Spring 2019

Congruence Relations Mod 2 For (2 x 4^t + 1)-Colored Partitions

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CONGRUENCE RELATIONS MOD 2 FOR $(2 \cdot 4t + 1)$-COLORED PARTITIONS

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Submitted in Partial Fulfillment
of the Requirements for
Graduation with Honors from the
South Carolina Honors College

May, 2019

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Abstract. Let $p_r(n)$ denote the difference between the number of r-colored partitions of n into an even number of distinct parts and into an odd number of distinct parts. Inspired by proofs involving modular forms of the Hirschhorn-Sellers Conjecture, we prove the following congruence. Let $t \geq 1, n \geq 0$, and p be an odd prime. Then for all $0 \leq k \leq p-1$ with $k \not\equiv \frac{3(p^2-1)}{8} \pmod{p}$, we have

$$p_{2\cdot 4^t+1}\left(\frac{p^{2^t}(8pn+8k+3) - \frac{2\cdot 4^t+1}{3}}{8}\right) \equiv 0 \pmod{2},$$

Using the Jacobi Triple Product identity, we discover a much stricter congruence for $p_3(n)$, which is the $t = 0$ case.

1. Introduction and statement of results

A *partition* of a positive integer n is a non-increasing sequence of positive integers whose sum is n. We denote by $p(n)$ the number of partitions of n. By convention, we set $p(0) := 1$. Euler showed that the generating function for $p(n)$ satisfies

$$\sum_{n=0}^{\infty} p(n)q^n = \prod_{n=1}^{\infty} \frac{1}{1-q^n}.$$

We call this a *generating function* because the function is generated by the power series coefficients of an infinite product. Relations like Euler's have become an important area of study in both combinatorics and number theory, offering nuanced interpretations for tracking objects like partitions and their relatives and generalizations. Gauss and Ramanujan are also well-known contributors to the field. Ramanujan in particular is known for contributing impressively concise congruences for $p(n)$, proven on the basis of *q-series* relations like Euler's. Known aptly as Ramanujan's congruences, he posited that for all integers $n \geq 0$, we have

$$\begin{aligned} p(5n+4) &\equiv 0 \pmod{5} \\ p(7n+5) &\equiv 0 \pmod{7} \\ p(11n+6) &\equiv 0 \pmod{11}. \end{aligned}$$

After Ramanujan, relatively few linear congruences for $p(n)$ beyond these had been discovered until the work of Ahlgren and Ono [2] from the late 1990's and early 2000's. Modern techniques involving modular forms have proven powerful in uncovering congruence relations of this kind. We make use of such tools here to prove congruence relations for *colored partitions*. Before introducing a definition, we require some notation. Let $n \geq 1$. We denote

Date: April 24, 2019.
2000 *Mathematics Subject Classification.* 11P83, 11F11.

the partition λ of n given by $n = \lambda_1 + \lambda_2 + ... + \lambda_t$ with $\lambda_1 \geq \lambda_2 \geq ... \geq \lambda_t \geq 1$ by $\lambda = (\lambda_1, \lambda_2, ..., \lambda_t)$. For all i, we say λ_i is a *part* of λ.

Definition 1.1. *For $r \geq 1$, a r-colored partition of n is a pair (α, c_α) where $\alpha = (\alpha_1, \alpha_2, ..., \alpha_l)$ is a partition of n and c_α is a function which assigns to each part of α one of the colors $1, 2, ..., r$ so that if $\alpha_i = \alpha_{i+1}$ then $c_\alpha(i) \leq c_\alpha(i+1)$. We say that $c_\alpha(i)$ is the color of the i-th part of α.*

For all $n \geq 1$, we let $p_r(n)$ be given by

$$p_r(n) = p_{e,r}(n) - p_{o,r}(n) \tag{1.1}$$

where $p_{e,r}(n)$ and $p_{o,r}(n)$ denote the number of r-colored partitions into an even (respectively, odd) number of distinct parts. We build our main theorem around this function.

