Spring 2023

Structure of Extremal Unit Distance Graphs

Kaylee Weatherspoon
University of South Carolina - Columbia

Director of Thesis: Joshua Cooper
Second Reader: Michael Filaseta

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Thesis Summary

This thesis begins with a selective overview of problems in geometric graph theory, a rapidly evolving subfield of discrete mathematics. We then narrow our focus to the study of unit-distance graphs, Euclidean coloring problems, rigidity theory and the interplay among these topics. After expounding on the limitations we face when attempting to characterize finite, separable edge-maximal unit-distance graphs, we engage an interesting Diophantine problem arising in this endeavor. Finally, we present a novel subclass of finite, separable edge-maximal unit distance graphs obtained as part of the author's undergraduate research experience.

Author's Note

This thesis was composed in hope that the majority of the text would be accessible to an undergraduate with similar background to the author when she first began this project (that is, with almost no background). By necessity, Sections 4, 5 and 6 are more technical. If anything in this thesis excluding the aforementioned sections is communicated as less than beautiful, accessible, and exciting, the fault is the author's alone.

Acknowledgements

While it would be impossible to thank everyone who supported my work on this thesis, I would like to express gratitude to

Dr. Cooper, for hours of valuable conversation and for treating me like a collaborator;
for knowing just when to push me, when to support me, and when to tell a well-timed story

Dr. Filaseta, for investing in me and infusing fun and humor into the research process

Ben, for fostering my curiosity and seeing potential

Pete, for allowing me to follow my curiosity

Dr. Jordán, for challenging me and calling out my strengths

Katie, for cheering me on... and turning off the lights the times I fell asleep working on this project

Amelia and Dan, for both understanding and appreciating

Grandmother, for teaching me discrete math before she'd ever heard the term

Jake, David, Anna, Joey, Luke, Carter, Maddie, and the others, for keeping the coffee coming

2 Introduction

In discrete math, a *graph* $G = (V, E)$ is a collection of vertices V and a set E of edges between them. Non-mathematicians and many applied mathematicians might call this object a network. While we often conceptualize graphs visually, a graph is an abstract object with multiple visual representations. For example, the graph defined as $G = (V, E)$, $V = \{1, 2, 3, 4\}$, $E = \{12, 23, 34, 14\}$ can be visualized in the plane as shown below, and in countless other ways.

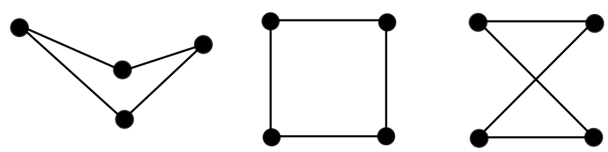


Figure 1: Three Depictions of a 4-cycle

We call the number of vertices in a graph the *order* of the graph. Given a vertex v, if uv is an edge of the graph for some other vertex u, then we say u is a *neighbor* of v. The *degree* of a vertex of a graph is the number of neighbors it has. A *complete graph* on n vertices is a graph in which every vertex is adjacent to every other vertex. That is, the edge set of a complete graph consists of all possible pairs of vertices and therefore has $\binom{n}{2}$ elements. A graph $H = (V', E')$ is a *subgraph* of a graph $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$.

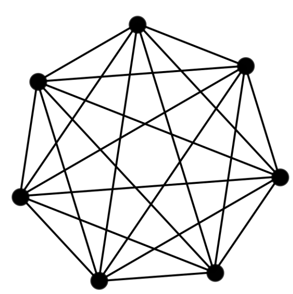


Figure 2: A Depiction of the Complete Graph on 7 Vertices

In the following section, we give an array of open problems in extremal graph theory, an active area of research in the study of graphs.

3 Context

3.1 Selected Extremal Problems in Geometric Graph Theory

Following the convention of János Pach in [56], we use the term *geometric graph* to refer to a graph drawn in the Euclidean plane with straight-line edges. Anyone who has used an atlas has intuition for geometric graphs, as the network of cities and distances between them often provided on a back page gives a familiar example of a such a graph (ignoring the fact that these distances are often based on non-straight line paths). Through the shift from atlases to GPS navigation, geometric graphs have only become more relevant: current interest in geometric graphs, or spatial networks, is driven in part by Vehicle Routing Problems, including efforts to develop efficient routes for mail delivery (see [47]).

Demand for efficient optimization algorithms on spatial networks has led to increased interest in geometric graphs. Solutions to extremal problems give insight into the best- and worst-case scenarios which these algorithms may face. In the following subsections, we explore areas of recent interest in extremal problems in geometric graph theory. Ultimately, the aim of this section is to contextualize the study of unit distance graphs, which we engage more directly in Section 3.2.

3.1.1 Forbidden Subgraph Characterizations

Given a graph $G = (V, E)$, we call a graph $G' = (V', E')$ a *subgraph* of G if $V' \subseteq V$ and $E' \subseteq E$. A graph $G[V']$ is an *induced subgraph* of G if it is a subgraph of G with vertex set $V' \subseteq V$ and an edge between two vertices of $G[V']$ if and only if there is an edge between them in G.

A *forbidden subgraph characterization* of a family of graphs $\mathcal{G}$ is a definition of $\mathcal{G}$ in terms of subgraphs which are not permitted in members of $\mathcal{G}$. If the characterization is *complete*, we can say that a graph G is a member of $\mathcal{G}$ *if and only if* G does not have one of the forbidden graphs as a subgraph.

Consider the bipartite graphs as a purely graph-theoretic example of a class with a forbidden subgraph configuration. It is straightforward to see that bipartite graphs have no K_3 subgraph, and in fact, a graph is bipartite if and only if it has no odd cycle as a subgraph (see Theorem 4 of [9]).

A *clique* is a complete graph, and an *independent set* is a set of vertices

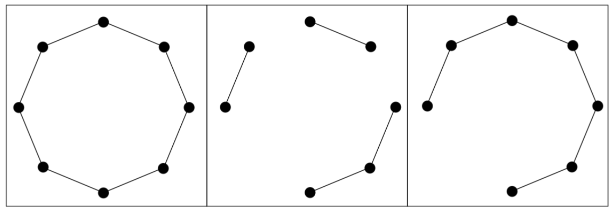


Figure 3: Example of Graph, Subgraph, and Induced Subgraph

which does not induce an edge. The following longstanding conjecture of Erdős and Hajnal has inspired significant study in the area of forbidden subgraphs:

Conjecture 1. (Erdős-Hajnal, [14]) For every graph H, there exists a $\delta(H) > 0$ such that every graph G with no induced subgraph isomorphic to H has either a clique or an independent set of size at least $|V(G)|^{\delta(H)}$.

Broadly, this conjecture suggests that graph classes defined in terms of a forbidden subgraph behave differently from almost all other classes of graphs, as there exist graphs on n vertices with *no* clique or independent set of size larger than $\mathcal{O}(\log n)$ [14],[23]. The perhaps more famous result below proves the conjecture for complete graphs K_n:

Theorem 1. (Ramsey's Theorem) For N sufficiently large, there exists c such that all K_r-free graphs on N vertices have an independent set of size $cN^{1/(r-1)}$.

We give an alternative statement with the more standard language of edge coloring:

Theorem 2. (Erdős-Szekeres Version of Ramsey's Theorem [16]) Every two-coloring of the edges of K_n has a monochromatic clique of order $(1/2)\log_2(n)$.

In this context, a *two-coloring* is an assignment of two colors to the edges of a graph. Observe that by taking any graph G on n vertices, coloring all edges of G blue, and constructing a red edge between any non-adjacent pair of vertices, we obtain a two-coloring of K_n. By Theorem 2, G has a monochromatic clique of order $(1/2)\log_2(n)$. That is, this graph has either a large enough red clique or a large enough blue clique, which implies that G

has either a large independent set (red clique in the coloring of K_n) or clique (blue clique in the coloring of K_n).

Just recently, it was shown that for $N \geq (4 - 2^{-7})^k$, any two-coloring of the edges of K_N has a monochromatic copy of K_k (see [11]. Equivalently, any K_k-free graph on $N \geq (4 - 2^{-7})^k$ vertices has an independent set of size k. This result represents an upper bound on the *Ramsey number* $R(k, k)$, which is the minimum number of vertices N for which any two coloring of the edges of K_N has a monochromatic K_k.

Lower bounds on $R(k, k)$ are proven by showing that there exist graphs on N vertices whose edges can be two-colored without a monochromatic K_k. All known lower bounds have been obtained probabilistically, that is, by showing that with high probability such graphs exist, without actually constructing one. See [23] for the first proof of a lower bound.

Developing a polynomial-time algorithm to construct graphs on N vertices with the property that for all two-colorings of the edges, there are no monochromatic $K_{c \log N}$ for c some constant is an open problem of significant interest (see [5]). Alternatively, we could state this problem as the task of finding a polynomial time algorithm to construct N-vertex graphs which are $K_{c \log N}$-free.

Forbidden subgraph characterizations are of interest in geometric graph theory, as well, because it is often more computationally taxing to verify that a graph is a member of a geometrically defined class than to check for the presence of a finite set of small subgraphs. For example, it is NP-hard to determine that a given graph is a unit distance graph, or a graph which can be drawn in the plane with all edges of length one (see 3.2 for formal definition). It is possible to determine slightly more quickly that a graph is *not* a unit distance graph by showing that it contains one of the forbidden subgraphs of the class of unit distance graphs.

Finding forbidden subgraphs of unit distance graphs is itself no easy task, however. For unit distance graphs on seven or fewer vertices, there are six forbidden subgraphs (see [13]). The authors of [30] obtain a forbidden subgraph characterization for unit distance graphs on up to 9 vertices, using SageMath, rigidity theory (see 3.4), and elementary geometry. Their characterization includes 74 minimal forbidden subgraphs; a forbidden subgraph characterization of unit distance graphs on 10 or more vertices may be prohibitively computationally taxing.

There are many similar characterizations in terms of substructures other than subgraphs. For example, Kuratowski's characterization of planar graphs

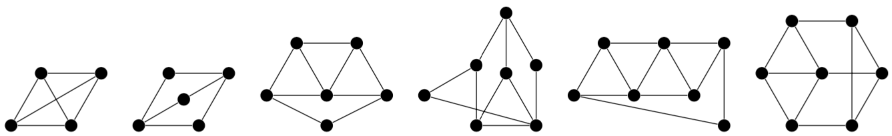


Figure 4: The Six Forbidden Subgraphs of Unit Distance Graphs on 7 or Fewer Vertices ([30], [13])

is stated in terms of forbidding subgraphs which are *homeomorphic* to two given subgraphs, rather than forbidding specific subgraphs themselves. We say a graph G is homeomorphic to a graph H if H is obtained from G by a sequence of edge subdivisions.

Theorem 3. (Kuratowski's Theorem, [6]) A graph G is non-planar if and only if it may be obtained from K_5 or $K_{3,3}$ by a sequence of edge subdivisions and edge/vertex additions.

Out of research on planar graphs, or graphs with no crossings, came questions about the minimum number of crossings in drawings of non-planar graphs. We give an overview of these problems in the next subsection.

3.1.2 Crossing Number Problems

Crossing number problems have their roots in sociometric research of the 1930s and 1940s. Seeking a visually communicative method for illustrating social networks, Moreno ([55],1934) and Bronfenbrenner ([10], 1944) aimed to find drawings of graphs which minimized edge crossings. Turán is credited with the first mathematical expression of this type of problem. His Brick Factory Problem asks for the minimum number of crossings possible in a drawing of a complete bipartite graph. Current best-known upper bounds are due to Zarankiewicz and Urbanik, who independently obtained the following result in 1955:

Proposition 1. (Zarankiewicz and Urbanik, 1955) The minimum number of crossings in a complete bipartite graph $K_{n,m}$ is at most

$$\left\lfloor \frac{n}{2} \right\rfloor \left\lfloor \frac{n-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor.$$

This bound is demonstrated by drawing the vertices of complete bipartite graph on the x and y axes of $\mathbb{R}^2$, with one class distributed as evenly as possible on either side of the origin on x-axis and the other class distributed as evenly as possible on either side of the origin on the y-axis.

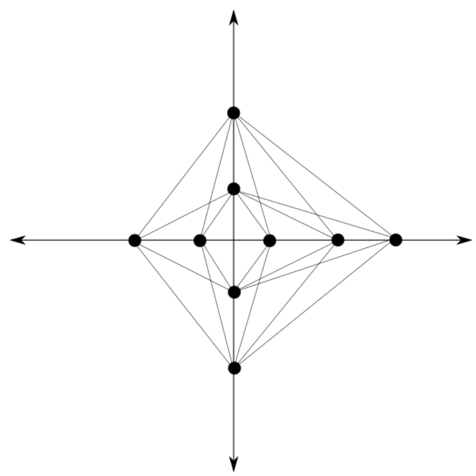


Figure 5: Zarankiewicz's Construction for Minimizing Bipartite Graph Crossings, shown for $K_{4,5}$

Work in this area has been generalized far beyond bipartite graphs, leading to the following definition and theorem:

Definition 1. (Crossing Number, [6]) The crossing number $\mathrm{cr}(G)$ of a graph G is the minimum number of edge crossings in any drawing of G in the plane.

Theorem 4. (Crossing Number Lemma, [6]) *Let G be a simple graph with n vertices and e edges. Then for some positive constant c (where we can take $c = 1/64$),*

$$\mathrm{cr}(G) \geq c\frac{e^3}{n^2} - n.$$

The proof of this result in [6] gives a valuable introduction to probabilistic proofs in geometric graph theory. Improvements on the bound given above typically involve increasing the constant c without significantly increasing lower bounds on the number of edges in a graph for which the inequality holds.

The Crossing Number Lemma is a critical piece of Székely's proof (see [66]) of the Szemerédi-Trotter Theorem, stated below. The Szemerédi-Trotter Theorem itself is a key fact in the study of the number of unit distances possible in a configuration of n points in the plane (see Section 3.2). The Szemerédi-Trotter Theorem is phrased in terms of incidences; we say a point is *incident* to a line if it lies on that line.

Theorem 5. (Szemerédi-Trotter Theorem, 1983) *Given a set of $n \geq 1$ points and $m \geq 1$ lines in the plane, the maximum number of incidences $I(n, m)$ is $\mathcal{O}(n^{2/3}m^{2/3} + n + m)$.*

Equivalently, there exists some positive constant c so that $I(n, m) \leq c(n^{2/3}m^{2/3} + n + m)$. Székely's proof defines a graph having as its vertex set the given n points and as its edge set the pairs of vertices which are incident to the same line and have no other point between them on that line. Counting the edges of this graph, upper bounding the number of crossings, and applying the Crossing Number Lemma yields the result.

We briefly present three variants of the crossing number problem, along with current bounds which may allow improvement.

I. *Rectilinear Crossing Number Problem ([58]):* The rectilinear crossing number of a graph is the minimum number of crossings in any drawing with each edge represented by a straight line segment.

   Although requiring edges to be straight lines often increases the number of crossings, the construction of Zarankiewicz for bipartite graphs has the same minimum number of crossings when we restrict to straight line edges. Therefore, if the associated upper bound is the true minimum number of crossings for bipartite graphs, the rectilinear crossing number of bipartite graphs is equal to the crossing number of bipartite graphs.

   As part of the Rectilinear Crossing Number Project ([63]), the rectilinear crossing number of all complete graphs K_n for $n \leq 17$ have been determined.

II. *Odd-Crossing Number Problem ([58]):* The odd crossing number of a graph is the minimum number of pairs of edges which cross each other an odd number of times in any drawing in the plane.

   By the following result, the odd crossing number of any non-planar graph is at least one.

Theorem 6. (Hanani-Tutte Theorem, 1970) Every drawing in the plane of a non-planar graph contains a pair of disjoint edges (no shared endvertices) that cross each other an odd number of times.

The authors of [41] show that the maximum number of edges $m_k^{\text{odd}}(n)$ in a graph on n vertices which can be drawn so that any edge is crossed an odd number of times by at most k other edges is $\sqrt{32kn}$. This article was published just this year, and both the limitations of the result and the renown of the mathematicians working on it should indicate how difficult results on this problem can be.

III. *Crossing Number on the Torus ([24]):* Given a graph G, the toroidal crossing number $\text{cr}_1(G)$ is the minimum number of crossings among all drawings of G on the surface of the torus. We see that the toroidal crossing number is at most the (planar) crossing number.

Early work, including bounds on the toroidal crossing number of K_n and $K_{m,n}$ are obtained in [33] and [32]. Exact toroidal crossing numbers for $K_{m,n}$ for $m, n > 6$ remain unknown ([72]).

Links between different crossing number variants and the torioidal crossing number are also of interest. Pach and Tòth established the following correspondence between the toroidal crossing number and planar crossing number:

Proposition 2. (Theorem 1 in [57]) Let G be a graph of n vertices with maximum degree Δ, and suppose G has a crossing-free drawing on the torus. Then $\text{cr}(G) \leq cdn$ for some constant c.

This bound is tight for all $d \geq 3$.

Broadly, questions stated in terms of graphs of a given *genus* are common in topological graph theory; we say a graph is of genus g if it can be embedded with no crossings on a sphere with g "handles" added. For example, graphs with toroidal crossing number 0 are of genus 1, since the torus is topologically equivalent to a sphere with one handle added (see Figure 6). For more problems dealing with graphs embedded in spaces other than $\mathbb{R}^d$, see 3.1.3.

