Biplanar Crossing Numbers of Bipartite Graphs

Sydney Miyasaki
University of South Carolina - Columbia

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By

Sydney Miyasaki

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Approved:

Eva Czabarka

Eva Czabarka
Major Professor

Laszlo Szekely

Laszlo Szekely
Second Reader

Steven Lynn
For South Carolina Honors College

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Thesis Summary

This thesis will explore the concept of a biplanar crossing number of a graph. A graph is a collection of nodes, or vertices, connected by edges. In this thesis, we draw graphs by drawing all of the nodes twice, once each on two non-intersecting planes. We draw each edge once by picking a plane to draw the edge on. When we do this, we often have points where the edges meet each other (other than where they must meet at the edges). These points are called crossings. Drawing a graph in this way is useful when designing circuitry, among other things. For circuitry, the vertices represent the components of the circuit and the edges represent wires, which may be placed on either side of the circuit board. In this application, it is desirable to minimize the number of crossings. The biplanar crossing number is the minimum number of crossing needed to draw the graph, and so computing the biplanar crossing number is an important, but difficult problem. This thesis investigates various methods of computing lower and upper bounds on the crossing number for a special family of graphs called complete bipartite graphs. Complete bipartite graphs are formed by dividing the vertices into two sets and drawing an edge between two vertices if, and only if, they are not in the same set. To compute upper bounds, we start by drawing a few graphs with few vertices and then using those graphs to build drawings of graphs with more vertices in a systematic way. To compute lower bounds, we take two approaches. One asks about what sorts of patterns must emerge on the planes, and the other uses a computer to derive new lower bounds from known existing lower bounds. The first method provides interesting insights, but the second method yields an improvement over the best known lower bound.

Abstract

The goal of this thesis is to compute upper and lower bounds on the biplanar crossing numbers of complete bipartite graphs. The concept of a biplanar crossing number was first introduced by Owens (Owens 1971) as an optimization problem in circuit design. To prove upper bounds, we follow a method used by Czabarka et. al. (Czabarka et al. 2006), in which they start from an optimal drawing of a small bipartite graph and use it to generate drawings of larger bipartite graphs. We explore several possibilities for computing lower bounds. One is using Ramsey theory, via the Bipartite Ramsey Number and the Connected Bipartite ramsey Number. We prove that these numbers are equal for complete bipartite graphs, except in a few trivial cases. The other method we use is a heavily computer-aided derivation, based on the counting method, of lower bounds for small complete bipartite graphs. This is the method used in Shavali and Zarrabi-Zadeh (Shavali and Zarrabi-Zadeh 2019). We present a slight improvement over their results.

Chapter 1

Introduction

We will define a bipartite drawing of G following the definition of a drawing in Székely (Székely 2004). A *drawing* of a graph G consists of a mapping ϕ from the vertices of G to a set of points in the plane, and a mapping of the edges to simple planar curves in the plane, such that the arc corresponding to the edge uv has $\phi(u)$ and $\phi(v)$ as endpoints, and no other points in the image of ϕ lie on the arc. The crossing number of a drawing in the sum of the number of non-vertex points of intersection in each unordered pair of edges. The crossing number $\text{cr}(G)$ of graph G is minimum crossing number of any drawing of the graph.

A *biplanar drawing* of a graph G is a partition of the edges of G into two sets E_1 and E_2, a drawing of the graph $G_1 = (V(G), E_1)$ on a plane p_1, and a drawing of the graph $G_2 = (V(G), E_2)$ on a plane p_2 which is disjoint from p_1. The biplanar crossing number $\mathcal{D}$ of a biplanar drawing $\text{cr}_2(\mathcal{D})$ is the sum of the crossing number of the drawing of G_1 on plane p_1 and the sum of the crossing number of the drawing of G_2 on plane p_2. The biplanar crossing number $\text{cr}_2(G)$ of a graph G is minimum biplanar crossing number of any biplanar drawing of the graph. Equivalently, the biplanar crossing number can be defined as $\text{cr}_2(G) = \min_{H \cup K} \text{cr}(H) + \text{cr}(K)$, where $H \cup K$ ranges over pairs of disjoint subgraphs of G whose union is G.

A drawing is *nice* if it satisfies the following five conditions:

i Any two of the curves have finitely many points in common

ii No two curves have a point in common tangentially iii No point in the plane belongs to the interior of three curves

iv No two adjacent edges cross

v Any two edges cross at most once

For any graph G, there is a nice drawing $\mathcal{D}$ of G such that $\mathrm{cr}(\mathcal{D}) = \mathrm{cr}(G)$. (Székely 2004)

A nice biplanar drawing is a biplanar drawing such that the drawing on each plane is nice. Likewise, for any graph G, there is a nice biplanar drawing $\mathcal{D}$ of G such that $\mathrm{cr}_2(\mathcal{D}) = \mathrm{cr}_2(G)$.

The crossing number of a graph was first introduced by Paul Turán. According Turán himself (Turán 1977) , the idea came to him while transporting bricks in a forced work camp. The bricks were transported on carts on rails and would fall off whenever the rails crossed each other. Wondering how the rails could be routed with fewer crossings and thus fewer hurdles, Turán posed the question we would now phrase as "What is $\mathrm{cr}(K_{m,n})$ for natural numbers m and n?" This problem is known as the Brick-Factory Problem, and was thought to be solved be Zarankiewicz. The claimed solution was

$$\mathrm{cr}(K_{m,n}) = \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor \left\lfloor \frac{m}{2} \right\rfloor \left\lfloor \frac{m-1}{2} \right\rfloor .$$

Although an error in the proof was later found, Zarankiewicz's claim remains the best-known upper bound and remains open as a conjectured solution. It is known that the smallest counter-example, if it exists, must be $K_{m,n}$ where m and n are odd, and that it holds for $\min\{m, n\} \leq 6$ (Kleitman 1970).

The biplanar crossing number and biplanarity were introduced by Owens, who framed the problem in relation to minimizing the number of vias needed to print a circuit (Owens 1971). A related concept is the *thickness* of a graph, which is the minimum number of planar subgraphs whose union is the graph. This was introduced by Tutte (Tutte 1963) before the work of Owens. The two concepts are related in that biplanar graphs are precisely the graphs with thickness of at most two.

The primary results on the the biplanar crossing number of the $K_{m,n}$ are due to Czabarka, Sýkora, Székely, and Vrt'o in Biplanar Crossing Numbers I (Czabarka et al. 2006), and these results were extended by Shavali and Zarrabi-Zadeh (Shavali and Zarrabi-Zadeh 2019). We extend the works of these two papers.

Chapter 2

Upper Bounds

In this chapter, we will develop the theory of $\mathcal{D}$-drawings and derive the Crossing Counting Formula, which will allow us to prove upper bounds on the bipartite crossing number using drawings of small graphs. We will then provide examples of such proofs. This is a formalization of a proof technique used in Czabarka et. al. (Czabarka et al. 2006) and we will use the Crossing Counting Formula to restate some proofs from their paper.

2.1 The Crossing Counting Formula

Suppose we have a graph G and a drawing $\mathcal{D}$ of G across two planes. Enumerate the vertices of G as $V = \{v_1, \cdots, v_n\}$. Let G' be a graph obtained by cloning the vertices of G. By this, we mean that we let $\{S_v : v \in V\}$ be a collection of disjoint sets, let $E' = \{xy : x \in S_v, y \in S_w, vw \in E(G)\}$, and let $G' = (\cup_{v \in V} S_v, E')$. Call the sets S_v the *arcs* of G' and for each $v, w \in V$ where $vw \in E(G)$, let the *edges of type* $v - w$ be the edges $\{xy : x \in S_v, y \in S_w\}$.

Any drawing of G' constructed from $\mathcal{D}$ in the following manner is called a *$\mathcal{D}$-drawing* of G. For each vertex v, take a circle centered at v which is small enough that

1. The circle and its interior do not contain any other vertices or any edge-crossings

2. Each edge which crosses the circle is incident to v and only crosses the circle once

Order the edges incident to v clockwise according to their intersection clockwise according to their intersection with the circle. Pick an arbitrary edge to start at. Partition these edges such that each partition is contiguous, and as equal to the other as possible. These partitions are the *sides* of v. Divide the circle into two semicircles such that the intersections corresponding the vertices of one side of v are precisely the intersection on one of the semicircles, and intersections corresponding to the vertices of the other side of v are precisely the intersections on the other arc. Choose a simple continuous curve whose endpoints are the endpoints of these semicircles. Place the vertices of S_v along this curve in any order. We allow arbitrary choices because none of our proofs will rely on how these choices are made.

Proposition 1. *Any $\mathcal{D}$-drawing of G' is a biplanar drawing of G'*

Proposition 2. *Assume $\mathcal{D}$ is nice. The crossings in a $\mathcal{D}$-drawing of G' can be classified as*

1. *crossings between two edges of the same type*
2. *crossings between a $v - w$ type edge and a $v - u$ type edge where $u \neq w$*
3. *crossings between a $v - w$ type edge and a $x - y$ type edge, where v, w, x and y are all distinct and vw and xy cross in $\mathcal{D}$.*

Furthermore, these classes are disjoint.