Theorem 1.2. *Let p be and odd prime. Then for all $t \geq 1$ and $n \geq 0$, we have*

$$p_{2\cdot 4^t+1}\left(\frac{p^{2^t}(8pn + 8k + 3) - \frac{2\cdot 4^t+1}{3}}{8}\right) \equiv 0 \pmod{2},$$

where $0 \leq k \leq p-1$ has $k \neq \frac{yp-3}{8}$, where y is given by

$$y = \begin{cases} 3 & \text{if } p \equiv 1 \pmod{8} \\ 1 & \text{if } p \equiv 3 \pmod{8} \\ 7 & \text{if } p \equiv 5 \pmod{8} \\ 5 & \text{if } p \equiv 7 \pmod{8}. \end{cases}$$

Remark 1.3. *We note that $0 \leq y \leq 7$ in the theorem has $y \equiv 3p \pmod{8}$. We also observe that $0 \leq k \leq p-1$ in the theorem has $k \not\equiv \frac{3(p^2-1)}{8} \pmod{p}$. Lastly, we observe a stricter congruence for the $t = 0$ case. For all $n \geq 0$, we have $p_3(n) \equiv 1 \pmod{2}$ if and only if there exists $m \in \mathbb{Z}$ such that $n = \frac{m^2+m}{2}$, i.e., if and only if n is a triangular number.*

We note that (1.1) offers a corollary to this theorem.

Corollary 1.4. *Let p be and odd prime. For n, t, and k as in Theorem 1.2, we have*

$$p_{e,2\cdot 4^t+1}\left(\frac{p^{2^t}(8pn + 8k + 3) - \frac{2\cdot 4^t+1}{3}}{8}\right) \equiv p_{o,2\cdot 4^t+1}\left(\frac{p^{2^t}(8pn + 8k + 3) - \frac{2\cdot 4^t+1}{3}}{8}\right) \pmod{2}.$$

Furthermore, we have

$$p_{e,3}(n) \equiv p_{o,3}(n) \pmod{2}$$

if and only if n is not a triangular number.

In the next section we will introduce concepts about modular forms that will be used to prove Theorem 1.2 in Section III. In Section IV we discuss replicas of this theorem for other generating functions.

2. Modular forms and partitions

The language here will move a good distance from congruence theory. This is evidence to a poignant quality of modular form theory, that it is known to almost overstep its bounds, delivering proofs in cases of mathematics sometimes far removed from its home in function theory. We let $\mathbb{H} = \{z \in \mathbb{C} : im(z) > 0\}$.

Definition 2.1. *Let $C^{\#} = \mathbb{C} \cup \{\infty\}$ and $\mathbb{H}^* = \mathbb{H} \cup \mathbb{Q} \cup \{\infty\}$ where $\{\infty\}$ is the point at infinity. A function $f : \mathbb{H} \mapsto \mathbb{C}^{\#}$ is a modular form of weight $k \in \mathbb{Z}$ on the modular group $\Gamma(1) = SL_2(\mathbb{Z})$ if and only if:*

(1) *$f(Mz) = f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma(1)$ and,*

(2) *f is holomorphic on $\mathbb{H}$ and at the cusp ∞ of $\mathbb{H}^*/\Gamma(1)$.*