A very thorough, regularly updated survey of variants of the crossing number problem is provided in [62].

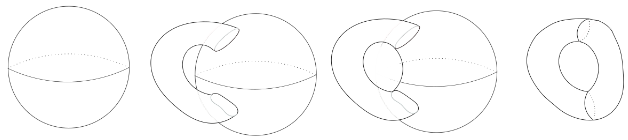


Figure 6: The torus is topologically equivalent to the sphere with an added handle

3.1.3 Edge-Maximal Graphs on Surfaces

After briefly sparking interest in the 1970s, the topic of edge-maximal graphs on surfaces has seen a resurgence of interest in the past decade. Given a graph class $\mathcal{G}$, a member G is *edge-maximal* if adding any edge to G would cause it to exit the class $\mathcal{G}$. A graph G is *embeddable* in a surface Σ if it can be represented so that the vertices of G are distinct points on Σ and no two edges intersect other than at shared endvertices ([69]).

Euler's formula relates the number of vertices v, edges e, and faces f of a graph embedded on a surface of *genus* g, or onto a surface with g "holes."

Theorem 7. (Euler's Formula, [61]) Let G be a connected graph drawn on a genus g surface with every face 2-dimensional. Then $v - e + f = 2 - 2g$.

The plane has genus 0, so the number of edges in a graph on v vertices which is embeddable in the plane is $e = v + f - 2$, which is maximized when f is as large as possible. That is, the edge-maximal graphs in the plane are triangulations. Many of the edge-maximal graphs in higher-genus surfaces are triangulations, as well. On the Klein bottle and the torus, the edge-maximal graphs are almost entirely accounted for by triangulations and complete graphs:

Theorem 8. *([19]) With the exception of $K_7 - e$, every edge-maximal graph embeddable on the Klein botle either triangulates the surface or is complete.*

Theorem 9. *([19]) With the exception of $K_8 - E(C_5)$, every edge-maximal graph embeddable on the torus either triangulates the surface or is complete.*

It is worth noting that determining whether a graph G triangulates a surface is NP-complete ([69]).

We say a graph class $\mathcal{G}$ is *pure* if all edge-maximal graphs in $\mathcal{G}$ on n vertices have the same number of edges ([54]). Theorem 9 above shows that

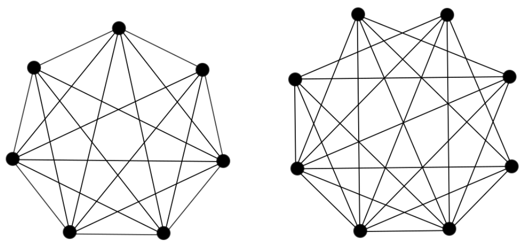


Figure 7: $K_7 - e$ and $K_8 - E(C_5)$

the class of graphs which are embeddable on the torus is not pure, since $K_8 - E(C_5)$ does not have the same number of edges as a triangulation on the torus. This discrepancy occurs for other graph classes as well, and much of the research in this area is directed toward studying the "impurity" of certain graph classes.

Following the convention of [54], say a graph class $\mathcal{G}$ is *k-impure* if for any pair of edge-maximal graphs G, H in $\mathcal{G}$ with the same number of vertices, the number of edges in G and H differ by at most k. The impurity of a graph class is tied to the genus of the surface being studied:

Theorem 10. *[54] The class of graphs embeddable in a surface of genus g has impurity at most cg for some constant c.*

As we noted in Section 3.1.1, graph classes defined in terms of forbidden substructures often behave strangely; we observe this phenomenon when studying impurity, too. Define $\mathcal{G}_H$ as the class of graphs not having H as a *minor*.

Definition 2. A graph H is a *minor* of a graph G if H can be obtained from G by removing edges and vertices and contracting edges.

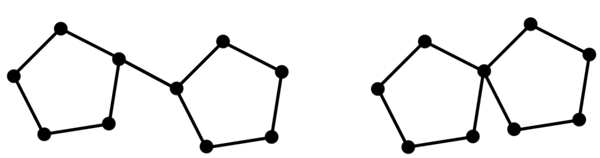


Figure 8: An example of an edge contraction

In their sweeping result provided in [53], McDiarmid and Przykucki prove that that the only connected graphs H such that $\mathcal{G}_H$ is pure are K_2, K_3, K_4, and P_4. That is, for any other connected graph H, $\mathcal{G}_H$ is impure.

There remain open problems in the area of determining just how impure these graph classes are on various surfaces. Additionally, the conjecture below remains open, and would prove the more famous Cycle Double Cover Conjecture if true (see [61] and [12]). The conjectures below refers to 2-connected (biconnected) graphs; these graphs require the removal of at least 2 vertices in order to disconnect them.

Conjecture 2. (Strong Embedding Conjecture, as in [61]) Let G be a 2-connected graph. Then there exists an embedding of G onto a genus g surface for some g so that the boundary of every face in the embedding is a cycle in G.

Conjecture 3. (Cycle Double Cover Conjecture, as in [61]) Let G be a 2-connected graph. Then there exists a collection of cycles in G such that every edge of G is used in the set of cycles exactly twice.

3.2 Unit Distance Graphs

One of the most natural restrictions to place on a geometric graph is the requirement that all edges have the same length. We call a graph with this property a unit distance graph. To avoid degeneracy arising from multiple vertices being placed at the same location in the plane, we define unit distance graph as follows:

Definition 3. (Unit Distance Graph) A graph G is a *unit distance graph* in the plane if there exists an injective homomorphism $f : V(G) \to \mathbb{R}^2$ so that $|f(v) - f(u)| = 1$ for all $uv \in E(G)$ under the usual Euclidean metric.

For ease of notation, denote by Γ the unit distance graph in $\mathbb{R}^2$. We say a unit distance graph G is edge-maximal if for any $u, v \in V(G)$, $G' = (V(G), E(G) \cup uv)$ has no injective homomorphism into Γ.

3.2.1 Foundational Results

When faced with a new class of graphs, a standard question to ask is "how many edges do its most dense members have?" The following version of this question for unit distance graphs is due to Erdős: "What is the maximum number of times that a unit distance can occur among n points in $\mathbb{R}^2$?" The result below is an asymptotic answer to this question whose proof hinges on the Szemerédi-Trotter Theorem (see Theorem 5).

Theorem 11. (Spencer-Szemerédi-Trotter Upper Bound, 1984) [65] A set of n points in the plane induces $\mathcal{O}(n^{4/3})$ unit distances, or $\leq cn^{4/3}$ for some constant c.

Since the publication of this seminal result, the upper bound on the number of unit distances on n points has been improved only in the constant term. However, it is widely conjectured that this bound is not best possible, especially in light of the size of the gap between this upper bound and known lower bounds.

Theorem 12. (Erdős Lower Bound, 1946) It is possible to construct a set of n points which induce $n^{1+c/(\log \log n)}$ unit distances for some constant c.

To incentivize work on this problem, Erdős offered \$500 for a proof that the true upper bound is equal to his proposed lower bound. This prize has yet to be claimed.

The work presented above and the work it inspired nearly answers the following question: "Given $d = 2$, what are the (dense) unit distance graphs in $\mathbb{R}^d$?" A dual problem has also been studied: "Given a graph, what is the smallest d for which it can be represented as a unit distance graph in $\mathbb{R}^d$?" Maehara and Rödl give an upper bound for this quantity:

Theorem 13. (Maehara & Rödl, 1990) If a graph has maximum degree d, then it can be represented as a unit distance graph in $\mathbb{R}^{2d}$.

In the following subsection, we return to unit distance graphs in $\mathbb{R}^2$, exploring a few classes of these graphs and variations on the standard definition of unit distance graphs.

3.2.2 Interesting Classes & Variants

Classes

I. *Generalized Petersen Graphs* were introduced as early as 1950. In [18], they are presented as an unnamed class of graphs defined as the union of a cycle graph and a star cycle graph with each vertex of the cycle adjacent to one vertex of the star cycle, and vice versa. As defined formally in [71], for integers n, k such that $1 \leq k \leq n-1$ and $2k \neq n$, a Generalized Petersen Graph $G(n, k)$ has vertex set

$$V(G(n,k)) = \{u_0, u_1, \ldots, u_{n-1}, v_0, v_1, \ldots, v_{n-1}\}$$

and edge set

$$\{u_1u_{i+1}, u_iv_i, v_iv_{i+k} | i \in \mathbb{Z}, u_i, v_i \in V(G(n,k))\}$$

where all subscripts are considered modulo n. We write the canonical Petersen Graph as $G(5, 2)$ using this notation.

In [76], it is shown that all Generalized Petersen Graphs are unit distance graphs. Named examples of Generalized Petersen Graphs include the Nauru Graph and Desargues Graph, pictured below as unit distance graphs:

Recent work on these graphs has included undergraduate projects games on Generalized Petersen Graphs [52], master's thesis work on Hamilton paths of Generalized Petersen Graphs [60], and journal papers on

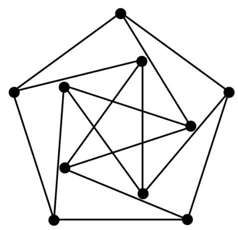


Figure 9: The Petersen Graph, $G(5,2)$

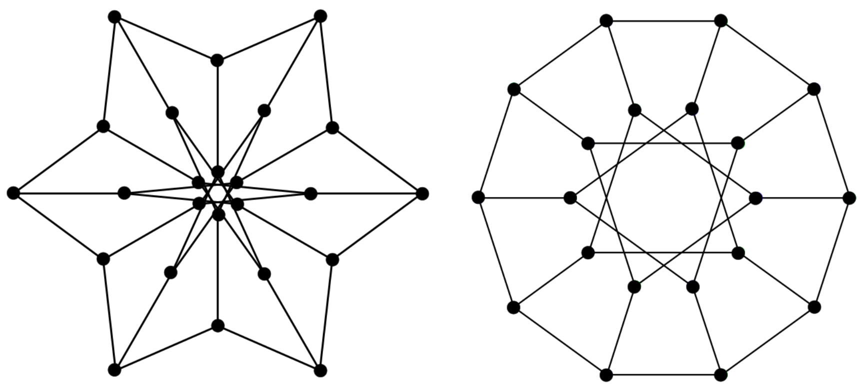


Figure 10: Nauru Graph $G(12,5)$ (adapted from [20]), Desargues Graph $G(10,3)$

colorings [46], spectral properties [29], and the underlying groups [27] of Generalized Petersen Graphs.

II. *Cartesian Products of Unit Distance Graphs*:

The Cartesian product of two graphs G, H is defined as $V(G\square H) :=$ $V(G) \times V(H)$ and $E(G\square H) :=$

$$\{(u,u')(v,v')|u = v \text{ and } u'v' \in E(H) \text{ or } u' = v' \text{ and } uv \in E(G)\}.$$

The Cartesian product of two unit distance graphs is a unit distance graph, both in the plane and in higher-dimensional space (see [35]). This fact yields an algorithm for constructing an infinite class of unit distance graphs. Several named classes of graphs are confirmed as unit distance graphs by virtue of being Cartesian products of unit distance graphs, as well. For example, all Hamming graphs $H(d, 3)$ are unit distance graphs, as the Cartesian products of d copies of K_3. Similarly, the Hypercube Graphs can be written as the Cartesian products of graphs composed of disjoint edges.

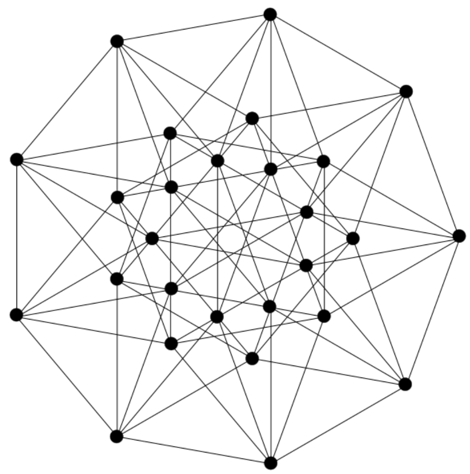


Figure 11: The Hamming Graph $H(3, 3)$ [68]

III. *Lower Bound Construction for the Erdős Unit Distance Problem*

Erdős proved the lower bound in Theorem 12 using a construction informed by a Diophantine problem. Beginning with an $\sqrt{n} \times \sqrt{n}$ subset of the integer lattice, we find m so that the graph with an edge between any two lattice points at distance $\sqrt{m}$ from each other has the maximum number of edges. Naïvely, we would seek to maximize the number of points incident to the circle of radius $\sqrt{m}$ about any given point. However, this approach does not necessarily maximize the number of edges on the $\sqrt{n} \times \sqrt{n}$ lattice, since distance $\sqrt{m}$ neighbors of a lattice point may not be included in the chosen $\sqrt{n} \times \sqrt{n}$ subset of the lattice.

In [22], Erdős considers this m as a solution to the Diophantine equation $u^2 + v^2 = m$. To find the value of m which maximizes the number of edges on the $\sqrt{n} \times \sqrt{n}$ integer lattice, we rewrite m as $m = p_1^{f_1} \ldots p_r^{f_r} m'$ where each p_i is prime and $p_i \equiv 1 \pmod 4$. Such p_i factor as $p_i =$

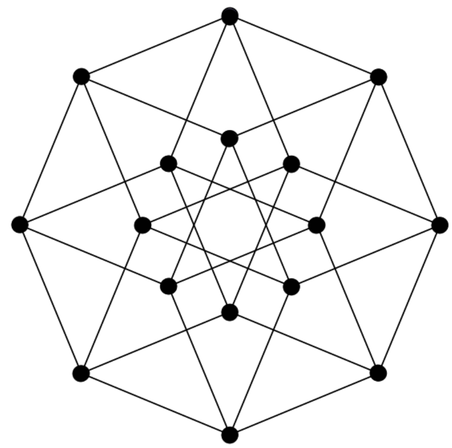


Figure 12: The Hypercube Graph Q_4

$(a_i + jb_i)(a_i - jb_i)$ in the Gaussian integers, so we have the following:

$$\begin{aligned} m &= \left(\prod_{k=1}^{t}(a_{i_k} + jb_{i_k})\right)\left(\prod_{k=1}^{t}(a_{i_k} - jb_{i_k})\right)m' \\ &= (a + jb)(a - jb)m' \\ &= a^2(m')^2 + b^2(m')^2 \\ &= (\pm am')^2 + (\pm bm')^2. \end{aligned}$$

We observe that for every prime $p_i \in \{p_1, \ldots, p_r\}$, we have a unique pair a_i, b_i which correspond to eight points on the circle of radius $\sqrt{m}$, unless one of a_i, b_i is 0. To maximize the number of edges in G, we seek m so that m has as many prime factors of the form $4k + 1$ as possible, while requiring that $\sqrt{m} \leq \sqrt{n}$.

For $\sqrt{n} \in [2, 9]$, $m = 1$ is best. For $\sqrt{n} \in [10, 49]$, choosing $m = 25$ yields the greatest number of edges. We note that for $n = 225$ $m = 25$ gives a graph with more edges than $m = 65$, even though 65 has more prime factors of the form $4k + 1$. With a smaller distance of $\sqrt{m} = 5$, more lattice points in the 15×15 lattice have all of their distance-5 neighbors included in the grid.

Asymptotically, without this additional consideration of points whose distance $\sqrt{m}$ neighbors fall outside the lattice still attains the bound.

Whenever $\sqrt{m} < \sqrt{n}/10$, the majority of the neighbors of any lattice point are contained within the $\sqrt{n} \times \sqrt{n}$ lattice.

Variants

Definition 4. (Strict Unit Distance Graph) A *faithful unit distance graph* or *strict unit distance graph* is a unit distance graph in which a pair of vertices are adjacent if and only if the Euclidean distance between them is 1.

In their beautiful paper "Two Notions of Unit Distance Graphs" ([4]), Alon and Kupavskii give an introduction to this class of graphs along with several criteria for determining whether a graph is realizable as a strict unit distance graph. The following results give a sense of what is being proven on this topic:

Theorem 14. (Theorem 1.1.3 in [4]) Any bipartite graph with maximum degree at most d in one of its parts so that no three vertices of degree d in this part have exactly the same set of neighbors is realizable as a strict unit distance graph in $\mathbb{R}^d$.

Theorem 15. (Theorem 1.4 in [4]) For any $d \geq 4$, the minimum number of edges in a bipartite graph which is *not* realizable as a strict unit distance graph in $\mathbb{R}^d$ is at least $\binom{d+2}{2}$ and at most $\binom{d+3}{2} - 6$.

Theorem 16. (Proposition 1.8 in [4]) For any $g \in \mathbb{N}$, there exists a sequence of strict unit distance graphs in $\mathbb{R}^d$ with girth greater than g such that the chromatic number of the sequence grows as $\Omega(d/(\log d))$, where the constant depends on g.

The extremal problems below remain open:

Question 1. (Problem 1 in [4]) What is the minimum number of edges $g(d)$ of a graph which is not realizable as a strict unit distance graph in $\mathbb{R}^d$?

Question 2. (Problem 4 in [4]) Define the strict unit distance Ramsey number $R_{SUD}(v, t, d)$ as the minimum integer such that for every graph G on $R_{SUD}(v, t, d)$ vertices, either G contains an induced v-vertex subgraph isomorphic to a strict unit distance graph in $\mathbb{R}^d$ or $\overline{G}$ contains an induced t-vertex subgraph isomorphic to a strict unit distance graph in $\mathbb{R}^d$. What are the best possible bounds on $R_{SUD}(v, t, d)$?