Proof. Consider two crossing edges in G', say e_1 and e_2. Either e_1 and e_2 are the same type, or they aren't. If they are, then the crossing they form is in class 1. Henceforth, assume they are of different types, say $v - w$ and $x - y$ respectively. Either the crossing they form is in class 2, or it isn't. Suppose that it isn't. Then v, w, x and y are all distinct. If the edges vw and xy do not cross in $\mathcal{D}$, then the minimum distance between any point on vw and any point on xy is non-zero. Thus, by taking "near" and "close" to be sufficiently small, we can ensure that the minimum distance between any point on an edge of type $v-w$ and any point on an edge of type $x-u$ is also non-zero, thus eliminating any crossing between edges of these two types. By our definition of a $\mathcal{D}$-drawing, we may assume that this occurs in our $\mathcal{D}$-drawing. Since we know that an edge of type $v-w$ and an edge of type $x-y$ cross in our $\mathcal{D}$-drawing, this contrapositively implies that e_1 and e_2 cross in $\mathcal{D}$. Therefore, the crossing formed belongs to class 3. □

A *bipartite drawing* of $K_{p,q}$ is formed by placing p vertices on one line segment and q vertices on a parallel line segment and drawing straight lines from each vertex on one line segment to each vertex on the other.

Proposition 3. *Let vw be an edge in G. Then the number of crossings between edges of type v-w in a $\mathcal{D}$-drawing is*

$$\binom{|S_v|}{2}\binom{|S_w|}{2}$$

Proof. The crossings, which occur near one of the arcs by construction, can be drawn in a manner similar to a bipartite drawing of $K_{|S_v|,|S_w|}$, thus yielding $\binom{|S_v|}{2}\binom{|S_w|}{2}$ crossings. □

Proposition 4. *In a $\mathcal{D}$-drawing, the number of crossings in Class 1 is*

$$\sum_{vw\in E(G)}\binom{|S_v|}{2}\binom{|S_w|}{2}$$

Proof. For each edge type,Proposition 3 counts the number of crossings. To determine the number of Class 1 crossings, we sum up those counts, which gives the claimed result. □

Say that two arcs S_v and S_w neighbor each other in plane p if vw is an edge in plane p.

Proposition 5. *Consider a nice $\mathcal{D}$-drawing of G'. Fix an arc S_v. Let $\mathcal{A}$ be a collection of arcs neighboring S_v, all in the same plane p and on the same side of*

S_v. The number of crossings between edges of type $v - w$ and type $v - u$ where $S_w, S_u \in \mathcal{A}$ is

$$\sum_{\{S_w, S_u\} \in \binom{\mathcal{A}}{2}} \binom{|S_v|}{2} |S_w||S_u|$$

Proof. Fix $\{S_w, S_u\} \in \binom{\mathcal{A}}{2}$. For each pair $\{x, y\} \in \binom{S_v}{2}$, the set $\{x, y\} \cup S_w \cup S_u$ induces a drawing of a complete bipartite graph. Note that every crossing between edges of the type $v-w$ and $v-u$ occurs as a crossing in such a graph. This complete bipartite graph is drawn in the form of a bipartite drawing, so it has $|S_w||S_u|$ crossings. Therefore, there are

$$\sum_{\{S_u, S_w\} \in \binom{\mathcal{A}}{2}} \binom{|S_v|}{2} |S_w||S_u|$$

crossings between edges of type $v - w$ and type $v - w$ where $S_u, S_w \in \mathcal{A}$. □

Let $\mathcal{N}(S_v)$ be the set of pairs $\{S_u, S_w\}$ such that S_u and S_w are both neighbors of S_v on the same plane, and are both on the same side of S_v.

The following proposition follows immediately from these definitions and Proposition 5.

Proposition 6. *Let $\mathcal{D}$ be a nice drawing of G and consider a $\mathcal{D}$-drawing of G'. The number of crossings in Class 2 is*

$$\sum_{v \in V(G)} \binom{|S_v|}{2} \sum_{\{S_u, S_w\} \in \mathcal{N}(S_v)} |S_u||S_w|$$

Proposition 7. *If, in $\mathcal{D}$, the edges vw and xy cross, then in a $\mathcal{D}$-drawing there are $|S_v||S_w||S_x||S_y|$ crossings between edges of type v-w and edges of type x-y. Otherwise, there are no crossings between such edges.*

Let $\chi(\mathcal{D})$ be the set of unordered crossing pairs of edges in $\mathcal{D}$.

Proposition 8. *The number of crossings in Class 3 of a $\mathcal{D}$-drawing is*

$$\sum_{\{vw, xy\} \in \chi(\mathcal{D})} |S_v||S_w||S_x||S_y|$$

Proposition 9 (Crossing Counting Formula)**.** *If $\mathcal{D}$ is nice, then the number of crossings in a $\mathcal{D}$-drawing is*

$$\sum_{vw \in E[G]} \binom{|S_v|}{2}\binom{|S_w|}{2} + \sum_{v \in V(G)} \binom{|S_v|}{2} \sum_{\{S_u, S_w\} \in \mathcal{N}(S_v)} |S_u||S_w| + \sum_{\{vw, xy\} \in \chi(\mathcal{D})} |S_v||S_w||S_x||S_y|$$

2.2 Applications of Crossing Counting Formula

We will now use the Crossing Counting Formula to prove upper bounds on the biplanar crossing number of certain families of bipartite graphs. The figures in this section have vertices labelled like $x : y$. This indicates that the vertex x lies in this position on plane p_1 and the vertex y lies in this position on plane p_2.

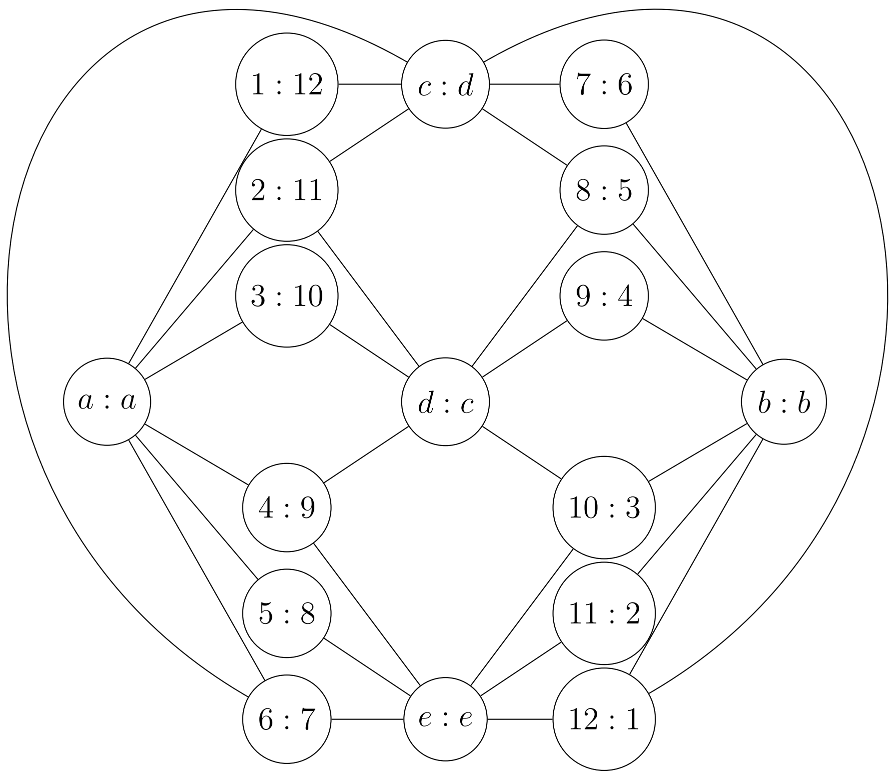


Figure 2.1 Biplanar Drawing of $K_{5,12}$

Proposition 10. *For $p \geq 5$ and $q \geq 12$, we have*

$$\mathrm{cr}_2(K_{p,q}) \leq 51 \left\lceil \frac{p}{5} \right\rceil^2 \left\lceil \frac{q}{12} \right\rceil^2 - 21 \left\lceil \frac{p}{5} \right\rceil^2 \left\lceil \frac{q}{12} \right\rceil - 45 \left\lceil \frac{p}{5} \right\rceil \left\lceil \frac{q}{12} \right\rceil^2 + 15 \left\lceil \frac{p}{5} \right\rceil \left\lceil \frac{q}{12} \right\rceil$$

Proof. Let $\mathcal{D}$ be the drawing shown in Figure 2.1. Create a $\mathcal{D}$-drawing of $K_{p,q}$ in the following manner. Partition the p vertices into 5 nearly equal size arcs S_a, S_b, S_c, S_d, S_e and the q vertices into 12 nearly equal size arcs $S_1, \cdots, S_{12}$. These arcs have at most $\left\lceil \frac{p}{5} \right\rceil$ vertices and $\left\lceil \frac{q}{12} \right\rceil$ vertices, respectively. At the location of each vertex in $\mathcal{D}$, place the vertices of the arc labeled with that vertex in a short straight line. This short line is vertical if the vertex is in $\{1, \cdots, 12\}$ or in $\{c, d, e\}$ and horizontal if the vertex is a or b. Connect the edges according to the definition of a $\mathcal{D}$-drawing. Note that $\mathcal{D}$ is nice.

