A New Improvement of a Fibonacci Inequality

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This paper provides an improvement to an inequality involving the Fibonacci sequence contained in [1,2].

1 Introduction

Throughout this article, the symbol F_n is the nth Fibonacci number $(n \geq 2)$. In [2] the following inequality is given

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4)\left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right) > 81. \tag{1}$$

In [1], an improvement to the inequality (1) is given as follows

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4)\left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right) > 100. \tag{2}$$

In this article, we present an improvement to the inequality (2) as follows

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4)\left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2}\right) > 158. \tag{3}$$

2 Some lemmas

In this section, we introduce a few lemmas aimed at obtaining (3).

Lemma 1. $\frac{F_{n+1}}{F_n} > 1$.

Proof. We have

$$\frac{F_{n+1}}{F_n} = \frac{F_n + F_{n-1}}{F_n} > 1.$$

□

Lemma 2. $\frac{F_n}{F_{n+1}} > \frac{1}{2}$.

Proof. We have

$$\frac{F_n}{F_{n+1}} = \frac{F_n}{F_n + F_{n-1}} > \frac{F_n}{2F_n} = \frac{1}{2}.$$

□

Lemma 3. $\frac{F_{n+2}}{F_n} > 2$.

Proof. We have
$$\frac{F_{n+2}}{F_n} = \frac{F_{n+1}+F_n}{F_n} = \frac{2F_n+F_{n-1}}{F_n} > 2.$$
□

Lemma 4. $\frac{F_{n+2}^2}{F_nF_{n+1}} > 4.$

Proof. By Cauchy-Swartz inequality, we have
$$\frac{F_{n+2}^2}{F_nF_{n+1}} = \frac{(F_n+F_{n+1})^2}{F_nF_{n+1}} > 4.$$
□

Lemma 5 (Extension of Candido's identity)**.** Let F_n be nth Fibonacci number. Then
$$F_n^2F_{n+1}^2 + F_{n+1}^2F_{n+2}^2 + F_{n+2}^2F_n^2 = \frac{F_n^4+F_{n+1}^4+F_{n+2}^4}{2}.$$

Proof. From Candido's identity (see [3, 4, 5]), we have
$$(F_n^2+F_{n+1}^2+F_{n+2}^2)^2 = 2(F_n^4+F_{n+1}^4+F_{n+2}^4)$$
which is equivalent to
$$2(F_n^2F_{n+1}^2 + F_{n+1}^2F_{n+2}^2 + F_{n+2}^2F_n^2) = F_n^4+F_{n+1}^4+F_{n+2}^4$$
or
$$F_n^2F_{n+1}^2 + F_{n+1}^2F_{n+2}^2 + F_{n+2}^2F_n^2 = \frac{F_n^4+F_{n+1}^4+F_{n+2}^4}{2}.$$
□

3 Main proof

From Lemma 5, we have
$$\begin{aligned}
&2(F_n^4+F_{n+1}^4+F_{n+2}^4)\left(\frac{1}{F_n^2}+\frac{1}{F_{n+1}^2}+\frac{1}{F_{n+2}^2}\right)^2 \\
&= 2(F_n^4+F_{n+1}^4+F_{n+2}^4)\frac{(F_n^2F_{n+1}^2 + F_{n+1}^2F_{n+2}^2 + F_{n+2}^2F_n^2)^2}{F_n^4F_{n+1}^4F_{n+2}^4} \\
&= \frac{(F_n^4+F_{n+1}^4+F_{n+2}^4)^3}{2F_n^4F_{n+1}^4F_{n+2}^4}.
\end{aligned} \tag{4}$$

Using the expansion
$$\begin{aligned}
&\frac{(a+b+c)^3}{abc} \\
&= \frac{a^3+b^3+c^3+6abc+3ab(a+b)+3bc(b+c)+3ca(c+a)}{abc} \\
&= 6+\left(\frac{a^2}{bc}+\frac{b^2}{ca}+\frac{c^2}{ab}\right)+3\left(\frac{a}{b}+\frac{b}{a}+\frac{b}{c}+\frac{c}{b}+\frac{c}{a}+\frac{a}{c}\right).
\end{aligned}$$

Take $a = F_n^4, b = F_{n+1}^4, c = F_{n+2}^4$, we get

$$\begin{aligned}
&\frac{(F_n^4 + F_{n+1}^4 + F_{n+2}^4)^3}{F_n^4 F_{n+1}^4 F_{n+2}^4} \\
&= 6 + \left(\left(\frac{F_n^2}{F_{n+1}F_{n+2}} \right)^4 + \left(\frac{F_{n+1}^2}{F_nF_{n+2}} \right)^4 + \left(\frac{F_{n+2}^2}{F_nF_{n+1}} \right)^4 \right) + \\
&+ 3 \left(\frac{F_n^4}{F_{n+1}^4} + \frac{F_{n+1}^4}{F_n^4} + \frac{F_{n+1}^4}{F_{n+2}^4} + \frac{F_{n+2}^4}{F_{n+1}^4} + \frac{F_n^4}{F_{n+2}^4} + \frac{F_{n+2}^4}{F_n^4} \right).
\end{aligned} \tag{5}$$

From Lemma 4, we have

$$\left(\frac{F_{n+2}^2}{F_nF_{n+1}} \right)^4 > 4^4 = 256. \tag{6}$$

From Lemma 3, we have

$$\left(\frac{F_{n+2}^2}{F_n} \right)^4 > 2^4 = 16. \tag{7}$$

From Lemma 1 and Lemma 2, we have

$$\left(\frac{F_{n+1}^2}{F_n} \right)^4 > 1, \ \left(\frac{F_{n+2}^2}{F_{n+1}} \right)^4 > 1, \ \left(\frac{F_n^2}{F_{n+1}} \right)^4 > \frac{1}{16}, \left(\frac{F_{n+1}^2}{F_{n+2}} \right)^4 > \frac{1}{16}. \tag{8}$$

Using (6), (7), (8) to (5), we get

$$\frac{(F_n^4 + F_{n+1}^4 + F_{n+2}^4)^3}{F_n^4 F_{n+1}^4 F_{n+2}^4} > 6 + 256 + 3 \left(16 + 1 + 1 + \frac{1}{16} + \frac{1}{16} \right) = \frac{2531}{8}. \tag{9}$$

Thus, from (9) and (4), we obtain

$$2(F_n^4 + F_{n+1}^4 + F_{n+2}^4) \left(\frac{1}{F_n^2} + \frac{1}{F_{n+1}^2} + \frac{1}{F_{n+2}^2} \right)^2 > \frac{2531}{16} > 158.$$

We complete the proof.

References

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