Partial results on Question 2 can be found in [43].

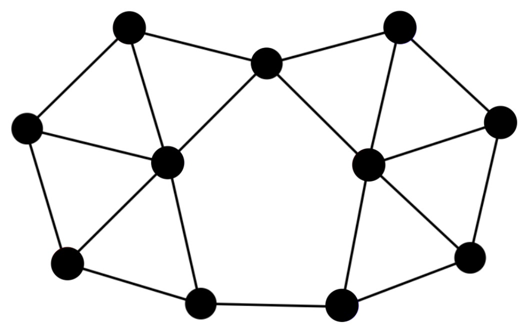


Figure 13: Example of a Penny Graph

Definition 5. (Matchstick Graph) A *matchstick graph* is a planar unit distance graph, that is, a unit distance graph which can be drawn in $\mathbb{R}^2$ with no crossing edges.

As recently as 2022, the maximum number of edges in a matchstick graph on n vertices was proven to be $\lfloor 3n - \sqrt{12n-3} \rfloor$ (see [45]). Their proof relies on Euler's Formula and the Isoperimetric Inequality, stated below:

Proposition 3. (The Isoperimetric Inequality) For any simple polygon (no self-crossings, no holes) of perimeter b and area A, $4\pi A < b^2$.

Alluding to the property that edge-maximal planar general graphs are triangulations, the authors introduce the term "lattice component" to refer to maximal biconnected subgraphs which can be embedded on a triangular lattice. The authors of [45] reach their result by proving results about the number of edges in lattice components.

Definition 6. (Penny Graphs) A penny graph has vertices corresponding to the centers of a set of non-overlapping circles of unit radius and edges between the centers of any two circles which are tangent to each other.

Question 3. *Are all planar unit distance graphs penny graphs (where we define planar unit distance graphs as having a unit distance representation with no crossing edges)? If not, what is the minimum number of distinct coin diameters needed so that all planar unit distance graphs can be represented by a set of coins?*

Theorem 17. (Koebe Representation Theorem, 1935) Every planar graph can be represented as the tangency graph of a family of nonoverlapping circular discs.

In the following generalization of Penny Graphs, we exit the class of unit distance graphs, but gain many applications to so-called "real-world problems".

Definition 7. (Unit Disk Graphs) A *unit disk graph* has vertices corresponding to the centers of circles of unit radius and edges between the centers of any two circles whose intersection is nonempty.

An early mention of this class of graphs is found in the 1980 Proceedings of the Institute for Electrical and Electronics Engineers ([34]), evidencing the importance of these graphs in engineering. At the time, research in unit disk graphs contributed to the optimization of radio frequency assignment; the problem of minimizing interference between stations is easy to translate into the language of unit disk graphs. Broadly, this type of optimization remains an active area of research, as engineers aim to both minimize interference and maximize coverage when dealing with cellular networks (see [1] for a recent overview of this).

The earliest publicly available work on this class in a mathematical journal is due to Clark, Colbourn, and Johnson ([15]) and is highly cited. In [15], the authors provide three conceptualizations of unit disk graphs:

I. Intersection Model: Given a family of sets $S_{i\,i=1}^{\,n}$ where n may be infinite, we define the vertex set of an *intersection graph* as $\{v_i\}_{i=1}^n$ and edge set as
$$\{v_iv_j | i \neq j, S_i \cap S_j \neq \emptyset\}.$$
We define the intersection graph of a set of circles as $G = (V, E)$ where V is the set of circles, and two distinct vertices are adjacent when the corresponding circles intersect or are tangent to each other. Unit disk graphs are the intersection graphs of unit circles.

II. Containment Model: Given n unit circles in the plane, an n-vertex graph with an edge between two vertices if one of the corresponding circles contains the center of the other is a unit disk graph.

III. Proximity Model: Given a set of n points in the plane, a graph obtained by constructing an edge between any two points whose pairwise distance is at most 1 is a unit disk graph.

In [7], Atminas and Zamarev find infinitely many forbidden subgraphs of unit disk graphs, including $\overline{K_2 \cup C_{2k+1}}$ for integers $k \geq 1$ and $\overline{C_{2k}}$ for integers $k \geq 4$. As shown in Figure 14, the complement of $K_2 \cup C_3$ is $K_{2,3}$, also a well-known forbidden subgraph of unit distance graphs (see the second graph from left in Figure 4).

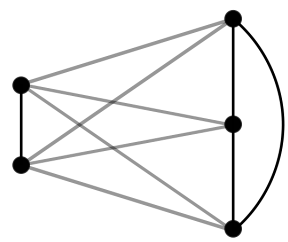


Figure 14: $K_2 \cup C_3$ (black) and $\overline{K_2 \cup C_3}$ (gray)

In Section 6, we introduce an original class of highly symmetric, separable edge-maximal unit distance graphs.

3.3 Euclidean Coloring Problems

As said by Fields Medalist Tim Gowers, "Combinatorics is the field of easy-to-ask questions." This is especially true in Euclidean Ramsey Theory, where problems are often stated without reference to any formal definitions, but rather in terms of coloring. The Four Color Theorem is a popular result in this area, known even among the general public. Informally, it shows that any map can be colored so that no two territories which share a border are the same color. We include a formal statement below.

Theorem 18. (Four Color Theorem) Any planar graph has a 4-coloring.

More challenging results are equally easy to state. For example, the venerable Hadwiger-Nelson problem, first published in 1950, asks for the number of colors required to color the plane so that no two points at unit distance from each other have the same color. This quantity is known as the Chromatic Number of the Plane. Even after several decades, this question remains open.

Until 2018, the state of the problem was that at least four and at most seven colors are needed. Lower bounds on the chromatic number of the plane are proven by constructing a finite subset of $\mathbb{R}^2$, specifically a unit distance graph, which requires at least k colors to avoid having two vertices of the same color at unit distance from each other. We call this number k the *chromatic number* of a unit distance graph G and define chromatic number for general graphs below:

Definition 8. A graph G has chromatic number k if there exists a function $f : V(G) \to [k]$ so that for all $uv \in E(G)$, $f(u) \neq f(v)$.

Interest in the Chromatic Number of the Plane was rekindled in 2018, when an amateur mathematician published an example of a unit distance graph of chromatic number 5. His graph and non-human verifiable proof can be found in *Geombinatorics*, a primary venue for results in Euclidean Ramsey Theory. Composed of many copies of the two graphs initially used to show that the Chromatic number of the plane is at least 4 (figure 15), this graph has 1581 vertices. As the saying goes, "seven is already infinite;" it is nearly impossible to obtain a human-verifiable proof that such a large graph is, in fact, of chromatic number 5. Since this initial result, considerable effort has been devoted to finding both a smaller 5-chromatic unit distance graph and

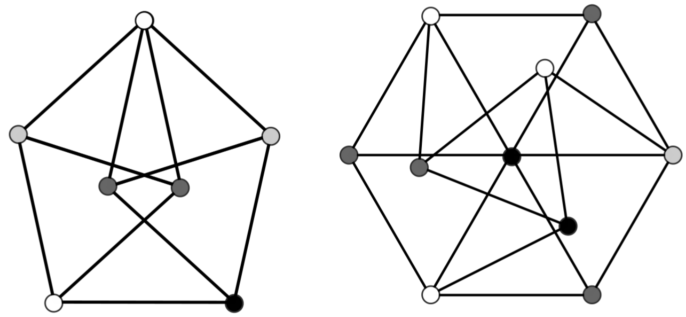


Figure 15: The Moser Spindle and Golomb Graph

ultimately a human-verifiable proof that the chromatic number of the plane is at least 5.

Substantial progress toward this end has been made in the Polymath 16 project, one of many highly collaborative online forums composing the ever-growing body of Polymath projects. A top contributor to Polymath 16, Jaan Parts, published a human-verifiable proof that the chromatic number of the plane is at least 5 in 2020 (see [59]). The accompanying graph has 481 vertices.

Perhaps more clearly reflecting the connection between Euclidean Ramsey Theory and Classical Ramsey Theory, the study of red-blue colorings (2-colorings) of Euclidean space which forbid blue unit distances and copies of a chosen graph with all vertices red are also of interest. Problems of this type are thoroughly introduced in [21]. Intricate connections between red-blue coloring problems and k-coloring problems ($k > 2$) were illuminated by an undergraduate researcher in 1999 in the following:

Proposition 4. (Szlam, 1999, [67]) If $A \subseteq \mathbb{R}^d$, a set closed under vector addition, can be colored with red and blue so that no two points of A are at unit distance from each other and for some k-element subset K of A, no translate of K in A has all its vertices red, then A can be k-colored so that no two points of A at unit distance are the same color.

This result was generalized in [3], which extends the connections between Szlam's Lemma and the Chromatic Number of the Plane. Specifically, thanks to the aforementioned result, showing that for any four-coloring of the plane which forbids unit distances for three colors and some two-point set which is not monochromatic in the fourth color is equivalent to showing the chromatic number of the plane is at least 6. For more details on this, see [38].

A seemingly endless set of problems in this area can be obtained by generalizing to higher dimensions or working with a non-Euclidean metric. The ε-chromatic number of the plane problem, is an interesting open question, too, asking for the number of colors necessary to color the plane so that monochromatic distances $d \in [d - \varepsilon, d + \varepsilon]$ for some small $\varepsilon > 0$. This quantity is known to be either 6 (see [2]) or 7 (see [26]).

3.4 Combinatorial Rigidity

Combinatorial Rigidity Theory encompasses questions relating to the "structural soundness" of graphs. More formally, we say a graph is *rigid* if it admits no continuous deformation in a chosen space. Intuitively, we are concerned with whether we can "push" on a graph, changing the location of one or more of its vertices, while maintaining the lengths of all edges in some geometric environment. The chosen space is most commonly the 2-dimensional Euclidean plane $\mathbb{R}^2$, and Laman's Theorem offers a characterization of minimal rigidity in $\mathbb{R}^2$. Currently, problems concerning rigidity properties of graphs in $\mathbb{R}^3$ are of great interest, with promising avenues of research offered to brave explorers.

Advances in rigidity theory contribute to applied mathematics, particularly in the study of rigid graphs which remain rigid even when "under attack". The study of k-vertex rigid graphs, or graphs which maintain rigidity upon the removal of any set of $k-1$ vertices, is relevant to sensor network localization. It is advantageous to guarantee that even if k sensors are lost, the relative distances of each to the other can still be interpolated. More details can be found in [25]. Often, so-called "applications" of mathematics are applications to other branches of pure mathematics, and rigidity theory has plenty of this type of application, as well.

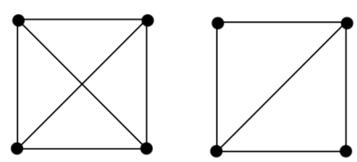


Figure 16: An edge-transitive redundantly rigid graph G (left) and $G-e$ (right)

Two foundational results in Rigidity Theory are Laman's characterization of minimal rigidity [44] and Hendrickson's characterization of global rigidity.

Theorem 19. (Laman's Theorem, 1970) *A graph $G=(V,E)$ is minimally rigid in $\mathbb{R}^2$ dimensions if and only if $|E|=2|V|-3$ and $i_G(X)\leq 2|X|-3$ for every $X\subseteq V$ of order at least 2.*

We say a graph is *redundantly rigid* if for all $e\in E(G)$, $G-e$ is rigid. Generalizing this, we say a graph G is k-edge rigid if for any set S of at most $k-1$ edges in $E(G)$, $G' := (V(G), E(G)\setminus S)$ is rigid.

Theorem 20. (Hendrickson's Characterization, 1992) *If G is a globally rigid graph in $\mathbb{R}^d$, then either G is a complete graph on at most $d+1$ vertices, or G is $(d+1)$-connected and redundantly rigid in $\mathbb{R}^d$.*

As is the case above, characterizations of rigidity properties are often given in terms of edge counts or connectivity. Connectivity and size are much easier to check computationally than rigidity properties, so these results contribute to fast algorithms for checking rigidity properties. For more examples of this type of characterization, see [40], [39], and [42].

There are copious open problems in this area. The author would be particularly interested in answers to the following:

Question 4. *What are the rigid graphs whose complement is also rigid?*

Note that by Laman's Theorem, such a graph must have at least 8 vertices. It may be advantageous to consider minimally rigid graphs. Variants of this problem can be created by replacing "rigid" with k-vertex (globally) rigid or k-edge (globally) rigid.

Question 5. *What are the minimally rigid graphs in $\mathbb{R}^3$ for which the deletion of a single vertex yields a minimally rigid graph in $\mathbb{R}^2$? Do such graphs exist?*

Other open problems are nested in Technical Reports of the Egerváry Research Group on Combinatorial Optimization and in various journals.

3.4.1 Rigidity and Unit Distance Graphs

Moments of coalescence of the study of unit distance graphs and rigidity theory have lead to interesting results. The Beckman-Quarles Theorem, stated below in both discrete and continuous forms, represents a fundamental example of a result at the intersection of these areas.

Theorem 21. (Continuous Beckman-Quarles, 1953) *For $2 \leq d < \infty$, any transformation $f : \mathbb{R}^d \to \mathbb{R}^d$ which preserves all unit distances must be a rigid motion of $\mathbb{R}^d$ onto $\mathbb{R}^d$.*

Proof. See [8] for a relatively elementary proof. □

Theorem 22. (Discrete Beckman-Quarles, Maehara [50]) *For any two points p and q at an algebraic distance α from each other, there exists a finite rigid unit distance graph G so that in any unit distance embedding of G into $\mathbb{R}^2$, the distance between p and q is α.*

Rigid unit distance graphs are easy to obtain by connecting copies of K_3 so that each copy shares an edge with at least one other copy. Triangle-free rigid unit distance graphs are more elusive, but a few have been discovered in [48] and [49]. Within the last five years, Solymosi and White [64] have found a general construction for triangle-free (infinitesimally, see [36] for definition) rigid unit distance graphs in $\mathbb{R}^d$. Their proof relies on computations involving the rigidity matrix, which is defined clearly in [36]. As is often the case in problems at the intersection of graph theory and geometry, lemmas making use of geometric constraints decrease the complexity of the problem and enable a highly computational proof.

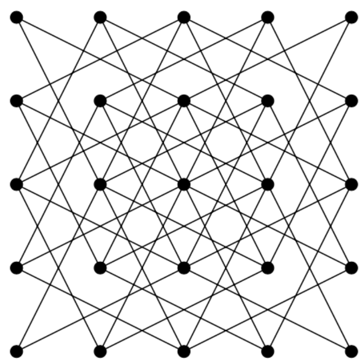


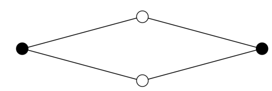
Figure 17: Example of an infinitesimally rigid K_3-free unit distance graph [64]

The following question is left as an open problem:

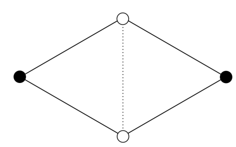
Question 6. [64] What is the minimum number of edges in an infinitesimally rigid unit-bar graph of girth $g \geq 4$?

3.4.2 Rigidity and Edge-Maximality

In the study of edge-maximal distance graphs, notions of rigidity are ever-present. Consider the cycle graph C_4. Is it an edge-maximal unit distance graph? In the embedding into $\mathbb{R}^2$ represented below, it appears to be, as there is no pair of vertices at unit distance from each other which do not also have an edge between them.



Yet C_4 is not rigid; it admits a continuous deformation (push on the two black vertices) into the representation below:



Because there exists an embedding of C_4 with a pair of vertices at unit distance having no edge between them, C_4 is not an edge-maximal unit distance graph. Had C_4 been rigid, we would not have been concerned with missing edges in some other embedding of the graph. Especially when the graph of interest is much larger than C_4, non-rigidity can greatly complicate the study of edge-maximality.

In this light, global rigidity may also yield insight into the study of edge-maximal distance graphs. A graph is globally rigid if the edge lengths determine the distance between all vertices, not just between adjacent vertices. That is, G is globally rigid in $\mathbb{R}^2$ if, given all edge lengths, there is a unique representation of G up to isometries of the plane (translations, rotations, reflections).

Observe that a graph may be rigid but not globally rigid, so that if given a representation of a supposedly edge-maximal rigid graph, one must still check that there is not some alternative embedding in which another edge could be constructed. On the other hand, if G is globally rigid and there does not exist a pair of vertices which could but do not have an edge between them under the given geometric definition of G, then G is edge-maximal in the given context.

In Section 4.1, we prove that the smallest of four parent graphs of a highly symmetric class of edge-maximal unit distance graphs is globally rigid in the plane.

Note: Questions about edge-minimality of rigid unit distance graphs have also generated recent interest, especially in recreational mathematics. For example, the study of braced polygons (see [28], [73], [51]) hinges on questions like the following: given a regular polygon in $\mathbb{R}^2$, what is the minimum number of vertices and unit distance edges which must be added to the polygon to obtain a rigid graph? This question is interesting both when asked in terms of planar unit distance graphs and in terms of graphs with crossings allowed. Current best known solutions of the latter variant are reported as sequence A218537 in the Online Encyclopedia of Integer Sequences ([74]).