Because each edge in $K_{5,12}$ has one endvertex from $\{1, \cdots, 12\}$ and one from $\{a, b, c, d, e\}$, our assumption about the sizes of the arcs implies that

$$\binom{|S_v|}{2}\binom{|S_w|}{2} \leq \binom{\left\lceil \frac{p}{5} \right\rceil}{2}\binom{\left\lceil \frac{q}{12} \right\rceil}{2}$$

for each edge $vw \in E(K_{5,12})$. There are 60 edges in $K_{5,12}$. Therefore,

$$\sum_{vw \in E(K_{5,12})} \binom{|S_v|}{2}\binom{|S_w|}{2} \leq 60\binom{\left\lceil \frac{p}{5} \right\rceil}{2}\binom{\left\lceil \frac{q}{12} \right\rceil}{2}$$

If $v \in \{a, b, c, d, e\}$, then

$$\sum_{\{S_u, S_w\} \in \mathcal{N}(S_v)} |S_u||S_w| \leq 12\left\lceil \frac{q}{12} \right\rceil^2$$

by our assumption about the cardinality of the arcs. If $v \in \{1, \cdots, 12\}$, then

$$\sum_{\{S_u, S_w\} \in \mathcal{N}(S_v)} |S_u||S_w| \leq \left\lceil \frac{p}{5} \right\rceil^2$$

Therefore,

$$\sum_{v \in V(G)} \binom{|S_v|}{2} \sum_{\{S_u, S_w\} \in \mathcal{N}(S_v)} |S_u||S_w| \leq 60\binom{\left\lceil \frac{p}{5} \right\rceil}{2}\left\lceil \frac{q}{12} \right\rceil^2 + 12\binom{\left\lceil \frac{q}{12} \right\rceil}{2}\left\lceil \frac{p}{5} \right\rceil^2$$

Since $\mathcal{D}$ has no crossings,

$$\sum_{vw, xy \in \chi(\mathcal{D})} |S_v||S_w||S_x||S_y| \leq 0.$$

Therefore, by the Crossing Counting Lemma, the $\mathcal{D}$-drawing has at most

$$60\binom{\lceil\frac{p}{5}\rceil}{2}\binom{\lceil\frac{q}{12}\rceil}{2}+60\binom{\lceil\frac{p}{5}\rceil}{2}\left\lceil\frac{q}{12}\right\rceil^2+12\binom{\lceil\frac{q}{12}\rceil}{2}\left\lceil\frac{p}{5}\right\rceil^2$$
$$=15\left(\left\lceil\frac{p}{5}\right\rceil^2-\left\lceil\frac{p}{5}\right\rceil\right)\left(\left\lceil\frac{q}{12}\right\rceil^2-\left\lceil\frac{q}{12}\right\rceil\right)+30\left(\left\lceil\frac{p}{5}\right\rceil^2-\left\lceil\frac{p}{5}\right\rceil\right)\left\lceil\frac{q}{12}\right\rceil^2+6\left(\left\lceil\frac{q}{12}\right\rceil^2-\left\lceil\frac{q}{12}\right\rceil\right)\left\lceil\frac{p}{5}\right\rceil^2$$
$$=51\left\lceil\frac{p}{5}\right\rceil^2\left\lceil\frac{q}{12}\right\rceil^2-21\left\lceil\frac{p}{5}\right\rceil^2\left\lceil\frac{q}{12}\right\rceil-45\left\lceil\frac{q}{5}\right\rceil\left\lceil\frac{q}{12}\right\rceil^2+15\left\lceil\frac{p}{5}\right\rceil\left\lceil\frac{q}{12}\right\rceil$$

crossings. □

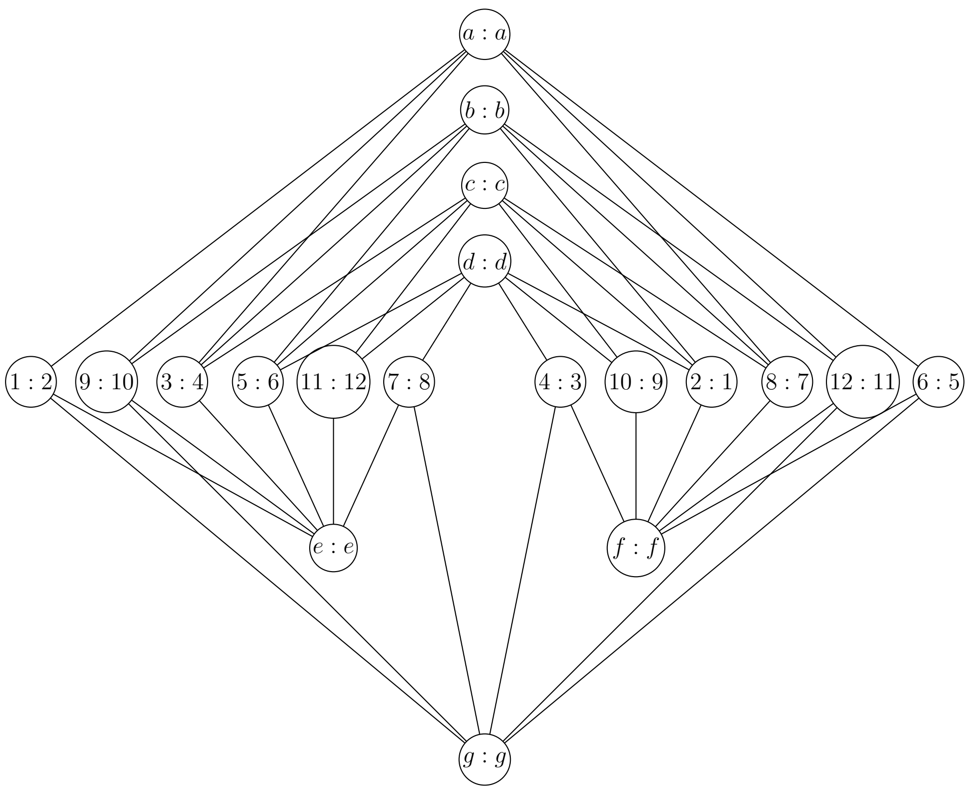


Figure 2.2 A biplanar drawing of $K_{7,12}$

Proposition 11.

$$\mathrm{cr}_2(K_{7,q})\leq 34\left\lceil\frac{q}{12}\right\rceil^2-22\left\lceil\frac{q}{12}\right\rceil$$

Proof. Consider the drawing in Figure 2.2. Divide the q-vertex partition as evenly as possible into the arcs $S_1,\cdots,S_{12}$. Label the remaining seven vertices $a,b,c,d,e,f,$

and g and place them on the corresponding points. For each edge $vw \in E(G)$, either $|S_v| = 1$ or $|S_w| = 1$, and so

$$\sum_{vw \in E(G)} \binom{|S_v|}{2}\binom{|S_w|}{2} = 0.$$

Now we will examine Class 2 crossings. If $v \in \{a, b, c, d, e, f, g\}$ then $|S_v| = 1$ and so $\binom{|S_v|}{2} = 0$. If $v \in \{1, \cdots, 12\}$ then $|S_v| \leq \left\lceil \frac{q}{12} \right\rceil$. According to the definition of a $\mathcal{D}$-drawing, the vertices of S_v are placed in such a way that $|\mathcal{N}(S_v)| = 3$ for all $v \in \{1, \cdots, 12\}$. Thus

$$\binom{|S_v|}{2} \sum_{\{S,T\} \in \mathcal{N}(S_v)} |S||T| \leq 36 \binom{\left\lceil \frac{q}{12} \right\rceil}{2}$$

There are 16 crossings, and for each edge vw, we have $|S_v||S_w| \leq \left\lceil \frac{q}{12} \right\rceil$. Thus there are at most

$$16 \left\lceil \frac{q}{12} \right\rceil^2$$

Class 3 crossings. Therefore, by the Crossing Counting Formula,

$$\text{cr}_2(K_{7,q}) \leq 36 \binom{\left\lceil \frac{q}{12} \right\rceil}{2} + 16 \left\lceil \frac{q}{12} \right\rceil^2 = 34 \left\lceil \frac{q}{12} \right\rceil^2 - 18 \left\lceil \frac{q}{12} \right\rceil$$

□

Proposition 12. *For all natural numbers $q \geq 6$*

$$\text{cr}_2(K_{5,q}) \leq \left\lfloor \frac{q}{6} \right\rfloor (q - 3 \left\lfloor \frac{q}{6} \right\rfloor - 3)$$

