curriculum and Evaluation Standards for School Mathematics* (*Standards)*. This document presented curriculum standards for what math students should know from kindergarten to twelfth grade as well as standards for assessing students and school math programs. The K-12 standards are divided into standards for grades K-4 (13 standards), 5-8 (13 standards), and 9-12 (14 standards). The fourth standard for each strand is labeled Mathematical Connections. In the introduction to the document, the NCTM noted this because:

> This label emphasizes our belief that although it is often necessary to teach specific concepts and procedures, mathematics must be approached as a whole. Concepts, procedures, and intellectual processes are interrelated. In a significant sense, "the whole is greater than the sum of its parts." Thus, the curriculum should include deliberate attempts, through specific instructional activities, to connect ideas and procedures both among different mathematical topics and with other content areas (NCTM, 1989).

The NCTM continued this call for focus on teaching mathematics with understanding with its 2000 publication *Principles and Standards for School Mathematics* (PSSM) (considered in the introductory chapter of this paper).

The National Research Council (NRC) report, *Adding It Up: Helping Students Learn Mathematics* (2001) was written for the purpose of, "attempt[ing] to address the conflicts in current proposals for changing school mathematics by giving a more rounded

portrayal of the mathematics children need to learn, how they learn it, and how it might be taught to them effectively" (p. xiv). This proposal was written in part to address:

> ...consistent and compelling weaknesses in the mathematical performance of U.S. students. State, national, and international assessments conducted over the past 30 years indicate that, although U.S. students may not fare badly when asked to perform straightforward computational procedures, they tend to have a limited understanding of basic mathematical concepts (p. 4).

In describing the "expertise, competence, knowledge, and facility in mathematics" (p. 5) students successful in mathematics possess, the term *mathematical proficiency* is used. Mathematical proficiency includes five "strands" that are intertwined:

> • *conceptual understanding*—comprehension of mathematical concepts, operations, and relations
>
> • *procedural fluency*—skill in carrying out procedures flexibly, accurately, efficiently, and appropriately
>
> • *strategic competence*—ability to formulate, represent, and solve mathematical problems
>
> • *adaptive reasoning*—capacity for logical thought, reflection, explanation, and justification
>
> • *productive disposition*—habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy (p. 5).

In presenting this definition of mathematical proficiency, the report emphasized that "***the five strands are interwoven and interdependent in the development of proficiency in mathematics"*** (p. 116).

In discussing the importance of conceptual understanding to learning mathematics, *Adding It Up* makes the following points:

- learning with understanding is more powerful than simply memorizing because the organization improves retention, promotes fluency, and facilitates learning related material
- having a deep understanding requires that learners connect pieces of knowledge, and that connection in turn is a key factor in whether they can use what they know productively in solving problems
- when students have acquired conceptual understanding in an area of mathematics, they see the connections among concepts and procedures and can give arguments to explain why some facts are consequences of others
- conceptual understanding frequently results in students having less to learn because they can see the deeper similarities between superficially unrelated situations (NRC, 2001, p. 118-120).

In the concluding remarks in the chapter describing mathematic proficiency, the NRC states that "many people in the United States consider procedural fluency to be the heart of the elementary school mathematics curriculum" (p. 144) and ends with this acknowledgement:

> We conclude that during the past 25 years mathematics instruction in U.S. schools has not sufficiently developed mathematical proficiency in the sense we have defined it. It has developed some procedural fluency, but it clearly has not helped students develop the other strands very far, nor has it helped them connect the strands. Consequently, all strands have suffered (p. 145).

Both the NRC *Adding It Up* (2001) and the NCTM *Principles and Standards* (2000) are cited in the *Common Core State Standards for Mathematics* (CCSSM) (CCSSO/NGA, 2010) (considered in the introductory chapter of this paper). Daro, McCallum, and Zimba (2010), members of the CCSSM working group, stated the CCSSM delineate both skills to master and concepts to understand. They also commented that "conceptual understanding intertwines with procedural skill to develop mathematics achievement. To make solid progress, students need not only skills to tackle mathematics problems, but also the mathematical concepts that give coherence and substance to the subject" (p. 285).

This section presented a brief historic outline of the call for emphasizing the teaching of conceptual understanding in school mathematics. Evidence suggesting teaching concepts leads to improved mathematical learning from Brownell (1935, 1947) and Thiele (1938) were presented as well as reports grounded in research that emphasized the teaching of concepts (NRC, 2000; NCTM, 1989, 1980, 1989, 2000; CCSSO/NGA 2010). Despite continued recommendations, reports and research proclaim that typical United States school math experiences continue to focus on the acquisition of procedural skills (i.e. Cangelosi, 2003; NRC, 2000; NCTM, 1989, 1980, 1989, 2000; CCSSO/NGA 2010). The work of Hiebert and Lefevre (1986) was considered for the importance the

constructs of *conceptual knowledge* and *procedural knowledge* as they apply to the teaching and learning of mathematics. Their work is credited with giving the field of mathematics education terminology with which to express these ideas, as well as the importance of each in developing mathematical knowledge.

Technology, Applets, and Mobile Technologies

The NCTM *Agenda for Action* called for "mathematics programs take full advantage of the power of calculators and computers at all grade levels" (1980). In developing the *Principles and Standards for School Mathematics* the NCTM included as one of its five 'principles' The Technology Principle that begins by stating: "*Technology is essential in teaching and learning mathematics; it influences the mathematics that is taught and enhances students' learning"* (2000). In 2010, the NGA/CSSO wrote the *Common Core State Standards for Mathematics* to include the use of technology within the standards without the need for a special statement – the assumption being technology is available for student use. The challenge now is that access to technological tools is easier and software development moves more quickly than our ability to evaluate how to use these tools effectively (Fey, Hollenbeck, & Wray, 2010). As Zbiek (2003) wrote in regard to the proliferation of computer algebra systems (CAS), "CAS research has lagged behind the implementation of CAS projects. As a result, the emerging pool of research is minimal and fractured" (p. 197).

> The wide range of [technological] tools available for...mathematics learning include tools for data collection and visualization, modeling programs, simulation programs, multi-user virtual environments, online search tools, communication technologies, course management tools, handheld mobile technologies,

probeware, dynamic geometry software, calculators, interactive whiteboards, virtual manipulatives, and online publishing (Dani & Koenig, 2008; Heid, 2005; Wofford, 2009). [in Donnelly & Mikusa, 2010]

In this environment of exponential growth in technology, *applets* have found a particular favor. *Applets* are software applications executed in the context of another program. Applets are very flexible and can be used on a variety of technologies, including computers, smart-phones, and tablet technologies. Two properties that make applets appealing for use in education settings are visualization and manipulation. Graphics can be built into applets that bring concepts to students in a visual medium. Manipulation of variables, inputs, graphics, etc. can also help students understand concepts (Trigo, Olguin, & Matiai, 2010).

D. Young (2006) described some benefits for the use of applets in mathematics education, including: (1) they are easily availability on the internet; (2) their focus on specific concepts; (3) applets allow students to engage with math in ways that are impossible with physical manipulatives; (4) applets provide instantaneous and corrective feedback; (5) applets allow for multiple representations of mathematical ideas; (6) they may be helpful for students with disabilities; and (7) applets increase motivation and attention in students and teachers.

Studies reviewed regarding applet use in teaching mathematics include:

- Doorman, Drijvers, Gravemeijer, Boon, and Reed (2012) investigation into the conceptual development of functions using the computer applet, AlgebraArrows, determined that the representations of the applet led students to increased levels of understanding of functions.

- Ke (2008) used math applets, a series of games called ASTRA EAGLE, during a summer math camp. Results indicated improved student attitude toward mathematics and increased mathematical engagement of students using 'situated learning' applets (problems embedded within a storyline).
- Hoffkamp's (2010) investigation included the use of two applets to determine the development of understanding of concepts regarding the fundamental theory of calculus and properties of functions. In this study, Hoffkamp used the phrase *interactive visualization* to denoted functions of applets that allowed students to manipulate the visual representations presented; a trait that allowed for the development of greater understanding of the mathematical concepts considered in the study.
- Heck, Boon, Bokhove, and Koolstra (2007) described the GALIOS project in which the authors were involved in developing and implementing applets in school settings to teach concepts of algebra and calculus in secondary schools. Findings included: increased student motivation; ability to address individual student needs; interactive and dynamic features promote understanding of concepts; and the development of student creativity with math.

Studies considered that investigated the use of 'mobile' technologies, included:

- Franklin and Peng (2008) presented a case study in which middle school students used and iPod Touch learning the algebra concepts. Using this technology, students made videos to represent math concepts. The authors noted one benefit of using this technology was the ability to continue math learning outside of the classroom.

- Dahar (2009) investigated students' perceptions in learning mathematics while using mobile phones versus using applets on a computer. Findings included student preference for using mobile phones based on its size (portability) and communication capability.
- Stickel and Hum (2008) presented findings from the introduction of 'tablets' into their college classrooms. PowerPoint presentations embedded with animations, figures, and videos were the primary method for instruction delivery in the investigators' differential equations and linear algebra (Stickel) and electromagnetics (Hum) courses. Student surveys indicated that there may be a correlation between learning style and the effectiveness of using tablet technology in learning.

Another study reviewed was Garrett's (2010) dissertation investigating how the use of mathematics technology affects the internal mathematical representations possessed by adult developmental mathematics students (p. ii). In this study, Garrett used a teaching experiment that included written and computer exercises with Geometer's Sketchpad software to determine the students' internal representations for the concept of functions.

This section informs the research of this investigation by documenting the need for investigating technologies being used in mathematics education. As noted by the NCTM (1989, 2000) and the NGA/CSSO (2010), technology is necessary for the teaching and learning of mathematics and the availability of technological tools is now assumed. However, Fey et al. (2010) and Zbiek (2003) are among the researchers that call for the need to investigate new technologies as their development and

implementation outpace the investigation of their effectiveness. Work by Trigo, Olguin, & Matai (2010) and Young (2006) highlighted some of the characteristics of applets and the benefits of using them for teaching and learning mathematics. This section concluded with a small sample of research that is representative of the nature of investigations that have been conducted using applets and various mobile technologies.

Instrumental Approach Framework

The genesis of the instrumental approach is attributed to L.S. Vygotsky's 1930 talk introducing "the instrumental method in psychology" (Wertsch, 2002; Drijvers & Trouché, 2008). He offered that, "by being included in the process of behavior, the psychological tool alters the entire flow and structure of mental functions. It does this by determining the structure of a new instrumental act, just as a technical tool alters the process of a natural adaptation by determining the form of labor operations" (Vygotsky, 1981, p. 137). The psychological tool referred to in this passage was natural language, however, he further implied this designation applies to "various systems of counting; mnemonic techniques; algebra symbol systems; works of art; writing; schemes; diagrams, maps, and mechanical drawings; all sorts of conventional signs; and so on" (p. 137), thus to Vygotsky, tools can be comprised of either psychological or physical. The "instrumental act" includes a problem that needs solving, the mental process for solving, and the (psychological) tools used during this procedure; hence a tool only becomes an instrument in the act/process of being used. Note that to Vygotsky, the tool being used can alter the mental structure/function as the user constructs ways in which the tool may be applied to a given situation.

Rabardel and colleagues elaborated Vygotsky's ideas by distinguishing artifacts from instruments (Rabardel, 2002; Vérillon & Rabardel, 1995). Rabardel designated an artifact as the "bare tool", either a material or abstract object, available to a person for the purpose of sustaining a particular type of activity, but may be rendered useless without the knowledge of the type of tasks the artifact can be used for, or the ways in which the artifact may be applied to a giving situation. When the user becomes aware of how the artifact may be applied and used to a given task, and once the user develops the means of using the artifact, then the artifact becomes an instrument.

Drijvers and Trouché (2008) extended Rabardel's ideas to describe an instrument as follows:

> Following Rabardel, we speak of an instrument when there exists a meaningful relationship between the artifact and the user for dealing with a certain type of task – in our case a mathematical task – which the user has intended to solve. As the interaction between the user and artifact requires mental processes, we see that the main "players" here, the mental processes of the user, the artifact, and the task, are the same as was the case for Vygotsky's previously described instrumental act. Particularly for mathematical tools, which can be considered "extensions of the mind" rather than extensions of the body, these mental processes are essential. Therefore, the instrument consists of both the artifact and the accompanying mental schemes that the user develops to use it for performing specific kinds of tasks (p. 367).

Drijvers and Trouché summarize their definition of instrument in two ways, with the equation: "Instrument = Artifact + Scheme" for a particular task; and by further

emphasizing that an “artifact develops into an instrument only in combination with the development of mental schemes” (p. 368).

Instrumental genesis refers to the process by which an artifact becomes an instrument. This process requires the user to develop mental schemes involving knowing how to use the artifact appropriately and understanding for which circumstances the artifact is useful. Instrumental genesis, and hence the instrumental approach, considers the interaction between the user and the artifact. *Instrumentation* concerns the effect the artifact has on the user’s thinking: “the possibilities and constraints of the artifact shape the techniques and the conceptual understanding of the user” (p. 368-9). *Instrumentalization* refers to how the artifact is shaped by the user: “the conceptions and preferences of the user change the ways in which he or she uses the artifact, and may even lead to changing or customizing it” (p. 369). Instrumentation and instrumentalization is a bidirectional interaction in which a student’s thinking is shaped by an artifact, but that thinking also shapes the artifact (Hoyles & Noss, 2003).

At the heart of instrumental genesis is the development of mental schemes. These schemes organize problem-solving strategies, including relevant concepts that form the basis of such strategies. Drijvers & Trouché consider a *utilization scheme*, in the context of solving mathematics problems with a tool, as: “a mental scheme that involves the global solution strategy, the technical means that the artifact offers, and the mathematical concepts that underpin the strategy” (p. 369). They distinguish between two types of utilization schemes, *usage schemes* and *instrumented action schemes*. Usage schemes are elementary schemes and are often direct functions of the artifact. An example may be using a graphing function on a calculator. Usage schemes are the building blocks for

higher order schemes, instrumented action schemes. Instrumented action schemes are built from utilization schemes through the instrumental genesis. Goldberg (1988) offered an example of such an instrumental action scheme in describing the mental process involved in changing the viewing window on a graphing calculator. While the technical skill and input required are not overly difficult, the corresponding mental schemes needed to understand that the viewing window represents only part of the graph and the ability to determine the appropriate viewing area for a given problem/exercise requires schemes of higher-order. "In the case of mathematical information technology tools, the conceptual part of utilization schemes therefore includes both mathematical objects and insight into the 'mathematics behind the machine.' As a consequence, seemingly technical obstacles that students experience while using a computerized environment for mathematics often turn out to have an important conceptual background" (Drijvers & Trouché, 2008, p. 371).

A number of researchers have invoked the instrumental approach as the framework for their investigations into the use of technology for learning mathematics. Haspekian (2003) offered an instrumental approach toward investigating building relationships between arithmetic and algebra via the use of spreadsheets. Hollebrands, Laborde, and Sträβer (2008) surveyed a number of studies that investigated the instrumentation of the "drag" operation in interactive geometry software. They cited studies into the ways students used the "drag" operation including the work of Arzarello, Micheletti, Olivero, Robutti, Paola, and Gallino (1998), Arzarello, Olivero, Paola, and Robutti (2002), Olivero (2002), Olivero and Robutti (2002), and Smith (2002). Hollebrands et al. (2008) also cited work of Holzl (1995 and 1996) documenting shifts in

student schemes for using the “drag” function from wandering toward a more purposeful use of constructing hypotheses; and the work of Talmon and Yerushalmy (2004) in using “dragging” to make predictions. Doorman, Drijvers, Gravemeijer, Boon, and Reed (2012) used aspects of tool and instrumentation theory to frame their investigation into the conceptual development of functions using the computer applet, AlgebraArrows. Their results suggested that the “relationship between tool use and conceptual development benefited from tools offering representations that allow for a progressively increasing levels of reasoning” (p. 1).

One study that outlines the relevance of the instrumental approach to this study comes from Drijvers (2003). He utilized an instrumental approach for investigating how the use computer algebra systems (CAS) promoted the understanding of the mathematical concept of parameter. This research included observing students solving a problem involving a parameter while using a TI-89 symbolic calculator, a handheld device that offers graphing capabilities as well as symbolic manipulation. The exercise offered students a set of graphs for the quadratic $y = x^2 + b\,x + 1$ and required students to express the coordinates of the extreme value for a given “family member” determined by the variable *b*. In discussing the observations made while watching one student use the calculator and analyzing their written result, Drijvers was able to identify: the artifact as the algebraic application within the TI-89; the instrumented action as solving the parametric equation using the artifact; the elementary usage schemes of using the “solve” command and use of a formula for the graph of a quadratic; and the instrumental action scheme as the combination of the technical ability and the conceptual components needed to complete the task. He concludes that the instrumental genesis of the scheme for the

application of the “solve” command in the CAS interacted with several aspects of the student’s understanding of parametric equations. These interfered with each other, and the resulting conflict resulted in a co-development (of schema for applying the command and solving parametric equations) during the instrumental genesis that forced the student to extend her conception of solving parametric equations.

This section outlined the development of the instrumental approach as an accepted framework for investigations concerning the use of technology by students for learning mathematics. Drijvers and Trouché’s (2008) work outlining the theory as it related to individual student learning is of particular importance to this study, as their vision emphasizes the development of schemes users create while engaged with technology. Drijvers’ (2003) study informs this research by providing an example for understanding the relationship between students, technology, and how the process of instrumental genesis can lead to greater understanding of a mathematical concept.

Think Aloud Method

The development of the think aloud method has been traced to the early 1900’s psychological practice of introspection (Crutcher, 1994; van Someren, Barnard, & Sandberg, 1994; Ericsson & Simon, 1980). Introspection practices involved training participants to interpret their own thinking and report their observations, verbally, to specialists (psychologists) to record and interpret these reports. Some psychologists claimed these records represented cognitive thoughts that could be used as data (i.e. Tichener, 1929). However, other psychologist questioned the chain between the subjects’ thoughts, the subjects’ process of interpreting and verbalizing their thoughts, and using the specialists’ record as cognitive data to be analyzed, thus questioning the validity of

the introspection method (i.e. Lashley, 1923). Later, some researchers modified this method to emphasize reporting thoughts as opposed to reporting the interpretations of those thoughts. For example, Dewey asked subjects to report their thoughts, not their interpretation of them, just a recollection of their thoughts retrospectively after recent thinking episodes (Aanstoos, 1985; Ericsson & Crutcher, 1991). Concerns for the validity of this method focus on the lapse in time between the thinking episode and the report of thoughts, as the verbal reports were open to the subjects' interpretation of the thinking.

Newell and Simon's (1972) research into problem-solving led to further development of the think aloud method. By using the verbalizations of subjects involved in a task and computer models of problem solving processes, they were able to construct detailed models of the human problem solving process. This work was influential to the use of verbal reports in research, as Newell and Simon were able to explain protocol data from a theory of human memory and assumptions about the knowledge subjects were able to invoke while involved in problems solving. (van Someren, Barnard, & Sanberg, 1994, p. 31)

Ericsson and Simon (1980, 1984) are often credited formally proposing and defending the think aloud method as a means for collecting and analyzing verbal data (Crutcher, 1994; K.A. Young, 2005). Using human-information processing theory as a framework, they contended that only information in short term memory is accessible to a subject without changing thought processes; because of this, thoughts accessed while a subject is engaged in mental activities incumbent on the use of short term memory, i.e. problem solving tasks. Ericsson and Simon posed that a subjects' unencumbered

verbalizations of their thoughts while involved in a task, all of which occurs in short term memory, precludes a subject from interpreting their thinking. Procedure for collecting data, as outlined by van Someren, Barnard, and Sanberg (1994), involve minimal observer intrusion. Subjects are given instructions to the task they will be expected to perform and then instructed to say out loud what comes to mind as they engage with the task. Should a subject go quiet during the process, the observer's prompt should be a short, non-leading phrase such as "keep talking", as opposed to a question like "what are you thinking?" to which a subject may stop and offer a reflection or interpretation (p. 42-43).

K. A. Young (2005) believes that think aloud data can be especially beneficial to research that examines student learning in a technological environment. As an example, Young cited her own investigation aiming to identify the types of learning taking place while working in a web-based environment. During this investigation, she asked the students, all in Grade 5 (Australia), to think aloud as they used a search engine (Google) to answer a student generated research question. One example provided involved a participant named Liz investigating her own question about the history of field hockey. Young transcribed the audio and synchronized it to concurrent observations from the video recording. In analyzing these data, Young was able to determine strategies the students developed, their use of knowledge in entering keywords into Google or the decision to enter particular websites suggested, and limitations of the search engine for finding specific information.

Perrenet and Kaasenbrood (2006) used think aloud methodology to investigate the level of understanding of the concept of algorithm with computer science students. They

used this qualitative study, in part, to validate results of a previous quantitative study that defined four levels of abstraction for the concepts related to the study of algorithms (Perrenet, Groote, & Kaasenbrood, 2005). In this study, computer science students were given a questionnaire about six concepts related to algorithms (i.e. the complexity of a problem is independent the choice of algorithm used to solve it) and asked to respond, in writing, while thinking aloud. The investigators goal here was to investigate the extent students really understood the computer science terms they listed in their written responses. Both the written and transcribed think aloud responses of participants were evaluated to determine their level of understanding of both the concept being asked on the questionnaire and their use of specific computer science terms. Their results emphasized that most students' responses indicated understanding of the computer science terms being used in their responses and intermediate levels of understanding for the concepts related to algorithms.

Ke (2008) employed the think aloud method in a mixed-method case study of fifteen 4^{th}-5^{th} grade students using a series of computer games designed to reinforce mathematics standards in Pennsylvania. Ke's case study involved observing students using five different computer games, in a series of games called ASTRA EAGLE, during a five-week summer math camp. The researcher's goal was to assess how the use of computer games affected math achievement, meta-cognitive awareness, attitude toward math, and engagement. Pre and post tests were given to assess differences in achievement. Students were asked to think aloud during gaming sessions. Observations records, as well as participants' game-playing records (kept track by the computer program) were also collected. Results of the observations and think aloud data indicated

an increase in students' positive attitudes toward learning math, and that the games that involved situated learning (problem solving within a game 'story') engaged students more readily than those games without situations.

Another study that used think aloud methods was Jocobse and Harskamp's (2009) investigation of the usefulness of working with a computer program with meta-cognitive hints for enhancing meta-cognitive skills and problem solving in Grade 5 students (Netherlands). The computer program, developed and modified by the investigating team, was designed to enhance the meta-cognitive skills of users by providing word-problems and asking users to respond with steps leading to the solution of the problem. Each step is support by hints, which the user could choose to view or neglect. In assessing students' meta-cognitive skills, the researchers used think aloud methods as part of a pre- and posttest measure. Ten random students were selected from the classroom that employed the computer program as part of the instruction for solving word problems. These students were asked to think aloud while writing their solution to a word task and the transcribed protocols reviewed by two evaluators who ranked the students' meta-cognitive skill according to a schemata posed by Veenman, Kerseboom, and Imthorn (2000). These scores were then analyzed using a paired-samples *t*-test that revealed that these students, who were a part of the experimental group, showed a significant increase in meta-cognitive skill.

This section informs the research proposed here by documenting the history and acceptance of the use of verbal data and the development of the think aloud method as valid tools for investigating cognitive processes. A review of recent articles in which the think aloud method was employed as part of the data collection informs this study in the

following ways: (1) think aloud data has been used to determine levels of student understanding (Jacobse & Harskamp, 2009; Perrenet & Kaasenbrood, 2006); (2) this method has been employed in studies conducted with students engaged in a computer environment (Young, 2006; Ke, 2008); and (3) the use of video recording during think aloud data collection (Young, 2006; Jacobse & Harskamp, 2009, Ke, 2008); and (4) information regarding methods and procedures used for collecting and analyzing think aloud data.

Hierarchy of Understanding and Continuity

Cottrill et al. (1996) presented Actions-Processes-Objects-Schema (APOS) theory as a perspective for investigating the concept of limits. The acronym, APOS, was formed from their view that there are three types of mathematical knowledge: *actions, processes, and objects*, which are organized into structures or *schemas.* Actions refer to physical or mental transformation of objects to obtain other objects. Processes are similar to actions in that objects are transformed, however, the defining characteristic is that an individual has control over the transformation and can think and reflect on the process; whereas actions are reactive. An object is constructed when an individual collectively thinks of a process, and the steps in the process, as its own entity. Important here is that objects can be broken down to obtain the processes and that an individual can move back and forth between object and process concepts of a mathematical idea. Schema is the collection of actions, processes, objects and other schema that are purposely linked. A scheme itself can become a new object; hence processes and schemas can bear new objects that can motivate new actions and processes resulting in a “spiraling” iterative process.

The van Hieles (1958, 1984, and 1986) described five levels of thinking for geometric thinking. As interpreted by Fuys, Geddes, and Tichner (1988), these levels are sequential; one cannot achieve a higher level of thinking without having passed through a previous one.

Level 0: The student identifies, names, compares and operates on geometric figures (e.g., triangles, angles, intersecting or parallel lines) according to their appearance.

Level 1: The student analyzes figures in terms of their components and relationships among components and discovers properties/rules of a class of shapes empirically (e.g., by folding, measuring, using a grid or diagram.

Level 2: The student logically interrelates previously discovered properties/rules by giving or following informal arguments.

Level 3: The student proves theorems deductively and establishes interrelationships among networks of theorems.

Level 4: The student establishes theorems in different postulational systems and analyzes/compares these systems (Fuys et al., 1988, p. 5).

Hoffer (1981) further described these levels as: visualization, analysis, abstraction, deduction, and rigor. The van Hieles (1958) study noted that learning is not a continuous process. In their observations, students did not transition to higher levels of thinking smoothly but through jumps; it was these leaps that lead to the development of the "levels".

Drijvers' (2003) (in the study considered in the section on instrumentation) used the level theory of van Hiele (1986) to study the levels of understanding of the concept of parameter. Drijvers presented the framework of Figure 2.1. In this, the ground level represents a visual/concrete understanding of a parameter as a placeholder. Second level understanding occurs when a parameter is seen as having the properties of a changing quantity, an unknown quantity, and/or a generalizer; at this stage of the model, parameter becomes an *object*. Third level thinking in this model is considered when these properties are subjected to a logical structure and relationships between the properties are formed (see Figure 2.1). Drijvers' summarized the use of this model by stating, "In this study, we use van Hiele's level theory to specify the intended level-raising of the understanding of the concept of parameter" (p. 71). In addition to mapping the concept of parameter using van Hiele's levels, Drijvers mentions that levels of understanding, particularly the formation of objects at the second level, can become the ground level for another concept and that this process is, "relative and iterative" (p. 71). This observation suggests a connection to APOS theory.

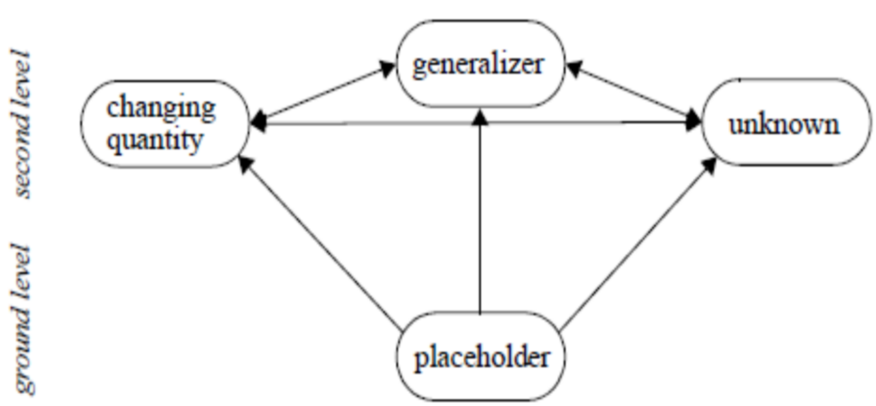

Figure 2.1 Drijvers' levels of understanding of the concept of parameter (2003, p. 71).

Tall (2008) provides a framework to represent the transition in thinking from "school mathematics" (elementary and secondary school) to "pure" or "formal" mathematics represented axiomatic systems and mathematical proof (university and research level). In developing this model, Tall indicates that three mental actions shape learning and thinking about mathematics: (1) *recognition* of patterns and similarities/differences; (2) *repetition* of sequences of actions until they become automatic; and (3) *language* to describe and refine the way we think about things (p. 6). The process of maturing in mathematical thinking involves using recognition, repetition, and language to "construct three interrelated sequences of development that blend together to build a full range of mathematical thinking" (p. 7). Collectively, these sequences were entitled "the *Three Worlds* of mathematics".