4 Early Work

Motivated by the areas of research described above and the search for unit distance graphs of high chromatic number in particular, the author and advisor of this thesis set out to characterize separable, edge-maximal unit distance graphs. After obtaining initial results relatively quickly, it seemed surprising that such a characterization had not yet been obtained. The geometric constraints implied by these results led to an interesting Diophantine Problem, solved by the second reader and presented in Section 5.

Once the second reader observed an error in the early arguments leading to the aforementioned results, it became clear that characterizing separable edge-maximal unit distance graphs was a more formidable task than we had believed. In the following subsections, we explore the boundaries of this problem, expounding on several approaches which ultimately did not lead to our desired graph class, but may still prove useful to the study of unit distance graphs. We also introduce a highly symmetric subclass of finite separable edge-maximal unit distance graphs around which we center our results in Section 6.

4.1 Finite, Separable, Edge-Maximal Unit Distance Graphs

We say a graph is *separable* if there is some vertex whose removal disconnects the graph. Such a vertex is called a *cutvertex*. A graph is *biconnected* if we must remove at least two vertices to disconnect the graph. Given a finite separable edge-maximal unit distance graph G, consider two distinct nonempty biconnected components G_i, G_j and select some shared cutvertex v. Both G_i and G_j must have a vertex at Euclidean distance 1 from v, simply by virtue of being non-empty connected unit distance graphs which share v.

Rotate G_i relative to G_j so that the distance between v_i, the vertex of G_i satisfying $|v - v_i| = 1$ and v_j, the vertex of G_j satisfying $|v - v_j| = 1$, are at unit distance from each other. Denote by t_θ the homomorphism which rotates G_i by θ. We rotate G_i by an angle θ so that $|t_\theta(v_i) - v| = 1$.

Under the assumption that G is an edge-maximal unit distance graph, the edge v_iv_j should be present in $E(G)$, or we arrive at a contradiction. However, if this edge is present in $E(G)$, then we contradict the assumption that G_i and G_j are distinct biconnected components with a cutvertex v shared between them. Therefore, adding the edge v_iv_j must cause G to exit the class of unit distance graphs. That is, $G+\{v_iv_j\}$ must not be a unit distance graph. Recall the nuanced definition of *unit distance graph* provided in 3.2; a graph is a unit distance graph if it has a drawing in the plane so that all edges are unit length *and* no two vertices are drawn at the same location. We (wrongly) conclude that there must have been another vertex v_j' of G_j at the location of $t_\theta(v_i)$ and the accompanying edge v_jv_j'. This statement is incorrect, since under the given assumptions, there needs to be a vertex of G_j at the same location as *some* vertex of $t_\theta(G_i)$, not necessarily at the location of $t_\theta(v_i)$.

Definition 9. A real number $r \geq 0$ is *gonal* if it is 0 or has the form $r = \csc(\theta)/2$ for θ some rational multiple of π.

Definition 10. Real numbers $r \geq 0$ and $q \geq 0$ are *co-gonal* if they satisfy $r^2 + q^2 - 2rq\cos\theta = 1$ for θ some nonzero rational multiple of π.

Definition 11. Define Γ' as the unit distance graph in $\mathbb{R}^2$ with all points at distance $d \in (0, 1/2)$ from the origin removed.

Requiring that the necessary failure of injectivity occurs at the location of $t_\theta(v_i)$ implies certain geometric constraints, namely, that within an annulus about a cutvertex, all radii are gonal, and pairs of radii within 1 of each other are co-gonal. We translate these properties into a Diophantine problem which we solve in Section 5. With these results in hand, we obtain that the biconnected components induced on Γ' of finite, separable, edge-maximal unit distance graphs with our additional injectivity rule are subgraphs of graphs defined as follows on just four sets S_1, S_2, S_3, S_4.

Definition 12. For a finite set S of reals greater than 1 and $R = \{\csc(\pi/s)/2 : s \in S\} \cup \{0\}$, define $Q(S) = (V, E)$ as the graph on $V \subset \mathbb{R}^2$ with $(0,0) \in V$ and V and E minimal with respect to the following properties:

I. For all $\rho \in R$, there is some $x \in V$ so that $|x| = \rho$.

II. For all $x \in V$, $y \in \mathbb{R}^2$ so that $|x - y| = 1$ and $|y| \in R$, $y \in V$.

III. For all $x, y \in V$ so that $|x - y| = 1$, $xy \in E$.

The full argument for this is presented in Section 6, where we indirectly require that the failure of injectivity occurs at $t_\theta(v_i)$, the location of the vertex that participated in the proposed unit distance. In the following sections, we explain some early routes we explored in an effort to accomplish that goal–namely, to find a set of graph properties which together bring us back to our class of edge-maximal separable graphs from Definition 12.

4.2 Finite Separable Nested Edge-Maximal Unit Distance Graphs

Definition 13. A graph G is *nested* if the biconnected components of G are linearly ordered by the (unlabeled) subgraph relation.

Initially, this additional graph-theoretic constraint appeared to have potential to imply gonality and co-gonality of the radii achievied by G. However, the following graph (Figure 18) is a finite separable nested edge-maximal unit distance graph, but achieves radius $\sqrt{3}$, which is not gonal.

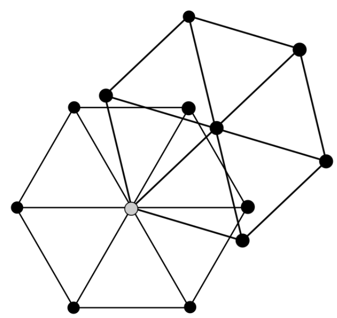


Figure 18: An example of a finite, separable, nested edge-maximal UDG which is not in our class.

Notably, the non-gonal radius occurs outside the *largest shared radius* $\tau(G)$ of the two biconnected components, or the largest radius achieved by both components. As a result, this counterexample shows only that there may exist non-gonal radii in a finite separable nested edge-maximal unit-distance graph, not that there may exist a non-gonal radius $< \tau(G)$.

Examples like the above can also be easily avoided by requiring that there exists an isomorphism between any two biconnected components which sends the cutvertices to each other. Even this requirement is not sufficient to recover gonality and co-gonality, however. We still cannot determine where the failure of injectivity occurs.

Definition 14. Given a unit distance graph G with a vertex $v \in V(G)$, a homomorphism $f : G \to \Gamma'$, and a set $X \subset V(G)$, we say that "X achieves the v-radius r" if there exists an $x \in X$ with $|f(x) - f(v)| = r$. If $v = (0, 0)$, we simply say "X achieves r". We often denote the radius of a vertex x under a homomorphism f as $|f(x)|$ and suppress the f when doing so does not cause ambiguity.

Furthermore, using the convention set in the definition above, $|f(v_j')| = |f(v_i')|$ may be distinct from $|f(v_i)| = |f(v_j)|$, so we cannot guarantee the existence of another vertex achieving the radius of $f(v_i)$. By all arguments currently known to the authors, "nestedness" is insufficient to recover the geometric properties of the graphs in Definition 12, because the pair (v_i, v_j) which corresponds to a failure of injectivity under f is relatively unrestricted.

As a somewhat more restricted subcase, consider finite separable edge-maximal unit distance graphs with isomorphic biconnected components. In this setting, the vertex pair $(v_i, v_j) \in V(G_i) \times V(G_j)$ for which $f(v_i) = f(v_j)$ must have that the neighborhood of v_i is identical to the neighborhood of some vertex of G_j *and* that the neighborhood of v_j is identical to the neighborhood of some vertex of G_i. For similar reasons to the above, this constraint is still not strong enough to guarantee multiple vertices at a given radius.

4.3 Critically Separable Subcovers

Let G and H be arbitrary graphs and Γ' the subset of the unit distance graph defined previously.

Definition 15. A graph G is *critically separable* with respect to a property P if for any edge e not in $E(G)$, if $G + e$ is in P then $G + e$ has fewer biconnected components than G.

Definition 16. Given a function $f : V(G) \to V(H)$, write f^* for the function $E(G) \to \binom{V(H)}{2}$ defined by $f^*(xy) = f(x)f(y)$ for each $xy \in E(G)$.

Definition 17. A function $f : V(G) \to V(H)$ is a *homomorphism from G to H* if $f^*(E(G)) \subseteq f^*(E(H))$.

Often, we simply write $f : G \to H$ for homomorphisms.

Definition 18. A homomorphism $f : V(G) \to V(H)$ is a *cover of H* if $f^*(E(G)) \supseteq E(H)$. (Equivalently: $f^*(E(G)) = E(H)$.)

Definition 19. A graph G is a *subcover of H* if there exist $f : V(G) \to V(H)$ and a set $S \subseteq V(H)$ so that f is a cover of $H[S]$.

Definition 20. Given a function $f : V(G) \to V(H)$, write G^f for the graph with vertex set $V(G)$ and an edge between $x, y \in V(G)$ whenever $f(x)f(y) \in E(H)$.

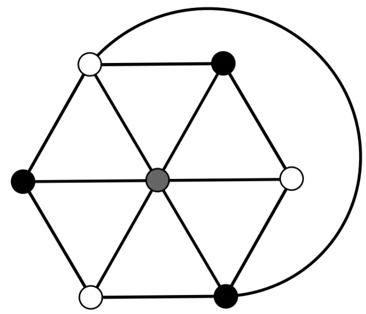


Figure 19: A finite critically separable subcover of Γ' which is not a unit distance graph

By requiring that for any edge uv of H there must exist an edge of G whose endvertices are sent to the same location as u, v under f, critically separable subcovers more closely approach a geometric definition. However, the class of critically separable subcovers of the unit distance graph includes graphs which we would not typically consider unit distance graphs, including the structure pictured in Figure 19 below:

The homomorphism sending white vertices to white vertices, black vertices to black vertices, and the gray vertex to itself is a cover, and the graph above is a subcover of Γ', on the vertices of the unit triangle. In this case and especially for larger graphs, the cover could be defined in various ways, reducing the strength of the claims we can make about subcovers of Γ'.

We conclude that finite critically separable subcovers are not the class of objects we seek, as they include graphs which do not satisfy our geometric conditions.

4.4 Finite Edge-Maximal Non-Rigid Unit Distance Graphs

As discussed in Section 3.4, rigidity and edge-maximality of geometric graphs are connected. Separable edge-maximal graphs exhibit an interesting type of non-rigidity, allowing full rotation about a cutvertex and possibly continuous deformations in a biconnected component.

To avoid dealing with deformations within a biconnected component, consider non-rigid edge-maximal unit distance graphs with globally rigid biconnected components. In the following result, we show that one of the four

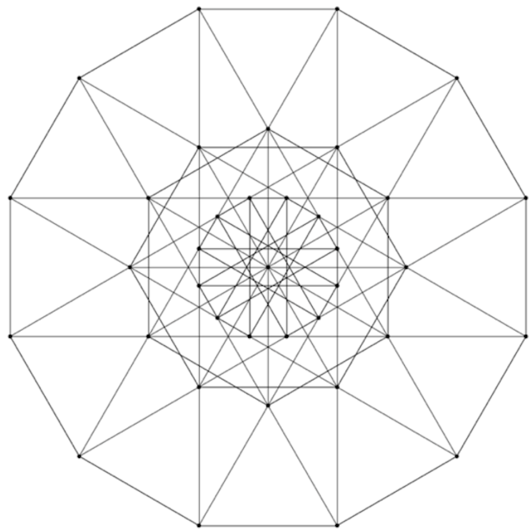


Figure 20: The Graph $Q(12/5, 6, 12)$

$Q(S)$ graphs is, in fact, globally rigid.

Proposition 5. *The graph* $Q(12/5, 6, 12)$ *as defined earlier in this subsection is globally rigid.*

Proof. As defined by Jackson, Servatius, and Servatius in [37] a graph G is *essentially 6-connected* if

I. G is 4-connected

II. for all pairs of subgraphs G_1, G_2 of G so that $G = G_1 \cup G_2$, $|V(G_1) - V(G_2)| \geq 3$, and $|V(G_2) - V(G_1)| \geq 3$, we have $|V(G_1) \cap V(G_2)| \geq 5$, and

III. for all pairs of subgraphs G_1, G_2 of G such that $G = G_1 \cup G_2$, $|V(G_1) - V(G_2)| \geq 4$ and $|V(G_2) - V(G_1)| \geq 4$, we have $|V(G_1) \cap V(G_2)| \geq 6$.

It is proven in [37] if a graph is essentially 6-connected then it is redundantly rigid. Furthermore, we have that $Q(12/5, 6, 12)$ is 4-connected and by a result of Hendrickson, Jackson, and Jordán that any 3-connected, redundantly rigid graph is globally rigid. We therefore set out to show that $Q(12/5, 6, 12)$, hereafter referred to as $Q(S)$, is essentially 6-connected.

Suppose toward a contradiction that $Q(S)$ is not essentially 6-connected. Then $Q(S)$ must contradict one of the requirements of essential 6-connectedness. We verified computationally using SageMath (see 8) that $Q(S)$ is 4-connected, so $Q(S)$ must violate either the second or third requirement of essential 6 connectedness in one of the following ways:

I. There exist a pair of subgraphs G_1, G_2 of G such that while $G = G_1 \cup G_2$, $|V(G_1) - V(G_2)| \geq 3$, and $|V(G_2) - V(G_1)| \geq 3$, we have $|V(G_1) \cap V(G_2)| < 5$.

II. There is a pair of subgraphs G_1, G_2 of G such that while $G = G_1 \cup G_2$, $|V(G_1) - V(G_2)| \geq 4$, and $|V(G_2) - V(G_1)| \geq 4$, we have $|V(G_1) \cap V(G_2)| < 6$.

For ease of discussion, we say vertices of G_1 are colored red and vertices of G_2 are colored blue, where a vertex may be colored both red and blue. The vertices of $G_1 \cap G_2$ are precisely the vertices which are both red and blue. By the requirement that $G_1 \cup G_2 = G$, we also require that the endvertices of any edge of G have the same color in any partition. (One may be red and blue while the other is only red; this counts as two vertices having the same color.)

We fail to satisfy the requirements of essential 6-connectedness if (1) it is possible to color a set X of at most 4 vertices both red and blue so that $G - X$ is disconnected and both connected components have size at least 3 or (2) it is possible to color a set Y of order at most 5 both red and blue so that $G - Y$ is disconnected and both connected components have order at least 4. These conditions are equivalent to the conditions given in the definition of essential 6-connectedness: if $G - X$ or $G - Y$ is connected, then G_1, G_2 do not satisfy $G_1 \cup G_2 = G$, as $G_1 \cup G_2$ must omit at least one edge of G, namely an edge with one end-vertex blue and the other red. Furthermore, both of the disconnected components must have size at least 3 or 4 (respectively) to ensure that the requirement on the number of vertices in exclusively one of G_1, G_2 is satisfied.

Thus, it suffices to show that upon the removal of any set of 4 (resp. 5) vertices of $Q(S)$, there do not exist two disconnected components each having order at least 3 (resp. 4). If this were the case, then there would exist a choice of red-and-blue set which would contradict that $Q(S)$ is essentially 6-connected.

Using SageMath, we check this computationally, removing all possible sets of 4 vertices (there are $\binom{37}{4}$) and, separately, all possible sets of 5 vertices and examining them, flagging any which have two sufficiently large disconnected components. Among the few choices which disconnect the graph, none have sufficiently large disconnected components. We conclude that there are no pairs G_1, G_2 of subgraphs which contradict the requirements of essential 6-connectedness, so $Q(S)$ is essentially 6-connected.

By the reasoning given at the outset of this proof, we may now conclude that $Q(S)$ is redundantly rigid and 3-connected, and therefore globally rigid. $\square$

Ultimately, the class of finite edge-maximal non-rigid unit distance graphs includes far too many elements beyond those composed from our $Q(S)$ graphs. The graph below, composed of a triangulated annulus and a non-rigid path, is also another variety of finite edge-maximal non-rigid unit distance graph.

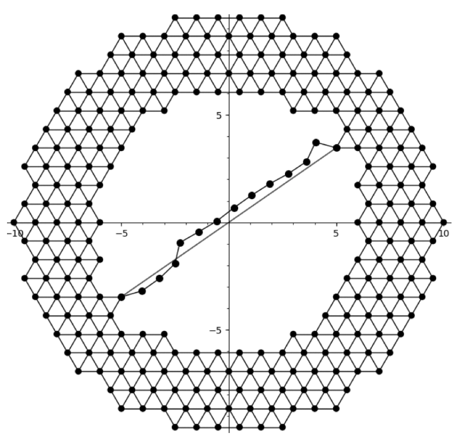


Figure 21: A graph which is a finite, separable, nested, edge-maximal unit distance graph but not in our class.

Even if we require that the graph be sufficiently nonrigid to permit a full revolution of one vertex of a component about a vertex of another component, we obtain graphs outside of our desired class. For example, a graph composed of two edge-maximal components and a set of parallel "struts" between them is finite edge-maximal and non-rigid. We conclude that non-rigidity and edge-maximality are not stringent enough constraints to eliminate strange graphs which are not members of the class we seek.

5 A Related Diophantine Problem

The work in this section is due to this thesis' second reader, Professor Michael Filaseta, who has graciously given his permission to include it here.