Proof. Consider Figure 2.3. For each $v \in \{a, b, c, d, e\}$, we have $|S_v| = 1$. Let $q = 6x + y$ where x and y are integers such that $x \geq 0$ and $5 \geq y \geq 0$. In arcs S_1 through S_y, place $x + 1$ vertices each. In the remaining $6 - y$ arcs, place x vertices. Thus,

$$\sum_{vw \in E(G)} \binom{|S_v|}{2}\binom{|S_w|}{2} = 0.$$

Since there are no crossings in $\mathcal{D}$,

$$\sum_{vw, xy \in \chi(\mathcal{D})} |S_v||S_w||S_x||S_y| = 0.$$

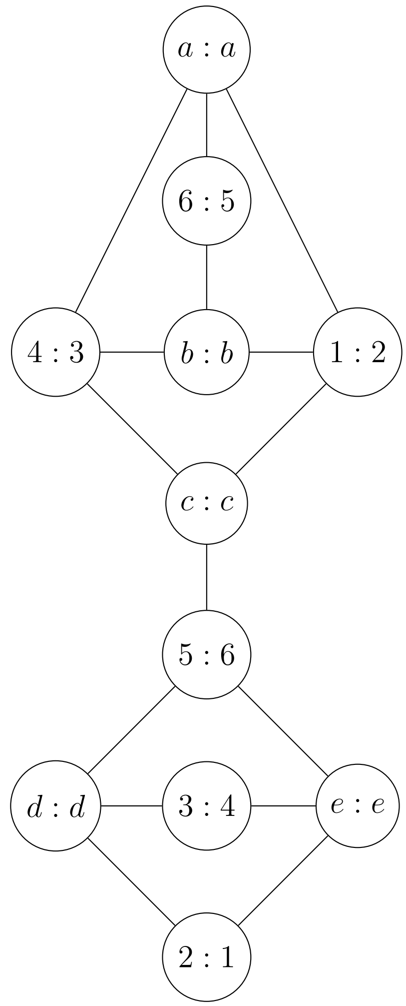


Figure 2.3 A biplanar drawing of $K_{5,6}$

For Class 2 crossings, we have

$$\begin{aligned}\sum_{v\in V(G)}\binom{|S_v|}{2}\sum_{\{S_u,S_w\}\in\mathcal{N}(S_v)}|S_u||S_w| &= \sum_{w\in\{1,\cdots,6\}}\binom{|S_v|}{2}\sum_{\{S_u,S_v\}\in\mathcal{N}(S_w)}1\\ &= \sum_{w\in\{1,\cdots,6\}}\binom{|S_v|}{2}|\mathcal{N}(S_w)|\\ &= y\binom{x+1}{2}+(6-y)\binom{x}{2}\end{aligned}$$

Observe that

$$
\begin{aligned}
y\binom{x+1}{2} + (6-y)\binom{x}{2} &= (q - 6\left\lfloor\frac{q}{6}\right\rfloor)\binom{\left\lfloor\frac{q}{6}\right\rfloor + 1}{2} + \left(6 - q + 6\left\lfloor\frac{q}{6}\right\rfloor\right)\binom{\left\lfloor\frac{q}{6}\right\rfloor}{2} \\
&= \frac{1}{2}\left\lfloor\frac{q}{6}\right\rfloor\left[\left(q - 6\left\lfloor\frac{q}{6}\right\rfloor\right)\left(\left\lfloor\frac{q}{6}\right\rfloor + 1\right) + \left(6 - q + 6\left\lfloor\frac{q}{6}\right\rfloor\right)\left(\left\lfloor\frac{q}{6}\right\rfloor - 1\right)\right] \\
&= \frac{1}{2}\left\lfloor\frac{q}{6}\right\rfloor\left[6\left(\left\lfloor\frac{q}{6}\right\rfloor - 1\right) + \left(q - 6\left\lfloor\frac{q}{6}\right\rfloor\right)\left(\left\lfloor\frac{q}{6}\right\rfloor + 1 - \frac{q}{6} + 1\right)\right] \\
&= \frac{1}{2}\left\lfloor\frac{q}{6}\right\rfloor\left[6\left(\left\lfloor\frac{q}{6}\right\rfloor - 1\right) + 2\left(q - 6\left\lfloor\frac{q}{6}\right\rfloor\right)\right] \\
&= \left\lfloor\frac{q}{6}\right\rfloor\left(q - 3\left\lfloor\frac{q}{6}\right\rfloor - 3\right)
\end{aligned}
$$

□

Chapter 3

Lower Bounds

Proposition 13 (The Counting Method)**.** *If H is a subgraph of G, let A be the number of copies of G in G and B be the max number of copies of H that a crossing in G can belong to. Then*

$$\operatorname{cr}_2(G) \geq \frac{A}{B}\operatorname{cr}_2(H)$$

Proof. There are A copies of H in G, and so there are A drawings of H in any drawing of G, and each drawing contains a minimum of $\operatorname{cr}_2(H)$ crossings in any bipartite drawing of G. Thus, if we sum the number of crossings in a copy of H across the copies of H in G, we have at least $A\operatorname{cr}_2(H)$ crossings. However, some of the crossings may be counted multiple times. Since each crossing can occur in B copies of H, the aforementioned sum can only overcount by a factor of B at most. Hence, there are at least $\frac{A}{B}\operatorname{cr}_2(H)$ crossings in any bipartite drawing of G. □

Proposition 14. *If $p, q > 2$, then*

$$\operatorname{cr}_2(K_{p,q}) \geq \max\left\{\left\lceil \frac{p}{p-2}\operatorname{cr}_2(K_{p-1,q})\right\rceil, \left\lceil \frac{q}{q-2}\operatorname{cr}_2(K_{p,q-1})\right\rceil\right\}$$

Proof. By symmetry, it suffices to show $\operatorname{cr}_2(K_{p,q}) \geq \left\lceil \frac{p}{p-2}\operatorname{cr}_2(K_{p-1,q})\right\rceil)$. Let $G = K_{p,q}$ and $H = K_{p-1,q}$. Each vertex in the p-partition of $K_{p,q}$ corresponds to a copy of H, namely the copy which results from deleting that vertex. So $A = p$. Suppose we have a crossing between edges ax and by where a and b are vertices belong to the p-partition. Then the only copies of H which do not contain this crossing are the copy which results from a deleting a and the copy which results from deleting b. Hence $B = p - 2$. Applying the counting method gives the claimed bound, without the ceiling. We take the ceiling because the biplanar crossing number must be an integer, we may round up. □

Using the exact value for $\mathrm{cr}_2(K_{5,q})$ provided by Czabarka et. al. (Czabarka et al. 2006) , we can prove a lower bound on $\mathrm{cr}_2(K_{5,q})$.

Proposition 15. *For* $p \geq 5$

$$\mathrm{cr}_2(K_{p,q}) \geq \frac{p(p-1)}{20} \left\lfloor \frac{q}{12} \right\rfloor \left(q - 6 \left\lfloor \frac{q}{12} \right\rfloor - 6\right)$$

Proof. By inductively applying Proposition 14 for $p \geq 5$, we have

$$\begin{aligned} \mathrm{cr}_2(K_{p,q}) &\geq \frac{p}{p-2} \cdot \frac{p-1}{p-3} \cdots \frac{6}{4} \mathrm{cr}_2(K_{5,q}) \\ &= \frac{p(p-1)}{20} \mathrm{cr}_2(K_{5,q}) \\ &= \frac{p(p-1)}{20} \left\lfloor \frac{q}{12} \right\rfloor \left(q - 6 \left\lfloor \frac{q}{12} \right\rfloor - 6\right) \end{aligned}$$

□

The best known lower bound on $\mathrm{cr}_2(K_{p,q})$ is due to Shavali and Zarrabi-Zadeh (Shavali and Zarrabi-Zadeh 2019):

Proposition 16. *For* $p, q \geq 21$*,*

$$\mathrm{cr}_2(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{216}$$

This bound is not tight for all p and q. We present a method for computing lower bounds of $\mathrm{cr}_2(K_{p,q})$ for particular values of p and q. We will need the following exact results from (Czabarka et al. 2006) for small cases:

Proposition 17. *If* $p < 5$ *or* $q < 5$ *then* $\mathrm{cr}_2(K_{p,q}) = 0$.

Proposition 18. *For any* $q \geq 1$*, we have*

$$\mathrm{cr}_2(K_{5,q}) = \left\lfloor \frac{q}{12} \right\rfloor \left(q - 6 \left\lfloor \frac{q}{12} \right\rfloor - 6\right)$$

Proposition 19. *For $1 \leq q \leq 16$ we have*

$$\mathrm{cr}_2(K_{6,q}) = 2\left\lfloor \frac{q}{8} \right\rfloor \left(q - 4\left\lfloor \frac{q}{8} \right\rfloor - 4\right)$$

Proposition 20. *For any $1 \leq q \leq 12$ we have*

$$\mathrm{cr}_2(K_{8,q}) = 4\left\lfloor \frac{q}{6} \right\rfloor \left(q - 3\left\lfloor \frac{q}{6} \right\rfloor - 3\right)$$

Now, we use the following algorithm to compute the lower bound of $\mathrm{cr}_2(K_{p,q})$:

If $\mathrm{cr}_2(K_{p,q})$ can be computed exactly, then let $b_1 = \mathrm{cr}_2(K_{p,q})$. Otherwise, let b_1 be the bound computed by Proposition 14. Note that this requires us to recursively compute $\mathrm{cr}_2(K_{p-1,q})$ and $\mathrm{cr}_2(K_{p,q-1})$.

If $p, q \geq 21$, let b_2 be the Shavali and Zarrabi-Zadeh bound and let $b = \max\{b_1, b_2\}$. The maximum of b_1 and b_2 is a lower bound on $\mathrm{cr}_2(K_{p,q})$ since both b_1 and b_2 are lower bounds. Otherwise, let $b = b_1$. The lower bound value output by this algorithm is b.