The first of these, the *concept-embodied world*, is based on the perception of and reflection on properties of objects. Initially sensed physically, these properties become part of a mental image (embodied). It refers to the perceptual representations of concepts. Tall uses van Hiele's (1986) "levels" as the manner in which an individual matures in mathematical thinking in the concept-embodied world. Growth from perception and description (level 0) to deduction and establishing relationships between theorems (level 3) can be accomplished by physical embodiment of concepts. However, it is only when an individual makes the shift to working with axioms and developing systems based on these axioms (level 4) that full maturation to the "formal" level of mathematical thinking is considered.

The *proceptual-symbolic world*, "grows out of the embodied world through action (such as counting) and is symbolized as thinkable concepts (such as number) that

function both as processes to do and concepts to think about (procepts)" (Tall, 2008, p. 7). Key to this world is the use of mathematical symbols that can represent a process to be carried out, or the concept that process represents; the fusion of these two terms, *procept*, is attributed to Gray and Tall (1994). Tall credits Cottrill et al. (1996) work on APOS theory as a model for the growth of mathematical thinking in the perceptual symbolic world. APOS theory, as used by Tall (2008), models the *compression* (the brain's ability to cope with multiple ideas at the same time by connecting and organizing multiple ideas into one concept) of concepts, highlighted by the use of symbols. APOS also provides context for blending embodiment and symbolism in the development in sophistication of mathematical thinking.

The *axiomatic-formal world* represents mathematics based on set-theoretical definitions of concepts. Knowledge and properties about these concepts are represented by theorems (axioms) and are developed through the use of formal proof.

Tall's model allows for integration between the *Three Worlds*. As such, he contends that a path toward formal thinking is possible through the embodied world, the symbolic world, or a combination/interaction of these two worlds, embodied symbolic. (For the sake of simplicity in discussing the worlds, Tall shortened the names to embodied, symbolic, and formal.) Figure 2.2 outlines the *Three Worlds* framework.

Tall's *Three Worlds* of mathematical thinking framework is indicated for this study by the work of Núñez et al. (1999) in describing the difficulty students have understanding the concept of continuity of functions. Continuity can be thought of informally, referred to as *natural continuity*, as a function without breaks, jumps, or holes. It can be manifested physically by the characterization that a continuous function

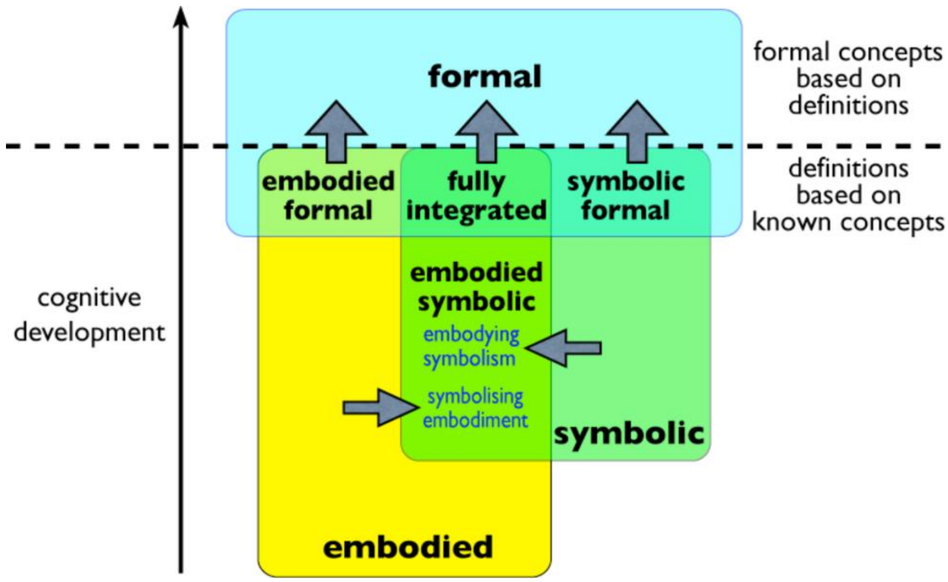

Figure 2.2 Tall's *Three Worlds* model of cognitive development of mathematics concepts. (2008, p.

is one that can be drawn without lifting a pen/pencil from the paper. The context for this understanding of the limit concept is *embodied cognition*. Núñez et al. describe the process of learning continuity in school settings as one that often begins with establishing natural continuity, but then introducing a formal definition for continuity (they use the Cauchy-Weierstrass definition) that supposedly embodies the ideas of natural continuity. A textbook example of this definition is:

> A function *f* is continuous at a number *a* if the following three conditions are satisfied:
>
> 1. *f* is defined on an open interval containing *a*;
>
> 2. $\lim_{x \to a} f(x)$ exists; and
>
> 3. $\lim_{x \to a} f(x) = f(a)$ (Simmons, 1985).

Disconnect between these two definitions, Núñez et al. contend, occurs because the two definitions rely on different cognitive contexts - essentially they are two different concepts that students have to learn (1999, p. 55). However, in light of Tall's *Three*

Worlds, the symbolic nature of the 'formal' definition, which itself describes a process of determining continuity (proceptual in nature), can lead toward an interpretation that this is a proceptual-symbolic world understanding of the concept.

In an earlier study, Núñez and Lakoff (1998) described the historic foundations for the formal definition of continuity that relied on limits, the ε-δ definition. Prior to the late nineteenth century, mathematicians such as Kepler, Leibniz, Newton, and Euler based continuity on the motion of a physical object with a definite direction and speed. Motion seen to continue without gaps or interruptions was considered to be continuous. They based further developments in their mathematical work using this 'natural' definition of continuity. However, ideas that regarded functions as naturally continuous curves changed in the late 1800's as mathematicians explored new concepts, which did not fit into the scheme of continuity generally accepted at the time. A more formal and precise definition of continuity was developed to encompass all cases. Núñez and Lakoff attribute this to the cultural values of the mathematics community of the time that emphasized "secure and rigorous" foundations using symbolic notation and logic. Núñez and Lakoff (1998) and Núñez, et al. (1999) call for a return to teaching continuity using natural continuity definitions and descriptions.

Tall and Vinner (1981) described this difference as one of *concept image*, the embodied idea of continuity representing a function whose graph has no gaps, and the *concept definition*, or formal definition of continuity. In an investigation of first year university students, they presented five questions/functions to students and asked them whether or not the function was continuous and give reasons for their answers. In analyzing the written responses, it was found that most students evoked the concept

image of continuity that implied a ‘natural’ definition scheme. By making instructors aware of these students’ concept image of continuity, Tall and Vinner suggest that discourse with students may resolve this cognitive conflict between natural and formal continuity.

Bezuidenhout (2001) also investigated nature and characteristics of students’ concept images of continuity. In this study, fifteen students were selected to be involved in ‘task-based’ interviews. These first year calculus students had been part of a larger study that analyzed students written results on items regarding limits and continuity. The interviews were prepared for each student based on students’ answers on the written test. During the interviews, students were asked to explain their written answers; when necessary, the interviewer asked follow-up questions. Two test items discussed during interviews focused on students’ conception of the formal definition of continuity. One misunderstanding documented was the erroneous student belief that the existence of a limit at a point implies continuity at the point. Other conceptual errors discussed involved relationships between limits, continuity, and differentiability. Bezuidenhout concludes that student understanding for continuity (as well as limits and differentiability) is dependent on isolated facts and procedures (in this he blames teaching approaches that emphasize procedures) without regard to the relationships and concepts inherent to continuity.

Chan’s (2011) master’s thesis investigated the differences in conceptual knowledge about continuity and derivatives between two groups of college freshmen. This study compared the written performance of students enrolled in an Emerging Scholars Program (ESP) with non-ESP students enrolled in the same calculus course.

Using a written assessment modified from Tall and Vinner (1981), the investigator devised a rubric to grade responses to each of five items for accuracy and conceptual understanding. Of import to this study, Chan's development of a conceptual understanding rubric (CUR), modified from the Mathematics Problem Solving Official Scoring Guide from the Oregon Department of Education (2008) and the Quasar General Rubric (Lane, 1993), credits conceptual understanding of items on a five-point scale. An item assessing continuity of a piecewise function and accompanying CUR is provided in Figure 2.3. The rubric reflects a combination of the levels presented by van Hiele (1986) and ideas of proceptual-symbolic world developments in cognitive thinking in a combined way. This study also prescribed the use of items that presented the graphs of functions in addition to items in which students were asked to only consider the definition of a function in determining continuity. Similar to Drijvers' (2003), Chan used this scale to document a level of understanding (see Figure 2.3).

Other studies of continuity were reviewed for this investigation, including: Takači, Pešić, and Tartar's (2003) investigation use of visual presentations in teaching about continuity using the computer program Scientific Workplace; Vela's (2011) thesis investigating concept image and the concept definition of continuity in high school students; Ko and Knuth's (2009) article using qualitative methods to determine the abilities and misunderstandings of student proofs involving continuity concepts; and Takači, Pešić,, and Tatar's (2006) analysis of high school students' theoretical and visual knowledge of continuity.

2. Are the following functions continuous on the interval $[-2, 2]$? Explain why.

c. $h(x) = \begin{cases} 0, & x < 1 \\ x - 1, & x \geq 1 \end{cases}$

5	➢ Mentions all of the following: a) $h(x_0)$ is defined, so that x_0 is in the domain of h b) $\lim_{x \to x_0} h(x)$ exists for x in the domain of h c) $\lim_{x \to x_0} h(x) = h(x_0)$ ➢ Identifies that both parts of the piecewise are continuous and verifies that the point at which the function definition switches is also continuous
4	➢ The function can be drawn without lifting a pencil ➢ Identifies that the function has no breaks/holes/jumps/discontinuities ➢ Identifies that both parts of the piecewise are continuous, and attempts to check the point at which the function definition switches
3	➢ In regards to a, b, c from above, the student either mentions b or c alone, or any combination of 2 of a, b, or c ➢ Restricts a, b, c from above to only a particular point ➢ States that $\lim_{x \to 1^-} h(x) = \lim_{x \to 1^+} h(x)$ same (uses reference to left and right hand limits but only in regards to the "problem point")
2	➢ Uses vague reference to L/R hand limits ➢ States that the function is defined for all reals, or is never undefined ➢ Uses factual criteria about the function to show that the function is not continuous ➢ Uses the function's differentiability in their justification
1	➢ Only mentions the nature of the point at which the function definition switches (i.e. "values are equal when the function changes") ➢ States that the function is continuous on the interval without any justification ➢ Uses nonfactual criteria or simply states that the function is not continuous everywhere/on the other interval

Figure 2.3 Item 2.c. and CUR from Chan's assessment on continuity (2011, p. 67 and 76)

This section documented the development of Tall's three-worlds of mathematics as a lens for viewing the conceptual understanding. APOS theory and van Hiele's level theory were considered, but the fusion and extension that Tall provides applies to understanding the concept of continuity as indicated by Núñez et al. Their work emphasized the difficulty in understanding continuity as being a cognitive division of two separate concepts, natural and formal continuity. Tall's framework provides provisions for both in the development toward formal thinking of continuity. Investigations of continuity that contribute to this study included: Núñez and Lakoff's (1998) historical

considerations of the formal definition of continuity; Tall and Vinner's (1981) investigation of concept image and concept definitions of continuity; Bezuidenhout's (1999) investigations into student misunderstandings of continuity; and Chan's (2011) investigation of differences in levels of understanding of continuity between groups of college freshmen. These investigations inform this study by providing framework for developing a measure assessing understanding of continuity concepts.

Summary

This review of relevant research informed this study by considering: the history of mathematics educators' emphasis on teaching mathematics for understanding, reports calling for the inclusion of technology in the curriculum, and examples of studies of applets and mobile technologies. The literature reviewed also allowed for a description of the development and theory that provided a basis for the instrumental approach, think aloud methodology, and three-worlds of mathematics. As continuity is the mathematical concept considered, relevant research regarding continuity was also reviewed.

III. Methodology

Introduction

This study sought to determine the properties of applets and the actions of students while using applets that foster the development of conceptual understanding for mathematics, by considering the following objectives:

1. Determine the particular characteristics of the Maplets for Calculus applets that promote student understanding of the mathematical concept of continuity of a function.
2. Determine the particular actions and strategies a student develops while using the Maplets, which promote the understanding of continuity.

Questions guiding this investigation included:

1. The Maplets for Calculus that present continuity exercises include interactive graphics, hints, “check” answer, and other features. To what degree do each of these features help promote conceptual understanding of continuity with respect to Tall’s *Three Worlds* (embodied, symbolic, and formal)?
2. Maplets on continuity also allow students to use multiple features simultaneously. Are there particular combinations of features, e.g. utilization schemes, students develop that lead to a more ‘formal’ understanding of continuity? Are there utilization schemes that inhibit understanding of continuity?

3. In addition to the computer and Maplet software, students were allowed the use of paper, pencil, and a calculator. Are there any other patterns of behavior or thought that students exhibited while engaged with the Maplets that promote/inhibit the development of conceptual understanding?

This chapter outlines the case study methodology that was used to investigate these questions.

Sample

Qualitative investigations generally involve in-depth study of relatively small samples selected *purposefully*. In this, Patton (2002) provides information guiding the selection of subjects for this investigation. Purposeful sampling includes determining those *information-rich cases* that yield a great amount of information relevant to the study being conducted. A *typical case sampling* includes representatives of a population that indicate average cases. Subjects are often selected with the cooperation of key informants who help identify who is typical.

A sample of seven high school AP Calculus (AB) students, who volunteered from each of two high schools in the northern region of South Carolina, participated in this study. Three students were enrolled at the high school where the researcher teaches; the other four from a high school in a neighboring district. The design of this investigation called for selecting students from two separate schools to ensure that at least half of the subjects were unfamiliar with and had not been taught by the investigator. The investigator had taught the three students enrolled at the school he teaches, but was not their teacher during the semester of data collection. The four participants from the neighboring district had no prior experience with the investigator. The most likely effect

of the inclusion of students from the same school was the ability to build rapport quickly; the inclusion of students from a second school allowed for some measure of control to effects that might be tied to investigator/student relationship.

Each teacher asked for student volunteers to participate in this study. From this group, the investigator, with the guidance of the teachers, was to select participants who represented a sampling of ‘typical’ students enrolled, as recommended by Patton (2002). However, upon asking for volunteers, only three students from the investigator’s school and four from the neighboring school expressed a desire to participate. The investigator decided to include all seven students in case one or more decided to remove themselves from the study. Additionally, the investigator decided to provide incentive for participants to remain in the study, a $25 gift card to a local department store.

The students included from the investigator’s school included two young men and one young woman. All three of these students were seniors. One student reported earning an ‘A’ the other two ‘B’s’ when asked about the grade they had earned in their pre-calculus class of the previous school year. The AP Calculus class in which these students were enrolled consisted of seven students; their teacher reported that the grades of these three students compared favorably with those of the other students in the class. The teacher also reported these three students, comparatively, represent a ‘typical’ sample of the class with regard to: male/female ratio, race, free/reduced meal status, and mathematics ability. The students included from the neighboring school consisted of two young women and two young men; three seniors and one junior. All four students had taken pre-calculus the previous school year: three of these students reported earning an ‘A’ and one a ‘B’. The class in which these students were enrolled consisted of 24

students; their teacher reported that the grades of these students were above average compared to those of other students in the class. The teacher also reported these students represented a ‘typical’ sample of the class with regard to: male/female ratio, race, free/reduced meal status, but were above average in math ability. As the intent of this study was to document individual changes in understanding the concepts of continuity while using Maplets for Calculus, the effects of the students’ school, gender, race, economic status, etc. while documented above, appear to be negligible to the results of the study. The use of seven students provided sufficient variability among the participants and yet allowed for the extensive data transcription and analysis needed to address this study’s research questions. Including more participants in this study may have been desirable, but financial costs and time considerations prevented this.

Application to the Institutional Review Board (IRB) of the University of South Carolina, and approval for the use of human subjects in this research was granted prior to the collection of data (Appendix A). The students who agreed to participate in this study were informed of the purpose of the study; consent of the students’ and their parent/guardians, in writing, was granted by each (Appendix B). Student participation was voluntary; they or their parent/guardian could choose to withdraw from the study at any time. Additionally, permission to conduct research in their facility and with their students was sought and granted by the principal of each school (Appendix C).

Data Collection

As Drijvers and Trouché (2008) explained, the challenge of investigating student understanding and of the development of mental schemes is that we cannot observe these directly; we are dependent on the interpretation of actions, oral reports, or written data

provided by the student (p. 371). In this, study, the think aloud method was used as the primary means of collecting data.

Other methods of collecting verbal data were considered for this study; however, each has inherent concerns for the validity of the data collected by the method. *Introspection* methods, in which a subject is asked to report on her or his cognitive processes, poses a problem to validity because subjects report of her or his thoughts post process. Concerns for validity of these verbal reports include memory errors and incorrect editing by the subject (van Someren et al., 1994, p.23). *Retrospection*, in which a subject is asked to verbalize her or his thoughts after completing a task, is offered as an alternative to the think aloud method. A benefit of retrospection is that it does not result in validity issues that may occur during concurrent methods because it does not disturb or interrupt the subject engaged in a task (p. 22). However, Wade (1990) discussed that problems associated with memory failure may result when verbal data is collected after completion of a task. *Prompting* involves asking the subject questions as to "why" they are using particular strategies, processes, or methods; it allows the investigator to explore specific aspects of a subjects' knowledge state at a given moment. A concern for the use of prompting is that the prompts require interpretation, which affects the problem solving process (Chi, Hutchinson, & Robin, 1989; Ferguson-Hessler & de Jong, 1990). Data collected during a *dialogue observation* can be voluminous; however, such data may be incomplete, as subjects may not discuss everything they are thinking during a conversation and that the discussion is not necessarily led by the participant (van Someren, et al., 1994).

The decision to forgo these other methods of data collection in favor of the think aloud method predicates an acknowledgement for the limitations of the method and measures needed to ensure the validity of data collected. K. A. Young (2005) presented issues regarding the use of think aloud data and ways in which these effects may be reduced. *Reactivity issues* refer to three consequences of asking a person to think aloud: 1) the ability to talk aloud and attend to a task simultaneously; 2) the effect of talking while performing a task normally done in silent; and 3) the effect of drawing a participants' attention to cognitive processes underlying the task being performed. Another possible limitation is the verbal ability of participants and/or their ability to verbalize their thoughts. Finally, the issues of validity of think-aloud data: does the data provided accurately reflect the thinking of the subject. Of particular concern in discussing validity is that while think aloud does reflect conscious thinking, it cannot reflect cognitive processes that never reach the level of consciousness (Wilson, 1994). Measures that can be taken to minimize these concerns include: practicing thinking aloud with a subject prior to the task that data will be drawn from; building rapport with the participant; subject selection; and combining data collection methods. Young advocates for the collection of both verbal and video data as a way to minimize concerns for validity due to incomplete information:

> This is where I have found the use of a combination of data methods essential. When one notes what appears to be a critical moment, a post-activity interview allows the researcher to delve into the moment to gain further insight...Although the learner may, at times, still be unable to provide further insight, it has proved for me to be a useful tool to tap into what may be unconscious during the time the

action/behavior is engaged but can be brought to consciousness when one is specifically asked to discuss the action/behavior. (p. 25-26)

While Young advocates recording video of subjects while responding, the researcher decided that videotaping subjects had the potential of inhibiting students' responses, by making them self-consciousness or more cautious. Furthermore, the researcher determined that data collected from video recording of students would not be relevant to the research questions. Heeding the concerns for validity of Young, this study did employed screen capture recording as well as audio recording while the think aloud method was employed.

The procedures outlined here are based on those recommended by van Someren et al. (1994), and were employed during sessions in which students worked with the continuity Maplets (full details of the session protocol is given in Appendix D). Sessions began with the student being: introduced to the observer, explained purpose of the research, and asked for permission to continue. Next, the student was introduced to the think aloud method and a "practice exercise" performed. Performed on paper, the observer asked the student to think aloud while solving a problem involving adding two fractions (e.g. $\frac{3}{4}+\frac{1}{3}$). This portion of the session was not recorded. The student was instructed to "say what you are thinking" while solving the problem and told that if they remained silent for twenty seconds they would be asked to "please keep talking." After practicing the think aloud method, the student was provided introduction and instruction to the Maplet: a description of the exercises of the Maplet, where the student was to input responses, and description of the features available to the student (i.e. 'hint', 'show', etc.).

Upon completion of these introductory steps, students were expected to complete the tasks presented by the Maplet while thinking aloud and recording began. As some of these tasks involved computation, paper and pencil was provided for the students as well as a graphing calculator. The researcher monitored the student in order to use the prompt, "please keep talking" when necessary, and to note (by indicating the time during the recording session) "critical moments" during the session using criteria described by van Someren et al. (outlined in the next paragraph). Consistent with recommendations by K. A. Young (2005), these sessions were recorded to capture student oral data and student actions on the computer screen. For this study Snagit, a screen capture software tool that both records spoken words and simultaneously records video of the computer's screen was used (TechSmith Corporation, 2013, March 27). The researcher also included notes of students' actions that related to the study's research questions (e.g., using paper and pencil to jot down ideas) and collected the written data students provided.

K. A. Young (2005), van Someren et al. (1994), and Ericsson & Simon (1980, 1984) suggest that a post-activity follow-up interview can provide meaningful and valid data if it is conducted soon after the activity session, preferably immediately afterwards. In particular, van Someren et al. (1994) describe using this combination of think aloud method to inform a retrospective interview for the purpose of illuminating: "pauses in the think aloud session or on fragments of the think aloud session that sounded incomprehensible, very incomplete or very odd. If possible this should be done directly after the think aloud session" (p. 27). Follow up interviews were recorded to ask for clarification about student thinking and to solicit the students' opinions about the features and experience of working with Maplets for Calculus.

Description of Maplets

Maplets for Calculus (M4C) is a collection of over 140 Maple applets (applets built using the Maple software) that provide "interactive graphical user interfaces for typical examples and exercises on a variety of topics in single-variable calculus" (Meade & Yasskin, 2012). Five Maplets have been developed for exercises about continuity of functions: Left and Right Limits and Continuity, given a Graph (*Continuity using a Graph*); Left and Right Limits and Continuity, given a Formula (*Continuity given a Piecewise Function*); Left and Right Limits and Continuity, given Numerical Data (*Continuity given a Black Box Function*); Continuity of Piecewise Defined Functions (*Finding the Value of C*); and The Epsilon-Delta Definition of Continuity (*Epsilon-Delta Continuity*). All five continuity Maplets were used in this study.

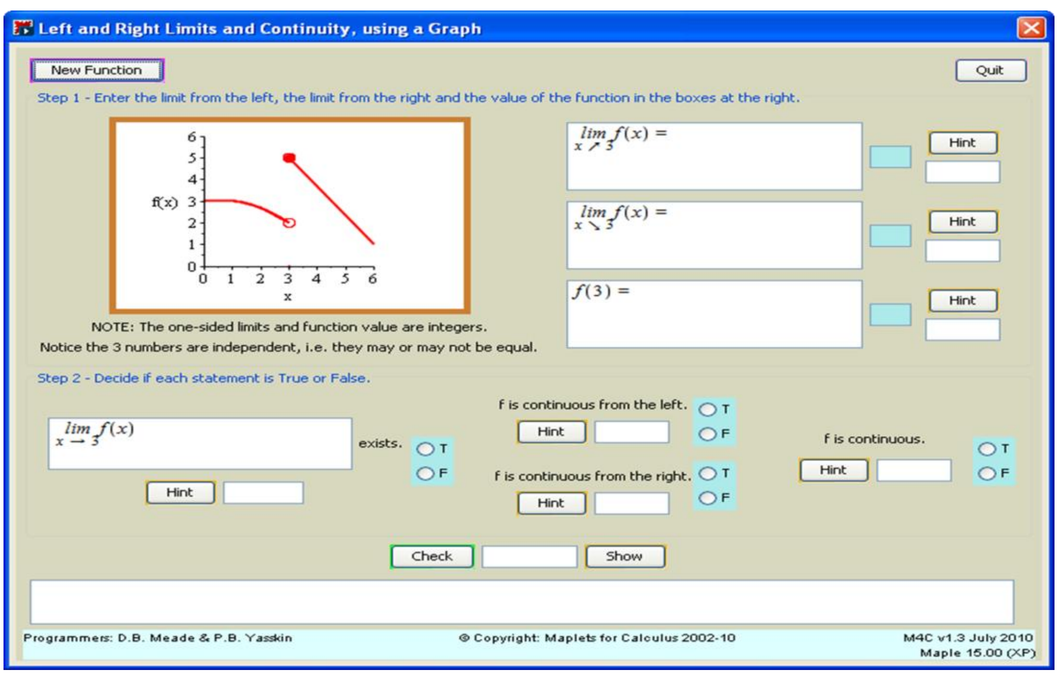

Figure 3.1 Beginning screen shot of *Continuity given a Graph* Maplet.

The *Continuity using a Graph* Maplet (Figure 3.1) provides the user with the graph of a piecewise function. In the first step, the user is expected to input the left limit,

the right limit, and the value of the function for a particular value of *x* (top half of screen shot presented in Figure 3.1). In the second step, the user is expected to answer a series of true/false questions regarding the limit of the function at the value of *x* indicated (usually where a break in the graph occurs), and the continuity of the function from the left, right, and overall (bottom half of screen shot in Figure 3.1).

Users can solicit help in answering a particular item by clicking the "hint" key. For the problem presented by Figure 3.1, the hint provided for finding the right limit of the function indicated, at the bottom of the screen, "The limit from the right is the height the graph approaches as x approaches 3 from the right" (Figure 3.2). Another feature of Maplets is that after entering responses to all questions, the user can check their answers. By clicking 'check, correct answers are highlighted in green and the word "correct" is displayed in a box near the answer; incorrect answers are highlighted in red with the word "incorrect" displayed (Figure 3.2). With Maplets that involve a number of answers, an overall evaluation is given at the bottom of the screen (in this example "incorrect" appears between the "check" and "show" buttons) only when all items are correct does the Maplet mark a particular exercise as correct and an affirming comment is printed in the text box at the bottom of the screen. (One example: "You're a genius. On to the next problem.") Users are able to use the "hint" and "check" buttons in combination, as shown in Figure 3.2. Another button available to users is the "show" button. Doing this provides the user correct answers to all questions and ends the exercise. The exercise also ends when the user correctly answers all items, the clicks the "New Function" button (presenting a new exercise), or "quits" the exercise.

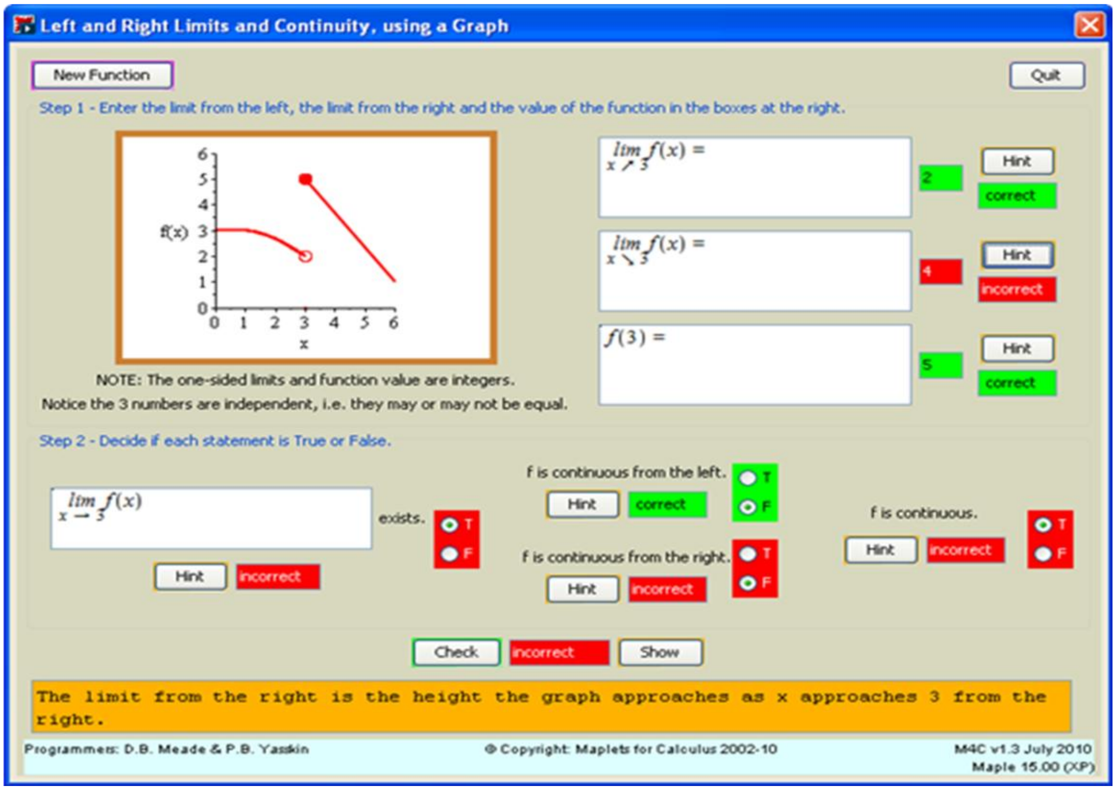

Figure 3.2 Screen shot of *Continuity given a Graph* Maplet after using 'check' and 'hint' for right

Descriptions of the other Maplets used in this study are provided in Appendix E.