In this section, for each $s \in \{6, 10, 12, 15\}$, we determine all rational $x > 1$ and $z > 1$ satisfying

$$r^2 + q^2 - 2rqt = 1,$$

where $r = \csc(\pi/x)/2$, $q = \csc(\pi/s)/2$, and $t = \cos(2\pi/z)$. Note that necessarily x is not the reciprocal of an integer so that r is defined. We set $u = \sin(\pi/x)$ and still use the notation for q and t above. Then we obtain

$$\frac{1}{4u^2} + q^2 - \frac{qt}{u} = 1,$$

or, equivalently,

$$4u^2(q^2 - 1) - 4qut + 1 = 0. \tag{1}$$

Before proceeding, we note that the computations throughout this section were done using Maple 2019.2. Where numerical estimates are preformed, we set the computations to 100 digit accuracy.

We will be interested in the exact value of q for each $s \in \{6, 10, 12, 15\}$. The value of $\cos(2\pi/5)$ can be determined by noting that it is a root of $T_5(x) - 1$ where $T_5(x)$ is the fifth Chebyshev polynomial of the first kind. To clarify, we have

$$T_5(x) - 1 = (x - 1)(4x^2 + 2x - 1)^2,$$

so $\cos(2\pi/5)$ is the only positive real root, namely $(\sqrt{5}-1)/4$, of the quadratic shown. The remaining trigonometric values required to evaluate q for $s \in \{6, 10, 12, 15\}$ can be obtained using classical trigonometric identities. For $s = 6, 10, 12$ and 15, we have respectively

$$q = 1, \quad q = \frac{1 + \sqrt{5}}{2}, \quad q = \frac{\sqrt{2} + \sqrt{6}}{2},$$

and

$$q = \frac{2}{\sqrt{7 - \sqrt{30 - 6\sqrt{5}} - \sqrt{5}}}.$$

Now, we return to (1) using the above values of q for each s. For $s = 6, 10, 12$ and 15, respectively, we obtain from (1) that

$$-4ut + 1 = 0, \tag{2}$$

$$2\big(\sqrt{5}+1\big)u^2 - 2\big(\sqrt{5}+1\big)ut + 1 = 0, \tag{3}$$

$$4\big(\sqrt{3}+1\big)u^2 - 2\big(\sqrt{2}+\sqrt{6}\,\big)ut + 1 = 0, \tag{4}$$

$$4\beta\big(\alpha+\sqrt{5}-3\big)u^2 + 8\big(\alpha+\sqrt{5}-7\big)ut + \beta\big(7-\alpha-\sqrt{5}\,\big) = 0 \tag{5}$$

where, in the last equation above, we have set

$$\alpha = \sqrt{30-6\sqrt{5}} \quad \text{and} \quad \beta = \sqrt{7-\alpha-\sqrt{5}}$$

and multiplied through by $\beta^{3/2} = \beta\big(7-\alpha-\sqrt{5}\,\big)$.

As it will be more convenient to work with polynomials in u and t which have integer coefficients, we adjust the equations for each $s \in \{10, 12, 15\}$. The basic idea is to replace each of the equations for these values of s with a "norm" taken over some number field. For example, for (3), we take the product of the expression on the left of the equation with the same expression but with $\sqrt{5}$ replaced by $-\sqrt{5}$. In other words, we multiply both sides of (3) by

$$2\big(-\sqrt{5}+1\big)u^2 - 2\big(-\sqrt{5}+1\big)ut + 1.$$

We obtain for $s = 10$ that

$$-16\,u^4 + 32\,u^3t - 16\,u^2t^2 + 4\,u^2 - 4\,ut + 1 = 0. \tag{6}$$

For $s = 12$, let L_{12} denote the left-hand side of (4). We write $\sqrt{6} = \sqrt{2}\sqrt{3}$ in L_{12} and multiply both sides of (4) by three expressions, the first the same as L_{12} but with $\sqrt{2}$ replaced by $-\sqrt{2}$, the second L_{12} with $\sqrt{3}$ replaced by $-\sqrt{3}$, and the third L_{12} with both $\sqrt{2}$ replaced by $-\sqrt{2}$ and $\sqrt{3}$ replaced by $-\sqrt{3}$. This gives for $s = 12$ the equation

$$1024\,u^8 - 1024\,u^6t^2 + 256\,u^4t^4 - 512\,u^6 + 256\,u^4t^2 - 64\,u^2t^2 + 16\,u^2 + 1 = 0. \tag{7}$$

For $s = 15$, let L_{15} denote the left-hand side of (5). We first multiply both sides of (5) by the expression one gets by replacing β by $-\beta$ in L_{15}. The resulting expression on the left, say L'_{15}, only has coefficients which are linear combinations of 1, α and $\sqrt{5}$ over $\mathbb{Q}$. Then we multiply both sides of the resulting equation by the expression one gets by replacing α by $-\alpha$ in L'_{15}. The new resulting expression on the left, say L''_{15}, only has coefficients which are linear combinations of 1 and $\sqrt{5}$ over $\mathbb{Q}$. Then we multiply both sides of this last equation by the expression one gets by replacing $\sqrt{5}$ by $-\sqrt{5}$ in L''_{15}. After dividing both sides by 2^{24}, we obtain

$$\begin{aligned}
&65536\,u^{16} - 524288\,u^{14}t^2 + 917504\,u^{12}t^4 - 458752\,u^{10}t^6 \\
&\quad + 65536\,u^8t^8 - 32768\,u^{14} + 524288\,u^{12}t^2 - 557056\,u^{10}t^4 \\
&\quad + 98304\,u^8t^6 - 28672\,u^{12} - 245760\,u^{10}t^2 + 139264\,u^8t^4 - 28672\,u^6t^6 \\
&\quad + 16384\,u^{10} + 61440\,u^8t^2 - 14336\,u^6t^4 + 2560\,u^8 - 3840\,u^6t^2 \\
&\quad + 3584\,u^4t^4 - 2176\,u^6 - 512\,u^4t^2 + 128\,u^4 - 128\,u^2t^2 + 32\,u^2 + 1 = 0.
\end{aligned} \tag{8}$$

We denote the left-hand sides of equations (2), (6), (7) and (8) by $f_6(u,t)$, $f_{10}(u,t)$, $f_{12}(u,t)$ and $f_{15}(u,t)$, respectively. Thus, for $s \in \{6, 10, 12, 15\}$, we have deduced from (1) that $f_s(u,t) = 0$ where $u = \sin(\pi/x)$ and $t = \cos(2\pi/z)$ for rational numbers x and z to be determined.

A significant piece of information that we will use is that for each $s \in \{6, 10, 12, 15\}$, we have $f_s(0,t) = 1 \neq 0$. As $t = \cos(2\pi/z)$, we know $|t| \leq 1$, so we can deduce for each such s that $|u|$ cannot be too small. We seek then to obtain a lower bound on $|u|$ given $f_s(u,t) = 0$ and $|t| \leq 1$. Such a bound is possible by considering the Lagrange multiplier problem of minimizing the value of $u \in [0,1]$ and maximizing the value of $u \in [-1,0]$ given the constraints $f_s(u,t) = 0$ and $-1 \leq t \leq 1$. We first estimated such a lower bound on $|u|$, and then proceeded with providing an argument which uses exact arithmetic based on this estimate to establish the bound we want. In the way of an example, we consider $f_{15}(u,t)$. The estimate we obtained from looking at a Lagrange multiplier argument is a lower bound on u of $0.1468486\ldots$ where we restrict to $0 \leq u \leq 1$ and $-1 \leq t \leq 1$. To clarify, this lower bound occurs at $t = \pm 1$. Note that $f_{15}(u,t)$ is an even function in both u and t, so this estimate also serves as a lower bound on $|u|$ for $-1 \leq u \leq 1$. For what follows, we will use the lower bound on $|u|$ to obtain an upper bound on n, and the slightly smaller bound of $7/48 = 0.1458333\ldots$, which we will use next, will give us the same bound on n.

For the lower bound of $7/48$ on $|u|$ satisfying $f_s(u,t) = 0$ with $|t| \leq 1$, we define the rectangular region

$$\mathcal{R} = \{(u,t) : 0 \leq u \leq 7/48, 0 \leq t \leq 1\},$$

noting again that $f_s(u,t)$ is a polynomial in u^2 and t^2. It suffices then to show that the minimum value of $f_s(u,t)$ is greater than 0 on $\mathcal{R}$. So we compute

$F_u(u,t) = \partial f_s/\partial u$ and $F_t(u,t) = \partial f_s/\partial t$. We want to know the points (u,t) inside $\mathcal{R}$ where both these partial derivatives are 0. If (u_0, t_0) is such a point, we can fix $u = u_0$ and view the partial derivates $F_u(u_0, t)$ and $F_t(u_0, t)$ as polynomials in t that have a common root at $t = t_0$. As such, the resultant of these polynomials, $\operatorname{Res}(F_u(u_0,t), F_t(u_0,t))$, as polynomials in t must be 0. To obtain possible u_0, we therefore can compute $\operatorname{Res}(F_u(u,t), F_t(u,t))$ with respect to the variable t to obtain a polynomial in u, say $R(u)$, and determine its roots. However, we have chosen $\mathcal{R}$ conveniently so that $R(u)$ has no roots in the interval $(0, 7/48)$. This can be verified by using a Sturm sequence using exact arithmetic; in fact, one can verify $R(u)$ has no roots in the larger interval $(0, 1/4)$. We deduce that the minimum value of $f_s(u,t)$ on $\mathcal{R}$ occurs on an edge of $\mathcal{R}$. As $f_s(0,t) = 1$ for all t, the minimum value along the edge $u = 0$ and $0 \le t \le 1$ is 1. We proceed by looking at each of the other edges where either $u = 7/48$, $t = 0$ or $t = 1$ and $f_s(u,t)$ is a polynomial in only one unknown. For these, we merely need to look at the resulting polynomial in one variable, either u or t, and use a Sturm sequence to check that there are no roots in the interval $0 < u < 7/48$ or $0 < t < 1$, whichever applies. Given that the value of $f_s(u,t)$ is positive at each of the four corners of $\mathcal{R}$, we deduce that $f_s(u,t)$ is positive on the boundary of $\mathcal{R}$ where the minimum occurs. Thus, $f_s(u,t)$ is positive on all of $\mathcal{R}$ and $f_s(u,t) = 0$ for $|t| \le 1$ implies $|u| > 7/48$.

Very similar calculations can be performed with $f_s(u,t)$ and $s = 12$ with $0 \le u \le 15/88$ and $0 \le t \le 1$. The lower bound of $1/4$ on $|u|$ in the case that $s = 6$ is more straightforward. For $s = 10$, a modified argument from the case $s = 15$ can be done. Note that $f_{10}(u,t)$ is not an even function of u or t. We take

$$\mathcal{R}_1 = \{(u,t) : 0 \le u \le 4/21, -1 \le t \le 1\}$$

and

$$\mathcal{R}_2 = \{(u,t) : -4/21 \le u \le 0, -1 \le t \le 1\}.$$

For a lower bound on $|u|$, we are interested in finding the minimum value of u for points (u,t) in $\mathcal{R}_1$ given the constraint $f_s(u,t) = 0$ and finding the maximum value of u for points (u,t) in $\mathcal{R}_2$ given the constraint $f_s(u,t) = 0$. This can be viewed as two Lagrange multiplier problems, one for each $\mathcal{R}_j$. Even ignoring $\partial f_s/\partial u$, we see that in both cases, if $f_s(u_0, t_0)$ is a minimum or maximum with (u_0, v_0) inside but not on the boundary of $\mathcal{R}_j$, then we must have both

$$f_s(u_0, t_0) = 0 \quad \text{and} \quad \frac{\partial f_s}{\partial t}(u_0, t_0) = -4u_0(8t_0u_0 - 8{u_0}^2 + 1) = 0.$$

Thus, the number u_0 is a common root of $f_s(u, t_0)$ and $(\partial f_s/\partial t)(u, t_0)$, and hence u_0 is a root of the polynomial

$$\operatorname{Res}\left(f_s(u,t), \frac{\partial f_s}{\partial t}(u,t) \right) = 1280u^4,$$

where the resultant is taken with respect to t. Thus, the only possibility for u_0 is 0, so there are no maximum nor minimum values of f_s at points (u_0, v_0) inside but not on the boundary of $\mathcal{R}_j$. Thus, the maximum and minimum values of $f_s(u,t)$ on $\mathcal{R}_1$ and $\mathcal{R}_2$ occur on the boundaries of these rectangles. Using Sturm sequences, one can verify that there are no zeroes of $f_s(u,t)$ on the boundaries of $\mathcal{R}_1$ and $\mathcal{R}_2$, and furthermore, $f_s(u,t)$ is positive at the corners of $\mathcal{R}_1$ and $\mathcal{R}_2$. Thus, $f_s(u,t)$ has positive values for the maxima and minima for (u,t) on $\mathcal{R}_1$ and $\mathcal{R}_2$, and hence $f_s(u,t)$ cannot be 0 in these rectangles, implying what we wanted. Table 1 indicates the bounds obtained on $|u|$ through this analysis.

s	lower bound on $\lvert u\rvert$ with $f_s(u,t) = 0$ given $\lvert t\rvert \le 1$
6	$1/4 = 0.25$
10	$4/21 = 0.1904761\ldots$
12	$15/88 = 0.1704545\ldots$
15	$7/48 = 0.1458333\ldots$

Table 1: Lower bounds on the absolute value of u

Recall that $u = \sin(\pi/x)$ and $t = \cos(2\pi/z)$ where $x > 1$ and $z > 1$ are rational numbers. We write $1/x = a/n$ and $2/z = b/m$ with n, m, a and b are positive integers and $\gcd(a,n) = \gcd(b,m) = 1$. Since x and z are > 1, we have that $a < n$ and $b < 2m$. Let $\zeta = \zeta_{4nm} = e^{2\pi i/(4nm)}$. Recall $\cos(\theta) = (e^{i\theta} + e^{-i\theta})/2$ and $\sin(\theta) = (e^{i\theta} - e^{-i\theta})/(2i)$. Also, $i = \zeta_4 = \zeta^{nm}$.

Thus,

$$
\begin{aligned}
u = \sin(\pi/x) &= \frac{e^{i\pi a/n} - e^{-i\pi a/n}}{2i} \\
&= \frac{e^{2\pi i\cdot 2ma/(4nm)} - e^{-2\pi i\cdot 2ma/(4nm)}}{2i} = \frac{\zeta^{2ma} - \zeta^{-2ma}}{2\,\zeta^{nm}}
\end{aligned}
$$

and

$$
\begin{aligned}
t = \cos(2\pi/z) &= \frac{e^{i\pi b/m} + e^{-i\pi b/m}}{2} \\
&= \frac{e^{2\pi i\cdot 2nb/(4nm)} + e^{-2\pi i\cdot 2nb/(4nm)}}{2} = \frac{\zeta^{2nb} + \zeta^{-2nb}}{2}.
\end{aligned}
$$

Substituting these expressions for u and t into $f_s(u,t)$ for $s \in \{6, 10, 12, 15\}$ and multiplying by an appropriate power of ζ, we see that we may view each $f_s(u,t)$ times a power of ζ as a polynomial in ζ and, hence, as a vanishing sum of roots of unity. There is a standard idea given by J. H. Conway and A. J. Jones [17] for determining all solutions to such an equation (see [31] for more details). However, a bit of work is involved to manage such an approach, particularly on the resulting expression in ζ for $f_{15}(u,t)$. Nevertheless, realizing that $f_s(u,t)$ times a power of ζ is a polynomial in a root of unity plays a valuable role in determining the solutions of $f_s(u,t) = 0$ in rational x and z in what follows.

Our next goal is to determine an upper bound on n. Set

$$
a' = a + 2^k \Bigg(\prod_{\substack{p \mid m \\ p \nmid a}} p \Bigg) n, \quad \text{where } k \in \{1, 2\}.
$$

Observe that a and a' differ by a multiple of $2n$ so that

$$
u = \sin(\pi/x) = \sin(a\pi/n) = \sin(a'\pi/n).
$$

Thus, we may replace a with a' above, noting however that we may have $a' > n$.

We consider different possibilities for a prime q dividing $4nm$ to determine whether such q can divide a'. If a prime q divides n, then since $\gcd(a,n) = 1$, we see that $q \nmid a'$. If a prime q satisfies $q \mid m$ and $q \nmid a$, then q divides the product above and not a so that again $q \nmid a'$. If an *odd* prime q satisfies $q \nmid n$, $q \mid m$ and $q \mid a$, then q does not divide the second term in the expression for a' and so again $q \nmid a'$. This leaves us with the case that $q = 2$, n is odd, and m and a are even. If $2 \| a$ (that is, $2 \mid a$ but $4 \nmid a$), then we take $k = 2$ in our expression for a' and see that $a' \equiv 2 \pmod 4$ so that $2 \| a'$. If $4 \mid a$, then we take $k = 1$ in our expression for a' and see again that $a' \equiv 2 \pmod 4$ so that $2 \| a'$. We deduce then that $d = \gcd(a', 4nm) \in \{1, 2\}$ provided we choose $k \in \{1, 2\}$ appropriately.