Using this algorithm, we obtain the following bound:

$$\mathrm{cr}_2(K_{39,39}) \geq 10264.$$

For comparison, the Shavali and Zarrabi-Zadeh bound is

$$\mathrm{cr}_2(K_{39,39}) \geq 10169.$$

Now, using the counting method, we have

$$\begin{aligned}
\mathrm{cr}_2(K_{p,q}) &\geq \frac{\binom{p}{39}\binom{q}{39}}{\binom{p-2}{37}\binom{q-2}{37}} \mathrm{cr}_2(K_{39,39}) \\
&\geq \frac{\frac{p!}{39!(p-39)!} \cdot \frac{q!}{39!(p-39)!}}{\frac{(p-2)!}{37!(p-39)!}} \cdot 10264 \\
&= \frac{10264}{39^2 \cdot (2 * 19)^2} p(p-1)q(q-1) \\
&= \frac{2566}{549081} p(p-1)q(q-1) \\
&\geq \frac{p(p-1)q(q-1)}{214}
\end{aligned}$$

Thus

Proposition 21. *For all $p, q \geq 39$,*

$$\mathrm{cr}_2(K_{p,q}) \geq \frac{p(p-1)q(q-1)}{214}$$

which improves upon the best published bound.

The Python 3 code used to compute these bounds is provided below.

Chapter 4

Structural Results

Ramsey's Theorem is a fundamental theorem about substructures of graphs.

Proposition 22 (Ramsey's Theorem)**.** *Let a and b be natural numbers. There exists a natural number N such that for any graph G on $n \geq N$ vertices, either G contains a subgraph isomorphic to K_a or the complement $\overline{G}$ contains a subgraph isomorphic to K_b.*

The minimum number N satisfying Ramsey's Theorem is the *Ramsey Number* $R(a, b)$. A consequence of Ramsey's Theorem is that any biplanar drawing of K_n, where $n \geq R(a, b)$, either contains a drawing of K_a on one plane or a drawing of K_b on the other. Thus, if $n \geq R(a, a)$, then $\mathrm{cr}_2(K_n) \geq \mathrm{cr}(K_a)$. Since a drawing of K_{n+1} contains a sub-drawing of K_n, $\mathrm{cr}(K_n)$ is non-decreasing in n. Therefore, Ramsey's Theorem allows us to prove lower bounds on $\mathrm{cr}_2(K_n)$.

There is a bipartite analog to Ramsey's Theorem (Zhang, Sun, and Wu 2013). In this proposition, and in the remainder of this chapter, if G is a spanning bipartite subgraph of $K_{p,q}$, then $\overline{G}$ will refer to the complement of G relative to $K_{p,q}$, i.e. $\overline{G} = K_{p,q} - E(G)$. This is called the bipartite complement. From here on, if we refer only to the complement, this is the graph to which we refer.

Proposition 23 (Bipartite Ramsey's Theorem)**.** *Let a and b be natural numbers. There exists a natural number N such that for any bipartite graph G with $n \geq N$ vertices per partition, either G contains a subgraph isomorphic to $K_{a,a}$ or $\overline{G}$ contains a subgraph isomorphic to $K_{b,b}$.*

The minimum number N satisfying the Bipartite Ramsey's Theorem is called the Bipartite Ramsey Number $BR(a, b)$. This theorem implies (by analogy with Ramsey's Theorem)

$$\text{cr}_2(K_{n,n}) \geq \text{cr}(K_{a,a})$$

where a is the largest number such that $n \geq BR(a, a)$.

Now, consider the following proposition:

Proposition 24. *$BR(n, n) \geq 2^{n/2}$*

Proof. Let $N = 2^{n/2}$. Color each edge of $K_{N,N}$ either red or blue, with a probability of $\frac{1}{2}$ for red and independently from the other edges. There are $m = \binom{N}{n}^2$ subgraphs of $K_{N,N}$ that are isomorphic to $K_{n,n}$. Let $H_1, \cdots, H_m$ be an enumeration of these subgraphs and let $X_i = 1$ if H_i is monochromatic, and 0 is otherwise. The probability that $X_i = 1$ is $p = \frac{2}{2^{n^2}}$. Then $\mathbb{E}(X_i) = p$. Let $Y = \sum_{i=1}^{m} X_i$. Then Y counts the number of monochromatic $K_{n,n}$'s in a random coloring of $K_{N,N}$. By the linearity of expectation,

$$\mathbb{E}(Y) = \sum_{i=1}^{m} \mathbb{E}(X_i) = mp.$$

Thus

$$\mathbb{E}(Y) = \binom{N}{n}^2 \frac{2}{2^{n^2}}$$

Since $N < 2^{n/2}$, we have $N^{2n} < 2^{n^2}$. Since $n \geq 2$, $n!^2 > 2$. Thus,

$$\binom{N}{n}^2 = \left(\frac{N!}{(N-n)!n!}\right)^2 = \left(\frac{N(N-1)\cdots(N-n+1)}{n!}\right)^2 \leq \left(\frac{N^n}{n!}\right)^2 < \frac{2^{n^2}}{2} = 2^{n^2-1}$$

Hence,

$$\mathbb{E}(X) < 2^{n^2-1}\frac{2}{2^{n^2}} = 1$$

. If every random coloring of $K_{N,N}$ contains a monochromatic $K_{n,n}$, then X is bounded below by 1. This implies $\mathbb{E}(X) \geq 1$. Therefore, since $\mathbb{E}(X) < 1$, there must be a random coloring of $K_{N,N}$ without a monochromatic $K_{n,n}$. Hence $BR(n, n) > N = 2^{n^2}$. $\square$

So, $n \geq BR(a,a)$ implies $n \geq 2^{a/2}$ and $a \leq 2\log_2(a)$. From the Zarankiewicz drawing, we see that the lower bound provided by the method is, at best,

$$\left(\left\lfloor \frac{2\log_2 n}{2} \right\rfloor \left\lfloor \frac{2\log_2 n - 1}{2} \right\rfloor\right)^2.$$

Therefore, the best lower bound that we can prove using the value of $BR(a,a)$ is asymptotically $\Omega(\log(n)^4)$. As we have already shown lower bounds which are asymptotically $\Omega(n^2)$, this method seems to be a dead-end.

There might, however, be a way to salvage this approach. Let P a property of graphs. Suppose that there is an optimal drawing of $K_{n,n}$ such the two graphs drawn on the two planes both have property P. Define the *property P bipartite Ramsey number* $BR_P(a,b)$ be the least number such that

1. G contains a subgraph isomorphic to $K_{a,a}$ or does not have property P or
2. The bipartite complement $\overline{G}$ contains a subgraph isomorphic to $K_{b,b}$ or does not have property P

By its definition, we see that $BR_P(a,b) \leq BR(a,b)$. Knowing that our drawing must satisfying property P on both planes, we know that if $n \geq BR_P(a,b)$, then either one plane contains a $K_{a,a}$ or the other plane contains a $K_{b,b}$. Thus, we have

$$\mathrm{cr}_2(K_{n,n}) \geq \mathrm{cr}(K_{a,a})$$

where a is the largest number such that $n \geq BR_P(a,a)$. If $BR_P(a,a)$ is significantly less than $BR(a,a)$, this may allows us to prove better bounds. This method presents two new problems: finding a property P which appears on both planes in some optimal biplanar drawing and determining bounds on $BR_P(a,a)$.

In the remainder of this chapter, we will investigate connectedness as our property P. We will show that connectedness fails to be a useful property, but showing this is an interesting exercise in its own right.

The first thing we must show is

Proposition 25. *If $n, m \geq 4$ or $\{n, m\} = \{3, 4\}$, then there exists a drawing of $K_{n,n}$ across two planes, with G on one plane and $\overline{G}$ on the other, which witnesses $\mathrm{cr}_2(K_{n,n})$ such that G and $\overline{G}$ are both connected.*

In order to prove this, we first must prove several lemmas

Lemma 1. *If G is connected graph and $x_0x_1 \cdots x_kx_0$ is a cycle in G, then $G - x_0x_1$ is connected.*

Proof. Let G and $x_0x_1 \cdots x_k$ be as stated. Suppose that u and v are vertices in $G - x_ox_1$. Since G is connected, there is a path $u = u_0u_1 \cdots u_\ell = v$ from u to v in G. Let $u_0 \cdots u_\ell$ have the minimum number of occurrences of x_0x_1 among u-v paths in G. For the sake of contradiction, suppose $u_iu_{i+1} = x_0x_1$ for some i. Assume $u_i = x_0$ and $u_{i+1} = x_1$. Then $u_1 \cdots u_ix_kx_{k-1} \cdots x_1u_{i+1} \cdots u_\ell$ is a walk in G and has fewer occurrences of x_0x_1 than $u_0 \cdots u_\ell$. Hence there is a u-v path with fewer occurrences of x_0x_1 than $u_0 \cdots u_\ell$. This contradicts the claim that $u_0 \cdots u_\ell$ has the minimum number of occurences of x_0x_1, so no such i exists and therefore $u_0 \cdots u_\ell$ does not contain the edge x_0x_1. Thus, $u_0 \cdots u_\ell$ is a path from u to v in $G - x_0x_1$. Since u and v are arbitrary, $G - x_0x_1$ is connected. □

Lemma 2. *If G is a (not necessarily bipartite) $K_{a,a}$-free graph, $a \geq 2$, and C, D are two different components of G, then $G + e$ is $K_{a,a}$-free for all edges of the form $e = uv$ where $u \in C$ and $v \in D$.*

Proof. In $G + e$, $(C \cup D) + e$ is a single component and e is not a part of any cycle. Since every edge of $K_{a,a}$ is on a cycle, e is not in any subgraph $H \cong K_{a,a}$ of $G + e$. Thus if $G + e$ contains a subgraph $H \cong K_{a,a}$, then H is a subgraph of G. Since G is $K_{a,a}$-free, $G + e$ is $K_{a,a}$-free. □

Proposition 26. *Let $n, m \geq 4$ be integers, and G be a connected spanning subgraph of $K_{n,m}$. If $\overline{G}$ is not connected, then one of the following holds*

i There is an edge $e \in E(G)$ which is not a bridge in G and which connects two components of $\overline{G}$

ii There are edges $e \in E(G)$ and $f \in E(\overline{G})$ such that $\overline{G} + e - f$ has fewer components than $\overline{G}$ and has a bridge e and $G - e + f$ is connected and has a bridge f.