Procedure for the Epsilon-Delta Continuity Maplet

The *Epsilon-Delta Definition of Continuity* Maplet provided a unique challenge to the investigator. As Maplets are intended for use as a support of classroom instruction, at the time of this study, none of the student volunteers had been taught the epsilon-delta definition of either limits or continuity. The investigator consulted with University of South Carolina professors Dr. Douglas Meade, mathematics, and Dr. Ed Dickey, mathematics education, to develop the protocols and activities used with this Maplet.

Student sessions began with a preview sheet that presented a cursory introduction to the epsilon-delta definition of continuity (Appendix F). The procedure included allowing students time to review this sheet. While this part of the sessions was recorded

(for audio) only students' questions regarding the information on the preview sheet were recorded. For consistency, the investigator declined to answer these student questions because the research design called for the instruction to come exclusively from the Maplet and instructions for the investiator would compromise the design. Students were then introduced to the *Epsilon-Delta Continuity* Maplet.

Figure 3.3 presents a 'screen-shot' of the *Epsilon-Delta* Maplet. Students were instructed to find a value of delta that satisfies the given epsilon condition for the limit provided by either moving the delta slider or entering values for delta into the accompanying 'δ=' box. The graph accompanying the diagram adjusts the vertical, epsilon, rectangle accordingly. Students can check their input for delta, as shown in Figure 3.3, and continue to enter values for delta. Upon finding a value of delta that satisfied the epsilon condition, the investigator asked students to find the largest value of delta that satisified the limit given by the Maplet. This process, finding a delta, then

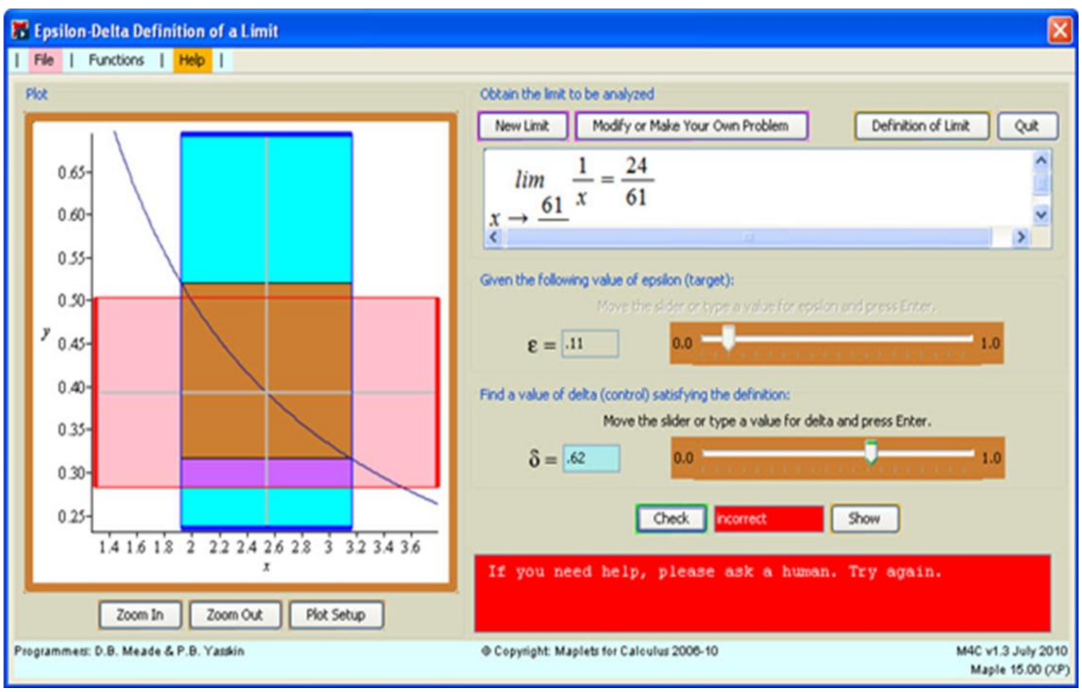

Figure 3.3 Screen shot of the Epsilon-Delta Maplet after 'check' for δ = .62.

finding the largest value of delta, was repeated for a second limit – one generated by the Maplet (using the ‘new limit’ button) but different from the first. In all cases, the first limit exercise used a limit whose graph appeared to be linear; the second limit appeared to be a curve in the graph feature of the Maplet.

The follow-up activity (Appendix F) presented the participants with a graph of a piecewise function with a break in the graph between the defined point (2, 5) and the open point (2, 3) (Figure 3.4). Students were asked to explain why the function was not continuous at $x = 2$ by using the epsilon-delta method of the Maplet. During this portion of the session, students had access to the preview sheet, the Maplet, the follow-up activity sheet, and writing instrument.

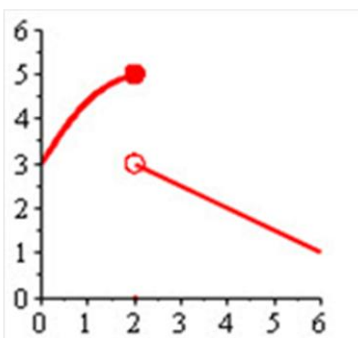

Figure 3.4 Graph used in the *Epsilon-Delta Continuity* follow up

All portions of the *Epsilon-delta Continuity* sessions were recorded, transcribed, and coded similar to the other Maplets. A separate analysis of these Maplet sessions was conducted to determine the features of the Maplet and other ‘tools’ (calculator, paper/pencil) used and understanding demonstrated for epsilon-delta continuity concepts.

Developing a Measure for Understanding Continuity

The *Three Worlds* of mathematics inform the development of a scale or rubric for determining student understanding of the concepts of continuity. Calculus topics are

blended in embodiment, symbolic, and formal worlds as was discussed by Tall (2008), which he exemplified with the concepts of limit and derivative. In this discussion, Tall presented a structure for analyzing understanding of derivative concepts developed by Hahkiöniëmi (2006). This structure provides for embodied and symbolic formation of the derivative concept and interactions involved in moving toward more formal understandings of the concept (Figure 3.5). Important to this study was the framing of understandings and the schemes necessary for developing greater understanding of the concepts of continuity. Developing a diagram for continuity similar to Hahkiöniëmi's for derivatives provided the basis upon which the verbal data collected was evaluated in determining students' understanding.

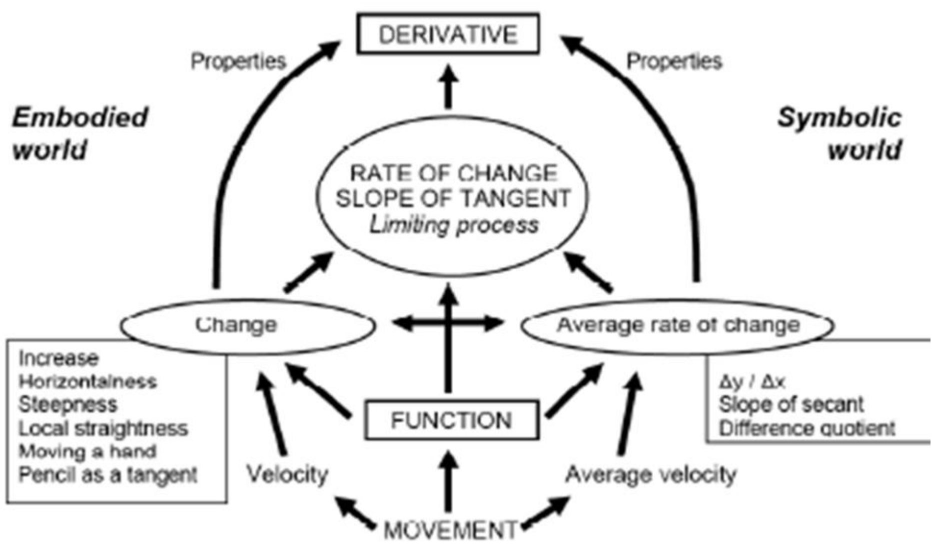

Figure 3.5 Hahkiöniëmi's hypothesized learning framework for derivative (2006).

Starting with the framework presented by Tall's diagram (Figure 2.2), the researcher, informed by Núñez et al.'s *natural continuity* as an embodied understanding and *formal continuity* as the symbolic understanding for continuity, and further informed by Chan's conceptual understanding rubric of continuity (Figure 2.3); the researcher

selected quotes or paraphrased examples from the student participants' verbal data to provide examples typical of the varying levels of understanding of continuity concepts (Figure 4.1). While the researcher has taught and studied calculus and continuity, to provide for the validity of this measurement tool, the investigator had this rubric reviewed for accuracy of continuity concept levels and accuracy of student verbalization indicative of understanding at these levels. Dr. Douglas Meade, professor of mathematics at the University of South Carolina, and Ms. Paula Adams, doctoral candidate (secondary mathematics education) at the University of South Carolina and AP Calculus teacher reviewed the rubric diagram and agreed that the draft presented was an appropriate representational framework for understanding the concept of continuity.

Analyzing the Data

Think aloud sessions were transcribed for oral data, coordinated computer activity, and written/calculator activity. For example:

Student: "Lets' see...if I follow this part of the graph...the limit appears to be 4" [input] left limit = 4

Or,

Student: "I think I need to compute the value at 3," computes function value at 3 on paper, "It's 5. I'll enter that." [input] right limit = 5

After transcribing, the researcher *segmented* the transcripts. Segmenting refers to the process of breaking up the transcript into segments or units. Van Someren et al. (1994) suggest that the combination of pauses and linguistic structure provide a general method for segmenting think aloud protocols (p. 120). Chi (1997) advises that segmenting can be performed based on non-content features, such as those suggested by van Someren et al.,

or on semantic features of the transcript, such as: argument chains, impasses met while problem solving; change in topic; etc. Combining segments along semantic lines of a particular idea or thought is referred to as an *episode* and considered to be a single element (van Someren et al., 1994, p. 120). This episodic approach was used in segmenting the transcribed data of this study.

Coding of the segments refers to labeling the units of the transcripts by categories and determining which segments will constitute evidence for a particular code. When used to study think aloud data the *coding scheme* represents the collection of all codes used to represent the model of cognition being studied (van Someren, et al., 1994; Chi, 1997). According to Chi (1997) this is dependent on the researcher's theoretical orientation, the questions being asked, the task, and the content domain (p. 12). Van Someren et al. (1994) suggest researchers develop a coding scheme for problem solving by taking every process described by the model and "state how you would expect these processes to appear in the protocols" (p. 121). Chi (1997) similarly states that codes can be developed from a taxonomic categorical scheme (p. 12). Chi and VanLehn (1991) developed a coding scheme of this type for investigating student use of textbook examples for solving physics problems. In this study, codes included: *concepts* to describe mentions of mass, weight, etc.; *principles* to code mentions of entities related by Newton's laws; *systems* for comments regarding the interaction of objects; and *technical knowledge* for algebraic manipulations of vectors. They used this coding based on their hypothesis that students could learn to solve problems without much understanding for the underlying principles or concepts. These categories were chosen to isolate the learning of concepts, principles, systems, and knowledge. Another study that involved

developing a coding scheme based on this deconstruction of a model for knowledge was Chi, de Leeuw, Chiu, and LaVancher's (1994) investigation to determine the mental models students have regarding the human circulatory system. The researchers coded based on the model of how the system works. Later in the study, they developed criteria for determining a student's mental model for the circulatory system based on collective oral data. For example, researchers inferred a student had a "Single Loop" model if a student made mention of *all* of the following during their verbal description: 1.) Blood is primarily contained in blood vessels; 2.) Blood is pumped from the heart to the body; and 3.) Blood returns to the heart from the body (p. 468). These models were hierarchical, with the complete model represented by evidence of all facts for the circulatory system.

These studies informed the construction and use of the model for understanding the concept of continuity (Figure 4.1) and served as the categorization and labels for coding student understanding in this study. Furthermore, Chi, de Leeuw, et al.'s (1994) study informs this research by the inclusion of criteria that must be present in order for a particular mental conception (model) to be evident in the verbal data.

In addition to determining student understanding of continuity, this study also sought to determine the actions or usage schemes developed while using Maplets that contribute to understanding continuity. Hoffkamp (2010) provided guidance for using protocol data to re-construct the development of mathematical understanding in students by viewing the data in chronological order. Working in pairs, students performed exercises related to the functions using applets in a dynamic geometry software package. Student groups were video recorded while engaged in the exercises to monitor both conversation and computer activity. By sequential analysis of the texts (transcripts) after

coding, Hoffkamp sought to determine, "the interpretation of the actions and conversations and the re-construction of meaning...based on the chronology of their appearance" (p. 15). Using this approach to analysis, Hoffkamp was able to document how students used their knowledge of functions while working on the computer activities to develop concepts of calculus; and to detect some obstacles in learning while working in the computer environment. Similar to Hoffkamp, this study sought to determine how the use of computer applets can contribute to the understanding of calculus concepts.

The research questions of this study required the coding of data for two particular situations: 1) the Maplet features used; and 2) the utilization schemes or strategies students used while working with the Maplets.

The coding schemata developed for the features used by students included documenting and labeling the feature used by a student during the Maplet exercises. For example, during the course of reviewing and transcribing the screen capture recordings, the investigator noticed that students often used a graph feature in a particular manner that involved using the cursor to 'trace' the graph. The investigator decided to label the use of the graph feature in this manner as GR-T for *graph-trace.* In coding the transcripts for the use of features, the following segment is representative of those labeled GR-T:

> [New Function] *graph w/ break*, *f defined on right side of graph*
>
> Uses cursor to trace graph from left and right to the break
>
> [input] left limit = 4, right limit = 5

This pattern of using the graph feature by tracing with the cursor was repeated by all students and was used across different Maplets. In similar fashion, other codes were identified and used. For example, the code GR-I was an abbreviation for Graph-

Interpretive that referred to the student use of the graph feature of a Maplet by mentioning it verbally in a response or in reasoning. Similarly HT-RL is coding for using the 'hint' feature for the right limit item of a Maplet. A complete list of codes for the features used by students appears in Appendix G.

Coding schemes for the utilization schemes students developed were determined by reviewing the screen capture recordings and transcripts and noting the patterns in which students combined and used features of the Maplets or combined Maplet features with the other 'tools' provided (paper/pencil and graphing calculator). Informed by Drijvers and Trouché (2008), these coding schemes were categorized as either *usage* (elementary) or *instrumented action* (more complex). *Usage schemes* were identified by the elementary use of one or two features in combination. For example, during review of the screen capture recordings and transcripts, the investigator would notice that after using the 'check' feature, a participant might then change an incorrect responses a true/false item without orally explaining the reason for the change. Also, there would be no apparent screen capture evidence of any further work. This pattern of using the 'check' feature of the Maplets followed by a 'change' in response was thus labeled CK-CG for *check-change* strategy. *Instrumented action schemes* move beyond the elementary use of features to a complexity that can be arrived at by combining features with reasoning, combining usage schemes, or combining features of the Maplets with other tools provided the students (paper, pencil, or calculator). An example of the development of an instrumented action scheme might involve the investigator noticing students on the screen capture recordings or from the transcripts, using the 'check' feature, being notified that an answer was incorrect, and then performing an intermediate

action before changing the input. These intermediate actions had either visual evidence on the recording, or the stated action of working with the other tools provided (students were asked to verbalize their actions while using paper, pencil, or calculator). One episode from the transcripts serves as another example of *Instrumental Action* that also illustrates the development of the investigator's utilization coding scheme is:

> [check] left limit, incorrect, "It's not -10"
>
> [hint] left limit, reads aloud, *The limit from the left…*
>
> Moves cursor to graph, "So the left is over here…you follow it," tracing graph left to right w/cursor, "as it approaches 2…so it's going down…it looks like it's towards -10", moves cursor to left limit, "but maybe it's just less than…-5"
>
> [input] left limit = -5

This episode was originally coded *check-hint-graph-change* to accommodate the intermediate actions and use of features by the student between the 'check' and changing of the response. However, after coding similar episodes with other intermediate student acts (e.g. *check-graph-change, check-calculator-change*, etc.), the investigator decided to collapse these codes to *check-rework-change* using the abbreviation CK-RW-CG. A complete list and description of utilization codes appears in Appendix G.

The examples from the previous paragraph detail the process used by the investigator to determine codes and apply the codes to episodes of the transcribed data. To ensure accurate coding, the investigator first encoded the data for features of the Maplets then encoded for utilization schemes. All encodings were based directly on the study's research questions. In reviewing transcripts and screen capture recording evidence for coding the features used, the investigator noted when a feature was used and

also provided more detail on the usage if needed (e.g. *hint→right limit*). The identification and coding of utilization schemes was determined by observing patterns of repeated use of combinations of features and strategies in the data. The repeated use of particular Maplet features, tools, and oral reasoning by students always caused the investigator to identify the episode with a particular utilization code. As with the feature codes, utilization codes described the sequence of features and/or actions of an episode, similar to the *check-hint-graph-change* sequence of the previous paragraph. Labels and description of codes developed for documenting the utilization schemes developed by students are presented in Appendix G.

Hoffkamp's (2010) work informed the analysis of the data. Using the framework developed for continuity in Tall's *Three Worlds* model, students' oral data was analyzed to document growth in understanding of continuity concepts. For example, one student initially described her understanding of continuity by referring to a graph or visual representations and this was interpreted as embodied understanding. Later, the same student described continuity in terms of the left/right limits, a shift to symbolic understanding. The investigator then reviewed the transcripts and screen capture evidence prior to this shift to note the features and strategies used by students during this episode. In reviewing the evidence from all students, the investigator analyzed the data to determine gains in understanding across all Maplets and to list the features and strategies that appeared to contribute to the particular instance of understanding (see Findings in Chapter IV).

In assessing and evaluating qualitative research, Merriam (2002) describes internal validity with the questions, "How congruent are one's findings with reality?"

(p. 25). Likewise, Lincoln and Guba (1985) discuss internal validity in terms of trustworthiness or credibility in using the qualifying question, "Do the findings capture what is really there?" (p. 290). Both suggest the internal validity of qualitative research can be improved through peer review. According to Merriam, "a thorough peer review examination would involve asking a colleague to scan some of the raw data and assess whether the findings are plausible based on the data" (p. 26). In addition to the issue of internal validity, both Merriam and Lincoln and Guba address the issue of reliability. In qualitative research, both refer to this issue as one of dependability or consistency, that is, "whether the results are consistent with the data collected" (Merriam, p. 27). Merriam and Lincoln and Guba again suggest that reliability can be enhanced through peer review and an *audit trail*. "An audit trail in a qualitative study describes in detail how data were collected, how categories were derived, and how decisions were made throughout the inquiry" (Merriam, p. 27). As Dey suggests, with qualitative research "we cannot expect others to replicate our account, the best we can do is explain how we arrived at our results" (1993, p. 251).

Dr. Jan Yow, Assistant Professor, University of South Carolina and Dr. Robert Petrulis, Principal Consultant, Evaluation, Policy and Research in Education Consulting each conducted a peer review of this study. Dr. Yow and Dr. Petrulis were asked to review this study based on their experience with qualitative research methods – including original research, officiating studies, and advising dissertations that employed qualitative methods. Each reviewed and provided the investigator with comments about representative excerpts from the raw data (screen capture recordings) to transcribed data and the development of coding schemes. They also reviewed the findings based on the

data for trustworthiness and consistency. Based on their review of the methods employed; representative excerpts from the raw data, transcripts, and developed codes; and their review of the investigator's findings based on the data, each concluded the study met their expectations for trustworthiness and consistency.

Summary

This chapter presented the research methods employed in this study. The methods included the selection of seven subjects from separate high schools enrolled in AP Calculus. All students who volunteered were included in the study; consultation with their teachers deemed these students as 'typical' representatives of students enrolled in their class. The method of data collection was documented to include use of, and validity of, think aloud protocols while students engaged in the Maplets for Calculus activities. Collection of data included the use of the screen capture software SnagIt to record student computer activity as well as oral data, including the follow-up interviews to Maplet sessions. A description of the Maplet for Calculus applets was provided using the Maplet *Continuity given a Graph.* As the Maplet *Epsilon-Delta Continuity* used a concept that had not been taught to students, a protocol and activities for these student sessions were developed with the help of mathematics and education experts. The process used to develop and validate a rubric for determining student understanding of continuity concepts with respect to Tall's *Three Worlds* Model was discussed. This rubric appears in the results of Chapter IV (Figure 4.1). Theory and examples from literature regarding the selection and application of codes and analysis of data was presented to justify the decision to code and analyze the transcribed data episodically for: demonstrated understanding of continuity using the three world rubric. Analysis of the data, similar to

Hoffkamp's, was conducted to determine the features and strategies used prior to students' documented gains in understanding.

IV. Results

Introduction

This study sought to determine the properties of applets and the actions of students while using applets that foster the development of conceptual understanding of mathematics, by considering the following objectives:

1. Determine the particular characteristics of the Maplets for Calculus applets that promote student understanding of the mathematical concept of continuity of a function.
2. Determine the particular actions and strategies a student develops while using the Maplets, which promote the understanding of continuity.

Questions guiding this investigation included:

1. The Maplets for Calculus that present continuity exercises include interactive graphics, hints, “check” answer, and other features. To what degree does each of these features help promote conceptual understanding of continuity with respect to Tall’s *Three Worlds* (embodied, symbolic, and formal)?
2. Maplets on continuity also allow students to use multiple features simultaneously. Are there particular combinations of features, e.g. utilization schemes, students develop that lead to a more ‘formal’ understanding of continuity? Are there utilization schemes that inhibit understanding of continuity?

3. In addition to the computer and Maplet software, students were allowed the use of paper, pencil, and a calculator. Are there any other patterns of behavior or thought that students exhibit while engaged with the Maplets that promote/inhibit the development of conceptual understanding?

These research questions provide the outline for the presentation of the results and findings in this chapter. First, as the conceptual understanding of continuity based on Tall's (2008) *Three Worlds* model is foundational to determining the influence of applets and strategies, the development of a comparable model specific to continuity concepts will be presented and justified based on the analysis of the data gathered in this study. Next, an analysis of student use of Maplet features, the frequency with which features are used, and examples of how the features are used will be presented. As suggested by Djivers and Trouché's (2008) *instrumental approach*, students involved in this investigation processed the features, functions, and "tools" of the Maplets and developed particular strategies, *utilization schemes*, while using these features for completing the exercises. The third section of this chapter will document these *usage schemes* and *instrumented action schemes*, based on the researcher's analysis of the data. In working with the Maplets during the recorded sessions, students were given access to paper, pencil, and a graphing calculator. Strategies and uses of these 'other' tools will also be included in the presentation of these utilization schemes. Each of the first four Maplet sessions concluded with an interview of the subjects. One objective of these interviews, suggested by K. A. Young (2005), was to clarify, for the interviewer, student thinking during particular instances of the 'think aloud' exercises. Another objective of these interviews was to ask the students for their perceptions and opinions of the Maplet and its

features. Student responses regarding features of the Maplets will be presented in a separate section. Maplets for Calculus are intended to be used as a support for classroom instruction. As the protocol for conducting the sessions with the *Epsilon-Delta Definition of Continuity* differed from the other Maplets used in this study, results from these sessions will be presented. The final section of this chapter presents the findings based on the data analysis.

Documenting Student Understanding of Continuity Concepts

The development of the model by which student conceptual understanding of continuity concepts is derived from research (i.e. Tall, 2008; Núñez et al., 1999; Núñez & Lakeoff,1998; and Chan, 2011) and from the verbal data provided by students as they thought out loud. The model developed provided the basis for evaluating the level of student understanding in Tall's *Three Worlds*: *embodied, symbolic and formal,* on the basis of verbal data given. First, the *Three Worlds* model and diagram of Tall (2008, p. 9, presented in the literature review) was re-drawn and revised to include the continuity concepts discussed by Núñez et al., Núñez & Lakeoff, and Chan. Figure 4.1 contains this diagram. The diagram features the categorization of the *natural limit* classification given by Núñez et al. as those understandings of continuity concepts that appear to be *conceptual* and *embodied.* Examples here include students stating a function is not continuous because of a break or jump in the graph. Primarily, student development of continuity concepts in the *embodied world* was documented in comments that primarily discussed the physical aspects of the graph of a function. In this model, Núñez et al.'s description of *definition continuity* categorizes the *proceptual-symbolic* representations of continuity based on limits and function values. Student data such as, "the function is

continuous because the left and right limit and the function value are all the same," suggests this student is using limits and function value as distinct items, *procepts,* to determine continuity. Chan's rubric for analyzing student responses to continuity items on a test provide for increasing levels of understanding of continuity concepts in *embodied, symbolic,* and *blended* worlds. The diagram in Figure 4.1 represents the culmination of the review of both the theory and analysis of the student data to present a model for understanding student responses of conceptual understanding of continuity within the three world's model.

Data Indicating Development in the Embodied World

Núñez et al. noted that advances in understanding can occur in the *embodied world.* In particular, Núñez & Lakeoff, in their presentation of the history of continuity, stated that prior to the late nineteenth century, mathematicians based continuity on motion of objects and based their advanced mathematical work on natural definitions of continuity. Analysis of the evidence suggests similar growth in reasoning while working with Maplets. One student who progressed into the *axiomatic-formal* areas of Tall's model related most all of his exercises to the physical and *natural* descriptions of continuity. For example, while using the Maplet *Continuity using a Piecewise Function*, this student proceeded in the following manner:

Given the piecewise function:

$$f(x) = \begin{cases} 5 - x & x < 4 \\ 3 & x = 4 \\ 3 & 4 < x \end{cases}$$

Student graphs $y = 5\text{-}x$ on graphing calculator and traces to $x = 4$,

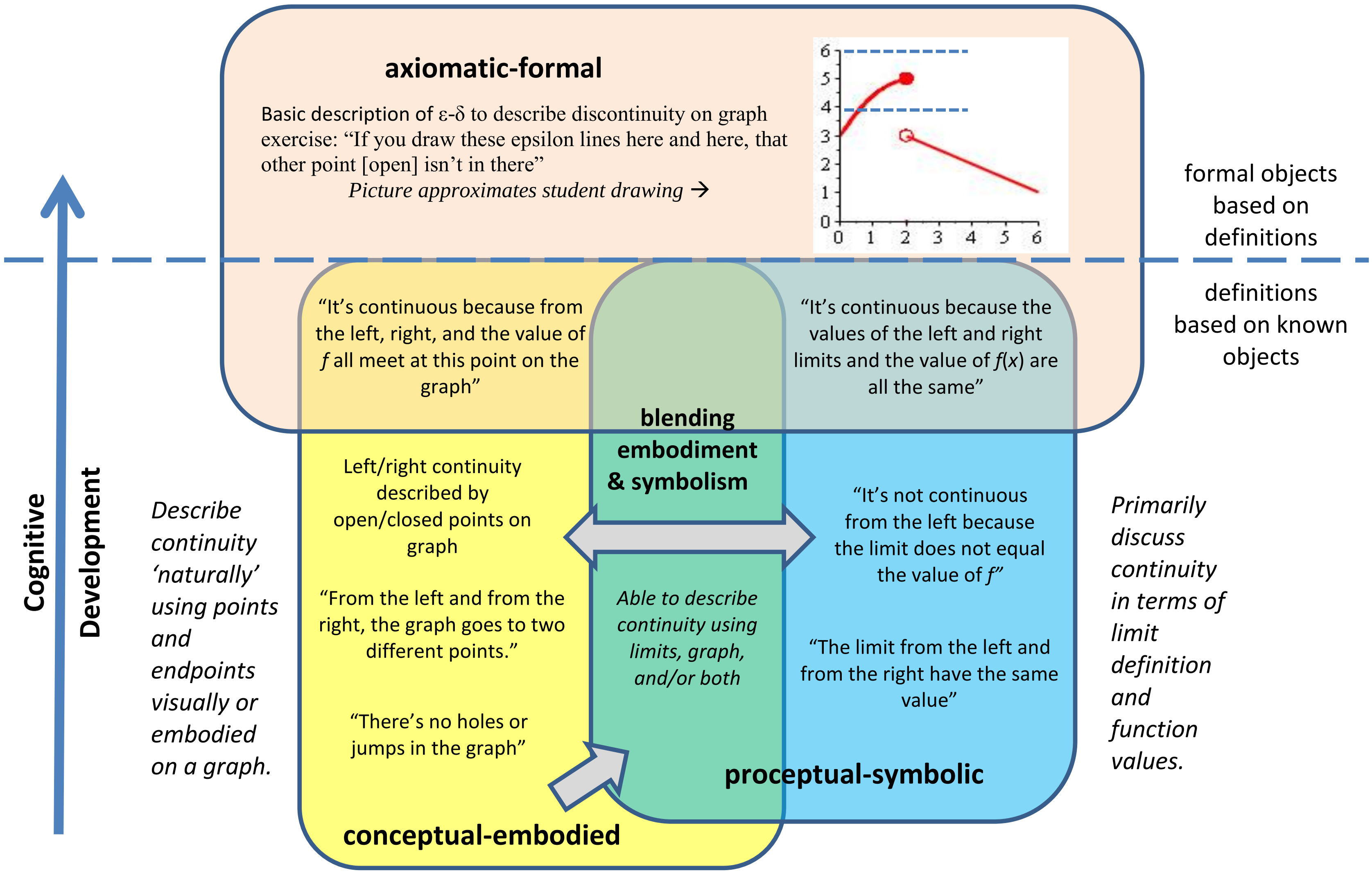

Figure 4.1 Students' level of understanding of continuity within David Tall's *Three Worlds* model.