Essentially, for a function to be *modular* in our setting, it must satisfy a transformation law over the special linear group $SL_2(\mathbb{Z})$ with respect to k, its weight, and it must be differentiable on $\mathbb{H}$ while also having a finite limit as it approaches the cusp point ∞ (*holomorphic*). The defining properties of a modular form $f(z)$ imply that we may write $f(z) = \sum_{n=0}^{\infty} a(n)e^{2\pi izn}$. We make the change of variable $q = e^{2\pi iz}$ to obtain f as a q-series. We want to make use of q-series similar to Euler's which are also modular forms. Famous examples of note are the *Eisenstein series* and the subsequent *modular discriminant*. Let $k \geq 4$ and let $\sigma_k(n) = \sum_{0<d|n} d^k$. The Eisenstein series of weight k is given by

$$E_k(z) := 1 - \frac{2k}{B_k} \sum_{n=1}^{\infty} \sigma_{k-1}(n)q^n, \tag{2.1}$$

where B_k is the k^{th} Bernoulli number. In contrast, the modular discriminant holds a fixed weight of 12 and has a remarkable q-series interpretation:

$$\Delta(z) := \frac{E_4^3 - E_6^2}{1728} = q\prod_{n=1}^{\infty}(1-q^n)^{24} = \sum_{n=1}^{\infty} \tau(n)q^n. \tag{2.2}$$

The generating function $\tau(n)$ is known as the Ramanujan τ-function. Ramanujan's $\Delta(z)$ is in further contrast to the Eisenstein series in that it belongs to a stricter class of modular forms known as cusp forms. Cusp forms vanish at the cusp ∞, meaning they tend to zero as the function approaches infinity on the complex axis. We define S_k and M_k to be the respective sets of cusp and modular forms weight k in $SL_2(\mathbb{Z})$. We note that $S_k \subset M_k$ and that there are no modular forms of $SL_2(\mathbb{Z})$ with weights $k < 0$, k odd, or $k = 2$ besides the zero function. Juxtaposing all these weighty definitions, we now recall the convenient fact that M_k and S_k are finite-dimensional $\mathbb{C}$-vector spaces. It turns out that there are linear operators on these spaces whose action recombines modular form coefficients in useful ways. Our proof makes use of the linear *Hecke operator* because it is known to be locally nilpotent modulo 2 for modular forms on $SL_2(\mathbb{Z})$ with coefficients reduced modulo 2.

Definition 2.2. *For every prime p, there is a Hecke operator $T_{p,k} : M_k \mapsto M_k$ whose action is given by*

$$\left(\sum_{n=0}^{\infty} a(n)q^n\right) \mid T_{p,k} = \sum_{n=0}^{\infty} \left(a(pn) + p^{k-1}a\left(\frac{n}{p}\right)\right) q^n.$$

We note that if $p \nmid n$, then we have $a\left(\frac{n}{p}\right) = 0$.

Definition 2.3. *Suppose that $f(z) \in M_k$ is a modular form with $f(z) \not\equiv 0 \pmod 2$. We say that f has degree of nilpotency $i \geq 1$ if there are odd primes $p_1, p_2, ..., p_{i-1}$ for which*

$$f(z)|T_{p_1,k}|T_{p_2,k}|...|T_{p_{i-1},k} \not\equiv 0 \pmod 2$$

and for every collection of distinct odd primes $\ell_1, \ell_2, ..., \ell_i$

$$f(z)|T_{\ell_1,k}|T_{\ell_2,k}|...|T_{\ell_i,k} \equiv 0 \pmod 2.$$

A very recent discovery by Jean-Louis Nicolas and Jean-Pierre Serre has shown that linear combinations of $\Delta(z)$ modulo 2 have a degree of nilpotency which is easily computable.

Theorem 2.4 (Nicolas-Serre [7])**.** *Let $g(f)$ denote the degree of nilpotency of a cusp form $f \in \mathbb{F}_2[\Delta]$. Then we have $f \equiv \sum_{i=1}^{\ell} c_i \Delta^i(z) \pmod 2$. Let $\ell = \sum_{i=0}^{\infty} \beta_i 2^i$ be the binary expansion of ℓ. We define $h(\ell) := \sum_{i=0}^{\infty} \beta_{2i+1} 2^i + \sum_{i=0}^{\infty} \beta_{2i+2} 2^i$. Then for all $\ell \geq 1$, we have*

$$g(\Delta(z)^\ell) = 1 + h(\ell).$$

This has proven to be a powerful tool for proving congruence relations modulo 2 for modular forms on $SL_2(\mathbb{Z})$. As an inspiration for our proof, we look to work done on a conjecture of Hirschhorn and Sellers from the late 1990's.