Proof. Denote the complement of G relative to $K_{n,n}$ by G_1 and denote G as G_2. We will denote the two independent sets of n vertices in $K_{n,n}$ with X and Y. Let C be a component of G_1 with the minimum number of vertices. Without loss of generality we assume $|V(C) \cap X| \leq |V(C) \cap Y|$. Let $D = G_1 \setminus C$.

Case 1: Suppose $|V(C) \cap X| \geq 2$. We have $|V(C) \cap Y| \geq 2$, so $|V(C)| \geq 4$. Since G_1 is not connected, D must contain at least one component of G_1. Thus, by the minimality of C, we have $|V(D)| \geq 4$. If $|V(D) \cap X| \geq 2$, then the vertices of $V(C) \cap Y$ and $V(D) \cap X$ induce a 4-cycle in G_2, since D and C are different components in G_1. Therefore, there is some edge $e \in E(G_2)$ on this cycle, necessarily of the form $e = uv$ with $u \in C$ and $v \in D$.

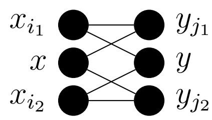
By Lemma 1, e is not a bridge in G_2 and connects two components in G_1.

Case 2: Suppose $|V(C) \cap X| = 1$. Since $n \geq 4$ and $|X| = |V(C) \cap X| + |V(D) \cap X|$, $|V(C) \cap X| = 1$ implies $|V(D) \cap X| \geq 4$. Suppose $|V(C) \cap Y| \geq 2$. Then the subgraph of G_2 induced by $V(C) \cap Y$ and $V(D) \cap X$ contains a 4-cycle. By Lemma 1, this yields our desired edge e. Suppose $|V(C) \cap Y| < 2$. If $|V(C) \cap Y| = 0$ then $|V(C) \cap X| = 0$ by $|V(C) \cap X| \leq |V(C) \cap Y|$. This contradicts $|V(C)| > 0$ so $|V(C) \cap Y| \geq 1$. Therefore, $|V(C) \cap Y| < 2$ implies $|V(C) \cap Y| = 1$ and so $|V(D) \cap Y| \geq 3$.

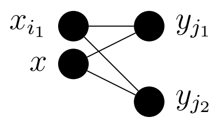
Let $V(C) \cap X = \{x\}$, $V(C) \cap Y = \{y\}$, $V(D) \cap X = X \setminus \{x\} = \{x_1, \cdots, x_{n-1}\}$ and $V(D_) \cap Y = Y \setminus \{y\} = \{y_1, \cdots, y_{m-1}\}$. Note that for all indexes i and j, we have

$yx_i, xy_j \in E(G_2)$.

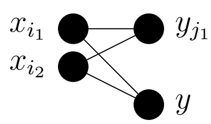
In order for G_2 to be connected, $E(G_2)$ must contain an edge of the form x_iy_j Suppose that there are two distinct edges of the form $x_iy_j \in E(G_2)$, say $x_{i_1}y_{j_1}$ and $x_{i_2}y_{j_2}$. If $i_1 \neq i_2$ and $j_1 \neq j_2$ then $yx_{i_1}y_{j_1}xy_{j_2}x_{i_2}y$ is a cycle in G_2. If $i_1 = i_2$ then



$j_1 \neq j_2$ and $xy_jx_{i_1}x_{j_2}x$ is a cycle in G_2. If $j_1 = j_2$ then $i_1 \neq i_2$ and $yx_{i_1}y_{j_1}x_{i_2}y$ is a



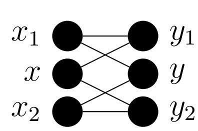
cycle in G_2. For each of these cycles, we may repeat the logic at the end of Case 1 in



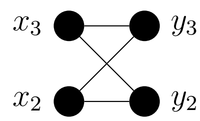
order to obtain an edge e.

On the other hand, suppose that there is only one edge of the form x_iy_j in $E(G_2)$. Since $|V(D) \cap Y| \geq 3$ and $|V(D) \cap X| \geq 3$, there exist distinct vertices $x_1, x_2, x_3 \in X$ and $y_1, y_2, y_3 \in Y$. Without loss of generality, we may assume that $x_iy_j = x_1y_1$. Let $G_2' = (G_2 + x_2y_2) - x_2y$. Note $G_1' = \overline{G_2'} = (G_1 - x_2y_2) + x_2y$.

Since $xy_2x_2yx_1y_1x$ is a cycle in $G_2 + x_2y_2$, G_2' is connected.



Since $x_2y_2x_3y_3x_2$ is a cycle in G_1, $G_1 - x_2y_2$ has exactly the same components as G_1, by Lemma 1. Thus, letting $e = x_2y$ and $f = x_2y_2$, we see that $G_2 + f - e$ is connected and $G_1 + e - f$ has fewer components than G_1.



Case 3: Suppose $|V(C) \cap X| = 0$. If $|V(C) \cap Y| \geq 2$ then there are two distinct vertices $y_1, y_2 \in V(C) \cap Y$. Since C is connected, there must be a path from y_1 to y_2 in C and since G is bipartite, this path must contain a vertex in X. Thus, by the contrapositive, $|V(C) \cap X| = 0$ implies $|V(C) \cap Y| \leq 1$. Since $V(C)$ must be non-empty, we have $|V(C) \cap Y| = 1$. Let $V(C) = V(C) \cap Y = \{y\}$ and enumerate X as $\{x_1, \cdots, x_n\}$and $Y \setminus \{y\} = \{y_1, \cdots, y_{n-1}\}$. Since C is a component in G_1, all edges of the form yx_i are edges in G_2.

Consider G_2. Since it is connected, each y_i has at least one neighbor in X. If at least $n-1$ vertices in X have two neighbors in $Y \setminus \{y\}$, then some y_i has at least two neighbors in X (by the Pigeonhole Principle, since $2(n-1) > n$ for $n > 2$). Thus we have the following cases: there is a y_k with two neighbors in X, or each vertex of $Y \setminus \{y\}$ has exactly one neighbor in X, some x_i (without loss of generality x_n) has no neighbors in $Y \setminus \{y\}$ and at least one x_i (without loss of generality x_{n-1}) has exactly one neighbor in $Y \setminus \{y\}$.

Case 3a: Suppose there are two distinct vertices x_i and x_j such that $x_i y_k \in E(G_2)$ and $x_j y_k \in E(G_2)$ for some k. Then $yx_iy_kx_jy$ is a cycle in G, and by Lemma 1 $e = yx_i$ satisfies condition (i).

Case 3b: Without loss of generality, suppose x_n has no neighbors in $Y \setminus \{y\}$ in G_2 and x_{n-1} has one neighbor in $Y \setminus \{y\}$ in G_2. We may also assume, without loss of generality, that y_1 and y_2 are not neighbors of x_{n-1} in G_2. Let $G_2' = (G_2 + x_n y_1) - x_n y$ and $G_1' = (G_1 - x_n y_1) + x_n y$. In G_2, y_1 is not adjacent to x_n or x_{n-1}. Since G_2 is connected, y_1 must have some neighbor $x_i \in X \setminus \{x_n, x_{n-1}\}$. Thus $x_n y_1 x_i y x_n$ is a cycle in $G_2 + x_n y_1$, and thus G_2' is connected by Lemma 1. Since x_n is connected to y_1 and y_2 in G_1 and x_{n-1} is connected to y_1 and y_2 in G_1, $G_1 - x_n y_1$ has the same components as G_1. Since $\{y\}$ is a component in both, $x_n y$ connects two components in $G_1 - x_n y_1$. Therefore, letting $e = x_n y$ and $f = x_n y_1$, we see that $G_1 + e - f$ has fewer components than G_1 and $G_2 - e + f$ is connected.