Since $\gcd(a', 4nm) = d$, there exist integers v and w such that $a'v + 4nmw = d$. With v and w so chosen, we see that $a'v \equiv d \pmod{4nm}$. We claim that $\gcd(v, 4nm) = 1$. If q is an odd prime dividing v and $4nm$, then $q \mid (a'v + 4nmw)$ contradicting that $a'v + 4nmw = d \in \{1, 2\}$. On the other hand, if $2 \mid v$, then $a'v + 4nmw = d \in \{1, 2\}$ implies $d = 2$. Since $d = \gcd(a', 4nm)$, we deduce a' is even and then that $a'v + 4nmw$ is divisible by 4, contradicting that $a'v + 4nmw = d \in \{1, 2\}$. We deduce that v is odd and $\gcd(v, 4nm) = 1$.

We now consider our equations $f_y(u, t) = 0$ times a power of ζ (depending on s) as polynomial equations in ζ. Since $\gcd(v, 4nm) = 1$ and $\zeta = \zeta_{4nm}$, the mapping $\phi_v(\zeta) = \zeta^v$ is an automorphism of the field $\mathbb{Q}(\zeta)$ that fixes $\mathbb{Q}$ (see [70]). Observe that

$$\phi_v(u) = \phi_v\big(\sin(\pi/x)\big) = \phi_v\left(\frac{\zeta^{2ma'} - \zeta^{-2ma'}}{2\,\zeta^{nm}}\right) = \frac{\zeta^{2ma'v} - \zeta^{-2ma'v}}{2\,\zeta^{vnm}}.$$

Since $a'v \equiv d \pmod{4nm}$ and $\zeta^{4nm} = 1$, we see that $\zeta^{2ma'v} = \zeta^{2md} = \zeta_{2n}^{d}$ and $\zeta^{-2ma'v} = \zeta_{2n}^{-d}$. Also, v is odd, so $\zeta^{vnm} = \zeta_4^v = \pm i$, where here and in what follows the $\pm$ sign indicates that what holds is for one of $+$ and $-$ and not for both. Therefore,

$$\phi_v(u) = \pm\frac{\zeta_{2n}^{d} - \zeta_{2n}^{-d}}{2i} = \pm\frac{e^{2\pi i d/(2n)} - e^{-2\pi i d/(2n)}}{2i} = \pm\sin(\pi d/n).$$

Similarly,

$$\phi_v(t) = \phi_v\big(\cos(2\pi/z)\big) = \phi_v\left(\frac{\zeta^{2nb} + \zeta^{-2nb}}{2}\right) = \frac{\zeta^{2nbv} + \zeta^{-2nbv}}{2} = \cos(\pi bv/m).$$

We deduce that

$$f_s\big(\pm\sin(\pi d/n), \cos(\pi bv/m)\big) = f_s\big(\phi_v(u), \phi_v(t)\big) = \phi_v\big(f(u,t)\big) = \phi_v(0) = 0. \tag{9}$$

We are now ready to use our information from Table 1, applied to (9). Note that $|\cos(\pi bv/m)| \leq 1$. Recall that $d \in \{1,2\}$. We therefore see that if $n \geq 3$, then

$$|\pm\sin(\pi d/n)| \leq |\sin(2\pi/n)|.$$

On the other hand, for each $s \in \{6,10,12,15\}$, Table 1 provides a lower bound on the value of $|\pm\sin(\pi d/n)|$. We deduce that

$$|\sin(2\pi/n)| \geq \begin{cases} 1/4 = 0.25 & \text{if } s = 6 \\ 4/21 = 0.1904761\ldots & \text{if } s = 10 \\ 15/88 = 0.1704545\ldots & \text{if } s = 12 \\ 7/48 = 0.1458333\ldots & \text{if } s = 15. \end{cases}$$

As $|\sin(2\pi/n)|$ decreases as n increases on $[4,\infty)$, we obtain with a calculation that

$$n \leq \begin{cases} 24 & \text{if } s = 6 \\ 32 & \text{if } s = 10 \\ 36 & \text{if } s = 12 \\ 43 & \text{if } s = 15. \end{cases} \tag{10}$$

Now that we have an upper bound on n, we will obtain information on the value of m. We will make use of the following (see [75, p. 13]).

Lemma 1. *Let n and m be positive integers. Then $\mathbb{Q}(\zeta_n,\zeta_m) = \mathbb{Q}(\zeta_{\mathrm{lcm}(n,m)})$.*

Writing $u = (\zeta_{2n}^a - \zeta_{2n}^{-a})/(2i)$, we see that $u \in \mathbb{Q}(\zeta_{2n}, i) \subseteq \mathbb{Q}(\zeta_{4n})$, where the containment follows since $\zeta_{4n}^n = i$. We can therefore view $f_s(u,t)$ as a polynomial in t with coefficients from $\mathbb{Q}(\zeta_{4n})$. Let δ_s be the degree of the polynomial $f_s(u,t)$ viewed as a polynomial in t. Since $t = \cos(\pi b/m) = (\zeta_{2m}^b + \zeta_{2m}^{-b})/2$, the polynomial

$$h_s(\eta) = \eta^{\delta_y} f_y\big(u, (\eta+\eta^{-1})/2\big).$$

satisfies $h_s(\zeta_{2m}^b) = 0$. Also, $h_s(\eta) \in \mathbb{Q}(\zeta_{4n})[\eta]$. Furthermore the degree of $h_s(\eta)$ in η is $2\delta_s$. Since ζ_{2m}^b is a root of $h_s(\eta)$, we see that ζ_{2m}^b is in an extension of degree $\leq 2\delta_s$ over $\mathbb{Q}(\zeta_{4n})$. Since $\gcd(b,m) = 1$, if b is odd, then $\mathbb{Q}(\zeta_{2m}^b) = \mathbb{Q}(\zeta_{2m})$. On the other hand, if b is even, then $\mathbb{Q}(\zeta_{2m}^b) = \mathbb{Q}(\zeta_m)$. But in this case $\gcd(b,m) = 1$ implies m is odd and, hence, $\mathbb{Q}(\zeta_m) = \mathbb{Q}(\zeta_{2m})$ (the former field is contained in the latter and they both have the same degree over $\mathbb{Q}$ since m odd implies $\varphi(2m) = \varphi(m)$). Thus, in either case, $\mathbb{Q}(\zeta_{2m}^b) = \mathbb{Q}(\zeta_{2m})$. By Lemma 1, we have

$$\mathbb{Q}(\zeta_{2m}^b, \zeta_{4n}) = \mathbb{Q}(\zeta_{\mathrm{lcm}(2m,4n)}).$$

Thus, we can write $\mathbb{Q}(\zeta_{2m}^b, \zeta_{4n}) = \mathbb{Q}(\zeta_{4nm'})$, where $m' = \mathrm{lcm}(2m, 4n)/(4n) = \mathrm{lcm}(m, 2n)/(2n)$. We use that Euler's totient function φ satisfies the inequality $\varphi(4nm') \geq \varphi(4n)\varphi(m')$ regardless of the positive integer values of n and m', as can be seen by writing $\varphi(4nm')$, $\varphi(4n)$ and $\varphi(m')$ in terms of the prime factorizations of $4n$ and m'. Since $[\mathbb{Q}(\zeta_{4nm'}) : \mathbb{Q}(\zeta_{4n})] = [\mathbb{Q}(\zeta_{2m}^b, \zeta_{4n}) : \mathbb{Q}(\zeta_{4n})] \leq 2\delta_s$ and

$$[\mathbb{Q}(\zeta_{4nm'}) : \mathbb{Q}(\zeta_{4n})] = \frac{[\mathbb{Q}(\zeta_{4nm'}) : \mathbb{Q}]}{[\mathbb{Q}(\zeta_{4n}) : \mathbb{Q}]} = \frac{\varphi(4nm')}{\varphi(4n)} \geq \frac{\varphi(4n)\varphi(m')}{\varphi(4n)} = \varphi(m'),$$

we see that $\varphi(m') \leq 2\delta_s$. The definition of m' now implies that m divides $2nm'$, for some m' satisfying $\varphi(m') \leq 2\delta_s$. For each $s \in \{6, 10, 12, 15\}$, Table 2 indicates the information we use for obtaining choices for n and m.

s	$\delta_s = \deg_t f_s$	upper bound on n	condition of m
6	1	$n \leq 24$	$m \mid (2nm')$ where $\varphi(m') \leq 2$
10	2	$n \leq 32$	$m \mid (2nm')$ where $\varphi(m') \leq 4$
12	4	$n \leq 36$	$m \mid (2nm')$ where $\varphi(m') \leq 8$
15	8	$n \leq 43$	$m \mid (2nm')$ where $\varphi(m') \leq 16$

Table 2: Conditions on n and m

For each $s \in \{6, 10, 12, 15\}$, we use the `InverseTotient` function in Maple to determine the possible choices for m' as in the last column of Table 2. With such an m' fixed, we consider each positive integer n satisfying the upper bound in the third column of Table 2. Then we determine the possibilities for m dividing $2nm'$. We further need only consider choices for m for which $2nm' = \mathrm{lcm}(m, 2n)$. With now s, n, m' and m fixed we proceed as follows. With $\delta = \delta_s$ equal to the degree of the polynomial $f_s(u, t)$ as a polynomial in t as before and with $\delta' = \delta'_s$ equal to the degree of the polynomial $f_s(u, t)$ as a polynomial in u, we define

$$H_s(\xi, \eta) = \xi^{\delta'} \eta^{\delta} f_s\big((\xi - \xi^{-1})/(2i), (\eta + \eta^{-1})/2\big).$$

Note that $H_s\bigl(\zeta_{2n}^a, \zeta_{2m}^b\bigr) = 0$. For $y \in \{12, 15\}$, the degree in u of the terms in $f_s(u,t)$ are all even, so $H_s(\xi, \eta)$ is a polynomial in ξ and η with rational coefficients. For $s \in \{6, 10\}$, there are terms in $f_s(u,t)$ having odd degree in u, so $H_s(\xi, \eta)$ is a polynomial in ξ and η with coefficients from $\mathbb{Q}(i)$. Define $\overline{H}_s(\xi, \eta)$ in this case to be the polynomial $H_s(\xi, \eta)$ with every occurrence of i replaced by $-i$. For $s \in \{6, 10, 12, 15\}$, define

$$G_s(\xi, \eta) = \begin{cases} H_s(\xi, \eta) & \text{if } s \in \{12, 15\} \\ H_s(\xi, \eta)\overline{H}_s(\xi, \eta) & \text{if } s \in \{6, 10\}. \end{cases}$$

Recall $2nm' = \operatorname{lcm}(m, 2n)$. Then $G_s(\xi, \eta)$ is a polynomial in ξ and η with rational coefficients satisfying

$$G_s\bigl(\zeta_{4nm'}^{2m'a}, \zeta_{4nm'}^{2nm'b/m}\bigr) = G_s\bigl(\zeta_{2n}^a, \zeta_{2m}^b\bigr) = 0.$$

In other words, $\zeta_{4nm'}$ is a root of the polynomial $G_s\bigl(\gamma^{2m'a}, \gamma^{2nm'b/m}\bigr)$ in γ with coefficients in $\mathbb{Q}$.

With now s, n, m' and m fixed, we consider all integers $a \in [1, n)$ and $b \in [1, 2m)$, with $\gcd(a, n) = \gcd(b, m) = 1$. We want to check whether the polynomial $G_s\bigl(\gamma^{2m'a}, \gamma^{2nm'b/m}\bigr)$ in γ is divisible by $\Phi_{4nm'}(\gamma)$. However, we note first that the expression $\xi - \xi^{-1}$ in $H_s(\xi, \eta)$ has the same value for $\xi = \zeta_{2n}^a$ and for $\xi = \zeta_{2n}^{n-a}$ since $\zeta_{2n}^n = \zeta_2 = -1$. Thus, it suffices to consider $a \in [1, n/2]$. Similarly, $H_s(\xi, \eta)$ has the same value for $\eta = \zeta_{2m}^b$ and for $\eta = \zeta_{2m}^{2m-b}$ since $\zeta_{2m}^{2m} = 1$. Thus, it suffices to consider $b \in [1, m]$. This corresponds to restricting our solutions to $x = n/a \geq 2$ and $z = 2m/b \geq 2$. For solutions to $x > 1$ and $s > 1$, one can then replace each $x \geq 2$ with $x/(x-1)$ and each z with $z/(z-1)$.

If $G_s\bigl(\gamma^{2m'a}, \gamma^{2nm'b/m}\bigr)$ is divisible by $\Phi_{4nm'}(\gamma)$, then we have possible $x = n/a$ and $z = 2m/b$ giving rise to a solution to (1). As the n, a, m and b which occur leading to $\Phi_{4nm'}(\gamma)$ dividing $G_s\bigl(\gamma^{2m'a}, \gamma^{2nm'b/m}\bigr)$ are fairly small, Maple is able to verify directly whether (1) holds. Thus, throughout the computations, only exact arithmetic is used. In this way, we were able to determine a complete list of rational solutions in $x > 1$ and $z > 1$ to (1). The solutions with $x \geq 2$ and $z \geq 2$ are provided in Table 3.

s	$x \geq 2$	$z \geq 2$
6	6	6
6	10	10
6	10/3	5
6	12	24
6	12/5	24/5

s	$x \geq 2$	$z \geq 2$
10	6	10
10	10	10
10	15	20
10	15/4	20

s	$x \geq 2$	$z \geq 2$
12	6	24
12	12	12
15	10	20
15	15	15

Table 3: All solutions to (1) with $x \geq 2$ and $z \geq 2$

6 Relative Rigidity

6.1 Gonality and Co-gonality in G_{n-1}

A key problem arising in the study of finite, separable, edge-maximal unit distance graphs is that while our argument implies that if there exist $u \in G_i, v \in G_j$ so that for some embedding f, $|f(u) - f(v)| = 1$, then there must exist a pair of vertices $u_0 \in G_i, v_0 \in G_j$ so that $f(u_0) = f(v_0)$, it does not allow us to determine any significant further information about u, v. For example, we know that $f(u_0) = f(v_0)$, but we do not know the coordinate location of $f(u_0) = f(v_0) \in \mathbb{R}^2$.

To circumvent this issue, we coined the term "relatively rigid," defined below:

Definition 21. A finite, edge-maximal unit distance graph G is *relatively rigid* if 1) G is separable, 2) the set of biconnected components can be linearly ordered $G_1 \subseteq G_2 \subseteq \dots$ and 3) if $f(v_0) = f(u_0)$ for any $u_0 \in G_i$, $v_0 \in G_j$, $i < j$, and homomorphism $f : G \to \Gamma'$ which is injective on each G_i (but not necessarily on G), then for all $u \in G_i$, there exists $v \in G_j$ so that $f(v) = f(u)$.

Before proceeding, we present and reintroduce a few important definitions, including Γ' as used above:

Definition 22. We call a graph G a *unit distance graph* if it admits an injective homomorphism into $\Gamma = (\mathbb{R}^2, \{xy : |x - y| = 1\})$.

Definition 23. Define Γ' as the unit distance graph in $\mathbb{R}^2$ with all points at distance $d \in (0, 1/2)$ from the origin removed.

Definition 24. Given a unit distance graph G with a vertex $v \in V(G)$, a homomorphism $f : G \to \Gamma'$, and a set $X \subset V(G)$, we say that "X achieves the v-radius r" if there exists an $x \in X$ with $|f(x) - f(v)| = r$. If $v = (0, 0)$, we simply say "X achieves r". We often denote the radius of a vertex x under a homomorphism f as $|f(x)|$ and suppress the f when doing so does not cause ambiguity.

Definition 25. We denote by $\tau(G)$ the largest value of $|f(v')|$ for any $v' \in V(G_{n-1})$ under any $f : G \to \Gamma'$ which is injective on the biconnected components of G. For any separable unit distance graph, $\tau(G) \geq 1$.

We turn our attention to finite, relatively rigid edge-maximal unit distance graphs and show that if a graph G falls in this class, nearly all the biconnected components of G are subgraphs of $Q(S)$.

Throughout this section, we assume that G is a relatively rigid, edge-maximal finite unit distance graph with a cut-vertex v which separates the graph into subgraphs $G_1, \ldots, G_k$ all containing v, and v is fixed at the origin by every homomorphism into $\mathbb{R}^2$ we consider. Furthermore, for such a homomorphism f of a graph G, a subset $X \subset V(G)$ and a real number θ, write f_θ to agree with f on $V(G) \setminus X$ and $f_\theta(u) = t_\theta(f(u))$ for $u \in X$. Then an injective f_θ is also a unit distance embedding of G if X is a biconnected component of G.

In the following lemmas, we assemble the results necessary to prove the theorem(s) below.

Theorem 1. *If G is a relatively rigid, edge-maximal finite unit distance graph on Γ' with biconnected components $G_1 \subseteq \cdots \subseteq G_n$, then all radii achieved by one of $G_1, \ldots G_{n-1}$ are gonal, and any two radii achieved by some element(s) of $\{G_1, \ldots G_{n-1}\}$ which differ by at most 1 are co-gonal.*

From this point forward, we fix an arbitrary homomorphism $f : G \to \Gamma'$, which is injective on the biconnected components of G. For ease of notation, set $G := f(G)$ and $G_i := f(G_i)$ for $i \in [n]$. For a vertex $v_i \in V(G_i)$ and $v_i \in V(G_j)$, we say $v_i = v_j$ if $f(v_i) = f(v_j)$.