Conjecture (Hirschhorn-Sellers [5])**.** *For all $t \geq 1$, let $a_t(n)$ be the generating function for the number of t-core partitions of $n \geq 1$. Then for $t \geq 2$, $n \geq 0$, and $k = 0, 2$, we have*

$$a_{2^t}\left(\frac{3^{2^{t-1}}(24n + 8k + 7) - \frac{4^t - 1}{3}}{8}\right) \equiv 0 \pmod 2.$$

Work of Boylan in 2000 [3] showed how to frame this conjecture in the context of modular forms and local nilpotency modulo 2. Later work by Chen in 2015 used Theorem 2.4 to build on Boylan's ideas and prove the conjecture. While we are not interested in t-core partitions here, the works of Boylan and Chen on this conjecture inform our approach to proving Theorem 1.2 on $p_r(n)$. Its generating function,

$$\sum_{n=0}^{\infty} p_r(n) q^n := \prod_{n=1}^{\infty} (1 - q^n)^r, \tag{2.3}$$

reveals an exploitable connection to $\Delta(z)$. We will move into the proof of our main theorem by establishing a congruence between r-colored partitions and $\Delta(z)$, proceeded by an application of nilpotency to uncover the key property that the theorem tracks. The process concludes with a tidying up of divisibility properties.

3. The proof of Theorem 1.2

Proof. We begin by demonstrating the congruence relation mod 2 between (2.2) and (2.3). We first take (2.3) and replace r with $2 \cdot 4^t + 1$:

$$\sum_{n=0}^{\infty} p_{2\cdot 4^t+1}(n) q^n = \prod_{n=1}^{\infty} (1 - q^n)^{2\cdot 4^t + 1}. \tag{3.1}$$

By (2.2), if $\ell_t = \frac{2\cdot 4^t+1}{3}$ then we have

$$\Delta(z)^{\ell_t} = q^{\ell_t}\prod_{n=1}^{\infty}(1-q^n)^{8(2\cdot 4^t+1)} := \sum_{n=0}^{\infty} b_t(n)q^n.$$

Thus if we raise (3.1) to the power of 8 and multiply it by q^{ℓ_t}, it follows from Fermat's Little Theorem that

$$\sum_{n=0}^{\infty} p_{2\cdot 4^t+1}(n)q^{8n+\ell_t} \equiv \Delta(z)^{\ell_t} \equiv \sum_{n=0}^{\infty} b_t(n)q^n \pmod 2.$$

Note that $p_{2\cdot 4^t+1}(n)$ is equivalent to either 0 or 1 mod 2. Raised to any power, both these values are preserved. Thus $p_{2\cdot 4^t+1}(n)^8 \equiv p_{2\cdot 4^t+1}(n) \pmod 2$. Comparing coefficients reveals that

$$b_t(n) \equiv p_{2\cdot 4^t+1}\left(\frac{n-\ell_t}{8}\right) \pmod 2.$$

Now let p be an odd prime with $p \nmid n$, and let $k_t = 12\ell_t$. We define integers $\beta_i(n)$ by

$$\begin{aligned}
\sum_{n=0}^{\infty}\beta_1(n)q^n &:= \Delta(z)^{\ell_t}|T_{p,k_t}\\
\sum_{n=0}^{\infty}\beta_2(n)q^n &:= \Delta(z)^{\ell_t}|T_{p,k_t}|T_{p,k_t}\\
\sum_{n=0}^{\infty}\beta_3(n)q^n &:= \Delta(z)^{\ell_t}|T_{p,k_t}|T_{p,k_t}|T_{p,k_t}\\
&\vdots\\
\sum_{n=0}^{\infty}\beta_{2^t}(n)q^n &:= \Delta(z)^{\ell_t}|T_{p,k_t}|T_{p,k_t}|...|T_{p,k_t}.
\end{aligned}$$