In any case, either condition (i) or there are edges e and f such that $G_1 + e - f$ has fewer components than G_1 and $G_2 - e + f$ is connected. If condition (i) fails, then it must be that latter statement holds. In this case, $G_2 - e$ must not be connected, or e would be an edge satisfying condition (i). Hence f must be a bridge in $G_2 - e + f$. In order for $G_1 + e - f$ to have fewer components than G_1, the edge e must be a bridge in $G_1 + e - f$. Hence, if condition (i) fails, then condition (ii) holds. □

Lemma 3. *If $\mathcal{D}$ is a drawing of G and e is a new edge which connects two components of G, then $G + e$ can drawn with no more crossings than $\mathcal{D}$.*

Proof. Let $e = uv$, let C be the component of G containing u, and let D be the component containing v. $\mathcal{D}$ contains a subdrawing of C. Using stereographic projection as discussed before, C be redrawn so that u is on the boundary of the outer face. Likewise, the subdrawing of D can be redrawn so that v is on the boundary of the outer face. Redrawing this way yields an equivalent drawing, and so the number of crossings does not increase. Since the vertices u and v are on the boundary of the outer face, e can be drawn without crossing another edge. □

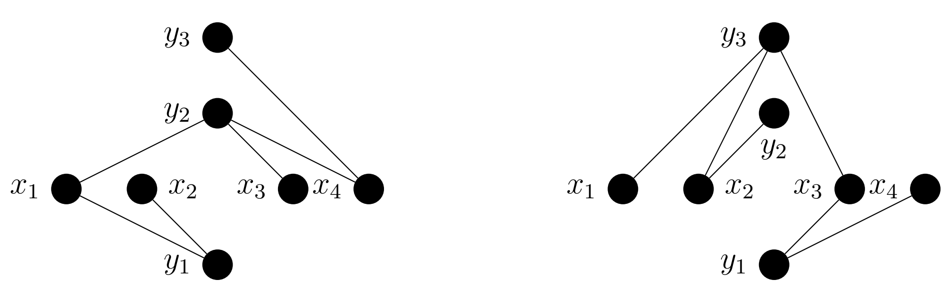


Figure 4.1 A biplanar drawing of $K_{3,4}$

Now we may prove Proposition 25, which we restate here.

Proposition. *If* $n, m \geq 4$ *or* $\{n, m\} = \{3, 4\}$*, then there exists a drawing of* $K_{n,n}$ *across two planes, with* G *on one plane and* $\overline{G}$ *on the other, which witnesses* $\mathrm{cr}_2(K_{n,n})$ *such that* G *and* $\overline{G}$ *are both connected.*

Proof. The case where $\{n, m\} = \{3, 4\}$ is show in Figure 4.1. As such, assume $n, m \geq 4$. Consider a drawing $\mathcal{D}$ of $K_{n,m}$ which witnesses $\mathrm{cr}_2(K_{n,m})$ and let G and $\overline{G}$ be as stated. If necessary, use Lemma 3 and translations to ensure that, for each component C of G, all other components of G are on the outside face of C and C has a vertex on the boundary of its outside face. These operations do not increase the crossing number of the drawing. There must be two components with a vertex u in one and v in the other, such that uv is an edge in $K_{n,m}$. We may assume that the components are redrawn such that u and v are on the boundaries of the outside faces of their respective components. Thus, we may add uv to G without adding crossings. Continue this process and redraw as before when necessary. In the resulting drawing, G is connected and has no more crossings that in the original drawing. The number of crossings in $\overline{G}$ does not increase since we are removing edges from that drawing. Thus, there is a biplanar drawing which witnesses the crossing number in which G is connected.

We may now assume that G is connected. If $\overline{G}$ is connected, then we are done. As such, we assume that $\overline{G}$ is not connected. Since $n, m \geq 4$, Proposition 26 applies.

Suppose that condition (i) of Prop. 26 applies. Let $e = xy$, let C be the component of $\overline{G}$ containing x, and let D be the component of $\overline{G}$ containing y. The edge e is not a bridge in G by condition (i), so removing e from G and adding it to $\overline{G}$ neither disconnects G nor increase the number of crossings in its drawing. Using Lemma 3 and translations, the components C and D can be redrawn in such a way that x and y are on the boundary of the outside face of their respective components. This redrawing does not increase the number of crossings in the drawing of $\overline{G}$. Now, e can be a dded to $\overline{G}$ without increasing the number of crossings.

On the other hand, suppose that condition (ii) of Prop. 26 holds. By the reasoning of the previous case, $\overline{G} + e - g$ and $\overline{G} - f$ can be drawn with the same number of crossings since e is a bridge in $\overline{G} + e - f$. Furthermore, since $\overline{G} - f$ is a subgraph of $\overline{G}$, $\overline{G} - f$ can be drawn with as few crossings as $\overline{G}$. The same reasoning holds for $G - e + f$ and G.

Thus, in either case, we can connect two components of $\overline{G}$ without disconnection G or increasing the number of crossings. Redrawing as necessary, this can be continued until both G and $\overline{G}$ are connected. The resulting drawing will witness $\mathrm{cr}_2(K_{n,m})$. □

Now we need to investigate $BR_C(a, b)$. First, we establish a few terms. A *bipartite Ramsey graph for* (a, b) is a spanning subgraph G of $K_{p,q}$ (where p and q are chosen so that G is spanning) such that G is $K_{a,a}$-free and the bipartite complement G is $K_{b,b}$-free. A *bipartite Ramsey critical graph for* (a, b) is bipartite Ramsey graph for (a, b) that is a spanning subgraph of a $K_{BR(a,b)-1,BR(a,b)-1}$

Lemma 4. *If $a, b \geq 2$, then there is a bipartite Ramsey critical graph G for (a, b) such that G is connected.*

Proof. Let G be a bipartite Ramsey critical graph for (a, b) which is minimal in the class of bipartite Ramsey critical graphs for (a, b) with respect to the number of components in G. For the sake of contradiction, suppose that G has at least two components. Then there is an edge uv with u in one component of G and v in a different component of G than u. Without loss of generality u and v are in different partitions. This is because either the component has vertices in both partitions, or it is a singleton vertex and you may choose any vertex in the other partition. By Lemma 2, the graph $G + uv$ is $K_{a,a}$-free. Since $\overline{G + uv} = \overline{G} - uv$, we have $E(\overline{G + uv}) \subset E(\overline{G})$ and therefore $\overline{G + uv}$ is $K_{b,b}$-free. Hence $G + uv$ is a bipartite Ramsey critical graph for (a, b) with fewer components than G, which violates the minimality of G. Hence G is connected. □

Finally, we will prove $BR_C(a,b) = BR(a,b)$ for a and b greater than 3. Since $BR_C(a,b) \leq BR(a,b)$, we need to show $BR_C(a,b) \geq BR(a,b)$, which the following proposition does:

Proposition 27. *If $a, b \geq 2$ then there is a bipartite Ramsey critical graph G for (a,b) such that G is connected and the bipartite complement $\overline{G}$ is connected.*

Proof. Let G be a connected bipartite Ramsey critical graph for (a,b); by Lemma 4, such a G exists. It suffices to show that if $\overline{G}$ is not connected, then there is a connected bipartite Ramsey critical graph G' such that $\overline{G'}$ such that $\overline{G'}$ has fewer components than $\overline{G}$. Note that since $a, b \geq 2$, G must have at least 4 vertices per partition (Zhang, Sun, and Wu 2013). By Prop. 26, either condition (i) or (ii) holds. If condition (i) holds, then $G - e$ is $K_{a,a}$-free since it is a subgraph of G. Since $b \geq 2$, $K_{b,b}$ is bridgeless and so any $K_{b,b}$ in $\overline{G - e} = \overline{G} + e$ cannot contain e and therefore must be a $K_{b,b}$ in $\overline{G}$. Thus, $\overline{G - e}$ is $K_{b,b}$-free. Hence, letting $G' = G + e$ suffices.

Suppose condition (ii) holds. Then $G - e + f$ has a $K_{a,a}$ if and only if $G - e$ has $K_{a,a}$ since f is a bridge. Since $G - e$ is a subgraph of G and G is $K_{a,a}$-free, so is $G - e$ and consequently $G - e + f$ is as well. Likewise, $\overline{G - e + f} = \overline{G} + e - f$ is $K_{b,b}$-free. Therefore, letting $G' = G - e + f$ suffices. □

Since $BR_C(a,b) = BR(a,b)$, the property of connectedness fails to prove any new lower bounds.