[input] left limit = 1

Graphs $y = 3$ on calculator, traces to $x = 4$ and [input] right limit = 3

This student further went on to explain that the function had continuity from the right because the graph "would have a closed hole at that point," but left continuity did not exist because there was, "an open hole at that point." All of his responses were correct for understanding the concepts of left and right limits as well as continuity, however, whereas most students would use the function definition given to compute the *values* of the limits from the left and right by simply substituting into the appropriate formula of the piecewise function given, this particular student continued to, in his words, "visualize the graphs" even when asked to perform the exercise above without using the graphing function on his calculator.

Data Indicating Development in Symbolic World

The use of the actual values of the limits, indicate development in the *proceptual-symbolic* world of thinking. Comparing the values and equality and how these indicate whether or not continuity exists fall into Núñez et al.'s description of *formal definition* of continuity based on the Cauchy-Weierstrass definition. Student evidence of growth in this domain focused on the verbalizations that eschewed the physical descriptions for reasoning based on limits, their values, and at times, equality of these. One example, from a student working with the Maplet *Continuity given a Black-Box Function:*

Student moves cursor to 'limit of *f* exists' item.

"So the limit does exist since both [the left and right] limits are the same,"

[input] limit *f* exists = True

“Continuous from the left and right is false because they [the left and right limits] don’t equal what f (2) equals,”

[input] f is continuous from the left = False

[input] f is continuous from the right = False

Another example from the evidence indicating student understanding of continuity in the symbolic world is taken from student data while working with the Maplet, *Finding the Value of C:*

New function:

$$f(x) = \begin{cases} x^2 & x \leq 2 \\ 2x^3 + 2Cx & 2 < x \end{cases}$$

Student substitutes and computes limits on paper.

[input] Left limit = 4; [check] correct

[input] Right limit = -16-4*C; [check] correct

Interviewer: “I’m going to ask you to try this one without manipulating the graph or using the slider.”

Student: Hesitates, and then says, “I’m going to figure out what C is so that way they [the left and right limits] both equal 4.”

Writes equation -16-4C = 4 on paper and solves to get C = -5.

[input] C = -5 [check] correct

“So that’s how you do it without using the graph.”

In this episode, the student data indicates an understanding of the definition of continuity in that the left limit must equal the right limit. Using this, the student set both limits equal, wrote the equation -16 – 4*C = 4, and solved for the value of C.

Data Indicating Development Blending Embodied and Symbolic Worlds

Núñez et al. (1999) contend that difficulties in developing understanding of continuity concepts arise from their assertion that natural continuity and formal continuity definitions represent two different cognitive contexts – which in turn represent two different sets of concepts that students have to learn (p. 55). Tall maintained the *Three Worlds* model is particularly well suited for investigating calculus concepts, as most of these concepts have embodied and symbolic components. Predating Tall's *Three Worlds* model, Tall and Vinner (1981) discussed the difference indicated by Núñez et al. as that of *concept image,* the embodied idea of continuity representing a graph with no gaps, and *concept definition*, the formal definition of continuity based on limits. Tall's *Three Worlds* model accounts for the ability to work in, or move between the embodied and symbolic worlds – a section of his model between the two that he termed "blending embodiment and symbolism." Evidence gathered in this investigation that demonstrates a blending of embodied and symbolic descriptions for continuity included student statements invoking both a limit definition of continuity and physical descriptor of the same within a single statement or reasoning for action while engaged in a Maplet activity. One such example, from a student transcript for the Maplet *Continuity given a Graph*:

> [input] limit $f(x)$ exists = False, "because there's a jump from…the limit from the left and right, from 3 to 5"

In this particular example, the left limit of the function was 3, the right limit was 5, and the value of the function was also 5. In describing the right continuity of this example, this student explained:

[change] right continuity = True, "Actually, it is continuous from the right because the limit of *f(x)* as it approaches 3 from the right is 5 and the value *f*(3) is also equal to 5. So it is continuous from the right."

This student's response in reasoning for the existence of the limit at $x = 3$ and the continuity of the function from the right demonstrates an ability to move between both the embodied, "there's a jump", and the symbolic, "the limit of *f*(x)…is 5 and the value of *f*(3) is also 5", with regard to reasoning for these continuity concepts.

Data Indicating Development of Formal Thinking

Formal thinking as described in Tall's worlds may take form in either the embodied or symbolic world. This thinking presents itself when knowledge and properties about concepts are represented by axiomatic thinking. For this study, formal thinking was indicated in the evidence when students used the definitions, knowledge, or properties of continuity to explain reasoning for a conclusion they made while engaged in the Maplet exercises. One example of such reasoning from a student discussing the continuity for a particular function from the transcripts while using *Continuity using a Piecewise Function*:

$$f(x) = \begin{cases} 5 - \frac{1}{5}x & x < 5 \\ 4 & x = 5 \\ 13 - 2x & 5 < x \end{cases}$$

Reads function formulas, substitutes/computes/inspects aloud,

[input] left limit = 4, right limit = 3, *f*(5) = 4

[input] limit exist = False, "because the limit from the left and right don't equal"

{*Move to more formal.}*

[input] *f* is continuous from the left = True,

[input] *f* is continuous from the left = False,

[input] *f* is continuous = False, "because left and right aren't [continuous]"

{*Move to using left and right continuity to determine overall continuity of f.*}

In this episode, the student exhibited formal thinking for the concept of the overall continuity of the function as a self-developed axiom that suggests that a function is continuous, only if it is continuous from the left and from the right. This differs from the definition given by the Maplet hint for this overall continuity that states, "The function is continuous when the limit from the left, the limit from the right and the value are all equal." (Notice the hint describes continuity in terms of its limits, whereas the student described continuity in terms of left and right continuities.)

Another student example of formal thinking is indicated by the following example from the epsilon-delta follow-up activity:

"If epsilon was 3," draws horizontal line at y = 2 (Figure 4.2), "the graph would be continuous since the right part of the graph," points to open point (2, 3) on graph, "would be within [the epsilon width] of the left part," points to the closed point (2, 5).

"But for it to be continuous, it would have to work for all values of epsilon, no matter how small.

This example demonstrates this student's ability to formally describe the discontinuity of this example in a manner consistent with the epsilon-delta definition. Furthermore, it provides evidence of Tall's and Núñez et al.'s contention that formal thinking can be

developed in the conceptual-embodied world and is not necessarily dependent on proceptual-symbolic thinking.

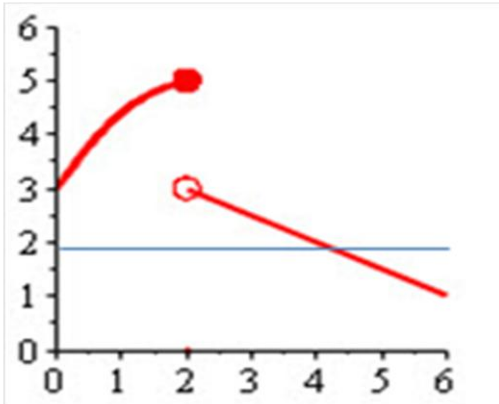

Figure 4.2 Graph from epsilon delta follow-up activity with example of student drawn line at $y = 2$.

Synthesis of Student Data for Understanding Concepts

Figure 4.1 provides a rubric for understanding student data regarding their understanding of the concepts of continuity in Tall's *Three Worlds*. This model includes examples of student responses that are indicative of levels of understanding of continuity concepts as outlined by Núñez and Chan within Tall's *Three Worlds* model to consider movement towards formal thinking.

The diagram in Figure 4.1 served as the rubric upon which the verbal data collected was evaluated for understanding of continuity concepts, as well as a chart of the progression or growth for understanding toward higher levels of comprehension related to the concept of continuity. The sections that follow will present data about the features students used and the strategies students developed while working with Maplets about continuity. While the data listed is relevant to answering the research questions, the model developed in this section informs and organizes the contribution to understanding that these features and strategies provide. The Findings section concluding this chapter

presents the evidence of understanding of continuity that students developed, using the rubric based on Tall's *Three Worlds* model along with evidence of features and strategies that appeared most helpful in developing these understandings

Features Used by Students Working With Maplets

The Maplets for Calculus that provide exercises with regard to continuity include numerous features and function operators that students may use while completing the exercises given. This section will present data based on the observations of the features students used as analyzed from the frequency with which these features/functions were employed while using the first four of the continuity Maplets. Table 4.1 summarizes these frequencies.

Table 4.1

Frequency of Use of Maplet Features

Feature/Function	Occurrences
Check	322
Graph/Slider/Black Box	193
Change	99
Hint	31
Show	24

Note. Data compiled for 7 students completing 158 exercises.

The most frequently used feature of the Maplets was the 'check' feature. This feature allowed students to determine if their input for answers to the exercises were correct. Upon selecting the 'check' button, the Maplet would display the word "correct"

highlighted in green or “incorrect” in red next to the student responses. The first three Maplets used in this study were designed so that the students could only use the ‘check’ feature after completing all seven parts of the exercise. For the fourth, finding the value of C, students could check each of the three inputs separately: left and right limit and value of C. In addition to displaying “correct” or “incorrect”, a message would be displayed in a box at the bottom of the screen. Correct messages included statements such as: “You hit the nail on the head. On to the next question.”; “Perfect. You're unstoppable. Try another function”; and “Cool beans! Try another step.” Incorrect messages included: “It takes a lot of wrongs to make a right. Please try again.”; “You're colder than a polar bear's toenail. Please try again.”; and “Sorry Charlie. Study the hints and answer again.” As seen in these few examples, the incorrect messages included suggestions to ‘try again’ and encouraged the students to ‘study the hints’.

The second most frequently used features were specific to each of the Maplets. This classification of features/function included the use of graphs, sliders, or the ‘black box’. Graphs were featured in the Maplets *Continuity using a Graph* and *Finding the Value of C*. Graphs provided for the cursor becoming a cross-hair when moved on the graph section of the computer screen, though it did not provide coordinates for the cross-hair (Figure 4.3).

Visual data (from screen capture recordings) that indicated a student used the graph included either using the cursor to move to a specific point on a graph or using the cursor to trace the graph, such as this example from the transcripts:

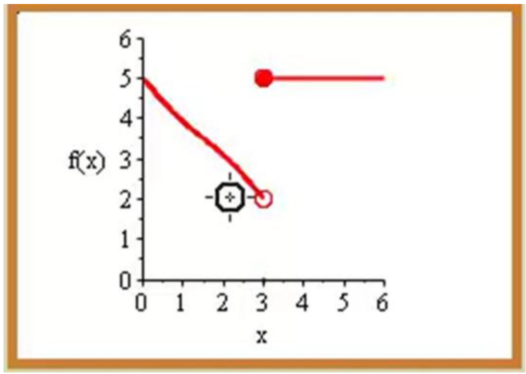

Figure 4.3 Screen shot of Maplet graph feature with cross-hair.

> [Student] moves cursor to the right limit, reads, "limit f(x) as x approaches 2 from the right," moves cursor to graph and traces from R to L, "follow this line.
>
> x = 4,"
>
> [input] right limit = 4.

Verbal data also indicated the use of the graph, as exemplified here:

> [check] right continuity is incorrect
>
> "Oh, it is continuous because there's a closed dot,"
>
> [change] right continuity = True, "that's why."

A slider was included in the *Finding the Value of C* Maplet. Movement of the slider would move one section of a piecewise function so that the two parts of the graph would become continuous. This allowed the user to estimate the value of *C* that would make the function continuous. The *Continuity using a Black Box Function* allowed a user to input a numeric value for the *x* variable and would output the value of the function. That these features were used so frequently appears to be indicative of the nature of the Maplet in which each was presented – answering the questions for limits and continuity depended on these features, thus students needed to use them.

Another key feature of these four Maplets was the ability to continue working on the same exercise and re-enter answers if the students' initial input was incorrect. In transcription, this was denoted as [change] to indicate a student changed a previous response. As indicated from the incorrect answer messages above, the Maplets encourage students to continue working on an exercise until it is completed correctly. One example of a student using the 'check' and then re-entering input, from the transcribed data using the continuity given a graph Maplet, is shown here:

[check] left and right limit are incorrect

"Ok, so I just got these mixed up."

[change] left limit = 1, right limit = 4

[check] all correct

"There we go."

The 'check' and 'change' features were used together in this episode and did become part of strategies developed by students. Data suggests that changing responses occurred more frequently early in the use of each Maplet, particularly the first three exercises or problems, because as students became more proficient in their responses, the less they needed to change their responses.

Data suggest the other two functions predominately placed on each Maplet screen, the 'hint' and 'show' buttons, were not frequently used. The 'hint' function presented users with a message in the box at the bottom of the screen, often a definition for the question they sought the hint for. An example of a hint for the question: "f is continuous from the left" (a true/false item):

> *The function is continuous from the left when the limit from the left is equal to the value of the function.*

As implied by its name, the ‘show’ function would display the answer to all items, in the case of the first three continuity Maplets, or a particular item in the case of the *Finding the Value of C* Maplet. One instance of student use of the ‘show’ function:

> “And then the limit x approaches 2…” moves cursor to right limit, “so that would be…”
>
> [hint] right limit, reads aloud, “The limit from the right is the value the function approaches as x approaches 2 from the right.”
>
> “So if C,” moves cursor briefly to graph, “Hmm…”
>
> [show] right limit = -8-2*C
>
> “Huh? Ok, what?”
>
> [hint] right limit, reads aloud, reads, “-8-2*C. How…does that work? …I have no idea…huh…so, let’s see”

Eventually this student used the ‘show’ function to develop an understanding that the limit needed to be an expression with *C* instead of a numeric value. Results on the use of these two features appeared mixed – in some instances they provided clarity and insight into how to proceed in an exercise, as was the case in this last example; in others, accessing the featured added to the confusion the student appeared to be experiencing when used. In follow up interviews, students who did not find the hints helpful responded in a manner similar to this student:

> “…they [the hints] do not help me at all, with the way they are worded. I was looking for a step by step procedure instead of a definition.”

While the ability to record instances for use of each of the above features is an advantage of screen capture and transcription review, one feature not mentioned previously was discussed by some of the students during interviews concluding each session. Five students reported the layout of the Maplet as contributing to their understanding. These students cited the order and organization of the problems from limits at the top (left/right and function value) and continuity on the bottom of the screen. “It was easy to follow,” noted one student. Another student commented, “The problems are done in steps and the questions organized in a way that was helpful to understand.”

Two other features of the Maplets appear to contribute to student understanding: the variety of problems presented by the Maplets and the directions/prompts given to students. Using the ‘new function’ or ‘new limit’ button presents the user with a new problem selected at random from a data base. Because students cannot determine which problem is presented, this feature was not included in the frequency chart in Table 4.1. However, the variety of problems does appear to contribute to the development of students’ understanding of continuity. The directions appear in blue type on each Maplet screen to guide the user through the activity. The prompts are presented in the ‘message box’ at the bottom of the screen after use of the ‘check’ or ‘hint’ feature. Reading and following the directions, as well as reading and following the suggestions of the prompts appear to contribute to understanding. As the directions and prompts could not be selected by the students, they were not included in Table 4.1.

In summary, analysis of data for student use of Maplet features has shown that the ‘check’ function was, by far, the most frequently employed by students while engaged in the exercises and problems presented by the Maplets, over one and a half times as much

as the next most frequent graphs, sliders, and black box function. The change feature was employed with less frequency as students progressed in their understanding of each exercise. Hints and the show functions were rarely used by students. Post session interviews with students indicated that in addition to these features, the layout/presentation of the Maplet problems and questions was another feature they considered important. Finally, analysis of the transcripts suggests that the variety of problems and the directions/prompts of the Maplets are two other features that may contribute to student understanding.

Utilization Schemes Used by Students Working with Maplets

Drijvers and Trouché summarized their instrumental approach framework with the equation: "instrument = artifact + scheme". In this, an artifact is the tool or technology that can be made into an instrument given an accompanying scheme for employing the artifact in use for solving a problem. In the case of Maplets for calculus, the artifacts can be considered the features and functions of the Maplets that may be used during the course of solving the mathematical problems that are posed. This section also includes the use of tools other than the Maplet: paper, pencil, and graphing calculator. Furthermore, Drijvers and Trouché distinguish between two types of *utilization* schemes: *usage schemes* and *instrumented action schemes.* In making this distinction, they describe a *usage scheme* as elementary level schemes that are often direct functions of the artifact. One example with Maplets may be the use of the check 'button' to determine if inputs are correct. An *instrumented action scheme* is a more complex scheme involving either multiple usage schemes or the incorporation of a particular usage scheme as part of a larger strategy for solving a problem. For example: one student used the hint feature of

the left and right limits whenever she was unclear about which limit the ‘arrow’ notation indicated. In this case, each hint started with the phrase, “The left limit is the limit of *f(x)* as …” that she used to determine where to input her response to the left limit items. In this use, the hint feature was part of a larger instrumented action strategy, which employed using the graph, formula rules, or the black-box function (this student used the hint for this purpose in each of the first three Maplets) to solve the problem of computing the left limit. (Note: this student did not use this strategy for the right limit, as once the left was known; the other limit *had* to be the right.)

Analysis of the data determined that students employed both utilization schemes in solving the exercises in the Maplets. Descriptions and examples of usage and instrumented action schemes will be presented in this section. Utilization schemes will be presented by the number of students employing each strategy. In addition to organizing this section by usage and instrumented approach schemes, some schemes appeared to be employed across multiple Maplets. Others appeared to be utilized in individual Maplets only. These divisions within the usage and instrumented action schemes were also documented during the analysis of the utilization schemes. A summary of schemes used by students while working with the first four Maplets is presented in Table 4.2.

Usage Schemes – Multiple Maplets

Data indicate that usage schemes employed by students across various Maplets included: tracing a given graph with the cursor to find limits; use of pencil and paper to

Table 4.2

Summary of Utilization Schemes Developed by Students Working with Continuity Maplets

Scheme	Usage (U) or Instrumented Action (IA)	Defining Characteristics	Number of students employing scheme (n = 7)
Graph-tracing	U	Trace graph with cursor	7
Paper/pencil	U	Use of paper and pencil to compute, make chart, etc.	7
Check-change	U	Use of check feature to change response without intermediate action by student.	6
Calculator-compute	U	Use of graphing calculator to compute values.	4
Slider to find C	U	Use slider feature to find C value before or after finding limits, or to check value of C.	6
BB-decimals	U	Input decimal entries into the black box function.	4
Use inequality symbols to determine L/R continuity	U	Reason that the strict inequality symbols presented in the *Piecewise* Maplet made left/right continuity false	3
Check-rework-change	IA	After checking, visual evidence of student work or use of features before changing input.	7
Check-reflect-change	IA	After checking, auditory evidence of student reflection of features or inputs before changing response.	5

Scheme	Usage (U) or Instrumented Action (IA)	Defining Characteristics	Number of students employing scheme (n = 7)
Prompts/directions	IA	Following specified directions and prompts of the Maplets.	3
Check-guess	IA	Checking and guessing answers repeatedly, without intermediate reflection/action.	3
Calculator-graph	IA	Use graphing calculator to graph formulas of *Piecewise* Maplet.	4
Hint-show	IA	Use of hint followed by use of show feature in *Finding Value of C* Maplet.	3
BB-whole numbers	IA	Input whole number entries into the black box function.	4
Function value – all true	IA	Input same response for limit and function value (from black box) and responding True to all continuity items.	3

compute, make charts, or notes; using the check function to change true/false responses (without reflection) and exchanging left/right limit responses that appeared due to the unique notation employed by the Maplets; and using the provided calculator to compute limit values.

Tracing the given graph to find the values of the left/right limits and the value of *f(x)* was employed by all seven students on two of the Maplets: *Continuity given a Graph* and *Finding the value of C.* Students used the given graphs (see Figure 3.1) in a manner exemplified in this data from the *Continuity given a Graph* Maplet:

[New Function]

Graph w/ break, f defined on right side of graph by closed point (4, 5).

Uses cursor to trace graph from the left and right to the break.

[input] Left limit = 4, Right limit = 5

[input] f(4) = 5, "Because of the open hole," circles the point (4, 4) on the graph w/cursor.

{*Appears to mean that because (4, 4) is open, f is defined by closed point at (4, 5).}*

All seven students used paper and pencil at some point during their sessions. Elementary uses included: computing limit values from piecewise function formulas (*Piecewise Function* and *Value of C*); solving equation for C (*Value of* C); making a table of values (*Black Box*); and writing notes regarding left/right limit notation (one student's attempt to keep the arrow notation correct). While these uses were elementary, some became part instrumented action schemes. Students used the 'check' feature in elementary fashion when employing this feature to interchange answers to left and right limit items, and to switch true/false responses without reason. This evidence suggests both uses of the check function:

[check] left and right limit; right continuity are incorrect

[change] exchanges left and right limit answers

"Man, I keep getting these mixed up!"

[change] right continuity from True to False

[check] all correct

This student exchanged left and right limit answers three times during the session from which this evidence was gathered, implicating difficulty with the 'arrow' notation used to represent left and right limits (this student also commented about this unique notation in

the follow up interview – “I’ve never seen that before.”). Additionally, this episode demonstrates the use of the check function for changing true/false items without explanation. In these instances (employed by six of the students) the change in response, appeared to be a ‘process of elimination’ – if one wasn’t right, the other was. More complex strategies involving true/false responses included student verbal reasoning for the change.

Finally, students used the calculator provided within the Maplet to determine the values of left/right limits in the *Piecewise Function* and *Value of C* Maplets. Data indicate that four students used the calculator to substitute values of x into piecewise formulas given in both Maplets and compute the values that determined the left/right limits.

Usage Schemes – Maplet Specific

Analysis of the data indicated the following usage schemes dependent to the Maplet employed: using the slider to find the value of C; entering decimals into the black box function; and using $f(x)$ defining inequalities to determine the left/right continuity in *Piecewise Function*.

Evidence indicated three distinct usage schemes for using the slider to find the value of C: use before computing the left and right limits (six students); use after computing the left/right limits (three students did this without setting the left/right limits equal); and using the slider to check the value of C computed by setting the left and right limits equal (two students). The graph/slider feature of this Maplet has directions above instructing, “Step 1: Estimate C by moving the slider” (appendix n). However, students used this feature to estimate the value of C as well as to estimate the left and right limits

of the function (outlined in the previous section). Usage schemes here are indicated by either manipulating the slider feature of the graph, or by entering values into the "*C =*" box below the graph. Changing the value of *C* shifts one side of the piecewise graph so that it either moves closer to the other side, eventually becoming continuous, or moving that piece further away and still discontinuous. Data indicating use of the slider to estimate *C* prior to finding the limits had students manipulating the slider and/or the "*C =*" entries until the graph became continuous. Data indicated that three of the students used the slider to determine the value of *C*, even after correctly inputting the values of the left and right limit. One such example from the data:

[New Function]

$$f(x) = \begin{cases} -3 - 3C & x \leq -1 \\ x & -1 < x \end{cases}$$

Substitutes 1 into each formula and compute, on paper, then

[input] Left limit = 3 + C [check] correct

[input] Right limit = -1 [check] correct

Moves slider and enters values into slider to estimate C = -4 makes graph continuous

[input] C = -4 [check] correct

In this example, the student used the slider to determine the value of *C*. Directions for the exercise state, "Step 3. Equate and Solve for C exactly." This student eschewed the directions and opted to use the slider as the primary method for computing the value of C. (Note: In subsequent problems, the investigator asked the student to complete the exercise without using the slider or graph.) One student, (not the one from the above example) never did use the limits to compute the value of C. The following example,

used by two students, indicates the use of the slider to check the value of C after setting the limits equal and computing as per the directions:

[New Function]

$$f(x) = \begin{cases} 2x^3 + Cx & x \leq -1 \\ -3x & -1 < x \end{cases}$$

Substitutes and evaluates left and right limits on paper

[input] Left limit = -2 - C [check] correct

[input] Right limit = 3 [check] correct

Sets limits equal and computes, on paper C = -5

[enter] C slider = -5, “to check,” student says.

[input] C = -5 [check] correct

Five of the students in this study used the Black Box function provided by the Maplet with decimals to determine the left and right limit values of *f(x).* Three of these students did so immediately upon starting the exercise. All three of these students were enrolled in the same class and indicated that they had completed similar exercises recently. Their data indicate the use of the black box to enter values for *x* (see appendix n) successively closer to the one indicated by the limit (i.e. for limit as *x* approaches 2, students entered 1.9, 1.99, and 1.999). Two of these students used paper to construct a chart, again similar to exercises completed in their class, to determine the left and right limits. Two students from the other school also used this strategy: one after asking, “Can I use decimals here?” and inputting a decimal; the other by ‘just trying’ a decimal input when frustrated with the exercise.

Three students used the inequalities of the formulas for the *Piecewise Function* to respond to the true/false questions, “*f* is continuous from the left/right”. Data suggest that

these students used the ‘strict’ inequalities in the formulas defining the function to determine that open points existed on the graph (a graph was not presented in this Maplet):

> “*f* continuous from the left, I would say false to that because they,” moving cursor between inequalities in function definition, “do not have the ‘equal to’ sign under them,” [input] Left/Right continuity = False.

Instrumented Action Schemes – Multiple Maplets

Data analysis revealed that the following instrumented action schemes were used by students in more than one Maplet: *check-rework-change, check-reflect-change, prompts/directions,* and *guess-check.*

The *check-rework-change* scheme presented itself in the data when students used the check feature followed by intermediate work, computation, tracing of the graph, employed a hint, etc. before changing an incorrect input. All seven of the students employed this scheme in some fashion during their Maplet sessions. This strategy considers the use of a variety of Maplet features. One example of a student using the black box function to rework a problem (Note: BB represents an entry into black box function; the arrow represents the output of the function.):

> [New Function] *limit as x approaches 5*
>
> [enter] BB = 5 → 5, [input] *f* (5) = 5
>
> “Now one below,” [enter] BB = 4 → 4, “So, still approaching 5,”
>
> [input] Left limit = 5

Later during this episode:

> [check] Left limit is incorrect

"Oh, ok, let me check."

[enter] 4 → 4

"Oh, I just checked the point; I didn't check everything above it."

"But I will check 4.9," [enter] 4.9 → 4, "Now it's getting closer to 4 instead."

[input] Left limit = 4

[check] all correct

"Ok, now I feel like I'm getting the hang of it."

This second example includes the application of a hint as part of the *rework* phase of this strategy:

*[check] Left/Right continuity are both incorrect.

Moves cursor to Left continuity.

Reads aloud the prompt at bottom of screen, "*I don't know where you went wrong. Study the hints and answer again.*"

[hint] Left continuity, reads aloud, "The function is continuous from the left when the limit from the left is equal to the value of the function."