Applying Theorem 2.4, we observe that

$$\ell_t = \frac{2\cdot 4^t+1}{3} = 1 + 2\left(\frac{4^t-1}{3}\right) = 1 + 2(1 + 2^2 + 2^4 + ... + 2^{2t-2} = 1 + 2 + 2^3 + ... + 2^{2t-1},$$

$$\text{and hence that } h(\ell_t) = 1 + 2 + 2^2 + ... + 2^{t-1} = 2^t - 1.$$

Thus $\Delta(z)^{\ell_t}$ has degree of nilpotency 2^t, which means $\beta_{2^t}(n) \equiv 0 \pmod 2$. It follows from Definition 2.2 that

$$\sum_{n=0}^{\infty}\beta_{2^t}(n)q^n \equiv \sum_{n=0}^{\infty}\beta_{2^t-1}(n)q^n\big|T_{p,k_t} \equiv \sum_{n=0}^{\infty}\left(\beta_{2^t-1}(pn) + p^{k_t-1}\beta_{2^t-1}\left(\frac{n}{p}\right)\right)q^n \pmod 2.$$

For all n with $p \nmid n$, we have $\beta_{2^t}(n) \equiv \beta_{2^t-1}(pn) \pmod 2$. We iterate this process to find, for all n with $p \nmid n$, that

$$\beta_{2^t}(n) \equiv b_t(p^{2^t}n) \equiv p_{2\cdot 4^t+1}\left(\frac{p^{2^t}n-\ell_t}{8}\right) \equiv 0 \pmod 2.$$

It is known that $\Delta(z)^{2j} \equiv \Delta(2z)^j \pmod 2$, thus it suffices to consider t for which ℓ_t is odd. Observe, for all $t \geq 0$, that we have

$$2 \cdot 4^t + 1 \equiv 2 \cdot 1 + 1 \equiv 0 \pmod 3,$$

and

$$\ell_t \equiv 1 \pmod 2.$$

Therefore ℓ_t is an odd integer for all $t \geq 0$.

Now we want to modify n so that the argument of $p_{2\cdot 4^t+1}$ is always an integer. It is easy to show, for all $t \geq 1$, that $\ell_t \equiv 3 \pmod 8$. Thus we require, for all $t \geq 1$, that $p^{2^t}n \equiv \ell_t \equiv 3 \pmod 8$. Note, for all x odd, $x^2 \equiv 1 \pmod 8$. Thus for all t, we have $p^{2^t}n \equiv n \equiv 3 \pmod 8$.

Now if we want our variable n to be without restrictions, we must account for $n \not\equiv 0 \pmod p$. The Chinese Remainder Theorem generates $p-1$ solutions modulo $8p$. Since $n \equiv 3 \pmod 8$, there exists $x \in \mathbb{Z}$ with $n = 8x+3$. We let $y = \frac{8x+3}{p}$ where $0 \leq x \leq p-1$, and let $p \equiv i \pmod 8$ such that $p = 8m+i$ for some integer m. Thus

$$x = \frac{8ym+iy-3}{8} \leq p-1 = 8m+i-1,$$

$$\text{therefore } \; y \leq \frac{64m+8i-5}{8m+i} < \frac{64m+8i}{8m+i} = 8.$$

Observe $x \in \mathbb{Z}$ if and only if $iy \equiv 3 \pmod 8$. Since $1 \leq y < 8$, there exists a unique y such that $y \equiv 3i^{-1} \pmod 8$. Furthermore, there exists a unique x with $0 \leq x \leq p-1$ such that $\frac{8x+3}{p} = y \in \mathbb{Z}$ where

$$y = \begin{cases} 3 & \text{if } \; p \equiv 1 \pmod 8 \\ 1 & \text{if } \; p \equiv 3 \pmod 8 \\ 7 & \text{if } \; p \equiv 5 \pmod 8 \\ 5 & \text{if } \; p \equiv 7 \pmod 8. \end{cases}$$