Chapter 5

Miscellaneous Observations

Proposition 28. *Consider an optimal drawing of $K_{7,13}$. Let $\{1, \cdots, 13\}$ be the partition of $K_{7,13}$ with 13 vertices. For each $n \in \{1, \cdots, 13\}$, let c_n denote the number of crossings in which one of the crossing edges has n as an endvertex. If $\mathrm{cr}_2(K_{7,13}) = 19$, then $c_n = 2$ for exactly one value of n, and $c_n = 3$ for all others.*

Proof. Since we are considering an optimal drawing of $K_{7,13}$, each crossing is between two edges, and each of those edges has exactly one endvertex from $\{1, \cdots, 13\}$, and these endvertices are different for the two edges. Hence,

$$\sum_{n=1}^{13} c_n = 2\,\mathrm{cr}_2(K_{7,13}).$$

Suppose $\mathrm{cr}_2(K_{7,13}) = 19$ and consider a solution $(c_1, \cdots, c_{13})$ to

$$\sum_{n=1}^{13} c_n = 38.$$

By the definition of c_n, each c_n is a non-negative integer. Suppose $c_n \geq 4$ for some n. If we remove the vertex n and all connected edges from our optimal drawing, we have a drawing of $K_{7,12}$ with no more than 15 crossings, as we have removed c_n crossings by removing n. Since $\mathrm{cr}_2(K_{7,12}) = 16$, such a drawing could not exist. Therefore, for each n, we have $0 \leq c_n \leq 3$. Under this constraint, the only solutions to

$$\sum_{n=1}^{13} c_n = 38.$$

are ones in which $c_n = 2$ for exactly one value of n, and $c_n = 3$ for all other values. □

As an example of how this fact can be used, consider Figure 2.2. We cannot add a 13th vertex to each of the planes in that drawing and get a drawing of $K_{7,13}$ with 19

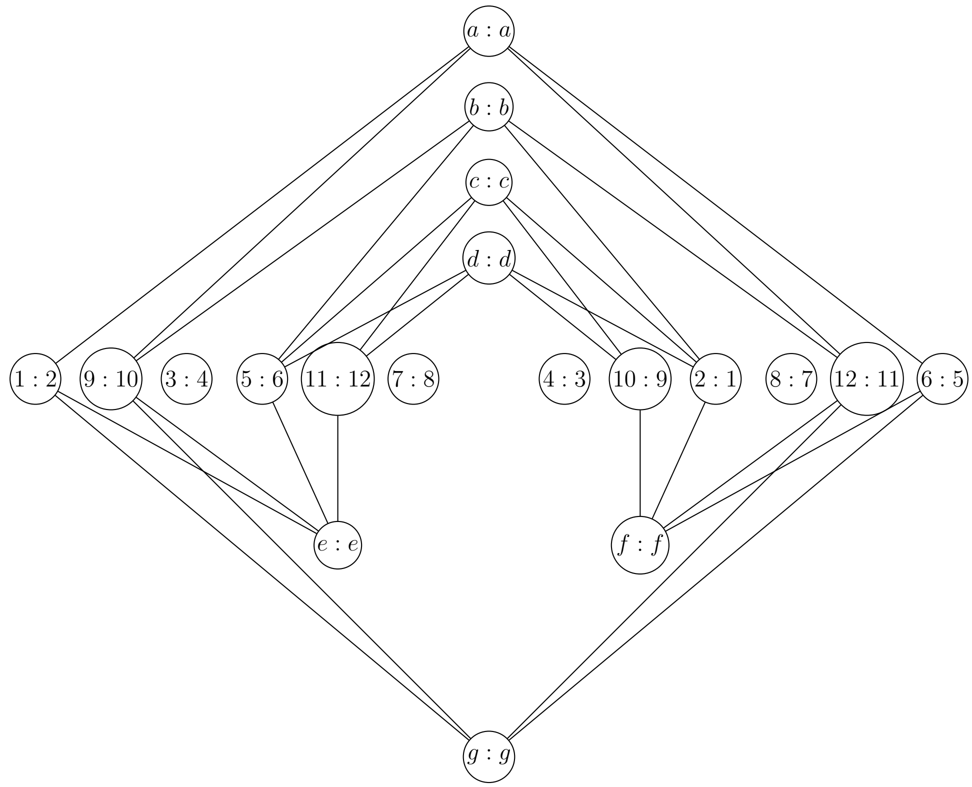


Figure 5.1

crossings. For the sake of contradiction, suppose we add a vertex labeled v on each of the planes, and draw edges adjacent to v such that the resulting drawing is a drawing of $K_{7,13}$ with 19 crossings. Computing the value of c_n for each $n \in \{1, \cdots, 12\}$, we see that $c_n = 3$ for certain vertices n. If we consider every edge from Figure 2.2 which is adjacent to such a vertex n, we divide the plane into various regions. Call these regions *permissible regions*. If any edge adjacent to v crosses the boundary of one of a permissible region, then it crosses an edge connected to some vertex n where $c_n = 3$ in Figure 2.2, and so in the $K_{7,13}$ drawing, we have $c_n = 4$ for that particular n. By the proposition we just proved, this implies our $K_{7,13}$ drawing has more than 19 crossings. Since we supposed otherwise, we must conclude that no edge adjacent to v crosses the boundary of any of these regions. Let R be an arbitrary permissible region. Then there is a subset $V(R)$ of $\{a, b, c, d, e, f, g\}$ consisting of all the vertices in $\{a, b, c, d, e, f, g\}$ which can be connected to a point inside of R by an edge that does not cross the boundary of R. In order to add v to the $K_{7,12}$ in the manner we supposed, then we must pick two permissible regions R_1 and R_2 such that $V(R_1) \cup V(R_2) = \{a, b, c, d, e, f, g\}$. By inspection (see Figure 5.1), we cannot do so. Hence, we cannot add a vertex v to Figure 2.2 in such way that the resulting drawing has 19 crossings and is a drawing of $K_{7,13}$.

Proposition 29. *If* $\mathrm{cr}_2(K_{7,13}) = 19$*, then there are at most* 10 *vertices which have at most one incident edge that crosses another edge in the drawing*

Proof. Suppose that $\mathrm{cr}_2(K_{7,13}) = 19$. Consider an optimal drawing and consider the partition with 13 vertices. For each vertex we can count the number of crossings in the ideal drawing which contain an edge that has the given vertex as an endpoint. Suppose that, for 11 of the 13 vertices, we count 1 or fewer crossings. Then, for the remaining 2 vertices in that partition, we must count the remaining 7 crossings. Hence, by the Pigeonhole Principle, one of these vertices must have at least 4 crossings. If we remove that vertex, we have a drawing of $K_{7,12}$ with only 15 crossings. Since $\mathrm{cr}_2(K_{7,12}) = 16$, this cannot be the case. Therefore, under the assumption that $\mathrm{cr}_2(K_{7,13}) = 19$, there can be at most 10 vertices which have 1 or fewer crossings. □

Appendix A

Lower Bound Algorithm implemented in Python 3

```
import math


# Shavali and Zarrabi-Zadeh bound
def general_bound(p, q):
    return math.ceil(p*(p-1)*q*(q-1)/216)


# The formula used in the novel method
def derived_bound(p, prev):
    return math.ceil(prev * (p / (p - 2)))


# These functions are for known exact results
def bound_5(q):
    return math.floor(q / 12) * ( q - 6 * math.floor(q / 12) - 6)


def bound_6(q):
    return 2 * math.floor(q / 8) * (q - 4 * math.floor(q / 8) - 4)


def bound_7(q):
    if q == 7:
```

```
        return 1
    elif q == 8:
        return 4
    elif q == 9:
        return 7
    elif q == 10:
        return 10
    elif q == 11:
        return 13
    elif q == 12:
        return 16
    elif q == 13:
        return 19

def bound_8(q):
    if q == 8:
        return 8
    elif q == 9:
        return 12
    elif q == 10:
        return 16
    elif q == 11:
        return 20
    elif q == 12:
        return 24
    elif q == 13:
        return 29
```

```
def bound(p, q, left, up):
    b = 0
    # These inequalities are testing for cases where we know
    # exact values of the crossing number
    if p < 5 or q < 5:
        b = 0;
    elif p == 5:
        b = bound_5(q)
    elif p == 6 and q <= 16:
        b = bound_6(q)
    elif p == 7 and q <= 13:
        b = bound_7(q)
    elif p == 8 and q <= 13:
        b = bound_8(q)
    else:
        # This is the only case where we actually have to do
        # interesting work
        # Using the counting method lower bound techniques, we can
        # derive two lower bounds for the crossing number at (p,q)
        # One comes from (p-1, q)
        b1 = derived_bound(p, up)
        # The other comes from (p, q-1)
        b2 = derived_bound(q, left)
        # Figure out which is better and record where it came from
        if b1 < b2:
            # up < left
```

```
            b = b2
        else:
            b = b1

    return (b, m)

def derive_coefficient(p,q,b):
    if (b == 0):
        return 0
    return (p * (p-1) * q * (q-1)) / b

# Array where we will hold our values
# arr[p][q] is the biplanar crossing number of K_{p,q}
arr = []

P_RANGE = 50 #p will range from 0 <= p < P_RANGE
Q_RANGE = 50 #q will range from 0 <= q < Q_RANGE
for p in range(P_RANGE):
    temp = []
    for q in range(Q_RANGE):
        # If p == 0, then we cannot use the value at (p-1,q)
        if (p == 0):
            up = 0
        else:
            up = arr[p-1][q]

        # If q == 0, then we cannot use the value at (p,q-1)
```

```
        if (q == 0):
            left = 0
        else:
            left = temp[q-1]

        # Exploiting symmetry
        if (q < p):
            b = arr[q][p]
        else:
            # Compute bound using the novel method
            bnd = bound(p,q,left,up)
            b = bnd[0] # Actual number
            m = bnd[1] # Method

        # best general bound requires p >= 21 and q >= 21
        if (p >= 21 and q >= 21):
            b = max(b, general_bound(p,q))
            temp.append(b)
        else:
            temp.append(b)

    arr.append(temp)
```

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