[change] Left continuity = False,

"That would be false because the left limit equals 3 and the function equals 5."

[change] Right continuity = False, "And that would be false for the same reason."

[check] all correct

(*Note: This transcript also provides example of the *prompts/directions* strategy described below.)

The *check-rework-change* scheme presented itself in the data through the visual evidence obtained in the recordings: actual student actions could be seen via the use of functions, cursor movements, Maplet 'buttons' being selected, etc.

The *check-reflect-change* scheme considered the verbal evidence in the absence of visual evidence that indicated students were engaged in reflection upon their work or features of the Maplets in the interim between the check and change. Analysis of the data showed that five of the students employed this strategy while working with the Maplets. One example from *Continuity given a Graph*:

> [new function] *graph w/ break, closed point on right, open on left*
>
> "This is one of those jumping ones. So I know now that it doesn't exist,"
>
> [input] limit exists = False
>
> Moves cursor from Left and Right sides of graph.
>
> [input] Left limit = 2, Right limit = 5, f(3) = 5
>
> "It is not continuous," [input] Left/Right/*f* cont. = False, "at all"
>
> [check] Right continuity is incorrect
>
> "Oh! Okay! I see. There's no hole there. Okay, I got it now."
>
> [change] Right continuity = True
>
> [check] all correct
>
> {*The graph had a closed point at (3, 5) and an open one at (3, 2). She appears to realize how Left/Right continuity are defined on the graph.}*

Key to identifying this strategy is the verbal evidence, without visual evidence, which suggests student reflection prior to changing a response. In this example, the student's verbalization regarding the "hole" indicates the student reflected upon the graph feature

to determine that the closed point was indicative of continuity from the right side of the graph.

The *prompts/directions* scheme describes a strategy employed by three of the students who used either the directions provided in the Maplet exercises or the prompts given to them by hints or incorrect answers and followed the directives precisely. This evidence from the transcripts previously presented in discussing the *check-rework-change* strategy, includes use of the *prompts/directions* scheme. (Note: See previous transcript denoted with a *.) In five of the six documented occurrences, students employing this strategy correctly answered Maplet items shortly thereafter, as indicated in the example provided above. This strategy and example above also demonstrate Drijvers and Trouché's contention that instrumented action schemes can be included or combined to form other instrumented action schemes. Here, the use of prompts and directions became part of the *check-rework-change* scheme. Inclusion of the *prompts/directions* scheme is indicated by the evidence of its apparent effectiveness.

Finally, the *check-guess* strategy was employed by three students. This strategy presented itself in the evidence as guessing when a student appeared to be close to a correct answer, and when students appeared not to understand or reason at all regarding the answer. Most instances of using this strategy in situations regarding closeness occurred while using the Maplet *Finding the Value of C* in which students initially used the graph to estimate either the left or right limit, but because the scale of the y-axis prevented accurate assessment of the limit value, students estimated then used the check feature repeatedly to determine the limit. One student employed the guess and check strategy for finding the left and right limits during the *Black Box* session. Unsure of how

to proceed with the black box function, she realized the limits were positive whole numbers (from previous Maplet sessions) and systematically guessed (1, 2, 3 …) until use of the check feature confirmed a correct response.

Instrumented Action Schemes – Maplet Specific

Instrumented action schemes specific to Maplets include: *calculator-graph, hint-show, BB function value-all true,* and *BB with whole numbers.*

Calculator-graph scheme title is used for the strategy that involved students using a graphing calculator (provided during sessions) to graph the formulas that defined the *Piecewise Function* Maplet exercises. Four of the students used this as a beginning strategy for completing the limit and continuity questions. All students employing this strategy did so in a manner similar to this example:

[New Function]

$$f(x) = \begin{cases} 5 - x & x < 4 \\ 3 & x = 4 \\ 3 & 4 < x \end{cases}$$

Graphs y = 5-x on graphing calculator and traces to x = 4 to estimate value

[input] Left limit = 1

Graphs y = 3 on calculator, traces, and [input] R limit = 3

[input] f(4) = 3, "Because of the equal 4…3," moves cursor over middle formula in function.

Upon checking, the limit values and function value were correct. All four of the students who employed this strategy correctly determined the left and right limits. The investigator asked all students employing this method to work, "without using the calculator to graph" after they had completed one or two exercises in this fashion.

The *hint-show* scheme was used by three of the students during their sessions with the *Finding the Value of C* Maplet. This strategy consisted of using the hint feature followed by the show feature ('shows' the correct response). Each instance of the use of this strategy occurred while students attempted to compute the limit in which the C variable was part of the correct response, such as in this episode:

[New Function]

$$f(x) = \begin{cases} 5x & x < -2 \\ 2x - 2Cx & -2 \leq x \end{cases}$$

[input] Right limit = -37

[hint] appears to read silently, "Take the limit of the formula which is correct to the right of -2."

Traces graph w/cursor from R to L. Moves slider, then [enter] slider C = -1.5.

Graph appears to be continuous, "So that would be C, right?"

Uses cursor to circle -1.5 on C = slider

[input] C= -1.5

Moves cursor to Right limit, "I still don't know why that," moves slider to C = -5.

[show] Right limit = -6 + 6*C

"Ok. I haven't done this yet {*in class}*…it looks like they want it in…with C included in it…um…so…try the next one."

As this data indicates, the student did not realize that the limit included the variable C and the employ of this *hint-show* strategy presented this to the students.

Two particular strategies were developed by students while engaged in the *Black Box Function* Maplet. All four of the students from just one of the schools/classes

included in this study used the first of these schemes. *BB- whole numbers* is the moniker given to the strategy in which students used whole numbers as inputs into the black box function. Two of the students employing this strategy used the black box outputs to discern patterns that they used to determine the left and right limits. Of these students, one eventually asked, “Can I use decimals?” and when told “yes” by the investigator, the student quickly proceeded to use decimal entries progressively closer to the value the limit approached. The second of these two students continued to use the whole number and pattern strategy throughout the session and did so successfully. A sample of this student’s work:

[New Function] *limit of f(x) as x approaches 5*

After some initial ‘discovery’ activity to determine how to use the black box function, the student proceeded:

[enter BB] 4 → 11/5; 3→12/5; 2→13/5

“So it goes down by 1/5…so it’d be heading towards…4 is 11/5 and it be heading towards 10/5, which is 2,” [input] Left limit = 2, “I think.”

“So if we go to 5 from the right, that would be greater than 5. So if we go 6,”

[enter] 6→5; 7→7; 8→9,

“So it would be heading towards 3, if the pattern is the same.”

[input] Right limit = 3

Data indicates the other two students never gained proficiency for using the black box function or successfully completing the Maplet exercises, even though both eventually experimented with using decimal entries in the black box function. In follow up

interviews, each student indicated that, unlike the students from the other school, they had never done problems similar to this in class.

Three students employed the strategy of using the black box to determine the *f(x)* value for the given limit and using that as input for the left and right limits and the value of *f(a)* as well as responding with True to all continuity questions. This strategy, which was titled *function value-all true* by the investigator, developed as a beginning 'guess' for students that appeared unsure of how to proceed with the black box function:

[New Function] *limit of f(x)as x approaches 1*

[enter BB] 4 → 16/5, 8 → 4/5, 15 → -17/5, 2 → 22/5, 1 → 5

[input] Left limit = 5, Right limit = 5, *f(1)* = 5

"Does the limit of *f* exist? I guess so," [input] limit exists = True, "since left and right [limits] match up."

[input] Left continuity = True, Right continuity = True, "because all are 5."

{Appears to mean left/right limit and f(1) values.}

[check] all correct

That all of this student's responses were correct appeared to be a coincidence of the function given by this exercise. The same student employed the same strategy in the next problem set attempted with different results (some responses were incorrect).

Summary of Utilization Schemes

Analysis of the data for the first four Maplets used by students in this study revealed the strategies presented in this section. Using Drijvers and Trouché's instrumentation approach theory, both elementary, usage, and more complex, instrumented action, schemes were identified within the transcripts of students' verbal

and visual action. Definitions derived from the data, as well as exhibits of each from the data were presented in this section and are summarized in Table 4.2. Definitions were developed for strategies used across multiple Maplets as well as those developed for specific Maplets; the strategies also included the use of the calculator, paper, and pencil provide during the Maplet interview sessions. Results for the *Epsilon-Delta Continuity* Maplet will be considered in a separate section that follows the data on student interviews.

Student Interview Responses to Maplet Features

At the conclusion of each Maplet session, student interviews were conducted. Two of the questions posed to students during these interviews asked them to consider the features of the Maplet just completed. This section presents results of student responses to the questions:

- What features of the Maplet were beneficial to you?
- What features of the Maplet hindered you?

These questions, asked in order to provide feedback to the Maplet developers about students' impressions of the software, did provide data relevant to the research questions regarding the features of Maplets that contribute to understanding of continuity. These results are organized by those features that students reported as beneficial/hindered them in multiple Maplets (*General Maplet*), followed by the features of specific Maplets (*Maplet Specific*) student reported as beneficial/hindering.

General Maplet Features Beneficial/Hindering Student Understanding

Features students reported helpful:

Check Answer – All students reported this being a feature that helped them while using the Maplets (all students used this feature extensively during their sessions). One student commented use of this feature increased understanding of continuity from the left and right. The findings suggest this as an important feature of Maplets that contributes to understanding.

Layout/Organization – Five students reported the layout of the Maplets as contributing to their understanding, citing the order and organization of the problems from limits at the top (left/right and function value) and continuity on the bottom of the screen. One student responded, "The questions were organized in a way that was helpful to understand; I was able to connect the first three answers (limits) to get the answers on the bottom (continuity)."

Hints – Three students commented the hints were useful. One student commented, "At first I didn't know how to begin; I wasn't sure what to do with the numbers. But once I read the hints and thought about it, I realized I had to plug the numbers into the formula."

Features students felt hindered them:

'Arrow' Notation for left/right limit – Five students expressed displeasure with the 'arrow' notation used to express the left and right limits; most of these reported they would prefer it to be similar to their textbooks (superscript + or -). Students said that the notation led to confusion in determining where to enter their answer for the left and right limit.

Hints – Two students responded that the 'terminology' of the hints was confusing. One student responded that the hints didn't help, "They weren't clear and didn't tell me

what to do. They gave me a definition, but not an explanation about what to do. The hints in all of these (Maplets) make me mad!"

Check answer on True/False questions – One student responded that they really didn't help her understand why the problem was wrong; she just changed to the other response and re-checked. She suggested adding another option or using multiple choices.

Maplet Specific Features Beneficial/Hindering Understanding

Features students reported as beneficial:

Graph – Four students commented that the graph feature of the *Continuity using a Graph* and the *Finding the Value of C* Maplets was helpful. One student liked the ability to move the cursor over the graph and that it helped in determining the left and right limits of the function.

Show feature - Three students mentioned the 'show' button as helping them while using the *Finding the Value of C* Maplet. "It allowed me to figure out that *C* needed to be included in the limits, and that I needed to set the limits equal to each other," said one student. Another added, "It helped me to see that I had to plug the numbers into the equations."

The Black Box function – Three students commented positively about the 'black box' function. Students liked that the function computed values for them and that they could input multiple values prior to answering the questions. One student liked the challenge of the black box, "it really forced me to think about [the limits] in terms of function and not really try to think about a graph."

One student also commented about the variety of problems for the *Continuity Using a Piecewise Function* Maplet and one student commented on the error messages of the *Black Box* Maplet as beneficial.

Features students reported as hindering their understanding while using specific Maplets:

The order of inequality – Two students expressed confusion that the last inequality of the *Piecewise Function* and *Finding the Value of C* Maplets were expressed in reverse order. One student used an example in which the last formula of the function was expressed as $x^2 – 2x\ 4 < x$. This student reported mistaking the '4' in the inequality as being a part of the formula and used it in computing the value of that formula.

Instructions and directions – Four students commented about their confusion for not understanding how to use the *Black Box* and *Finding the Value of C* Maplets. One student commented on the misunderstanding of the type of numbers that could be inputted in the black box function. Another commented she couldn't figure out what to do without extensive use of the hints and an error message.

The next three features students responded as hindering their understanding are specific to the *Finding the Value of C* Maplet:

Syntax for input of C expression – Three students commented about the formatting for the limit expressions that included *C*: "I got confused by the capital '*C*' versus the little '*c*' and the "*" for multiplication," (the Maplet only accepted the capital form of 'C'.) and; "Even though I had the expression right, it told me it was wrong."

Graph - Five students made comments about the graph. These included complaints about the scaling: "The graph was pretty big, but I couldn't tell when the

different parts met up," "At times it looked like the graph was continuous, but it wasn't." Others expressed concern for the relationship between graph and the problem: "I didn't understand what [the graph] had to do with finding these limits" and; "The graph didn't help me to find the limits at all. Even when I got the two graphs together, I didn't know what it wanted me to do."

Table 4.3

Summary of student responses to interview questions about Maplet features.

Maplet	Beneficial Features	Features that Hinder
Continuity Using a Graph	• Check answer (7) • Organization/layout (5) • Hints – availability (3) • Graph (2)	• 'Arrow' notation of limits (5) • Hints – wording (2) • Check for true/false items (1)
Continuity Using a Piece-wise Function	• Check answer (3) • Organization/layout (3) • Hints – availability (1) • Variety of problems (1)	• 'Arrow' notation of limits (2) • Hints – wording (2) • No graph (2) • Check for true/false items (1) • Order of inequality (1)
Continuity Using a Black Box Function	• Check answer (5) • Black box function (3) • Hints – availability (2) • Error message (1)	• 'Arrow' notation of limits (2) • Instructions/directions (3) • Check for true/false items (1) • Hints – wording (1) • No graph (2)
Finding the Value of C that Makes a Piece-wise Function Continuous	• Graph and slider (4) • Check individual answers (3) • Show answers (3) • Hints – availability (2)	• Syntax of *C* expressions (3) • Graph (5) • Slider (7) • Instructions/directions (4) • Hints – wording (3) • Order of inequality (2)

Note. Parentheses indicate number of students reporting. ($N = 7$)

Slider – All seven students expressed concern for the slider. Their comments are best summarized by the student who stated: "I started by using the slider and just trying

to find the value of *C* on the graph in order to make it continuous. Eventually I figured it out that the left and right limits had to be the same. But with the *C* value in there, that threw me off for quite a few examples, I wasn't sure what to do. The values that I could input for *C*, I kept putting values for *C* in the blue box and knew eventually I could find the right value. If it wasn't for the slider, I would've focused on setting the limits equal to each other quicker. The slider handicapped me."

Two students also reported that the lack of a graph in the *Piecewise Function* and *Black Box* Maplets hindered their understanding.

Summary

This section presented evidence and analysis of student data gathered from interviews conducted at the conclusion of each Maplet session with the subjects. The presentation of this section has been organized by the student responses to features they found as beneficial to, or hindering of their completion of each Maplets exercises. Table 4.3 summarizes the student responses to the Maplet feature questions.

Results from the Epsilon-Delta Maplet

As described in Chapter III, protocol for student sessions with the *Epsilon-Delta Continuity* Maplet differed from that of the other four Maplets. In these sessions, students were asked to review an information sheet (Appendix F) prior to engaging in the Maplet exercise. Students were then prompted to use the delta slider to determine two or three values satisfying the epsilon condition for the given limit. After completing this exercise, the investigator then asked the students to use the Maplet to determine the largest value of delta that would satisfy a given epsilon condition for a limit. These Maplet exercises were then repeated with a second limit. Upon completing the Maplet

activities, students were given a follow-up activity prompting them to use the epsilon-delta method of the Maplet to explain why a piecewise function was not continuous (Appendix F). Data presented here includes analysis of responses while students were engaged in the Maplet and follow-up activity.

The data indicate that all seven students, using a guessing and checking strategy were able to find an initial value for delta that satisfied the epsilon condition. Students were then asked to find another value of delta, as instructed by the Maplet (Figure 4.4), and analysis shows that all seven were able to determine, within two attempts, a second delta (Note: not all prompts gave instructions to try smaller values of delta.). Additionally, data analyzed during this phase of the session shows that six of the seven students mentioned or made use of, via cursor movements, the graph feature of the Maplet – either noting the change in the delta band/rectangle on the graph, or the darker box defined by the upper and lower bounds of delta changing as the slider entries were changed.

The investigator then asked students to find the largest value of delta, to the nearest tenth, which satisfied the epsilon condition. All seven students successfully completed this task and were able to state that once finding the largest value of delta, any value of delta equal to or less than would satisfy the epsilon condition. Analysis indicate that five students stated that the largest value of delta coordinated with the 'shaded box' fitting within the epsilon band; furthermore, three students indicated that the function or graph of the function played a part in this value, as evident in this data:

> "It's [the graph of the function] close to a…line, which based on what we did with the first one…it [the maximum delta value] should be close to what epsilon is"

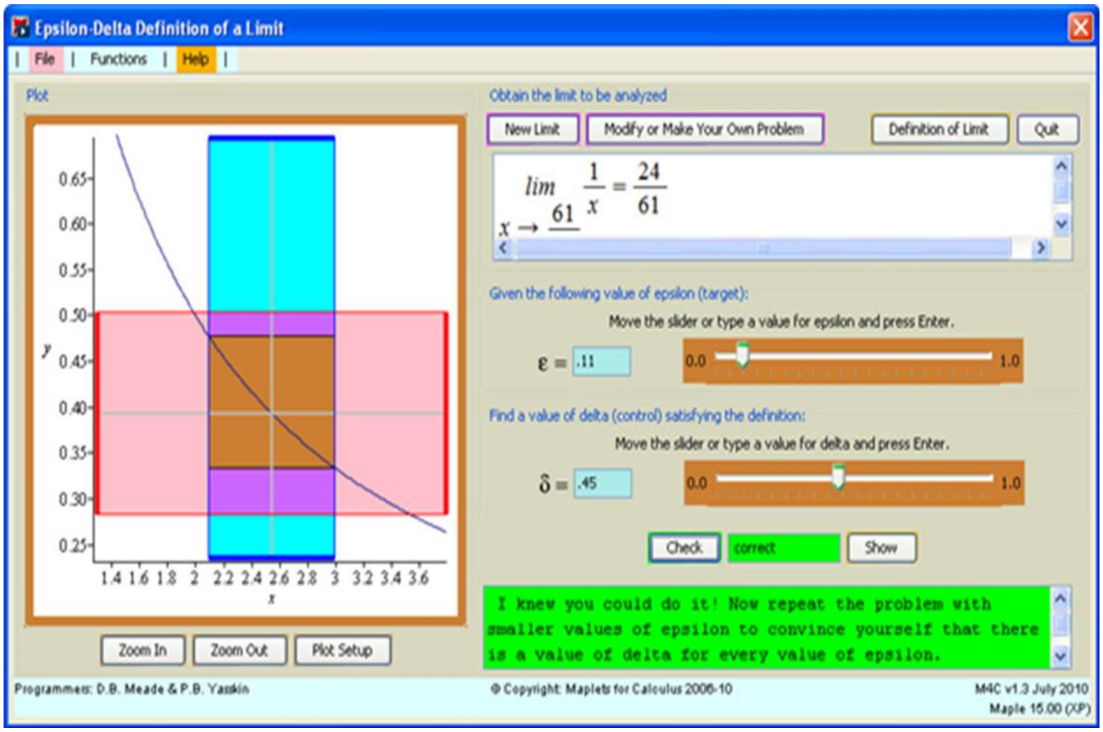

Figure 4.4 Epsilon-Delta Continuity Maplet after correct input of delta

The follow-up activity presented the participants with a graph of a piecewise function with a break in the graph (depicted in Figure 3.4); in which students were asked to explain why the function was not continuous at $x = 2$ by using the epsilon-delta method of the Maplet. During this portion of the session, students had access to the preview sheet, the Maplet, the follow-up activity sheet, and writing instrument. In analyzing this data, the students who referred or used each during their descriptions were noted. This data is summarized in Table 4.4.

Table 4.4

Items referred to or used in explaining discontinuity during Epsilon-Delta follow-up activity.

Item	Subjects employing item during explanation.						
	A1	A2	A3	A4	B1	B2	B3
Epsilon-Delta Maplet	Yes			Yes		Yes	Yes
Preview Sheet Graph	Yes	Yes		Yes		Yes	
Preview Sheet Definitions		Yes				Yes	
Draw ε or δ Bands on Follow-Up Sheet Graph	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Wrote Inequalities on Follow-Up Sheet		Yes	Yes				

Analysis of the follow-up activity also identified statements and/or actions that indicated levels of conceptual understanding of the use of the epsilon-delta method for describing continuity. Subjects' indicators of conceptual understanding included:

- Ability to describe or draw δ or ε bands on graph to include correct orientation of each (δ with *x*-axis and ε with *y*-axis).
- Draw δ or ε bands on graph and *include* numeric values for bands.
- Describe, by drawing on the graph, discontinuity in terms of the δ and ε bands not overlapping or "meeting up", similar in describing the 'shaded box' on Maplet graph.

- Describe, by drawing and explanation the discontinuity in terms of one point, usually (2, 3), not falling within the ε band of the other point.

Student use of these descriptions is presented in Table 4.5.

Table 4.5

Indicators of student understanding of continuity using epsilon-delta method.

Indicator	Subjects using indicator during explanation.						
	A1	A2	A3	A4	B1	B2	B3
Describe/draw δ or ε bands on graph	Yes	Yes	Yes	Yes	Yes	Yes	Yes
Draws δ or ε bands with values		Yes			Yes		Yes
Describe discontinuity by δ -ε regions	Yes					Yes	
Describe discontinuity using ε band and proximity of points		Yes		Yes	Yes		Yes

The last two of these indicators provide evidence of student understanding, if even at a basic level, for continuity in terms of the epsilon-delta method of the Maplet. The two students providing evidence for this understanding using epsilon-delta regions used reasoning similar to student B2:

> B2: “From what I see on this graph here,” pointing to the ‘shaded box’ on the graph on the Maplet/computer screen, “it’s looking for one specific region that satisfies where δ and ε can be; where they are ascribed to. And on this,” pointing to follow-up graph, “you’d have to account for two separate values for epsilon.”
>
> Interviewer: “What do you mean by that?”

B2: "I mean that you would have your epsilon regions right here and right here," draws two epsilon bands: above and below (2, 5); and above and below (2, 3), "and delta could remain the same, but your epsilon would have two separate values in accordance with that *x*."

Evidence indicating understanding of the discontinuity by using the epsilon-delta bands along with the proximity of the two key points on the graph include this response from student A4:

A4: "It could be that the closed hole is further away from the open hole if you looked at it as epsilon and delta."

Interviewer: "What do you mean?"

A4: "For example, in the Maplet in this graph," moves cursor to graph on screen, "epsilon and delta are usually within tenths or hundredths of each other. You won't find one that's from the Maplet, from what I've seen so far, you wouldn't find one," pointing to the graph on the follow-up activity, "that's this wide."

{Meaning between the two points – (2, 3) and (2, 5).}

Student B3 also used proximity in describing the discontinuity:

B3: "But on this," follow-up activity graph, "there's a gap. So if you made the epsilon, if you draw the epsilon like that," draws horizontal lines at $y = 5.5$ and $y = 3.5$, "This one," points to the open point (2, 3), "would be left out."

In summary, analysis of the data gathered while students engaged in the *Epsilon-Delta Continuity* Maplet and follow-up activity reveal students' thinking and understanding about this very abstract mathematical construct. Student results of working with the Maplet indicated all were able to determine values of delta that satisfied

the epsilon condition, as well as determine the largest value of delta that satisfied a given epsilon. Analysis of the follow-up activity included the tools and actions used by students while explaining the discontinuity of the graph, summarized in Table 4.4. Finally, data and analysis indicating conceptual understanding of epsilon-delta continuity was explained, examples given, and students demonstrating each understanding was presented in Table 4.5.

Findings

Excluding the epsilon-delta Maplet, 28 sessions were conducted with students engaged in Maplet for Calculus applets about continuity; 25 of those ended with students demonstrating proficiency in completing the Maplet exercises – proficiency being measured by the students' ability to complete three successive problems without error. Of the three cases in which students did not demonstrate proficiency, two involved the *Finding the Value of C* Maplet. In both of these, the students appeared to reach a point of frustration with the exercise and the investigator decided to end the session. The third occurred with the *Black Box Function* in which the student (one of the two who experienced difficulty with the *Value of C* Maplet) reached a level of frustration with the exercise. This student reported, "We haven't done anything like this in class," during the interview following the session. As students did not necessarily understand all parts of the exercises prior to beginning, development of understanding did occur during the Maplet sessions. In this section, the understandings students developed of continuity, using the rubric based on Tall's *Three Worlds*' model are presented along with evidence of features and strategies that appeared most helpful in developing these understandings.

In addition to these, findings regarding the development of procedural understanding and the slider-graph feature of the *Finding the Value of C* Maplet is also presented.

Student Understanding of Left/Right Continuity

One of the primary understandings gained from the first Maplet, *Continuity Using a Graph,* was the concept of left and right continuity. None of the students answered all items correctly during their first three problems in sessions with this Maplet. The most frequently missed items were the '*f is continuous from the left'* and *'f is continuous from the right'*; both were true/false items. Use of the 'check' feature showed that students' missed one or both items during their first three problems with the *Using a Graph* Maplet. Initially, students switched their response from the incorrect to correct answer, and then continued to the next problem. However, second and third attempts and errors in responding prompted students to investigate their reasoning by using the graph feature or the 'hint' feature to determine how to respond or correct their incorrect responses. All seven of the students eventually demonstrated understanding of left and right continuity; six employed embodied reasoning in concluding in a manner similar to the following student:

> [New Function] *Graph w/jump, closed at one end, open on other*
>
> Traces graph w/cursor
>
> "I'll start with that one," [input] f(2) = 4.
>
> "And, arrow's going up, so that makes this one 2, since it's coming up there," traces graph left to right w/cursor, then [input] left limit = 2. "And that makes this one 4," [input] right limit = 4.
>
> "They do not match up, so all these will be false," [input] left/right/f cont. = false

"And if I'm right, this is false as well," [input] limit exists = false

[check] right continuity is incorrect, pause

"Oh, it is continuous because there's a closed dot," [change] right continuity = true, "that's why."

This student, and others employed the 'check' function, along with an inspection of the graph, to 'change' the incorrect answer to the correct response by employing a developing understanding of left/right continuity in the embodied world by describing that the points on the graph were either undefined (open), indicating no continuity from that side, or defined (closed) indicating that continuity from that side existed. One student employed the 'hint' feature, as documented in this episode:

[check] left/right continuity are incorrect

Reads prompt at bottom of screen, "*I don't know where you went wrong. Study the hints and answer again.*"

[hint] left continuity, reads aloud, "*The function is continuous from the left when the limit from the left is equal to the value of the function.*"

[change] left continuity = false, "That would be false because the left limit equals 3 and the function equals 5."

[change] right continuity = false, "And that would be false for the same reason."

[check] all correct

This student, like the others, employed the 'check' feature, but instead of reviewing the graph, employed the 'hint' feature, then used the values of the limits to determine whether or not continuity from the left or right existed. This demonstrates development in the symbolic world for the use of limits in determining the left/right continuity of the

function. This student also employed a *check-rework-change* strategy in employing the hint.

The researcher noted that in both situations, the six students employing the graph to come to the embodied understanding and the one using limits; all seven needed at least three problems to become proficient in determining these answers. This suggests that the variation of problems also played a part in the development of this understanding – as some students answered correctly to these items in one problem, only to answer incorrectly to left/right continuity items in a following exercise. Only when the student gained understanding of these continuities, either embodied or symbolic, did they gain proficiency in completing the exercises.

Developing Understanding of Continuity in the Symbolic World

The sequence, layout, and variety of the Maplet exercises appear to be helpful in developing understanding of continuity in the symbolic world. The Maplets *Continuity Given a Piecewise Function* and *Continuity Given a Black Box Function* do not provide students with a graph. However, this did not prevent four of the students from beginning their session with the *Piecewise Function* Maplet by graphing the function on a graphing calculator to allow them to determine the left and right limits, an indication of these students' preference for working in the embodied world. After one or two problems, these students were asked to stop graphing on the calculator and they, along with the other three students, used the function to compute the left and right limits. Student explanations for their responses to the 'limit exists' and continuity questions either continued in the embodied world, or moved to blended explanations – incorporating

symbolic and physical descriptions. Data show that four students continued explaining in embodied terms and three used blended explanations such as the following:

> [input] true for all continuity items
>
> "Because there's only one value that *f* [and the left/right limits are] approaching, so obviously, there's no holes, or asymptotes, or jumps, or anything like that if you were to graph this."