Lemma 2. *For G_i, G_i biconnected components of G with $G_i \subseteq G_j$, any radius achieved by G_i is also achieved by G_j.*

Proof. Both G_i and G_j have at least one vertex achieving radius 1. Therefore, there must exist an angle θ so that $t_\theta(x) = y$ where $x \in G_i, y \in G_j$ are vertices which achieve radius 1. By relative rigidity, for any other vertex $v \in G_i$ achieving radius r, there must exist a vertex $u \in G_j$ so that $t_\theta(v) = u$. That is, $r = |t_\theta(v)| = |u|$, so G_j achieves radius r. $\square$

Lemma 3. *For any r_i achieved by $v_i \in V(G_i)$ and r_j achieved by $v_j \in V(G_j)$ satisfying $|r_i - r_j| \leq 1$, there exists an angle θ so that $|v_j - t_\theta(v_i)| = 1$.*

Proof. Consider the angle ϕ for which v, $t_\phi(v_i)$ and v_j fall along a ray extending from $v = (0, 0)$. Then $|t_\phi(v_i) - v_j| = |r - r'| \leq 1$. Furthermore, $|t_{\phi+\pi}(v_i) - v_j| = |r + r'| \geq 1/2 + 1/2 = 1$, since we exclude points at distance less than $1/2$ from v in our definition of Γ (22). Because the distance between $t_{\phi+q\pi}(v_i)$ and v_j is continuous as we increase q along $\mathbb{R}$ from 0 to 1, by the Intermediate Value Theorem there must be some q at which $|t_{\phi+q\pi}(v_i) - v_j| = 1$. $\square$

Definition 26. A real number $r \geq 0$ is *gonal* if it is 0 or has the form $r = \csc(\theta)/2$ for θ some rational multiple of π.

Definition 27. A *p/q-gon* is a graph (V, E) with $V = \mathbb{Z}/p\mathbb{Z}$ and $E = \{uv | u \equiv v + q \pmod{p}\}$.

Lemma 4. *For any pair of biconnected components G_i, G_j with $G_i \subseteq G_j$, all radii achieved by G_i are gonal.*

Proof. Take a pair G_i, G_j where $i < j$, and an arbitrary vertex v_i of G_i. By Lemma 2, there exists $v_j \in G_j$ so that $|v_i| = |v_j|$. Furthermore, by Lemma 3 there exists some angle θ so that $|t_\theta(v_i) - v_j| = 1$.

Since G is an edge-maximal unit distance graph, the edge $v_i v_j$ should be in $E(G)$, contradicting that G_i, G_j are distinct biconnected components, unless $(G \setminus G_i) \cup t_\theta(G_i)$ has two vertices at the same location in $\mathbb{R}^2$.

That is, a pair $x_i \in G_i, y_j \in G_j$ satisfying $t_\theta(x_i) = y_i$ must exist. By the relative rigidity of G, for all vertices $x \in G_i$, there must be a vertex y of G_j so that $t_\theta(x) = y$. Specifically, there exists $v'_j \in G_j$ so that $v'_j = t_\theta(v_i)$, $|v'_j - v_j| = 1$, and $v_j v'_j \in E(G)$.

Equipped with the same reasoning, we have that $|t_{2\theta}(v_i) - v'_j| = 1$, so as before, we must have $x'_i \in V(G_i), y'_i \in V(G_j)$ so that $t_{2\theta}(x'_i) = y'_j$. Then, by the relative rigidity of G, there exists $v''_j \in G_j$ so that $t_{2\theta}(v_i) = v''_j$, $|v''_j - v'_j| = 1$, and $v''_j v'_j \in E(G)$.

Since $|V(G)| < \infty$, this process must terminate. That is, there must be some sufficiently large α so that the vertex $v_j^{\alpha-1}$ satisfying $|t_{\alpha\theta}(v_i) - v_j^{\alpha-1}| = 1$ is equal to $v_j^{\beta-1}$ $\beta < \alpha$, implying that for all $k \in \mathbb{Z}^+$, $v_j^{\alpha+k} = v_j^{\beta+k}$.

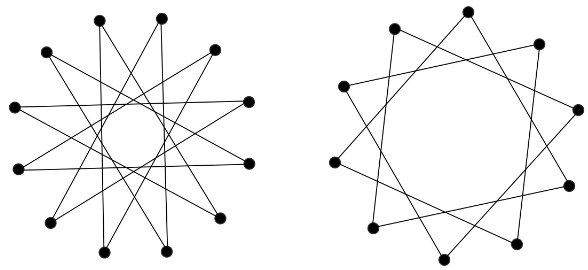


Figure 22: The $\frac{12}{5}$-gon and $\frac{10}{3}$-gon

Equivalently, we must be able to take finitely many unit-chord steps about the circle of radius r and return to our initial vertex. To satisfy this requirement, r must be the circumradius of a polygon of unit side length, where we generalize the notion of "polygon" to include "p/q-gons". That is, there exists $m \in \mathbb{Q}$ so that $r = \csc(m\pi)/2$.

□

Lemma 5. *For any radius $r \leq \tau(G)$ achieved by G_i, there exists a vertex $v_j \in V(G_j)$, $i \neq j$ achieving radius r' such that $0 < |r - r'| \leq 1$.*

Proof. Take a vertex $v_i \in V(G_i)$ achieving some radius $r \leq \tau(G)$. As before, we call the cutvertex v. By definition of $\tau(G)$, there exists a vertex $v_j \in V(G_j)$ which achieves radius $\tau(G)$.

Given that G_j is a connected unit distance graph, there must be a path P composed of unit-length steps from v to v_j. As a result, there must be some vertex of P in the annulus about v with radii $|v_j| - 1, |v_i|$, or else P is not a connected unit distance path. That is, there must be a point $v'_j \in G_j$ satisfying $0 < |v'_j| - |v_i| \leq 1$. That is, for any vertex $v_i \in G_i$ with $|v_i| \leq \tau(G)$, there exists a vertex $v'_j \in G_j$ achieving radius r' where $0 < |r - r'| \leq 1$. □

Definition 28. Real numbers $r \geq 0$ and $q \geq 0$ are *co-gonal* if they satisfy $r^2 + q^2 - 2rq\cos\theta = 1$ for θ some nonzero rational multiple of π.

Lemma 6. *Any two radii r, r' achieved by G_{n-1} satisfying $|r - r'| \leq 1$ are co-gonal.*

Proof. Suppose G is a finite relatively rigid edge-maximal unit distance graph with biconnected components $G_1 \subseteq \cdots \subseteq G_n$.

By Lemma 2 and the assumptions above, G_n achieves r. Take v_n to be a vertex of G_n achieving radius r and v_{n-1} to be a vertex of G_{n-1} achieving radius r'. Consider the representation of G in which v_n, v_{n-1}, and the origin are collinear. By lemma 3, there exists θ so that $|v_n - t_\theta(v_{n-1})| = 1$. Then, edge-maximality requires that $v_n v_{n-1} \in E(G)$, contradicting that G_n and G_{n-1} are distinct biconnected components, unless there exist $x_n \in G_n, y_{n-1} \in G_{n-1}$ so that $t_\theta(y_{n-1}) = x$.

Given $y_n \in G_n, x_{n-1} \in G_{n-1}$ so that $t_\theta(x_{n-1}) = y_n$, by the relative rigidity of G there exists $y \in G_n$ so that $t_\theta(x) = y$ for each $x \in G_{n-1}$. Specifically, there exists v'_n so that $v'_n = t_\theta(v_{n-1})$, $|v'_n - v_n| = 1$, and $v'_n v_n \in E(G_n)$.

Now, $|t_{-\theta}(v_{n-1}) - v_n| = 1$, and by the previous argument there must exist $y'_n \in G_n, x'_{n-1} \in G_{n-1}$ such that $t_{-\theta}(x'_{n-1}) = y'_n)$. Again by relative rigidity, this implies that there exists v''_n so that $v''_n = t_{-\theta}(v_{n-1})$, $|v_n - v''_n| = 1$, and $v''_n v_n \in E(G_n)$.

By assumption, G_{n-1} also achieves radius r'. Denote some such vertex by v'_{n-1}. By Lemma 3, there exists some angle ϕ so that $|v'_n - t_\phi(v'_{n-1})| = 1$. Edge-maximality requires that $v'_n v'_{n-1} \in E(G)$, contradicting that G_n and G_{n-1} are distinct biconnected components, unless there exist $x'_n \in G_n, y'_{n-1} \in G_{n-1}$ so that $t_\theta(y'_{n-1}) = x'_n$.

Given $y'_n \in G_n, x'_{n-1} \in G_{n-1}$ so that $t_\theta(x'_{n-1}) = y'_n$, by the relative rigidity of G there exists $y \in G_n$ so that $t_\theta(x) = y$ for each $x \in G_{n-1}$. Specifically, there exists v'''_n so that $f(v'''_n) = t_\theta(v'_{n-1})$, $|v'''_n - v''_n| = 1$, and $v'''_n v''_n \in E(G_n)$.

This process must terminate, since $|V(G)| < \infty$. That is, it must be possible to take finitely many unit-length steps between vertices at radius $|f(v_n)| = r$ and $|f(v_{n-1})| = r'$ alternatingly and obtain a cycle. To satisfy this requirement, triangle with side lengths $r, r', 1$ must have that the angle θ formed by the sides of length r and r' is a rational multiple of π. By the Law of Cosines, r and r' satisfy

$$1 = r^2 + r'^2 - 2rr'\cos(\theta) \text{ where } \theta \in \pi\mathbb{Q}.$$

Note: It is not possible to have a path between the cutvertex and any other vertex composed entirely of vertices achieving radii which are not co-gonal to each other, because any path must admit a vertex at radius 1, which is gonal.

$\square$

6.2 Failure of Gonality and Co-gonality in G_n

In a finite relatively rigid edge-maximal unit distance graph G with biconnected components $G_1 \subseteq \cdots \subseteq G_n$, the radii achieved by G_n need not be gonal or co-gonal to each other. The proof that radii of G_{n-1} are gonal (see Lemma 4) requires that all radii achieved by G_{n-1} are achieved by at least two components, as proven in Lemma 2. The largest component, G_n, may have radii which are not achieved by any other component of G. For example, the graph consisting of the two biconnected components represented below admits a non-gonal radius in the larger component.

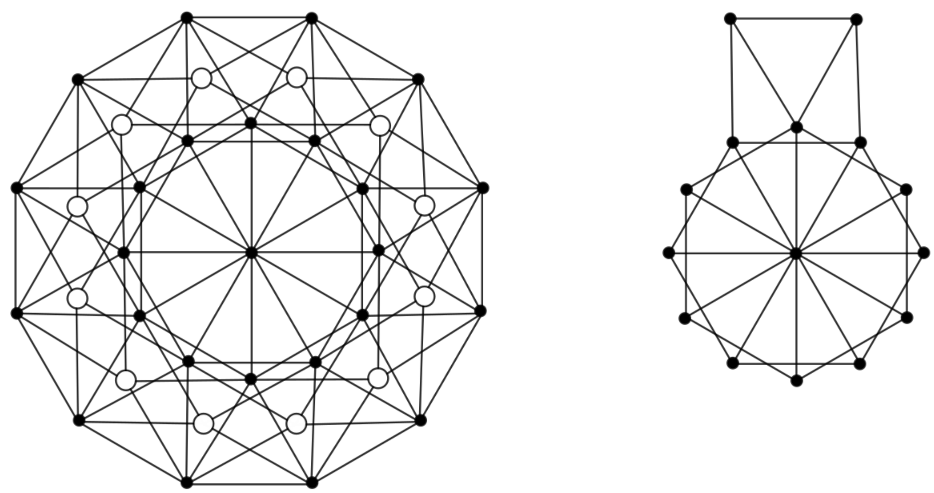


Figure 23: Example of a Finite Relatively Rigid Edge-Maximal Graph with Largest Component Achieving a Non-Gonal Radius

The white vertices achieve radius $\sqrt{2}$, which is not gonal but is co-gonal to $1/(2\sin(\pi/12))$ and 1. The proposed graph is finite and edge-maximal; adding any edge would cause the graph to have no unit distance embedding. It is also relatively rigid; we have that for any angle θ with $v_i \in V(G_i), v_j \in (G_j)$ so that $t_\theta(v_i) = v_j$, , there exists a $y \in V(G_j)$ for every $x \in (G_j)$ so that $t_\theta(x) = y$.

Furthermore, G_n may achieve radii which are neither gonal nor co-gonal to the nearby radii also achieved by G_{n-1}. With high probability, any choice of radius r in $(1/2, \tau(G))$ has θ_1, θ_2 irrational in the following equations, where r_0 is a nearby radius achieved by G_{n-1}:

$$
\begin{aligned}
1 &= r^2 + r_0^2 - 2rr_0\cos(\theta_1) \\
1 &= 2r\sin(\theta_2)
\end{aligned}
$$

We claim that the largest biconnected component G_n of a finite relatively rigid edge-maximal unit distance graph G may achieve such a radius. Choose such an r, and add copies of G_{n-1} rotated relative to each other at irrational multiples of π so that r has two neighbors achieving each radius of G_{n-1} which differ from r by at most 1. To maintain that G_n is biconnected, we require at least two vertices achieving radius r, each with at least one neighbor in each copy of G_{n-1}. A graph constructed this way is edge-maximal and finite. Under any rotation θ satisfying $t_\theta(x_0) = y_0$ for some $y_0 \in V(G_n)$ and $x_0 \in V(G \setminus G_n)$, there also exists a $y \in V(G_n)$ for every $x \in V(G \setminus G_n)$ so that $t_\theta(x) = y$. Therefore, a graph constructed as described above is also relatively rigid.

We conclude that a finite relatively rigid edge-maximal unit distance graph may admit vertices outside of gonal radii and pairs of radii which are not co-gonal, but such radii are restricted to the largest biconnected component, G_n.

7 Results

In this section, we utilize elementary geometry, the number theoretic results of Section 5, and the results of Section 6 to prove the theorem below.

Theorem 1. *If G is a relatively rigid edge-maximal finite unit distance graph on Γ' with biconnected components $G_1 \subseteq \cdots \subseteq G_n$, then G_{n-1} is contained in $Q(S)$ for*

$$S \in \{\{12/5, 6, 12\}, \{10/3, 6, 10, 15\}, \{12/5, 6, 10, 15\}, \{10/3, 6, 12\}\}.$$

Proof. Suppose G is a relatively rigid edge-maximal finite unit distance graph on Γ' with nonempty biconnected components $G_1 \subseteq \cdots \subseteq G_n$. By virtue of being a nonempty connected unit distance graph, any biconnected component G_i, $i < n$ achieves radius 1. By the result of 6, any other radius $r \in (1/2, 2)$ achieved by G_i must be gonal and co-gonal to 1.

As given in Section 5, there are six gonal radii co-gonal to 1, namely the radius of the unit p/q-gon for $p/q \in \{12/5, 10/3, 6, 10, 12\}$. Yet, we claim that there are only four graphs whose constituent polygons are compatible for coexistence in a finite, relatively rigid edge-maximal unit distance graph.

For ease of notation, we use $r_{p/q}$ to refer to the radius of the unit p/q-gon. Assume that G_i achieves r_{12}. Since $r_{12/5}$ is more than one less than r_{12}, we allow G_i to achieve $r_{12/5}$, as well, without concerns regarding co-gonality. As shown in Section 5, the only rational number co-gonal to $r_{12/5}$ is 1. Likewise, no rational number greater than r_{12} is co-gonal to r_{12}, so we obtain $S_1 = \{12/5, 6, 12\}$.

Now assume that G_i achieves r_{10}, instead. The radius of the 10/3-gon is exactly one less than the radius of the 10-gon, so it is also possible for S to contain both 10 and 10/3 in a finite graph $Q(S)$. In this scenario, $Q(S)$, being edge-maximal, has each vertex at r_{10} adjacent to one vertex at $r_{10/3}$. Furthermore, the radius r_{10} is co-gonal to the radius r_{15}, and r_{15} is co-gonal only to itself and r_{10}, yielding $S_2 = \{10/3, 6, 10, 15\}$.

Since r_{12} is not co-gonal to r_{10}, no G achieving r_{12} in any component may achieve r_{10}, Similarly, since $r_{10/3}$ is not co-gonal to $r_{12/5}$, no G achieving $r_{10/3}$ may achieve $r_{12/5}$. However, G could achieve $r_{12/5}$ and r_{10} or $r_{10/3}$ and r_{12} simultaneously, since these radii differ by more than 1 and are therefore not subject to co-gonality. We thus obtain $S_3 = \{10/3, 6, 12\}, S_4 = \{12/5, 6, 10, 15\}$.

In the argument below, we determine the number of vertices at each radius in $Q(S_i)$, $i \in [4]$. That is, we find the multiplicity of of each polygon in the four graphs.

Consider a component G_i $1 < i < n$ achieving $r_{12/5}$ and r_6. By the arguments of Section 6, any vertex of G_i achieving radius $r_{12/5}$ must have two neighbors achieving radius 1. Denote by $v_{12/5}$ a chosen vertex of G_i achieving radius $r_{12/5}$. There are two vertices v, v' achieving radius 1 which are at unit distance from $v_{12/5}$. The angle $\angle vv_{12/5}v'$ is $5\pi/6$, which is not an integer multiple of $\pi/3$, the smallest angle between any two points of the unit hexagon. We therefore require two copies of the hexagon in any biconnected component G_i which achieves radius $r_{12/5}$.