From this, we conclude our main formula:

$$p_{2\cdot 4^t+1}\left(\frac{p^{2^t}(8pn+8k+3)-\ell_t}{8}\right) \equiv 0 \pmod 2 \;\; \forall t \geq 1$$

where $0 \leq k \leq p-1$ with $k \neq \frac{yp-3}{8}$.

We note that this restriction on k may be simplified to $k \not\equiv \frac{3(p^2-1)}{8} \pmod p$. We observe that our requirement $8x+3 \not\equiv 0 \pmod p$ implies that $x \not\equiv -3(8^{-1})$. Since p is odd, we have $\frac{p^2-1}{8} \in \mathbb{Z}$. We observe that

$$\frac{p^2-1}{8} \equiv (p^2-1)8^{-1} \equiv -8^{-1} \pmod p,$$

$$\therefore x \not\equiv -3(8^{-1}) \longrightarrow x \not\equiv \frac{3(p^2-1)}{8} \pmod p.$$

It remains to show that the case of Remark 1.3 holds as well. We take (2.3) and replace r with 3. It is known to follow from the Jacobi Triple Product Theorem that

$$\sum_{n=0}^{\infty} p_3(n)q^n = \prod_{n=1}^{\infty}(1-q^n)^3 = \sum_{m=0}^{\infty}(-1)^m(2m+1)q^{\frac{m^2+m}{2}}.$$

We reduce modulo 2, revealing that

$$\sum_{n=0}^{\infty} p_3(n)q^n \equiv \sum_{m=0}^{\infty} q^{\frac{m^2+m}{2}} \pmod 2.$$

The remark follows by comparing coefficients.

□

4. Other Congruent Generating Functions

It should be stressed just how flexible this process is for generating functions with similar q-series. We demonstrate this in the following remark.

Remark 4.1. *Let $\sum_{n=0}^{\infty} v(n)q^n := \prod_{n=1}^{\infty}(1-q^{t_1 n})^{r_1}(1-q^{t_2 n})^{r_2}...(1-q^{t_k n})^{r_k}$ with $r = r_1 + r_2 + ... + r_k$. Then $v(n) = v_e(n) - v_o(n)$ where $v_e(n)$ and $v_o(n)$ denote the number of r-colored partitions into an even (respectively, odd) number of distinct parts with the following restrictions:*

(1) *parts of the first r_1 colors are multiples of t_1*
(2) *parts of the next r_2 colors are multiples of t_2*
$\vdots$
(k) *parts of the last r_k colors are multiples of t_k.*

Now observe that, following from Fermat's little theorem, we have

$$\begin{aligned}
\sum_{n=0}^{\infty} u(n)q^n &\equiv \prod_{n=1}^{\infty}(1-q^{4^t n})^2(1-q^n) \\
\sum_{n=0}^{\infty} w(n)q^n &\equiv \prod_{n=1}^{\infty}(1-q^{2n})^{4^t}(1-q^n) \\
&\equiv \prod_{n=1}^{\infty}(1-q^n)^{2\cdot 4^t}(1-q^n) \\
&\equiv \prod_{n=1}^{\infty}(1-q^n)^{2\cdot 4^t+1} \pmod 2 \\
&\equiv \sum_{n=0}^{\infty} p_{2\cdot 4^t+1}(n)q^n \pmod 2.
\end{aligned}$$

Thus we can produce the same formula as in Theorem 1.2 for stricter forms of colored partitions. For example, $u(n)$ represents 3-colored partitions where the parts of the first 2 colors are multiples of 4^t and $w(n)$ represents $(4^t + 1)$-colored partitions where the parts of the first 4^t colors are even.

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