Whether or not a student started by graphing the piecewise rules did not appear to determine whether or not students used embodied or symbolic descriptions as two of the students who started by graphing used blended descriptions, while the other two used an embodied description.

By the end of the *Black Box Function* Maplet, all seven students used either symbolic or blended descriptions for stating their reasons for answering the 'limit exists' and continuity questions. In reviewing the data to determine the features and strategies that promoted this change, four of the students had employed a 'hint' regarding continuity or the existence of the limit during their sessions with the first three Maplets. In two of the remaining three cases, it appeared that students connected their 'visualizations' of the piecewise and black box functions to the values of the left/right limits and the function. Each of these two students apparently were using the information from the piecewise and black box function visualize a graph of the function in a manner similar to this evidence from a student using the *Black Box* Maplet:

> "This makes me begin to suspect that it is going to look something like a piece-wise function…maybe approaching from the left at 5 and from the right at 4."
>
> [input] left limit = 5, right limit = 4.

[enter] BB = 3 → 5, [input] f (3) = 5.

"So I'm going to say the limit does not exist because I have a feeling that it's a piecewise based on what I've entered."

[input] limit exists = False

"I'm going to say it is continuous from the left because the limit as it approaches from the left and $f(x)$ are equal."

This episode documented the transition from the embodied to the symbolic use of the limits in determining the continuity of this function. Neither of these two students used the 'hint' feature – however, examination of their transcripts showed use of the 'check' feature as well as the strategy of *check-reflect-change* in addition to the variety of exercises leading to the development of this understanding. The seventh student demonstrated symbolic understanding of continuity and review of this student's *Piecewise* and *Black-Box* transcripts provided evidence that this student came to describe continuity symbolically through the use of the 'check' feature combined with the strategy of *check-reflect-change* and by completing more exercises than any other student using the *Piecewise* Maplet. This student attempted eleven problems using the *Piecewise* Maplet, four more than the most attempted by other students, even after being asked if she wanted to stop at an earlier point. Eventually, during the tenth problem of this session, the student determined:

[input] left limit = 2, right limit = 4, f(5) = 4.

"They're not continuous," [input] left/right/function continuity = False, "because they're not equal to," moves the cursor under the inequality 5<x.

{*Student appears to mean that the discontinuities are from the inequality being strict, and as stated by the student previously, 'there's a hole there'.*}

"And I think it exists," [input] limit exists = True

[check] limit exists, right continuity are incorrect

"Oh!", moves cursor to right continuity problem, "I understand the continuity part!" [changes] right continuity = True, and moves the cursor between the middle and bottom formulas, "because the 4 does not exist at 4."

[changes] limit exist = F, "because there's two of them." Moves cursor between the two 4's in the function, "Which is why you can't have …," long pause, "I'm pretty sure…now I understand why I got that one wrong."

[check] all correct

Investigator: asks about the reason for changing the right continuity response

"Because up here," the student moves cursor between the two 4's in the formula of *f*, "there's two points where it equals 4. And one of them, it does exist, which would be at x = 5," moves cursor over middle formula, "So it's continuous up until this point, because the hole, I guess, is filled."

This episode demonstrated that the student eventually came to the understanding that continuity from the left or right depends on the value of the limit being equal to the function value. In explaining how the, "hole, I guess, is filled," an embodied statement, the student demonstrated the ability to work in the symbolic world and explain in the embodied world. After this, the student quickly and correctly completed an eleventh exercise, and then asked to be done with the session.

Proceptual-Symbolic Use of Continuity to Find C

The first three Maplets required students to determine the continuity of a function using the left and right limits of the function. The *Finding the Value of C* Maplet worked from the premise that the function was to be continuous and students needed to determine the value of the variable C that made it so. This required a proceptual-symbolic understanding that if a function is continuous the conditions of the definition are also true. In particular, to complete this Maplet exercise, students had to come to the understanding that continuity implies that the left and right limits of the function are equal. Six of the seven students came to understanding – one that is necessary to find the value of *C* in this Maplet the way the developers intended. An example from one student's experience:

> Computes left limit by substituting x = -2
>
> [input] left limit = 2, [check] correct
>
> Computes right limit by substitution with x = -2
>
> [input] right limit =4*C + 24, [check] correct
>
> "So you set 4*C + 24 = 2," verbally solves the equation to get -11/2
>
> [input] C = -11/2, [check] correct

This episode is representative of the experience of the six students who found the value of the *C* variable by setting the left and right limits equal and solving.

The one student who did not use this approach, successfully found the value of *C* for all problems she attempted (seven) by using the slider provided. While the investigator eventually asked the other six students to work the problems without using the slider (discussed further in another section), this student was the first to complete the

Finding the Value of C Maplet session and used the slider during the entire session. By the end of the session, this student was able to determine correctly the left and right limits; however, the student did not set these limits equal to each other to find the *C* variable.

Formal Thinking: Development of a Rule for Overall Continuity

The first three Maplets presented to the students: *Continuity Using a Graph, Continuity Using a Piecewise Function,* and *Continuity Using a Black Box Function* all posed the same questions. In each, students were instructed to find the limit of a function from the left and right, and the value of the function. The Maplets then instructed students to answer true/false items regarding the existence of the limit of the function and continuity from the left, right, and the overall continuity of the function. As discussed in the previous sections, during the course of working through the first three, all students moved from an embodied view of continuity to either a blended or symbolic perspective by the end of the third session. However, during their work with these Maplets, students demonstrated formal thinking in their understanding of continuity and its properties.

Four students verbalized their thinking for the overall continuity of a function in terms of the continuity from the left and right. The operational definition for continuity, as given by the Maplet hint for the '*f is continuous'* prompt is: "*The function is continuous when the limit from the left, the limit from the right and the value are all equal."* This implies checking the values of the left and right limits and comparing them to the value of the function. However, these four students verbalized their understanding of continuity in terms of the left and right continuities, similar to this student:

[input] left/right continuous = True

"Continuous from the left would be true, and the right would also be true. Which means [the function] is continuous."

[input] f continuous = True

Similarly, another student reasoned a discontinuity in terms of left and right continuities:

"It is continuous from the right but not from the left,"

[input] left continuity = False, right continuity = True

"Which means it's not continuous overall."

[input] f continuous = False

Review of transcripts and strategy analysis showed that development of this rule for determining the overall continuity of a function appears to have come from students using the 'check' feature with the continuity items, as well as *check-reflect-change* with a variety of problems to determine that the item '*f is continuous'* was true only when '*f is continuous from the left'* and '*f is continuous from the right'* were also true.

Formal Thinking: Embodied Understanding of Epsilon-Delta Continuity

As noted in the presentation of data with the *Epsilon-Delta Continuity* Maplet, six of the students demonstrated the ability to describe continuity using epsilon-delta 'bands' similar to those presented in the Maplet. This suggests a formal thinking in the embodied world by being able to describe the discontinuity in terms of 'nearness' or 'closeness' of the left and right sides of the function presented in the follow up activity for this Maplet. All students used the graph/slider and the 'check' feature while working with the Maplet, as well as the *check-reflect-change* strategy when the investigator asked the students to find the largest delta that satisfied the given epsilon condition.

The Development of Procedural Understanding

Many of the strategies and feature of the Maplets helped students develop understanding of the procedures necessary for solving the Maplet exercises. These understandings fell into two categories: notation and formatting, and understanding procedures for solving the problems.

Various notations and conventions used in the Maplet presentation of the problems initially confused students, but understanding them was essential for students to complete the Maplet exercises. The 'arrow' notation used in the first three Maplets for left and right limits was one example of this. Most student errors for left and right limit were a result of putting the value of the left limit in the right limit answer space and vice-versa. Students used the 'check' feature and the 'hint' feature in determining the correct response space for left and right limits. Once familiar with this convention, most student errors because of this confusion dissipated, however, students still responded negatively to this notation in the interviews. Another notation that proved difficult for students was the presentation of inequalities in the formulas for the piecewise functions. In particular, the last rule in each function used inequalities of the form "$a < x$". Students reported this notation interfered with their progress in two ways. First, the notation led to confusion that this rule was the one needed to determine the right limit; and second, the numeric part of the notation, the 'a' number, was used by three students in computing the value of the right limit. In this, the spacing between the function symbolic statement and inequality for some of the exercises was so close that students mistook the number for part of the function statement. Use of the 'check' and 'show' feature combined with the *check-rework-change,* as well as the *check-reflect-change* enabled students to understand

this notation. All students experienced difficulty with the variable expression required for one of the limits during their use of the *Finding the Value of C* Maplet. In completing this Maplet, all seven students eventually determined the correct left and right limits, however, all struggled to determine that one of the limits would be an expression with the variable *C* in it. Furthermore, once students determined the variable expression, they all experienced difficulty in the formatting required by the Maplet. In determining that a variable expression was required, all seven students used the 'hint' feature. Of these students, five also used the 'show' feature using the *hint-show* strategy. This enabled the students to understand these answers would be expressions as opposed to values, as demonstrated here:

> [show] right limit = -6 + 6*C
>
> "Ok. I haven't done this yet [in class]…it looks like they want it in…with *C* included in it…um…so…try the next one."

A second challenge students experienced with these responses was the particular format of the answer. As seen in the example above, the program was particular in the way the operators were expressed and in the variable needing to be capitalized. During the investigation, two of the students used the 'show' feature to determine the appropriate format after their 'correct' responses were 'incorrect' according to the Maplet, with the help of the investigator. The investigator assisted the other students in entering their answers in the proper form.

Each of the Maplets included in this study required students to provide answers to problems about continuity. In the solving of these problems, most of the features and strategies used by students impacted the procedural understanding of 'how' to determine

the correct solutions on a consistent basis. As noted earlier, the 'check' feature and the ability to 'change' answers for each of the problems provided students unlimited opportunity to try different answers and strategies. Those students who understood the 'hints' quickly understood how to proceed with the items for which a hint was accessed. As mentioned in the previous section regarding the *Finding the Value of C* Maplet, the 'hint' combined with 'show' features enabled students to determine the correct form of the answer. This combination also provided students with the procedure for finding the limits from the left and right with the instruction from the hint, "*Take the limit of the formula which is correct to the left of -2.*" After using this hint students were soon able to determine that the procedure was to substitute *x* with -2 and compute the function value for that 'piece'. Many of the examples provided in the presentation of the *Features Used by Students* and the *Utilization Schemes Used by Students* sections exemplify the use of features and strategies for developing procedural understanding of the continuity problems given by the Maplets.

Graph and Slider – A Feature that Prevented Understanding

All seven students struggled to complete the *Finding the Value of C* Maplet. One of the reasons for this difficulty, outlined in the previous section, was that one limit was a variable expression that included the *C* variable. This difficulty was easily overcome once the students understood: 1) the correct answer was a variable expression; and 2) the correct formatting of the expression required by the Maplet. However, the graph and slider feature presented in this Maplet inhibited student understanding at the beginning of each of their sessions.

Directions for the Maplet required students to use the slider to estimate the value of *C* that makes the function continuous. From there, students were to use the given piecewise function to determine the left and right limits of the function, and then equate the two limits and solve for the *C* variable. All students began as instructed and were able to estimate the value of *C* that made the two sides of the graph come together in one continuous graph. However, after this, most students continued to use the *graph-trace* strategy to determine the left and right limits, as exemplified here:

> Reads left limit problem, moves cursor to graph, traces graph from L to R, "So it goes over here to 2, it seems to go down, and this is a -50," {*reading y-axis scale on graph}* "So it goes by tens. So we're looking at 2," moves cursor from y-axis, right, to f graph, "so it looks to be about -10"
>
> [input] left limit = -10
>
> [check] left limit, incorrect, "It's not -10
>
> [hint] left limit, reads aloud
>
> Moves cursor to graph, "So the left is over here…you follow it," tracing graph left to right with cursor, "as it approaches 2…so it's going down…it looks like it's towards -10", moves cursor to left limit, "but maybe it's just less than…-5"
>
> [input] left limit = -5, [check] incorrect
>
> "No."
>
> [hint] reads aloud, "I don't know."
>
> [show] left limit = -6, "Oh, so its close, you just can't really tell,"

This student used the *graph-tracing* strategy; however, the scale of the y-axis prevented an accurate determination of the value of the left limit. Other students used a similar

approach, even after repeated attempts at using the graph and continually getting feedback that there answers were incorrect.

One student did realize that use of the slider prohibited understanding of the procedure needed to successfully complete this exercise:

> Moves cursor to graph, “Approaching from the left, it would be at,” moves cursor along graph from left to right, “50”.
>
> [input] left limit = 50
>
> “Approaching from the right,” tracing graph w/cursor from right to left. Stops. Pauses.
>
> Moves cursor to top formula in f, “The C is in the top one now...,” moves cursor to left limit, “Yeah, it’s not right.” [deletes] left limit = 50.
>
> “Maybe if I try not using the graph.”
>
> {*Awareness that graph is not helping}*
>
> On paper, substitutes 2 into x for top formula and computes to get 12-2C.
>
> [input] left limit = 12-2*C {*with formatting help from investigator}*
>
> [check] left limit is correct
>
> “Ok, I’m going to solve this one on paper too,” moves cursor to bottom formula in f, “because I can’t tell what number that is.” Moves cursor to R limit on graph.
>
> Computes by replacing x with 2 to get -20
>
> [input] right limit = -20, [check] correct

After this problem, the student correctly used substitution to find the values of the left and right limits, however, the student continued to use the slider to answer for the value of *C* in all problems. This student was the first to complete the *Finding the Value of C*

Maplet session. In sessions with the other students, the investigator initially allowed students to complete the Maplet as presented, however, after two or three problems, he asked students to complete the exercise without using the slider or graph. Once the feature was removed, all students, through ‘hints’, ‘show’, and using the strategies of *check-reflect-change* and *check-rework-change* came to understand not only how to find the limits by substitution, but also how to determine the value of C by equating the limits.

Summary

The results presented in this chapter began with evidence and analysis that led to the development of a rubric for analyzing student understanding of continuity concepts with respect to Tall’s *Three Worlds* model while engaged in Maplets for Calculus exercises and activities (Figure 4.1). Oral data obtained from students during recording sessions provided examples of evidence indicative of particular levels of understanding. Next, an analysis of Maplet features used by students was presented to include the features used, evidence of how these features were used, and a record of the frequency of their use. Students used the ‘check’ feature most frequently (Table 4.1). The work of Drijvers and Trouché informed data analysis for the development of utilization schemes employed by students as they worked with Maplets. These schemes were presented as *usage schemes* and *instrumented actions schemes.* Descriptions and evidence from the data were provided for each as well as analysis for the number of students who employed each scheme (Table 4.2). From this data, it was shown that all seven participants in this study employed *Graph-tracing, Paper and pencil,* and *Check-rework-change* strategies. Student interviews conducted after Maplet recording sessions elicited responses from students to questions about the features they felt helped them or hindered them while

completing the exercises as well as providing insights into how the feature contributed to understanding of continuity. Response data to these questions was presented, along with the summary of Table 4.3. Analysis of this data showed that the *Check answer* and the availability of *Hints* features were the most frequently cited to be helpful while the *'Arrow' notation* and the wording of the *Hints* were most commonly cited as hindering their progress. The *Epsilon-Delta Definition of Continuity* Maplet contained mathematical content students had not been taught prior to the time this study was conducted, so results from the investigator developed activities for using this Maplet were presented in a special section of the results. This data analysis included: student use of the Maplet, engagement of tools used in answering the follow-up activity question, and presentation of student data and analysis indicating understanding obtained for the epsilon-delta definition of continuity.

The last section of this chapter described the findings of this study. These included: evidence about the features of Maplets and strategies used that developed students' embodied and symbolic understanding of left and right continuity; evidence for the sequencing of Maplets along with the features and strategies that contribute to understanding of continuity in the symbolic world; the development of proceptual-symbolic understanding of the definition of continuity to find the value of *C*; evidence of students using the concepts of left and right continuity develop a formal 'rule' for determining the overall continuity of a function; development of formal thinking in the embodied world for epsilon-delta continuity; the contributions of Maplets to procedural understanding; and the evidence supporting the finding of the graph/slider feature of the

Finding the Value of C inhibiting the development of understanding of continuity concepts.

V. Conclusion

Introduction

This final chapter begins with a summary of this study, then presents conclusions and discussion about the findings. Recommendations for the use and development of applets for teaching and learning mathematics will follow. Suggestions for future research conclude this chapter.

Summary

The recent development of the Common Core State Standards for Mathematics continues the advocacy of almost a century of mathematics educators' research that informs the community that teaching mathematics for understanding is beneficial to students. Additionally, technological advances of the late 20th and early 21st centuries have made computers and more specifically, mobile technologies readily available to students and teachers. As these technologies are dependent on applications, "applets," for human interaction, the problem statement for this investigation asked, "Is it possible to determine the characteristics of applets that lead students toward greater understanding of mathematical concepts? Can we determine specific actions and strategies learners develop while using applets that increase their understanding? In particular, which features of Maplets for Calculus lead students toward greater understanding of continuity of functions? Can we determine specific actions and strategies students develop while using Maplets that increase their understanding?" Student understanding of the concept of continuity, an under-represented topic in mathematics education literature, was the

topic considered in this investigation as students used the collection of computer applets Maplets for Calculus. The research questions guiding this study addressed the features of the Maplets for Calculus that students used and impact of those features on students' understanding of continuity, as well as the effectiveness of strategies students developed while working with these applets.

The literature review for this study presented historic evidence of the call for teaching mathematics for understanding. From Brownell's early study (1929) with "students having 'special difficulty' in arithmetic" in which it was documented that students taught number strategies and properties improved their performance on arithmetic tasks, to the NCTM reports of the 1980's and 90's, and more recent NRC and CCSSM, the emphasis on teaching mathematics for conceptual understanding continues. The influence of technology on teaching and learning mathematics, including the endorsement of NCTM, NRC, and CCSSM for using technology in mathematics education, as well as recent studies regarding the potential of mobile technologies for improving conceptual understanding were also presented. The instrumental approach theory, as developed by Drijvers and Trouché, provided the theoretical background for investigating the applet features and the strategies used by students working with the Maplets for Calculus. Development of this theory and studies engaging the instrumental approach were reviewed. Ericsson and Simon's formal development of the 'think aloud' method for collecting cognitive data and the protocols developed by van Someren, Barnard, and Sandberg provided the basis for data collection of this study. David Tall's '*Three Worlds*' model for the development of formal thinking in mathematics provided a theoretical background for investigating students' conceptual understanding of

continuity.   Research by Núñez et al. and others regarding the challenges of learning continuity concepts as well as studies of continuity concluded the literature review.

The methods employed during this investigation included recruiting seven high school students from two schools in northern South Carolina. These subjects were selected with the help of their AP Calculus instructors and the students were rewarded with a gift card for their participation. Each student met with the investigator for five sessions, working with Maplet for Calculus applets about continuity. During these sessions, screen capture software recorded the computer activity and oral data as students thought aloud while completing the Maplet activities on a laptop computer. Protocol for the student session with the *Epsilon-Delta Continuity* Maplet was modified, as it became apparent to the investigator that students had not been introduced to the epsilon-delta definition of continuity by their classroom teachers (Maplets are designed to support classroom instruction). Recordings were transcribed to record students' verbal data as well as their computer activity. The episodes were coded by the research for: Maplet features used, strategies used, and conceptual understanding. Analysis of the data included: development of rubric for documenting conceptual understanding of continuity concepts based on student verbalizations; frequency of Maplet features used by students; identification and description of utilization schemes, usage and instrumented action, and student use; and student interview reports regarding the benefits of Maplet features.

The data presentation included the rubric developed from the student verbal protocols to document growth in understanding of continuity in Tall's *Three Worlds*: embodied, symbolic, and formal. This data was summarized into a rubric diagram that was presented in Figure 4.1. Examples from the data regarding the use of Maplet

features were presented as well as the determination that the 'check' feature was most frequently used by students (Table 4.1). Further examples from the data were used to describe and identify both usage and instrumented action schemes developed by students. In addition to this identification, the number of students employing each scheme was presented (Table 4.2). Student interviews were conducted at the end of each Maplet session and students were asked for their opinion about Maplet features that were beneficial or detrimental to their learning. Students reported the 'check' and the 'hints' as most beneficial and the 'arrow' notation for left/right limits and the wording of the hints as most detrimental. These results are summarized in Table 4.3. Data and analysis of the 'epsilon-delta' Maplet showed that all students used the slider and 'check' features to successfully find values of delta that satisfied the given epsilon for the Maplet exercises and that all students used these features successfully to find the largest value of delta that satisfied epsilon condition for any given limit. Student use of 'tools' for answering the epsilon-delta follow-up activity were presented (Table 4.4). Analysis for student understanding of continuity using epsilon-delta revealed that all students expressed basic understanding of epsilon-delta continuity. The findings of this study included documentation of the gains in student understanding of continuity within the framework of Tall's *Three Worlds* along with the features and strategies used prior to student verbal expression of the understanding. These findings included documentation of: understanding of left/right continuity; understanding of continuity in symbolic world; proceptual-symbolic use of continuity to find *C* variable; formal thinking in developing a rule for overall continuity; formal thinking in embodied understanding of epsilon-delta

continuity; development of procedural understanding; and graph-slider feature of the *Finding the Value of C* Maplet inhibiting understanding.

Conclusions and Discussion

The research questions of this study were:

1. The Maplets for Calculus that present continuity exercises include interactive graphics, hints, “check” answer, and other features. To what degree do each of these features help promote conceptual understanding of continuity with respect to Tall’s *Three Worlds* (embodied, symbolic, and formal)?
2. Maplets on continuity also allow students to use multiple features simultaneously. Are there particular combinations of features, e.g. utilization schemes, students develop that lead to a more ‘formal’ understanding of continuity? Are there utilization schemes that inhibit understanding of continuity?
3. In addition to the computer and Maplet software, students were allowed the use of paper, pencil, and a calculator. Are there any other patterns of behavior or thought that students exhibited while engaged with the Maplets that promote/inhibit the development of conceptual understanding?

This section will answer these questions in light of the findings presented in Chapter IV and the research and theory informing this study presented in Chapter II.

Features of Maplets that Promote Understanding

From the findings, it is evident that the ‘check’ feature and the ‘change’ feature contributed to conceptual and procedural understanding of continuity concepts. The instrumental genesis for use of these features developed quickly in students’ Maplet

sessions. Each student used the features once or twice before the use of the features reached the 'instrument' level. As discussed by Drijvers and Trouché (2008) an artifact, in this case the features of 'check' and 'change,' only becomes an instrument when combined with a scheme (p. 368). Once acclimated to how these features worked, students used them repeatedly in their attempts to answer the exercises correctly. That these two features also became part of more complex instrumented action schemes provided evidence that the features were important to students. The use of these two features apparently allowed students to try different strategies, and encouraged them to either reflect or change course of action when their answers were incorrect.

The 'hint' feature promoted understanding of concepts and procedures. However, the hints did not always help. In interviews, some students expressed concern about the wording of the hints as being "too formal" and difficult to understand. Students who did understand the hints, appeared to gain in procedural and conceptual understanding, as following the use of the hint, they were able to proceed correctly and express orally understanding of the question, item, or concept for which they sought the hint, as exemplified in this portion of data presented previously in Chapter 4:

> [hint] left continuity, reads aloud, "*The function is continuous from the left when the limit from the left is equal to the value of the function."*
>
> [change] left continuity = false, "That would be false because the left limit equals 3 and the function equals 5."

This student gained the symbolic understanding of left/right continuity by using the hint for left continuity, as well as procedural understanding needed for finding the correct answers to subsequent items. Even students who failed to understand and therefore did

not use the hints reported (in interviews) that they liked the idea of having hints available. In this situation, students understood the potential for the use of the 'hint' feature, however, they lacked the understanding required to apply this tool to the given problem situation. In terms of Vygotsky's "instrumental method," a tool only becomes an instrument in the act of being used (1981, p. 137); for students who did not understand them, the hints remained an unused tool.

The variety of problems presented by the Maplets contributed to student understanding of the concepts of continuity. This appears to be the most important factor in students developing an understanding of the concept of left and right continuity. As presented in the findings, all seven students needed at least three exercises before becoming proficient in answering these exercises. While not a 'tool' that could become an 'instrument' to the students, the effect of this Maplet feature forced students to reconsider their hypothesis for answering the continuity items by presenting functions with different left/right continuities.

The graphs, black box function, and sliders features contributed to understanding of continuity concepts in three of the Maplets: *Continuity Using a Graph*, *Continuity Using a Black Box Function,* and *Epsilon-Delta Continuity*. Gains in all three situations depended on the Maplet being used at the time. For *Continuity Using a Graph*, the graph feature (Figure 3.1) allowed student to develop understanding of continuity concepts and procedures in what Tall (2008) referred to as a concept-embodied world, in which physically sensed properties (visually in this case) become part of a mental image. Procedurally, students used the graphing feature to find the left/right limits and the value of the function. Conceptually, students described continuity from the left/right by using

the open or closed points of the graph to describe why the left/right continuity items were true/false. The *Black Box Function* provided students with another method to determine the left/right limits and the value of a function – helping students understand the procedure. In providing only numeric outputs, this function allowed students to grow in a proceptual-symbolic sense, as they were expected to apply the definition of left/right and overall continuity in order to answer these true/false items correctly. The graph/slider of the *Epsilon-Delta Continuity Maplet* provided a basis for student conceptual understanding in the axiomatic-formal world by developing an embodied sense of epsilon-delta continuity.

The 'show' feature, used primarily in the *Finding the Value of C* Maplet contributed to student understanding of the procedure needed to find the value of *C* by presenting students the variable expression required for the left/right limits and also displaying the exact symbols required for the variable expression. From this, students appeared to be able to determine that the values needed to substitute into the formulas of the function to solve the equation. While not used extensively, the researcher believes this is a feature students could be encouraged to employ more frequently, especially when students become frustrated in trying to determine correct ways to answer items.

Utilization Schemes that Promote Understanding

The findings show that the strategies of *check-reflect-change* and *check-rework-change* promoted growth in conceptual understanding of continuity concepts. These two strategies are both what Drijvers and Trouché (2008) classify as *instrumented action schemes* in that they incorporate a combination of tools or strategies. The *check-reflect-change* strategy indicates students did not perform any intermediate work prior to

changing an incorrect response, instead, the student verbalized internal thinking about the problem or the process used to solve. The investigator intentionally identified the strategy *check-rework-change* to include the various other ‘tools’ and strategies students employed while using this scheme. Features of the Maplets included in the ‘rework’ phase of this strategy included: graphs/slider/black box function, hints, and show. Tools other than the ones provided by the Maplets used in this strategy included: paper, pencil, and a calculator. Strategies incorporated in the *check-rework-change* scheme included: *graph-tracing, paper/pencil, calculator-compute, slider to find C, BB-decimals, calculator-graph,* and *hint-show.* These two strategies accounted for the growth in understanding of left/right continuity when using the graph feature for an embodied understanding, and accounted for symbolic representation/verbalization of these continuities when using the piecewise and black box functions. The growth in symbolic understanding of continuity by finding the value of *C* came from the *check-rework-change*, as students used the *hint-show* during the rework phase to determine the form of the left/right limits as well as understanding that these limits needed to be equated in order to solve for *C*, thus promoting the proceptual-symbolic understanding of the definition of continuity (i.e. that a function is continuous implies the left limit equals the right limit). Both strategies were enabled by the feature of the Maplets that allowed students to continue working on a problem indefinitely.

Though employed by only three students, the *prompts/directions*, an instrumented action scheme, appeared to have potential for helping students with their understanding of continuity concepts. Each Maplet featured directions, numbered, for users to follow in completing the Maplet exercises. Upon answering a question incorrectly, a prompt

would appear in a box at the bottom of the screen suggesting a next step to the user (i.e. "*Try looking at the hints and answering again."*). Evidence of students using this strategy included verbalization of the direction or prompt, followed by student action suggested by the prompt. Only one student consistently read the directions and prompts; two other students did so occasionally. Here is an example one student's use of directions and the prompt given by a hint:

> [new function] *open on left, open on right, and closed 3*rd *point defining f at 3*
>
> Reads direction at top of screen, "Step 1. Enter the limit from the left, the limit from the right, and the value of the function in the boxes at the right."
>
> Moves cursor to left limit, "So the limit of f(x) as x goes to 2,"
>
> Moves cursor to left part of graph, "from the right would be…goes to 2 at x = 3."
>
> Moves cursor to left limit, "that's from the left."
>
> [hint] Appears to read silently, *The limit from the left is the height the graph approaches as x approaches 3 from the left.*
>
> "Ok, from the left, which would be 3."
>
> [input] left limit = 3
>
> {*Appears to have used hint to clarify this was the left limit.*}

This episode, along with the episode presented in Chapter IV (pages 106-7) of this student using a prompt from an incorrect answer to check the hints for left continuity, provides evidence of the potential use of the directions and prompts in developing understanding of procedures and concepts.