As shown in Figure 20, two copies of the hexagon rotated at an angle of $\pi/6$ relative to each other are sufficient to ensure that each point of a unit 12-gon has two neighbors at radius 1. We conclude that the graph $Q(12/5, 6, 12)$ has one 12/5-gon, two 6-gons, and one 12-gon.

For $S_2 = \{10/3, 6, 10, 15\}$, we consider the graph G_i achieving only $r_{10/3}$ and r_6. The angle witnessing co-gonality between r_6 and $r_{10/3}$ is $2\pi/5$. That is, there must be points at radius 1 differing by an angle measure of $4\pi/5$. To achieve increments of $\pi/5$ with hexagons, we need 5 copies of the hexagon. We thus obtain 30 vertices at radius 1, corresponding to 30 vertices at radius $r_{10/3}$, obtained from 3 copies of the 10/3-gon. Now consider the case in which G_i achieves $r_{10/3}, r_1$, and r_{10}. Because $|r_{10/3} - r_{10}| = 1$, we need only one neighbor at radius r_{10} for each vertex achieving radius $r_{10/3}$. Furthermore, the angle witnessing co-gonality between r_6 and r_{10} is $\pi/5$, which is accounted for by our 5 copies of the unit hexagon. Similarly, given two copies of the 15-gon, each vertex of the 10-gons has two neighbors achieving radius r_{15}, and vice versa. We conclude that the biconnected graph $Q(10/3, 6, 10, 15)$ has 3 copies of the 10/3-gon, 5 copies of the 6-gon, 3 copies of the 10-gon, and 2 copies of the 15-gon.

Likewise in the case of $Q(12/5, 6, 10, 15)$, we see that in a component G_i achieving only 12/5 and 6, we require two copies of the 6-gon. The angle witnessing co-gonality between r_6 and r_10 is $\pi/5$. The minimum angle between points we currently have at radius 1 is $\pi/6$, so we require enough polygons to give that the angle between points at radius 1 is $\pi/(\mathrm{lcm}(6, 5) = 30)$. That is, we require 60 points achieving radius 1, corresponding to 5 copies of the graph we obtained considering only 12/5 and 6. The 60 evenly spaced points at radius r_{10} are sufficient to ensure that given four copies of the 15-gon rotated so that the 60 corresponding points are evenly spaced, each vertex has two neighbors achieving radius r_{10}. We conclude that the graph $Q(12/5, 6, 10, 15)$ has 5 copies of the 12/5-gon, 10 copies of the 6-gon, 6 copies of the 10-gon, and 4 copies of the 15-gon.

Finally, we consider $Q(10/3, 6, 12)$. Given only $r_{10/3}$ and r_6, we see by the argument for $Q(10/3, 6, 10, 15)$ that there must be 3 copies of the 10/3-gon and 5 copies of the 6-gon. However, the smallest angle by which two points of these 6-gons differ is $\pi/15$, while the co-gonal angle between r_6 and r_12 is $\pi/12$. The least common multiple of 12 and 15 is 60, so there must be 10 copies of the 6-gon, along with 6 copies of the 10/3-gon and 5 copies of the 12-gon.

$\square$

8 Appendix A: Sage Code

```
# 5-checker

verts=[]
for i in range(0,37):
    verts.append(i)
S = Subsets(verts, 5, submultiset=False);
fivesets=S.list()


badgraphs=[]

for i in range(0, len(fivesets)):
    C=Q1.copy()
    C.delete_vertices(fivesets[i])
    #if C.is_connected()==True:
        #print ('Q1 is connected.')
    if C.is_connected()==False:
        #print('this is graph number', i)
        badgraphs.append(i)
        conncomps=C.connected_components()
        if (len(conncomps)==2 and len(conncomps[0])<4
            or len(conncomps[1])<4):
            badgraphs.remove(i)
        if len(conncomps)>2:
            print('Graph', i, 'has more than two
                  connected components after removal of 5-set.')
            print('___________________________________')
print(badgraphs)

#4-Checker

verts=[]
for i in range(0,37):
    verts.append(i)
S = Subsets(verts, 4, submultiset=False);
```

```
foursets=S.list()


badgraphs=[]

for i in range(0, len(foursets)):
    C=Q1.copy()
    C.delete_vertices(foursets[i])
    #if C.is_connected()==True:
        #print ('Q1 is connected.')
    if C.is_connected()==False:
        #print('this is graph number', i)
        badgraphs.append(i)
        conncomps=C.connected_components()
        if (len(conncomps)==2 and len(conncomps[0])<3
            or len(conncomps[1])<3):
            badgraphs.remove(i)
        if len(conncomps)>2:
            print('Graph', i, 'has more than two
                  connected components after removal of 5-set.')
            print('___________________________________')
print(badgraphs)
```

References

[1] YA Adediran, H Lasisi, and OB Okedere. “Interference management techniques in cellular networks: A review”. In: *Cogent Engineering* 4.1 (2017), p. 1294133.

[2] Amelia Shapiro Alan Li Maxwell Plummer and Kaylee Weatherspoon. “Epsilon Chromatic Number of the Plane is at Least 6”. In: *preprint* (2021).

[3] Kaylee Weatherspoon Alan Li Amelia Shapiro. “A Generalization of Szlam’s Lemma”. In: *Geombinatorics* 32.2 (2022), p. 75.

[4] Noga Alon and Andrey Kupavskii. “Two notions of unit distance graphs”. In: *Journal of Combinatorial Theory, Series A* 125 (2014), pp. 1–17.

[5] Noga Alon et al. “Crossing patterns of semi-algebraic sets”. In: *Journal of Combinatorial Theory, Series A* 111.2 (2005), pp. 310–326.

[6] Gergely Ambrus. *Lecture Notes in Discrete and Convex Geometry*. May 2022.

[7] Aistis Atminas and Viktor Zamaraev. “On forbidden induced subgraphs for unit disk graphs”. In: *Discrete & Computational Geometry* 60.1 (2018), pp. 57–97.

[8] Walter Benz. “An elementary proof of the Theorem of Beckmann and Quarles.” In: *Elemente der Mathematik* 42 (1987), pp. 4–9.

[9] Béla Bollobás. *Modern graph theory*. Vol. 184. Springer Science & Business Media, 1998.

[10] Urie Bronfenbrenner. “The graphic presentation of sociometric data”. In: *Sociometry* 7.3 (1944), pp. 283–289.

[11] Marcelo Campos et al. “An exponential improvement for diagonal Ramsey”. In: *arXiv preprint arXiv:2303.09521* (2023).

[12] Melody Chan. “A survey of the cycle double cover conjecture”. In: *Princeton University* (2009).

[13] Kiran B Chilakamarri and Carolyn R Mahoney. “Unit-distance graphs, graphs on the integer lattice and a Ramsey type result”. In: *aequationes mathematicae* 51.1-2 (1996), pp. 48–67.

[14] Maria Chudnovsky. “The Erdös–Hajnal conjecture—a survey”. In: *Journal of Graph Theory* 75.2 (2014), pp. 178–190.

[15] Brent N Clark, Charles J Colbourn, and David S Johnson. “Unit disk graphs”. In: *Discrete mathematics* 86.1-3 (1990), pp. 165–177.

[16] David Conlon, Jacob Fox, and Benny Sudakov. “Two extensions of Ramsey’s theorem”. In: (2013).

[17] John Conway and A Jones. “Trigonometric diophantine equations (On vanishing sums of roots of unity)”. In: *Acta Arithmetica* 30 (1976), pp. 229–240.

[18] Harold SM Coxeter. “Self-dual configurations and regular graphs”. In: *Bulletin of the American Mathematical Society* 56.5 (1950), pp. 413–455.

[19] James Davies and Florian Pfender. “Edge-maximal graphs on orientable and some nonorientable surfaces”. In: *Journal of Graph Theory* 98.3 (2021), pp. 405–425.

[20] David Eppstein. “The many faces of the Nauru graph”. In: *h ttp://11011110. livejournal. com/124705. html* (2007).

[21] P Erdos et al. *Euclidean Ramsey theorems. III, Infinite and finite sets (Colloq., Keszthely, 1973; dedicated to P. Erdos on his 60th birthday), Vol. I.* 1975.

[22] Paul Erdös. “On sets of distances of n points”. In: *The American Mathematical Monthly* 53.5 (1946), pp. 248–250.

[23] Paul Erdös. “Some remarks on the theory of graphs”. In: (1947).

[24] Paul Erdös and Richard K Guy. “Crossing number problems”. In: *The American Mathematical Monthly* 80.1 (1973), pp. 52–58.

[25] Tolga Eren et al. “Rigidity, computation, and randomization in network localization”. In: *IEEE INFOCOM 2004*. Vol. 4. IEEE. 2004, pp. 2673–2684.

[26] Geoffrey Exoo. “ε-Unit Distance Graphs”. In: *Discrete and Computational Geometry* 33 (2005), pp. 117–123.

[27] Roberto Frucht, Jack E Graver, and Mark E Watkins. “The groups of the generalized Petersen graphs”. In: *Mathematical Proceedings of the Cambridge Philosophical Society*. Vol. 70. 2. Cambridge University Press. 1971, pp. 211–218.

[28] Martin Gardner. *Three-Dimensional Maze. § 6.3 in The Sixth Book of Mathematical Games from Scientific American/M. Garner*. 1984.

[29] Ralucca Gera and Pantelimon Stanica. *The spectrum of generalized Petersen graphs*. Tech. rep. NAVAL POSTGRADUATE SCHOOL MONTEREY CA DEPT OF APPLIED MATHEMATICS, 2010.

[30] Aidan Globus and Hans Parshall. "Small unit-distance graphs in the plane". In: *arXiv preprint arXiv:1905.07829* (2019).

[31] Andrew Granville and Zeev Rudnick. "Torsion points on curves". In: *Equidistribution in Number Theory, An Introduction*. Vol. 237. Springer, Dordrecht, 2007, pp. 85–92.

[32] Richard K Guy and TA Jenkyns. "The toroidal crossing number of Km, n". In: *Journal of Combinatorial Theory* 6.3 (1969), pp. 235–250.

[33] Richard K Guy, Tom Jenkyns, and Jonathan Schaer. "The toroidal crossing number of the complete graph". In: *Journal of Combinatorial Theory* 4.4 (1968), pp. 376–390.

[34] William K Hale. "Frequency assignment: Theory and applications". In: *Proceedings of the IEEE* 68.12 (1980), pp. 1497–1514.

[35] Boris Horvat and Tomaž Pisanski. "Products of unit distance graphs". In: *Discrete mathematics* 310.12 (2010), pp. 1783–1792.

[36] Bill Jackson. "Notes on the rigidity of graphs". In: *Levico Conference Notes*. Vol. 4. 2007.

[37] Bill Jackson, Brigitte Servatius, and Herman Servatius. "The 2-dimensional rigidity of certain families of graphs". In: *Journal of Graph Theory* 54.2 (2007), pp. 154–166.

[38] Peter D. Johnson. *Sample Problems: Coloring Problems, Euclidean and Otherwise*. 2022. URL: `https://cws.auburn.edu/shared/files?id=268&filename=REU%5C%20Sample%5C%20Problems%5C%201-31-22.pdf`.

[39] Tibor Jordán. *Minimum size highly redundantly rigid graphs in the plane*. Tech. rep. TR-2020-21. `egres.elte.hu`. Egerváry Research Group, Budapest, 2020.

[40] Tibor Jordán et al. *Four-regular graphs with extremal rigidity properties*. Tech. rep. TR-2022-13. `egres.elte.hu`. Egerváry Research Group, Budapest, 2022.

[41] János Karl and Géza Tóth. "Crossing lemma for the odd-crossing number". In: *Computational Geometry* 108 (2023), p. 101901.

[42] Viktória Kaszanitzky and Csaba Király. *On minimally highly vertex-redundantly rigid graphs*. Tech. rep. TR-2014-08. `egres.elte.hu`. Egerváry Research Group, Budapest, 2014.

[43] Andrey B Kupavskii, Andrei M Raigorodskii, and Maria V Titova. "New bounds for the distance Ramsey number". In: *Discrete Mathematics* 313.22 (2013), pp. 2566–2574.

[44] Gerard Laman. "On graphs and rigidity of plane skeletal structures". In: *Journal of Engineering mathematics* 4.4 (1970), pp. 331–340.

[45] Jérémy Lavollée and Konrad J Swanepoel. "Bounding the number of edges of matchstick graphs". In: *SIAM Journal on Discrete Mathematics* 36.1 (2022), pp. 777–785.

[46] Yanyi Li and Lily Chen. "Injective edge coloring of generalized Petersen graphs". In: *AIMS Mathematics* 6.8 (2021), pp. 7929–7943.

[47] Yuchen Luo. "Mathematical Programming and Heuristic Approaches for Three Novel Vehicle Routing Problems". PhD thesis. 2022.

[48] H Maehara. "A rigid unit-bar-framework without triangle". In: *Mathematica Japonica* 36.4 (1991), pp. 681–683.

[49] H Maehara and K Chinen. "An infinitesimally rigid unit-bar-framework in the plane which contains no triangle". In: *Ryukyu Math. J* 8 (1995), pp. 37–41.

[50] Hiroshi Maehara. "Distances in a rigid unit-distance graph in the plane". In: *Discrete applied mathematics* 31.2 (1991), pp. 193–200.

[51] Curiosa Mathematica. *Braced Regular Polygons*. 2020.

[52] Bryan Lu Maxwell Fogler Katherine Levandosky and Jacob K Porter. *Modified Lights Out on Generalized Petersen Graphs*. Jan. 2023.

[53] Colin McDiarmid and Michał Przykucki. "On the purity of minor-closed classes of graphs". In: *Journal of Combinatorial Theory, Series B* 135 (2019), pp. 295–318.

[54] Colin McDiarmid and David R Wood. "Edge-maximal graphs on surfaces". In: *Canadian Journal of Mathematics* 70.4 (2018), pp. 925–942.

[55] Jacob Levy Moreno. "Who shall survive?: A new approach to the problem of human interrelations." In: (1934).

[56] János Pach. “The beginnings of geometric graph theory”. In: *Erdős Centennial* (2013), pp. 465–484.

[57] János Pach and Géza Tóth. “Crossing number of toroidal graphs”. In: *Topics in Discrete Mathematics: Dedicated to Jarik Nešetřil on the Occasion of his 60th Birthday*. Springer. 2006, pp. 581–590.

[58] János Pach and Géza Tóth. “Thirteen problems on crossing numbers”. In: *Geombinatorics* 9.4 (2000), pp. 194–207.

[59] Jaan Parts. “The Chromatic Number of the Plane is at Least 5: a Human Verifiable Proof”. In: *Geombinatorics* 30.2 (1986), p. 77.

[60] William Pensaert. “Hamilton paths in Generalized Petersen graphs”. MA thesis. University of Waterloo, 2002.

[61] Mary Radcliffe. *Embedding Graphs in Surfaces*. Sept. 2016.

[62] Marcus Schaefer. “The graph crossing number and its variants: A survey”. In: *The electronic journal of combinatorics* (2012), DS21–Apr.

[63] Institute for Software Technology. *The Rectilinear Crossing Number Project*. Available at `http://dist.ist.tugraz.at/cape5/` (2023/03/27).

[64] József Solymosi and Ethan White. “On Rigidity of Unit-Bar Frameworks”. In: *Graphs and Combinatorics* 35.5 (2019), pp. 1147–1152.

[65] Joel Spencer, Endre Szemerédi, and William T Trotter. “Unit distances in the Euclidean plane”. In: *Graph theory and combinatorics*. Academic Press, 1984, pp. 294–304.

[66] László A Székely. “Crossing numbers and hard Erdős problems in discrete geometry”. In: *Combinatorics, Probability and Computing* 6.3 (1997), pp. 353–358.

[67] Arthur D Szlam. “Monochromatic translates of configurations in the plane”. In: *Journal of Combinatorial Theory, Series A* 93.1 (2001), pp. 173–176.

[68] Taxel and Pegg. *Hamming Graph H(3,3)*. Dec. 2020.

[69] Carsten Thomassen. “Embeddings of graphs”. In: *Discrete Mathematics* 124.1-3 (1994), pp. 217–228.

[70] Lawrence C. Washington. *Introduction to cyclotomic fields*. Vol. 83. Graduate Texts in Mathematics. Springer-Verlag, New York, 1982.

[71] Mark E Watkins. "A theorem on Tait colorings with an application to the generalized Petersen graphs". In: *Journal of Combinatorial Theory* 6.2 (1969), pp. 152–164.

[72] Eric W Weisstein. "Toroidal Crossing Number". In: *https://mathworld. wolfram. com/* (1999).

[73] David G Wells. *The Penguin dictionary of curious and interesting geometry*. Penguin Mass Market, 1991.

[74] Arkadiuz Wesolowski. *Minimum number of Rods Needed to Construct Rigid Regular n-gons: OEIS A218537*. Nov. 2012.

[75] Tom Weston. *Algebraic number theory*. Viewed 05/10/2021. URL: `https://people.math.umass.edu/~weston/cn/notes.pdf`.

[76] Arjana Zitnik, Boris Horvat, and Tomaz Pisanski. "All generalized Petersen graphs are unit-distance graphs". In: *Journal of the Korean Mathematical Society* 49.3 (2012), pp. 475–491.