The *hint-show* strategy, an instrumented action scheme combining the use of a 'hint' followed by the 'show' feature, promoted procedural understanding required for

finding the left/right limits of the *Finding the Value of C* Maplet. Use of this strategy allowed students to recognizing the *C* variable expression needed in the correct response to left/right limit items. Students used the ‘show’ feature to determine that the limit expression included the *C* variable as part of the piecewise function presented in the problem. Soon after this realization, students determined that substituting the given value of *x* into this formula yielded the correct response.

The following schemes used by students while working with Maplets for Calculus did not appear to either contribute to, or prevent the development of understanding of continuity concepts: *check-change, check-guess, BB-whole numbers,* and *function value – all true*. Each of these strategies, used by at least three students, were documented, however, data suggest these schemes as ineffective – the researcher found no visual or verbal data to suggest these strategies contributed to understanding.

Discussion of the Graph-Slider Feature

The first three Maplets used in this study required students to determine the continuity of a function by leading them through a series of exercises motivated by the definition of continuity. The *Finding the Value of C* Maplet develops the understanding that if a function is continuous, the left limit and right limit are equal; a reverse premise of the first three continuity Maplets. The exercise within this Maplet (Figure 5.1) requires students to use the slider feature to estimate the find values of C, a task all students in this study accomplished with ease. Student difficulty with this Maplet occurred as they attempted to find the left and right limits. All the students initially tried to use the *graph-trace* strategy, using the cursor to trace the graph, to find these limits – similar to the strategy they all used in the *Continuity Using a Graph* Maplet.

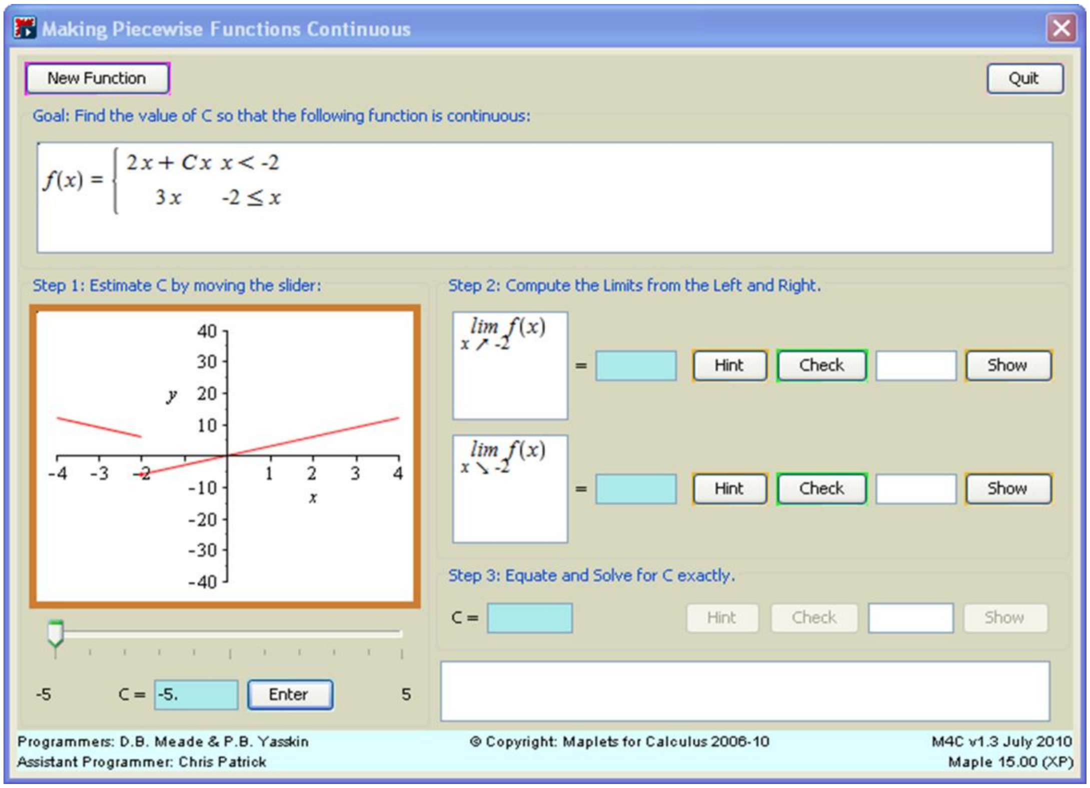

Figure 5.1 Screen shot of *Finding the Value of C* Maplet

The two problems students encountered using this strategy were: 1) the scale of the y-axis varied between problems and determining values between the intervals listed challenged students (most ended up guessing and checking values); and 2) this strategy was successful for determining value of the limit expressed numerically, but prevented understanding that the other was an expression of *C*. In these two situations, it appears the students used their experiences of the *Using a Graph* Maplet, repeatedly, even after many failed attempts. Eventually, all students used the ‘hint’ or ‘show’ feature to determine one limit was a variable expression with *C*; and from there used substitution to find the value of *C* required for the limit to exist. Another way the slider hindered the development of understanding was that the slider could provide the value of *C*. Confusion arose because students either used their estimate from beginning slider

exercise or returned to using the slider after computing the correct limits to answer the final "*C* =" question, without following the directions "Equate and Solve for C exactly" given in Step 3. This prevented them from the intended understanding of both procedure and concept that required the limits to be equated. In addition to student success with the *Using a Graph* Maplet of the first session, the motivation for continued use of the graph/slider feature may have come from students' preference for working in the embodied world. As described in the Findings section of Chapter IV, the progression of Maplet exercises led to development of understanding in the symbolic world, but in this Maplet, it appeared students wanted to 'see' or 'find' the answers on the graph or with the slider. Finally, as the graph/slider feature is prominent, taking up almost one-quarter of the window area, students may have been compelled to use it – why would it be there if it wasn't useful? Its inclusion in this Maplet, while well intended for showing how the correct value of *C* would make the function continuous, seemed to inhibit the development of student understanding of the properties of continuity.

Recommendations

The research presented in this study has implications for researchers investigating learners' understanding of mathematics, researchers investigating the use of technology for building understanding, software/applet developers, and high school/college instructors.

Researchers Investigating Learners' Understanding of Mathematics

Grounded in the theory of Tall's *Three Worlds* Model (2008), this study applied the work of Núñez et al. (1999) to continuity concepts to create a learning framework similar to those created by Drijvers (2003) levels of understanding of parameter (Figure

2.1) and Hahkiöniëmi's (2006) framework for derivative (Figure 3.5). One result of this study included a similar framework (Figure 4.1). Whether presented as a diagram or as rubrics similar to van Hiele's (1986) levels of geometric thinking and Chan's (2011) conceptual understanding rubric of continuity (Figure 2.3), the development of such frameworks is important for determining levels of understanding. This knowledge of learners' understanding of mathematics concepts becomes more important when investigating learners' transitioning from embodied to symbolic mathematical understanding. As Tall stated, and Núñez et al. specified with continuity concepts, calculus is a subject that exists in both the embodied and symbolic worlds. However, as proof in mathematics is most often conducted in the axiomatic-formal world with proceptual-symbolic objects and ideas, documenting learners' development and understanding in the symbolic world is important. Developed frameworks similar to the one used in this study can provide guidance for determining learners' understanding of mathematics concepts and for developing tools, exercises, or test items for doing so.

The procedure used to develop the framework of Figure 4.1 can also inform researchers considering learners' understanding of mathematics. The combination of theory, concept, instructor consultation, and student evidence provides guidance for the development of such frameworks for other topics. If, as suggested in the previous paragraph, frameworks such as the one developed in this study can be used for developing tools and test items, which could be used in quantitative studies, it may be of benefit to prepare such frameworks prior to an investigation. In considering a qualitative study, such a framework forces a researcher to determine and validate what qualifies as evidence of one level of understanding versus another. The use of student verbal data in

this study necessitated the development and use of such a framework and can inform other researchers.

Researchers Investigating the Use of Technology for Building Understanding

Drijvers and Trouché (2008) stated the *instrumental approach*, "…stresses the subtle relationship between machine technique and mathematical insight, and provides a conceptual framework for investigating the development of schemes…" (p. 375). They also observed, "A difficulty is that we cannot observe mental schemes directly. Our observations are limited to techniques students carry out with the artifact, and to the way they report on this in written or oral form." (p. 371) In defending the 'think aloud' method of collecting data, Ericsson and Simon (1980, 1984) contended that information in short term memory is accessible to a subject without changing thought processes, and that engagement in a problem solving task prevents a subject from interpreting the reasons for their decisions. The method of data collection (oral and screen capture recording) combined with the instrumental approach of Drijvers and Trouché enabled the investigator to determine features of the Maplets and strategies students developed that promoted conceptual understanding of continuity concepts. Additionally, the use of qualitative methods allowed the researcher to determine individual strategies (such as the *prompts-direction*) that, while not used by the majority of the students, contribute to student understanding of mathematics concepts.

Software/Applet Developers

As noted by the NCTM (1989, 2000) and the NGA/CSSO (2010), technology is necessary for the teaching and learning of mathematics and the availability of technological tools is now assumed. Fey et al. (2010) and Zbiek (2003) stated the need to

investigate new technologies as their development and implementation outpace the investigation of their effectiveness. The findings of this study inform developers of software/applets by providing a method for 'field-testing' their products prior to mass distribution. Though only seven participated in this study, the recorded sessions of these students provided a depth and breadth of data that developers are likely to find sufficient for informing decisions about software improvements and development. The benefit to developers for recording and documenting a few cases of users engaged in their technology prior to general release would be invaluable.

The findings of this study inform applet developers about the features found to promote understanding. Overt features of the Maplet software, the ability to 'check' and 'change' responses engaged students and helped them develop understanding of continuity concepts. The inclusion and effective use of a 'hint' feature, supports the claim of Jocobse and Harskamp (2009) that the inclusion of cognitive hints in technology help promote understanding of mathematics concepts. The findings for the effectiveness of the 'subtle' features, such as the presentation and layout of the Maplet exercises, the order of the Maplets, and variation of problems can help developers by understanding the contributions of each to user learning. Awareness that some features provide nothing more than a distraction (the graph/slider feature) or that particular features may not work with some students (hints) inform developers of the need to either improve or remove features.

High School/College Instructors

The findings of this study provide evidence of an increase in students' understanding of continuity concepts through the use Maplets for Calculus. However, as

Meade and Yasskin (2008) noted, the Maplets are intended as a supplements to classroom instruction. Student experience with the *Epsilon-Delta Continuity* Maplet lends further emphasis to this point. Having prior classroom or study experience with the concept of epsilon-delta limits and continuity, as provided with the preview sheet given students, is essential for ensuring that students gain further understanding while working with the Maplets. Maplets are not a replacement for classroom or lecture hall instruction. While Maplet software applets can increase the understanding of a concept, Meade and Yasskin designed and promote Maplets as a 'tutoring' software package to supplement instruction (2008, December).

The number of Maplets, over 140 as of this writing, and applets and various other software choices available to instructors and students can be overwhelming. This study provides data to inform choices in both the selection of applets for use as well as the methods of use. The features of the Maplet tutorial applets that predicated student understanding – the ability to check and change responses, availability of cognitive hints, variety of problems, a layout of problems leading to the development of a concept, etc. are features of Maplets that should be considered when determining a software/applet for use with students. Many tutorial applets and software packages present problems and give 'right/wrong' feedback to students without the ability to modify the response. The ability to check and modify or change responses may be the feature most responsible for the student gains in understanding demonstrated in this study.

In addition to informing the choice of applets for use in instruction, the utilization schemes developed by students while using Maplets have implications for their use in classroom instruction and by students. The intermediate steps of the *check-rework-*

change and *check-review-change* appeared to most influence the development of understanding of the continuity concepts when students used the Maplets. Instructors should consider demonstrating and encouraging these practices prior to, or during, students' use of these programs. Instructors can also inform students of other successful strategies, such as the use of the *prompts/directions* that lead to development in understanding. In this recommendation, the instructor should facilitate the experience of students working with applet technology. This facilitation could include: providing guidance for student using applets in a classroom setting; presenting a demonstration of the applet prior to student use; and leading a classroom discussion about strategies students developed that promoted understanding when using the applets.

The student difficulty that occurred while using the graph/slider feature of the *Finding the Value of C* Maplet demonstrated an unexpected situation that only became apparent as the students engaged with the software. While instructors may rely on the developers to provide them with a product that is ready to use, it may be advantageous for instructors to preview the software and use it with a few students prior to use with all students.

The evidence presented in Chapter IV in which one student attempted over 10 problems while working with the *Continuity given a Piecewise Function* Maplet, shows that Maplets may contribute to the CCSSM Practice calling for persistence in problem solving.

Suggestions for Future Research

The case study method of this investigation precludes the generalization of the findings to other student populations. The students included in this study consisted of

high school students; Maplets for Calculus were designed by college professors (Meade and Yasskin) for use by college students enrolled in calculus courses. While the opinion of this investigator is that the findings of this study are likely to produce similar results with college students (as well as AP Calculus students enrolled in other high schools), this opinion requires further investigation. One consideration of methodology for this study was a mixed methods investigation that would have included a pre- and post-evaluation of student understanding of continuity concepts with a hypothesis that students using Maplets for Calculus would experience higher gains in score on the evaluation when compared to students who did not use Maplets for Calculus.

Additionally, as this study used only five of the over 140 available Maplets, investigation involving other Maplets (or applets) is needed to determine if the features and strategies determined effective for developing understanding of continuity concepts are also effective in other applets and/or with other mathematical concepts.

One of the benefits to the teachers of the AP Calculus classes whose students participated in this study was license to use the Maplets for Calculus software with their classes after the data collection phase. Both of these instructors, as well as a pre-calculus teacher at one of the schools, indicated they had previewed other Maplets in the series and discussed which they might include in their instruction. This suggests a study in which the use of Maplets for instruction (as one teacher had done with the *Derivatives Using the Chain Rule* Maplet) and student use during a calculus class/course are documented to determine if Maplets (or other applet technologies) influence the achievement of students enrolled in these classes/courses. This also suggests an investigation into the choices an instructor makes about which Maplets (or applets) to use

with a class and the results of these selections on student learning. An investigation of this type has potential for adding to the knowledge base in a number of ways: comparing choices of two or more teachers teaching common course; comparing student outcomes for particular units of study between two teachers employing different applets; investigating different teaching methods of two teachers using the same applet; etc.

Any of the features and strategies frequently used by students could be investigated further to determine if their contribution to student understanding of mathematical concepts can be generalized to include other Maplet or applet software, or are these findings particular to the features and Maplets of this study. For example, the 'check' and 'change' features were shown to contribute to students' understanding of continuity concepts; is this necessarily true for other Maplets or applets that have these features? Is this necessarily true with mathematical concepts other than continuity? Furthermore, a new investigation might determine the characteristics of features that become *instruments* and contrast these with features that do not. Recall the *instrumental genesis* is the process by which a tool or artifact becomes and instrument (Drijvers and Trouché, 2008). This process requires the user to develop mental schemes involving knowing how to use the artifact appropriately and understanding for which circumstances the artifact is useful (p. 368) Are particular features better suited to become instruments and be used in strategies than others? Why? For example, all students in this study employed the *check-reflect-change* or *check-rework-change* strategies and became proficient in applying them. Why these features and these strategies? How come the 'hint' feature did not become an *instrument,* even though students said in interviews they wanted to use them? Is this an issue of students' being unable to develop an appropriate

strategy, or a characteristic of the feature? These questions are fertile ground for future investigations.

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Appendix A – USC IRB Approval Letter

UNIVERSITY OF SOUTH CAROLINA

OFFICE OF RESEARCH COMPLIANCE

August 28, 2012

Mr. Raymond Ellis Patenaude
College of Education
Instruction and Teacher Education
Wardlaw
Columbia, SC 29208

Re: **Pro00019806**
Study Title: *The Use of Applets for Developing Understanding in Mathematics: A Case Study Using Maplets for Calculus with Continuity Concepts*

FYI: University of South Carolina Assurance number: FWA 00000404 / IRB Registration number: 00000240

Dear Mr. Patenaude:

In accordance with 45 CFR 46.101(b)(**1**), the referenced study received an exemption from Human Research Subject Regulations on **8/28/2012.** No further action or Institutional Review Board (IRB) oversight is required, as long as the project remains the same. However, you must inform this office of any changes in procedures involving human subjects. Changes to the current research protocol could result in a reclassification of the study and further review by the IRB.

Because this project was determined to be exempt from further IRB oversight, consent document(s), if applicable, are not stamped with an expiration date.

Research related records should be retained for a minimum of three years after termination of the study.

The Office of Research Compliance is an administrative office that supports the USC Institutional Review Board. If you have questions, please contact Arlene McWhorter at arlenem@sc.edu or (803) 777-7095.

Sincerely,

Thomas A. Coggins
Director

cc: Edwin Dickey

University of South Carolina-Columbia, South Carolina – 803/777-7095 – Fax 803/576-5589

Appendix B – Parent Consent and Student Assent Forms

University of South Carolina
COLLEGE OF ARTS AND SCIENCES
MATHEMATICS

LeConte College, 1523 Greene Street, Columbia, SC 29208

Dear parent:

We would like to invite your child to participate in a study of an innovative calculus learning system, Maplets for Calculus. I am Professor Douglas Meade, the project's principal investigator. The study will be conducted by Mr. Ray Patenaude, a math teacher at South Pointe High School and a doctoral candidate at USC, under my supervision.

Your child's participation will require approximately three one-hour sessions, in which they will be asked to work several calculus problems using Maplets for Calculus. We will record him or her as they work the problems and "think aloud" so that we can follow their work. Analysis of the recordings will help us to understand how students perceive the system, and to improve the system's effectiveness.

Your child's participation in this project is voluntary. Your decision of whether to allow your child to participate in this study will not affect your child's grades, either negatively or positively. If you choose to allow your child to participate, you or the student may decide to terminate participation in the study at any time for any reason with no penalty. Participation in the study involves no foreseeable risks.

All data generated by this study will be kept confidential, and no personally identifiable information will be included in any research papers or other materials that may result from the study.

If you have any questions about the research, you may contact Mr. Patenaude at (803) 487-4048, email raypatenaude@comporium.net.   Alternatively, you may contact me at the University of South Carolina Department of Mathematics, LeConte College 300e, Columbia, SC 29208, phone (803) 622-1595, email MEADE@mailbox.sc.edu with any questions or concerns.

Finally, if you have any questions about your child's rights as a research subject, you may contact: Thomas Coggins, Director, Office of Research Compliance, University of South Carolina, Columbia, SC 29208, Phone - (803) 777-7095, Fax - (803) 576-5589, E-Mail - tcoggins@mailbox.sc.edu.

Thank you for taking time to consider this request. If you agree to allow your child to participate, please fill in the blanks in the attached form and return it to Mr. Patenaude.

Sincerely,

Dr. Douglas Meade

Maplets for Calculus parent consent form

I have read (or have had read to me) the contents of this consent form and have been encouraged to ask questions. I have received answers to my questions. I give my consent for my child to participate in this study, and I have been told that he or she may withdraw at any time without negative consequences. I may retain the attached explanation of the research for my records and future reference.

Student's name: ____

Parent/legal guardian's name: ____________________________________

Parent/legal guardian's signature:__________________________Date: __________

University of South Carolina
COLLEGE OF ARTS AND SCIENCES
MATHEMATICS
LeConte College, 1523 Greene Street, Columbia, SC 29208

Dear student:

We would like to invite you to participate in a study of an innovative calculus learning system, Maplets for Calculus. I am Professor Douglas Meade, the project's principal investigator. The study will be conducted by Mr. Ray Patenaude, a math teacher at South Pointe High School and a doctoral candidate at USC, under my supervision.

Your participation will require approximately three one-hour sessions, in which you will be asked to work several calculus problems using Maplets for Calculus. We will record you as you work the problems and "think aloud" so that we can follow your work. Analysis of the recordings will help us to understand how students perceive the system, and to improve the system's effectiveness.

Your participation in this project is voluntary. Your decision to participate in this study will not affect your grades or academic standing either negatively or positively. If you choose to participate, you may decide to terminate participation in the study at any time for any reason with no penalty. Participation in the study involves no foreseeable risks. All data generated by this study will be kept confidential, and no personally identifiable information will be included in any research papers or other materials that may result from the study.

If you have any questions about the research, you may contact Mr. Patenaude at (803) 487-4048, email raypatenaude@comporium.net. Alternatively, you may contact me at the University of South Carolina Department of Mathematics, LeConte College 300e, Columbia, SC 29208, phone (803) 622-1595, email MEADE@mailbox.sc.edu with any questions or concerns.

Finally, if you have any questions about your rights as a research subject, you may contact: Thomas Coggins, Director, Office of Research Compliance, University of South Carolina, Columbia, SC 29208, Phone - (803) 777-7095, Fax - (803) 576-5589, E-Mail - tcoggins@mailbox.sc.edu.

Thank you for taking time to consider this request. If you agree to participate, please fill in the blanks below and return this form to Mr. Patenaude.

Sincerely,

Dr. Douglas Meade

Maplets for Calculus Student Assent Form

I have read the description of the study in this form, and I have been told what the procedures are and what I will be asked to do in this study. Any questions I had have been answered. I have received permission from my parent(s) to participate in the study, and I agree to participate in it. I know that I can quit the study at any time. I will be given the attached explanation of the research for future reference.

Student's name: ____

Student's signature:______________________________ Date: __________

As a representative of this study, I have explained to the participant or the participant's legally authorized representative the purpose, the procedures, the possible benefits, and the risks of this research study; the alternatives to being in the study; the voluntary nature of the study; and how privacy will be protected.

Representative's signature:_________________________ Date: __________

Appendix C –High School Consent for Research Letters

The following is an example of the letter of consent signed by the principals of the high schools whose students participated in this study.

To Whom It May Concern:

Mr. Ray Patenaude, PhD candidate at the University of South Carolina, has my support to conduct research for his dissertation, the proposed entitled, "*The Use of Applets for Developing Understanding in Mathematics: A Case Study Using Maplets for Calculus with Continuity Concepts,"* at________ High School.

I understand his proposal will involve the participation of three AP Calculus students enrolled at _______ High School for three one-hour sessions. These sessions will involve the recording of the students' computer activity as well as audio recordings of the students as they 'think aloud' (talk aloud while solving problems) while using computer applets, Maplets for Calculus, about the continuity of functions.

Furthermore, I understand that prior to collecting any data at our school, Mr. Patenaude will ask for and receive the written consent of both the student participants and their parent/guardians and that participation is voluntary – student participants and/or their parents may choose to withdraw from this study at any time without fear of reprisal. Mr. Patenaude does not teach, nor does he hold a position of direct authority over the students who may participate in this study.

Mr. Patenaude has assured me that any data pertaining to students will be encoded to ensure the confidentiality of the students, and that any published or presentations regarding his findings will refer to ________________ High School using geographic descriptions, i.e. a high school in northern South Carolina.

Sincerely,

School Principal

Appendix D – Protocol for M4C Recording Sessions

1. Introduce observer
    a. My background as a teacher and student at USC

2. Describe purpose of recording session
    a. Working with team developing applets to teach math
    b. Want to watch students using software in order to improve applets

3. Consent
    a. Let student know that their participation is voluntary and they are free to stop at any time
    b. Ask permission to continue

4. Describe and practice Think-Aloud Method
    a. Want you to talk out loud while solving problems and working with software
    b. This will give us an idea of what is good/bad about the applets
    c. I will prompt you to “keep talking” if you go quiet for 20 seconds → not an insult/discipline, we need to know what you’re thinking, good or bad

5. Practice Problem
    a. Give student a practice math problem (not on computer) of relative ease (probably adding two fractions with different denominators) to practice “thinking aloud”
    b. Student will be provided paper and pencil or asked to work on whiteboard
    c. At conclusion, ask if student understands how we will proceed
    d. Ask for consent to continue

6. Intro to software/applet
    a. Introduce student to applet and explain features and how to use the applet

7. Ask student to continue working with applet while talking aloud
    a. Screen capture and oral recording of student using software
    b. Monitor student and end session when student either “masters” applet (consecutive problems answered correctly with little difficulty), or gets frustrated on three consecutive attempts, or in 20 minutes.

8. Short interview with subject
    a. Ask about particular episodes while working with Maplet (noted by observer)
    b. Ask about Maplet (for developers)
        i. What did you like about this Maplet?
        ii. What didn't you like?
        iii. Any other thoughts about using Maplet you'd like us to know?

9. Conclude session and thank participant.

Appendix E – Description of Maplets

Continuity using a Graph Maplet

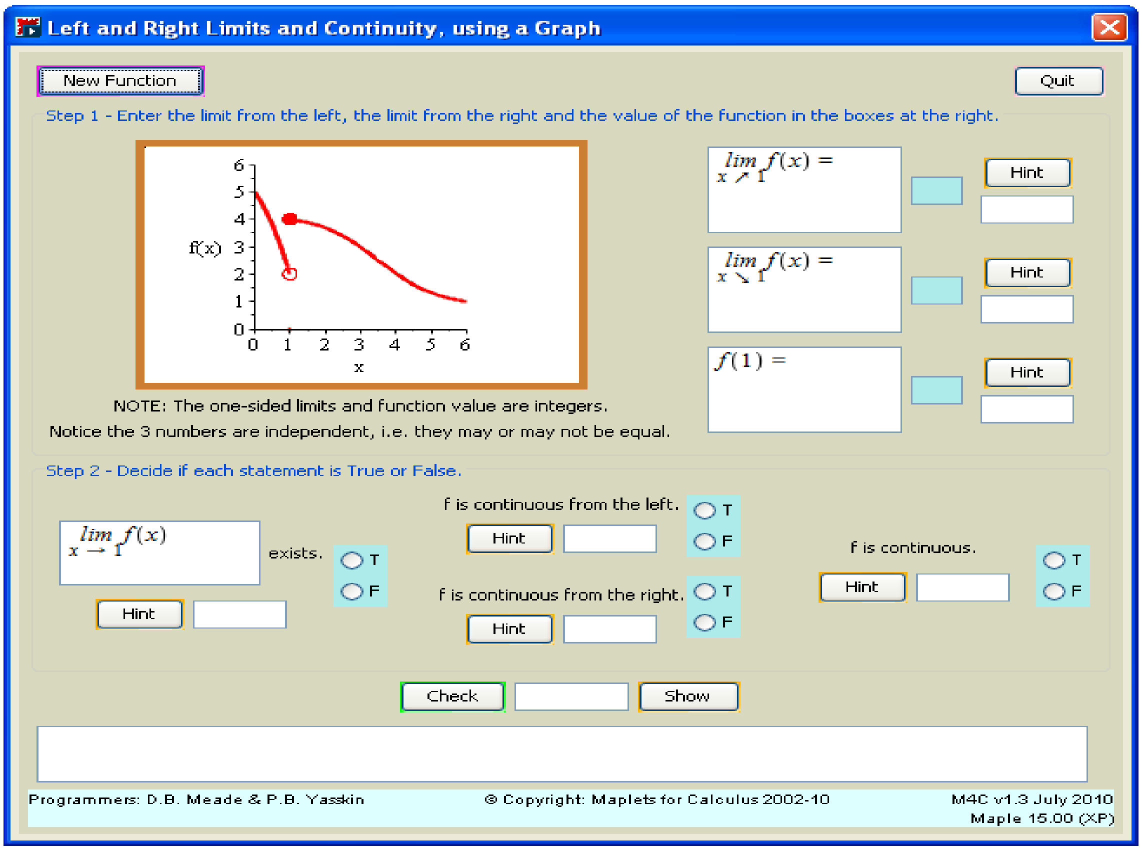

Figure E.1 Beginning screen shot of *Continuity using a Graph* Maplet.

The *Continuity using a Graph* Maplet (Figure E.1) provides the user with the graph of a piecewise function. The instructions require the user to input the left limit, the right limit, and the value of the function at a given value of *x*. The user is then directed to answer a series of true/false questions: does the limit of *f*(*x*) exist? Is the function continuous from the left and right? Is the function continuous?

Features available to the user include: 'check' answers and re-enter responses, obtain 'hints' for each item, and 'show' all answers. When complete, the 'new function' button gives a computer generated function and the exercise begins again.

Continuity using a Piecewise Function **Maplet**

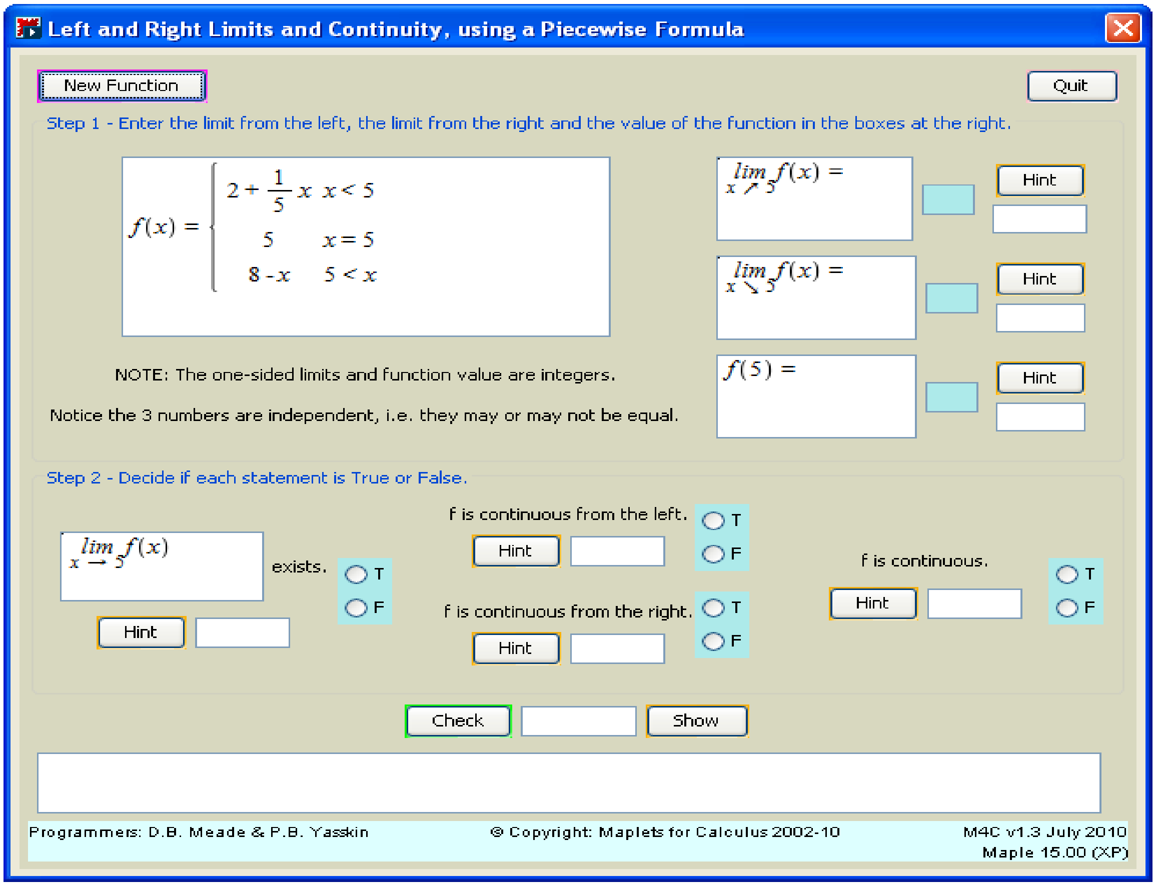

Figure E.2 Beginning screen shot of the *Continuity using a Piecewise Function* Maplet.

The *Continuity using a Piecewise Function* Maplet (Figure E.2) provides the user with the formula of a piecewise function. As with the *Given a Graph* Maplet, the user is expected to input the left limit, the right limit, and the value of the function. In the second step, the user is expected to answer the same true/false items: does the limit of $f(x)$ exist? Is the function continuous from the left and right? Is the function continuous?

Features available to the user include: ‘check’ answers and re-enter responses, obtain ‘hints’ for each item, and ‘show’ all answers. When complete, the ‘new function’ button gives a computer generated function and the exercise begins again.

Continuity given a Black Box Function Maplet

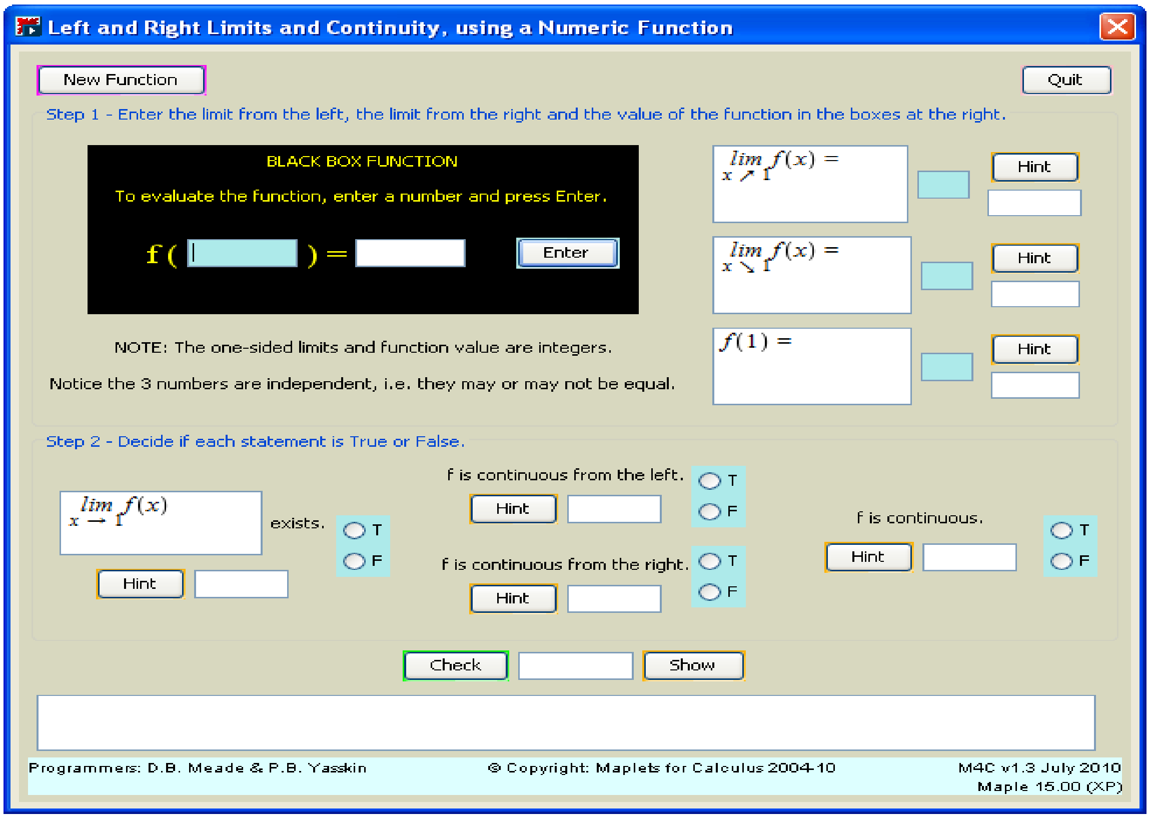

Figure E.3 Beginning screen shot of *Continuity using a Black Box Function* Maplet.

The *Continuity using a Black Box Function* Maplet (Figure E.3) provides the user with a *Black Box Function*. Users can values for *x* into the function, clicks ‘enter’, and the numeric value of the function are displayed. Using the function values, the user is directed to input the left limit, the right limit, and the value of the function. As with the previous Maplets, the user is then asked the true/false items: does the limit of *f*(*x*) exist? Is the function continuous from the left and right? Is the function continuous?

Features available to the user include: ‘check’ answers and re-enter responses, obtain ‘hints’ for each item, and ‘show’ all answers. When complete, the ‘new function’ button gives a computer generated function and the exercise begins again. The diagram of Figure E.4 shows a screen shot of this Maplet after using the ‘check’ feature.

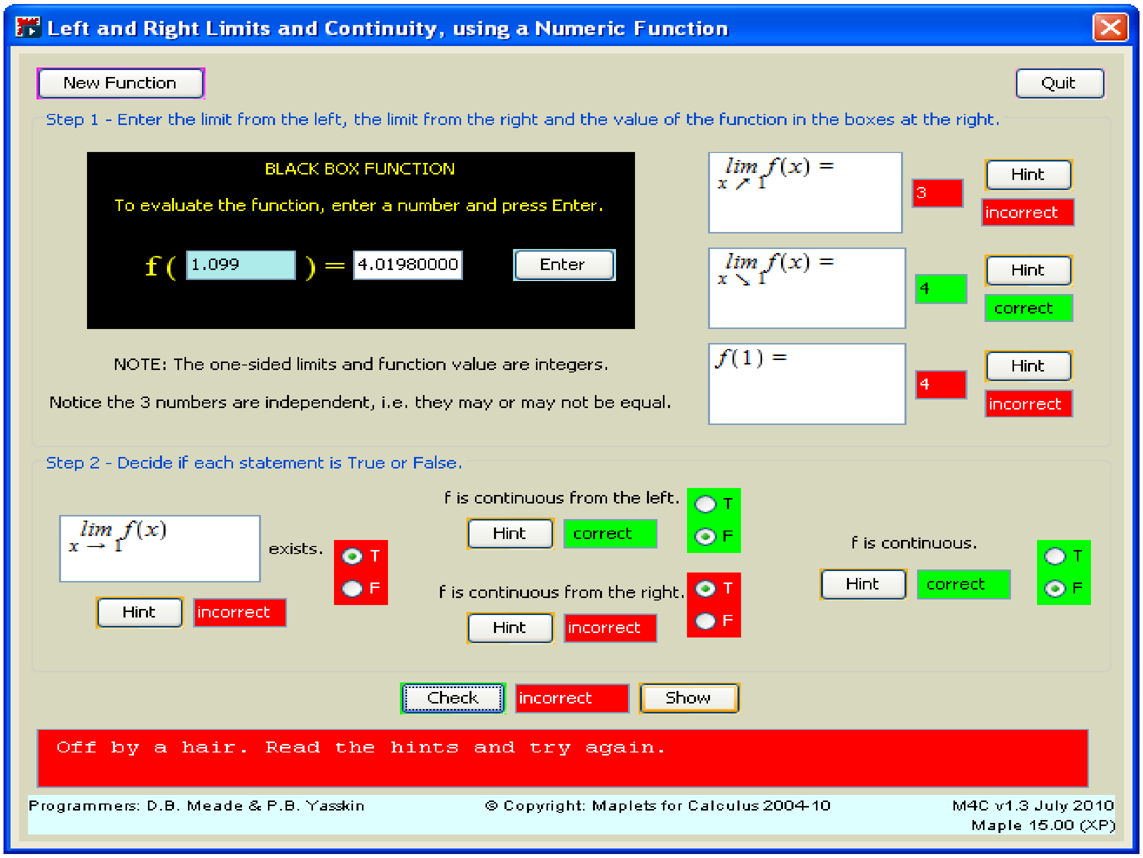

Figure E.4 Screen shot of *Black Box* Maplet after using the ‘check’ feature.

Finding the Value of C Maplet

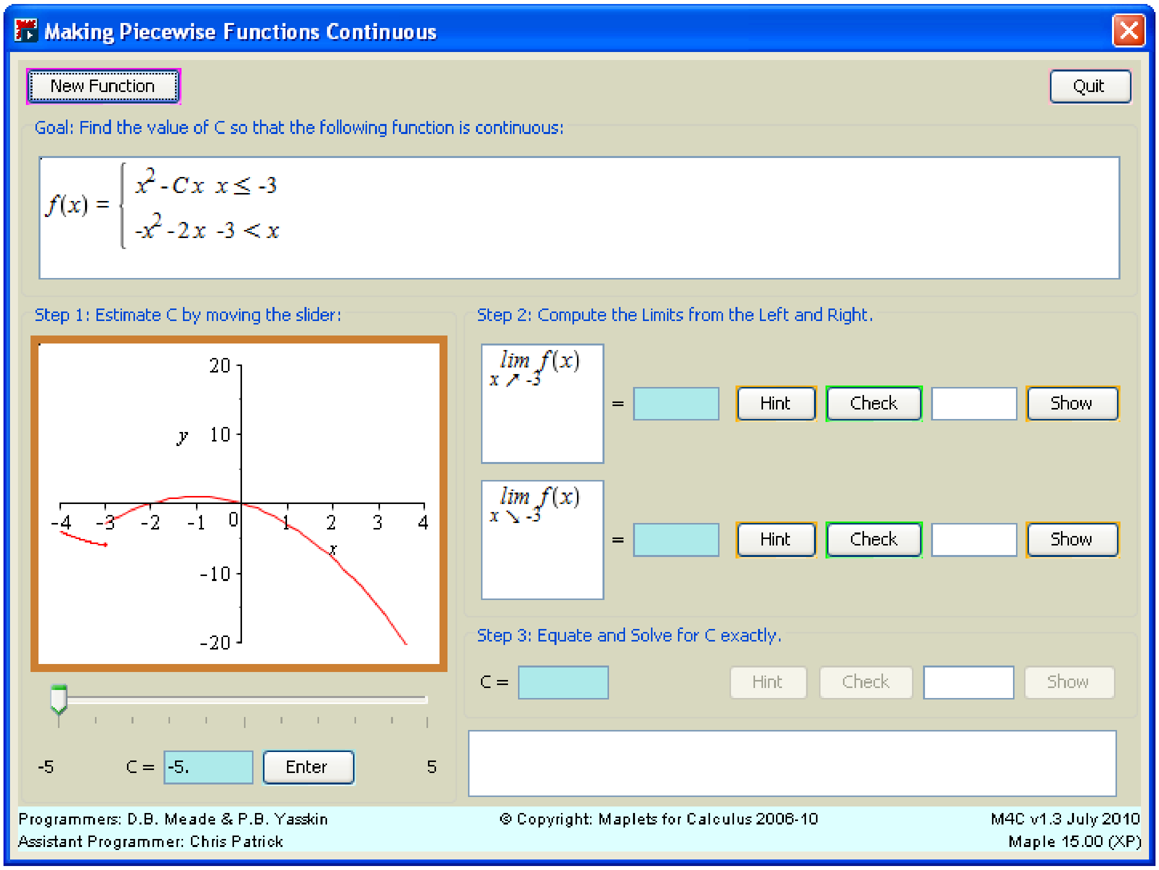

Figure E.5 Beginning screen shot of the *Finding the Value of C* Maplet

The *Finding the Value of C* Maplet (Figure E.5) presents the user with a piecewise function with the stated goal, "Find the value of C so that the following function is continuous." The user is then directed to use the graph/slider to estimate the value of *C*. The value of *C* can be entered either by using the slider or typing a value into the blue box. As the value of the slider approaches the actual value of *C*, the graph moves to become continuous (Figure E.6). Step 2 expects users to compute the left/right limits of the function. Unlike the previous Maplets, the user can 'check' individual limit responses and/or 'show' the correct answers to the individual limits. The 'hint' feature is available for both limits. Only when both limits are correct can the user input a response

for C in Step 3. Users may use the 'hint', 'check', and 'show' feature for this item as well. Figure E.6 shows a screen shot of this Maplet after: estimating C = - 4 in the slider feature, using 'show' for the left limit, and the 'check' feature for the right limit.

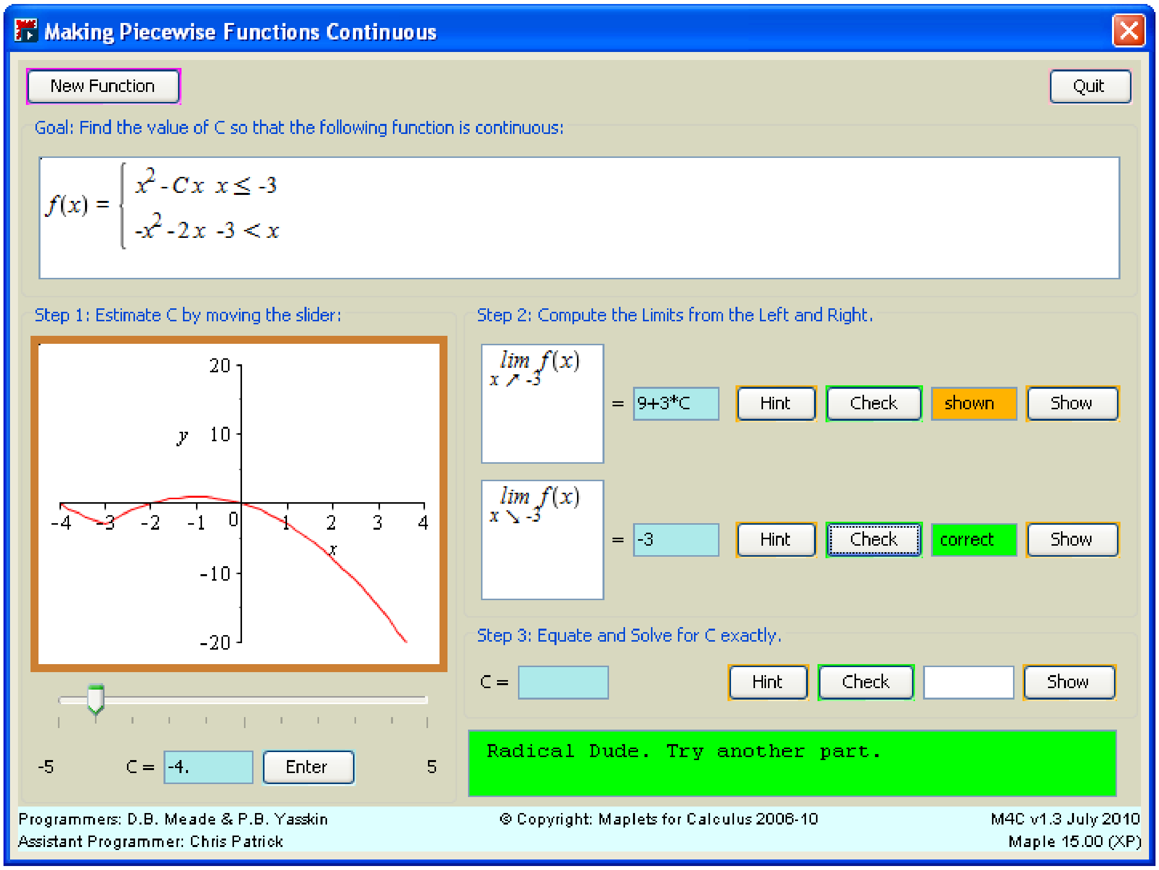

Figure E.6 Screen shot of the *Finding the Value of C* Maplet after using the 'show' and 'check' feature.

Epsilon-Delta Continuity Maplet

The *Epsilon-Delta Continuity* Maplet (Figure E.7) presents users with a limit and a given value for epsilon. By using the slider, or by typing in values for delta, the user is directed to find a value of delta that satisfies the given epsilon condition of the limit. As the value of delta decreases, the vertical, blue 'delta-band' on the graph narrows and the 'gold' box approaches the horizontal, pink 'epsilon-band'. Users can 'check' values of delta and 'show' correct values as well. Correct values of delta are ones in which the

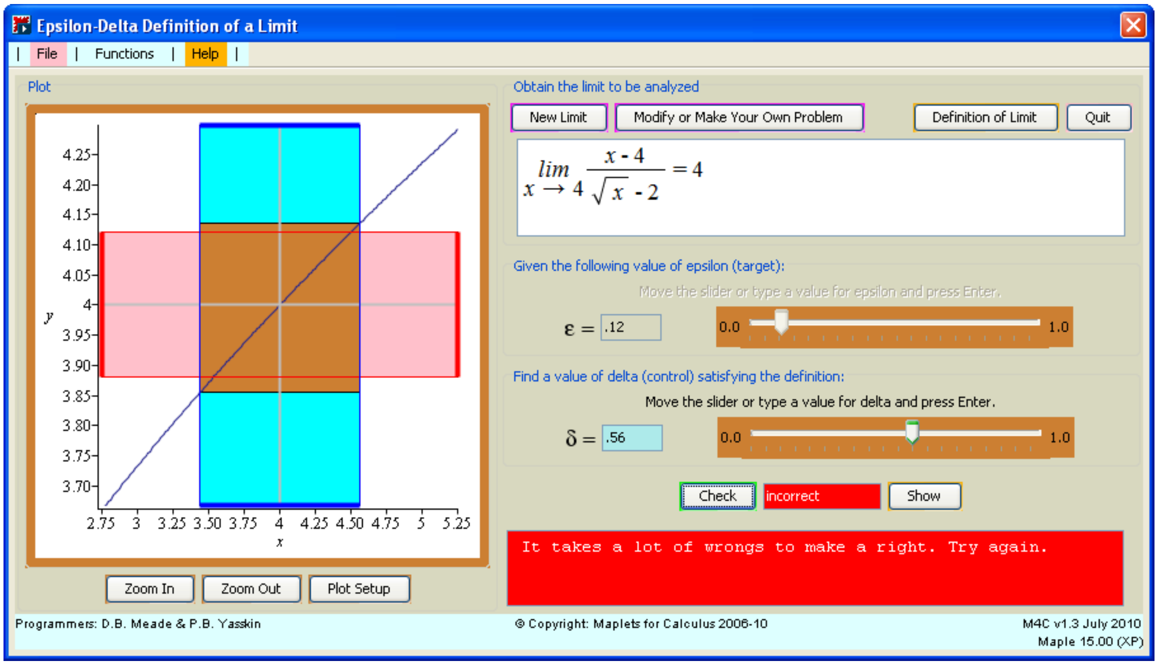

Figure E.7 Screen shot of the *Epsilon-Delta Continuity* Maplet after using 'check' feature

'gold' box is completely within the pink 'epsilon-band' (Figure E.8). After choosing a correct value for delta, the epsilon-slider can be moved to set another condition using the same limit. The 'new limit' feature provides a new exercise with a different function.

.

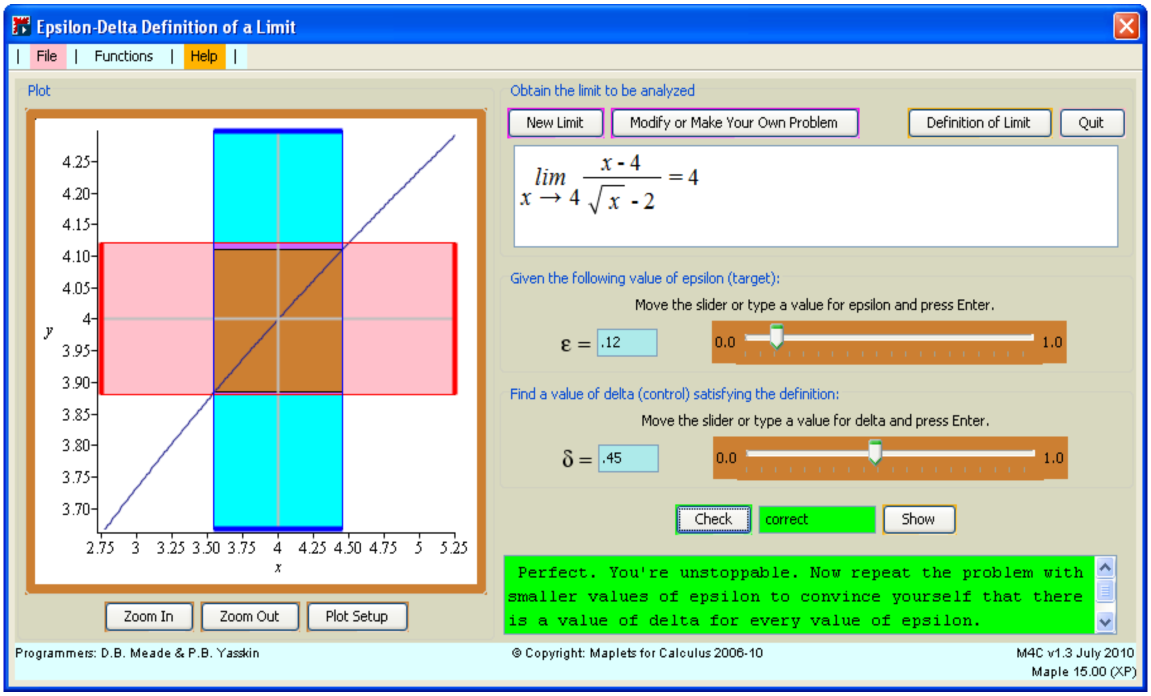

Figure E.8 Screen shot of the *Epsilon-Delta Continuity* Maplet after 'checking' a correct value of delta.

Appendix G – Coding Schemes

Features Coding Scheme

Table G.1 contains the abbreviation and descriptions of the codes used for analyzing the transcribed data to document the features used by students while working with Maplets for Calculus about continuity.

Table G.1

Schemata used for coding transcripts for features of Maplets used.

Code	**Feature**	**Description**
BB	Black Box	Use black box function of a Maplet
Calc-C	Calculator-Compute	Use TI-84 calculator to compute values.
Calc-GR	Calculator-Graph	Use graphing feature of TI-84 calculator.
CG	Change	Change a previously entered response to a Maplet item.
CK-C	Check-Correct	Use of check feature of a Maplet – receive feedback that all items are correct.
CK-I	Check-Incorrect	Use of check feature of Maplet – receive feedback of incorrect input.
CK-W	Check-Warning	Use check feature of a Maplet – receive warning message.
GR-I	Graph-Interpretive	Using graph feature of a Maplet with oral evidence referring to the graph.
GR-T	Graph-Trace	Using graph feature of a Maplet with visual evidence of tracing on screen with cursor.
HT	Hint	Use the hint feature of Maplet.
LC/RC	Left/Right Continuity	Used in combination with other codes to designate left or right continuity. For example HT-RC denotes use of hint feature for the right continuity item.

Code	Feature	Description
LL/RL	Left/Right Limit	Used in combination with other codes to designate left or right limit. For example HT-LL denotes use of hint feature for the left limit item.
P-C	Paper-compute	Use paper/pencil for computation
P-T	Paper-Table	Use paper/pencil to make a table of values
SH	Show	Use of the Maplet feature 'show'.
SL	Slider	Use of the slider feature of a Maplet.

Utilization Schemes Coding Scheme

Table G.2 contains the abbreviation and descriptions of the codes used for analyzing the transcribed data to document the strategies used by students while working with Maplets for Calculus about continuity.

Table G.2

Schemata used for coding transcripts for utilization strategies of students.

Code	Strategy	Description
BB-D	Black Box-Decimals	Use of the black box feature by entering decimal values.
BB-W	Black Box – Whole Numbers	Use of the black box feature by entering whole number values.
Calc-Comp	Calculator – Compute	Use of calculator for computational purposes.
Calc-GR-PW	Calculator-Graph-Piecewise	Use of graphing capability of graphing calculator to graph individual parts of piecewise function.
CK-CG	Check-Change	Use of check feature followed by change of incorrect response without visual or oral evidence of intermediate activity.
CK-HT-CG	Check-Hint-Change	Use of check feature followed by use of the hint feature prior to changing a response. This code collapsed into check-rework-change.
CK-RF-CG	Check-Reflect-Change	Use of the check feature followed by oral evidence indicating reflection or description of reasoning, followed by changing a response.

Code	Strategy	Description
CK-RW-CG	Check-Rework-Change	Use of check feature followed by visual evidence of working with either Maplet features or paper, pencil, or calculator prior to changing a response.
FV-T	Function Value – True	Use the value of the function given by the Black Box feature as response for left/right limit and *f(x)* values of limit items and answer TRUE to all continuity items of the *Black Box* Maplet.
GR-T	Graph-Trace	Use of the graphing feature and cursor to trace the graph
GU-CK	Guess-Check	Repeated oral or visual evidence of student guessing at correct response followed immediately by use of the check feature to determine if response was correct.
HT-SH	Hint-Show	Use of hint feature followed by the use of show feature of Maplet.
P-C	Paper-Compute	Use of paper and pencil to compute values or solve equations.
P-T	Paper-Table	Use paper and pencil to make a table of values.
PT-DR	Prompts – Directions	Oral evidence of reading the prompts or directions, followed by oral or visual evidence of student following the directions as stated.
SL-FC-AF	Slider – Find *C* – After	Use of the slider feature to determine the value of *C* after successful completion of the limit exercises.
SL-FC-BF	Slider-Find *C* - Before	Use of slider feature of the *Finding the Value of C* Maplet to determine the value of *C* prior to completing the limit exercises.